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AHLEM ALOUANI

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Lahcène Mezrag	**Prof.**	**Univ. Med Boudiaf- M'Sila**	**Président**
Dahmane Achour	**Prof.**	**Univ. Med Boudiaf- M'Sila**	**Rapporteur**
Pilar Rueda	**Prof.**	**Univ. de Valencia, Espagne**	**Co-rapporteur**
Elhadj Dahia	**MCA.**	**Ecole Normale Supérieure - Bou Saâda**	**Examinateur**
Amar Belacel	**MCA.**	**Univ. Amar Telidji-Laghouat**	**Examinateur**
Abdelkader Dehici	**Prof.**	**Univ. Mohamed Chérif Messaadia -Souk-Ahras**	**Examinateur**

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Contents

ملخص:

مضمون هذه الرسالة، يتمحور حول عديد من المفاهيم للمؤثرات الجمعية في الحالة الغير الخطية كالمؤثرات المتعددة الخطية و كثيرات الحدود للمؤثرات المتعددة الخطية. في الفصل الثاني قدمنا فئة المؤثرات المتعددة الخطية من نوع كوهان p-نكليار المعرفة بين فضاءات بناخ باعتبارها نسخة موسعة للحالة الخطية. هذا التنوع في المؤثرات المتعددة الخطية المتناظرة سمح لنا بتقديم صنف جديد كثير حدود p-نكليار هذا الأخير تم استخدمه كمثال توضيحي في الفصل 4 أهم النتائج المثبتة في هذا الفصل كالتالي : تطبيق لنظرية الهيمنة لبيتش و نظرية التفكيك على اعتبارها تمديد لنظرية التفكيك ل كوابن. - العلاقة مع المؤثرات المتعددة الخطية الجمعية. كما يجدر بالذكر ومتابعة للنتائج التي تم التوصل إليها، أثبتنا أن المؤثرات المتعدد الخطة من نوع كوهان p-نكليار ضعيفة التراص • الفصل الثالث يعالج فضاء المتتاليات من خلال المؤثرات المتعددة الخطية و كثيرات الحدود للمؤثرات المتعددة الخطية المتجانسة • مؤثر مثالي جمعي يمكن أن نميزه باستخدام الاستمرارية لمؤثر موتر مرفق له ,هذا المؤثر الموتر معرف بين فضاء بناخ للمتتاليات .هدفنا هو توفير دراسة شاملة لهذه المميزات للمؤثرات الجمعية حيث نعمل في نطاق أوسع يضم المؤثرات المتعددة الخطية المتناظرة. أمثلة تطبيقية قدمت • في الفصل الرابع أعطينا خاصية مكافئة لكثيرات الحدود للمؤثرات المتعددة الخطية المتجانسة المعرفة بواسطة السلاسل الجمعية والتي يكون المؤثر الخطي المرفق لها من نوع كوهان p-نكليار . نختم بإيجاد علاقة قوية مع كثيرات الحدود للمؤثرات المتعددة الخطية المتجانسة التكاملية من نوع قروتنديك.

كلمات مفتاحيه. المؤثرات المتعددة الخطية. الجداء الموتر. كثيرات الحدود للمؤثرات المتعددة الخطية. كثير حدود p-نكليار. كثير حدود p-نكليار التفكيكي.

Abstract

The present thesis is devoted to summing non linear operators. We focus our attention on introducing and studying polynomials and multilinear mappings that share good properties of summability with distinguished classes of summing linear operators. In the second chapter we introduce the class of Cohen p-nuclear m-linear operators between Banach spaces. This is the multilinear version of p-nuclear operators. The polynomial variant is obtained thanks to consider the symmetric multilinear mapping associated to the polynomial. This polynomial variant forms the p-nuclear polynomial, and it is used as an illustrative example also in Chapter IV. The main results proved in Chapter II are: a characterization in terms of Pietsch's domination theorem and the related factorization theorem, which is an extension to the multilinear setting of Kwapien's factorization theorem for dominated linear operators. Connections with the theory of absolutely summing m-linear operators are also established. It is worth mentioning that, as a consequence of our results, we show that every Cohen p-nuclear m-linear mapping on arbitrary Banach spaces is weakly compact. The third chapter deals with transformations of sequences via summing nonlinear operators. Operators Ṭthat belong to some summing operator ideal can be characterized by means of the continuity of an associated tensor operator Ṭthat is defined between tensor products of sequences spaces. Our aim is to provide a unifying treatment of these tensor product characterizations of summing operators. We work in the more general frame, provided by homogeneous polynomials, where an associated tensor polynomial which plays the role of T, needs to be determined first. Examples of applications are shown. In Chapter IV we characterize in terms of summability those homogeneous polynomials whose linearization is p-nuclear. This characterization provides a strong link between the theory of p-nuclear linear operators and the (non linear) homogeneous p-nuclear polynomials that significantly improves former approaches. The deep connection with Grothendieck integral polynomials is also analyzed.

Keywords: multilinear operator, tensor product, m-homogeneous polynomial, p-nuclear operator, factorable strongly p-nuclear nuclear.

Résumé

Cette thèse est consacrée aux opérateurs non linéaires sommant. Nous concentrons notre attention sur l'introduction et l'étude des polynômes et des applications m-linéaires partageant les bonnes propriétés de sommabilité avec les classes distinguées des opérateurs linéaires sommant. Dans le deuxième chapitre, nous introduisons la classe des opérateurs m-linéaires Cohen p-nucléaires entre les espaces de Banach, qui est la version multilinéaire des opérateurs p-nucléaires. La variante polynomiale est obtenue en tenant compte l'application m-linéaire symétrique associée au polynôme, cette variante forme le polynôme p-nucléaire, qui est également utilisé à titre d'exemple explicatif au chapitre IV; les principaux résultats démontrés au chapitre II sont: une caractérisation en termes du Théorème de Domination de Pietsch et du Théorème de Factorisation, qui est une extension du Théorème de Factorisation de Kwapien pour les opérateurs linéaires dominés. Une connexion avec les opérateurs m-linéaires absolument sommant est également établie. Il est important de mentionner que selon nos résultats et comme conséquence, nous montrons que tout operateur linéaire Cohen p-nucléaire ṃ-linéaire entre, les espaces de Banach est faiblement compact. On a traité dans le troisième chapitre les transformations des suites à travers les opérateurs non linéaires sommant. Un opérateur idéal sommant Ṭse caractérise par la continuité d'un opérateur tensoriel associé T, lequel est défini entre le produit tensoriel des espaces, de suites. Notre objectif est d'offrir un traitement uniforme pour ces caractérisations tensorielles des opérateurs sommant. On travail dans un cadre plus général fourni par les polynômes homogènes, où un polynôme tensoriel associé jouant le rôle de T, doit être déterminé préalablement. Des exemples d'applications sont présentés. Au chapitre IV, nous décrivons en termes de sommabilité les polynômes homogènes dont leurs linéarisations sont p-nucléaire. Cette caractérisation fournit un lien fort entre la théorie des opérateurs linéaires p-nucléaires et la théorie des polynômes homogènes p-nucléaires (non linéaires) qui améliore considérablement les anciennes approches. Une connexion distinguée avec les polynômes Grothendieck intégrale est également analysée.

Mots-clés: opérateur multilinéaire, produit tensoriel, polynôme m-homogène, opérateur p-nucléaire, operateur factorable fortement p-nucléaire.

Notation

$\mathbb{K}$	The field of real or complex numbers.
p^*	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p^*} = 1$
X^*	The topological dual of X.
B_X	The closed unit ball of X
$L(X;Y)$	The set of all linear operators.
$\mathcal{L}(X;Y)$	The set of all continuous linear operators.
w	The weak topology.
w^*	The weak * topology.
$\mathcal{I}$	Ideal of linear operator.
T^*	The adjoint operator of T.
T_L	The linearization of the operator T.
$\check{P}$	The associated symmetric m-linear operators of polynomial P.
$I_{p,\infty}$	The formal inclusion map defined between $L_\infty(\mu)$ and $L_p(\mu)$ $(1 \leq p < \infty)$.
i_X	The isometric embedding, $i_X : X \longrightarrow C(X^*)$ given by $i_X(x) := \langle x, . \rangle$.
$\mathcal{K}$	The set of all compact linear operators.
$\mathcal{W}$	The set of all weakly compact linear operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.
Π_p	The set of all linear p-summing operators $(1 \leq p < \infty)$.
$\mathcal{D}_p$	The set of all strongly linear p-summing operators $(1 < p \leq \infty)$.
$\mathcal{N}_p$	The set of all Cohen p-nuclear linear operators $(1 \leq p < \infty)$.
$\mathcal{N}_p^m$	The set of all Cohen p-nuclear multilinear operators $(1 \leq p < \infty)$.
$\mathcal{P}_{p,N}^c$	The set of all Cohen p-nuclear polynomials).
$\mathcal{P}_{F\ St,p}$	The set of all factorable strongly p-summing polynomials$(1 < p \leq \infty)$.
$\mathcal{L}_{p,N}^{fs}$	The set of all factorable strongly p-nuclear operators $(1 < p \leq \infty)$.
$\mathcal{P}_{p,N}^{fs}$	The set of all factorable strongly p-nuclear polynomials $(1 < p \leq \infty)$.

List of publications

1. D. Achour, A. Alouani, P. Rueda and K. Saadi, Factorable strongly p-nuclear m-homogeneous polynomial. Revista de la Real Academia de Ciencias Exactas,fisicas y Naturales Serie A.Mathematicas (2018), 10.1007/s13398-018-0530-z .

2. D. Achour, A. Alouani, P. Rueda and E.A. Sánchez-Pérez, Tensor Characterizations of Summing Polynomials. Mediterr. J. Math. 15 (2018), 127.

Introduction

The nonlinear theory of summability, specially addressed to the classes of multilinear and homogeneous polynomial mappings between Banach spaces, has been a permanent topic of investigation, developed by many authors as Alencar, Botelho, Cillia, Geiss, Gutiérrez, Matos, Pellegrino, Pérez-García, Pietsch, Villanueva... Actually it was Pietsch [52] who conceived the basis of a research program on summing multilinear mappings where he began this study under the inspiration of the ideas and techniques , derived from the theory of linear operators ideals. The 1989 paper by Alencar-Matos is other cornerstone in this line of thought. This approach turned out to be very successful and a number of operators ideals have been fruitfully generalized to the multilinear and polynomial settings. It must be clear that there is not a unique way to generalize a given operator ideal to polynomials and multilinear mappings. This has caused the appearance of several works that compare different classes of summing non linear ideals, as in [49], [54] or [26]. It is noteworthy that even if Pietsch deeply studied linear operator ideals without using tensor product terminology, tensor products have proved to be a useful tool for the theory of operator ideals. Indeed, the excellent monograph [29] deals with the tensor product point of view of the theory and provides many applications to the study of the structure of several spaces of summing linear operators. In the last decades this linear theory has spreaded to non-linear contexts that include multilinear mappings, polynomials, holomorphic functions or Lipschitz mappings among others. Transferring summability properties to nonlinear mappings is not an obvious task as shows the variety of different generalizations of several classes of summing operators to the multilinear case, and to hit the multilinear class that is closest, in some sense, to the original linear class is not trivial (see e.g. [20, 17, 49, 54]). Even more complicated is working with polynomials, where important lacks of basic results, as a Pietsch type factorization theorem for dominated polynomials, proved deep differences between the linear and the polynomial theories (e.g. [18, 15, 57]). The way that summing linear and multilinear mappings transform vector-valued sequences is the essence of the theory of summing operators. Botelho and Campos [14] show how these transformations can be treated from an unified point of view, and recover in detail some former inaccuracies that have appeared in the literature. If $S(X)$ denotes a X-valued sequence space, the summability with respect to S of an operator $T : X \to Y$ (being X and Y Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is related to the associated operator $\widehat{T} : S(X) \to S(Y)$ given by $\widehat{T}((x_i)_i) := (T(x_i))_i$. Indeed, Botelho and Campos [14] have unified these results by analyzing the transformations of vector-valued sequences by multilinear operators. In this thesis we provide the tensor product counterpart of this unifying approach in the context of polynomials, more precisely, the classes of p-dominated polynomials (see e.g. [13]), Cohen strongly p-summing polynomials introduced by Achour and Saadi in [7], and the class of Cohen p-nuclear polynomial defined by Achour, Alouani, Rueda and Sánchez-Pérez in [4]. One of the main problems when dealing with the nonlinear theory of summability is that linear properties are not inherited in general by non linear maps. This has caused the appearance of several different of approaching the linear theory to a non linear context. A good example of such a lack of multilinear extensions of linear properties concerning summability, are the well-known factorization theorems for summing linear operators, that have no counterpart in the polynomial setting. It was not until[54] with the introduction of factorable strongly summing multilinear mappings, and in general, homogeneous polynomials, that this objective was accomplished. This class of factorable strongly summing non linear operators is a distinguished subclass of p-summing multilinear operators/polynomials modeled in [57], and it shares the main linear properties at once, such as Grothendieck's Theorem, Pietsch Domination Theorem, Dvoretzky–Rogers Theorem, weak compactness and a natural factorization theorem through subspaces of L_p-spaces in the manner of the Pietsch factorization theorem. Inspired by this success, we apply the ideas of factorable strongly summing polynomials/multilinear mappings to improve the already known class of p-nuclear multilinear mappings (see [2, 6]) that generalize the p-nuclear linear operators introduced by Cohen in [28]. We will mainly work in the more general context of homogeneous polynomials rather than multilinear mappings. Of course all the results are true in the multilinear context. The symbiosis between factorable strongly and nuclear polynomials allows to get the fundamental link with the linear theory via the linearization of polynomials: a homogeneous polynomial is factorable strongly p-nuclear if and only if its linearization is a p-nuclear operator (Theorem 4.18). Now, new results merge in the non linear theory. For instance, the adjoint of a factorable strongly p-nuclear homogeneous polynomial is a p^*-nuclear operator, where p^* is the conjugate of p (*i.e.*, $\frac{1}{p} + \frac{1}{p^*} = 1$). Another consequence is that factorable strongly p-nuclear homogeneous polynomials are factorable strongly p-summing and so, weakly compact. We end this research showing the connection between nuclearity and integrability in this nonlinear context. Our main tool is the study of integrability that was done by Cilia and Gutiérrez in [25] in the frame of vector valued homogeneous polynomials. Integral polynomials with scalar values were introduced by Dineen when studying duality in the Theory of Holomorphic Types. There are two ways of considering integral polynomials in the vector case: Grothendieck-integral polynomials and Pietsch-integral polynomials (respectively G-integral and P-integral for short). The notion of P-integrability is stronger than the notion of G-integrability. The relation of P-integral polynomials with summing polynomials was studied in [41], where it was proved that the space of all scalar P-integral m-homogeneous polynomials is isometrically isomorphic to the space of all factorable strongly summing m-homogeneous polynomials. Some partial results were also proved for vector-valued homogeneous polynomials with domain in a $C(K)$ space. Now, we are interested in G-integral polynomials. We establish the link between integrability and p-nuclearity in the vector valued polynomial case. Grothendieck-integral polynomials are proved to be the main examples of factorable strongly p-nuclear polynomials and, in some cases, we get the equality between both classes of polynomials.

The thesis consists of four chapters. In the first Chapter we establish the notation of the thesis. We introduce some important results concerning sequences Banach spaces and we recall the main definitions and properties of nonlinear application more precisely multilinear/polynomial applications and m-fold tensor products of Banach spaces with $m > 2$, we recall the most important results of the theory of operator ideals that we will generalized later as the classes of the p-summing and strongly p-summing and p-nuclear linear mappings. The second chapter of this thesis is dedicated to a generalization of the class oh p-nuclear to the cases multilinear and polynomials. This chapter is divided into three parts. In the first part we recall the basic concepts on the theory of multi-ideals when we give some examples of multi -ideals as p-dominated, absolutely $(p; q_1, , q_m; r)$-summing multilinear operators and the last example see[2] is our generalization to the multilinear context of p-nuclear operators. In the second part we introduce and study a new concept of the summability applications which is a generalization of p-nuclear mapping to polynomial version, and we finish the chapter with the domination and factorization theorems. In the third chapter titled by "Tensor characterizations of summing polynomials" we give characterization of operators T that belong to some summing operator ideal, by means of the continuity of an associated tensor operator $\overline{T}$ that is defined between tensor products of sequences spaces. After developing some techniques for well defining an associated polynomial between tensor products of sequences spaces we finally give an application of this procedure on particular classes of polynomials. The end of this research and work is the fourth chapter, where we open the way to the linearization for nonlinear applications by means of the new concepts of factorable strongly p-summing and factorable strongly p-nuclear polynomials. It is of special relevance that an m-homogeneous polynomial is factorable strongly p-summing if and only if its linearization is absolutely p-summing. The same result is obtained for the class factorable strongly p-nuclear polynomials. In addition, more related results are proved. We begin introducing and analyzing the class of factorable strongly p-summing polynomials. The heart of this study is the theorem of linearization. In the Section 2, strongly p-nuclear and factorable strongly p-nuclear polynomials are introduced and analyzed. Connections to summing polynomials are established and some fundamental properties are obtained. Regardin the linearization of the polynomial, we prove that factorable strongly p-nuclear homogeneous polynomials are those that factor through p-nuclear linear operators. We also relate them with factorable strongly p-summing polynomials, and prove a domination inequality that characterize the class. In Section 3 we prove that a homogeneous polynomial is factorable strongly p-nuclear if, and only if, its adjoint is a p^*-nuclear operator, where p^* is the conjugate of p. We also describe the space of all factorable strongly p-nuclear m-homogeneous polynomials as the dual of a suitable tensor product space. We end this chapter with Section 4, where we characterize G-integral homogeneous polynomials as those that factor through an integral linear operator. As a consequence, we show that a m-homogeneous polynomial P is G-integral if, and only if, its adjoint P^* is G-integral. We use this result to prove that if the dual of the range space is a $\mathcal{L}_{p,\lambda}$ space then, the spaces of factorable strongly p-nuclear polynomials and the space of G-integral polynomials coincide. In particular, being G-integral or factorable strongly 2-nuclear is the same for any homogeneous polynomial with range in a Hilbert space. We also prove that every G-integral homogeneous polynomial is factorable strongly p-nuclear.

Chapter 1

Multilinear mappings and polynomials

In this chapter we will be concerned by the basic properties of various spaces with values in a normed linear space X used throughout the thesis. These sequence spaces will appear in the definitions of the both classes linear and nonlinear operators, studied and treated right here us first part. In the second part we will recall some fundamental definitions, properties in nonlinear mappings (multilinear, m-homogeneous polynomial) and in tensor product; we end this chapter by given various examples of ideals of linear summing operators.

1.1 Notation and background

Throughout this chapter X, Y will denoted vector spaces over a field $\mathbb{K}$ which may be either the real or complex numbers. A Banach space is a complete normed vector space. The closed unit ball of X is denote by B_X.

A linear map $T : X \to Y$ is continuous if and only if

$$\|T\| = \sup \{\|T(x)\| : \|x)\| \leq 1, x \in X\} < \infty,$$

this value is a norm on the vector space $L(X,\ Y)$ of all linear operators from X into Y. We denoted by $\mathcal{L}(X,\ Y)$ the space of continuous linear operators, and by X^* the dual space of X, the norm of $x^* \in X^*$ is given by

$$\|x^*\| = \sup \{| \langle x^*, x\rangle | : x \in B_X\}.$$

We will denoted by (Ω, Σ, μ) a measure space, if $\mu(\Omega) = 1$ then is (Ω, Σ, μ) called probability space. For any measure space (Ω, Σ, μ) we define the space $L_p(\mu) = L_p(\Omega, \Sigma, \mu)$,

$1 \leq p \leq \infty$, to be the space of Σ-measurable functions such that $\int |f(w)|^p \, d\mu(w) < \infty$, we mean $\sup ess |f(w)| < \infty$. We will use the notation

$$\|f\|_p = \left(\int |f(w)|^p \, d\mu(w) \right)^{\frac{1}{p}} \text{ for } p \geq 1, \text{ and } \|f\|_\infty = \sup ess |f(w)| .$$

We know that:

1. A linear operator $T : X \to Y$ is invertible if there exist a linear operator noted $T^{-1} : Y \to X$ such that $T^{-1} \circ T = id_X$, and $T \circ T^{-1} = id_Y$, where id_X (or id_Y) is identity operator on X (or on Y).

2. A linear operator $T : X \to Y$ between two normed spaces X and Y is isomorphism if T is a continuous bijection whose inverse T^{-1} is also continuous. In this case, the spaces X and Y are isomorphism in addition if $\|T(x)\| = \|x\|$, for all $x \in X$. Then T be isometric isomorphism.

3. Let $T : X \to Y$ be continuous linear operator. Then the continuous linear operator $T^* : Y^* \to X^*$ defined by

$$T^*(y^*)(x) = y^* T(x)$$

for every $y^* \in Y^*$ and $x \in X$ is called the adjoint of T and he have the property

$$\|T\| = \|T^*\| .$$

Also we have $(T \circ u)^* = u^* \circ T^*$ for all continuous linear operator u.

Let X be a Banach space and $1 \leq p \leq \infty$, p^* is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. We denote by $\ell_p(X)$, the space of all *absolutely p-summable sequences* in X; that is, sequences $(x_i)_i$ in X such that

$$\|(x_i)_i\|_p := (\sum_i \|x_i\|^p)^{1/p} < \infty,$$

if $1 \leq p < \infty$ or,

$$\|(x_i)_i\|_\infty := \sup_i \|x_i\|,$$

if $p = \infty$.
$\ell_p^w(X)$, the space of all *weakly p-summable sequences* in X; that is, sequences $(x_i)_i$ in X such that

$$\|(x_i)_i\|_{p,w} := \sup_{x^* \in X^*, \|x^*\| \leq 1} (\sum_i |x^*(x_i)|^p)^{1/p} < \infty,$$

if $1 \leq p < \infty$ or,

$$\|(x_i)_i\|_{\infty,w} := \sup_i \sup_{x^* \in X^*, \|x^*\| \leq 1} |x^*(x_i)| = \sup_i \|x_i\|,$$

if $p = \infty$,

where X^* denotes the topological dual of X. The closed unit ball of X will be denoted B_X.

Note that $\ell_p(X)$ is a linear subspace of $\ell_p^w(X)$ and

$$\|(x_i)_i\|_{p,w} \leq \|(x_i)_i\|_p \text{ for all } (x_i)_i \in \ell_p(X).$$

Then $\ell_p(X) = \ell_p^w(X)$ for some $1 \leq p < \infty$ if, and only if, $\dim(X) < \infty$. If $p = \infty$, we have $\ell_\infty(X) = \ell_\infty^w(X)$, also if we take $X = \mathbb{K}$ (or $n = 1$), then the spaces $\ell_p(\mathbb{K})$ and $\ell_p^w(\mathbb{K})$ coincide

A sequences $(x_i)_{i=1}^\infty$in X is said to be *unconditionally p-summable* if

$$\lim_{n \to \infty} \|(x_i)_{i=n}^\infty\|_{p,\omega} = 0.$$

We denote by $\ell_p\langle X\rangle$, the space of all *strongly p-summable sequences* in X; that is, sequences $(x_i)_i$ in X such that

$$\|(x_i)_i\|_{\ell_p\langle X\rangle} := \sup_{\|(x_i^*)_i\|_{p^*,\omega} \leq 1} \left| \sum_i x_i^*(x_i) \right| < \infty.$$

It well know that for $1 \leq p < \infty$ and $(\varphi_i)_{i=1}^n \in \ell_{p^*,\omega}^n(Y^*)$ we have

$$\|(\varphi_i)_i\|_{p^*,\omega} = \sup_{\psi \in B_{Y^{**}}} \left(\sum_{i=1}^n |\psi(\varphi_i)|^{p^*} \right)^{\frac{1}{p^*}} = \sup_{y \in B_Y} \|(\varphi_i(y))_i\|_{p^*}.$$

If u is a linear bounded operator from X into Banach Y for $\varphi_i \in Y^*$, $i = 1, ..., n$ we have

$$\|(\varphi_i \circ u)_i\|_{p,\omega} \leq \|u\| \|(\varphi_i)_i\|_{p,\omega}. \tag{1.1}$$

1.2 Multilinear mappings

Let $m \in \mathbb{N}$ and $X_j \, (1 \leq j \leq m)$ be a Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ ou $\mathbb{C}$). We consider the Cartesian product

$$X_1 \times ... \times X_m = \left\{ \left(x^1, ..., x^m \right) : x^j \in X_j, \ \forall 1 \leq j \leq m \right\},$$

which is a normed space equipped with the norm

$$\left\| \left(x^1, ..., x^m \right) \right\| := \max \left\{ \left\| x^j \right\| ; \ x^j \in X_j, \forall 1 \leq j \leq m \right\}. \tag{1.2}$$

Definition 1.1. *An application $T : X_1 \times ... \times X_m \to Y$ is called multilinear (or m-linear) if the mappings*

$$\begin{array}{rccl} T_j : & X_j & \to & Y \\ & x^j & \mapsto & T\left(x^1, ..., x^j, ..., x^m\right), \end{array}$$

are linear for each $x^k \in X_k$, $k \neq j$, in other words

$$T\left(x^1, ..., \lambda x^j + y^j, ..., x^m\right) = \lambda T\left(x^1, ..., x^j, ..., x^m\right) + T\left(x^1, ..., y^j, ..., x^m\right);$$

for all $\lambda \in \mathbb{K}$ and x^j, $y^j \in X_j$ $(1 \leq j \leq m)$, we denoted by $L(X_1, ..., X_m; Y)$ the space of all m-linear applications T from $X_1 \times ... \times X_m$ into Y. The set $\mathcal{S}$ of all vectors in Y of the form $T(x^1, ..., x^m)$, $x^j \in X_j$ $(1 \leq j \leq m)$ is not in general vector subspace of Y. (see [30, Section 1.1]).

Now we define the following linear operations

$$\begin{aligned} (S + T)\left(x^1, ..., x^m\right) &:= S\left(x^1, ..., x^m\right) + T\left(x^1, ..., x^m\right) \\ (\lambda T)\left(x^1, ..., x^m\right) &:= \lambda T\left(x^1, ..., x^m\right), \lambda \in \mathbb{K} \end{aligned}$$

which gives to $L(X_1, ..., X_m; Y)$ a structure of a vector space. If $Y = \mathbb{K}$, we write $L(X_1, ..., X_m)$.

Definition 1.2. *The multilinear application $T : X_1 \times ... \times X_m \to Y$ is continuous if it is continuous as a function between two normed spaces.*

As a consequence of this definition, and the following equality

$$T\left(x^1, ..., x^m\right) - T\left(y^1, ..., y^m\right) = T\left(x^1 - y^1, ..., x^m\right) + T\left(x^1, x^2 - y^2, ..., x^m\right) + ... + T\left(x^1, ..., x^m - y^m\right),$$

we have a result that gives a characterization of continuous m-linear mapping.

Proposition 1.3. *Let $X_1, ..., X_m$, Y be normed spaces. For all $T \in L(X_1, ..., X_m; Y)$, the following statements are equivalent*

(1) *T is continuous.*

(2) *T is continuous in $(0, ..., 0)$.*

(3) *There exists a constant $C > 0$ such that*

$$\left\|T\left(x^1, ..., x^m\right)\right\| \leq C \left\|x^1\right\| ... \|x^m\|, \tag{1.3}$$

for all $x^j \in X_j$, $(\forall 1 \leq j \leq m)$.

If X_j, $(\forall 1 \leq j \leq m)$ and Y are a normed spaces, then we provide the space $\prod_{J=1}^{m} X_j$ of topology of product vector spaces, and we denote by $\mathcal{L}(X_1, ..., X_m; Y)$ the vector space of all the continuous m-linear applications of $\prod_{J=1}^{m} X_j$ into Y. We write by $\mathcal{L}(^m X; Y)$ for $X_1 = ... = X_m = X$, and by $\mathcal{L}(X_1, ..., X_m)$ for $Y = \mathbb{K}$. It is obviously if $m = 1$ and $Y = \mathbb{K}$, then $\mathcal{L}(X; \mathbb{K}) = \mathcal{L}(X) = X^*$ the topological dual of X.

