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Faculty of Mathematics and Informatics
Department of Mathematics

Submitted for the degree OF MASTER

Field: MATHEMATICS AND INFORMATICS

Major: MATHEMATICS

Option: Partial differential equation and applications

Presented by
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Subject

Nonlinear weighted elliptic equations with L^1 data

Presented publicly in jun 2024

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Promotion: 2023/2024

Acknowledgement

I thank God Almighty first and foremost for the great grace that he has bestowed upon me, and i want to thank all the people who played a part to preparing this research.

First, I am deeply grateful to my supervisor ***Dr: Rabah MECHETER****, for his guidance, encouragement and patience during the preparation of my thesis.*

Also, my thanks go to the members of the jury who have honored me by examining this work.

I would also like to express my thanks to all the teachers in the department of mathematics, specially the teachers in partial differential equations and their applications.

Finally, i would like to thank my family, specially my mother for their love and encouragement.

Thanks

Dedication

I dedicate this modest work :

-To my parents,

-To my brothers and sisters,

-To my aunts,

-To all my family,

-To all friends and all my department family,

-To all my adorable ones that i have known during all my life ...

Djerida Chaima

Contents

Notation

We introduce the necessary notations and definition which are used in the sequel.

$\mathcal{H}$	Hilbert space.
$\Omega \subset \mathbb{R}^N$	Open set in $\mathbb{R}^N$
X'	Topological dual of X.
$\mathbb{R}^N$	Euclidean space of dimension N, where N is a nonzero natural number.
x	Vector in $\mathbb{R}^N$, $x = (x_1, x_2, ..., x_N)$, $x_i \in \mathbb{R}$, $1 \leq i \leq N$
dx	Lebesgue measure in N-dimensional space.
$\partial\Omega = \Gamma$	Boundary of Ω.
$\|E\|$	or mes (E) measure of the set E.
$B(x_0, r)$	Open ball of radius r centered at $x0$.
B_E	$= \Big\{x \in E; \|x\| \leq 1\}$.
$\|.\|$	Hilbert norm.
χ_E	Characteristic function of set E.
Ω	Open subset of $\mathbb{R}^N$.
$\langle ., . \rangle$	Duality bracket between X and its dual space.
$\int_\Omega f(x)\, dx$	Integral of f in Ω with respect to the Lebesgue measure.
supp u	Support of the function u.
$\frac{\partial u}{\partial n}$	Outward normal derivative.
∇u	$= (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N})$ Gradient of the function u.
div u	Divergence of the vector u, $divu = \frac{\partial u}{dx_1} + \frac{\partial u}{dx_2} + ... + \frac{\partial u}{dx_N}$
Δu	$= \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$ Laplacian of u.
$f_n \to f$	Denotes that the sequence $\{f_n\}$ converge to f.
$f_n \rightharpoonup f$	Denotes that the sequence $\{f_n\}$ converge weakly to f.
$L^p(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable and $\Big(\int_\Omega \|u(x)\|^p\Big)^{1/p} dx < +\infty$ such that $1 \leq p < \infty\}$.
$\|u\|_p$	$= \Big[\int_\Omega \|u(x)\|^p dx\Big]^{1/P} = \|u\|_{L^p}$.
$L^\infty(\Omega)$	$= \{u : \Omega \to \mathbb{R}$ measurable, $\exists M > 0 \mid \|u(x)\| \leq M$a.e.$\}$.

$\|u\|_{L^\infty}$ $= \inf\{C : |u(x)| \leq C$ a.e on $\Omega\}$.

q Hölder conjugate of $p :\ q = \frac{p}{p-1}$ if $p > 1$ and $q = \infty$ if $p = 1$

$L^q(\Omega) \subset L^p(\Omega), \quad \forall\, 1 \leq p \leq q < \infty$

$\|u\|_{C_c(\Omega)}$ $= \sup_{x\in\Omega} |u(x)| = \max_{x\in\Omega} |u(x)|$

$C_c(\Omega)$ Space of continuous functions with compact.

$\mathcal{D}(\Omega)$ Space of infinitely differentiable functions on Ω with compact support in Ω.

$W^{1,p}(\Omega)$ $= \Big\{u \in L^p(\Omega) \mid \nabla u \in (L^p(\Omega))^N\Big\}$.

$W_0^{1,p}(\Omega)$ $= \Big\{u \in W^{1,p}(\Omega), \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega\Big\}$.

$W^{1,p}(\Omega, w) = \Big\{u \in L^p(\Omega, w) \mid \nabla u \in (L^p(\Omega, w))^N\Big\}$.

$(a+b)^p \leq 2^{p-1}(a^p + b^p) \quad \text{for} \quad 1 \leq p < \infty \quad \text{and} \quad a, b \geq 0.$

a.e. Almost everywhere.

General Introduction

Partial differential equations, which will be abbreviated as "PDE" in the following, constitute an important branch of applied mathematics and this field is becoming increasingly important in modern times. EDPs have great utility in mod- delusion of many phenomena of different natures such as physics, chemistry, biology and other sciences. In other words, EDPs intervene in many other fields: in chemistry to model reactions, in economics to study market behavior, in finance to study derivative products and in image processing to restore damage. These problems boil down to mathematical models usually written in the form

$$L(u) = f, \qquad (1)$$

where L is a definite operator from a function space E to a function space F, u is the unknown function and f a given function. PDEs probably first appeared during the 17th century. In-the field of PDEs has been enriched as the sciences have developed and especially physics. However, the systematic study of PDEs is much more recent and it is only during of the 20th century that mathematicians began to develop and indeed this theory has recently experienced a great theoretical and practical advancement. The mathematical analysis of these partial differential equations requires an appropriate choice of functional spaces and a clear definition of the notion of solution (the existence and sometimes uniqueness). One of the things to keep in mind about PDEs is to ask the question: can we get solutions explicitly?. So what mathematics can do on the other hand, is to say if one or more solutions exist, and to describe sometimes very precisely- ment some properties of these solutions. So usually an exact solution of problem (1) is not easy to find and sometimes we cannot even find the explicit solution. This leads to the introduction of the notion of the weak solution which appeared in 1934 in the work of Jean Leray. therefore, in most cases it is very difficult, if not impossible, to exhibit the solutions of a PDE. In some cases we manage to try to show that the problem admits a unique solution (it is said to be well-posed). But sometimes we can calculate numerical approximations of the solutions.

Weighted elliptic equations have been widely studied in the literature, starting from the paper by Trudinger in the linear case $p = 2$ (see [6]), looking for solutions in weighted Sobolev spaces. For a comprehensive study of weighted Sobolev spaces, and its applications to the existence of solutions for weighted elliptic equations, see for example the book [8] A more direct approach, which does not use weighted Sobolev spaces, is the one of [5] (with weights in $L^r(\Omega)$,$r > 1$, which can be zero on a subset of) and [4], where strictly positive weights in $L^1(\Omega)$ have been considered

Moreover, the work presented in this dissertation concerns a particular case of the equations to partial derivatives of the elliptic type involving the divergent operator

$$Au = -\text{div}(S(x)|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty$$

We are interested in this thesis to study the following problem:

$$(P) \qquad \begin{cases} Au + e(x)|\nabla u|^{p-2}\nabla u &= f \ \text{ in } \ \Omega; \\ u &= 0 \text{ on } \ \partial\Omega, \end{cases}$$

Ω is a bounded open set of $\mathbb{R}^N$ with $N > 2$, we assume that $f \in L^1(\Omega)$.

We first illustrate the main difficulties that may arise when studying of the problem (P).

1) The non-linearity of the differential operator Au.

2) The irregularity in the case where $f \in L^1(\Omega)$ i.e. the second member of (P) does not belong to space $W^{-1,p'}(\Omega)$.

To solve this issue, we are going to approximate the problem (P), next, we will prove some uniform estimates on the sequence of approximate solutions, and we shall finally pass to the limit in the approximate problems to establish the existence of a weak solution for the problem (P).

Technique

This study is mainly based on the article [1]"L. Boccardo et al, Nonlinear weighted elliptic equations with Sobolev weights.Boll. Unione Mat. Ital. 15, No. 4, 503-514 (2022)"

We are going to prove the existence of the weak solution of the problem (P). To do this, we approximate the problem (P) by a sequence of approximate problems (P_n) given in L^∞ whose existence of the solution approximate is guaranteed (See [10]). Then we will prove some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives. Once this is done, the linearity of the operator with respect to the gradient as well as the boundedness and the continuity of $|\nabla u|^{p-2}\nabla u$ will make it possible to pass to the limit, thus finding the solution.

Thesis plan

This thesis is divided into three chapters as follows: The first chapter is dedicated to giving some basic definitions and results with functional analysis tools essential to the achievement of the objectives for the study of the (P) problem. For example we recall functional spaces (Lebesgue, Sobolev) and we give brief reminders of weak and weak star convergence.

In the second chapter we recall some definitions on the operators (Bounded, hime-continuous, Monotone and coercive). We also present the method of monotonic (pseudo-monotonic) operators in the general framework to prove the existence of a solution for the equation

$$Au = -\text{div}(S(x)|\nabla u|^{p-2}\nabla u) = f.$$

According to hypotheses on the functions $u \mapsto |\nabla u|^{p-2}\nabla u$ and f $(f \in W^{-1,p'}(\Omega))$ and for to prove the existence and the regularity of the solution we use the Theorem 2.1.

In the third chapter, we will study existence of solutions for the problem

$$(P) \qquad \begin{cases} -\mathrm{div}(S(x)|\nabla u|^{p-2}\nabla u) + e(x)|u|^{p-2}u &= f \quad \Omega; \\ u &= 0 \quad \partial\Omega, \end{cases}$$

with $S(x)$ such that (3.1) holds, and the functions $e(x)$ in $L^1(\Omega)$ and $f(x)$ in $L^1(\Omega)$ are such that there exists $k_0 > 0$ such that

$$|f(x)| \leq k_0 e(x)$$

For this equation we will prove existence of a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Chapter 1

Mathematical preliminaries

The purpose of this first chapter is to present a number of analytical tools from the theory of Sobolev spaces that will be used throughout this thesis. We will also take the opportunity to introduce the main notations.

