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# T H È S E

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Thème

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# Méthodes Computationnelles pour la Résolution des Équations Intégrales Non Linéaires

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# Dedication

♠ I thank my brothers, my sisters, and my extended
family for their continued support and encouragement.
And i owe a special thank to my parents ♠

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# List of symbols

The table below is a short list of symbols and notation used in this thesis

| | |
|---|---|
| $\mathbb{R}^n$ | : Set of $n$-tuples $x = (x_1, x_2, \ldots, x_n)$. |
| $\langle x, y \rangle$ | : Euclidian inner product in $\mathbb{R}^n$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$. |
| $X, Y, E$ | : Metric spaces, Banach or Hilbert spaces. |
| $X^*$ | : Dual space of $X$. |
| $\overline{\Omega}$ | : Closure of $\Omega$. |
| $C^k\left(\overline{\Omega}\right)$ | : Stands for $C^k\left(\overline{\Omega}; \mathbb{R}\right)$. |
| $C^k\left(\Omega; \mathbb{R}^n\right)$ | : Set of $k$-times continuously differentiable functions $\varphi : \Omega \longrightarrow \mathbb{R}^n, (\Omega \subset \mathbb{R}^n \text{ open})$. |
| $L^p\left(\Omega; \mathbb{R}^n\right)$ | : Space of all measurable functions $\varphi : \Omega \longrightarrow \mathbb{R}^n$, with $\int_\Omega \lvert\varphi(x)\rvert^p dx < \infty, (\Omega \subset \mathbb{R}^n \text{ open}, 1 \leq p < \infty)$. |
| $\lvert\varphi\rvert_\infty$ | : $\underset{x \in \overline{\Omega}}{Max} \lvert\varphi(x)\rvert, \left(\Omega \subset \mathbb{R}^n \text{ bounded open}, \varphi \in C^k\left(\overline{\Omega}; \mathbb{R}^n\right)\right)$. |
| $\lvert.\rvert_p$ | : Norm in $L^p\left(\Omega; \mathbb{R}^n\right)$, $\lvert\varphi\rvert_p = \left(\int_\Omega \lvert\varphi(x)\rvert^p dx\right)^{1/p}$. |
| $A^{-1}$ | : Inverse matrix of the matrix $A$. |
| $J(.)$ | : Jacobian matrix. |
| $O(.)$ | : Order of convergence. |
| $\kappa_n$, $N$, $n+1$ | : Number of discretization points. |
| NVIE | : Nonlinear Volterra integral equation. |
| NFIE | : Nonlinear Fredholm integral equation. |
| NK | : Newton-Kantorovich method. |
| NKSM | : Modified Newton-Kantorovich-Simpson method. |
| TCM | : Chebyshev collocation method. |
| LCM | : Legendre collocation method. |

# Introduction

Integral equations arise naturally in applications, in many areas of mathematics, science and technology such as the population dynamics, spread of epidemics, and semi-conductor devices, ... etc, see [55]. And the most problems can be formulated as either a differential or an integral equation, for example: chemical reactor model [12, 32] and pendulum problem [14].

This introduction contains a brief description of the objective, approach, and organization of the thesis. The objective of this work is to explain the most numerical methods which can be used to solve nonlinear integral equations of the second kind that cannot be solved analytically. The most general form of a nonlinear integral equation is

$$H\left(\varphi\right)\left(t\right)=\lambda\int_{G}K\left(t,x,\varphi\left(x\right)\right)dx,\ \ t,x\in G$$

where $H$ a measurable function given on $\mathbb{R}^n$, $\lambda$ a scalar given which can be real or complex, $K\left(t,x,\varphi\left(x\right)\right)$ a measurable function $G^2\times\mathbb{R}^n$ called the kernel of the integral equation and $\varphi\left(t\right)$ is the unknown function, see [27, 54, 58, 55, 52]. In this thesis we will consider Fredholm[1] and Volterra[2] of the second kind (only one-dimensional). Vito Volterra was the first to elaborate a general theory of integral equations, investigating the existence and uniqueness of the solutions and arriving at presenting a general method for their solution.

There exist a number of problems arising in different scientific and technical fields belong to a class of ill-posed problems, and nonlinear integral equation of the first kind is considered ill-posed problem. The general principles of regularization for these kinds of problems are known, in this thesis we analyze Tikhonov regularization method only for some nonlinear

[1] Erik Ivar Fredholm $(1866-1927)$ is best remembered for his work on integral equations and spectral theory. He was a Swedish mathematician who established the theory of integral equations and his paper in Acta Mathematica played a major role in the establishment of operator theory.