Proposition 1.4. *Let Y be a Banach space, the vector space $\mathcal{L}(X_1, ..., X_m; Y)$ is a Banach space endowed with the norm $\|T\|$ define by:*

$$\begin{aligned}\|T\| &= \sup_{\|x^j\| \leq 1,\ j=1,...m} \left\|T\left(x^1, ..., x^m\right)\right\| \\ &= \sup_{x^j \neq 0,\ j=1,...m} \frac{\left\|T\left(x^1, ..., x^m\right)\right\|}{\|x^1\| ... \|x^m\|} \\ &= \inf\left\{C : \left\|T\left(x^1, ..., x^m\right)\right\| \leq C \left\|x^1\right\| ... \|x^m\|\right\}.\end{aligned}$$

If $T \in \mathcal{L}(X_1, ..., X_m; Y)$ we define the adjoint operator of T by

$$T^* : Y^* \to \mathcal{L}(X_1, ..., X_m),\ y^* \mapsto T^*(y^*) : X_1 \times ... \times X_m \to \mathbb{K}$$

with

$$T^*(y^*)\left(x^1, ..., x^m\right) = y^*\left(T\left(x^1, ..., x^m\right)\right).$$

Now we define the multilinear mapping

$$K : X_1 \times ... \times X_m \to \mathcal{L}(X_1, ..., X_m)^*$$

with

$$K\left(x^1, ..., x^m\right)(\phi) = \phi\left(x^1, ..., x^m\right)$$

for all $x^j \in X_j$ and $\phi \in \mathcal{L}(X_1, ..., X_m)$. Then K is continuous and $\|K\| = 1$. A good result in multilinear operator theory is the isometric isomorphic identification described in the following proposition

Proposition 1.5. *[47, Theorem 1.10] Let $X_1, ..., X_m$ and Y be Banach spaces. Then we have the isometric isomorphism identification.*

$$\mathcal{L}(X_1, ..., X_m, Y) = \mathcal{L}(X_1, ..., X_m; Y^*). \tag{1.4}$$

Symmetric multilinear application

For all $m \in \mathbb{N}^*$, we denoted by $\mathcal{S}_m$ the set of all permutations of $\{1, ..., m\}$, in other word the set of all bijections

$$\sigma : \{1, ..., m\} \to \{1, ..., m\}.$$

Definition 1.6. *Let $m \in \mathbb{N}^*$ and let X, Y be a vector spaces over a field $\mathbb{K}$. A multilinear application $T : X \times ... \times X \to Y$, is symmetric if*

$$T\left(x^1, ..., x^m\right) = T\left(x_{\sigma(1)}, ..., x_{\sigma(m)}\right),$$

for every $x^1, ..., x^m \in X$ and every $\sigma \in \mathcal{S}_m$.
$\mathcal{L}_s({}^mX; Y)$ will denote the subspace of $\mathcal{L}({}^mX; Y)$ consisting of all the symmetric multilinear mappings from X^m into Y.

Proposition 1.7. *For all $T \in \mathcal{L}({}^mX; Y)$. Let T_s be the associate symmetric operator of T. Then, the following are checked*

(1) $T_s \in \mathcal{L}_s({}^mX; Y)$

(2) $T_s = T$ *if,and only if* $T \in \mathcal{L}_s({}^mX; Y)$

(3) $(T_s)_s = T_s$

(4) *If $x \in X$, then $T(x, ..., x) = T_s(x, ..., x)$.*

The next lemma shows the polarisation formula applied to the multilinear operator. For more details see [47].

Lemma 1.8. *Let X, Y be vector spaces and $T \in \mathcal{L}({}^mX; Y)$. Then*

$$T_s(x^1, ..., x^m) = \frac{1}{m!2^m} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 ... \varepsilon_m T\left(x^0 + \sum_{j=1}^{m} \varepsilon_j x^j + ... + x^0 + \sum_{j=1}^{m} \varepsilon_j x^j\right), \tag{1.5}$$

for every $x^0, x^1, ..., x^m \in X$. In particular, if T is symmetric, then it is determined by its values $T(x, ..., x)$, $x \in X$, along the diagonal.

1.3 m-homogeneous polynomials and tensor product

The letters X and Y correspond to Banach spaces over the same scalar-field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We work with $1 \leq p \leq \infty$, and p^* denotes the conjugate of p, *i.e.*, $\frac{1}{p} + \frac{1}{p^*} = 1$; if $p = 1$ we take $p^* = \infty$ and consider $\frac{1}{\infty} = 0$. A map $P : X \to Y$ is called m-homogeneous polynomial, if there exists a symmetric m-linear mapping $\check{P}$, such that $P(x) = \check{P}(x, \overset{m}{\ldots}, x)$ for all $x \in X$. In that case, we say that $\check{P}$ is the multilinear mapping associated with P.
Both mappings (m-homogeneous P and symmetric multilinear operator $\check{P}$) are related by polarization formula (for more details see [46]).

$$\check{P}\left(x^1, \ldots, x^m\right) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_m P(\varepsilon_1 x^1 + \cdots + \varepsilon_m x^m), \quad (x^j)_{j=1}^m \subset X. \tag{1.6}$$

where $x^1, \ldots, x^m \in X$.
The m-homogeneous polynomial, P :X→Y is continuous if there is a constant $C \geq 0$ with

$$\| P(x) \| \leq C \| x \|^m, \tag{1.7}$$

we put

$$\| P \| = \sup_{x \in B_X} \|P(x)\|. \tag{1.8}$$

Moreover, if P is bounded, then $\check{P}$ is also bounded, and

$$\| P \| \leq \| \check{P} \| \leq \frac{m^m}{m!} \| P \|. \tag{1.9}$$

Or equivalently the next diagram

$$\begin{array}{ccc} X & \overset{\Delta_m}{\to} & X \times \overset{m}{\ldots} \times X \\ & \underset{P}{\searrow} & \downarrow \check{P} \\ & & Y, \end{array} \tag{1.10}$$

is commutative, where Δ_m is the natural embedding called diagonal mapping of X into $X \times \overset{m}{\ldots} \times X$ which is defined as:

$$\begin{array}{rccc} \Delta_m : & X & \to & X \times \overset{m}{\ldots} \times X \\ & x & \mapsto & (x, \overset{m}{\ldots}, x). \end{array}$$

The space of all continuous m-homogeneous polynomials from X into Y is denoted by $\mathcal{P}(^m X; Y)$ ($\mathcal{P}(^m X)$ when $Y = \mathbb{K}$), and becomes a Banach space when endowed with the norm $\|P\|$. When $m=1$, the space $\mathcal{P}(^1X;Y)$ is nothing by $\mathcal{L}(X;Y)$, the space of all bounded linear operators from X to Y. For the general theory of homogeneous polynomials we refer to [30] and [47].

Example 1.9. *Let X, Y be normed spaces, $\varphi \in X^*$ and let $u \in \mathcal{L}(X,Y)$, $k \in \mathbb{N}$. We define the mapping P by*

$$P : X \to Y,\ P(x) = (\varphi(x))^{m-1} u(x).$$

Then P is a continuous m-homogeneous polynomial, and $\|P\| \le \|\varphi\|^{m-1}\|u\|$.

Example 1.10. *[34, Example 1.3.11] Let X and Y be normed spaces. The application given by*

$$Q_m : X \to (\mathcal{P}(^mX))^*, Q_m(x)(P) = P(x),$$

is a continuous m-homogeneous polynomial of norm 1.

Now draw attention towards tensor product. As usual, $X_1 \otimes \cdots \otimes X_m$ denotes the m-fold tensor product of $X_1, \ldots, X_m$, and if $X_1 = \cdots = X_m = X$ we use the shorter notation $\otimes^m X := X \otimes \cdots \otimes X$. The projective norm π on $X_1 \otimes \cdots \otimes X_m$ is defined by

$$\pi(u) := \inf\{\sum_{i=1}^{n} |\lambda_i| \|x_i^1\| \cdots \|x_i^m\| : u = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes \cdots \otimes x_i^m\},$$

where the infimum is taken over all representations of $u \in X_1 \otimes \cdots \otimes X_m$. The completion of $X_1 \otimes \cdots \otimes X_m$ when endowed with π is denoted by $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$.

It is well know that, if $T : X_1 \times \cdots \times X_m \to Y$ is a bounded m-linear operator then, there exists only one bounded linear operator

$$T_L : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m \to Y,$$

such that

$$T_L\left(x_i^1 \otimes \cdots \otimes x_i^m\right) = T(x_i^1, \ldots, x_i^m),$$

for all $x^j \in X_j, j = 1, \ldots, m$. Moreover, $\|T\| = \|T_L\|$. In other words, the mapping $T \to T_L$ from $\mathcal{L}(X_1, \ldots, X_m; Y)$ to $\mathcal{L}\left(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m; Y\right)$ is an isometric isomorphism. In particular, the vector spaces $\mathcal{L}(X_1, \ldots, X_m)$ and $\left(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m\right)^*$ are isometrically isomorphic. For more details refer to [59].

We denote by ϵ the injective norm on the tensor product $X_1 \otimes \cdots \otimes X_m$ given by

$$\epsilon(u) = \sup\left\{\left|\sum_{i=1}^{n} \langle x_i^1, x_1^* \rangle \cdots \langle x_i^m, x_m^* \rangle\right|; x_1^* \in B_{X_1^*}, \ldots, x_m^* \in B_{X_m^*}\right\},$$

$u \in X_1 \otimes \cdots \otimes X_m$, where $\sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m$ is any representation of u.

The completion of the space $X_1 \otimes \cdots \otimes X_m$ endowed with ϵ, is the injective tensor product $X_1 \widehat{\otimes}_\epsilon \cdots \widehat{\otimes}_\epsilon X_m$.

Consider now a Banach space X, the symmetric m-fold tensor product $\otimes^{m,s} X$, defined as the linear span of the elements in $\otimes^m X$ of the form $x \otimes \cdots \otimes x$, $x \in X$.

We will require two norms on this space:

1. The projective s-tensor norm π_s given by

$$\pi_s(z) = \inf\left\{\sum_{i=1}^{k} |\lambda_i| \, \|x_i\|^m : \ z = \sum_{i=1}^{k} \lambda_i x_i \otimes \cdots \otimes x_i\right\},$$

for $z \in \otimes^{m,s} X$.

2. The injective s-tensor norm ϵ_s given by

$$\epsilon_s(u) = \sup\left\{\left|\sum_{i=1}^{n} \lambda_i \langle x_i, x^* \rangle \cdots \langle x_i, x^* \rangle\right| : x^* \in B_{X^*}\right\},$$

being the supremum independent of the representation of u as $\sum_{i=1}^{n} \lambda_i x_i \otimes \cdots \otimes x_i$.

By $\widehat{\otimes}_{\pi_s}^{m,s} X$ and $\widehat{\otimes}_{\epsilon_s}^{m,s} X$ we mean the completion of $\otimes_{\pi_s}^{m,s} X$ and $\otimes_{\epsilon_s}^{m,s} X$ respectively.

We associated to all m-homogeneous polynomial $P \in \mathcal{P}(^m X; Y)$ an unique linear operator $P_L \in \mathcal{L}(\widehat{\otimes}_{\pi_s}^{m,s} X, Y)$ such that:

$$P(x) = P_L(x \otimes ... \otimes x), \text{for all } x \in X.$$

The linear operator P_L called linearization of P . Ryan [59] proved that the correspondence $P \leftrightarrow P_L$ establishes an isometric isomorphism between the space $\mathcal{P}(^m X)$ and the dual of $\widehat{\otimes}_{\pi_s}^{m,s} X$. Consider the canonical m-homogeneous polynomial $\delta_m : X \to \otimes_{\pi_s}^{m,s} X$ defined by

$\delta_m(x) = x \otimes \cdots \otimes x$. Then, the diagram

$$\begin{array}{ccc} X & \xrightarrow{P} & Y \\ {\scriptstyle \delta_m}\downarrow & \nearrow_{P_L} & \\ \otimes_{\pi_s}^{m,s} X & & \end{array} \tag{1.11}$$

is commutative, that is $P = P_L \circ \delta_m$.

The scalar case deserves special attention: when $Y = \mathbb{K}$ we call $\Delta_m : \mathcal{P}(^m X) \to \left(\widehat{\otimes}_{\pi_s}^{m,s} X\right)^*$ the isometric isomorphism $\Delta_m(P) = P_L$ for all $P \in \mathcal{P}(^m X)$. Note that $\Delta_m^{-1} = \delta_m^*$.

We can associated to P other operator P^* the adjoint of P define from Y^* into $\mathcal{P}(^m X)$ given by $P^*(y^*)(x) := y^*(P(x))$, for $y^* \in Y^*$ and $x \in X$. It is well known that, $(u \circ P)^* = P^* \circ u^*$, for all $u \in \mathcal{L}(Y; Z)$, where Z is a Banach space.

1.4 Normed operator ideals

A linear operator $u \in \mathcal{L}(X, Y)$ is said to have *finite rank* if $u(X)$ is finite dimensional. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(X, Y)$. This space is generated by the mappings of the special form

$$x^* \otimes y \mapsto x^*(x)\, y$$

i.e. if $u \in \mathcal{L}_f(X, Y)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$ (see [51, page 25]).

Definition 1.11. *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces X and Y its components $\mathcal{I}(X, Y) := \mathcal{L}(X, Y) \cap \mathcal{I}$ satisfy:*

(i) $\mathcal{I}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$ which contains the finite rank operators.

(ii) The ideal property: if $u \in \mathcal{L}(X, Z)$, $v \in \mathcal{I}(Z, K)$ and $w \in \mathcal{L}(K, Y)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(X, Y)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies

(i') $(\mathcal{I}(X, Y), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces X and Y,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') If $u \in \mathcal{L}(X,Z)$, $v \in \mathcal{I}(Z,K)$ and $w \in \mathcal{L}(K,Y)$,
$\|w \circ v \circ u\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\|$,
then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ for the sup norm. The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one (see [51, Theorem 1.2.2]).

- **The ideal of p-summing linear operators**

Let $1 \leq p < \infty$. A linear operator $u : X \longrightarrow Y$ between Banach spaces is said to be *absolutely p-summing* or just *p-summing* if it takes weakly p-summable sequences $(x_i)_{i=1}^{\infty}$ of X to absolutely p-summable sequences $(u(x_i))_{i=1}^{\infty}$ of Y. This means that $\widehat{u} : (x_i)_{i=1}^{\infty} \longmapsto (u(x_i))_{i=1}^{\infty}$ defines a linear mapping from $\ell_p^{\omega}(X)$ into $\ell_p(Y)$ that is bounded in view of the closed graph theorem (see[50],[31]). Hence there exists a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^{n} \|u(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{X^*} \leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}}, \tag{1.12}$$

for every finite family $(x_i)_{i=1}^{n} \subset X$. This inequality characterizes p-summing operators. The set of all p-summing operators, is denoted by $\Pi_p(X,Y)$, which constitute a Banach ideal under the ideal norm

$$\pi_p(u) := \inf \{C, \text{ for all } C \text{ verifying the inequality (1.12)}\},$$

moreover we have

$$\pi_p(u) = \|\widehat{u}\|.$$

The nowadays classical Pietsch's domination theorem characterizes the p-summability of an operator by means of a norm domination uniform inequality. Concretely, it says that the mapping $u \in \mathcal{L}(X,Y)$ is p-summing if and only if there exist a constant C and a regular Borel probability measure μ on B_{X^*} (with the weak star topology) such that

$$\|u(x)\| \leq C \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^p \, d\mu(x^*)\right)^{\frac{1}{p}}, \quad x \in X. \tag{1.13}$$

In this case, $\pi_p(u)$ is the least of all the constants C so that (1.13) holds. This inequality also provides a factorization of u through the natural mapping $C(B_{X^*}) \to L^p(\mu)$.

• **The ideal of strongly p-summing linear operators**

Pietsch has shown that the identity mapping from ℓ_1 into ℓ_2 is 2-summing but the adjoint operator is not 2-summing, for this reason the concept of strongly p-summing operators ($1 < p \leq \infty$) appears as characterization of absolutely p^*-summing operators (see [28]).

Definition 1.12. *Let $1 < p \leq \infty$. A linear operator u between two Banach spaces X and Y is strongly p-summing if there is a positive constant C such that for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $y_1^*, \ldots, y_n^* \in Y^*$ we have*

$$\left\| \left(\langle u(x_i), y_i^* \rangle \right)_{i=1}^n \right\|_1 \leq C \left\| (x_i)_{i=1}^n \right\|_p \left\| (y_i^*)_{i=1}^n \right\|_{p^*,\omega} . \tag{1.14}$$

In other terms u is strongly p-summing if and only if the operator

$\widehat{u} : \ell_p(X) \to \ell_p\langle Y \rangle$ is continuous.

The set of all strongly p-summing linear operators, denoted by $\mathcal{D}_p(X,Y)$, is a Banach ideal with the ideal norm

$$d_p(u) := \inf \left\{ C > 0 : \ C \text{ verifying the inequality } (1.14\ ) \right\}.$$

For $p = 1$ we have $\mathcal{D}_1(X,Y) = \mathcal{L}(X,Y)$.

The following result due to J.S. Cohen [28, Theorem 2.2.2].

Theorem 1.13. *i) Let $1 \leq p < \infty$. The linear operator u belongs to $\Pi_p(X,Y)$ if and only if the adjoint operator u^* belongs to $\mathcal{D}_{p^*}(Y^*, X^*)$. In this case $\pi_p(u) = d_{p^*}(u^*)$.*

ii) Let $1 < p \leq \infty$. The linear operator u belongs to $\mathcal{D}_p(X,Y)$ if and only if the adjoint operator u^* belongs to $\Pi_{p^*}(Y^*, X^*)$. In this case $d_p(u) = \pi_{p^*}(u^*)$.

Remark 1.14. *According to (ii) in the previous theorem we obtain $\mathcal{D}_p = \Pi_{p^*}^{dual}$, thus $(\mathcal{D}_p, d_p)$ is Banach operator ideal.*

• **The ideal of Cohen p-nuclear linear operators**

The concept of Cohen p-nuclear introduced by Cohen in [28]

Definition 1.15. *Let $1 < p < \infty$. A linear operator u between two Banach spaces X and Y is Cohen p-nuclear if there is a positive constant C such that for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $y_1^*, \ldots, y_n^* \in Y^*$ we have*

$$\left|\sum_{i=1}^{n}\left\langle T\left(x_{i}\right), y_{i}^{*}\right\rangle\right| \leq C\left\|\left(x_{i}\right)_{i=1}^{n}\right\|_{p,\omega}\left\|\left(y_{i}^{*}\right)_{i=1}^{n}\right\|_{p^{*},\omega}$$

The smallest constant C which is denoted by $n_p(u)$, such that the above inequality holds, is called the Cohen p-nuclear norm on the space $\mathcal{N}_p(X,Y)$ of all Cohen p-nuclear operators from X into Y which is a Banach space. For $p = 1$ and $p = \infty$ we have respectively $\mathcal{N}_1(X,Y) = \Pi_1(X,Y)$, and $\mathcal{N}_\infty(X,Y) = \mathcal{D}_\infty(X,Y)$ (for $1 < p \leq \infty$, $\mathcal{D}_p(X,Y)$ the Banach space of all strongly p-summing linear operators, see [10]).

Cohen in [28, Theorem 2.1.3] gives a characterization of Cohen p-nuclear operator using the tensor product

Theorem 1.16. *Let X and Y be Banach spaces and let $1 < p < \infty$. An operator u is Cohen p-nuclear if and only if the mapping*

$$I \otimes u : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_\pi Y$$

is continuous.

The following theorem establishes a direct link between the Cohen p-nuclear operator u and his adjoint operator u^*.

Theorem 1.17. *[28, Theorem 2.2.4] Let $1 < p < \infty$. An operator $u \in \mathcal{N}_p\left(X,Y\right)$ if and only if $u^* \in \mathcal{N}_{p^*}\left(Y^*, X^*\right)$. Moreover*

$$n_p(u) = n_{p^*}(u^*).$$

Chapter 2

Multilinear and polynomial Cohen p-nuclear operators

Since 1983 Pietsch's paper [52], the ideal of multilinear mappings (multi-ideals) and homogeneous polynomials (polynomial ideals) between Banach spaces have been studied as natural extension of the successful theory of operator ideals, several ideals and several generalizations of absolutely summing operators to the multilinear (multi-ideals) setting have been investigated, in this chapter and as first part we highlight some examples of multi-ideals we count: The ideal of Cohen strongly p-summing multilinear operators introduced by Achour and Mezrag [6], the ideal of absolutely $(p; q_1, ..., q_m; r)$-summing represented by Achour in [1], and the last example which is our generalization (Achour and the author see [2]) in multilinear version of the concept of p-nuclear introduced in [28], this generalization called Cohen p-nuclear multilinear operators . The second part of the chapter is dedicated to an other generalization of p-nuclear linear operators to the polynomial version, we finish the chapter with the domination and factorization theorems and some interesting properties. The connection with a linear/multilinear mappings is given .

2.1 Cohen p-nuclear multilinear operators and kwapien's factorization theorem

Let $m \in \mathbb{N}$ and $X_1, ..., X_m, Y$ be Banach spaces over $\mathbb{K}$ (real or complex scalars field), and let $\mathcal{L}(X_1, ..., X_m; Y)$ the Banach space of all continuous m-linear mappings from $X_1 \times ... \times X_m$

to Y. We denote by $\mathcal{L}_f(X_1,...,X_m;Y)$, the space of all m-linear mappings of finite type, which is generated by the mappings of the special form

$$T_{y\otimes_{j=1}^m x_j^*} = x_1^* \otimes ... \otimes x_m^* \otimes y : \left(x^1,...,x^m\right) \to x_1^*\left(x^1\right)...x_m^*\left(x^m\right).y,$$

for some non-zero $x_j^* \in X_j^*\,(1 \leq j \leq m)$ and $y \in Y$.

According to Pietsch in [52]. An ideal of multilinear mappings (or multi-ideal) is a subclass $\mathcal{M}$ of all continuous multilinear mappings between Banach spaces such that for all $m \in \mathbb{N}$, Banach spaces $X_1,...,X_m$ and Y, the components

$$\mathcal{M}(X_1,...,X_m;Y) := \mathcal{L}(X_1,...,X_m;Y) \cap \mathcal{M}$$

satisfy:
(i) $\mathcal{M}(X_1,...,X_m;Y)$ is a linear subspace of $\mathcal{L}(X_1,...,X_m;Y)$ which contains the m-linear mappings of finite type.
(ii) The ideal property: If $T \in \mathcal{M}(G_1,...,G_m;F)$, $u_j \in \mathcal{L}(X_j,G_j)$ for $j = 1,...,m$ and $v \in \mathcal{L}(F,Y)$, then $v \circ T \circ (u_1,...,u_m)$ is in $\mathcal{M}(X_1,...,X_m;Y)$.
If $\|.\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^+$ satisfies
(i') $(\mathcal{M}(X_1,...,X_m;Y), \|.\|_{\mathcal{M}})$ is a normed (Banach) space for all Banach spaces $X_1,...,X_m$ and Y and all m,
(ii") $\|T^m : \mathbb{K}^m \to \mathbb{K} : T^m\left(x^1,...,x^m\right) = x^1...x^m\|_{\mathcal{M}} = 1$ for all m,
(iii'") If $T \in \mathcal{M}(G_1,...,G_m;F)$, $u_j \in \mathcal{L}(X_j,G_j)$ for $j = 1,...,m$ and $v \in \mathcal{L}(F,Y)$, then $\|v \circ T \circ (u_1,...,u_m)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| ... \|u_m\|$, then $(\mathcal{M}, \|.\|_{\mathcal{M}})$ is called a normed (Banach) multi-ideal.

A polynomial $P \in \mathcal{P}(^mX;Y)$ is called of finite type, if there exists $k \in \mathbb{N}$ such that

$$P(x) = \sum_{j=1}^{k} (\varphi_j(x))^m b_j, \tag{2.1}$$

where $\varphi_j \in X^*$, $b_j \in Y$ and $j = 1,...,k$. It is easy to shows that the subset of all continuous m-homogeneous polynomials of finite type is a vectorial subspace of $\mathcal{P}(^mX;Y)$, which will be noted by $\mathcal{P}_f(^mX;Y)$, for more details see [46]. An ideal of homogeneous polynomials (or polynomial ideal) is a subclass $\mathcal{Q}$ of the class of every continuous homogeneous polynomials between Banach spaces such that, for all $m \in \mathbb{N}$ and any Banach spaces X and Y the components $\mathcal{Q}(^m X;Y) = \mathcal{P}(^m X;Y) \cap \mathcal{Q}$ satisfy:

(i) $\mathcal{Q}(^m X;Y)$ contains the m-homogeneous polynomials of finite type;

(ii) $\mathcal{Q}$ has the ideal property: if $u \in \mathcal{L}(E;X), P \in \mathcal{Q}(^m X;Y)$ and $t \in \mathcal{L}(Y;F)$, then $t \circ P \circ u$ belongs to $\mathcal{Q}(^m E;F)$.

$(\mathcal{Q}; \|\cdot\|_{\mathcal{Q}})$ is a normed (Banach) polynomial ideal if

(i') $(\mathcal{Q}(^m X;Y); \|\cdot\|_{\mathcal{Q}})$ is a normed (Banach) space for all X, Y;

(ii') $\|id_{\mathbb{K}^m} : \mathbb{K} \to \mathbb{K} : id_{\mathbb{K}^m}(x) = x^m\|_{\mathcal{Q}} = 1$ for all m;

(iii') If $u \in \mathcal{L}(E;X), P \in \mathcal{Q}(^m X;Y)$ and $t \in \mathcal{L}(Y;F)$, then

$$\|t \circ P \circ u\|_{\mathcal{Q}} \leq \|t\| \, \|P\|_{\mathcal{Q}} \, \|u\|^m .$$

We begin by presenting different classes of ideals of multilinear mappings related to the concept of absolutely summing operators:

• **Cohen strongly p-summig m-linear operators**

The notion of Cohen strongly p-summig m-linear operator was introduced by Achour and Mezrag in [6]. Consider m in $\mathbb{N}$. An m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is Cohen strongly p-summing $(1 < p \leq \infty)$ if, and only if, there exists a constant $C > 0$ such that for any $x_1^j, ..., x_n^j \in X_j$, $(1 \leq j \leq m)$, and any $y_1^*, ..., y_n^* \in Y^*$, we have

$$\left| \sum_{i=1}^{n} \langle T(x_i^1, ..., x_i^m), y_i^* \rangle \right| \leq C \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \left\| \left(x_i^j \right) \right\|^p \right)^{\frac{1}{p}} \|(y_i^*(y))_i\|_{p^*,\omega}$$

The class of all Cohen strongly p-summing m-linear operators from $X_1 \times ... \times X_m$ into Y, which is denoted by $\mathcal{D}_p^m(X_1, ..., X_m; Y)$ is a Banach space with the norm $d_p^m(.)$ which is the smallest constant C such that the above inequality holds. For $p = 1$, we have $\mathcal{D}_1^m(X_1, ..., X_m, Y) = \mathcal{L}(X_1, ..., X_m; Y)$.

It well know (see [1, Theorem 3.6],[6]) that,

(i) T is Cohen strongly p-summing $(1 < p \leq \infty)$ if, and only if, there exists a constant $C > 0$ and a Radon probability measure μ on $B_{Y^{**}}$ such that for all $(x^1, ..., x^m) \in X_1 \times ... \times X_m$ and $y^* \in Y^*$, we have

$$\left|\left\langle T\left(x^{1},...,x^{m}\right),y^{*}\right\rangle\right| \leq C\prod_{j=1}^{m}\left\|x^{j}\right\|\left(\int_{B_{Y^{**}}}\left|y^{*}(y^{**})\right|^{p^{*}}d\mu\right)^{\frac{1}{p^{*}}}. \tag{2.2}$$

(ii) For $1 < p \leq \infty$, T is Cohen strongly p-summing m-linear operator if and only if T_L is strongly p-summing linear operator.

(iii) $T \in \mathcal{D}_p^m(X_1, ..., X_m; Y)$, if and only if there exist Banach space G, strongly p-summing linear operator $u \in \mathcal{L}(G, Y)$ and a continuous m-linear operator $S \in \mathcal{L}(X_1, ..., X_m; G)$ so that

$$T = u \circ S,$$

$\left(i.e., \mathcal{D}_p^m = \mathcal{D}_p \circ \mathcal{L} \text{ holds isometrically}\right)$.