1.1 Reminders and some definitions

Throughout this chapter Ω will denote a bounded domain in $\mathbb{R}^N$,$N \geq 1$, i.e.a connected and bounded open subset of $\mathbb{R}^N$. Its boundary will be denoted by Γ or $\partial\Omega$ and its closure by $\overline{\Omega}$.

Let $u = u(x_1, ..., x_N)$ be a function defined in $\Omega \subset \mathbb{R}^N$. Assuming that exists, the gradient of u at point x is defined as the vector:

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x)\right)$$

The Euclidean norm of ∇u is denoted by $|\nabla u|$:

$$|\nabla u| = \left[\left|\frac{\partial u}{\partial x_1}\right|^2 + ... + \left|\frac{\partial u}{\partial x_N}\right|^2\right]^{1/2}$$

Definition 1.1. ***(Space of test functions)*** *The space $\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$ is the set of infinitely differentiable functions $\varphi : \Omega \longrightarrow \mathbb{R}$ (of class $C^\infty(\Omega)$) with compact support in Ω. $\mathcal{D}(\Omega)$ is a vector space, and each element of this space is called a test function.*

- *supp* $\varphi = \overline{\{x \in \Omega,\ \varphi(x) \neq 0\}}$,
- $\mathcal{D}(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$.

Definition 1.2 (**Separable Spaces**). *A Banach space E is said to be separable if there exists a countable set that is dense in E.*

Definition 1.3 (**Dual Space**). *The topological dual of normed space E over the field $\mathbb{K}$, (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is defined as the set of continuous linear forms on E,i.e. continuous linear maps form E to $\mathbb{K}$. We denote by $E' = \mathbb{L}(E, \mathbb{K})$.*
E' equipped with the norm $\|.\|$ defined by :

$$\|u\|_{E'} = \sup_{x \in E, \|x\|=1} |u(x)| = \sup_{x \in E, \|x\| \leq 1} |u(x)| = \sup_{x \in E, \|x\| \neq 0} \frac{|u(x)|}{\|x\|}$$

Definition 1.4. ***(Reflexive space)***: *The normed space E is a said to be reflexive if the canonical injection $i : E \longrightarrow E"$ is surjective,i.e. $i(E) = E"$.*
when E is reflexive, $E"$ is usually identified with E; ($E" = E$).

Proposition 1.1. *1. $E_1 \subset E_2$ so $E_2' \subset E_1'$,*

2. $\langle u, x \rangle_{E'E} \leq \|u\|_{E'} \|x\|_E$.

1.2 Functional spaces

1.2.1 L^p Spaces

Let Ω be an open subset of $\mathbb{R}^N$ and let $p \in \mathbb{R}$ with $1 \leqslant p < \infty$ we set :

$$L^p(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \quad \text{is measurable and} \quad \int_\Omega \mid u(x) \mid^p < \infty \Big\}.$$

With

$$\|u\|_{L_p(\Omega)} = \|u\|_p = \Big(\int_\Omega |u(x)|^p \Big)^{\frac{1}{p}}, \quad p \geq 1.$$

By $L^\infty(\Omega)$ we set

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable}, \exists M > 0 \mid |u(x)| \leq M \quad \text{a.e.on} \quad \Omega \}$$

With

$$\|u\|_{L^\infty(\Omega)} = \|u\|_\infty = \inf \quad \{ M > 0 : \mid u \mid \leq M \quad \text{a.e.on} \quad \Omega \}$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u,v)_{2,\Omega} = \int_\Omega uvdx.$$

Remark 1.1. *Let $1 \leq p \leq \infty$, be denoted by p' the conjugate exponent of p*

$$i.e. \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p' = \frac{p}{p-1}.$$

Let $1 \leq p < \infty$. then the dual space of $L^p(\Omega)$ is $L^{p'}(\Omega)$.

Theorem 1.1.
- *$L^p(\Omega)$ is a banach space for all $1 \leq p \leq \infty$*
- *$L^p(\Omega)$ space is a séparale space for all $1 \leq p < \infty$*
- *$L^p(\Omega)$ space is a réflexif space for all $1 < p < \infty$*

1.2.2 Main inequalities

Let $1 \leq p \leq q \leq \infty$ be two real numbers, and let $p^{'}$ be the conjugate exponent of p.

Lemma 1.1 (Hölder's inequality)**.** *Assum that $f \in L^p(\Omega)$ and $g \in L^{p^{'}}(\Omega)$ with $1 \leq p \leq \infty$, then $f \cdot g \in L^1(\Omega)$ and*

$$\| f \cdot g \|_{L^1(\Omega)} \leq \| f \|_{L^p(\Omega)} \| g \|_{L^{p'}(\Omega)}$$

If more $| \Omega | < \infty$ and $f \in L^q(\Omega)$, so $f \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leq | \Omega |^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(\Omega)}$$

In particular,

$$L^q(\Omega) \subset L^p(\Omega), \quad \forall\, 1 \leq p \leq q < \infty$$

Lemma 1.2. *Let $p_i \in [1, +\infty]$ be exponents with $1 \leq i \leq k$ such that: $1/p = 1/p_1 + ... + 1/p_k \leq 1$. Then, for all functions $f_i \in L^{p_i}(\Omega)$, We have $f = f_1 ... f_k \in L^p(\Omega)$ and generalized Hölder inequality*

$$\|f\|_p \leq \| f_1 \|_{p_1} \dots \| f_k \|_{p_k}.$$

Lemma 1.3 (Young's inequality). *For all $a, b \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have:*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad p' = \frac{p}{p-1}. \tag{1.1}$$

Corollary 1.1. *Generalized Young's inequality: for $1 < p < \infty$, $p' = \frac{p}{p-1}$ and any positive ϵ we have:*

$$ab \leq \epsilon^p \frac{a^p}{p} + \frac{1}{\epsilon^{p'}} \frac{b^{p'}}{p'}.$$

Proposition 1.2. *we have the following two inequalities*

- *if* $1 < p < 2$

$$|| a |^{p-2} a - | b |^{p-2} b | \leq c_p | a - b |^{p-2}$$

- *if* $p \geq 2$

$$|| a |^{p-2} a - | b |^{p-2} b | \leq c_p(|a|^{p-2} - | b |^{p-2}) | a - b |$$

$\forall c_p > 0$ *depends on* p *and* $\forall a, b \in \mathbb{R}^N$

Lemma 1.4 (**Divergence and Green's formulas**). *Let Ω be a domain in $\mathbb{R}^N$, and $n(x)$ its exterior normal such that $\|n(x)\| = 1$. Let u and v be two regular functions, and w be a vector field defined on Ω. Then,*

$$\int_\Omega \text{div} w \quad dx = \int_{\partial\Omega} w \cdot n d\sigma \quad \textit{(divergence formula)},$$

$$\int_\Omega (\Delta u) v \quad dx = -\int_\Omega \nabla u \cdot \nabla v \quad dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \quad d\sigma \quad \textit{(green formula)}.$$

1.2.3 Convergence theorems

Definition 1.5. *Let (u_n) be a sequence of measurable functions on Ω and u a measurable function on Ω.*

- *The sequence* (u_n) *converges almost everywhere on* Ω *to* u *if and only if*

$$meas\{x \in \Omega : u_n(x) \text{ does not converge to } \quad u(x)\} = 0,$$

- *The sequence* (u_n) *is said to be Cauchy in measure if for every* $\varepsilon > 0$ *and every* $\eta > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $m, n \geq n_0$*, so*

$$meas\{x \in \Omega : |u_n(x) - u(x)| > \eta\} \leq \varepsilon$$

Lemma 1.5. *[9] Let* (u_n) *be a sequence of measurable functions of in* Ω *and* $\mathbb{R}^N$*. If* (u_n) *of Cauchy in measure then there exists a sub-sequence of* (u_n) *converge almost everywhere*

Lemma 1.6. *[9] Let* f *be a strictly positive measurable function. Then, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every measurable* $A \subset \Omega$*, we have*

$$\int_A f dx \leq \delta \quad \Rightarrow \quad meas(A) \leq \varepsilon. \tag{1.2}$$

Theorem 1.2 (Lebesgue's Dominated Convergence Theorem[3])**.** *Let* (f_n) *be a sequence of functions in* $L^1(\Omega)$*. We suppose that*

- $f_n(x) \to f(x)$ *a.e in* Ω

- *there exists is a function* $g \in L^1(\Omega)$ *such that for every* n *,* $|f_n(x)| \leq g(x) \quad a.e \text{ in } \quad \Omega$

We have $f \in L^1(\Omega)$ *and*

$$\|f_n - f\|_{L^1(\Omega)} \to 0 \quad as \quad n \to \infty.$$

Theorem 1.3 (Inverse Theorem)**.** *Let* (f_n) *be a sequence in* $L^p(\Omega)$ *and let* $f \in L^p(\Omega)$ *be such that* $\quad \|f_n - f\|_{L^p(\Omega)} \to 0$
Then ,there exist a subsequence (f_{n_k}) *and a function* $h \in L^p(\Omega)$ *such that*

- $f_{n_k}(x) \to f(x)$ *a.e in* Ω

- $|f_{n_k}(x)| \leq g(x) \quad a.e \text{ in } \quad \Omega, \quad \forall k.$

1.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.6. *The Sobolev space $W^{1,p}(\Omega)$, is defined by*

$$W^{1,p}(\Omega) = \Big\{ u \in L^p(\Omega), |\ \nabla u \in (L^p(\Omega))^N \Big\}$$

Remark 1.2. *The space $W^{1,p}(\Omega)$ is equipped with the norm*

$$\| u \|_{W^{1,p}(\Omega)} = \| u \|_p + \| \nabla u \|_p \ . \tag{1.3}$$

For $p = \infty$

$$\|u\|_{W^{1,\infty}} = \max(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}).$$

For $p = 2$ we denote by $H^1 = W^{1,2}$ which is a Hilbert [1] space equipped with the scalar product.