[2] Vito Volterra $(1860-1940)$ is Italian mathematics, he started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908.

integral equations. This method transforms a first kind equation to a second kind equation. For more details, see [30, 53, 48] and others.

In general, it is not possible to solve an integral equation analytically. However there are some special cases for relatively simple equations where certain tricks can be used to arrive at a solution, degenerate kernel method, we refer to Golberg [24], Brunner [13], and Wazwaz [55], successive approximation method, we refer to [45, 5, 33, 55], and others. These are some analytical methods to solve the integral equation of the second kind with continuous kernel.

In this work we study the numerical solutions of some nonlinear integral equations with continuous kernel. This thesis consists of introduction, four chapters, and references.

**The two first chapters** will present few basic concepts from general theoretical framework, such as compactness in metric spaces and compact of nonlinear operators in Banach spaces and theory of fixed point, we refer to Dedieu [17], Granas et al. [20], Precup [42], Brown [12], and Agarwal et al. [2, 1]. It also contains some classification of nonlinear integral equations, and existence and uniqueness theorems of the nonlinear integral equations (for more details, see, [28, 11, 8, 42, 18]), relation between nonlinear differential and these equations (for more details, see, [44, 2, 58]), and some regularization principles.

**In the chapter three** we will investigate some analytical and numerical methods together with their convergence properties, such as successive approximations method, degenerate kernel method, projection method (For more details, see, [23, 5, 6, 24, 13, 34, 7]), Nyström method (the references, [5, 6, 29, 39]) and the combination of the Newton-Kantorovich and modified Simpson method for obtaining approximate solution of the nonlinear Urysohn integral equations (the references, [5, 57, 45, 33, 55, 42, 3]), this our results have been published in [38] with M. Nadir. All these numerical methods used for integral equations make use of numerical integration, for approximating the integral in the equations.

**In the last chapter** we try to apply some of the numerical methods illustrated in chapter three to approximate the solution of the some nonlinear integral equations and give an experimental comparison between these numerical methods. The chapter also contains some physical problems are modeled in the form of Volterra or Fredholm integral equations.

The main goal of this thesis is to give a detailed description of solution methods for the nonlinear integral equations of Volterra and Fredholm of the second kind. For the computer programming, we used the preferred language in numerical analysis is Matlab.

# Chapter 1

# Recalls and basic concepts

The purpose of this chapter is to present some of the notation that will be used throughout the thesis, and state some definitions and results from the literature that will be required later. It is necessary that we have compactness criteria for the various spaces in which we wish to work.

## 1.1 Compactness

An important notion on which a lot of powerful tools are based is compactness, for more details, see, [36, 5, 22, 42, 19, 35]. Let $(X, d)$ be a metric space. Recall that a subset $S$ of $X$ is called compact if every open cover[1] of $S$ has a finite subcover. Equivalently, a subset $S$ of $S$ is compact if every sequence in $S$ contains a convergent subsequence with a limit in $S$.

**Definition 1.1.** ***(Compact operator).*** *Let $X, Y$ be two Banach spaces and let $S$ be a subset of $X$. We say that $T : S \longrightarrow Y$ is compact, if it is continuous and for every bounded set $B \subseteq S$, the set $\overline{T(B)}$ is compact in $Y$. We denote the set of compact maps by*

$$K(S; Y).$$

Another notion involving compactness is given in the next definition:

**Definition 1.2.** *Let $X, Y$ be two Banach spaces and $T : S \subset X \longrightarrow Y$. The operator $T$ is said to be completely continuous if it is continuous and maps any bounded subset of $S$ into a relatively compact subset of $Y$.*

[1] A class of subsets $H = \{U_j\}$ of $X$ is called a *cover* of a set $S$ if we have: $S \subset \underset{J}{\cup} U_j$.

There are other definitions for a compact operator, but the above is the one used most commonly. It is clear that a continuous operator $T : S \subset X \longrightarrow Y$ is completely continuous if and only if for every bounded sequence $(\varphi_k)$ with $\varphi_k \in S$, the sequence $(T(\varphi_k))$ has a convergent subsequence. Notice that any completely continuous operator is a bounded operator.