- **Absolutely** $(p; q_1, ..., q_m; r)$**-summing multilinear operators**

The notion of Absolutely $(p; q_1, ..., q_m; r)$-summing multilinear operators was introduced by Achour in [1]. A bounded multilinear operator T is absolutely $(p; q_1, ..., q_m; r)$-summing for all $0 < p, q_1, ..., q_m, r < \infty$ with $\frac{1}{p} \leq \frac{1}{q_1} + ... + \frac{1}{q_m} + \frac{1}{r}$ if there is a constant $C \geq 0$ such that

$$\left(\sum_{i=1}^{n}\left|\varphi_i\left(T\left(x_i^1,...,x_i^m\right)\right)\right|^p\right)^{\frac{1}{p}} \leq C\prod_{j=1}^{m}\left\|\left(x_i^j\right)_{1\leq i\leq n}\right\|_{q_j,\omega}\left\|(\varphi_i)_{1\leq i\leq n}\right\|_{r,\omega} \tag{2.3}$$

with $\varphi_i \in Y^*$and $x_i^j \in X_j$, for all $n \in \mathbb{N}$, $i = 1, ..., n$ and $j = 1, ..., m$.
For $p \geq 1$, the space $\left(\mathcal{L}_{as(p;q_1,...,q_m;r)}(X_1, ..., X_m; Y), \pi^m_{(p;q_1,...,q_m;r)}(.)\right)$ is a Banach multi-ideal, where $\pi^m_{(p;q_1,...,q_m;r)}(.)$ is the smallest constant $C > 0$ such that the inequality 2.3 holds.

Now we give the definition of Cohen (mp, p^*)-nuclear multilinear operators this last is a particular case of the class of absolutely $(p; q_1, ..., q_m; r)$-summing multilinear operators

Definition 2.1. *Let $1 < p < \infty$. The multilinear operator $T : X_1 \times ... \times X_m \to Y$; $m \in \mathbb{N}$ is Cohen (mp, p^*)-nuclear if there is a positive constant C such that, for $x_1^j, ..., x_n^j \in X_j$; $1 \leq j \leq m$ and for $y_1^*, ..., y_n^* \in Y^*$we have :*

$$\sum_{i=1}^{n}\left|\left\langle T\left(x_i^1,...,x_i^m\right);y_i^*\right\rangle\right| \leq C\prod_{j=1}^{m}\left\|\left(x_i^j\right)_{1\leq i\leq n}\right\|_{mp,\omega}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega}. \tag{2.4}$$

We denote by $\mathcal{N}^m_{mp,p^*}(X_1,...,X_m;Y)$ *the Banach space of Cohen* (mp,p^*)*-nuclear multilinear operators with the norm*

$$n^m_{mp,p^*}(T) = \inf\{C \text{ verifying the inequality (2.4)}\}.$$

The concept of Cohen p-nuclear multilinear operators introduced in conjunction with supervisor and myself in 2010 (see [2]). This concept is different to the concept of Cohen (mp,p^*)-nuclear in the multilinear case.

Definition 2.2. *An* m*-linear operator* $T: X_1 \times ... \times X_m \longrightarrow Y$ *is Cohen* p*-nuclear* $(1 < p < \infty)$ *if there is a constant* $C > 0$ *such that for any* $x_1^j,...,x_n^j \in X_j$, $(1 \leq j \leq m)$, *and any* $y_1^*,...,y_n^* \in Y^*$, *we have*

$$\left|\sum_{i=1}^{n}\left\langle T(x_i^1,...,x_i^m), y_i^*\right\rangle\right| \leq C \sup_{x^{*j}\in B_{X_j^*}}\left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left|x^{*j}\left(x_i^j\right)\right|^p\right)^{\frac{1}{p}}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega}. \quad (2.5)$$

Again the class of all Cohen p*-nuclear* m*-linear operators from* $X_1 \times ... \times X_m$ *into* Y*, which is denoted by* $\mathcal{N}^m_p(X_1,...,X_m;Y)$*, is a Banach space with the norm* $n^m_p(.)$*, which is the smallest constant* C *such that the inequality (2.5) holds.*

Remark 2.3. *For every* $T \in \mathcal{N}^m_p(X_1,...,X_m;Y)$*,* T *is continuous and* $\|T\| \leq n^m_p(T)$*.*

We recall that from [20]. $T \in \mathcal{L}(X_1,...,X_m;Y)$ is *semi*-integral m-linear operators if there exists a constant $C \geq 0$ and a regular probability measure μ on the Borel $\sigma-$algebra $C(B_{X_1^*}\times ... \times B_{X_m^*})$ of $B_{X_1^*}\times ... \times B_{X_m^*}$ endowed with the weak star topologies $\sigma(X_j^*, X_j), 1 \leq j \leq m$, such that

$$\|T(x^1,...,x^m)\| \leq C(\int_{B_{X_1^*}\times...\times B_{X_m^*}} |\varphi_1(x^1)...\varphi_m(x^m)|^p \, d\mu(\varphi_1,...,\varphi_m)^{\frac{1}{p}}$$

for every $x^j \in X_j$ and $j = 1,...,m$. This class of operators will denoted by $\mathcal{L}_{si}(X_1,...,X_m;Y)$.

So, it is clear that

Proposition 2.4. *a) For* $p = 1$*, we have* $\mathcal{N}^m_1(X_1,...,X_m;Y) = \mathcal{L}_{si,1}(X_1,...,X_m;Y)$*.*
b) For $p = \infty$*, we have* $\mathcal{N}^m_\infty(X_1,...,X_m;Y) = \mathcal{D}^m_\infty(X_1,...,X_m;Y)$*.*

Remark 2.5. *The inequality* (2.5) *equivalent to*

$$\sum_{i=1}^{n}\left|\left\langle T(x_i^1,...,x_i^m),y_i^*\right\rangle\right| \leq C \sup_{x^{*j}\in B_{X_j^*}}\left(\sum_{i=1}^{n}\prod_{j=1}^{m}\left|x^{*j}\left(x_i^j\right)\right|^p\right)^{\frac{1}{p}}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega}. \tag{2.6}$$

Example 2.6. *Let K be a compact Hausdorff space, let μ be a positive regular Borel measure on K and let $1 \leq p < \infty$. Each $g \in L_p(\mu)$ defines an m-linear multiplication operator $T_g \in \mathcal{L}(^mC(K); L_1(\mu))$, $T_g(f^1,...,f^m) = g\cdot f^1\cdot ...\cdot f^m$. This map is Cohen p-nuclear and $n_p^m(T_g) = \|g\|_{L_p(\mu)}$.*

Example 2.7. *[25] Every integral m-linear operator is Cohen p-nuclear.*
We recall that $T\in \mathcal{L}(X_1,....,X_m;Y)$ is integral if there exists a constant $C\geq 0$ such that for every $m\in N$, and all families $(x_i^j)_{1\leq i\leq n}\subset X_j$ and $(y_i^*)_{1\leq i\leq n}\subset Y^*$, we have

$$\left|\sum_{i=1}^{n}\left\langle T\left(x_i^1,...,x_i^m\right),y_i^*\right\rangle\right| \leq C \sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\leq j\leq m}}\left\|\sum_{i=1}^{n}x^{1*}\left(x_i^1\right)...x^{m*}(x_i^m)\,y_i^*\right\|_{Y^*}.$$

Indeed, let $T\in\mathcal{L}(X_1,...,X_m;Y)$. If T is integral, we can use Hölder's inequality in order to write

$$\begin{aligned}&\left|\sum_{i=1}^{n}\left\langle T\left(x_i^1,...,x_i^m\right),y_i^*\right\rangle\right|\\ \leq\ & \|T\|_I \sup_{\substack{x_j^*\in B_{X_j^*}\\ 1\leq j\leq m}}\Big(\sup_{y\in B_Y}\left|\sum_{i=1}^{n}x^{1*}\left(x_i^1\right)...x^{m*}(x_i^m)\,y_i^*(y)\right|\Big)\\ \leq\ & \|T\|_I \sup_{\substack{x_j^*\in B_{X_j^*}\\ 1\leq j\leq m}}\Big(\sum_{i=1}^{n}\left|x^{1*}\left(x_i^1\right)...x^{m*}(x_i^m)\right|^p\Big)^{\frac{1}{p}}\sup_{y\in B_Y}\Big(\sum_{i=1}^{n}\left|y_i^*(y)\right|^{p^*}\Big)^{\frac{1}{p^*}}.\end{aligned}$$

This T is Cohen p-nuclear.

The proof of the next proposition is straightforward and will be omitted

Proposition 2.8. *$(\mathcal{N}_p^m(X_1,...,X_m;Y), n_p^m(.))$ is a normed (Banach) multi-ideal.*

This class satisfies a Pietsch domination theorem which is the principal result of this section. For the proof we will use Ky Fan's lemma (see [31, p.190]).

Ky Fan's Lemma. *Let E be a Hausdorff topological vector space, and let $\mathcal{C}$ be a compact convex subset of E. Let M be a set of functions on $\mathcal{C}$ with values in $(-\infty,\infty]$ having the following properties:*

(a) *each $f\in M$ is convex and lower semicontinuous;*

(b) *if* $g \in conv(M)$*, there is an* $f \in M$ *with* $g(x) \leq f(x)$*, for every* $x \in \mathcal{C}$*;*

(c) *there is an* $r \in \mathbb{R}$ *such that each* $f \in M$ *has a value not greater than* r*.*

Then there is an $x_0 \in \mathcal{C}$ *such that* $f(x_0) \leq r$ *for all* $f \in M$*.*

Theorem 2.9. *For* $T \in \mathcal{L}(X_1, ..., X_m; Y)$ *and* $1 < p < \infty$*, the following conditions are equivalent:*

(i) The operator T *is Cohen* p*-nuclear.*

(ii) No matter how we choose finitely many vectors $x_1^j, ..., x_n^j$ *in* X_j $(1 \leq j \leq m)$ *and* $y_1^*, ..., y_n^*$ *in* Y^**, we have*

$$\sum_{i=1}^{n} |\langle T(x_i^1, ..., x_i^m), y_i^* \rangle| \leq n_p^m(T)(\sup_{\substack{x^{j*} \in B_{X_j^*} \\ 1 \leq j \leq m}} \sum_{i=1}^{n} \prod_{j=1}^{m} \left| \left\langle x_i^j, x^{j*} \right\rangle \right|^p)^{\frac{1}{p}} \sup_{y \in B_Y} \|(y_i^*(y))\|_{p^*}.$$

(iii) There exist Radon probability measures $\mu_j \in C(B_{X_j^*})^*$ $(1 \leq j \leq m)$ *and* $\lambda \in C(B_{Y^{**}})^*$ *such that for all* $(x^1, ..., x^m) \in X_1 \times .... \times X_m$ *and* $y^* \in Y^*$*,*

$$\left| \left\langle T\left(x^1, ..., x^m\right), y^* \right\rangle \right| \leq C \prod_{j=1}^{m} \|x_j\|_{L_p(B_{X_j^*}, \mu_j)} \|y^*\|_{L_{p^*}\left(B_{Y^{**}}, \lambda\right)}. \tag{2.7}$$

Proof. The implication (i)⇒(ii) is trivial . The proof is omitted.

The main point of the proof, the implication (ii)⇒ (iii) follows the ideas in [35] and [6]. We consider the sets $P(B_{X_j^*})$ $(1 \leq j \leq m)$ and $P(B_{Y^{**}})$ of probability measures in $C(B_{X_j^*})^*$ and $C(B_{Y^{**}})^*$, respectively. These are convex sets which are compact when we endow $C(B_{X_j^*})^*$ and $C(B_{Y^{**}})^*$ with their weak∗ topologies. We are going to apply Ky Fan's lemma with $E = C(B_{X_1^*})^* \times ... \times C(B_{X_m^*})^* \times C(B_{Y^{**}})^*$ and $\mathcal{C} = P(B_{X_1^*}) \times ... \times P(B_{X_m^*}) \times P(B_{Y^{**}})$.

Consider the set M of all functions $f : \mathcal{C} \to \mathbb{R}$ for which there exist $x_1^j, ..., x_n^j \in X_j$ $(j = 1, ..., m)$ and $y_1^*, ..., y_n^* \in Y^*$ such that

$$f(\mu_1, ..., \mu_m, \lambda) :=$$
$$\sum_{i=1}^{n} |\langle T(x_i^1, ..., x_i^m), y_i^* \rangle| - \tfrac{C}{p} \sum_{i=1}^{n} \prod_{j=1}^{m} \int_{B_{X_j^*}} \left| \left\langle x_i^j, x^{j*} \right\rangle \right|^p d\mu_j(x^{*j}) - \tfrac{C}{p^*} \sum_{i=1}^{n} \int_{B_{Y^{**}}} |\langle y_i^*, y^{**} \rangle|^{p^*} d\lambda(y^{**})$$

for all $(\mu_1, ..., \mu_m, \lambda) \in \mathcal{C}$. It is clear that all such f are continuous and affine and that the set M is a convex cone and consequently conditions (a) and (b) of Ky Fan's Lemma are satisfied.

For condition (c), since $B_{X_j^*}$ and $B_{Y^{**}}$ are weak∗ compact and norming, there exist for $f \in M$ elements $x_0^{*j} \in B_{X_j^*}$ and $y_0 \in B_{Y^{**}}$ such that

$$\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\leq j\leq m}} \sum_{i=1}^{n}\prod_{j=1}^{m}\left|\left\langle x_i^j, x^{j*}\right\rangle\right|^p = \sum_{i=1}^{n}\prod_{j=1}^{m}\left|\left\langle x_i^j, x_0^{j*}\right\rangle\right|^p$$

and

$$\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_{p^*}^n}^{p^*} = \sum_{i=1}^{n}|\langle y_i^*, y_0\rangle|^{p^*}$$

Using the elementary identity

$$\alpha\beta = \inf_{\epsilon>0}\left\{\frac{1}{p}\left(\frac{\alpha}{\epsilon}\right)^p + \frac{1}{p^*}(\epsilon\beta)^{p^*}\right\}, \quad \forall\alpha,\beta\in\mathbb{R}_+^* \tag{2.8}$$

we find by taking $\alpha = (\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\leq j\leq m}} \sum_{i=1}^{n}\prod_{j=1}^{m}\left|\left\langle x_i^j, x^{j*}\right\rangle\right|^p)^{\frac{1}{p}}, \beta = \sup_{y\in B_Y}\|(y_i^*(y))\|_{p^*}$ and $\epsilon = 1$

$$f\left(\delta_{x_0^{*1}},...,\delta_{x_0^{*m}},\delta_{y_0}\right) =$$
$$\sum_{i=1}^{n}|\langle T(x_i^1,...,x_i^m), y_i^*\rangle| - \tfrac{C}{p}(\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\leq j\leq m}} \sum_{i=1}^{n}\prod_{j=1}^{m}\left|\left\langle x_i^j, x^{j*}\right\rangle\right|^p) - \tfrac{C}{p^*}\sup_{y\in B_Y}\|(y_i^*(y))\|_{p^*}^{p^*}$$
$$\leq$$
$$\sum_{i=1}^{n}|\langle T(x_i^1,...,x_i^m), y_i^*\rangle| - C(\sup_{\substack{x^{j*}\in B_{X_j^*}\\ 1\leq j\leq m}} \sum_{i=1}^{n}\prod_{j=1}^{m}\left|\left\langle x_i^j, x^{j*}\right\rangle\right|^p)^{\frac{1}{p}}\sup_{y\in B_Y}\|(y_i^*(y))\|_{p^*},$$

where $\delta_{x_0^{*1}},...,\delta_{x_0^{*m}},\delta_{y_0}$ are the Dirac measures at $x_0^{*1},...,x_0^{*m},y_0$, respectively.

The last quantity is less than or equal to zero (by hypothesis (ii)) and hence condition (c) is verified by taking $r = 0$. By Ky Fan's lemma, there is $(\mu_1,...,\mu_m,\lambda)\in\mathcal{C}$ with $f(\mu_1,...,\mu_m,\lambda)\leq 0$ for all $f\in M$. Then, if f is generated by the single elements $(x^1,...,x^m)\in X_1\times...\times X_m$ and $y^*\in Y^*$,

$$|\langle T(x^1,...,x^m), y^*\rangle| \leq \tfrac{C}{p}\prod_{j=1}^{m}\int_{B_{X_j^*}}\left|\left\langle x_i^j, x^{*j}\right\rangle\right|^p d\mu_j(x^{*j}) + \tfrac{C}{p^*}\int_{B_{Y^{**}}}|\langle y_i^*, y^{**}\rangle|^{p^*} d\lambda(y^{**}).$$

Fix $\epsilon>0$. Replacing x^j by $\frac{1}{\epsilon^{\frac{1}{m}}}x^j$, y^* by ϵy^* and taking the infimum over all $\epsilon>0$ (using the elementary identity 2.8), we find

$$\begin{aligned}
& |\langle T(x^1,...,x^m), y^*\rangle| \\
\leq\ & C(\left[\tfrac{1}{p}(\prod_{j=1}^{m}\int_{B_{X_j^*}}\left|\left\langle x_i^j, x^{*j}\right\rangle\right|^p d\mu_j(x^{*j}))^{\frac{1}{p}}/\epsilon)^p\right. \\
& \left. + \tfrac{1}{p^*}\left((\epsilon(\int_{B_{Y^{**}}}|\langle y^*, y^{**}\rangle|^{p^*} d\lambda(y^{**}))^{\frac{1}{p^*}}\right)^{p^*}\right] \\
\leq\ & C\prod_{j=1}^{m}(\int_{B_{X_j^*}}|\langle x^j, x^{*j}\rangle|^p d\mu_j(x^{*j}))^{\frac{1}{p}}(\int_{B_{Y^{**}}}|\langle y^*, y^{**}\rangle|^{p^*} d\lambda(y^{**}))^{\frac{1}{p^*}}.
\end{aligned}$$

Now we prove that (iii) implies (i). Let $(x_i^1, ..., x_i^m) \in X_1 \times .... \times X_m$ and $y_i^* \in Y^*$ by 2.7, we have

$$|\langle T(x_i^1, ..., x_i^m), y_i^* \rangle| \leq C \prod_{j=1}^{m} \|x_i^j\|_{L_p(B_{X_j^*}, \mu_j)} \|y_i^*\|_{L_{p^*}(B_{Y^{**}}, \lambda)},$$

for all $1 \leq i \leq n$, then

$$\begin{aligned} \left| \sum_{i=1}^{n} \langle T(x_i^1, ..., x_i^m), y_i^* \rangle \right| &\leq \sum_{i=1}^{n} |\langle T(x_i^1, ..., x_i^m), y_i^* \rangle| \\ &\leq C \sum_{i=1}^{n} (\prod_{j=1}^{m} \|x_i^j\|_{L_p(B_{X_j^*}, \mu_j)} \|y_i^*\|_{L_{p^*}(B_{Y^{**}}, \lambda)}). \end{aligned}$$

We can use Hölder's inequality in order to write

$$\begin{aligned} &\left| \sum_{i=1}^{n} \langle T(x_i^1, ..., x_i^m), y_i^* \rangle \right| \\ &\leq C(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|_{L_p(B_{X_j^*}, \mu_j)}^p)^{\frac{1}{p}} (\sum_{i=1}^{n} \|y_i^*\|_{L_{p^*}(B_{Y^{**}}, \lambda)}^{p^*})^{\frac{1}{p^*}} \\ &= C(\sum_{i=1}^{n} \int_{B_{X_1^*} \times ... \times B_{X_m^*}} |x^{1*}(x_i^1) ... x^{m*}(x_i^m)|^p \, d(\mu_1 \otimes ... \otimes \mu_m)(x^{1*}, ..., x^{m*}))^{\frac{1}{p}}. \\ &\quad (\sum_{i=1}^{n} \int_{B_{Y^{**}}} |y_i^*(y^{**})|^{p^*} \, d\lambda(y^{**}))^{\frac{1}{p^*}} \\ &\leq C(\sup_{\substack{x^{j*} \in B_{X_j^*} \\ 1 \leq j \leq m}} \sum_{i=1}^{n} |x^{1*}(x_i^1) ... x^{m*}(x_i^m)|^p)^{\frac{1}{p}} \sup_{y \in B_Y} (\sum_{i=1}^{n} |y_i^*(y)|^{p^*})^{\frac{1}{p^*}}. \end{aligned}$$

Therefore T is Cohen p-nuclear and $n_p^m(T) \leq C$, as we wanted to prove. □

Kwapien's Factorization Theorem

Comparing the Theorem 2.9 with condition (b) of [29, *Corollary* 19.2], it is fair to say that Cohen p-nuclear multilinear operators are a generalization of $(p; p^*)$-dominated linear operators. Therefore the following theorem can be regarded as a multilinear version of Kwapien's factorization theorem.

Theorem 2.10. *Let $1 < p < \infty$. Then $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is Cohen p-nuclear if and only if there exist Banach spaces $G_1, ..., G_m$, absolutely p-summing linear operators $u_j \in \mathcal{L}(X_j, G_j)$ and a Cohen strongly p-summing m-linear mapping $S \in \mathcal{L}(G_1, ..., G_m; Y)$ so that $T = S(u_1, ..., u_m)$ Moreover,*

$$n_p^m(T) = \inf \left\{ d_p^m(S) \prod_{j=1}^{m} \pi_p(u_j) \;\; / \; T = So(u_1, ..., u_m) \right\}.$$

(i.e, $\mathcal{N}_p^m = \mathcal{D}_p^m \circ (\Pi_p, ..., \Pi_p)$ holds isometrically).

Proof. For the proof of the converse, it is fair to say that it follows from a straightforward combination of Proposition 2.8 with Pietsch's domination theorem for absolutely p-summing linear operators.

To prove the first implication, take $T \in \mathcal{N}_p^m(X_1, ..., X_m; Y)$. Then, by the inequality 2.7, there exist Radon probability measures $\mu_j \in C(B_{X_j^*})^*$ $(1 \leq j \leq m)$ *and* $\lambda \in C(B_{Y^{**}})^*$ such that for all $(x^1, ..., x^m) \in X_1 \times .... \times X_m$ and $y^* \in Y^*$

$$|\langle T(x^1, ..., x^m), y^* \rangle| \leq C \prod_{j=1}^{m} \|x_j\|_{L_p(B_{X_j^*}, \mu_j)} \|y^*\|_{L_{p^*}(B_{Y^{**}}, \lambda)}.$$

Let $(x^1, ..., x^m) \in X_1 \times .... \times X_m$. Define $u_j^0(x^j) := \langle ., x^j \rangle \in C(B_{X_j^*})$ and consider the diagram

$$\begin{array}{ccccccc}
X_1 & \times ... \times & X_m & & \overset{T}{\longrightarrow} & Y & \\
\downarrow i_{X_1} & & \downarrow i_{X_m} & & & S \uparrow & \\
i_{X_1}(X_1) & \times ... \times & i_{X_m}(X_m) & \underset{(I_1, ..., I_m)}{\longrightarrow} & G_1 & \times ... \times & G_m \\
\downarrow & & \downarrow & & \cap & & \cap \\
C(B_{X_1^*}) & \times ... \times & C(B_{X_m^*}) & \underset{(I_1, ..., I_m)}{\longrightarrow} & L_p(\mu_1) & \times ... \times & L_p(\mu_m)
\end{array}$$

where $I_j : C(B_{X_j^*}) \longrightarrow L_p(\mu_j)$ is the canonical injection, $i_{X_j} : X_j \longrightarrow C(K_j)$ is the natural isometric injection and G_j is the closure of the space $I_j o u_j^0(X_j)$, $u_j(x^j) := I_j(u_j^0(x^j))$. Since $\pi_p(I_j) = 1$ and $\|u_j^0\| = 1$, it follows that $\pi_p(u_j) \leq 1$.

The operator S is defined on $u_1(X_1) \times ... \times u_m(X_m)$; $u_j(X_j) = I_j(u_j^0(x^j))$ $(1 \leq j \leq m)$, by

$$S(u_1(x^1), ..., u_m(x^m)) := T(x^1, ..., x^m)$$

and this definition makes sense because

$$|\langle S(u_1(x^1), ..., u_m(x^m)), y^* \rangle| \leq n_p^m(T) \prod_{j=1}^{m} \|u_j(x^j)\|_{G_j} \Big(\int_{B_{Y^{**}}} |\langle y^*, y^{**} \rangle|^{p^*} d\lambda(y^{**}) \Big)^{\frac{1}{p^*}}.$$

It follows that S is continuous on $u_1(X_1) \times ... \times u_m(X_m)$ and has a unique extension to $\overline{u_1(X_1)} \times ... \times \overline{u_m(X_m)} = G_1 \times ... \times G_m$; moreover, the inequality implies that

$$\begin{aligned}
\|S^*(y^*)\| &= \sup \{ |\langle S^*(y^*), (u_1(x^1), ..., u_m(x^m)) \rangle| \ / \ \|u_j(x^j)\| \leq 1 \} \\
&\leq n_p^m(T) \Big(\int_{B_{Y^{**}}} |\langle y^*, y^{**} \rangle|^{p^*} d\lambda(y^{**}) \Big)^{\frac{1}{p^*}},
\end{aligned}$$

which means that S^* is absolutely p^*-summing. From [43, Theorem 2.7], S is a Cohen strongly p-summing m-linear operator and $d_p^m(S) = \pi_{p^*}(S^*) \leq n_p^m(T)$.

This end the proof. □

Proposition 2.11. *Let $1 < p < \infty$, let $X_1, ..., X_m, Y$ be Banach spaces and let $T : X_1 \times ... \times X_m \longrightarrow Y$ be a m-linear operator :*
If $T \in \mathcal{N}_p^m(X_1, ..., X_m; Y)$ then $T \in \mathcal{D}_p^m(X_1, ..., X_m; Y)$ and $d_p^m(T) \leq n_p^m(T)$.

Proof. If T is Cohen p-nuclear, then

$$
\begin{aligned}
& |\langle T(x^1, ..., x^m), y^* \rangle| \\
\leq \; & n_p^m(T) \prod_{j=1}^{m} (\int_{B_{X_j^*}} |\langle x^j, \xi^j \rangle|^p \, d\mu_j(\xi^j))^{\frac{1}{p}} (\int_{B_{Y^{**}}} |\langle y^{**}, y^* \rangle|^{p^*} \, d\lambda(y^{**}))^{\frac{1}{p^*}} \\
\leq \; & n_p^m(T) \prod_{j=1}^{m} (\sup_{\xi^j \in B_{X_j^*}} |\langle x^j, \xi^j \rangle|)(\int_{B_{Y^{**}}} |\langle y^{**}, y^* \rangle|^{p^*} \, d\mu(y^{**}))^{\frac{1}{p^*}} \\
\leq \; & n_p^m(T) \prod_{j=1}^{m} \|x^j\|_{X_j} (\int_{B_{Y^{**}}} |\langle y^{**}, y^* \rangle|^{p^*} \, d\mu(y^{**}))^{\frac{1}{p^*}}.
\end{aligned}
$$

Thus, by the Inequality (2.2) , T$\in \mathcal{D}_p^m(X_1, ..., X_m; Y)$ *and* $d_p^m(T) \leq n_p^m(T)$.

□

A multilinear mapping $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be weakly compact, in symbols $T \in \mathcal{L}_w(X_1, ..., X_m; Y)$, if T map $B_{X_1} \times \cdots \times B_{X_m}$ onto a relatively weakly compact subset of Y. This is equivalent to say that $(T(x_n^1, ..., x_n^m))_{n=1}^{\infty}$ has a weakly convergent subsequence for every bounded sequence $(x_n^j)_{n=1}^{\infty} \subset X_j \, (j = 1, ..., m)$. As a consequence of Proposition 2.11 and (iii) in [1, Theorem 3.6] we have

Corollary 2.12. *Every Cohen p-nuclear m-linear mapping defined from arbitrary Banach spaces is weakly compact.*

2.2 Cohen p-nuclear m-homogeneous polynomials

The polynomial version of the above concepts of multilinear Cohen p-nuclear and Cohen (mp, p^*)-nuclear operators are coincide. In this range we have the following definition.

Definition 2.13. *Fix $m \in \mathbb{N}$. Let $1 \leq p \leq \infty$ and let X, Y be Banach spaces. An m-homogeneous polynomial $P : X \to Y$ is Cohen p-nuclear, if there exists a positive constant C such that for any $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in Y^*$, we have*

$$\left| \sum_{i=1}^{n} \langle P(x_i), y_i^* \rangle \right| \leq C \left\| (x_i)_{1 \leq i \leq n} \right\|_{mp,\omega}^{m} \left\| (y_i^*)_{1 \leq i \leq n} \right\|_{p^*,\omega}. \tag{2.9}$$

The class of such polynomials is denoted by $\mathcal{P}^c_{p,N}\left({}^m X;Y\right)$ *witch is equipped with the norm* $\|P\|_{p,N}$, *i.e. The smallest constant* C *such that inequality (2.9) holds.*

We recall that:

an m-homogeneous polynomial $P\in\mathcal{P}({}^mX;Y)$ is *Cohen strongly p-summing* if there is a constant $C\geq 0$ such that

$$\left\|\left(P(x_i)\right)_{i=1}^n\right\|_{\ell_p\langle Y\rangle}\leq C\|(x_i)_{i=1}^n\|^m_{\ell_p(X)}$$

for all $x_1,\ldots,x_n\in X$ and all $n\in\mathbb{N}$. The infimum of all such $C>0$ defines a norm on the space $\mathcal{P}^c_{p,S}({}^mX;Y)$ of all strongly Cohen p-summing m-homogeneous polynomials from X to Y, that we denote $\|P\|_{p,S}$. For more information on Cohen strongly p-summing polynomials we refer to [7].