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$$

The associated norm

$$\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})^{1/2}$$

is a norm equivalent to that of $W^{1,2}(\Omega)$

Proposition 1.3.

- *The space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$.*
- *The space $W^{1,p}$ is a separable space for $1 \leq p < \infty$.*
- *The space $W^{1,p}$ is a reflexive space for $1 < p < \infty$.*
- *The space $W_0^{1,p}$ is a separable space . Also reflexive for $1 < p < \infty$.*
- *The space H_0^1 is a separable Hilbert space.*

[1] David Hilbert (1862-1943) is a German mathematician. He is often considered one of the greatest mathematicians of the 20th century.

Definition 1.7. *the space $D(\Omega)$ denotes the set of functions in $C^\infty(\Omega)$ with compact support in Ω, we define $1 \leq p < \infty$*

$$W_0^{1,p}(\Omega) = \Big\{ u \in W^{1,p}(\Omega), \quad with \quad u = 0 \quad on \quad \partial\Omega \Big\}.$$

Proposition 1.4. *Since, for $1 \leq p < \infty$, the space $D(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, we can identify the dual of $W_0^{1,p}(\Omega)$ with a subspace of the space of distributions $D'(\Omega)$. we denote this as*

$$W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$$

1.4 Sobolev injections

Let $(E, \| \, . \, \|_E), (F, \| \, . \, \|_F)$ Banach spaces

- E is injected continuously into F ,means that the canonical injection $j : E \longrightarrow F$ is continuous i.e, $\exists c > 0, \forall x \in E : \| \, x \, \|_F \leq c \, \| \, x \, \|_E$, and we denote by $E \hookrightarrow F$.
- E is injected in compact into F means that the canonical injection $j : E \longrightarrow F$ is compact i.e for all sequence bounded u_n in E we can extract subsequence u_{n_k} convergent in F and we denote by $E \hookrightarrow_c F$.
  If $1 \leq p < \infty$, the Sobolev exponent of p defined by $p^* = \frac{Np}{N-p}$ or $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

Theorem 1.4. *Let $1 \leq p < \infty$, we suppose that Ω is on open set class C^1 a bounded frontier,and we take $\Omega = \mathbb{R}_+^{\mathbb{N}}$*

- $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*] \quad if \;\; p < N.$
- $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, \infty] \quad if \;\; p = N.$
- $W^{1.p}(\Omega) \hookrightarrow L^\infty(\Omega) \quad if \;\; p > N.$

Theorem 1.5 (**Rellich-Kondrachon**). *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$be a bounded domain of class C^1*

- $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1, p^*] \quad if \;\; p < N.$

- $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [p, \infty] \quad if \;\; p = N.$
- $W^{1.p}(\Omega) \subset C(\overline{\Omega}) \quad if \;\; p > N$

Lemma 1.7 (**Poincare's inequality**). *Suppose that $1 \leq p < \infty$, and Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that*

$$\| u \|_{L^p(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)} . \tag{1.4}$$

Moreover , by the density of $\mathcal{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, the inequality (1.4) stay true for any function $u \in W_0^{1,p}(\Omega)$.

1.5 Some notions of convergence

1.5.1 Strong convergence

Definition 1.8. *convergence of a sequence $(x_n)n \in \mathbb{N}$ in a normed vector space $(E, \| \, . \, \|_E)$ to an element $x \in E$,defined in the following way :*

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow \| x_n - x \|_E \leq \epsilon.$$

Or

$$\lim_{n \longrightarrow +\infty} \| x_n - x \|_E = 0.$$

1.5.2 Weak convergence

Definition 1.9. *Let E be a Banach space, E' its dual and $\langle ., . \rangle$ the duality bracket EE'.*

We say that the sequence (x_n) in E weakly converges to $x \in E$ if and only if:

$$\langle f, x_n \rangle \longrightarrow \langle f, x \rangle, \quad \forall f \in E'.$$

For $n \longrightarrow +\infty$,this weak convergence is written as $x_n \rightharpoonup x$ in E

Proposition 1.5.
- *if $x_n \longrightarrow x$ strongly then $x_n \rightharpoonup x$ in E.*

- *if* $x_n \rightharpoonup x$ *in* E *then* $\|x_n\|_E$ *is bounded.*

- $$\begin{cases} If \quad x_n \rightharpoonup x, \text{ weakly in } E \\ And \quad f_n \longrightarrow f \text{ strongly in } E' \end{cases}$$

  then $\langle f_n, x_n\rangle \longrightarrow \langle f, x\rangle$ *as* $n \longrightarrow +\infty$.

Theorem 1.6. *Let* E *be a reflexive Banach space. Then, for any bounded sequence* $(x_n)_n \subset E$*, there exists a sub-sequence* $(x_{nk})_{nk}$ *that converges weakly in* E.

1.5.3 (weak*) convergence

Definition 1.10. *We say that the sequence* (f_n) *in* E' *weakly* converges to* $f \in E'$ *if and only if:*

$$\langle f_n, g\rangle \longrightarrow_{n\longrightarrow+\infty} \langle f, g\rangle, \quad \forall g \in E$$

we denote this as

$$f_n \rightharpoonup *f \quad in \quad E.$$

Proposition 1.6.

$$f_n \longrightarrow f \quad \text{strongly} \quad \Longrightarrow f_n \rightharpoonup f \quad \text{in } E'$$

For more details, you can refer to the book[3].

Chapter 2

weighted Space

In this chapter, we study weighted Lebesgue and Sobolev spaces defined in the domains of n-dimensional Euclidean space $\mathbb{R}^N$, $N \geq 2$. By a weight, we shall mean a locally integrable function w on $\mathbb{R}^N$ such that $w(x) > 0$ a.e.. Every weight w gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration.

2.1 Weighted Sobolev spaces

2.1.1 The weighted Lebesgue space

Definition 2.1. *Let Ω be a measurable subset of $\mathbb{R}^N, N \geq 2$, and $w : \mathbb{R}^N \to [0, +\infty)$ be a locally integrable nonnegative function, i.e., a weight. The weighted Lebesgue space $L^p(\Omega, w), 1 \leq p < \infty$, is defined as a Banach space of measurable functions $f : \Omega \to \mathbb{R}$ endowed with the following norm:*

$$\|f\|_{L^p(\Omega,w)} = \left(\int_\Omega |f(x)|^p w(x)dx\right)^{\frac{1}{p}}.$$

If $w = 1$, we write $L^p(\Omega, w) = L^p(\Omega)$.

2.1.2 The Weighted Sobolev spaces

Definition 2.2. *Let Ω be an open subset of $\mathbb{R}^N, N \geq 2$. We define the weighted Sobolev space $W^{1,p}(\Omega, w), 1 \leq p < \infty$, as a normed space of measurable functions $f : E \to \mathbb{R}$ endowed with the following norm:*

$$\|f\|_{W^{1,p}(\Omega;w)} = \|f\|_{L^p(\Omega;w)} + \|\nabla f\|_{L^p(\Omega;w)}.$$

Proposition 2.1. *We also define $W_0^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, w)$ and The dual space of $W_0^{1,p}(\Omega, w)$ is $W^{-1,p'}(\Omega, w)$. The spaces $W_0^{1,p}(\Omega, w)$ and $W^{1,p}(\Omega, w)$ are*

Banach spaces. When $w = 1$, these spaces will be denoted $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively.

Remark 2.1. *It is evident that the weight function w which satisfies*

$$0 < c_1 \leq w(x) \leq c_2, \quad \textit{for} \quad x \in \Omega, (c_1 \textit{ and } c_2 \quad \textit{positive constants}),$$

gives nothing new (the space $W_0^{1,p}(\Omega, w)$ is then identical with the classical Sobolev space $W_0^{1,p}(\Omega)$.

We now determine conditions on the weight w that guarantee that functions in $L^p(\Omega, w)$ are locally integrable on Ω. Let $1 \leq p < \infty$ and let w be a weight such that $w^{-1/(p-1)}$ is locally integrable , when $p > 1$, and such that w locally is bounded from below away from 0, when $p = 1, i.e.$,such that

$$ess \sup_{x\in B} \frac{1}{w(x)} < \infty.$$

for every ball B. Suppose that $f \in L^p(\Omega, w)$, and let $B \subset \Omega$ be a ball . If $1 < p < \infty$, then ,by Hölder's inequality,

$$\int_B |f|dx \leq \Big(\int_B w^{-1/(p-1)} dx\Big)^{1/p'} \Big(\int_B |f|^p w dx\Big)^{1/p}. \tag{2.1}$$

and if $p = 1$,

$$\int_B |f|dx \leq ess \sup_{x\in B} \frac{1}{w(x)} \int_B |f| w dx. \tag{2.2}$$

It follows that $L^p(\Omega, w) \subset L^1_{loc}(\Omega)$ and that convergence in $L^p(\Omega, w)$ implies local convergence in $L^1(\Omega)$. If Ω is bounded , one obtains in the same way that $L^p(\Omega, w)$ is continuously embedded in $L^1(\Omega)$.