**Remark 1.3.** *A completely continuous linear operator $T : X \longrightarrow Y$ is also known as Dunford-Pettis operator and is of course continuous. And in general the classes of compact maps and completely continuous maps are not comparable.*

**Remark 1.4.** *For linear operators, compactness implies complete continuity.*

**Proposition 1.5.** *If $X$ is a reflexive Banach space, $Y$ is a Banach space, $S \subset X$ is non-empty, closed, and convex, and $T : S \to Y$ is completely continuous, then $T$ is compact.*

*Proof.* (See, for example [36]). □

**Corollary 1.6.** *[36]. If $X$ is a reflexive Banach space, $Y$ is a Banach space, and $L : X \to Y$ is linear, then $L$ is compact if and only if $L$ is completely continuous.*

Several problems in science and engineering can be modeled and represented with the help of function spaces and integral operators on them, and in particular, in the spaces $C(G;\mathbb{R}^n)$ and $\mathbb{L}_p(G;\mathbb{R}^n)$, $G \subset \mathbb{R}^n$ is bounded closed and $1 \leq p, n < \infty$.

## 1.2 Compact integral operators on Banach spaces

In this section we present some results from nonlinear functional analysis that will be useful in the study of nonlinear integral equations. And we present four examples of nonlinear integral operators which are completely continuous on some spaces of continuous functions: The Fredholm integral operator, the Volterra integral operator, the Hammerstein integral operator, and a particular integral operator with delay. And we give some preliminaries for using in the proofs of the existence results: Since our Banach space is $C([a, b])$, then the following version of the *Arzelà -Ascoli's theorem* is very useful in proving the total boundedness of our proposed operator, see [55, 5].

**Theorem 1.7.** ***(Arzelà-Ascoli's theorem)***. *Assume that $G$ is a compact set in $\mathbb{R}^n$, $n \geq 1$. Then a set $S \subset C(G)$ is relatively compact in $C(G)$ if and only if the functions in $S$ are uniformly bounded and equicontinuous on $G$.*

- *To say that the functions in $S$ are uniformly bounded means that there exists a $M > 0$ such that*

$$\|\varphi\| = \max_{x \in G} |\varphi(x)| \leq M \text{ for all } x \in G \text{ and all } \varphi \in S.$$

- *To say that the functions in $S$ are equicontinuous on $G$ means that for every $\epsilon > 0$ there exists an $\delta > 0$ such that for every $x, y \in G$ and every $\varphi \in S$ we have*

$$|x - y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \epsilon.$$

**Definition 1.8.** *Let $p \in [1, \infty]$ and $q \in [1, \infty)$. A function $f : G \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is said $(p, q) -$ Carathéodory if the following condition is satisfied:*

$$\begin{cases} (a) \text{ if } 1 \leq p \leq \infty \text{ then } |f(x, z)| \leq g(x) + c\,|z|^{p/q} \\ \text{for a.e. } x \in G, \text{ all } z \in \mathbb{R}^m \text{ and some } g \in L^q(G; \mathbb{R}_+), \ c \in \mathbb{R}_+; \\ (b) \text{ if } p = \infty \text{ then for every } R > 0 \text{ there is a } g_R \in L^q(G) \text{ with} \\ |f(x, z)| \leq g_R(x) \text{ for a.e. } x \in G \text{ and all } z \in \mathbb{R}^m \text{ with } |z| \leq R. \end{cases}$$

Now, we present four examples of nonlinear integral operators which are completely continuous on Banach spaces:

**Theorem 1.9.** *[42]. Let $h : G^2 \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous mapping. Then the Fredholm operator associated to $h$, $T : C(G; \mathbb{R}^n) \longrightarrow C(G; \mathbb{R}^n)$ given by*

$$T(\varphi)(x) = \int_G h(x, y, \varphi(y))\, dy, \ \ x, y \in G$$

*is completely continuous.*

*Proof.* (For a proof of the **theorem 1.9**, see for example [42]). We first prove that $T$ is continuous. Let $\varphi \in C(G; \mathbb{R}^n)$ and choose any number $R > |\varphi_0|_\infty$. Let $\varepsilon > 0$, since $h$ is uniformly continuous on the compact set $G^2 \times \overline{B}_R(0; \mathbb{R}^n)$, there exists a $\delta_\epsilon > 0$ such that for every $\varphi \in C(G; \mathbb{R}^n)$ satisfying $|\varphi - \varphi_0|_\infty \leq \delta_\epsilon$ one has $\varphi(y) \in \overline{B}_R(0; \mathbb{R}^n)$ and