Proposition 2.14. *Let* $P\in\mathcal{P}\left({}^mX;Y\right)$. *Then*
(i) $\mathcal{P}^c_{p,N}\left({}^mX;Y\right)\subset\mathcal{P}\left({}^mX;Y\right)$ *with*

$$\|P\|\leq\|P\|^c_{p,N}\,,\ \textit{for all } P\in\mathcal{P}^c_{p,N}\left(X,Y\right).$$

(ii) $\mathcal{P}^c_{p,N}\left({}^mX;Y\right)\subset\mathcal{P}^c_{p,S}\left({}^mX;Y\right)$. *If* $p=\infty$ *we have* $\mathcal{P}^c_{\infty,N}\left({}^mX,Y\right)=\mathcal{P}^c_{\infty,S}\left({}^mX,Y\right)$.

Proof. (i) Let P be Cohen p-nuclear polynomial of X into Y. By definition, if $x\in X$ and $y^*\in Y^*$ we have

$$\begin{aligned}\left|\left\langle P\left(x\right),y^*\right\rangle\right| &\leq \|P\|_{p,N}\sup_{x^*\in B_{X^*}}\left|\left\langle x,x^*\right\rangle\right|^m\sup_{\psi\in B_{Y^{**}}}\left|\left\langle y^*,\psi\right\rangle\right|\\ &\leq \|P\|_{p,N}\left\|x\right\|^m\left\|y^*\right\|,\end{aligned}$$

passing to supremum over all $y^*\in B_{Y^*}$, we get

$$\|P\|\leq\|P\|_{p,N}\left\|x\right\|^m,$$

hence P is continuous, moreover $\|P\|\leq\|P\|_{p,N}$.
(iii) Let $P\in\mathcal{P}^c_{p,N}\left({}^mX;Y\right)$. Then

$$\begin{aligned}\sum_{i=1}^n\left|\left\langle P\left(x_i\right),y_i^*\right\rangle\right| &\leq \|P\|_{p,N}\sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^n\left|\left\langle x_i,x_i^*\right\rangle\right|^{mp}\right)^{\frac{1}{p}}\|(y_i^*)_i\|_{p^*,\omega}\\ &\leq \|P\|_{p,N}\left[\left(\sum_{i=1}^n\left\|x_i\right\|^{mp}\right)^{\frac{1}{mp}}\right]^m\|(y_i^*)_i\|_{p^*,\omega}.\end{aligned}$$

So, P is strongly p-summing, and $\|P\|_{p,S} \leq \|P\|_{p,N}$.

Now, let choose $p=\infty$. Then if P is strongly ∞-summing polynomial, hence :

$$\begin{aligned}\sum_{i=1}^{n}|\langle P(x_i), y_i^*\rangle| &\leq \|P\|_{\infty,S} \sup_i \|x_i\|^m \|(y_i^*)_i\|_{1,\omega} \\ &\leq \|P\|_{\infty,S} \sup_i \sup_{x^*\in B_{X^*}} |x_i^*(x_i)|^m \|(y_i^*)_i\|_{1,\omega} \\ &= \|P\|_{\infty,S} \sup_{x^*\in B_{X^*}} \sup_i |x_i^*(x_i)|^m \|(y_i^*)_i\|_{1,\omega},\end{aligned}$$

then, P is Cohen ∞-nuclear, and $\|P\|_{\infty,N} \leq \|P\|_{\infty,S}$. □

A famous example can be contained in our class

Example 2.15. *Let $1<p<\infty$; $m\in\mathbb{N}$ and let $u: X\to Y$ be an Cohen p-nuclear operator, where X, Y are two Banach spaces. For $\varphi\in X^*$ the mapping*

$$\begin{aligned}P &: X\to Y \\ \| P\|_{p,N} &= \varphi^{m-1}(x)\,u(x),\end{aligned}$$

is Cohen p-nuclear polynomial, moreover $\|P\| \leq n_p(u)\|\varphi\|^{m-1}$.

Indeed. For $x_1,...,x_n\in X$ and $y_1^*,...,y_n^*\in Y^*$, we have :

$$\begin{aligned}\left|\sum_{i=1}^{n}\langle P(x_i); y_i^*\rangle\right| &= \left|\sum_{i=1}^{n}\langle \varphi^{m-1}(x_i)\,u(x_i); y_i^*\rangle\right| \\ &= \left|\sum_{i=1}^{n}\langle u(\varphi^{m-1}(x_i)(x_i)); y_i^*\rangle\right| \\ &\leq n_p(u)\sup_{\xi\in B_{X^*}}\left(\sum_{i=1}^{n}|\langle\varphi^{m-1}(x_i)(x_i),\xi\rangle|^p\right)^{\frac{1}{P}}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega} \\ &= n_p(u)\sup_{\xi\in B_{X^*}}\left(\sum_{i=1}^{n}|\varphi^{m-1}(x_i)\,\xi(x_i)|^p\right)^{\frac{1}{P}}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega} \\ &\leq n_p(u)\|\varphi\|^{m-1}\sup_{\xi\in B_{X^*}}\left(\sum_{i=1}^{n}|\xi(x_i)|^{mp}\right)^{\frac{1}{P}}\left\|(y_i^*)_{1\leq i\leq n}\right\|_{p^*,\omega},\end{aligned}$$

hence, P is Cohen p-nuclear, furthermore $\| \mathrm{P}\|_{p,N} \leq n_p(u)\|\varphi\|^{m-1}$.

Proposition 2.16. *Let $P\in\mathcal{P}(^mX,Y)$, and let $v: l_p^n\to Y^*$ be bounded linear operator. Then the polynomial P is Cohen p-nuclear if*

$$\left|\sum_{i=1}^{n}\langle P(x_i), v(e_i)\rangle\right| \leq C\|(x_i)\|_{p,\omega}^m\|v\|.$$

Proof. Let $v : l_p^n \to Y^*$ be bounded linear operator such that

$$v(e_i) = y_i^*; \ ((i.e) \ v = \sum_{i=1}^{n} e_i \otimes y_i^*$$

where e_i is the canonical base of $l_{p^*}^n$). Since there is an isometric between the spaces $l_p^\omega(Y^*)$, and $\mathcal{L}(l_p; Y^*)$, in other word $\|v\| = \|y_i^*\|_{l_p^{n;w}}$. We will have the proof. □

Theorem 2.17. *Let P be an m-homogeneous polynomial between Banach spaces X and Y. P is Cohen p-nuclear if and only if his associated symmetric m-linear operator $\check{P} \in \mathcal{L}(^m X, Y)$ is Cohen p-nuclear, and*

$$\|P\|_{p,N} = n_p^m\left(\check{P}\right).$$

Proof. First we suppose that $\check{P}$ is a multilinear Cohen p-nuclear operator by definition we have:

For all $x_1, ..., x_n \in X$, and $y_1^*, ..., y_n^* \in Y^*$, then

$$\begin{aligned}
\left|\sum_{i=1}^{n} \langle P(x_i), y_i^* \rangle\right| &= \left|\sum_{i=1}^{n} \langle \check{P}(x_i, ..., x_i), y_i^* \rangle\right| \\
&\leq n_p^s\left(\check{P}\right) \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} \prod^{m} |\langle x_i, x^* \rangle|^p\right)^{\frac{1}{p}} \left\|(y_i^*)_{1 \leq i \leq n}\right\|_{p^*,\omega} \\
&= n_p^s\left(\check{P}\right) \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^{mp}\right)^{\frac{1}{p}} \left\|(y_i^*)_{1 \leq i \leq n}\right\|_{p^*,\omega}.
\end{aligned}$$

hence P is Cohen p-nuclear polynomial, and $\|P\|_{p,N} \leq n_p^s\left(\check{P}\right)$.

Conversely, let $(x_i^j)_{i=1}^n \in B_{l_{q,w}^n(X)}, j = 1, ..., m$ and let $y_1^*, ..., y_n^* \in Y^*$. By the triangle inequality,

$$\begin{aligned}
&\left\|\left(\epsilon_1 x_i^1 + \cdots + \epsilon_m x_i^m\right)_{i=1}^n\right\|_{mp,w} \\
&\leq \left\|\left(\epsilon_1 x_i^1\right)_{i=1}^n\right\|_{mp,w} + \cdots + \left\|(\epsilon_m x_i^m)_{i=1}^n\right\|_{mp,w} \leq m
\end{aligned}$$

for every $\epsilon_1, ..., \epsilon_m = \pm 1$.

Using the Polarization Formula (1.6) and (1.9) we have

$$\begin{aligned}
&\left\| \left(\, y_i^* \left(\hat{P} \left(x_i^1, ..., x_i^m \right) \right) \right)_{i=1}^n \right\| \\
&= \sum_{i=1}^n \left| \, y_1^* \left(\hat{P} \left(x_i^1, ..., x_i^m \right) \right) \right| \\
&\le \frac{1}{m! 2^m} \sum_{i=1}^n \left| \sum_{\epsilon_1, ..., \epsilon_m = \pm 1} \epsilon_1 \cdots \epsilon_m \; y_i^* \left(P \left(\sum_{j=1}^m \epsilon_j x_i^j \right) \right) \right| \\
&\le \frac{1}{m! 2^m} \sum_{i=1}^n \left(\sum_{\epsilon_1, ..., \epsilon_m = \pm 1} \left| \, y_i^* \left(P \left(\sum_{j=1}^m \epsilon_j x_i^j \right) \right) \right| \right) \\
&\le \frac{1}{m! 2^m} \sum_{\epsilon_1, ..., \epsilon_m = \pm 1} \left(\sum_{i=1}^n \left| \, y_i^* \left(P \left(\sum_{j=1}^m \epsilon_j x_i^j \right) \right) \right| \right) \\
&\le \frac{1}{m! 2^m} \|P\|_{\mathcal{P}_{p,N}^c} \sum_{\epsilon_1, ..., \epsilon_m = \pm 1} \left\| \sum_{j=1}^m \epsilon_j x_i^j \right\|_{mp,w}^m \|(\, y_i^*)_{i=1}^n \|_{p^*,w} \\
&\le \frac{1}{m! 2^m} \|P\|_{\mathcal{P}_{p,N}^c} \left(\sum_{\epsilon_1, ..., \epsilon_m = \pm 1} m^m \right) \|(\, y_i^*)_{i=1}^n \|_{p^*,w} \\
&\le \frac{\left(2^{\frac{1}{p}-1} m \right)^m}{m!} \|P\|_{p,N} \|(\, y_i^*)_{i=1}^n \|_{p^*,w} \, .
\end{aligned}$$

So, for every $(x_i^j)_{i=1}^n \in B_{l_{mp,w}^n(X)}$, $1 \le j \le m$, we have

$$\left\| \left(\, y_i^* \left(\check{P} \left(x_i^1, ..., x_i^m \right) \right) \right)_{i=1}^n \right\| \le \frac{\left(2^{\frac{1}{p}-1} m \right)^m}{m!} \|P\|_{p,N} \|(\, y_i^*)_{i=1}^n \|_{p^*,w} \, .$$

and for $(x_i^j)_{i=1}^n \in l_{mp,w}^n(X)$ with $x_i^j \neq 0$ and $j = 1, ..., m$,

$$\begin{aligned}
&\left\| \left(\, y_i^* \left(\check{P} \left(\frac{x_i^1}{\|(x_i^1)_{i=1}^n\|_{mp,w}}, ..., \frac{x_i^m}{\|(x_i^m)_{i=1}^n\|_{mp,w}} \right) \right) \right)_{i=1}^n \right\| \\
&\le \frac{\left(2^{\frac{1}{p}-1} m \right)^m}{m!} \|P\|_{p,N} \|(\, y_i^*)_{i=1}^n \|_{p^*,w} \, .
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\| \left(\, y_i^* \left(\check{P} \left(x_i^1, ..., x_i^m \right) \right) \right)_{i=1}^n \right\| \\
&\le \frac{\left(2^{\frac{1}{p}-1} m \right)^m}{m!} \|P\|_{p,N} \prod_{j=1}^m \left\| \left(x_i^j \right)_{i=1}^n \right\|_{mp,w} \|(\, y_i^*)_{i=1}^n \|_{p^*,w} \, .
\end{aligned}$$

Therefore, $\check{P}$ is Cohen p-nuclear and

$$n_p^m \left(\check{P} \right) \le \frac{\left(2^{\frac{1}{p}-1} m \right)^m}{m!} \|P\|_{p,N} \le \frac{m^m}{m!} \|P\|_{p,N}.$$

□

In a recent paper Achour and Bernardino [5] obtained an characterization of (q,r)-dominated polynomials: they first obtained donination/factorization theorems, where they prove that the polynomial $(q;r)$-dominated can be factorized by an absolutely q-summing operator and cohen strongly r^*-summing polynomial, also they have obtained some important consequences. Since our class of polynomials is an extension to the polynomial version of a particular case of multilinear class introduced by Achour and Bernardino in [5] (for $p=1$, $q=mp$ and $r=p^*$), so we summarize some results in the following

The characterization of p-nuclear polynomials is useful.

Proposition 2.18. *Let $1<p<\infty$ and P is Cohen p-nuclear polynomial, then their linearization operator P_L belongs to $\mathcal{N}_p\left(\widehat{\otimes}_{\pi_s}^{m,s}X,Y\right)$.*

Proof. Suppose that P be Cohen p-nuclear polynomial;
Let $x_i\otimes ...\otimes x_i\in\widehat{\otimes}_{\pi_s}^{m,s}X$ and $y_i^*\in Y^*$ then :

$$\begin{aligned}
\left|\sum_{i=1}^{n}\langle P_L\left(x_i\otimes ...\otimes x_i\right),y_i^*\rangle\right| &= \left|\sum_{i=1}^{n}\langle P\left(x_i\right),y_i^*\rangle\right| \\
&\leq \|P\|_{p,N}\sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^{n}|\langle x_i;x^*\rangle|^{mp}\right)^{\frac{1}{p}}\|(y_i^*)\|_{p^*,\omega} \\
&\leq \|P\|_{p,N}\sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^{n}|x^*\left(x_i\right)...x^*\left(x_i\right)|^{p}\right)^{\frac{1}{p}}\|(y_i^*)\|_{p^*,\omega} \\
&\leq \|P\|_{p,N}\sup_{\varphi\in\mathcal{L}(^mX)}\left(\sum_{i=1}^{n}|\varphi\left(x_i\otimes ...\otimes x_i\right)|^{p}\right)^{\frac{1}{p}}\|(y_i^*)\|_{p^*,\omega} \\
&\leq \|P\|_{p,N}\sup_{\psi\in(\otimes_{\pi s}^{m,s}X)^*}\left(\sum_{i=1}^{n}|\psi\left(x_i\otimes ...\otimes x_i\right)|^{p}\right)^{\frac{1}{p}}\|(y_i^*)\|_{p^*,\omega} \\
&\leq \|P\|_{p,N}\|(x_i\otimes ...\otimes x_i)\|_{p,\omega}\|(y_i^*)\|_{p^*,\omega}.
\end{aligned}$$

By the density of $\otimes_{\pi s}^{m,s}X$ in $\widehat{\otimes}_{\pi_s}^{m,s}$, we achieved the prove. □

The next proposition shows that the space of Cohen p-nuclear m-homogeneous polynomial is a Banach ideal

Proposition 2.19. *Let X, Y be Banach spaces, m a positive integer and $1\leq p\leq\infty$. Then*
(i) Every polynomial of finite type from X into Y is Cohen p-nuclear.
(ii) $\|\left(P:\mathbb{K}\rightarrow\mathbb{K}:P(x)=x^m\right\|_{p,N}=1$; for all $m\in\mathbb{N}$.

(iii) Let $P \in \mathcal{P}(^m X, Y)$, $S \in \mathcal{L}(Y;Z)$ *and* $T \in \mathcal{L}(E;X)$, *(where* Z *and* E *are a Banach spaces).* *If* P *is Cohen p-nuclear then the polynomial* $S \circ P \circ T$ *is p-nuclear; and*

$$\|S \circ P \circ T\|_{p,N} \leq \|S\| \, \|P\|_{p,N} \, \|T\|^m .$$

Proof. Let P be an m-homogeneous polynomial of finite type of X into Y, where X and Y are Banach spaces. It is clear that $\mathcal{P}^c_{p,N}(^m X; Y)$ is a subspace of $\mathcal{L}(X;Y)$, then :

$$P = \sum_{i=1}^{n} x_i^* (x)^m \, y_i.$$

The polynomial P can be written as the form $P(x) = x^{*m}(x)\, y$.

Just we prove that the form polynomials:

$$P = x^{*m}(.)\, y,$$

where $x^* \in X^*$ and $y \in Y$ are in the space $\mathcal{P}^c_{p,N}(^m X, Y)$.

Let $(x_i)_{1 \leq i \leq n} \subset X$, we have

$$\begin{aligned}
\left| \sum_{i=1}^{n} \langle P(x_i), y_i^* \rangle \right| &= \left| \sum_{i=1}^{n} \langle x^*(x_i)^m \, y, y_i^* \rangle \right| \\
&= \left| \sum_{i=1}^{n} x^*(x_i)^m \, \langle y, y_i^* \rangle \right| \\
&\leq \left(\sum_{i=1}^{n} |x^*(x_i)^m|^p \right)^{\frac{1}{p}} \left(\sum_{i\ =1}^{n} |\langle y, y_i^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq \|x^*\|^m \left(\sup_{\varphi \in B_{X^*}} \sum_{i=1}^{n} |\varphi(x_i)|^{mp} \right)^{\frac{1}{p}} \left\| (y_i^*(y_i))_{1 \leq i \leq n} \right\|_{p^*}
\end{aligned}$$

passing to the sup for $y \in B_Y$ we find:

$$\left| \sum_{i=1}^{n} \langle P(x_i), y_i^* \rangle \right| \leq \|x^*\|^m \left(\sup_{\varphi \in B_{X^*}} \sum_{i=1}^{n} |\varphi(x_i)|^{mp} \right)^{\frac{1}{p}} \|(y_i^*)_i\|_{\omega,p^*} ;$$

consequently

$$P \in \mathcal{P}^c_{p,N}(^m X, Y) .$$

(ii) Is easy.

(iii) Let $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in Y^*$. Then

$$\begin{aligned}\left|\sum_{i=1}^{n}\langle S\circ P\circ T\left(x_i\right), y_i^*\rangle\right| &= \left|\sum_{i=1}^{n}\langle P\circ T\left(x_i\right), S^*\left(y_i^*\right)\rangle\right| \\ &\leq \|P\|_{p,N}\sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^{n}\left|\langle T\left(x_i\right), x^*\rangle\right|^{mp}\right)^{\frac{1}{p}}\|R\|,\end{aligned}$$

with :

$$\begin{aligned}R\left(y\right) &= \sum_{i=1}^{n}\langle S^*\left(y_i^*\right), y\rangle e_i \\ &= \sum_{i=1}^{n}\langle y_i^*, S\left(y\right)\rangle e_i \\ &= \|S\left(y\right)\|\sum_{i=1}^{n}\left\langle y_i^*, \frac{S(y)}{\|S(y)\|}\right\rangle e_i,\end{aligned}$$

which gives us:

$$\|R\| \leq \|S\|\,\|(y_i^*)_i\|_{\omega,p^*}.$$

Returning

$$\begin{aligned}\left|\sum_{i=1}^{n}\langle S\circ P\circ T\left(x_i\right), y_i^*\rangle\right| &\leq \|P\|_{p,N}\sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^{n}\left|\langle T\left(x_i\right), x^*\rangle\right|^{mp}\right)^{\frac{1}{p}}\|S\|\,\|(y_i^*)_i\|_{\omega,p^*} \\ &\leq \|P\|_{p,N}\|T\|^m\|S\|\sup_{x^*\in B_{X^*}}\left(\sum_{i=1}^{n}\left|\langle x_i, x^*\rangle\right|^{mp}\right)^{\frac{1}{p}}\|(y_i^*)_i\|_{\omega,p^*};\end{aligned}$$

this implies that $S\circ P\circ T$ is Cohen p-nuclear polynomial. Moreover

$$\|S\circ P\circ T\|_{p,N} \leq \|S\|\,\|P\|_{p,N}\,\|T\|^m.$$

□

2.3 Domination and factorisation theorems

The polynomial version of the Pietsch Domination/Factorisation Theorem can be easy obtained as an application of Theorem 2.17 and [2, Theorem 2.4] or [5, Theorem 4.3]

Theorem 2.20. *An m-homogeneous polynomial $P \in \mathcal{P}(^m X; Y)$ is Cohen p-nuclear if and only if there exist Radon probability measures $\mu_1 \in C(B_{X^*})^*$, $\mu_2 \in C(B_{Y^{**}})^*$ and $C \geq 0$ such that, for all $x \in X$ and $y^* \in Y^*$,*

$$|\langle P(x), y^* \rangle| \leq C \left(\int\limits_{B_{X^*}} |x^*(x)|^{mp} d\mu_1(x^*) \right)^{\frac{1}{p}} \left(\int\limits_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\mu_2(y^{**}) \right)^{\frac{1}{p^*}}. \tag{2.10}$$

Moreover, in this case

$$\|P\|_{p,N} = \inf \{C : C \text{ verifies } 2.10\}.$$

Theorem 2.21. *Let $1 < p < \infty$, and let $P \in P(^m X; Y)$ the following conditions are equivalent:*

(i) P is Cohen p-nuclear.

(ii) There exist regular probability measures μ_1 on B_{X^*}, μ_2 on $B_{Y^{**}}$, Banach spaces $G \subset L_{mp}(B_{X^*}, \mu_1)$, H^* the dual of H subspace of $L_{p^*}(B_{Y^{**}}, \mu_2)$, absolutely mp-summing linear operator $u \in \mathcal{L}(X; G)$ and a Cohen strongly p-summing polynomial $Q \in \mathcal{P}(^m G; Y)$ such that $P = Q \circ u$. Moreover

$$\|P\|_{p,N} = \inf \{d_p(Q) \pi_{mp}(u)^m : P = Q \circ u\}$$

$\left(i.e., \mathcal{P}^c_{p,N} = \mathcal{P}^p_{coh} \circ \pi_{mp} \text{ holds isometrically} \right).$

Proof. First we prove the converse. Let $x \in X$ and $y^* \in Y^*$. Let P has the factorisation $Q \circ u$, since Q is Cohen strongly p-summing polynomial and u is absolutely mp-summing linear operator then, by [7, Theorem 2.3] and [50] we have

$$\begin{aligned} |\langle P(x), y^* \rangle| &= |\langle Q \circ u(x), y^* \rangle| \\ &\leq d_p(Q) \|u(x)\|^m \left(\int\limits_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\mu_2(y^{**}) \right)^{\frac{1}{p^*}} \end{aligned}$$

We know that see [50, Theorem 2], there is $\mu_1 \in C(B_{X^*})^*$, such that

$$\|u(x)\| \leq \pi_{mp}(u) \left(\int\limits_{B_{X^*}} |x^*(x)|^{mp} d\mu_1(x^*) \right)^{\frac{1}{mp}}.$$

Now we get

$$\left|\left\langle P\left(x\right),y^{*}\right\rangle\right|\leq d_{p}\left(Q\right)\pi_{mp}\left(u\right)^{m}\left(\int\limits_{B_{X^{*}}}\left|x^{*}\left(x\right)\right|^{mp}d\mu_{1}\left(x^{*}\right)\right)^{\frac{1}{p}}\left(\int\limits_{B_{Y^{**}}}\left|y^{**}\left(y^{*}\right)\right|^{p^{*}}d\mu_{2}\left(y^{**}\right)\right)^{\frac{1}{p^{*}}}.$$

Therefore P is p-nuclear and $\|P\|_{p,N}\leq d_{p}\left(Q\right)\pi_{mp}\left(u\right)^{m}$.

To prove the first implication, let P be Cohen p-nuclear polynomial. Then by the Inequality 2.10, *there exist Radon probability measures* $\mu_1\in C(B_{X^*})^*$ *and* $\mu_2\in C(B_{Y^{**}})^*$ *such that, for all* $x\in X$ *and* $y^*\in Y^*$,

$$\left|\left\langle P\left(x\right),y^{*}\right\rangle\right|\leq\|P\|_{p,N}\left(\int\limits_{B_{X^{*}}}\left|x^{*}\left(x\right)\right|^{mp}d\mu\left(x^{*}\right)\right)^{\frac{1}{p}}\left(\int\limits_{B_{Y^{**}}}\left|y^{**}\left(y^{*}\right)\right|^{p^{*}}d\mu_{2}\left(y^{**}\right)\right)^{\frac{1}{p^{*}}}.$$

Now, we consider the operators

$$\begin{aligned}\bar{u}&:X\rightarrow L_{mp}\left(\mu_{1}\right)\\ \bar{u}\left(x\right)&:=\left\langle x,.\right\rangle,\end{aligned}$$

and

$$\begin{aligned}\bar{h}&:Y^{*}\rightarrow L_{p^{*}}\left(\mu_{2}\right)\\ \bar{h}\left(y^{*}\right)&:=\left\langle y^{*},.\right\rangle;\end{aligned}$$

notice that

$$\begin{aligned}\left\|\bar{u}\left(x\right)\right\|&=\left(\int\limits_{B_{X^{*}}}\left|\left\langle x,x^{*}\right\rangle\right|^{mp}d\mu_{1}\left(x^{*}\right)\right)^{\frac{1}{mp}}\\&\leq\|x\|,\text{ for all }x\in X,\end{aligned}$$

and

$$\begin{aligned}\left\|\bar{h}\left(y^{*}\right)\right\|&=\left(\int\limits_{B_{Y^{**}}}\left|\left\langle y^{*},y^{**}\right\rangle\right|^{p^{*}}d\mu_{2}\left(y^{*}\right)\right)^{\frac{1}{p^{*}}}\\&\leq\|y^{*}\|,\text{ for all }y^{*}\in Y^{*}.\end{aligned}$$

Let G be the closer in $L_{mp}(\mu_1)$ of the range of $\bar{u}$ ($i.e.$, $G := \overline{\bar{u}(X)}^{L_{mp}}$) and let $u : X \to G$ be the induced operator and H be the closure in $L_{p^*}(\mu_2)$ of the range of $\bar{h}$ ($i.e.$, $H := \overline{\bar{h}(Y^*)}^{L_{p^*}}$) and $h : Y^* \to H$ be the induced operator .

Also we note that u is absolutely p-summing with $\pi_{mp}(u) \leq 1$, and h is absolutely p^*-summing with $\pi_{p^*}(h) \leq 1$.

According to Theorem 2.2.2 (i) in [28], we have $h^* : H^* \to Y^{**}$ is Cohen strongly p-summing operator and $\pi_{p^*}(h) = d_p(h^*)$.

Define the $(m+1)$-linear form T, on $\bar{u}(X) \times ... \times \bar{u}(X) \times \bar{h}(Y^*)$, by

$$T\left(u\left(x^1\right), ..., u\left(x^m\right), h\left(\varphi\right)\right) := \left\langle \varphi, \widehat{P}\left(x^1, ..., x^m\right)\right\rangle = \varphi\left(\widehat{P}\left(x^1, ..., x^m\right)\right), \tag{2.11}$$

where $x^j \in X$ $(j = 1, ..., m)$, $\varphi \in Y^*$.