2.2 Some notions and results on operators

2.2.1 Monotone Operators

Definition 2.3. *An operator $A : V \to V'$ is said to be:*

- *Monotone if* $\langle Au - Av, u - v \rangle \geq 0, \forall u, v \in V$
- *strictly monotone if* $\langle Au - Av, u - v \rangle > 0, \forall u, v \in V, u \neq v$

Example 2.1. *The operator* $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ *defined by* $Au = -\Delta u$ *is monotone, where* $H_0^1(\Omega)$ *is equipped with the norm of the gradient. Indeed, for* $u, v \in H_0^1(\Omega)$*,and we Use lemma 1.4*

we have:

$$\begin{aligned} \langle Au - Av, u - v \rangle &= \int_\Omega \nabla(u - v).\nabla(u - v) \\ &= \|u - v\|_{H_0^1(\Omega)}^2 \\ &\geq 0. \end{aligned}$$

2.2.2 Bounded Operator

Definition 2.4. *Let* V *and* W *be two Banach spaces and let* $A : V \to W$ *be an operator. We say that* A *is bounded if it maps every bounded set in* V *to a bounded set in* W*; i.e*

$$\forall \rho > 0, \quad \exists C_\rho > 0 : \quad A(B_V(0, \rho)) \subset B_W(0, C_\rho)$$

where $B_V(0, \rho)$ *designates the open ball in* V *with center* 0 *and radius* $\rho > 0$ *and* $B_W(0, C_\rho)$ *designates the open ball in* W *with center* 0 *radius* $C_\rho > 0$ *.*

2.2.3 Hemicontnuous Operators

Definition 2.5. *An operator* $A : V \to W$ *is said to be hemicontinuous at the point* u_∞ *in* V *if, for any sequence* $\{u_n\}$ *converging to* u_∞*, the sequence* $\{Au_n\}$ *converges weakly to* Au_∞ *in* W*, in other words :*

$$\forall v \in V, \quad \forall \{\lambda_n\} \subset \mathbb{R}, \quad \lambda_n \to 0, \quad A(u_\infty + \lambda_n v) \rightharpoonup Au_\infty.$$

If A *is hemicontnuous at every point in* V*, we say that it is hemicontinuous on* V*. In reflexive spaces and when* $W = V'$*, and passing from sequential to continuous, we can* *define hemicontinuitly on V by requiring that:*

$$\forall u, v, w \in V \quad \text{the map} \quad \mathbb{R} \ni \lambda \mapsto \langle A(u + \lambda v), w\rangle \in \mathbb{R}$$

is continuous .

Example 2.2. *The operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined by*

$Au = -\Delta u = -\text{div}(\nabla u)$ is hemicontinuous. Indeed, let $u, v, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$, we use lemma 1.4.

we have :

$$\begin{aligned}
\langle A(u + \lambda v), w\rangle &= \int_\Omega \nabla(u + \lambda v).\nabla w \\
&= \int_\Omega \nabla u.\nabla w + \lambda \int_\Omega \nabla v \nabla w \\
&= a + b\lambda.
\end{aligned}$$

This shows that $\lambda \longrightarrow \langle A(u + \lambda v), w\rangle$ is continuous.

2.2.4 Coercive Operator

Definition 2.6. *An operator $A : V \to V'$ is said to be coercive if :*

$$\lim_{|v|_V \to +\infty} \frac{\langle Av, v\rangle}{\|v\|_V} = +\infty$$

2.3 Pseudo-monotonic Operators

Definition 2.7. *An operator $A : V \to V'$*

i) We say that A is pseudo-monotonic (in sense 1) if it is

- *for all $u_n \rightharpoonup u$ in V with $\lim_{n\to\infty} \sup\langle Au_n, u_n - u\rangle \leq 0$*

we have

$$\lim_{n\to\infty} inf \langle Au_n, u_n - v\rangle \geq \langle Au_n, u - v\rangle, \quad \forall v \in V$$

ii)We say that A is pseudo-monotonic (in sense 2) if it is

- *for all $u_n \rightharpoonup u$ in V with $A(u_n) \rightharpoonup \xi$ in V' and*

$$\lim_{n\to\infty} \sup\langle Au_n, u_n\rangle \leq \langle \xi, u\rangle$$

We have

$$\xi = A(u) \quad et \quad \langle Au_n, u_n\rangle \to \langle Au, u\rangle$$

Proposition 2.2. *If A is bounded , then A is pseudo-monotonic in sense 1 if only if it is pseudo-monotonic in sense 2.*

Proof. Let us first assume that A is pseudo-monotonic in sens 1(but not necessarily bounded). Let u_n be a sequence such that $u_n \rightharpoonup u, \quad A(u_n) \rightharpoonup \xi$ and

$$\lim\sup\langle A(u_n), u_n\rangle \leq \langle \xi, u\rangle$$

We deduce that

$$\lim\sup\langle A(u_n), u_n - u\rangle = \lim\sup(\langle A(u_n), u_n\rangle - \langle A(u_n) - u\rangle) \leq 0.$$

So, by pseudo-monotony in sense 1,if follows that for all $v \in V$,

$$\lim\inf\langle A(u_n), u_n - v\rangle \geq \langle A(u), u - v\rangle,$$

hence by developing the left hook,

$$\lim\inf\langle A(u_n), u_n\rangle - \langle \xi, v\rangle \geq \langle A(u), u - v\rangle.$$

Taking $v = u$, we deduce that $\lim\inf\langle A(u_n), u_n\rangle \geq \langle \xi, v\rangle$,from where

$$\langle A(u_n), u_n\rangle \to \langle \xi, u\rangle$$

and by reporting in the inequality above

$$\langle \xi, u - v\rangle \geq \langle A(u), u - v\rangle.$$

Taking $v = u + w$, we deduce that $\xi = A(u)$ and that A is pseudo-monotonic in sense 2.

Now suppose that A is pseudo-monotonic in sense 2 and bounded. Let u_n be a sequence such that $u_n \rightharpoonup u$ and $\lim\sup\langle A(u_n), u_n - u\rangle \leq 0$.Let $v \in V$.Aa A is bounded,we can extract a sequence such that

$$A(u_{n'}) \rightharpoonup \xi \quad \text{et} \quad \langle A(u_{n'}), u_{n'} - v\rangle \to \lim\inf\langle A(u_n), u_n - v\rangle.$$

As $\langle A(u_{n'}), u\rangle \to \langle \xi, u\rangle$,we first have $\lim\sup\langle A(u_{n'}), u'_n\rangle \leq \langle \xi, u\rangle$.So,by pseudo-monotonicity in sense 2,if follows that $\xi = A(u)$ and $\langle A(u'_n), u'_n\rangle \to \langle A(u), u\rangle$.We therefore see that

$$\langle A(u'_n), u'_n - v\rangle \to \langle A(u), u - v\rangle$$

So A is pseudo-monotonic in sense 1. □

Proposition 2.3. *If $A : V \to V'$ is bounded , hemicontinuous and monotonic, then A is pseudo-monotonic (in sense 1).*

Proof. Let $\{u_j\}$ be a sequence that converges weakly to u in V.Suppose that

$$\lim_{j\to\infty} \sup\langle Au_j, u_j - u\rangle \leq 0$$

If A is monotonic ,we have

$$\lim_{j\to\infty} \quad \langle Au_j, u_j - u\rangle \quad \to \quad 0 \tag{2.3}$$

Indeed,the monotonicity of A and the weak convergence of $\{u_j\}$ to u implies that

$$\langle Au_j, u_j - u\rangle \geq \langle Au, u_j - u\rangle \to 0 \quad for \quad j \to \infty$$

And so

$$0 \geq \lim_{j\to\infty} \sup\langle Au_j, u_j - u\rangle \geq \lim_{j\to\infty} inf\langle Au_j, u_j - u\rangle \geq \lim_{j\to\infty} \sup\langle Au, u_j - u\rangle = 0$$

Hence (2.3)

b) four $v \in V$ and $t \in]0, 1[$,let $w = (1 - t)u + tv$. we have $\langle Au_j - Aw, u_j - w\rangle$ of fate that

:

$$t\langle Au_j, u - v\rangle \geq -\langle Au_j, u_j - u\rangle + \langle Aw, u_j - u\rangle - t\langle Aw, v - u\rangle.$$

Hence , thanks to (2.3):

$$t \lim_j inf\langle Au_j, u - v\rangle \geq -t\langle Aw, v - u\rangle,$$

hence , dividing by t and taking into account (2.3):

$$\lim_j inf\langle Au_j, u_j - v\rangle \quad \geq \quad \langle Aw, u - v\rangle \tag{2.4}$$

$$w = (1 - t)u + tv \quad \forall t \in]0, 1[$$

By finishing tend t to 0 in (2.4), and using hemicontinuity, we deduce

$$\lim_{j\to\infty} inf\langle Au_j, u_j - v\rangle \geq \langle Au, u - v\rangle, \quad \forall v \in V$$

Which means that A is pseudo-monotonic in sense 1 . □

2.4 Monotonic Operator Theorems

2.4.1 General result

Let V be a reflexive and separable Banach space, and $A : V \to V'$ be an operator.

Theorem 2.1 (Existence theorem). *Suppose that $A : V \longrightarrow V'$ is an operator :*

- *bounded*
- *hemicontinuous*
- *coercive*
- *monotone*

Let $f \in V'$. Then, there exists a function $u \in V$ and only one such that:

$$Au = f$$

For the proof this theorem, we refer the reader to the book [10].

2.5 The weighted p-Laplacian operator

Let $S : \mathbb{R}^N \to [0, +\infty)$ be a locally integrable nonnegative function, i.e., a weight. The operator

$$\triangle_{p,S}(u) = \operatorname{div}(S(x) \mid \nabla u \mid^{p-2} \nabla u), \quad 1 < p < \infty$$

is known as weighted p-Laplacian (see, for example,[7, 8]). The weighted p- Laplacian is a non-linear operator when $p \neq 2$ and is linear when $p = 2$.