$$|h(x, y, \varphi(y)) - h(x, y, \varphi_0(y))| \leq \varepsilon$$

for all $x, \varphi \in G$. Then

$$\begin{aligned} |T(\varphi)(x) - T(\varphi_0)(x)| &\leq \int_G |h(x, y, \varphi(y)) - h(x, y, \varphi_0(y))| \, dy \\ &\leq \varepsilon \mu(G) \end{aligned}$$

for every $x \in G$. Hence

$$|T(\varphi) - T(\varphi_0)|_\infty \leq \varepsilon \mu(G)$$

whenever $|\varphi - \varphi_0|_\infty \leq \delta_\epsilon$. Therefore $T$ is continuous at $\varphi_0$.

Next, given a bounded subset $Y$ of $C(G; \mathbb{R}^n)$, we shall prove that $T(Y)$ is relatively compact in $C(G; \mathbb{R}^n)$. According to the Ascoli-Arzelà theorem, we have to show that $T(Y)$ is bounded and equicontinuous.

Indeed, since $Y$ is bounded there exists a constant $c > 0$ such that

$$|\varphi_0|_\infty \leq c \text{ for all } \varphi \in Y.$$

It follows that for any $\varphi \in Y$ we have

$$|T(\varphi)|_\infty \leq M\mu(G),$$

where

$$M = \max_{G^2 \times \overline{B}_R(0; \mathbb{R}^n)} |h(x, y, z)|.$$

Hence the set $T(Y)$ is bounded in $C(G; \mathbb{R}^n)$.

On the other hand, using the uniform continuity of $h$ on the compact $G^2 \times \overline{B}_R(0; \mathbb{R}^n)$, for each $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that

$$|h(x, y, \varphi(y)) - h(x', y, \varphi(y))| \leq \varepsilon$$

for all $x, x', y \in G$ with $|x - x'| \leq \delta_\varepsilon$ and $\varphi \in Y$. This immediately yields

$$|T(\varphi)(x) - T(\varphi)(x')| \leq \varepsilon \mu(G),$$

for all $x, x' \in G$ with $|x - x'| \leq \delta_\varepsilon$ and $\varphi \in Y$. Thus $T(Y)$ is equicontinuous. □

**Theorem 1.10.** *[42]. Let $h : [a, b]^2 \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous. Then the Volterra operator associated to $h$, $T : C([a, b]; \mathbb{R}^n) \longrightarrow C([a, b]; \mathbb{R}^n)$ given by*

$$T(\varphi)(t) = \int_a^t h(t, y, \varphi(y)) \, dy, \ \ t \in [a, b]$$

*is completely continuous.*

*Proof.* Essentially the same reasoning as in the proof of the previous theorem establishes the result. □

Now, we examine the *Nemytskii operator*, which is an important nonlinear operator that arises in many applications and then we pass to the study of nonlinear integral operators. Let $f : G \times X \longrightarrow Y$ and consider the nonlinear operator

$$N_f(\varphi)(t) = f(t, \varphi(t)),$$

which to each function $\varphi : G \longrightarrow X$ assigns the $Y-$valued $t \longmapsto f(t, \varphi(t))$. This operator is known in the literature as the Nemytskii operator corresponding to the function $f$ (also known as the superposition operator of $f$, or the composition operator of $f$, or the substitution operator of $f$). See, for example [15, 22], or [36].

A Fredholm linear integral operator $K$ on $C(G)$ is defined by

$$(K\varphi)(t) := \int_G k(t, x)\varphi(x)\,dx, \quad t \in G$$

where $k$ is the kernel of the operator $K$. Let $T : C(G) \longrightarrow C(G)$ be defined by

$$(T\varphi)(t) := (KN_f\varphi)(t) = \int_G k(t, x) f(x, \varphi(x))\,dx, \tag{1.1}$$

is called a nonlinear integral operator of Hammerstein.