Using the inequality 2.10 and by the polarization formula we have:

$$\begin{aligned}
&\left|T\left(u\left(x^1\right), ..., u\left(x^m\right), h\left(\varphi\right)\right)\right| \\
&= \left|\varphi\left(\widehat{P}\left(x^1, ..., x^m\right)\right)\right| \\
&\leq \frac{m^m}{m!}\|P\|_{p,N}\left(\int_{B_{X^*}} \left|\left\langle x^1, x^*\right\rangle\right|^{mp} d\mu_1\left(x^*\right)\right)^{\frac{1}{mp}} \times ... \times \left(\left(\int_{B_{X^*}} \left|\left\langle x^m, x^*\right\rangle\right|^{mp} d\mu_1\left(x^*\right)\right)^{\frac{1}{mp}}\right) \\
&\times \left(\left(\int_{B_{Y^{**}}} \left|\left\langle \varphi, y^{**}\right\rangle\right|^{p^*} d\mu_2\left(y^*\right)\right)^{\frac{1}{p^*}}\right) \\
&= \frac{m^m}{m!}\|P\|_{p,N}\left\|u\left(x^1\right)\right\| \times ... \times \left\|u\left(x^m\right)\right\| \times \left\|h\left(\varphi\right)\right\|.
\end{aligned}$$

It follows that T is a continuous $(m+1)$-linear form T, on $\bar{u}(X) \times ... \times \bar{u}(X) \times \bar{h}(Y^*)$ and has unique extension $\tilde{T}$ to $G \times ... \times G \to H$, and naturally induces a continuous m-linear operator $\hat{S} : G \times ... \times G \to H^*$. Hence $\hat{S}$ is continuous and defines the continuous m-homogeneous polynomial S by $S(f) = \hat{S}(f, ..., f)$. Let $x \in X$ and $\varphi \in Y^*$, then

$$\begin{aligned}
\left\langle h^* S u\left(x\right), \varphi\right\rangle &= \left\langle h^* \hat{S}\left(u\left(x\right), ..., u\left(x\right)\right), \varphi\right\rangle \\
&= \left\langle \hat{S}\left(u\left(x\right), ..., u\left(x\right)\right), h\left(\varphi\right)\right\rangle \\
&= T\left(u\left(x^1\right), ..., u\left(x^m\right), h\left(\varphi\right)\right).
\end{aligned}$$

So, by equality (2.11) we have

$$\begin{aligned}\langle h^* Su(x), \varphi\rangle &= \left\langle k_Y \widehat{P}(x, ..., x), \varphi\right\rangle \\ &= \langle k_Y P(x), \varphi\rangle.\end{aligned}$$

Hence $h^* Su = k_Y P$. We notice the image of $h^* \circ S$ is in Y. According to [7, Corrollary 3.3] we have $Q = h^* \circ S$ belongs to $\mathcal{P}^p_{coh}({}^m G; Y)$, then $P = Q \circ u$. □

An immediate consequence of Theorem 2.21, and [7, Corrollary 3.5] is the following

Corollary 2.22. *Let* $1 < p < \infty$. *Then*

$$\mathcal{P}^c_{p,N} = \mathcal{D}_p \circ \mathcal{P} \circ \Pi_{mp}.$$

Chapter 3

Tensor characterizations of summing polynomials

The results in this chapter were published in collaboration with D. Achour, E. A. Sánchez Pérez and P. Rueda in the journal of "Mediterranean Journal of mathematics " [4]. The useful of the tensor products in the linear theory geos back to the early work of Grothendieck [36], where he defined in terms of tensor product the concepts of summing linear operators. Several authers emphasize the use of this tool as: linearizing, definition of duality theory of spaces of operator and his interisting role in the theory of operator ideals. We can see in the excellent monograph [29], recently the investigation is transferring summability property to non linear mapping which is not a trivial task. Botelho and Compos [14] show how these transformations can be treated from an unified point of view in parallel even the polynomial case the work is more complicated. In this chapter we work in more general frame provided characterization of summing polynomials where we can gives an associated polynomial that is defined between tensor product spaces of sequences.

The tensor product $\ell_p \otimes X$ can be seen as a subspace of $X^{\mathbb{N}}$ via the algebraic isomorphism $s_{p,X} : \ell_p \otimes X \to X^{\mathbb{N}}$ given by $s_{p,X}(\sum_{i=1}^{n}(a_{ij})_j \otimes x_i) := (\sum_{i=1}^{n} a_{ij}x_i)_j$. The following facts are well-known and can be found in [28].

- Let $1 \leq p \leq \infty$ and let p^* be the conjugate of p, i.e. $\frac{1}{p}+\frac{1}{p^*} = 1$. The space $\ell_p^w(X)$ induces the injective norm ε on $\ell_p \otimes X$, defined as $\varepsilon(u) := \sup_{\|x^*\|_{\ell_{p^*}} \leq 1, \|y^*\| \leq 1} \left| \sum_{i=1}^{n} x^*(x_i)y^*(y_i)\right|$, for any $u = \sum_{i=1}^{n} x_i \otimes y_i \in \ell_p \otimes X$.

- Let $1 \leq p \leq \infty$. The space $\ell_p(X)$ induces the Δ_p norm on $\ell_p \otimes X$, defined as $\Delta_p(\sum_{i=1}^n e_i \otimes x_i) = (\sum_{i=1}^n \|x_i\|^p)^{1/p}$.

- Let $1 \leq p < \infty$. The space $\ell_p\langle X\rangle$ induces the projective norm π on $\ell_p \otimes X$, defined as $\pi(u) := \inf \sum_{i=1}^n \|x_i\|_{\ell_p} \|y_i\|$ where the infimum is taken over all representations of $u = \sum_{i=1}^n x_i \otimes y_i \in \ell_p \otimes X$.

The following characterizations (see [28]) provide nice examples of how tensor products come into the theory of summing operators:

- *An operator $T : X \to Y$ is absolutely p-summing if and only if $I \otimes T : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_{\Delta_p} Y$ is continuous.*

- *An operator $T : X \to Y$ is strongly p-summing if and only if $I \otimes T : \ell_p \otimes_{\Delta_p} X \to \ell_p \otimes_\pi Y$ is continuous.*

- *Let $1 < p < \infty$. An operator $T : X \to Y$ is Cohen p-nuclear if and only if $I \otimes T : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_\pi Y$ is continuous.*

The interplay between tensor products and summing polynomials has been explored for a long time, for example in [40, 32, 25, 15, 21]. In this chapter we unify and characterize particular classes of summing polynomials by introducing an associated polynomial defined between tensor product subspaces of vector-valued sequences spaces. This approach can be applied to several classes of summing polynomials, as p-dominated polynomials, strongly Cohen p-summing polynomials or Cohen p-nuclear polynomials. Note that $\mathcal{P}(^1X; Y)$ coincides with the space of all continuous linear operators from X to Y endowed with the usual norm, and so, to unify the linear case it just suffices to take $m = 1$. In that case, $P = T$ is a continuous linear operator and the associated polynomial defined between tensor product spaces is nothing but the associated tensor product operator (see the next section for definitions). The concept of finitely determined sequence classes introduced in [14], that was the key of that study, also plays a fundamental role when dealing with the transformation of tensor product spaces by homogeneous polynomials.

3.1 Associated polynomials

Given a linear operator $T : X \to Y$, its associated tensor product operator $I \otimes T : \ell_p \otimes X \to \ell_p \otimes Y$ is defined by

$$I \otimes T(\sum_{i=1}^{n}(c_{ij})_j \otimes x_i) := \sum_{i=1}^{n}(c_{ij})_j \otimes T(x_i),$$

and this map is clearly linear. When dealing with a m-homogeneous polynomial $P : X \to Y$ one can be tempted naïvely to replace the T by the P in the above definition with the hope to have an associated polynomial. However, a quick look makes us to refuse such an approach as the resulting map is not even well defined when $m \geq 2$. So, some extra work is required to introduce an associated tensor polynomial that plays the role of $I \otimes T$.

This kind of tensor product of homogeneous polynomials has already been considered in the literature (see, e.g., [12, Section 6] and [11]). For our purposes we will consider the vector space ℓ_p^0 of all sequences in ℓ_p with all entries 0 but finitely many. Let e_j denote the canonical unit vector of ℓ_p^0 with 1 in the jth coordinate and 0 otherwise. Note that if u belongs to $\ell_p^0 \otimes X$ then there exist non-necessarily unique $(a_{ij})_{j=1}^{k_i} \in \ell_p^0$ and $x_i \in X$, $i = 1, \dots, n$, so that

$$u = \sum_{i=1}^{n}(a_{ij})_{j=1}^{k_i} \otimes x_i.$$

Adding 0 if necessary, we can assume that all the k_i are equal. Let us denote them by k. Now define the associated "tensor" polynomial $\overline{P} : \ell_{mp}^0 \otimes X \to \ell_p^0 \otimes Y$ by

$$\overline{P}(\sum_{i=1}^{n}(a_{ij})_{j=1}^{k} \otimes x_i) := \sum_{j=1}^{k} e_j \otimes P(\sum_{i=1}^{n} a_{ij}x_i).$$

To check that the map $\overline{P}$ is well defined we do

$$u = \sum_{i=1}^{n}(a_{ij})_{j=1}^{k} \otimes x_i = \sum_{i=1}^{n}\sum_{j=1}^{k} a_{ij}e_j \otimes x_j = \sum_{j=1}^{k} e_j \otimes y_j$$

where $y_j := \sum_{i=1}^{n} a_{ij}x_i$, $j = 1, \dots, k$. An easy calculation shows that the representation of an element $u \in \ell_p^0 \otimes X$ of the form $u = \sum_{j=1}^{k} e_j \otimes y_j$ with $y_j \in X$ is unique and so $\overline{P}$ is well defined.

When we take $k = n$ and $a_{ij} := 1$ if $i = j$ and 0 otherwise, in particular we get

$$\overline{P}(\sum_{i=1}^{n} e_i \otimes x_i) = \sum_{i=1}^{n} e_i \otimes P(x_i).$$

In [58] tensor products have been used to characterize summability properties of linear and multilinear operators by means of an “order reduction" procedure and the calculus of traced tensor norms. The map $\overline{P}$ is the restriction to the diagonal of the m-linear symmetric operator

$$\overline{T} : (\ell_{mp}^0 \otimes X) \times (\ell_{mp}^0 \otimes X) \times \cdots \times (\ell_{mp}^0 \otimes X) \to \ell_p^0 \otimes Y$$

defined as

$$\overline{T}((\sum_{i=1}^n e_i \otimes x_i^1, \ldots, \sum_{i=1}^n e_i \otimes x_i^m)) := \sum_{i=1}^n e_i \otimes T(x_i^1, \ldots, x_i^m),$$

where T is the unique symmetric m-linear operator such that $T(x, \ldots, x) = P(x)$. Therefore $\overline{P} : \ell_{mp}^0 \otimes X \to \ell_p^0 \otimes Y$ is a m-homogeneous polynomial and

$$\check{\overline{P}} = \bar{\check{P}}.$$

The class of all Banach spaces over $\mathbb{K}$ is denoted by BAN and if $X, Y \in$ BAN then $X \overset{1}{\hookrightarrow} Y$ means that X is a linear subspace of Y and $\|x\|_Y \leq \|x\|_X$ for all $x \in X$. The set of all sequences in X with all entries 0 but finitely many is denoted by $c_{00}(X)$. We take from [14] the following definition, that will be of interest in our study.

Definition 3.1. *A class of vector-valued sequences S, or simply a sequence class S, is a rule that assigns to each $X \in$ BAN a Banach space $S(X)$ of X-valued sequences such that*

$$c_{00}(X) \subset S(X) \overset{1}{\hookrightarrow} \ell_\infty(X) \text{ and } \|e_j\|_{S(\mathbb{K})} = 1 \text{ for every } j.$$

A sequence class S is finitely determined if for every sequence $(x_j)_{j=1}^\infty \in X^{\mathbb{N}}$, $(x_j)_{j=1}^\infty \in S(X)$ if and only if $\sup_k \|(x_j)_{j=1}^k\|_{S(X)} < +\infty$ and, in this case,

$$\|(x_j)_{j=1}^\infty\|_{S(X)} = \sup_k \|(x_j)_{j=1}^k\|_{S(X)}.$$

The sequences classes $\ell_\infty(\cdot)$, $\ell_p(\cdot)$, $\ell_p^w(\cdot)$ and $\ell_p\langle\cdot\rangle$ are finitely determined [14, Remark 1.3].

Given a m-homogeneous polynomial $P : X \to Y$, let us consider the associated m-homogeneous polynomial $\widehat{P} : X^{\mathbb{N}} \to Y^{\mathbb{N}}$ naturally defined by $\widehat{P}((x_i)_i) := (P(x_i))_i$.

The linear space $\ell_p^0 \otimes X$ can be seen as a vector subspace of $X^{\mathbb{N}}$ by means of the map

$$s_{p,X}(\sum_{i=1}^n (a_{ij})_{j=1}^k \otimes x_i) := (\sum_{i=1}^n a_{ij} x_i)_{j=1}^k,$$

which is an algebraic isomorphism into.

Lemma 3.2. *If $P : X \to Y$ is a m-homogeneous polynomial then*

$$\widehat{P} \circ s_{mp,X} = s_{p,Y} \circ \overline{P}.$$

Proof. For

$$\sum_{i=1}^{n} (a_{ij})_{j=1}^{k} \otimes x_i \in \ell_p^0 \otimes X$$

we have

$$\begin{aligned} \widehat{P} \circ s_{mp,X} \left(\sum_{i=1}^{n} (a_{ij})_{j=1}^{k} \otimes x_i \right) &= \widehat{P} \left(\Big(\sum_{i=1}^{n} a_{ij} x_i \Big)_{j=1}^{k} \right) = \left(P\Big(\sum_{i=1}^{n} a_{ij} x_i \Big) \right)_{j=1}^{k} \\ &= s_{p,Y} \left(\sum_{j=1}^{k} e_j \otimes P\Big(\sum_{i=1}^{n} a_{ij} x_i \Big) \right) \\ &= s_{p,Y} \circ \overline{P} \left(\sum_{i=1}^{n} (a_{ij})_{j=1}^{k} \otimes x_i \right). \end{aligned}$$

□

Lemma 3.3. *Let S_1 and S_2 be two finitely determined sequence classes. Let $P \in \mathcal{P}(^m X; Y)$ be such that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Then $c_{00}(X)$ is a norming set for $\widehat{P} : S_1(X) \to S_2(Y)$.*

Proof. Let us write $N(\widehat{P}) := \sup \|(P(x_i))_i\|_{S_2(Y)}$, where the supremum is taken over all $(x_i)_i \in c_{00}(X)$ with $\|(x_i)_i\|_{S_1(X)} \le 1$. Clearly $N(\widehat{P}) \le \|\widehat{P}\|$. If $N(\widehat{P}) = \infty$ there is nothing to be proved. If we assume that $N(\widehat{P}) < \|\widehat{P}\|$, there is $(x_i)_{i=1}^{\infty} \in S_1(X)$ with $\|(x_i)_{i=1}^{\infty}\|_{S_1(X)} \le 1$ such that $N(\widehat{P}) < \|(P(x_i))_{i=1}^{\infty}\|_{S_2(Y)}$. Since $S_1(X)$ and $S_2(Y)$ are finitely determined,

$$\|(x_i)_{i=1}^{N}\|_{S_1(X)} \le \|(x_i)_{i=1}^{\infty}\|_{S_1(X)} \le 1$$

for every $N \in \mathbb{N}$ and

$$\|(P(x_i))_{i=1}^{\infty}\|_{S_2(Y)} = \sup_N \|(P(x_i))_{i=1}^{N}\|_{S_2(Y)} \le N(\widehat{P}),$$

which is a contradiction. □

From now on we consider two classes of vector-valued sequences S_1 and S_2, and $P \in \mathcal{P}(^m X; Y)$ so that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Note that

$$s_{mp,X}(\ell_{mp}^0 \otimes X) \subset c_{00}(X) \subset S_1(X) \text{ and } s_{p,Y}(\ell_p^0 \otimes Y) \subset c_{00}(Y) \subset S_2(Y).$$

Therefore, the following diagram arises

$$\begin{array}{ccc} S_1(X) & \xrightarrow{\widehat{P}} & S_2(Y) \\ {\scriptstyle s_{mp,X}} \uparrow & & \uparrow {\scriptstyle s_{p,Y}} \\ \ell_{mp}^0 \otimes X & \xrightarrow{\overline{P}} & \ell_p^0 \otimes Y \end{array}$$

that, in virtue of Lemma 3.2, commutes.

Theorem 3.4. *Let X and Y be Banach spaces and let S_1 and S_2 be sequence classes. Let $P \in \mathcal{P}(^m X; Y)$ be so that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Let α and β be norms on $\ell_{mp}^0 \otimes X$ and $\ell_p^0 \otimes Y$ respectively so that $s_{mp,X} : \ell_{mp}^0 \otimes_\alpha X \to S_1(X)$ and $s_{p,Y} : \ell_p^0 \otimes_\beta Y \to S_2(Y)$ are continuous.*

1. *If $s_{p,Y}$ is an isometry into then $\overline{P} : \ell_{mp}^0 \otimes_\alpha X \to \ell_p^0 \otimes_\beta Y$ is continuous whenever $\widehat{P} : S_1(X) \to S_2(Y)$ is continuous. In this case $\|\overline{P}\| \leq \|\widehat{P}\| \|s_{mp,X}\|$.*

2. *If $S_1(X)$ and $S_2(Y)$ are finitely determined and $s_{mp,X}$ is an isometry into then $\widehat{P} : S_1(X) \to S_2(Y)$ is continuous whenever $\overline{P} : \ell_{mp}^0 \otimes_\alpha X \to \ell_p^0 \otimes_\beta Y$ is continuous. In this case $\|\widehat{P}\| \leq \|\overline{P}\| \|s_{p,Y}\|$.*

Proof. (1) It follows immediately from Lemma 3.2 and the hypothesis on $s_{p,Y}$ being an isometry.

(2) Since $S_1(X)$ and $S_2(Y)$ are finitely determined, by Lemma 3.3 $c_{00}(X)$ is a norming set for $\widehat{P} : S_1(X) \to S_2(Y)$. Take $(x_i)_{i=1}^n \in c_{00}(X)$ with $\|(x_i)_{i=1}^n\|_{S_1(X)} \leq 1$. Then,

$$\begin{aligned} \left\|\widehat{P}\big((x_i)_{i=1}^n\big)\right\|_{S_2(Y)} &= \left\|\big(P(x_i)\big)_{i=1}^n\right\|_{S_2(Y)} = \left\|\Big(s_{p,Y}\big(\sum_{i=1}^n e_i \otimes P(x_i)\big)\Big)\right\|_{S_2(Y)} \\ &= \left\|s_{p,Y}\Big(\overline{P}\big(\sum_{i=1}^n e_i \otimes x_i\big)\Big)\right\|_{S_2(Y)} \\ &\leq \|s_{p,Y}\| \beta\Big(\overline{P}\big(\sum_{i=1}^n e_i \otimes x_i\big)\Big) \\ &\leq \|s_{p,Y}\| \|\overline{P}\| \alpha\big(\sum_{i=1}^n e_i \otimes x_i\big)^m = \|s_{p,Y}\| \|\overline{P}\| \left\|(x_i)_{i=1}^n\right\|_{S_1(X)}^m. \end{aligned}$$

□

Corollary 3.5. *Let X and Y be Banach spaces and let S_1 and S_2 be sequence classes. Let $P \in \mathcal{P}(^m X; Y)$ so that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Let α and β be norms on $\ell_{mp}^0 \otimes X$ and $\ell_p^0 \otimes Y$ respectively so that $s_{mp,X} : \ell_{mp}^0 \otimes_\alpha X \to S_1(X)$ and $s_{p,Y} : \ell_p^0 \otimes_\beta Y \to S_2(Y)$ are isometries into. If $S_1(X)$ and $S_2(Y)$ are finitely determined then $\widehat{P} : S_1(X) \to S_2(Y)$ is continuous if and only if $\overline{P} : \ell_{mp}^0 \otimes_\alpha X \to \ell_p^0 \otimes_\beta Y$ is continuous. In this case $\|\widehat{P}\| = \|\overline{P}\|$.*

The next result is the polynomial version of [14, Proposition 1.4]. Note that although the proof cannot be adapted straightforwardly to polynomials (because it uses the closed graph theorem for multilinear operators), it still remains true.

Proposition 3.6. *Let $m \in \mathbb{N}$, $P \in ({}^mX;Y)$ and let S_1 and S_2 be sequence classes. The following are equivalent:*

1. *$(P(x_i))_{i=1}^\infty \in S_2(Y)$ whenever $(x_i)_{i=1}^\infty \in S_1(X)$.*
2. *The induced map $\widehat{P} : S_1(X) \to S_2(Y)$ is a well-defined continuous m-homogeneous polynomial.*

The conditions above imply condition (3) below, and they are all equivalent if the sequence classes S_1 and S_2 are finitely determined.

(3) There is a constant $C > 0$ such that $\left\|\left(P(x_i)\right)_{i=1}^n\right\|_{S_2(Y)} \leq C\|(x_i)_{i=1}^n\|_{S_1(X)}$ for all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$.

In this case, $\|\widehat{P}\| = \inf\{C : (1) \text{ holds }\}$.

Proof. (2) implies (1) clearly. Assuming (2), it is also clear that $\widehat{P}$ is well-defined and a m-homogeneous polynomial. Let us prove the continuity. It suffices to be proved that the associated m-linear symmetric operator $\check{\widehat{P}}$ is continuous. Consider the m-linear operator induced by $\check{P}$, that is, $\widehat{\check{P}} : S_1(X) \times \cdots \times S_1(X) \to S_2(Y)$ given by $\widehat{\check{P}}((x_i^1)_{i=1}^\infty, \ldots, (x_i^m)_{i=1}^\infty) := \left(\check{P}(x_i^1, \ldots, x_i^m)\right)_{i=1}^\infty$, $(x_i^j)_{i=1}^\infty \in S_1(X)$, $j = 1, \ldots, m$. By the polarization formula (see e.g. [47, Theorem 1.10]), for each $i \in \mathbb{N}$

$$\check{P}(x_i^1, \ldots, x_i^m) = \frac{1}{m!2^m} \sum_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \varepsilon_1 \cdots \varepsilon_m P(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m). \tag{3.1}$$

Since $S_1(X)$ is a Banach space, the sequence $(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m)_{i=1}^\infty$ belongs to $S_1(X)$. Then, by (1) the sequence $(P(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m))_{i=1}^\infty \in S_2(Y)$. The equality (3.1) gives now that the sequence $\left(\check{P}(x_i^1, \ldots, x_i^m)\right)_{i=1}^\infty \in S_2(Y)$. From Proposition 1.4 in [14] we get that $\widehat{\check{P}}$ is well-defined and continuous. Since $\check{\widehat{P}} = \widehat{\check{P}}$, it follows that $\widehat{P}$ is continuous and condition (2) is proved. We have actually shown that if P satisfies (1) then $\check{P}$ satisfies [14, Proposition 1.4.(a)], part (c) of that result gives easily (3). The rest of the proof follows the lines of [14, Proposition 1.4]. Immediately one gets (2) implies (3) and that $\|\widehat{P}\| \geq \inf\{C : (1) \text{ holds }\}$. We now assume (3) and that $S_1(X)$ and $S_2(Y)$ are finitely determined. Taking the supremum over n in (3) we get its equivalence with (2) and $\|\widehat{P}\| \leq \inf\{C : (1) \text{ holds }\}$. □

3.2 Applications to classes of summing polynomials

We apply now Theorem 3.4 and Proposition 3.6 to some classes of summing polynomials to get new characterizations in terms of tensor product transformations and also to recover probably known characterizations of these classes in terms of transformations of vector-valued sequences. With our approach, all the results are straightforward applications of Corollary 3.5 and Proposition 3.6, just using that the sequences classes $\ell_\infty(\cdot)$, $\ell_p(\cdot)$, $\ell_p^w(\cdot)$ and $\ell_p\langle\cdot\rangle$ are finitely determined [14, Remark 1.3] and that the maps $s_{p,X} : \ell_p \otimes_\varepsilon X \to \ell_p^w(X)$, $s_{p,X} : \ell_p \otimes_{\Delta_p} X \to \ell_p(X)$ and $s_{p,X} : \ell_p \otimes_\pi X \to \ell_p\langle X\rangle$ are isometries into.

3.2.1 p-dominated polynomials

Let $m \in \mathbb{N}$, $m \leq p < \infty$ and let X and Y be Banach spaces. A m-homogeneous polynomial $P \in \mathcal{P}(^m X; Y)$ is *p-dominated* if there is a constant $C \geq 0$ such that

$$\left\| \left(P(x_i)\right)_{i=1}^n \right\|_{\ell_{p/m}(Y)} \leq C \|(x_i)_{i=1}^n\|_{\ell_{p,w}(X)}^m$$

for all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$. The infimum of all such $C > 0$ defines a norm on the space $\mathcal{P}_{p,d}(^m X; Y)$ of all p-dominated m-homogeneous polynomials from X to Y, that we denote $\|P\|_{p,d}$. For more information on p-dominated polynomials we refer to [13] and the references therein.

Corollary 3.7. *Let $m \in \mathbb{N}$, $m \leq p < \infty$ and $P \in \mathcal{P}(^m X; Y)$. The following are equivalent:*

1. *P is p-dominated.*
2. *$(P(x_i))_{i=1}^\infty \in \ell_{p/m}(Y)$ whenever $(x_i)_{i=1}^\infty \in \ell_{p,w}(X)$.*
3. *The induced map $\widehat{P} : \ell_{p,w}(X) \to \ell_{p/m}(Y)$ is a well-defined continuous m-homogeneous polynomial.*
4. *The induced m-homogeneous polynomial $\overline{P} : \ell_{p,w}^0 \otimes_\varepsilon X \to \ell_{p/m}^0 \otimes_{\Delta_{p/m}} Y$ is continuous.*

In this case $\|P\|_{p,d} = \|\widehat{P}\| = \|\overline{P}\|$.

3.2.2 Cohen strongly p-summing polynomials

Let $m \in \mathbb{N}$, $1 < p \leq \infty$ and let X and Y be Banach spaces. An m-homogeneous polynomial $P \in \mathcal{P}(^m X; Y)$ is *Cohen strongly p-summing* if there is a constant $C \geq 0$ such that

$$\left\| \big(P(x_i)\big)_{i=1}^n \right\|_{\ell_p\langle Y\rangle} \leq C \|(x_i)_{i=1}^n\|_{\ell_p(X)}^m$$

for all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$. The infimum of all such $C > 0$ defines a norm on the space $\mathcal{P}_{p,S}^c(^m X; Y)$ of all strongly Cohen p-summing m-homogeneous polynomials from X to Y, that we denote $\|P\|_{p,S}$. For more information on Cohen strongly p-summing polynomials we refer to [7].

Corollary 3.8. *Let $1 < p \leq \infty$, $m \in \mathbb{N}$ and $P \in \mathcal{P}(^m X; Y)$. The following are equivalent:*

1. *P is Cohen strongly p-summing.*
2. *$(P(x_i))_{i=1}^\infty \in \ell_p\langle Y\rangle$ whenever $(x_i)_{i=1}^\infty \in \ell_p(X)$.*
3. *The induced map $\widehat{P} : \ell_p(X) \to \ell_p\langle Y\rangle$ is a well-defined continuous m-homogeneous polynomial.*
4. *The induced m-homogeneous polynomial $\overline{P} : \ell_p^0 \otimes_{\Delta_p} X \to \ell_p^0 \otimes_\pi Y$ is continuous.*

In this case $\|P\|_{p,S} = \|\widehat{P}\| = \|\overline{P}\|$.

3.2.3 Cohen p-nuclear polynomials

Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$ and let X and Y be Banach spaces. An m-homogeneous polynomial $P \in \mathcal{P}(^m X; Y)$ is *Cohen p-nuclear* if there is a constant $C \geq 0$ such that

$$\left\| \big(P(x_i)\big)_{i=1}^n \right\|_{\ell_p\langle Y\rangle} \leq C \|(x_i)_{i=1}^n\|_{\ell_{mp}^w(X)}^m$$

for all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$. The infimum of all such $C > 0$ defines a norm on the space $\mathcal{P}_{p,N}^c(^m X; Y)$ of all Cohen p-nuclear m-homogeneous polynomials from X to Y, that we denote $\|P\|_{p,N}$.

Clearly, P is Cohen p-nuclear if and only if the (unique) symmetric m-linear operator A given by $A(x, \ldots, x) = P(x)$, is either Cohen p-nuclear in the sense of [2] or absolutely $(1; mp, \ldots, mp, p^*)$-summing in the sense of [1].

Corollary 3.9. *Let $1 < p < \infty$, $m \in \mathbb{N}$ and $P \in \mathcal{P}(^m X; Y)$. The following are equivalent:*

1. *P is Cohen p-nuclear.*

2. *$(P(x_i))_{i=1}^{\infty} \in \ell_p\langle Y\rangle$ whenever $(x_i)_{i=1}^{\infty} \in \ell_{mp}^w(X)$.*

3. *The induced map $\widehat{P} : \ell_{mp}^w(X) \to \ell_p\langle Y\rangle$ is a well-defined continuous m-homogeneous polynomial.*

4. *The induced m-homogeneous polynomial $\overline{P} : \ell_{mp}^0 \otimes_\varepsilon X \to \ell_p^0 \otimes_\pi Y$ is continuous.*

In this case $\|P\|_{p,N} = \|\widehat{P}\| = \|\overline{P}\|$.