Lemma 2.1. *Let $p > 1$ and let $S(x)$ be a function in $W^{1,p}(\Omega)$ such that*

$$0 < \alpha \leq S(x) \leq \beta. \tag{2.5}$$

Then, the operator A defined by

$$Au = -\operatorname{div}(S(x) \mid \nabla u \mid^{p-2} \nabla u), \quad 1 < p < \infty$$

to verify all the hypotheses of the theorem 2.1, so

$$\forall f \in W^{-1,p'}(\Omega), \quad \exists u \in W_0^{1,p}(\Omega) \quad \text{such that} \quad Au = f$$

Proof of lemma 2.1. **Firstly.** We have that the weighted operator

$$Au = -\operatorname{div}(S(x) \mid \nabla u \mid^{p-2} \nabla u), p \neq 2.$$

is monotone from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$.

we recall that

$$\forall x, y \in \mathbb{R}^{\mathbb{N}} : (|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq \mu |x - y|^p, \mu > 0.$$

So, $\forall u, v \in V$ and we use lemmm 1.4

$$\begin{aligned}\langle Au - Av, u - v\rangle &= \int_\Omega S(x)\Big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\Big)\nabla(u-v) \\ &\geq \alpha\int_\Omega \mu|\nabla u - \nabla v|^p \\ &\geq 0\end{aligned}$$

Secondly From the expression of the norm in a dual space, Let ρ strictly positive and let $u \in B_{W_0^{1,p}(\Omega)}(0,\rho)$ then $\|u\|_{W_0^{1,p}(\Omega)} \leq \rho$, one can get

$$\| Au \|_{V'} = \sup_{\substack{\varphi\in V \\ \|\varphi\|\leq 1}} |\langle Au, \varphi\rangle| = \sup_{\substack{\varphi\in V \\ \|\varphi\|\leq 1}} \Big|\int_\Omega S(x)|\nabla u|^{p-2}\nabla u.\nabla\varphi\Big|.$$

Using the Hölder's inequality and (2.5), we obtain

$$\begin{aligned}\| Au \|_{V'} &\leq \sup_{\substack{\varphi\in V \\ \|\varphi\|\leq 1}} \int_\Omega |S(x)||\nabla u|^{p-1}.|\nabla\varphi| \\ &\leq \beta \sup_{\substack{\varphi\in V \\ \|\varphi\|\leq 1}} \Big(\int_\Omega |\nabla u||^p\Big)^{\frac{1}{p'}}\Big(\int_\Omega |\nabla\varphi|^p\Big)^{\frac{1}{p}} \\ &= \beta \sup_{\substack{\varphi\in V \\ \|\varphi\|\leq 1}} \| u \|^{p-1}_{W_0^{1,p}(\Omega)}\| \varphi \|_{W_0^{1,p}(\Omega)} \\ &\leq \beta\rho^{p-1}\end{aligned}$$

Which implies the boundedness of A.

Thirdly In fact, let $u, v, w \in W_0^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$. Let's show that the function of $\mathbb{R}$ in $\mathbb{R}$:

$$\lambda \mapsto \langle A(u+\lambda v), g\rangle = \int_\Omega S(x)|\nabla(u+\lambda v)|^{p-2}\nabla(u+\lambda v)\nabla g$$

is continuous.

Let $\lambda \in \mathbb{R}$ be fixed and let $\{\lambda_n\}$ be a sequence of $\mathbb{R}$ converging to λ. Since $\lambda_n \to \lambda$ in $\mathbb{R}$, we have

$$S(x)|\nabla(u+\lambda_n\nabla v)|^{p-2}\nabla(u+\lambda_n\nabla v)\nabla g \to^{p.p} S(x)|\nabla(u+\lambda\nabla v)|^{p-2}(\nabla(u+\lambda\nabla v))\nabla g$$

we use lemma 1.4,

$$\begin{aligned}\langle A(u+\lambda_n v), g\rangle &= \int_\Omega S(x)|\nabla(u+\lambda_n v)|^{p-2}\nabla(u_\infty+\lambda_n v)\nabla g\\ &= \int_\Omega S(x)|(\nabla u+\lambda_n\nabla v)|^{p-2}(\nabla u+\lambda_n\nabla v)\nabla g\end{aligned}$$

we can assume $|\lambda_n|\leq 1$, and $S(x)\leq\beta$,$\beta>0$ then

$$\begin{aligned}\Big|S(x)|\nabla(u+\lambda_n\nabla v)|^{p-2}(\nabla u+\lambda_n\nabla v)\nabla g\Big| &= S(x)|\nabla(u_\infty+\lambda_n\nabla v|^{p-1}|\nabla g|\\ &\leq \beta\Big[|\nabla u|+|\lambda_n||\nabla v|\Big]^{p-1}|\nabla g|\\ &\leq \beta\Big[|\nabla u|+|\nabla v|\Big]^{p-1}|\nabla g|\end{aligned}$$

we recall that

$$(\sum_{i=1}^{N} a_i)^\alpha \leq \max\{1, N^{\alpha-1}\}\sum_{i=1}^{N} a_i^\alpha, \quad \forall a_i\geq 0,\ \alpha>0 \tag{2.6}$$

Using Young's inequality (1.3) and (2.6), we can write

$$\begin{aligned}\Big|k(x)|\nabla(u+\lambda_n\nabla v)|^{p-2}(\nabla u+\lambda_n\nabla v)\nabla g\Big| &\leq \frac{\beta|\nabla g|^p}{p}+\frac{\beta\Big[|\nabla u+|\nabla v|\Big]^p}{p'}\\ &\leq \frac{\beta}{p}|\nabla g|^p+\frac{max(1,2^{p-1})}{p'}\Big(|\nabla u|^p+|\nabla v|^p\Big)\beta.\end{aligned}$$

the fact that $g, v, u\in W_0^{1,p}(\Omega)$,and $\beta>0$ implies that

$$\frac{\beta}{p}|\nabla g|^p+\frac{2^{p-1}}{p'}\Big(|\nabla u|^p+|\nabla v|^p\Big)\beta\in L^1(\Omega)$$

According to Lebesgue's dominated convergence theorem

$$\lim_{n\to+\infty}\langle A(u+\lambda_n v), g\rangle = \langle A(u+\lambda v), g\rangle$$

Which shows the hemicontinuity of A.

Finally For $v\in W_0^{1,p}(\Omega)$ and (2.5), we have

$$\begin{aligned}\lim_{|v|_V\to+\infty}\frac{\langle Av, v\rangle}{\|v\|_V} &= \lim_{|v|_V\to+\infty}\frac{\int_\Omega S(x)|\nabla v|^p}{\|v\|_V}\\ &\geq \lim_{|v|_V\to+\infty}\frac{\alpha\|v\|_V^p}{\|v\|_V}\\ &= \lim_{|v|_V\to+\infty}\alpha\|v\|^{p-1}=+\infty \quad car \quad \alpha>0 \quad and \quad 1<p<+\infty.\end{aligned}$$

□

Now, we have the following result.

Lemma 2.2. *Let $p > 1$, g belong to $L^\infty(\Omega)$ and let $S(x)$ be a function in $W^{1,p}(\Omega)$ such that*

$$0 < \alpha \leq S(x) \leq \beta. \tag{2.7}$$

Let v in $W_0^{1,p}(\Omega)$ be the weak solution of the equation

$$\begin{cases} -\mathrm{div}(S(x) \mid \nabla v \mid^{p-2} \nabla v) & = & g(x) \quad in\ \Omega; \\ v & = & 0 \quad on \quad \partial\Omega. \end{cases} \tag{2.8}$$

Then v is such that

$$\int_\Omega \mid \nabla v \mid^{p-2} \nabla v.\nabla\varphi = \int_\Omega G(x)\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

where $G(x)$ is the $L^1(\Omega)$ function

$$G(x) = \Big[\mid \nabla v \mid^{p-2} \nabla v.\nabla S + g(x)\Big] \frac{1}{S(x)}. \tag{2.9}$$

Remark 2.2. *Note that the solution v of (2.8) exists since the differential weighted operator*

$$A(v) = -\mathrm{div}(S(x) \mid \nabla v \mid^{p-2} \nabla v)$$

is both coercive and pseudomonotone on $W_0^{1,p}(\Omega)$, so that it is surjective on the dual of $W_0^{1,p}(\Omega)$ (see chapter 2).

Proof. Let φ belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Since $S(x)$ belongs to $W^{1,p}(\Omega)$, and $\frac{1}{S(x)}$ is a bounded function, we have that $\psi = \frac{\varphi}{S(x)}$ belongs to $W_0^{1,p}(\Omega)$, so that it can be chosen as test function in the weak formulation of (2.8). We obtain

$$\int_\Omega S(x) \mid \nabla v \mid^{p-2} \nabla v.\nabla \left(\frac{\varphi}{S(x)}\right) dx = \int_\Omega g(x) \frac{\varphi}{S(x)} dx$$

Using that

$$\nabla \left(\frac{\varphi}{S(x)}\right) = \frac{S(x)\nabla\varphi - \varphi\nabla S(x)}{S^2(x)},$$

we have

$$\int_\Omega \frac{S(x)}{S(x)} \mid \nabla v \mid^{p-2} \nabla v . \nabla \varphi - \int_\Omega \frac{S(x)}{S^2(x)} \mid \nabla v \mid^{p-2} \nabla v . \nabla S \varphi = \int_\Omega g(x) \frac{\varphi}{S(x)}.$$

which can be rewritten as

$$\int_\Omega \mid \nabla v \mid^{p-2} \nabla v . \nabla \varphi = \int_\Omega \mid \nabla v \mid^{p-2} \nabla v . \nabla S \frac{\varphi}{S(x)} + \int_\Omega g(x) \frac{\varphi}{S(x)} = \int_\Omega G(x) \varphi.$$

where in the last passage we have used the definition (2.9) of $G(x)$. Because that $S(x) \in W^{1,p}(\Omega)$ and the fact that $v \in W_0^{1,p}(\Omega)$, we have that $|\nabla v|^{p-2} \nabla v \nabla S$ is bounded in $L^1(\Omega)$, while the function g is bounded ($g \in L^\infty$). Furthermore, thanks to assumption (2.7), the term $\frac{1}{S(x)}$ is bounded from above by $\frac{1}{\alpha}$, so that $G(x)$ is bounded in $L^1(\Omega)$, as desired. □

Chapter 3

Nonlinear weighted elliptic equations with L^1 data

In this chapter, we study existence of weak solutions for a class of nonlinear boundary value problems of elliptic type associated with weighted p-Laplacian operator but with a non-regular second member. i.e, not belonging to a dual space. we consider second member in L^1.