**Theorem 1.11.** *Let $G \subset \mathbb{R}^n$ be open, $k : G^2 \longrightarrow \mathbb{R}$ and $f : G \times \mathbb{R}^n$. Let $p \in [1, \infty], q \in [1, \infty)$ and let $r \in (1, \infty]$ be the conjugate of $q$. Assume that the Fredholm linear integral operator $K : L^q(G; \mathbb{R}^n) \longrightarrow L^p(G; \mathbb{R}^n)$ of kernel $k$ is well defined and completely continuous. In addition assume that $f$ is a $(p, q) -$ Carathéodory function. Then the Hammerstein integral operator $T : L^p(G; \mathbb{R}^n) \longrightarrow L^p(G; \mathbb{R}^n)$ given by (1.1) is well defined and completely continuous.*

The following delay integral equation

$$\varphi(t) = \int_{t-\tau}^{t} f(x, \varphi(x))\,dx,$$

can be interpreted as a model for the spread of certain infectious diseases with a contact rate that varies seasonally. In this equation $\varphi(t)$ is the proportion of infectives in a population at time $t$, $\tau$ is the length of time an individual remains infectious, and $f(t, \varphi(t))$ is the proportion of new infectives per unit time, see [42].

Now, we study the complete continuity of delay integral equation on a given interval of time $[0, t_1]$.

**Theorem 1.12.** *[42]. Assume* $f \in ([-\tau, t_1] \times \mathbb{R}^n; \mathbb{R}^n)$, $u \in C([-\tau, 0]; \mathbb{R}^n)$ *and that*

$$u(0) = \int_{-\tau}^{0} f(x, u(x))\, dx.$$

*Then the delay integral operator* $T : D(T) \longrightarrow C([0, t_1]; \mathbb{R}^n)$ *given by*

$$T(\varphi)(t) = \int_{t-\tau}^{t} f(x, \widetilde{\varphi}(x))\, dx, \;\; t \in [0, t_1]$$

*where*

$$D(T) = \{\varphi \in C([0, t_1]; \mathbb{R}^n) : \varphi(0) = u(0)\}$$

*and*

$$\widetilde{\varphi}(t) = \begin{cases} u(t) \text{ for } t \in [-\tau, 0], \\ \varphi(t) \text{ for } t \in [0, t_1], \end{cases}$$

*is completely continuous.*

*Proof.* Use the Ascoli-Arzelà theorem (See, for example [42]). □

All the nonlinear integral equations take the form $\varphi = T\varphi$, $\varphi \in C(G)$, so that we are concerned with finding fixed points of the mapping $T$. We make use of the following well-known theory of the fixed points, (see [1]).

## 1.3 Fixed point theorems on Banach spaces

Many nonlinear equations are naturally formulated as fixed-point problems

$$\varphi = T(\varphi) \tag{1.2}$$

where $T$ is fixed point operator, may be nonlinear. A solution $\varphi^*$ of (1.2) is called a fixed point of the map $T$. Before discussing theorem of fixed point we make some notions and definitions. Let $(X, d)$ be a metric space, a map $T : X \longrightarrow X$ is said to be *Lipschitzian* if there exists a constant $\alpha \geq 0$ with

$$d(T(\varphi), T(\nu)) \leq \alpha d(\varphi, \nu) \text{ for all } \varphi, \nu \in X. \tag{1.3}$$

Notice that a Lipschitzian map is necessarily continuous. The smallest $\alpha$ for which (1.3) holds is said to be *the Lipschitz constant* for $T$ and is denoted by $L$. If $L < 1$ we say that $T$ is *a contraction*, whereas if $L = 1$, we say that $T$ is *non expansive*. The fixed-point iteration, which is given by

$$\varphi_{n+1} = T(\varphi_n).$$

This iterative method is also called *nonlinear Richardson iteration*, *Picard iteration*, or *the method of successive substitution.*

Fixed point theory plays a major role in many of our existence principles, therefore we state the following fixed point theorems:

The most well known result in the theory of fixed points is Banach's contraction mapping principle appeared in explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution for an integral equation [9]. For purposes we define $T^n(\varphi), \varphi \in X$ and $n \in \{0, 1, 2, ...\}$, inductively by $T^0(\varphi) = \varphi$ and $T^{n+1}(\varphi) = T\left(T^n(\varphi)\right)$.

**Theorem 1.13.** *Let $(X, d)$ be a complete metric space and let $T : X \longrightarrow X$ be a contraction with Lipschitzian constant $L$. Then $T$ has a unique fixed point $\varphi^* \in X$. Furthermore, for any $\varphi \in X$ we have*

$$\lim_{n\to\infty} T^n(\varphi) = \varphi^*,$$

*with*

$$d(T^n(\varphi), \varphi^*) \leq \frac{L^n}{1-L} d(T(\varphi), \varphi).$$

*Proof.* [1] □

The last result is known as theorem of *Banach* or theorem of *Picard-Banach-Caccioppoli* (also called *contraction mapping principle*).