Chapter 4

Factorable strongly p-nuclear m-homogeneous polynomials

In the last part of this thesis we characterize in terms of summability those homogeneous polynomials whose linearization is p-nuclear. This characterization provides a strong link between the theory of p-nuclear linear operators and the (non linear) homogeneous p-nuclear polynomials that significantly improves former approaches. The deep connection with Grothendieck-integral polynomials is also analyzed.

The chapter is organized as follows. Section 1 is devoted to fix the notation and to recall some definitions and basic facts. In Section 2 strongly p-nuclear multilinear and factorable strongly p-nuclear polynomials mappings (cf. Definition 2.4) are introduced and analyzed. Connections to summing polynomials are established and some fundamental properties are obtained. By passing to the linearization operator associated to the polynomial, we prove that factorable strongly p-nuclear homogeneous polynomials are those that factor through p-nuclear linear operators. We also relate them with factorable strongly p-summing polynomials, and prove a domination inequality that characterize the class. In Section 3 we prove that a homogeneous polynomial is factorable strongly p-nuclear if, and only if, its adjoint is a p^*-nuclear operator, where p^* is the conjugate of p. We also describe the space of all factorable strongly p-nuclear m-homogeneous polynomials as the dual of a suitable tensor product space. We end this chapter with Section 4, where we characterize G-integral homogeneous polynomials as those that factor through an integral linear operator. As a consequence, we show that a m-homogeneous polynomial P is G-integral if, and only if, its adjoint P^* is G-integral. We use this result to prove that if the dual of the range space is a $\mathcal{L}_{p,\lambda}$ space then, the spaces of factorable strongly p-nuclear polynomials and the space of G-integral polynomials coincide. In particular, being G-integral or factorable strongly 2-nuclear is the same for any homogeneous polynomial with range in a Hilbert space. We also prove that every G-integral homogeneous polynomial is factorable strongly p-nuclear.

4.1 Factorable strongly p-summing m-homogeneous polynomials

It is well know that the heart of the theory is the concept of summing linear operator. When moving to a non-linear context, several generalizations of (Cohen) strongly p-summing linear operators appear. This is why strongly p-summing and Cohen strongly p-summing polynomials will refer to different classes of polynomials. This does not mean any ambiguity as both concepts coincide in the linear case. In the following, we recall some known concepts related to summability of non-linear operators whose ideas were inspired by [57].

Let $1 < p \leq \infty$. A continuous m-linear operator T is *factorable strongly p-summing* if there exists $C \geq 0$ such that for every $\left(x_{i,k}^{j}\right)_{1\leq i\leq n_2, 1\leq k\leq n_1} \subset X_j$ and all positive integers n_1, n_2 we have

$$\left(\sum_{k=1}^{n_1}\left\|\sum_{i=1}^{n_2} T\left(x_{i,k}^{1}, ..., x_{i,k}^{m}\right)\right\|^{p}\right)^{\frac{1}{p}} \leq C \sup_{\|\varphi\|\leq 1,\ \varphi\in B_{\mathcal{L}(X_1\times....\times X_m)}} \left(\sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2}\varphi\left(x_{i,k}^{1}, ..., x_{i,k}^{m}\right)\right|^{p}\right)^{\frac{1}{p}}. \tag{4.1}$$

The set of all factorable strongly p-summing m-linear operator $T : X_1 \times .... \times X_m \to Y$ is denoted by $\mathcal{L}_{FSt,p}(X_1, ..., X_m; Y)$ and endowed with the norm $\|.\|_{FSt,p}$, where $\|T\|_{FSt,p}$ is given by the infimum of all constant C cheek the inequality 4.1. Note that if T is factorable strongly p-summing then making $n_2 = 1$ we have

$$\left(\sum_{k=1}^{n_1}\left\|T\left(x_{1,k}^{1}, ..., x_{1,k}^{m}\right)\right\|^{p}\right)^{\frac{1}{p}} \leq C \sup_{\|\varphi\|\leq 1,\ \varphi\in B_{\mathcal{L}(X_1\times....\times X_m)}} \left(\sum_{k=1}^{n_1}\left|\varphi\left(x_{1,k}^{1}, ..., x_{1,k}^{m}\right)\right|^{p}\right)^{\frac{1}{p}}, \tag{4.2}$$

i.e.,T is strongly p-summing introduced by Dimant in [32] . In particular every time $m = 1$, $\mathcal{L}_{FSt,p}(X_1; Y) = \Pi_p(X_1; Y)$, where $\Pi_p(X_1; Y)$ is the class of absolutely p-summing operator from X_1 to Y .

A polynomial $P \in \mathcal{P}(^m X; Y)$ is:

• *factorable strongly p-summing* $1 \leq p < \infty$ (see [54, Definition 4.1]) if there exists $C \geq 0$ such that for all n_1, n_2, all $(x_{i,k})_{1\leq i\leq n_2, 1\leq k\leq n_1} \subset X$ and all scalars λ_k^i, $1 \leq k \leq n_1$, $1 \leq i \leq n_2$ the following relation holds

$$\left(\sum_{k=1}^{n_1}\left\|\sum_{i=1}^{n_2}\lambda_k^i P\left(x_{i,k}\right)\right\|^p\right)^{\frac{1}{p}} \leq C \sup_{\substack{\|q\|\leq 1 \\ q\in\mathcal{P}(^m X)}}\left(\sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2}\lambda_k^i q\left(x_{i,k}\right)\right|^p\right)^{\frac{1}{p}}, \tag{4.3}$$

This class of polynomials is denoted by $\mathcal{P}_{F\ St,p}(X, Y)$ and endowed with the norm

$$\|P\|_{F-St,p} = \inf\left\{C \geq 0 : \ C \text{ satisfies the inequality (4.3)}\right\}.$$

• *strongly p-summing polynomial* (see [32]) if n_2 is required to exclusively take the value 1 in (4.3) (and the λ_k^1's are 1 in the real case). In this case, the class is denoted by $\mathcal{P}_{St,p}(^m X, Y)$ and its norm by $\|.\|_{St,p}$. It is not difficult to complete the following Proposition and show that factorable strongly m-homogeneous polynomials form an ideal of polynomials.

Proposition 4.1. *If $P \in \mathrm{P}_{F\ St,p}(X, Y)$ and $u : G \to X$, $v : Y \to Z$ are continuous linear operators then $v \circ P \circ u \in \mathrm{P}_{F\ St,p}(G, Z)$ and*

$$\|v \circ P \circ u\|_{F-St,p} \leq \|v\| . \|P\|_{F-St,p} \|u\|^m .$$

This new class of summing polynomials keep a big amount of the fundamental properties as a natural Pietsch Factorization type theorem. Pellegrino, Rueda and Sanchez Perez in [54] show that an homogeneous polynomials is Factorable strongly p-summing if and only if its associated multilinear map is factorable strongly p-summing or, equivalently , its linearization is absolutely p-summing. In addition they prove that this class can be obtained as the composition of ideal.

Proposition 4.2. *If $Q \in \mathcal{P}(G, X)$ and $u : X \to Y$ is an absolutely p-summing linear operator, then $u \circ Q \in \mathcal{P}_{F\ St,p}(G, Y)$*

$$\|u \circ Q\|_{F-St,p} \leq \pi_p(u) \|Q\| .$$

Proposition 4.3. *Let $P \in \mathcal{P}(X, Y)$. The following are equivalent:*
(i) $P \in \mathcal{P}_{F\ St,p}(X, Y)$
(ii) $P_{L,s}$ *is absolutely p-summing.*

(iii) $\check{P} \in \mathcal{L}_{F\ St,p}(X^m, Y)$.

In that case,

$$\|P\|_{F-St,p} = \pi_p \left(P_{L,s}\right).$$

4.2 Strongly p-nuclear multilinear mappings

We now proceed to introduce our distinguished class of p-nuclear m-linear operators. The study of this class and its polynomial version is an another main objective of the thesis as it recovers the essence of p-nuclear linear operators in a more accurate way than former classes. For the sake of clarity, we will give both definitions: for multilinear mappings and for homogeneous polynomials.

Definition 4.4. *Let $1 \leq p \leq \infty$, and let $m \geq 1$. A bounded m-linear operator $T : X_1 \times \cdots \times X_m \rightarrow Y$ is strongly p-nuclear if there exists a positive constant C such that for any $x_1^j, ..., x_n^j \in X_j\ (1 \leq j \leq m)$ and $y_1^*, ..., y_n^* \in Y^*$, we have*

$$\sum_{i=1}^{n} \left|\left\langle T\left(x_i^1, \ldots, x_i^m\right), y_i^*\right\rangle\right| \leq C \sup_{\varphi \in \mathcal{B}_{\mathcal{L}(X_1,\ldots,X_m)}} \left(\sum_{i=1}^{n} \left|\varphi\left(x_i^1, \ldots, x_i^m\right)\right|^p\right)^{\frac{1}{p}} \|(y_i^*)_i\|_{p^*,\omega}.$$

The corresponding class is denoted by $\mathcal{L}^s_{p,N}(X_1, \ldots, X_m; Y)$ and endowed with the norm $\pi^s_{p,N}(T)$, where $\pi^s_{p,N}(T)$ is given by the infimum of all constants $C \geq 0$ that satisfy the above inequality .

Remark 4.5. • *For $m = 1$, we have $\mathcal{L}^s_{p,N}(X_1; Y) = \mathcal{N}_p(X_1, Y)$, see [28] where $\mathcal{N}_p(X_1, Y)$ is the class of all p-nuclear operators from X_1 to Y. For more details refer to [28].*

• *By definition, if $p = 1$, we have $\mathcal{L}^s_{1,N}(X_1, \ldots, X_m; Y) := \mathcal{L}_{St,1}(X_1, \ldots, X_m; Y)$.*

The proof of the next proposition is quite simple.

Proposition 4.6. *Let $1 \leq p \leq \infty$, and let $T \in \mathcal{L}(X_1, \ldots, X_m; Y)$, then*

• *Every strongly p-nuclear multilinear operator is strongly p-summing, i.e.*

$$\mathcal{L}^s_{p,N}(X_1, \ldots, X_m; Y) \subset \mathcal{L}_{St,p}(X_1, \ldots, X_m; Y).$$

• *Every strongly p-nuclear multilinear operator is Cohen strongly p-summing*

$$\mathcal{L}^s_{p,N}(X_1, \ldots, X_m; Y) \subset \mathcal{D}^m_p(X_1, \ldots, X_m; Y).$$

As a straightforward consequence of the Proposition 4.6 and [1], we get

Corollary 4.7. *Every strongly p-nuclear multilinear operator is weakly compact.*

This class satisfies the analogue of Pietsch domination theorem, for the proof we use the full general Pietsch domination theorem recently presented by Pellegrino, Santos and S. Sepúlvzda in [55]

Let $X_1, ..., X_m, Y$ and $E_1, ..., E_k$ be (arbitrary) non-void sets, $\mathcal{H}$ be a family of mappings from $X_1 \times ... \times X_m$ to Y. Let also $K_1, .., K_t$ be compact Hausdorff topological spaces, $G_1, ..., G_t$ be Banach spaces and suppose that the maps

$$\begin{cases} R_j : K_j \times E_1 \times ... \times E_k \times G_j \to [0, +\infty)\ , j = 1, ..., t \\ S : \mathcal{H} \times E_1 \times ... \times E_k \times G_1 \times ... \times G_t \to [0, +\infty) \end{cases}$$

satisfy:

(1) For each $x^l \in E_l$ and $b \in G_j$, with $(j, l) \in \{1, ..., t\} \times \{1, ..., k\}$ the mapping $(R_j)_{x^1,...,x^k,b} : K_j \to [0, +\infty)$ defined by $(R_j)_{x^1,...,x^k,b}(\varphi_j) = R_j(\varphi_j, x^1, ..., x^k, b)$ is continuous.

(2) The following inequalities hold:

$$\begin{cases} R_j(\varphi_j, x^1, ..., x^k, \eta_j b^j) \leq \eta_j R_j(\varphi_j, x^1, ..., x^k, b^j) \\ S(f, x^1, ..., x^k, \alpha_1 b^1, ..., \alpha_t b^t) \geq \alpha_1 ... \alpha_t S(f, x^1, ..., x^k, b^1, ..., b^t), \end{cases}$$

for every $\varphi_j \in K_j, x^l \in E_l$ (with $l \in \{1, ..., k\}), 0 \leq \eta_j, \alpha_j \leq 1, b^j \in G_j$ with $j = 1, ..., t$ and $f \in \mathcal{H}$.

Definition 4.8. *If* $0 < p_1, ..., p_t, p < \infty$, *with* $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_t}$, *a mapping* $f : X_1 \times ... \times X_m \longrightarrow Y$ *in* $\mathcal{H}$ *is said to be* $R_1, ..., R_t$*-*S*-abstract* $(p_1, ..., p_t)$*-summing if there is a constant* $C > 0$ *so that*

$$\left(\sum_{i=1}^{n} S(f, x_i^1, ..., x_i^k, b_i^1, ..., b_i^t)^p \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{t} \sup_{\varphi \in K_j} \left(\sum_{i=1}^{n} R_j(\varphi_j, x_i^1, ..., x_i^k, b_i^j)^{p_j} \right)^{\frac{1}{p_j}},$$

for all $x_1^s, ..., x_n^s \in E_s, b_1^j, ..., b_n^j \in G_j, n \in \mathbb{N}$ *and* $(s, j) \in \{1, ..., k\} \times \{1, ..., t\}$.

Theorem 4.9. *[55]A map* $f \in \mathcal{H}$ *is* $R_1, ..., R_t$*-*S*-abstract* $(p_1, ..., p_t)$*-summing if and only if there is a constant* $C > 0$ *and Borel probability measures* μ_j *on* K_j *such that*

$$S(f, x^1, ..., x^k, b^1, ..., b^t) \leq C \prod_{j=1}^{t} \left(\int_{K_j} R_j(\varphi_j, x^1, ..., x^k, b^j)^{p_j} d\mu_j \right)^{\frac{1}{p_j}},$$

for all $x^l \in E_l$, $l \in \{1, ..., k\}$ *and* $b^j \in G_j$ *with* $j = 1, ..., t$.

As consequence of the Theorem 4.9 we get the following domination theorem

Theorem 4.10. *If* $1 < p < \infty$ *with* $1 = \frac{1}{p} + \frac{1}{p^*}$. *A continuous* m*-linear mapping* $T : X_1 \times ... \times X_m \longrightarrow Y$ *is strongly* p*-nuclear if and only if there is a constant* $C > 0$ *and Borel probability measures* μ_1 *on* $B_{\mathcal{L}(X_1,...,X_m)}$ $(1 \leq j \leq m)$ *and* μ_2 *on* $B_{Y^{**}}$*, so that for all* $(x^1, ..., x^m, y^*) \in X_1 \times .... \times X_m \times Y^*$ *the inequality*

$$|\langle T(x^1, ..., x^m), y^*\rangle| \leq$$

$$C(\int_{B_{\mathcal{L}(X_1,...,X_m)}} \left|\varphi\left(x^1, ..., x^m\right)\right|^p d\mu_1)^{\frac{1}{p}}(\int_{B_{Y^{**}}} |\phi(y^*)|^{p*} d\mu_2)^{\frac{1}{p*}}, \tag{4.4}$$

is valid.

Proof. Note that by choosing the parameters

$$\begin{cases} t = 2, r = m-1 \\ E_j = X_j, \ j = 1, ..., m-1 \\ G_1 = X_m, \text{ and } G_2 = Y^* \\ K_1 = B_{\mathcal{L}(X_1 \times .... \times X_m)} \text{ and } K_2 = B_{Y^{**}} \\ \mathcal{H} = \mathcal{L}(X_1, ..., X_m; Y) \\ p = 1, \ p_1 = p, \text{ and } p_2 = p^* \\ S(T, x^1, ..., x^m, y^*) = |\langle T(x^1, ..., x^m), y^*\rangle| \\ R_1(\varphi, x^1, ..., x^m) = |\varphi(x^1, ..., x^m)| \\ R_2(\varphi, x^1, ..., x^{m-1}, y^*) = |\phi(y^*)|, \end{cases}$$

we can easily conclude that $T : X_1 \times ... \times X_m \longrightarrow Y$ is strongly p-nuclear if and only if T is R_1,R_2-S abstract (p, p^*)-*summing.* Theorem 4.9 tells us that T is R_1,R_2-S abstract (p, p^*)-*summing* if and only if there is a $C > 0$ and there are probability measures μ_k *on* $K_k, k = 1, 2$, such that

$$S(T, x^1, ..., x^m, y^*)) \leq$$
$$C\left(\int_{K_1} R_1(\varphi, x^1, ..., x^m)^p d\mu_1\right)^{\frac{1}{p}} \times \left(\int_{K_2} R_2(\varphi, x^1, ..., x^{m-1}, y^*)^{p^*} d\mu_2\right)^{\frac{1}{p^*}},$$

i.e;

$$|\langle T(x^1, ..., x^m), y^*\rangle|$$
$$\leq C(\int_{B_{X_1^* \times ... \times X_m^*}} |\varphi(x^1, ..., x^m)|^p d\mu_1)^{\frac{1}{p}}(\int_{B_{Y^{**}}} |\langle y^*, \phi\rangle|^{p*} d\mu_2)^{\frac{1}{p*}},$$

and we recover the inequality 4.4. □

Proposition 4.11. *Let $1 \leq p \leq \infty$, and let $m \geq 1$. A bounded m-linear operator $T : X_1 \times \cdots \times X_m \to Y$ is strongly p-nuclear if there exists Banach space G, strongly p-summing linear operator $S \in \mathcal{L}(G; Y)$ and strongly p-summing multilinear operator $u \in \mathcal{L}(X_1 \times ... \times X_m; G)$ so that $T = S \circ u$.Moreover*

$$\pi^s_{p,N}(T) \leq \inf \{ d_p(S) \pi_{St,p}(u) : T = S \circ u \} \tag{4.5}$$

Proof. Let $S \in \mathcal{L}(G; Y)$ is strongly p-summing linear operator and $u \in \mathcal{L}(X_1 \times .... \times X_m; G)$ is strongly p-summing multilinear operator, where G is a Banach space. For all $x^j \in X_j \, (1 \leq j \leq m)$ and $y^* \in Y^*$, we suppose that the multilinear operator T has the factorisation $S \circ u$, by [50] and [32] we have

$$\begin{aligned} \left|\left\langle T\left(x^1, ..., x^m\right), y^*\right\rangle\right| &= \left|\left\langle S \circ u\left(x^1, ..., x^m\right), y^*\right\rangle\right| \\ &\leq d_p(S) \left\|u\left(x^1, ..., x^m\right)\right\| \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\mu_2(y^{**}) \right)^{\frac{1}{p^*}} \end{aligned}$$

We know that see [32, Proposition 1.2], there is $\mu_1 \in C\left(B_{\mathcal{L}(X_1 \times .... \times X_m)}\right)^*$, such that

$$\left\|u\left(x^1, ..., x^m\right)\right\| \leq \pi_{st,p}(u) \left(\int_{B_{\mathcal{L}(X_1 \times .... \times X_m)}} \left|\varphi\left(x^1, ..., x^m\right)\right|^p d\mu_1(\varphi) \right)^{\frac{1}{p}}.$$

Now we get

$$\left|\left\langle T\left(x^1, ..., x^m\right), y^*\right\rangle\right|$$

$$\leq d_p(S) \pi_{st,p}(u) \left(\int_{B_{\mathcal{L}(X_1 \times .... \times X_m)}} |\varphi(x^1, ..., x^m)|^p d\mu_1(\varphi) \right)^{\frac{1}{p}} \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\mu_2(y^{**}) \right)^{\frac{1}{p^*}}.$$

Therefore T is strongly p-nuclear and $\pi^s_{p,N}(T) \leq d_p(S) \pi_{st,p}(u)$.

□

Example 4.12. *Let $2 \leq p < \infty$. Consider the bilinear mapping*

$$T : \ell_2 \times \ell_2 \to \ell_\infty, \textit{given by}, T(x, y) = (x_n y_n)_n.$$

T is strongly p-nuclear. Recall that the operator $I : \ell_1 \to \ell_2$ is absolutely q-summing for $1 \leq q < \infty$; however the conjugate operator I^* mapping ℓ_2 into ℓ_∞ is strongly p-summing for $2 \leq p < \infty$. In the other hand, and according to [22] the bilinear operator

$$\begin{array}{rccl} S: & \ell_2 \times \ell_2 & \rightarrow & \ell_2 \\ & (x,y) & \longmapsto & (x_n y_n)_n\,, \end{array}$$

is strongly p-summing for every p. So, by the Proposition 4.11, the operator $T = I^* \circ S$ defined from $\ell_2 \times \ell_2$ into ℓ_∞ is strongly p-nuclear.

It is remarkable that some multilinear approaches are simple but there are several delicate, surprising and intriguing questions related to the multilinear extensions of absolutely summing operators. Indeed, let us now take the time to present examples of some surprising/challenging results/questions related to the multilinear setting which justify the efforts of the last years dedicated to the constructions of multilinear "prototypes" of absolutely summing operators. We know that every multilinear operator $T \in \mathcal{L}(X_1 \times .... \times X_m; Y)$ has an associated linear operator $T_L \in \mathcal{L}\left(X_1 \hat{\otimes} .... \hat{\otimes} X_m; Y\right)$. It is clear that if T_L is Cohen p-nuclear. Then T is strongly p-nuclear, Indeed.
For all $x_j \in X_j$ $(1 \leq j \leq m)$ and $y^* \in Y^*$, then $x_i^1 \otimes ... \otimes x_i^m \in X_1 \otimes .... \otimes X_m$, we have

$$\begin{aligned} \sum_{i=1}^{n} \left| \left\langle T\left(x_i^1, ..., x_i^m\right), y_i^* \right\rangle \right| &= \sum_{i=1}^{n} \left| \left\langle T_L\left(x_i^1 \otimes ... \otimes x_i^m\right), y_i^* \right\rangle \right| \\ &\leq n_p\left(T_L\right) \sup_{\varphi \in B_{\left(X_1 \hat{\otimes} .... \hat{\otimes} X_m\right)^*}} \left(\sum_{i=1}^{n} \left| \varphi\left(x_i^1 \otimes ... \otimes x_i^m\right) \right|^p \right)^{\frac{1}{p}} \left\| (y_i^*)_i \right\|_{p^*,\omega} \\ &= n_p\left(T_L\right) \sup_{\phi \in B_{\mathcal{L}\left(X_1 \times .... \times X_m\right)}} \left(\sum_{i=1}^{n} \left| \phi\left(x_i^1, ..., x_i^m\right) \right|^p \right)^{\frac{1}{p}} \left\| (y_i^*)_i \right\|_{p^*,\omega}. \end{aligned}$$

Hence T is strongly p-nuclear. Moreover $\pi_{p,N}^s(T) \leq n_p(T_L)$.

Open Problem. Does $T \in \mathcal{L}(X_1 \times .... \times X_m; Y)$ strongly p-nuclear imply $T_L \in \mathcal{L}\left(X_1 \hat{\otimes} .... \hat{\otimes} X_m; Y\right)$ is Cohen p-nuclear ?

If the answer of the problem is yes, we can define commutative diagrams and factorizations of strongly p-nuclear multilinear operators as in the linear case.

4.3 Factorable strongly p-nuclear m-homogeneous polynomials

Here, we introduce a new attempt of lifting properties of absolutely p-summing linear operator to the multilinear and polynomial cases. Factorable strongly p-nuclear multilinear

/polynomials mappings which is a subclass of strongly p-nuclear multilinear/ polynomials operators. Our motivation of this new definition, that trying to keep a big amount of the fundamental properties as a natural Pietsch Factorization type theorem or weak compactness and ensure the path to linearization

Definition 4.13. *Let $1 \leq p \leq \infty$, and let $m \geq 1$. A bounded m-linear operator $T : X_1 \times \cdots \times X_m \to Y$ is factorable strongly p-nuclear, if there exists $C \geq 0$ such that for all natural numbers n_1, n_2, all $\left(x_{i,k}^j\right)_{1\leq i\leq n_2, 1\leq k\leq n_1} \subset X_j$, $1 \leq j \leq m$, and all $(y_k^*)_{1\leq k\leq n_1} \subset Y^*$, the following holds*

$$\sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2}\langle T\left(x_{i,k}^1,\ldots,x_{i,k}^m\right),y_k^*\rangle\right| \leq C \sup_{\varphi\in\mathcal{B}_{\mathcal{L}(X_1,\ldots,X_m)}}\left(\sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2}\varphi\left(x_{i,k}^1,\ldots,x_{i,k}^m\right)\right|^p\right)^{\frac{1}{p}}\|(y_k^*)_k\|_{p^*,\omega},$$

where the supremum is taken over all m-linear functionals $\varphi : X_1 \times \cdots \times X_m \to \mathbb{K}$ with $\|\varphi\| \leq 1$.

The space of all factorable strongly p-nuclear operators is denoted by $\mathcal{L}_{p,N}^{fs}(X_1,\ldots,X_m;Y)$ and endowed with the norm $\pi_{p,N}^{fs}(T)$ where $\pi_{p,N}^{fs}(T)$ is given by the infimum of all constants $C \geq 0$ that satisfy the above inequality.

Note that if T is factorable strongly p-nuclear then :

- Making $n_2 = 1$ we have

$$\sum_{k=1}^{n_1}\left|\langle T\left(x_{1,k}^1,\ldots,x_{1,k}^m\right),y_k^*\rangle\right| \leq C\sup_{\|\varphi\|\leq 1}\left(\sum_{k=1}^{n_1}\left|\varphi\left(x_{1,k}^1,\ldots,x_{1,k}^m\right)\right|^p\right)^{\frac{1}{p}}\|(y_k^*)_k\|_{p^*,\omega}.$$

When T satisfies just this condition, we get T is *strongly p-nuclear*. The corresponding class is denoted by $\mathcal{L}_{p,N}^{s}(X_1,\ldots,X_m;Y)$ and endowed with the norm $\pi_{p,N}^{s}(T)$, where $\pi_{p,N}^{s}(T)$ is given by the infimum of all constants $C \geq 0$ that satisfy the above inequality. Therefore, $\mathcal{L}_{p,N}^{fs}(X_1,\ldots,X_m;Y) \subset \mathcal{L}_{p,N}^{s}(X_1,\ldots,X_m;Y)$ continuously.

- In particular for $m = 1$, we have $\mathcal{L}_{p,N}^{s}(X_1;Y) = \mathcal{L}_{p,N}^{fs}(X_1;Y) = \mathcal{N}_p(X_1,Y)$, where $\mathcal{N}_p(X_1,Y)$ is the class of all Cohen $p-$nuclear operators from X_1 to Y. For more details refer to [28].

Now we give the polynomial version.

Definition 4.14. *Let $1 \leq p \leq \infty$ and $P \in \mathcal{P}(^mX;Y)$. We say that P is strongly p-nuclear if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, all $x_1,\ldots,x_n \in X$ and all $y_1^*,\ldots,y_n^* \in$*

Y^*, the following holds

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \leq C \sup_{q \in B_{\mathcal{P}(^m X)}} \left(\sum_{i=1}^{n} |q(x_i)|^p \right)^{\frac{1}{p}} \|(y_i^*)_i\|_{p^*,\omega} .$$

We denote by $\mathcal{P}_{p,N}^s(^m X; Y)$ the class of all strongly p-nuclear polynomials from X into Y endowed with the norm $\pi_{p,N}^s(P)$ given by the infimum of all C's.

When $m = 1$, 1-homogeneous polynomials reduce to linear operators. In that case, we use the classical notation of Cohen p-nuclear linear operators $\mathcal{N}_p(X, Y)$ and n_p for the norm. For details we refer to [28] (see also [31, Page 187] and [29, 19.1]).