3.1 The problem (P)

We are interested in problems of the type

$$(P) \qquad \begin{cases} -\mathrm{div}(S(x)|\nabla u|^{p-2}\nabla u) + e(x)|u|^{p-2}u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded , open subset of $\mathbb{R}^N$,$N > 2$. $p \geq 1$, f belongs to $L^1(\Omega)$ and $S(x)$ is such that

$$S(x) \in W^{1,p}(\Omega) : S(x) \geq \alpha > 0. \tag{3.1}$$

for some α in $\mathbb{R}$ and the function $0 \leq e(x) \in L^1(\Omega)$. Even if $f \in L^1(\Omega)$, the assumption

$$\text{there exists} \quad k_0 > 0 \quad \text{such that} \quad |f(x)| \leq k_0.e(x). \tag{3.2}$$

implies the existence of a weak solution u belonging to $W_0^{1,p}(\Omega)$ and to $L^\infty(\Omega)$.

3.1.1 Weak solutions

Definition 3.1. *says that u is a weak solution to problem (P) if: $u \in W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$ and*

$$\int_\Omega S(x) \mid \nabla u \mid^{p-2} \nabla u \nabla\varphi \, dx + \int_\Omega e(x)|u|^{p-2}u\varphi dx \;\; = \;\; \int f\varphi dx, \; \forall\varphi \in W_0^{1,\infty}(\Omega). \tag{3.3}$$

Now, The result proved in this chapter is the following theorem

Theorem 3.1. *Let $p \geq 1$ and let $f \in L^1(\Omega)$. Then, the problem (P) has at least one weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the sense of Definition 3.1.*

The proof of the Theorem 3.1 needs several steps: First, we approximate the problem (P) with sequence of problems (P_n) having smooth solutions (u_n). Then, after deriving uniform estimates on u_n , we pass to the limit

3.1.2 The solution obtained by approximation

Consider the following approximated problem

$$(P_n) \qquad \begin{cases} -\mathrm{div}(S_n(x)|\nabla u_n|^{p-2}\nabla u_n) + e_n(x)|u_n|^{p-2}u_n = f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

i.e., satisfying

$$\int_\Omega S_n(x)|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx + \int_\Omega e_n(x)|u_n|^{p-2}u_n\varphi dx = \int_\Omega f_n\varphi dx, \tag{3.4}$$

for every $\varphi \in W_0^{1,\infty}(\Omega)$, with f_n, e_n be a sequence of bounded functions defined in Ω defined by

$$e_n(x) = \frac{e(x)}{1 + \frac{k_0 e(x)}{n}}, \quad f_n(x) = \frac{f(x)}{1 + \frac{|f(x)|}{n}}. \tag{3.5}$$

Note that the definition of e_n and f_n in (3.5) and since $\psi(s) = s(1 + \frac{s}{n})^{-1}$ is increasing, we deduce by (3.2) that

$$\begin{cases} \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \quad \|e_n\|_{L^1(\Omega)} \leq \|e\|_{L^1(\Omega)}, \\ |f_n| \leq n, \quad |e_n| \leq n. \end{cases}$$

and

$$|f_n(x)| = \frac{f(x)}{1 + \frac{|f(x)|}{n}} \leq \frac{k_0 e(x)}{1 + \frac{k_0 e(x)}{n}} = k_0 e_n(x) \tag{3.6}$$

and $S_n(x)$ be a sequence of bounded functions in $W^{1,p}(\Omega)$, such that

$$S_n(x) = \frac{S(x)}{1 + \frac{S(x)}{n}}. \tag{3.7}$$

Note that we have, thanks to (3.1),that

$$\frac{\alpha}{1+\alpha} \le S_n(x) \le n. \tag{3.8}$$

The operator $B_n : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$\int_\Omega \langle B_n v, \varphi \rangle dx = \int_\Omega S_n(x)|\nabla v_n|^{p-2}\nabla v_n)\nabla\varphi dx, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

is coercive due to the fact that $S_n(x) \ge \frac{\alpha}{1+\alpha} > 0$ and is pseudo-monotone on $W_0^{1,p}(\Omega)$(see Lemma 2.1). Thus, the existence of the approximate solution u_n is proved as in [10].

3.2 Uniform Estimates of Approximate solutions

As we only have $\|f_n\|_{L^1(\Omega)} \le C, \quad \forall n \ge 1$, in order to bound the term $\int_\Omega f_n\varphi$ uniformly with respect to n, need to choose a test function

$$\varphi_n = \quad \text{a function of } u_n \text{ such that} \quad \|\varphi_n\|_{L^\infty(\Omega)} \le C, \quad \text{with} \quad \varphi_n \in W_0^{1,p}(\Omega)$$

To choose the test functions φ, we use the following result due to Stampacchia :

Lemma 3.1 (Stampacchia)**.** *Let $T : \mathbb{R} \to \mathbb{R}$ be a globally lipschitz function,i.e.*

$$\exists C > 0 \quad \textit{such that} \quad |T(s) - T(t)| \le C|s-t|, \quad \forall s,t \in \mathbb{R},$$

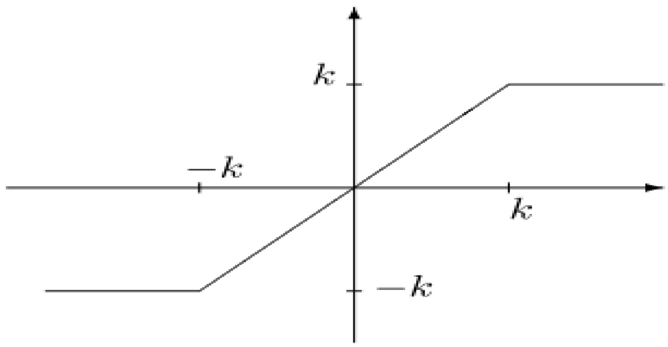
such that $T(0) = 0$. then, $\forall v \in W_0^{1,p}(\Omega)$ with $1 \le p \le \infty$ we have :

$$T(v) \in W_0^{1,p}(\Omega) \quad \textit{and} \quad \nabla T(v) = T'(v)\nabla v \quad \textit{in} \quad \mathcal{D}'(\Omega) \quad \textit{and} \quad \textit{almosteverywhere} \quad \textit{in} \quad \Omega$$

Example 3.1. *Let $k > 0$. The truncation at levels $-k$ and k is defined by the function T_k of $\mathbb{R}$ from $\mathbb{R}$ given by*

$$T_k(r) = \begin{cases} k, & \text{if } r \ge k, \\ r, & \text{if } |r| < k, \\ -k, & \text{if } r \le -k. \end{cases}$$

It can be verified that the function T_k is a globally Lipschitz function, The graph of T_k is: We can verify that the function T_k is a globally Lipschitz function satisfying $|T_k(r)| \le k$ and $|T_k(r)| \le |r|$.



Example 3.2. *Let $k > 0$. We will use the following function defined for $s \in \mathbb{R}$ by*

$$G_k(r) = r - T_k(r) = \begin{cases} r - k, & \text{if } r > k, \\ 0, & \text{if } |r| \leq k, \\ r + k, & \text{if } r < -k. \end{cases}$$

so,

$$G_k(u_n) = u_n - T_k(u_n) = \begin{cases} u_n - k, & \text{if } u_n > k, \\ 0, & \text{if } |u_n| \leq k, \\ u_n + k, & \text{if } u_n < -k. \end{cases}$$

and

$$\nabla G_k(u_n) = \begin{cases} \nabla u_n, & \text{if } |u_n| > k, \\ 0, & \text{if } |u_n| \leq k. \end{cases}$$

Lemma 3.2. *Let u_n be the solutions to problems (P_n). Then, there exists a positive constant C such that*

$$\|u_n\|_{L^\infty(\Omega)} \leq C, \quad \|u_n\|_{W_0^{1,p}(\Omega)} \leq C, \quad \forall n \in \mathbb{N}.$$

Proof. We choose $G_R(u_n)$ as test function in (P_n) with $R = K_0^{\frac{1}{p-1}}$, we get

$$\int_\Omega S_n(x)|\nabla u_n|^{p-2}\nabla u_n \nabla(G_R(u_n))dx + \int_\Omega e_n(x)|u_n|^{p-2}u_n G_R(u_n)dx = \int_\Omega f_n G_R(u_n)dx. \tag{3.9}$$

Since $|\nabla u_n|^{p-2}\nabla u_n.\nabla(G_R(u_n)) = |\nabla G_R(u_n)|^p$ and the fact that $|T_n(u_n)| \leq |u_n|$, we obtain

$$\frac{\alpha}{1+\alpha}\int_\Omega |\nabla G_R(u_n)|^p dx + \int_\Omega e_n(x)|u_n|^{p-2}u_n G_R(u_n)dx \leq \int_\Omega |f_n||G_R(u_n)|dx. \tag{3.10}$$

Using that $u_n G_R(u_n) \geq 0$ and (3.2), we obtain

$$\frac{\alpha}{1+\alpha}\int_\Omega |\nabla G_R(u_n)|^p dx + \int_\Omega e_n(x)(|u_n|^{p-1} - k_0)|G_R(u_n)|dx \leq 0.$$

Observing that the integrand in the second integral is zero if $|u_n(x)| \leq k_0^{\frac{1}{p-1}} = R$, so we deduce that

$$\frac{\alpha}{1+\alpha}\int_\Omega |\nabla G_R(u_n)|^p dx = 0,$$

which implies

$$|u_n| \leq k_0^{\frac{1}{p-1}} = R. \tag{3.11}$$

This implies that $(u_n)_n$ is bounded in $L^\infty(\Omega)$.