**Theorem 1.14.** ***(Schauder's fixed-point theorem)*** *Let $X$ be a Banach space and let $S \subset X$ be bounded, closed, and convex. Assume $T : S \longrightarrow S$ is a completely continuous operator. Then $T$ has at least one fixed point in the set $S$.*

The finite-dimensional version of Schauder's theorem, namely Brouwer's fixed point theorem.

**Theorem 1.15.** ***(Brouwer's fixed-point theorem)*** *Let $S \subset \mathbb{R}^n$ be a nonempty convex compact set and let $T : S \longrightarrow S$ be a continuous mapping. Then there exists at least one $\varphi \in S$ with $T(\varphi) = \varphi$.*

*Proof.* [42]. □

We conclude the note with some additional fixed point theorems. The first one, *Schaefer's* fixed point theorem, is a version of Schauder's theorem. Sometimes it is called the *Leray-Schauder* principle and is an example of the mathematical principle saying "apriori estimates implies existence". The second one, *Krasnoselskii's* fixed point theorem, is a mixture of Banach's and Schauder's fixed point theorems.

**Theorem 1.16.** ***(Schaefer)*** *Let $X$ be a Banach space, $S \subset X$ a closed convex subset, $U \subset S$ a bounded set, open in $S$ and $\varphi_0 \in U$ a fixed element. Assume that the operator $T : \overline{U} \longrightarrow S$ is completely continuous and satisfies the boundary condition*

$$\varphi \neq (1-\lambda)\,\varphi_0 + \lambda T(\varphi) \text{ for all } \varphi \in \partial U, \lambda \in (0,1) \tag{1.4}$$

*Then $T$ has at least one fixed point in $\overline{U}$.*

*Proof.* [42] □

In particular, note that to apply Schaefer's theorem we do not need to prove that a certain set is convex or compact. The problem is reformulated as to show certain a priori estimates for the operator $T$.

**Theorem 1.17.** ***(Krasnoselskii's fixed point theorem).*** *Assume that $S$ is a closed bounded convex subset of a Banach space $X$. Furthermore assume that $T_1$ and $T_2$ are mappings from $S$ into $X$ such that*

- $T_1(\varphi) + T_2(\nu) \in S$ *for all* $\varphi, \nu \in S$,
- $T_1$ *is a contraction,*
- $T_2$ *is continuous and compact.*

*Then $T_1 + T_2$ has a fixed point in $S$.*

For further studies on theory of the fixed point, we refer to Dedieu [17], Granas et al. [20], Precup [42], Brown [12], and Agarwal et al. [2, 1].

# Chapter 2

# Basics about nonlinear integral equations

A nonlinear integral equation is an integral equation in which the unknown function appears in the equation in a nonlinear manner. The nonlinearity may occur either inside or outside of the integrand or simultaneously in both of these locations. However, one of the most common categories is that of equations assuming the form

$$H(\varphi)(t) = \lambda \int_G K(t, x, \varphi(t), \varphi(x))\, dx,\ \ t, x \in G \tag{2.1}$$

where $H$ a measurable function given on $\mathbb{R}^n$, $\lambda$ a scalar given which can be real or complex, $K(t, x, \varphi(t), \varphi(x))$ a measurable function $G^2 \times \mathbb{R}^n \times \mathbb{R}^n$ called the kernel of the integral equation and $\varphi(t)$ is the unknown function.

Let $X = C(G; \mathbb{R}^n)$ be the Banach space of all continuous functions from $G$ to $\mathbb{R}^n$ $(1 \leq n < \infty)$, under the sup-norm $|.|_\infty$. We consider equations of the general form (2.1), but we are mainly concerned with equations of two special types.

## 2.1 Classification of nonlinear integral equations

To begin, we shall give a classification of some of the major types of nonlinear integral equations. The limits of the integral may be constants, in which case we have a *Fredholm* type of equation, whereas if one of the limits is the independent variable, then we have an equation of *Volterra* type.

An integral equation is said to be linear if all terms occurring in the equation are linear in the unknown function. Otherwise, the equation is said to be nonlinear. Solution of the last equations is obviously more difficult and in most of this work, we consider only these integral equations.