Remark 4.15. *The following relations follow easily:*

1. *Every Cohen p-nuclear polynomial is strongly p-nuclear, i.e.*

$$\mathcal{P}_{p,N}^c(^m X; Y) \subset \mathcal{P}_{p,N}^s(^m X; Y).$$

2. *Every strongly p-nuclear polynomial $P : X \to Y$ is Cohen strongly p-summing.*

3. *Every strongly p-nuclear polynomial is strongly p-summing, i.e.*

$$\mathcal{P}_{p,N}^s(^m X; Y) \subset \mathcal{P}_{St,p}(^m X, Y).$$

Pellegrino, Rueda and Sánchez-Pérez [54] have determined those strongly summing multilinear operators/polynomials that not only share the main properties of p-summing linear operators, as Grothendieck's Theorem, Pietsch Domination Theorem and Dvoretzky–Rogers Theorem, but also that have even better properties like weak compactness and a natural factorization theorem. These m-homogeneous polynomials are called *factorable strongly p-summing polynomials*. Even though these polynomials have been introduced by means of a summing inequality, they form the well-known class of composition with absolutely summing operators. This makes possible the characterization of a factorable strongly p-summing polynomial via the corresponding associated multilinear mapping and its linearization. In concrete, a m-homogeneous polynomial is factorable strongly p-summing if and only if its associated multilinear map is factorable p-summing or, equivalently, its linearization is p-summing. The multilinear version has been studied by Popa in [53]. Our objective is to apply these techniques to give an appropriate definition with the aim of characterizing those

p-nuclear polynomials whose linearization is Cohen p-nuclear. This will guaranty the best possible way to generalize linear results to the polynomial and multilinear setting.

Definition 4.16. *Let $P \in \mathcal{P}(^m X;Y)$ and $1 \leq p \leq \infty$. We say that P is factorable strongly p-nuclear if there exists a constant $C \geq 0$ such that for all $n_1, n_2 \in \mathbb{N}$, all $(x_{i,k})_{1\leq i\leq n_2, 1\leq k\leq n_1} \subset X$, all $(y_k^*)_{1\leq k\leq n_1} \subset Y^*$, and all scalars $\lambda_{i,k}$, $1 \leq i \leq n_2, 1 \leq k \leq n_1$, we have*

$$\sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2}\lambda_k^i\langle P\left(x_{i,k}\right), y_k^*\rangle\right| \leq C \sup_{\|q\|\leq 1, q\in\mathcal{P}(^m X)}\left(\sum_{k=1}^{n_1}\left|\sum_{i=1}^{n_2}\lambda_k^i q\left(x_{i,k}\right)\right|^p\right)^{\frac{1}{p}} \|(y_k^*)_k\|_{p^*,\omega}.$$

The class of all factorable strongly p-nuclear m-homogeneous polynomials from X to Y is denoted by $\mathcal{P}_{p,N}^{fs}\left(^m X;Y\right)$ and endowed with the norm $\pi_{p,N}^{fs}$ given by the infimum of all constants C as above.

Remark 4.17. *The following statements come immediately from the definition:*

1. *If we take $m = 1$ we get the class of Cohen p-nuclear linear operators $\mathcal{N}_p\left(X_1, Y\right)$.*
2. *Taking $n_2 = 1$ in the definition of factorable strongly p-nuclear polynomial, we get that every factorable strongly p-nuclear polynomial is strongly p-nuclear, i.e.*

$$\mathcal{P}_{p,N}^{fs}\left(^m X;Y\right) \subset \mathcal{P}_{p,N}^{s}\left(^m X;Y\right).$$

3. *When $p = 1$, we obtain $\mathcal{P}_{1,N}^{fs}\left(^m X;Y\right) = \mathcal{P}_{FSt,1}\left(^m X;Y\right)$.*

The following result justifies the introduction of factorable strongly p-nuclear polynomials as it establishes a direct connection with their linearization.

Theorem 4.18. *Let $1 < p \leq \infty$ and $P \in \mathcal{P}\left(^m X;Y\right)$. Then, the following assertions are equivalent*

1. *P is factorable strongly p-nuclear.*
2. *$P_L : \widehat{\otimes}_{\pi_s}^{m,s} X \to Y$ is Cohen p-nuclear.*
3. *There exist a Cohen strongly p-summing operator u and a factorable strongly p-summing polynomial Q such that $P = u \circ Q$, i.e.*

$$\mathcal{P}_{p,N}^{fs}\left(^m X;Y\right) = \mathcal{D}_p \circ \mathcal{P}_{F-St,p}\left(^m X;Y\right).$$

In that case, $\pi_{p,N}(P_L) = \pi^{fs}_{p,N}(P)$.

Proof. (1) $\Rightarrow$ (2) Suppose that P is a factorable strongly p-nuclear polynomial and take $(u_k)_{1\leq k\leq n_1} \subset \otimes^{m,s} X$ and $(y_k^*)_{1\leq k\leq n_1} \subset Y^*$. Then, completing the sum with zeros if necessary, there is a natural number n_2 so that we can write each $u_k = \sum_{i=1}^{n_2} \lambda_k^i x_{i,k} \otimes \cdots \otimes x_{i,k}$ for some $x_{i,k} \in X$ and scalars λ_k^i. Since P is factorable strongly p-nuclear it follows that

$$\begin{aligned}
\sum_{k=1}^{n_1} |\langle P_L(u_k), y_k^*\rangle| &= \sum_{k=1}^{n_1} \left| \sum_{i=1}^{n_2} \lambda_k^i \langle P\left(x_{i,k}\right), y_k^*\rangle \right| \\
&\leq \pi^{fs}_{p,N}(P) \sup_{\|q\|\leq 1, q\in\mathcal{P}(^m X)} \left(\sum_{k=1}^{n_1} \left| \sum_{i=1}^{n_2} \lambda_k^i q\left(x_{i,k}\right) \right|^p \right)^{\frac{1}{p}} \|(y_k^*)_k\|_{p^*,\omega} \\
&= \pi^{fs}_{p,N}(P) \sup_{\|q_L\|\leq 1, q_L\in\left(\widehat{\otimes}^{m,s}_{\pi_s} X\right)^*} \left(\sum_{k=1}^{n_1} \left| \sum_{i=1}^{n_2} \lambda_k^i q_L\left(x_{i,k}\otimes\cdots\otimes x_{i,k}\right) \right|^p \right)^{\frac{1}{p}} \|(y_k^*)_k\|_{p^*,\omega} \\
&= \pi^{fs}_{p,N}(P) \sup_{\|q_L\|\leq 1, q_L\in\left(\widehat{\otimes}^{m,s}_{\pi_s} X\right)^*} \left(\sum_{k=1}^{n_1} \left| q_L\left(u_k\right)\right|^p \right)^{\frac{1}{p}} \|(y_k^*)_k\|_{p^*,\omega}.
\end{aligned}$$

By density of $\otimes^{m,s} X$ in $\widehat{\otimes}^{m,s}_{\pi_s} X$ we conclude that the operator P_L is Cohen p-nuclear and

$$\pi_{p,N}\left(P_L\right) \leq \pi^{fs}_{p,N}\left(P\right).$$

(2) $\Rightarrow$ (1) Let us suppose that the linearization $P_L : \widehat{\otimes}^{m,s}_{\pi_s} X \to Y$ is Cohen p-nuclear. Take $(x_{i,k})_{1\leq i\leq n_2, 1\leq k\leq n_1} \subset X$, $(y_k^*)_{1\leq k\leq n_1} \subset Y^*$ and scalars $(\lambda_k^i)_{1\leq i\leq n_2, 1\leq k\leq n_1}$. Then, for each $1\leq k\leq n_1$ the element $u_k = \sum\limits_{i=1}^{n_2} \lambda_k^i x_{i,k}\otimes\cdots\otimes x_{i,k}$ belongs to $\widehat{\otimes}^{m}_{\pi,s} X$. Since $P_L : \widehat{\otimes}^{m,s}_{\pi_s} X \to Y$ is p-nuclear we have:

$$\begin{aligned}
\sum_{k=1}^{n_1} \left| \sum_{i=1}^{n_2} \lambda_k^i \langle P\left(x_{i,k}\right), y_k^*\rangle \right| &= \sum_{k=1}^{n_1} \left|\langle P_L\left(u_k\right), y_k^*\rangle\right| \\
&\leq \pi_{p,N}\left(P_L\right) \left\|(u_k)_{1\leq k\leq n_1}\right\|_{p,\omega} \|(y_k^*)_{1\leq k\leq n_1}\|_{p^*,\omega} \\
&= \pi_{p,N}\left(P_L\right) \sup_{\|q\|\leq 1, q\in\mathcal{P}(^m X)} \left(\sum_{k=1}^{n_1} \left| \sum_{i=1}^{n_2} \lambda_k^i q\left(x_{i,k}\right) \right|^p \right)^{\frac{1}{p}} \|(y_k^*)_{1\leq k\leq n}\|_{p^*,\omega}.
\end{aligned}$$

Hence P is factorable strongly p-nuclear and

$$\pi^{fs}_{p,N}\left(P\right) \leq \pi_{p,N}\left(P_L\right).$$

(2) $\Rightarrow$ (3) By [31, Theorem 9.7] (or [29, 19.3]) the p-nuclear operator P_L factors as $P_L = S\circ T$, where S is a Cohen strongly p-summing operator and T is a p-summing operator. By [54, Proposition 4.4] the composition $T\circ\delta_m$ is a factorable strongly p-summing polynomial, and

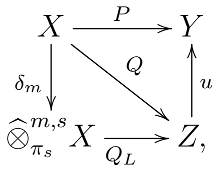
$$\|T\circ\delta_m\|_{F-St,p} \leq \pi_p\left(T\right)\|\delta_m\|.$$

Then

$$
\begin{aligned}
P &= P_L \circ \delta_m \\
&= S \circ T \circ \delta_m
\end{aligned}
$$

and (3) follows.

(3) $\Rightarrow$ (2) Let u be a Cohen strongly p-summing operator and Q be a factorable strongly p-summing polynomial such that $P = u \circ Q$. Then, the following diagram commutes:



i.e. $P = u \circ Q_L \circ \delta_m$. Since Q is factorable strongly p-summing polynomial, by [54, Proposition 4.4] its linearization Q_L is absolutely p-summing and, by [31, Theorem 9.7] the composition $P_L = u \circ Q_L$ is p-nuclear.

$\square$

An immediate consequence of Theorem 4.18, [15, Proposition 3.2 (b)] and [31, Theorem 9.7] is the following corollary.

Corollary 4.19. *(Factorization Theorem). Let X, Y be Banach spaces. Then*

i) $\mathcal{P}_{p,N}^{fs}\left({}^{m}X;Y\right) = \mathcal{N}_p \circ \mathcal{P}\left({}^{m}X;Y\right)$.

ii) $\mathcal{P}_{p,N}^{fs}\left({}^{m}X;Y\right) = \mathcal{D}_p \circ \Pi_p \circ \mathcal{P}\left({}^{m}X;Y\right)$.

Corollary 4.20. *Every factorable strongly p-nuclear homogeneous polynomial between arbitrary Banach spaces is factorable strongly p-summing.*

Proof. If $P \in \mathcal{P}({}^{m}X;Y)$ is factorable strongly p-nuclear then P_L is p-nuclear by Theorem 4.18. Then, by [28, Theorem 2.2.1] P_L is absolutely p-summing. Finally, [54, Proposition 4.8] gives that P is factorable strongly p-summing. $\square$

Corollary 4.21. *Every factorable strongly p-nuclear homogeneous polynomial between arbitrary Banach spaces is weakly compact.*

From Theorem 4.18, [54, Proposition 4.2], [15, Proposition 3.7 (b)] and Corollary 4.19, we can deduce that factorable strongly p-nuclear m-homogeneous polynomials form an ideal of polynomials.

Proposition 4.22. *The class of factorable strongly p-nuclear m-homogeneous polynomials constitutes a Banach ideal of m-homogeneous polynomials.*

Let us characterize factorable strongly p-nuclearity in terms of a domination inequality.

Proposition 4.23. *Given $P \in \mathcal{P}(^m X; Y)$ and $1 < p \leq \infty$. P is factorable strongly p-nuclear if and only if there are a constant $C \geq 0$ and regular probability measures μ on $B_{\mathcal{P}(^m X)}$ and λ on $B_{Y^{**}}$ (both endowed with the weak star topology) such that for all $(x_i)_{1\leq i\leq n} \subset X$ and $y^* \in Y^*$,the following relation holds*

$$\sum_{i=1}^{n} |\langle P(x_i), y^*\rangle| \leq C \left(\int_{B_{\mathcal{P}(^m X)}} \left| \sum_{i=1}^{n} q(x_i) \right|^p d\mu(q) \right)^{\frac{1}{p}} \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}}.$$

Moreover $\pi^{fs}_{p,N}(P) = \inf\{C > 0 \colon C$ *satisfies the above inequality* $\}$.

Proof. By Theorem 4.18 P is factorable strongly p-nuclear if and only if its linearization $P_L : \widehat{\otimes}^{m,s}_{\pi_s} X \to Y$ is p-nuclear. In this case $\pi_{p,N}(P_L) = \pi^{fs}_{p,N}(P)$. By the Pietsch domination theorem [31, Theorem 9.7 (iii)], there are $C > 0$ and regular probability measures μ on $B_{\left(\widehat{\otimes}^{m,s}_{\pi_s} X\right)^*}$ and λ on $B_{Y^{**}}$ such that for all $u \in \widehat{\otimes}^{m,s}_{\pi_s} X$ and $y^* \in Y^*$ we have

$$|\langle P_L(u), y^*\rangle| \leq C \left(\int_{B_{\left(\widehat{\otimes}^{m,s}_{\pi_s} X\right)^*}} |q(u)|^p d\mu(q) \right)^{\frac{1}{p}} \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}} \tag{4.6}$$

Moreover, $\pi_{p,N}(P_L)$ is the infimum of the constants C.

Let $(x_i)_{1\leq i\leq n}$ in X. Applying (4.6) to the tensor $u = \sum_{i=1}^{n} x_i \otimes \cdots \otimes x_i$, we get

$$\left| \sum_{i=1}^{n} \langle P(x_i), y^*\rangle \right| \leq C \left(\int_{B_{\mathcal{P}(^m X)}} \left| \sum_{i=1}^{n} q(x_i) \right|^p d\mu(q) \right)^{\frac{1}{p}} \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}}.$$

□

Theorem 4.18 has a natural multilinear counterpart. We just state the following characterization as we will make explicit use of it in Corollary 4.25.

Proposition 4.24. *Let $1 \leq p \leq \infty$. Let $X_1, \ldots, X_m, Y$ be Banach spaces and $T : X_1 \times \cdots \times X_m \to Y$ be an m-linear operator. Then T is factorable strongly p-nuclear if, and only if, its linearization $T_L : X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m \to Y$ Cohen is p-nuclear. Moreover, $\pi^{fs}_{p,N}(T) = n_p(T_L)$.*

Proof. Let us suppose that T is factorable strongly p-nuclear. Let $(u_k)_{1\leq k\leq n} \subset X_1 \otimes_\pi .... \otimes_\pi X_m$.Then there exists a natural number n_k and $\left(x^j_{i,k}\right)_{1\leq i\leq n_k} \subset X_j$ $(1 \leq j \leq m)$ such that

$$u_k = \sum_{i=1}^{n_k} x^1_{i,k} \otimes ... \otimes x^m_{i,k}. forall 1 \leq k \leq n,$$

we have

$$T_L(u_k) = \sum_{i=1}^{n_k} T_L\left(x^1_{i,k} \otimes ... \otimes x^m_{i,k}\right) = \sum_{i=1}^{n_k} T\left(x^1_{i,k}, ..., x^m_{i,k}\right)$$

and

$$\hat{\varphi}(u_k) = \sum_{i=1}^{n_k} \hat{\varphi}\left(x^1_{i,k} \otimes ... \otimes x^m_{i,k}\right) = \sum_{i=1}^{n_k} \varphi\left(x^1_{i,k}, ..., x^m_{i,k}\right),$$

where $\psi : X_1 \times .... \times X_m \to \mathbb{K}$ is a bounded m-linear functional.

Now since T is factorable strongly p-nuclear

$$\begin{aligned}
&\sum_{k=1}^{n} |\langle T_L(u_k), y^*_k\rangle| \\
&= \sum_{k=1}^{n} \left| \sum_{i=1}^{n_k} \langle T\left(x^1_{i,k}, ..., x^m_{i,k}\right), y^*_k\rangle \right| \\
&\leq \pi^{fs}_{p,N}(T) \sup_{\|\varphi\|\leq 1} \left(\sum_{k=1}^{n} \left| \sum_{i=1}^{n_k} \varphi\left(x^1_{i,k}, ..., x^m_{i,k}\right) \right|^p \right)^{\frac{1}{p}} \|(y^*_k)_k\|_{p^*,\omega} \\
&= \pi^{fs}_{p,N}(T) \sup_{\substack{\|\hat{\varphi}\|\leq 1 \\ \hat{\varphi}\in(X_1\otimes_\pi....\otimes_\pi X_m)^*}} \left(\sum_{k=1}^{n} \left| \sum_{i=1}^{n_k} \hat{\varphi}\left(x^1_{i,k} \otimes ... \otimes x^m_{i,k}\right) \right|^p \right)^{\frac{1}{p}} \|(y^*_k)_k\|_{p^*,\omega} \\
&= \pi^{fs}_{p,N}(T) \sup_{\substack{\|\hat{\varphi}\|\leq 1 \\ \hat{\varphi}\in(X_1\otimes_\pi....\otimes_\pi X_m)^*}} \left(\sum_{k=1}^{n} |\hat{\varphi}(u_k)|^p \right)^{\frac{1}{p}} \|(y^*_k)_k\|_{p^*,\omega}.
\end{aligned}$$

So we get

$$\sum_{k=1}^{n} |\langle T_L(u_k), y^*_k\rangle| \leq \pi^{fs}_{p,N}(T) \sup_{\|\hat{\varphi}\|\leq 1, \hat{\varphi}\in(X_1\otimes_\pi....\otimes_\pi X_m)^*} \left(\sum_{k=1}^{n} |\hat{\varphi}(u_k)|^p \right)^{\frac{1}{p}} \|(y^*_k)_k\|_{p^*,\omega}. \tag{4.7}$$

Since $X_1 \otimes_\pi .... \otimes_\pi X_m$ is (norm) dense in $X_1 \widehat{\otimes}_\pi .... \widehat{\otimes}_\pi X_m$ we deduce from 4.7 that for every $(v_k)_{1\leq k\leq n} \subset X_1 \widehat{\otimes}_\pi .... \widehat{\otimes}_\pi X_m$ the following relation holds

$$\sum_{k=1}^{n} |\langle T_L(v_k), y^*_k\rangle| \leq \pi^{fs}_{p,N}(T) \sup_{\|\hat{\varphi}\|\leq 1, \hat{\varphi}\in\left(X_1\widehat{\otimes}_\pi....\widehat{\otimes}_\pi X_m\right)^*} \left(\sum_{k=1}^{n} |\hat{\varphi}(v_k)|^p \right)^{\frac{1}{p}} \|(y^*_k)_k\|_{p^*,\omega}$$

hence T_L is Cohen p- nuclear, and

$$n_p\left(T_L\right) \leq \pi_{p,N}^{fs}\left(T\right). \tag{4.8}$$

Conversely. Now let suppose that $T_L : X_1\widehat{\otimes}_\pi ....\widehat{\otimes}_\pi X_m \to Y$ is Cohen p-nuclear. Let n, n_k be natural numbers and $\left(x_{i,k}^j\right)_{1\leq i\leq n_k, 1\leq k\leq n} \subset X_j, (1 \leq j \leq m)$. Then for each $1 \leq k \leq n$ the element $u_k = \sum\limits_{i=1}^{n_k} x_{i,k}^1 \otimes ... \otimes x_{i,k}^m \in X_1 \otimes_\pi .... \otimes_\pi X_m$ and
$T_L\left(u_k\right) = \sum\limits_{i=1}^{n_k} T\left(x_{i,k}^1, ..., x_{i,k}^m\right)$.
Since T_L is Cohen p- nuclear we have

$$\sum_{k=1}^{n} \left|\left\langle T_L\left(u_k\right), y_k^*\right\rangle\right| \leq n_p\left(T_L\right) \sup_{\|\hat{\varphi}\|\leq 1, \hat{\varphi}\in\left(X_1\widehat{\otimes}_\pi ....\widehat{\otimes}_\pi X_m\right)^*} \left(\sum_{k=1}^{n} \left|\hat{\varphi}\left(u_k\right)\right|^p\right)^{\frac{1}{p}} \left\|(y_k^*)_k\right\|_{p^*,\omega},$$

or equivalently,

$$\sum_{k=1}^{n} \left|\sum_{i=1}^{n_k} \left\langle T\left(x_{i,k}^1, ..., x_{i,k}^m\right), y_k^*\right\rangle\right| \leq n_p\left(T_L\right) \sup_{\|\varphi\|\leq 1} \left(\sum_{k=1}^{n} \left|\sum_{i=1}^{n_k} \varphi\left(x_{i,k}^1, ..., x_{i,k}^m\right)\right|^p\right)^{\frac{1}{p}} \left\|(y_k^*)_k\right\|_{p^*,\omega},$$

where $\varphi : X_1 \times \cdots \times X_m \to K$ is a m-linear functionals, with $\|\varphi\| \leq 1$.
which means that T is factorable strongly p-nuclear, moreover

$$\pi_{p,N}^{fs}\left(T\right) \leq n_p\left(T_L\right). \tag{4.9}$$

combining the inequalities 4.8and 4.9 we obtain

$$\pi_{p,N}^{fs}\left(T\right) = n_p\left(T_L\right).$$

□

Corollary 4.25. *Let $P \in \mathcal{P}\left({}^mX; Y\right)$. Then, P is factorable strongly p-nuclear if, and only if, its associated multilinear operator $\hat{P}$ is factorable strongly p-nuclear. In this case,* $\pi_{p,N}^{fs}\left(P\right) = \pi_{p,N}^{fs}\left(\hat{P}\right)$.

4.4 Duality

In this second part of the chapter we are interested to the adjoint of a m-homogeneous polynomial. While we inspired the idea of an old result of Cohen (see [28, Theorem 2.2.4]) states that a linear operator is p-nuclear if, and only if, its adjoint is p^*-nuclear. We start proving a polynomial variant of this theorem. After we move to the corresponding tensor representation of factorable strongly p-nuclear m-homogeneous polynomial space.

Theorem 4.26. *Let $1 < p \leq \infty$. A polynomial $P \in \mathcal{P}\left({}^{m}X;Y\right)$ is factorable strongly p-nuclear if, and only if, the adjoint operator $P^* : Y^* \to \mathcal{P}\left({}^{m}X\right)$ is p^*-nuclear.*

Proof. Assume first that $P : X \to Y$ is factorable strongly p-nuclear. By Theorem 4.18 its linearization $P_L : \widehat{\otimes}_{\pi_s}^{m,s} X \to Y$ is p-nuclear. By [28, Theorem 2.2.4] its adjoint $P_L^* : Y^* \to (\widehat{\otimes}_{\pi_s}^{m,s} X)^*$ is a p^*-nuclear operator. Consider the isometric isomorphism $\Delta_m : \mathcal{P}({}^{m}X) \to (\widehat{\otimes}_{\pi_s}^{m,s} X)^*$ given by $\Delta_m(P) = P_L$. Since $P = P_L \circ \delta_m$, by duality we get $P^* = \delta_m^* \circ P_L^* = \Delta_m^{-1} \circ P_L^*$. The ideal property ensures that P^* is p^*-nuclear.

Conversely, assume that P^* is p^*-nuclear. Then the equality $P_L^* = \Delta_m \circ P^*$ and the ideal property gives that P_L^* is p^*-nuclear. Again, [28, Theorem 2.2.4] gives that P_L is p-nuclear, and by Theorem 4.18 we conclude that P is factorable strongly p-nuclear. □

Combining [54, Proposition 4.8], [28, Theorem 2.2.2] and the ideal property we get the following result.

Proposition 4.27. *Let $1 \leq p < \infty$. A polynomial $P \in \mathcal{P}({}^{m}X;Y)$ is factorable strongly p-summing if, and only if, the adjoint operator P^* belongs to $\mathcal{D}_{p^*}\left(Y^*, \mathcal{P}\left({}^{m}X\right)\right)$.*

Cohen in [28, Lemma 2.5.1] proved that, the p-nuclear operators can be described in terms of a suitable tensor product. i.e;

$$\mathcal{N}_p\left(X,Y\right) = \left(X \otimes_{\omega_p} Y^*\right)^*; \tag{4.10}$$

where $\omega_p\left(.\right)$ is a reasonable cross norm on $X \otimes Y$ defined by:

$$\omega_p\left(u\right) = \inf\left\{\left\|(x_i)_{i=1}^{n}\right\|_{p,\omega}\left\|(y_i)_{i=1}^{n}\right\|_{p^*,\omega}\right\},$$

the infimum being taken over all representations of $u = \sum\limits_i x_i \otimes y_i$ in $X \otimes Y$.

From Theorem 4.18 and the isometric isomorphism (4.10) we extend this to m-homogeneous factorable strongly p-nuclear polynomials.

Corollary 4.28. *The spaces $\mathcal{P}_{p,N}^{fs}({}^{m}X;Y)$ and $\left(\widehat{\otimes}_{\pi_s}^{m,s} X \otimes_{\omega_p} Y^*\right)^*$ are isometrically isomorphic.*

4.5 Relation with G-integral polynomials

In [36], Grothendieck introduced the integral operators, which we call G-integral, between Banach spaces. This notion has been widely studied and applied by many authors (e.g. [23], [24], [63] and the references therein).

A polynomial $P \in \mathcal{P}\left({}^m X; Y\right)$ is *Grothendieck-integral* [8] (G-integral for short) if there exists a regular Y^{**}-valued Borel measure G of bounded variation on B_{X^*}, endowed with the weak star topology, such that

$$P(x) = \int_{B_{X^*}} \gamma(x)^m dG(\gamma)$$

for all $x \in X$. The space of m-homogeneous G-integral polynomials is denoted by $\mathcal{P}_{GI}\left({}^m X; Y\right)$ ($\mathcal{L}_{GI}(X;Y)$ when $m = 1$) and the integral norm of a polynomial $P \in \mathcal{P}_{GI}\left({}^m X; Y\right)$ is defined as

$$\|P\|_{GI} = \inf\left\{|G|\left(B_{X^*}\right)\right\},$$

where the infimum is taken over all measures G representing P.

In [24, Proposition 2.5] and [23, Proposition 1], the authors show that the linearization $P_L : \otimes^{m,s} X \to Y$ of any G-integral polynomial $P : X \to Y$ is continuous when $\otimes^{m,s} X$ is endowed with the s-injective norm ϵ_s. We shall denote $P_{L,\epsilon} : \widehat{\otimes}_{\epsilon_s}^{m,s} X \to Y$ the injective linearization of P, i.e. $P_{L,\epsilon}(x \otimes \cdots \otimes x) = P(x)$, for all $x \in X$. Indeed, they prove that the correspondence $P \leftrightarrow P_{L,\epsilon}$ between G-integral polynomials from X to Y and G-integral operators from $\widehat{\otimes}_{\epsilon_s}^{m,s} X$ to Y, determines an isometric isomorphism between the spaces $\mathcal{P}_{GI}\left({}^m X; Y\right)$ and $\mathcal{L}_{GI}\left(\widehat{\otimes}_{\epsilon_s}^{m,s} X; Y\right)$. We will need the variant of [25, Theorems 2.3 and 2.4] for G-integral polynomials. There, the authors show which modifications should be done in order to state Theorem 2.3 for G-integral polynomials. For the sake of clarity, we summarize these results for G-integral polynomials in the next theorem.

Theorem 4.29. *([25, Theorem 2.3] for G-integral polynomials) Let $P \in \mathcal{P}({}^m X; Y)$. The following are equivalent:*

1. *P is G-integral.*

2. *There are a compact Hausdorff space K, an embedding $h \in \mathcal{L}(X, C(K))$, and a regular countably additive, Y^{**}-valued Borel measure G of bounded variation on K such that*

$$P(x) = \int_K \left[h(x)(\omega)\right]^m dG(\omega), \ x \in X.$$

3. *There are a compact Hausdorff space K, an embedding $h \in \mathcal{L}(X, C(K))$, a finite non-negative countably additive, Borel measure μ on K, and an operator $A \in \mathcal{L}\left(L_1\left(K, \mu\right); Y^{**}\right)$, such that the following diagram is commutative*

$$\begin{array}{ccccc} X & \xrightarrow{P} & Y & \xrightarrow{K_Y} & Y^{**} \\ {\scriptstyle R}\downarrow & & & & \uparrow{\scriptstyle A} \\ C(K) & & \xrightarrow[j_1]{} & & L_1(K,\mu) \end{array}$$

where j_1 is the natural inclusion mapping, $K_Y : Y \to Y^{**}$ is the canonical isometric embedding and $R \in \mathcal{P}(^m X; C\left(K\right))$ is given by $R\left(x\right) := \left[h\left(x\right)\right]^m$, for all $x \in X$.