Now, to show that the sequence $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$, inserting u_n in (P_n), and from (3.11), we get

$$\frac{\alpha}{1+\alpha}\int_\Omega |\nabla u_n|^p dx + \int_\Omega e_n(x)|u_n|^p dx \leq \int_\Omega |f_n||u_n|dx,$$

dropping nonnegative terms and using (3.11), we can rewrite the above inequality as follows

$$\frac{\alpha}{1+\alpha}\int_\Omega |\nabla u_n|^p dx \leq R\|f_n\|_{L^1(\Omega)} \leq k_0^{\frac{1}{p-1}}\|f\|_{L^1(\Omega)}.$$

Since $f \in L^1(\Omega)$, we find

$$\int_\Omega |\nabla u_n|^p dx \leq C \quad ,C = \frac{(1+\alpha)k_0^{\frac{1}{p-1}}}{\alpha}\|f\|_{L^1(\Omega)}.$$

Thus,

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq C. \tag{3.12}$$

This implies that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. □

3.2.1 Passage to the limit

Proposition 3.1. *since $(u_n)_n$ is bounded in the space $W_0^{1,p}(\Omega)$, according to Rellich-Kondrachov theorem, we can extract from the sequence (u_n) a sub-sequence, also denote by (u_n) such that*

$$u_n \rightharpoonup u \quad \textit{weakly in} \quad W_0^{1,p}(\Omega), \tag{3.13}$$

$$u_n \longrightarrow u \quad strongly\ in \quad L^p(\Omega). \tag{3.14}$$

Therefore

$$u_n \longrightarrow u \quad a.e\ in \quad \Omega, \tag{3.15}$$

Lemma 3.3. *Lets K_n be a sequence of functions in $L^1(\Omega)$, $p > 1$ and let S_n be a sequence of functions in $W^{1,p}(\Omega)$ such that:*

1. *the sequence $\{K_n\}_n$ is bounded in $L^1(\Omega)$;*

2. *the sequence $\{S_n\}_n$ is bounded in $W^{1,p}(\Omega)$;*

3. *for every n in $\mathbb{N}$ there exists $\lambda > 0$ and $\beta_n > 0$ such that*

$$\lambda \le S_n(x) \le \beta_n. \tag{3.16}$$

4. *there exists a weak solution u_n in $W_0^{1,p}(\Omega)$ of*

$$(*) \qquad \begin{cases} -div(S_n(x)|\nabla u_n|^{p-2}\nabla u_n) = K_n, & in\, \Omega \\ u_n = 0, & on\ \partial\Omega. \end{cases}$$

5. *the sequence u_n is bounded in $W_0^{1,p}(\Omega)$;*

Then, we have

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi = \int_\Omega H_n\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

where H_n is bonded in $L^1(\Omega)$. Furthermore,

$$\nabla u_n \longrightarrow \nabla u \ \ a.e.\ \ in\ \ \Omega. \tag{3.17}$$

where u is the weak limit of the sequence u_n in $W_0^{1,p}(\Omega)$.

Proof. Thanks to Lemma 2.2 applied to the equations solved by u_n, we have that (3.17) holds, with (see (2.9))

$$H_n(x) = \frac{1}{S_n(x)}\left(|\nabla u_n|^{p-2}\nabla u_n \nabla S_n + K_n\right). \tag{3.18}$$

Thanks to (3.8), the assumptions (2) of Lemma 3.3 holds, and the sequence $\{\frac{1}{S_n(x)}\}_n$ is bounded by $\frac{1+\alpha}{\alpha}$. furthermore, since $|\nabla S_n(x)| \leq |\nabla S(x)|$, we have

$$\begin{aligned}\int_\Omega |H_n(x)|dx &\leq \frac{1+\alpha}{\alpha}\int_\Omega |\nabla u_n|^{p-2}|\nabla u_n||\nabla S(x)|dx + \frac{1+\alpha}{\alpha}\int_\Omega |K_n|dx \\ &\leq C_1\int_\Omega |\nabla u_n|^{p-1}|\nabla S(x)|dx + C_1\int_\Omega |K_n|dx. \end{aligned} \tag{3.19}$$

By application of Young's inequality, we have

$$\int_\Omega |H_n(x)|dx \leq \frac{C_1}{p'}\int_\Omega |\nabla u_n|^p dx + \frac{C_1}{p}\int_\Omega |\nabla S(x)|^p dx + C_1\int_\Omega |K_n|dx.$$

By assumption (2), the $L^{p(\cdot)}(\Omega)$-bound on ∇u_n (since(5)) and the fact that the sequence $\{K_n\}_n$ is bounded in $L^1(\Omega)$ (since (1)), we conclude that H_n is bounded in $L^1(\Omega)$. Now, We show that (∇u_n) is Cauchy in measure, which implies that $\nabla u_n \longrightarrow \nabla u$ almost everywhere, for a sub-sequence. This consists of proving that

$$\forall\delta > 0, \forall\varepsilon > 0, \exists n_0 \text{ such that } \quad \forall p, q \geq n_0 \quad \text{meas}\{x \in \Omega|\ |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq \varepsilon$$

To do this, let us fix $\delta > 0$ and $\varepsilon > 0$, and note that for $\lambda > 0$ and $\eta > 0$, we have

$$\{x \in \Omega|\ |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \subset E_1 \cup E_2 \cup E_3 \cup E_4$$

where

$$E_1 = \{x \in \Omega|\ |\nabla u_p| \geq \lambda\}, \quad E_2 = \{x \in \Omega|\ |\nabla u_q| \geq \lambda\}$$

$$E_3 = \{x \in \Omega|\ |u_p - u_q| \geq \eta\}$$

and

$$E_4 = \{|\nabla u_p - \nabla u_q| \geq \delta,\ |\nabla u_p| \leq \lambda,\ |\nabla u_q| \leq \lambda,\ |u_p - u_q| \leq \eta\}.$$

And since (5), the sequence (∇u_n) is bounded in $L^p(\Omega)$, by choosing λ large, we can make meas(E_1) and meas(E_2) arbitrarily small. For example

$$\text{meas}(E_1) = \int_{E_1} 1dx = \frac{1}{\lambda}\int_{E_1} \lambda dx \leq \frac{1}{\lambda}\int_{E_1} |\nabla u_p|dx \leq \frac{1}{\lambda}\int_\Omega |\nabla u_p|dx \leq \frac{C}{\lambda}.$$

Which yields,

$$\text{meas}(E_1) \leq \frac{C}{\lambda} \to 0 \quad \text{as} \quad \lambda \to +\infty, \tag{3.20}$$

hence, we deduce that for all $\varepsilon > 0$, there exists $n_0 > 0$ such that

$$\text{meas}(E_1) \leq \frac{\varepsilon}{4}.$$

for meas(E_3), we have

$$\text{meas}(E_3) \leq \frac{1}{\eta} \int_{E_3} |u_p - u_q| dx \leq \frac{1}{\eta} \int_{\Omega} |u_p - u_q| dx$$

Since (u_n) is a Cauchy sequence in $L^1(\Omega)$ [see (3.14)], then for fixed $\eta > 0$, we see that

$$\text{meas}(E_3) \to 0 \quad \text{if} \quad p, q \to +\infty$$

hence,

$$\text{meas}(E_3) \leq \frac{\varepsilon}{4}.$$

It remains to control meas(E_4). We Define the vector-valued function $\widehat{a}(x, \xi) : \Omega\mathbb{R}^N$, where $\widehat{a}(x, \xi) = |\xi|^{p-2}\xi$. Because the set $\{(\xi_1, \xi_2)|,\ |\xi_1| \leq \lambda,\ |\xi_2| \leq \lambda,\ |\xi_1 - \xi_2| \leq \delta\}$ is a compact set and

$$\widehat{a} : (x, \nabla v_n) \longmapsto \widehat{a}(x, \nabla v_n) = |\nabla v_n|^{p-2} \nabla v_n \qquad \text{(see chapter 2)},$$

is continuous in ξ for a.e $x \in \Omega$, the quantity

$$(\widehat{a}(x, \xi_1) - \widehat{a}(x, \xi_2))(\xi_1 - \xi_2) > 0$$

reaches its minimum value on this compact set, and we will denote it by $\mu(x)$ such that

$$(\widehat{a}(x, \xi_1) - \widehat{a}(x, \xi_2))(\xi_1 - \xi_2) \geq \mu(x) > 0.$$

Therefore, by (1.2) for any $\tau > 0$ there exists $\tau' > 0$ such that

$$\int_{E_4} \mu(x) dx \leq \tau' \Rightarrow meas(E_4) \leq \tau. \tag{3.21}$$

To obtain $meas(E_4) \leq \tau$, it suffices to show that $\int_{E_4} \mu(x)dx \leq \tau'$. Using the definition of $\mu(x)$ and E_4, we can write

$$\int_{E_4} \mu(x)dx \leq \int_{E_4} \Big(|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_q|^{p-2}\nabla u_q\Big)\nabla(u_p - u_q)\mathbf{1}_{\{|u_p-u_q|\leq\eta\}}dx.$$

Moreover, the integral term is positive and $\nabla T_\eta(u_p - u_q) = \nabla(u_p - u_q)\mathbf{1}_{\{|u_p-u_q|\leq\eta\}}$, so we have

$$\int_{E_4} \mu(x)dx \leq \int_{E_4} \Big(|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_q|^{p-2}\nabla u_q\Big)\nabla T_\eta(u_p - u_q)dx. \tag{3.22}$$

Where T_η is the truncation function at levels η and T'_η is defined by

$$T'_\eta(\sigma) = \begin{cases} 1, & |\sigma| \leq \eta; \\ 0, & |\sigma| > \eta. \end{cases}$$

By taking $T_\eta(u_p - u_q)$ as a test function in $(*)$ for u_p and u_q, we have

$$\int_\Omega S_n(x)|\nabla u_p|^{p-2}\nabla u_p \nabla T_\eta(u_p - u_q)dx = \int_\Omega K_n T_\eta(u_p - u_q)dx \tag{3.23}$$

and

$$\int_\Omega S_n(x)|\nabla u_q|^{p-2}\nabla u_q \nabla T_\eta(u_p - u_q)dx = \int_\Omega K_n T_\eta(u_p - u_q)dx. \tag{3.24}$$

Then, subtracting the inequality resulting from (3.23) and (3.24), we obtain

$$\int_\Omega S_n(x)(|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_q|^{p-2}\nabla u_q)\nabla T_\eta(u_p - u_q)dx = 0.$$

Using (3.8), we can write that

$$\begin{aligned} &\int_\Omega (|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_q|^{p-2}\nabla u_q)\nabla T_\eta(u_p - u_q)dx \\ &\leq \eta \to^{\eta\to 0} 0. \end{aligned} \tag{3.25}$$

For η sufficiently small, (3.22) and (3.25) imply

$$\int_{E_4} \mu(x)dx \leq \eta = \tau',$$

and also by (3.21) we have $meas(E_4) \leq \tau = \frac{\varepsilon}{4}$. Therefore for all $p, q \geq n_0$ we have

$$meas\{x \in \Omega |\, |(\nabla u_p - \nabla u_q)(x)| \geq \delta\} \leq \varepsilon.$$

Thus, we have the convergence of ∇u_n to ∇u in measure, as well as the lemma 1.5 (after extraction of a sequence)

$$\nabla u_n \to \nabla u \quad \text{almost everywhere in} \quad \Omega.$$

□

3.2.2 The end of the proof of Theorem 3.1

For $\varphi \in W_0^{1,\infty}(\Omega)$ (See (3.4)), we have

$$\int_\Omega S_n(x)|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx + \int_\Omega e_n(x)|u_n|^{p-2}u_n\varphi dx = \int_\Omega f_n\varphi dx,$$

1)Passage to the limit in $\int_\Omega f_n\varphi dx$

Using that $f_n \to f$ strongly in $L^1(\Omega)$, we have $\forall \varphi \in W_0^{1,p}(\Omega)$

$$\lim_{n\to+\infty}\int_\Omega f_n\varphi dx = \int_\Omega f\varphi dx.$$

2)Passage to the limit in $\int_\Omega e_n(x)|u_n|^{p-2}u_n\varphi dx$.

thanks to (3.11), we have

$$\int_\Omega e_n(x)|u_n|^{p-2}u_n dx \leq \int_\Omega e_n(x)|u_n|^{p-1}dx \leq k_0\int_\Omega e_n(x)dx. \tag{3.26}$$

Since $e_n(x) \in L^\infty(\Omega)$, the sequence $\{e_n(x)|u_n|^{p-2}u_n\}_n$ is bounded in $L^1(\Omega)$ and thanks to (3.15), we have that

$$e_n(x)|u_n|^{p-2}u_n \to e(x)|u|^{p-2}u \quad \text{strongly in} \quad L^1(\Omega). \tag{3.27}$$

So, we have

$$\begin{aligned}\left|\int_\Omega e_n(x)|u_n|^{p-2}u_n\varphi - \int_\Omega e_n(x)|u|^{p-2}u\varphi\right| &= \left|\int_\Omega (e_n(x)|u_n|^{p-2}u_n - e(x)|u|^{p-2}u)\varphi\right| \\ &\leq C\|e_n(x)|u_n|^{p-2}u_n - e(x)|u|^{p-2}u\|_{L^1(\Omega)} \to 0,\end{aligned}$$

as $n \to +\infty$, ensure that

$$\lim_{n\to+\infty} \int_\Omega e_n(x)|u_n|^{p-2}u_n\varphi dx = \int_\Omega e(x)|u|^{p-2}u\varphi dx.$$

3)Passage to the limit in $\int_\Omega S_n(x)|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx$.

It is clear that assumption (1) in the Lemma 3.3 is satisfied because that the sequence $\{K_n\}_n$ where

$$K_n(x) = f_n(x) - e_n(x)|u_n|^{p-2}u_n,$$

is bounded in $L^1(\Omega)$. We can apply the Lemma 3.3 to have that $\{\nabla u_n\}$ converges almost everywhere to ∇u in Ω. Indeed, thanks to the boundedness of the sequence $(u_n)_n$ in $W_0^{1,p}(\Omega)$, we have that

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in} \quad L^{p'}(\Omega). \tag{3.28}$$

Since the sequence $(S_n)_n$ strongly converges to $S(x)$ in $L^p(\Omega)$ (being strongly convergent in $W^{1,p}(\Omega)$), we can use (3.28) and by the dominated convergence Theorem, we get

$$S_n(x)|\nabla u_n|^{p-2}\nabla u_n \to S(x)|\nabla u|^{p-2}\nabla u \quad \text{in} \quad L^1(\Omega).$$

So,

$$\lim_{n\to+\infty} \int_\Omega S_n(x)|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi dx = \int_\Omega S(x)|\nabla u|^{p-2}\nabla u\nabla\varphi dx.$$

Finally, we pass to the limit in the formulation (3.4), we obtain

$$\int_\Omega S(x)|\nabla u|^{p-2}\nabla u\nabla\varphi dx + \int_\Omega e(x)|u|^{p-2}u\varphi dx = \int_\Omega f\varphi dx,$$

for every $\varphi \in W_0^{1,\infty}(\Omega)$. Therefore. we have to prove that u is a solution to problem (P). This finishes the proof of theorem 3.1.

Conclusion

In this thesis, we have focused our attention on a class of nonlinear elliptic equations with degenerate coercivity of the form

$$(P) \qquad \begin{cases} Au + e(x)|\nabla u|^{p-2}\nabla u &= f \text{ in } \Omega; \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where the differential operator

$$Au = -\text{div}(S(x)|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty$$

and the right-hand f in $L^1(\Omega)$.

This study is mainly based on the article [1]"L. Boccardo et al, Nonlinear weighted elliptic equations with Sobolev weights.Boll. Unione Mat. Ital. 15, No. 4, 503-514 (2022)"

We are going to prove the existence of the weak solution of the problem (P). To do this, we approximate the problem (P) by a sequence of approximate problems (P_n) given in L^∞ whose existence of the solution approximate is guaranteed (See [10]). Then we will prove some estimates uniform on the sequence of solutions of these problems (P_n) and their partial derivatives. Once this is done, the linearity of the operator with respect to the gradient as well as the boundedness and the continuity of the function $u \mapsto |\nabla u|^{p-2}\nabla u$ will make it possible to pass to the limit, thus finding the solution.

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Abstract

in this work, we prove the existence of a weak solution of elliptic problem (P) defined by

$$(P) \qquad \begin{cases} -\text{div}(S(x)|\nabla u|^{p-2}\nabla u) + e(x)|u|^{p-2}u &= f \text{ in } \Omega; \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$. The weighted p-Laplacian operator $Au = -\text{div}(S(x)|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty$ is a pseudo-monotone operator on $W_0^{1,p}(\Omega)$ despite being well-defined between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$. The method of solving our problem consist of obtaining local estimates for suitable approximate problems and then passing to the limit.

keywords: Weighted Sobolev spaces, pseudo-monotone, operator nonlinear, elliptic equation, weak solution.

Résumé

Dans ce travail, nous etudions l'existence d'une solution faible d'un problème elliptique non régulière définie par

$$(P) \qquad \begin{cases} -\text{div}(S(x)|\nabla u|^{p-2}\nabla u) + e(x)|u|^{p-2}u &= f \text{ in } \Omega; \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$. L'opérateur $Au = -\text{div}(S(x)|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty$ est un opérateur pseudo-monotone. Les étapes principales de la preuve consister à approcher par une suite de problèmes à donnée dans L^∞, ensuite obtenir des estimations uniformes et locales pour la suite des solutions approchées u_n et ∇u_n, puis le passage à la limite.

mots-clés: Espace de Sobolev avec poids, pseudo-monotone, opérateur non linéaire, équation elliptique, solution faible .

ملخص

في هذه المذكرة نسعى الى اثبات وجود الحلول الضعيفة للمعادلة التناقصية وغير الخطية المعرفة كما يلي :

$$\begin{cases} -\mathrm{div}(S(x)|\nabla u|^{p-2}\nabla u) + e(x)|u|^{p-2}u &= f \quad \Omega; \\ u &= 0 \quad \partial\Omega, \end{cases} \qquad (P)$$

تحت الشرط التالي $u = 0$ **على الحافة حيث ان الطرف الايمن** f **ينتمي الى الفضاء** $L^1(\Omega)$ **المؤثر** $Au = -\mathrm{div}(S(x)|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty$ **معرف بشكل جيد بين الفضاء** $W_0^{1,p}$**وفضاءه الثنوي، الخطوات الرئيسية للاثبات تتمحور فيمايلي:**

نقرب الجملة (P) **بمتتالية جمل**(P_n) **ثم نبرهن ان الجملة المقربة تتمتع بحل ضعيف محدود** u_n **ثم الحصول على التقديرات لمتتالية الحلول التقريبية** u_n **و** ∇u_n. **وفي الاخير نمر بالنهاية في المسالة التقريبية للحصول على حل للجملة** (P)

كلمات مفتاحية: معادلات تناقصية غير خطية ، مؤثر غير خطي، فضاء سوبولاف، حل ضعيف