4. *There are a finite measure space $\left(\Omega, \Sigma, \mu\right)$, an operator $A \in \mathcal{L}\left(L_1\left(K, \mu\right); Y^{**}\right)$, and an embedding $h \in \mathcal{L}(X, L_\infty\left(\Omega, \mu\right))$ such that the following diagram is commutative*

$$\begin{array}{ccccc} X & \xrightarrow{P} & Y & \xrightarrow{K_Y} & Y^{**} \\ {\scriptstyle R}\downarrow & & & & \uparrow{\scriptstyle A} \\ L_\infty(\Omega,\mu) & & \xrightarrow[i_1]{} & & L_1(\Omega,\mu) \end{array}$$

where i_1 is the natural inclusion mapping and $R \in \mathcal{P}(^m X; C\left(K\right))$ given by $R\left(x\right) = \left[h\left(x\right)\right]^m$, for all $x \in X$.

5. *There are a finite measure space $\left(\Omega, \Sigma, \mu\right)$, an operator $B \in \mathcal{L}\left(\widehat{\otimes}_{\epsilon_s}^{m,s} X; L_\infty\left(\Omega, \mu\right)\right)$ and $A \in \mathcal{L}\left(L_1\left(\Omega, \mu\right); Y^{**}\right)$ such that the following diagram is commutative*

$$\begin{array}{ccccc} \widehat{\otimes}_{\epsilon_s}^{m,s} X & \xrightarrow{P_{L,\epsilon}} & Y & \xrightarrow{K_Y} & Y^{**} \\ {\scriptstyle B}\downarrow & & & & \uparrow{\scriptstyle A} \\ L_\infty(\Omega,\mu) & & \xrightarrow[i_1]{} & & L_1(\Omega,\mu) \end{array}$$

In this case, the operator $P_{L,\epsilon}$ is G-integral.

As an application of Theorem 4.29, we can prove the following result, that will be used afterwards.

Proposition 4.30. *Let $P \in \mathcal{P}(^m X; Y)$. Then, the following are equivalent:*

(i) P is G-integral

(ii) $P_{L,\epsilon}$ is G-integral.

(iii) There exists a Banach space Z such that $P = T \circ Q$, for some $Q \in \mathcal{P}(^m X; Z)$ and $T \in \mathcal{L}_{GI}(Z; Y)$.

(iv) The polynomial $K_Y \circ P \in \mathcal{P}(^m X; Y^{**})$ is G-integral.

If one (and then all) of these assertions holds, then we have

$$\|P\|_{GI} = \|P_{L,\epsilon}\|_{GI} = \|K_Y \circ P\|_{GI}.$$

Proof. The equivalence between (i) and (ii) follows from Theorem 4.29.

(ii) implies (iii) follows from the factorization $P = P_{L,\epsilon} \circ \delta_m$.

Assume (iii), i.e. $P \in \mathcal{P}(^m X; Y)$ factors as $P = T \circ Q$, with $Q \in \mathcal{P}(^m X; Z)$ and $T \in \mathcal{L}_{GI}(Z; Y)$. By the ideal property, $T \circ Q_{L,\epsilon}$ is G-integral. Then, $(T \circ Q)_{L,\epsilon} = T \circ Q_{L,\epsilon}$ is G-integral. Hence, by the equivalence between (i) and (ii) $P = T \circ Q$ is G-integral.

The equality of the norms $\|P\|_{GI} = \|P_{L,\epsilon}\|_{GI}$ follows from combining Theorem 2.3 and Theorem 2.4 of [25].

(i) implies (iv) follows from the ideal property. Besides,

$$\|K_Y \circ P\|_{GI} \leq \|P\|_{GI}. \tag{4.11}$$

Let us prove (iv) implies (i). Let $P \in \mathcal{P}(^m X; Y)$ be such that $K_Y \circ P \in \mathcal{P}_{GI}(^m X; Y^{**})$. Then by Theorem 4.29 (5) there are finite measure space (Ω, Σ, μ), an operator $A \in \mathcal{L}(L_1(\Omega, \mu), Y^{****})$, and an embedding $h \in \mathcal{L}(X, L_\infty(\Omega, \mu))$ such that $K_{Y^{**}} \circ (K_Y \circ P) = A \circ i_1 \circ R$, where $R(x) = [h(x)]^m$, for all $x \in X$.

Since $[K_{Y^*}]^* \circ K_{Y^{**}} = I_{Y^{**}}$, we have

$$\begin{aligned} K_Y \circ P &= [[K_{Y^*}]^* \circ K_{Y^{**}}] \circ K_Y \circ P \\ &= [K_{Y^*}]^* \circ A \circ i_1 \circ R \\ &= A' \circ i_1 \circ R, \end{aligned}$$

where $A' = [K_{Y^*}]^* \circ A$. Hence, we obtain the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{P} & Y & \xrightarrow{K_Y} & Y^{**} \\ {\scriptstyle R}\downarrow & & & & \uparrow{\scriptstyle A'} \\ L_\infty(\Omega, \mu) & & \xrightarrow[i_1]{\hspace{2em}} & & L_1(\Omega, \mu) \end{array}$$

Therefore, by Theorem 4.29 (4), P is G-integral and (i) is proved.

Besides,

$$\|P\|_{GI} \leq \|R\| \, \mu(K).$$

Then, taken the infimum over all factorizations we get

$$\|P\|_{GI} \leq \|K_Y \circ P\|_{GI}. \tag{4.12}$$

So, combining (4.11) and (4.12) we get

$$\|P\|_{GI} = \|K_Y \circ P\|_{GI}.$$

$\square$

Theorem 4.31. *Let $P \in \mathcal{P}(^m X; Y)$. The following assertions are equivalent:*

(i) $P \in \mathcal{P}_{GI}(^m X, Y)$.

(ii) $P^* \in \mathcal{L}_{GI}(Y^*, \mathcal{P}(^m X))$.

(iii) $P^{**} \in \mathcal{L}_{GI}((\mathcal{P}(^m X))^*, Y^{**})$.

Proof. $(i) \Rightarrow (ii)$. Let $P \in \mathcal{P}_{GI}(^m X, Y)$. By Theorem 4.29 (4) there are a probability measure μ, a bounded linear operator $A : L_1(\mu) \longrightarrow Y^{**}$ and a continuous m-homogeneous polynomial $R : X \longrightarrow L_\infty(\mu)$ such that $K_Y \circ P = A \circ i_1 \circ R$.

From

$$(K_Y \circ P)^* = P^* \circ K_Y^* \text{ and } K_Y^* \circ K_{Y^*} = id_{Y^*},$$

we obtain

$$P^* = R^* \circ i_1^* \circ A^* \circ K_{Y^*}.$$

Then, there is a decomposition

$$K_{\mathcal{P}(^m X)} \circ P^* : Y^* \xrightarrow{K_{Y^*}} Y^{***} \xrightarrow{A^*} L_1(\mu)^* \xrightarrow{i_1^*} L_\infty(\mu)^* \xrightarrow{R^*} \mathcal{P}(^m X) \xrightarrow{K_{\mathcal{P}(^m X)}} \mathcal{P}(^m X)^{**}$$

Now, factoring $i_1^* = K_{L_1(\mu)} \circ i_1 : L_\infty(\mu) \xrightarrow{i_1} L_1(\mu) \xrightarrow{K_{L_1(\mu)}} (L_1(\mu))^{**}$, we obtain the following commutative diagram

$$\begin{array}{ccccc} Y^* & \xrightarrow{P^*} & \mathcal{P}(^m X) & \xrightarrow{K_{\mathcal{P}(^m X)}} & \mathcal{P}(^m X)^{**} \\ \downarrow b & & & & \uparrow a \\ L_\infty(\mu) & & \xrightarrow[i_1]{} & & L_1(\mu), \end{array}$$

where $a = K_{\mathcal{P}(^m X)} \circ R^* \circ K_{L_1(\mu)}$, and $b = A^* \circ K_{Y^*}$. Therefore, P^* is G-integral.

The equivalence $(ii) \Leftrightarrow (iii)$ can be found in [31, Theorem 5.15].

$(iii) \Rightarrow (i)$ Assume now that $P^{**} \in \mathcal{L}_{GI}((\mathcal{P}(^m X))^*, Y^{**})$. We define the map $K : X \to \mathcal{P}(^m X)^*$ by

$$[K(x)](Q) := Q(x),$$

for $x \in X$ and $Q \in \mathcal{P}(^m X)$. It is easy to see that K is a continuous m-homogeneous polynomial with $\|K\| = 1$, and

$$K_Y \circ P = P^{**} \circ K.$$

Using Proposition 4.30 we get that $P^{**} \circ K$, and so $K_Y \circ P$, is G-integral. Again by Proposition 4.30, P is G-integral. □

Remark 4.32. *Note that since the space $\mathcal{P}(^m X)$ is a dual space, the bidual $\mathcal{P}(^m X)^{**}$ can be avoided in the factorization whenever the adjoint P^* is G-integral. That is, P^* factors as $P^* : Y^* \xrightarrow{b} L_\infty(\mu) \xrightarrow{i_1} L_1(\mu) \xrightarrow{a'} \mathcal{P}(^m X)$, where $a' = R^* \circ K_{L_1(\mu)}$.*

Next, using the concept of G-integral polynomial, we give some examples of factorable strongly p-nuclear polynomials.

Theorem 4.33. *Every G-integral m-homogeneous polynomial is factorable strongly p-nuclear.*

Proof. Let $P \in \mathcal{P}(^m X; Y)$ be a G-integral m-homogeneous polynomial. By Theorem 4.29 (5) (or Proposition 4.30) its linearization $P_{L,\varepsilon} : \widehat{\otimes}^{m,s}_{\epsilon_s} X \to Y$ is G-integral, hence p-nuclear by [28, Theorem 3.3.3]. Using the ideal property, the operator $P_L = P_{L,\varepsilon} \circ i : \widehat{\otimes}^{m}_{\pi_s} X \to Y$ is p-nuclear, where $i : \widehat{\otimes}^{m,s}_{\pi_s} X \to \widehat{\otimes}^{m,s}_{\epsilon_s} X$ is the canonical continuous inclusion. As a consequence of Theorem 4.18, P is factorable strongly p-nuclear. □

By Theorem 4.33 and [23, Lemma 4 and Remark 6], we conclude that all polynomials defined as below are factorable strongly p-nuclear.

Example 4.34. *Let (Ω, Σ, μ) be a finite measure space and $G : \Sigma \to X$ a vector measure which is absolutely continuous with respect to μ. Then, the polynomial P_0 given by*

$$P_0(f) = \int_\Omega f^m(\omega)\, dG(\omega) \tag{4.13}$$

is factorable strongly p-nuclear m-homogeneous polynomial on $L_\infty(\Omega, \mu)$ with

$$\pi^{fs}_{p,N}(P) \le |G|.$$

Also, for any compact hausdorff space K and any regular, Borel measure G on K, the polynomial on $C(K)$ given in (4.13) is factorable strongly p-nuclear, with $\pi_{p,N}^{fs}(P) \leq |G|$.

Let us now show that, for a wide class of range spaces Y both classes, G-integral homogeneous polynomials and factorable strongly p-nuclear homogeneous polynomials, coincide. For the notion and main properties of $\mathcal{L}_{p,\lambda}$ spaces, we refer the reader to [38].

Theorem 4.35. *Suppose that Y^* is an $\mathcal{L}_{p,\lambda}$-space. Then*

$$\mathcal{P}_{p,N}^{fs}(^mX;Y) = \mathcal{P}_{GI}(^mX;Y).$$

Proof. Let $P \in \mathcal{P}_{p,N}^{fs}(^mX;Y)$. By Theorem 4.26 the adjoint $P^* : Y^* \to \mathcal{P}(^mX)$ is a p^*-nuclear linear operator. Since Y^* is a $\mathcal{L}_{p,\lambda}$ space, by [28, Theorem 3.3.3] the operator P^* is G-integral. Then, by Theorem $P : X \to Y$ is G-integral. The converse follows from Theorem 4.33. □

Since every Hilbert space is a $\mathcal{L}_{2,\lambda}$-space for all $\lambda > 1$, the above theorem gives the following consequence.

Corollary 4.36. *If Y is Hilbert space then $\mathcal{P}_{GI}(^mX;Y) = \mathcal{P}_{2,N}^{fs}(^mX;Y)$.*

Bibliography

[1] D. Achour, *Multilinear extensions of absolutely $(p; q; r)$-summing operators*, Rend. Circ. Mat. Palermo. **60** (2011), 337–350.

[2] D. Achour and A. Alouani, *On the multilinear generalizations of the concept of nuclear operators*, Colloquium Math. **120** (2010), 85–102.

[3] D. Achour, A. Alouani, P. Rueda and K. Saadi, *Factorable strongly p-nuclear m-homogeneous polynomials*, Revista de la Real Academia de Ciencias Exactas,fisicas y Naturales Serie A. Mathematicas. (2018), https://doi.org/10.1007/s13398-018-0530-z .

[4] D. Achour, A. Alouani, P. Rueda and E.A. Sánchez-Pérez, *Tensor representations of summing polynomials*, Mediterr. J. Math. 15 (2018), 127.

[5] D. Achour and A.T. Bernardino, *$(q; r)$-Dominated holomorphic mappings*, Collect. Math. **59** (2012), 877–897.

[6] D. Achour and L. Mezrag, *On the Cohen strongly p-summing multilinear operators*, J. Math. Anal. Appl. **327** (2007), 550–563.

[7] D. Achour and K. Saadi, *A polynomial characterization of Hilbert spaces*, Collect. Math. (3) **61** (2010), 291–301.

[8] R. Alencar, *On reflexivity and basis for $\mathcal{P}(^m X)$*, Proc. Roy. Irish. Acad. Sect. A **85** (1985), 131–138.

[9] R. Alencar and M. Matos, *Some classes of multilinear mappings between Banach spaces*, Publ. Dep. Analisis Mat. Univ. Complut. Madrid **12** (1989).

[10] H. Apiola, *Duality between spaces of p-summable sequences, (p, q)-summing operators and characterizations of nuclearity*, Math. Ann. **219** (1976), 53–64.

[11] R. M. Aron, P. Rueda, *p-Compact homogeneous polynomials from an ideal point of view. Function spaces in modern analysis*, Contemp. Math. 547, Amer. Math. Soc, Providence, RI, (2011), 61–71.

[12] G.Botelho, *Type, cotype and generalized Rademacher functions*, Rocky Mountain J. M. **28** (1998), 1227–1250.

[13] G. Botelho, *Ideals of polynomials generated by weakly compact operators*, Note Mat. **25** (2005/2006), 69–102.

[14] G. Botelho and J. Campos, *On the transformation of vector-valued sequences by linear and multilinear operators*, Monatsh. Math. (3) **183** (2017), 415–435.

[15] G. Botelho, D. Pellegrino and P. Rueda, *On composition ideals of multilinear mappings and homogeneous polynomials*, Publ. RIMS, Kyoto Univ. **43** (2007), 1139–1155.

[16] G. Botelho, D. Pellegrino and P. Rueda, *Pietsch's factorization theorem for dominated polynomials*, J. Funct. Anal. (1) **243** (2007), 257–269.

[17] G. Botelho, D. Pellegrino, P. Rueda, *A unified Pietsch domination theorem*, J. Math. Anal. Appl. (1) **365** (2010), 269–276.

[18] G. Botelho, D. Pellegrino and P. Rueda, *Preduals of spaces of homogeneous polynomials on L_p-spaces*. Linear Multilinear Algebra, 60, no. **5** (2012), 565–571 .

[19] G. Botelho, D. Pellegrino and P. Rueda, *On Pietsch measures for summing operators and dominated polynomials*, Lin. Multilin. Alg. (7) **62** (2014), 860–874.

[20] E. Çaliskan, D. M. Pellegrino, *On the multilinear generalizations of the concept of absolutely summing operators*, Rocky Mountain J. Math. **37** (2007), 1137–1154.

[21] E. Çalişkan, P.Rueda, *On distinguished polynomials and their projections*, Ann. Acad. Sci. Fenn. Math. **37** (2012), 595–603.

[22] D. Carando and V. Dimant, *On summability of bilinear operators* . **259** (2003), 3–11.

[23] D. Carando and S. Lassalle, *Extension of vector-valued integral polynomials*, J. Math. Anal. Appl. **307** (2005), 77–85.

[24] R. Cilia, M. D'Anna and J.M. Gutiérrez, *Polynomial characterization of $\mathcal{L}_\infty$-spaces*, J. Math. Anal. Appl. **275** (2002), 900–912.

[25] R. Cilia and J. M. Gutiérrez, *Ideals of integral and r-factorable polynomials*, Bol. Soc. Mat. Mexicana. **14** (2008), 95–124.

[26] E. Çaliskan, D. Pellegrino, *On the multilinear generalizations of the concept of absolutely summing operators*, Rocky Mountain J. Math. **37** (2007), 1137–1154.

[27] J.S. Cohen, *Absolutely p-summing, p-nuclear operators and their conjugates*, Dissertation, Univ. of Md, College Park, Md, Jan, 1970.

[28] J.S. Cohen, *Absolutely p-summing, p-nuclear operators and their conjugates*, Math. Ann. **201** (1973), 177–200.

[29] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland, Amsterdam, 1993.

[30] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London, 1999.

[31] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge University Press, Cambridge, 1995.

[32] V. Dimant, *Strongly p-summing multilinear operators*, J. Math. Anal. Appl. **278** (2003), 182–193.

[33] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note Mat. **17** (1997), 153–188.

[34] L. Garcia, *O adjunto de um polinômio homogêneo contínuo entre espaços de Banach*, Tese de Doutorado, Universidade federal de Uberlândia, 2013.

[35] S. Geiss, *Ideale multilinearer abbildungen*, Diplomarbeit, 1984.

[36] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo **8** (1956), 1–79.

[37] J. Gutiérrez and I. Villanueva, *Extensions of multilinear operators and Banach space properties*, Proc. Royal. Soc. Edinburgh Series A. **133** (2003).

[38] J. Lindenstrauss and H. P Rosenthal, *The $\mathcal{L}_p$ space*, Israel J. Math. **7** (1969), 325–349.

[39] M. C. Matos, *On multilinear mappings of nuclear type*, Rev. Mat. Comput, **6** (1993), 61–81.

[40] M. Matos, K. Floret, *Application of a Khintchine inequality to holomorphic mappings*, Math. Nachr. **176** (1995), 65–72.

[41] M. Mastylo, P. Rueda and E. A. Sánchez-Pérez, *Factorization of (p, q)-summing polynomials through Lorentz spaces*, J. Math. Anal. Appl. (1) **449** (2017), 195–206.

[42] Y. Meléndez and A. Tonge, *Polynomials and the Pietsch domination theorem*, Proc. Roy. Irish Acad. Sect. A **99** (1999), 195–212.

[43] L. Mezrag, *On strongly ℓ_p-summing m-ltilinear operators*, Clloq. Math. **111** (2008), 59–70.

[44] M. Y. Miyamura, *Reflexividade de Espacos de Operadores Lineares e Espaços de Polinômios Homogêneos*, Tese de Doutorado, Universidade Estadual de Campinas, 2008.

[45] J. Mujica, *Aplicações $\tau(p;q)$-somantes e $\sigma(p)$-nucleares*, Tese de Doutorado, Universidade Estadual de Campinas, 2006.

[46] J. Mujica, *$\tau(p;q)$-summing mappings and the domination theorem*, Port. Math. (2) **65** (2008), 211–226.

[47] J. Mujica, *Complex analysis in Banach spaces*, Dover Publications, Dover, 2010.

[48] A. Pełczyński, *On weakly compact polynomial operators on B-spaces with Dunford-Pettis property*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 371–378.

[49] D. Pérez-García, *Comparing different classes of absolutely summing multiliner operators*, Arch Math. (Basel) (3) **85** (2005), 285–267.

[50] A. Pietsch, *Absolute p-summierende Abbildungen in normierten Räumen*, Studia Math. **28** (1967), 333–353.

[51] A. Pietsch, *Operator ideals*, Deutsch. Verlag Wiss, Berlin, 1978; North-Holland, Amsterdam-London-New York-Tokyo, 1980.

[52] A. Pietsch, *Ideals of multilinear functionals (designs of a theory)*, Proceedings of the Second International Conference on Operator Algebras, Ideals, and their Applications in Theoretical Physics (Leibzig). Teubner-Texte (1983), 185–199.

[53] D. Popa, *A note the concept of factorable strongly p-summing operators*, Revista de la Real Academia de ciencias Exactas, Físicas y Naturales Serie A. Matemáticas. **111** (2016), 465–471.

[54] D. Pellegrino, P. Rueda and E. A. Sánchez-Pérez, *Surveying the spirit of absolute summability on multilinear operators and homogeneous polynomials*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales Serie A. Matematicas, **110** (2016), 285–302.

[55] D. Pellegrino, J. Santos and J. B. Seoane-Sepúlvzda, *Some techniques on nonlinear analysis and applications*, Sciverse ScienceDirect advaces in Mathematics , **229** (2012), 1235–1265.

[56] M. S. Ramanujan and E. Schock, *Operator ideals and spaces of bilinear operators*, Lin. Multilin. Alg. **18** (1985), 307–318.

[57] P. Rueda and E. A. Sánchez-Pérez, *Factorization of p-dominated polynomials through L^p-spaces*, Michigan Math. J. (2) **635** (2014), 345–353.

[58] P. Rueda, E.A. Sánchez-Pérez and A. Tallab, *Traced tensor norms and multiple summing multilinear operators*, Linear Multlinear Algebra. (4) **65** (2017), 768–786.

[59] R. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy*, Ph.D. thesis, Trinity College, Dublin. **15**, 1980.

[60] R. Ryan, *Introduction to tensor product of Banach Spaces*, Springer-Verlag, London, 2002.

[61] B. Schneider, *On absolutely p-summing and related multilinear mappings*, Wissenchaftliche Zeitschrift der Brandenburger Landeshochochule, **35** (1991), 105–117.

[62] A.R. Silva, *Linearização de aplicações multilineares contínuas entre espaços de Banach e multi-ideais de composição*, Dissertação de Mestrado, Universidade Federal de Uberlândia, 2010.

[63] I. Villanueva, *Integral mappings between Banach spaces*, J. Math. Anal. Appl. **279** (2003), 56–70.

ملخص:

مضمون هذه الرسالة، يتمحور حول عديد من المفاهيم للمؤثرات الجمعية في الحالة الغير الخطية كالمؤثرات المتعددة الخطية و كثيرات الحدود للمؤثرات المتعددة الخطية. في الفصل الثاني قدمنا فئة المؤثرات المتعددة الخطية من نوع كوهان p-نكليار المعرفة بين فضاءات بناخ باعتبارها نسخة موسعة للحالة الخطية. هذا التنوع في المؤثرات المتعددة الخطية المتناظرة سمح لنا بتقديم صنف جديد كثير حدود p-نكليار هذا الأخير تم استخدمه كمثال توضيحي في الفصل 4 أهم النتائج المثبتة في هذا الفصل كالتالي : تطبيق لنظرية الهيمنة لبيتش و نظرية التفكيك على اعتبارها تمديد لنظرية التفكيك ل كوابن. - العلاقة مع المؤثرات المتعددة الخطية الجمعية. كما يجدر بالذكر ومتابعة للنتائج التي تم التوصل إليها، أثبتنا أن المؤثرات المتعدد الخطة من نوع كوهان p-نكليار ضعيفة التراص ◦ الفصل الثالث يعالج فضاء المتتاليات من خلال المؤثرات المتعددة الخطية و كثيرات الحدود للمؤثرات المتعددة الخطية المتجانسة ◦ مؤثر مثالي جمعي يمكن أن نميزه باستخدام الاستمرارية لمؤثر موتر مرفق له ,هذا المؤثر الموتر معرف بين فضاء بناخ للمتتاليات .هدفنا هو توفير دراسة شاملة لهذه المميزات للمؤثرات الجمعية حيث نعمل في نطاق أوسع يضم المؤثرات المتعددة الخطية المتناظرة. أمثلة تطبيقية قدمت ◦ في الفصل الرابع أعطينا خاصية مكافئة لكثيرات الحدود للمؤثرات المتعددة الخطية المتجانسة المعرفة بواسطة السلاسل الجمعية والتي يكون المؤثر الخطي المرفق لها من نوع كوهان p-نكليار . نختم بإيجاد علاقة قوية مع كثيرات الحدود للمؤثرات المتعددة الخطية المتجانسة التكاملية من نوع قروتنديك.

كلمات مفتاحيه. المؤثرات المتعددة الخطية. الجداء الموتر. كثيرات الحدود للمؤثرات المتعددة الخطية. كثير حدود p-نكليار. كثير حدود p-نكليار التفكيكي.

Abstract

The present thesis is devoted to summing non linear operators. We focus our attention on introducing and studying polynomials and multilinear mappings that share good properties of summability with distinguished classes of summing linear operators. In the second chapter we introduce the class of Cohen p-nuclear m-linear operators between Banach spaces. This is the multilinear version of p-nuclear operators. The polynomial variant is obtained thanks to consider the symmetric multilinear mapping associated to the polynomial. This polynomial variant forms the p-nuclear polynomial, and it is used as an illustrative example also in Chapter IV. The main results proved in Chapter II are: a characterization in terms of Pietsch's domination theorem and the related factorization theorem, which is an extension to the multilinear setting of Kwapien's factorization theorem for dominated linear operators. Connections with the theory of absolutely summing m-linear operators are also established. It is worth mentioning that, as a consequence of our results, we show that every Cohen p-nuclear m-linear mapping on arbitrary Banach spaces is weakly compact. The third chapter deals with transformations of sequences via summing nonlinear operators. Operators Ṭthat belong to some summing operator ideal can be characterized by means of the continuity of an associated tensor operator Ṭthat is defined between tensor products of sequences spaces. Our aim is to provide a unifying treatment of these tensor product characterizations of summing operators. We work in the more general frame, provided by homogeneous polynomials, where an associated tensor polynomial which plays the role of T, needs to be determined first. Examples of applications are shown. In Chapter IV we characterize in terms of summability those homogeneous polynomials whose linearization is p-nuclear. This characterization provides a strong link between the theory of p-nuclear linear operators and the (non linear) homogeneous p-nuclear polynomials that significantly improves former approaches. The deep connection with Grothendieck integral polynomials is also analyzed.

Keywords: multilinear operator, tensor product, m-homogeneous polynomial, p-nuclear operator, factorable strongly p-nuclear nuclear.

Résumé

Cette thèse est consacrée aux opérateurs non linéaires sommant. Nous concentrons notre attention sur l'introduction et l'étude des polynômes et des applications m-linéaires partageant les bonnes propriétés de sommabilité avec les classes distinguées des opérateurs linéaires sommant. Dans le deuxième chapitre, nous introduisons la classe des opérateurs m-linéaires Cohen p-nucléaires entre les espaces de Banach, qui est la version multilinéaire des opérateurs p-nucléaires. La variante polynomiale est obtenue en tenant compte l'application m-linéaire symétrique associée au polynôme, cette variante forme le polynôme p-nucléaire, qui est également utilisé à titre d'exemple explicatif au chapitre IV; les principaux résultats démontrés au chapitre II sont: une caractérisation en termes du Théorème de Domination de Pietsch et du Théorème de Factorisation, qui est une extension du Théorème de Factorisation de Kwapien pour les opérateurs linéaires dominés. Une connexion avec les opérateurs m-linéaires absolument sommant est également établie. Il est important de mentionner que selon nos résultats et comme conséquence, nous montrons que tout operateur linéaire Cohen p-nucléaire ṃ-linéaire entre, les espaces de Banach est faiblement compact. On a traité dans le troisième chapitre les transformations des suites à travers les opérateurs non linéaires sommant. Un opérateur idéal sommant Ṭse caractérise par la continuité d'un opérateur tensoriel associé T, lequel est défini entre le produit tensoriel des espaces, de suites. Notre objectif est d'offrir un traitement uniforme pour ces caractérisations tensorielles des opérateurs sommant. On travail dans un cadre plus général fourni par les polynômes homogènes, où un polynôme tensoriel associé jouant le rôle de T, doit être déterminé préalablement. Des exemples d'applications sont présentés. Au chapitre IV, nous décrivons en termes de sommabilité les polynômes homogènes dont leurs linéarisations sont p-nucléaire. Cette caractérisation fournit un lien fort entre la théorie des opérateurs linéaires p-nucléaires et la théorie des polynômes homogènes p-nucléaires (non linéaires) qui améliore considérablement les anciennes approches. Une connexion distinguée avec les polynômes Grothendieck intégrale est également analysée.
Mots-clés: opérateur multilinéaire, produit tensoriel, polynôme m-homogène, opérateur p-nucléaire, operateur factorable fortement p-nucléaire.