Each of these can be further classified into different kinds. For example if $G = [a, b]$

$$0 = \int_a^b k(t, x, \varphi(t), \varphi(x))dx + f(t), \tag{2.2}$$

$$\varphi(t) = \int_a^b k(t, x, \varphi(t), \varphi(x))dx + f(t), \tag{2.3}$$

$$g(t)\,\varphi(t) = \int_a^b k(t, x, \varphi(t), \varphi(x))dx + f(t), \tag{2.4}$$

are *Fredholm* equations of the first, second and third kind, respectively. These equations are supposed to hold for all values of $t$ in $[a, b]$. Here $\varphi(t)$ is the unknown function which is to be determined, while $f(t), g(t)$ and $k(t, x, \varphi(t), \varphi(x))$ are known functions can be a nonlinear functions of its arguments.

If the upper limit $b$ in the equations (2.2), (2.3) and (2.4) is replaced by $t$, then we obtain the *Volterra* equations of the first, second and third kind. Equations of Volterra type can be considered as a special case of Fredholm type, with kernel $k_*(t, x, \varphi(t), \varphi(x))$ defined by

$$k_*(t, x, \varphi(t), \varphi(x)) = \begin{cases} k(t, x, \varphi(t), \varphi(x)) & \text{for } t \geq x \\ 0 & \text{for } t < x, \end{cases}$$

further, solving a Volterra equation is usually simpler than solving a Fredholm equation.

Particular examples of integral equations of second kind which occur frequently in practice are the *Volterra-Urysohn* (or Volterra) equation

$$\varphi(t) = \int_a^t k(t, x, \varphi(x))dx + f(t),$$

the *Urysohn* (or Fredholm) equation

$$\varphi(t) = \int_a^b k(t, x, \varphi(x))dx + f(t),$$

the *Hammerstein* equation

$$\varphi(t) = \int_a^b k(t, x)g(x, \varphi(x))dx + f(t),$$

and the *Hammerstein-Volterra* equation

$$\varphi(t) = \int_a^t k(t, x)g(x, \varphi(x))dx + f(t).$$

The free term $f(t)$ is assumed to be complex-valued and continuous on the interval $[a,b]$. If $f(x)=0$ on the interval $[a,b]$, then the integral equation is called *homogeneous*; otherwise, it is called *inhomogeneous.*

**Definition 2.1.** ***(Singular integral equation)*** *An integral equation may be called singular if either:*

- *its kernel contains a singularity, or*
- *the range of integration is infinite,*

*and it is said to be weakly-singular if the kernel becomes infinite at* $x=t$.

**Definition 2.2.** ***(Mixed integral equation)*** *The Volterra-Fredholm integral equations appear in the literature in two forms, namely*

$$\varphi(t)=f(t)+\lambda_1\int_a^t k_1(t,x,\varphi(x))\,dx+\lambda_2\int_a^b k_2(t,x,\varphi(x))\,dx,\ \ \lambda_1,\lambda_2\in\mathbb{R}$$

*and*

$$\varphi(t,x)=f(t,x)+\lambda\int_0^t\int_\Omega F(t,x,\xi,\tau,\varphi(\xi,\tau))\,d\xi d\tau,\ \ (t,x)\in\Omega\times[0,T]$$

*where* $k_1,k_2$ *are two nonlinear functions of* $\varphi(t)$, $f(t)$, $f(t,x)$ *and* $F(t,x,\xi,\tau,\varphi(\xi,\tau))$ *are analytic functions on* $D=\Omega\times[0,T]$, *and* $\Omega$ *is a closed subset of* $\mathbb{R}^n$, $n=1,2,3$.

**Definition 2.3.** ***(Integro-differential equation)*** *The nonlinear integro-differential equation appears in the form:*

$$\varphi^{(n)}(t)=f(t)+\lambda\int_G K(t,x,\varphi(x))\,dx,\ \ x,y\in G$$

*and the standard form of the nonlinear integro-differential equation of the first kind is given by*

$$\int_G K_1(t,x,\varphi(x))\,dx+\int_G K_2\left(t,x,\varphi^{(n)}(x)\right)dx=f(t),\ \ K_2\left(t,x,\varphi^{(n)}(x)\right)\neq 0,$$

*where* $\varphi^{(n)}$ *indicates the* $n$*th derivative of* $\varphi(x)$, *the kernels* $K,K_1,K_2$ *and the function* $f(x)$ *are given real valued functions. The Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators.*

## 2.2 Some examples of nonlinear integral equations

Integral equations are an important part of many areas of mathematics, from the population dynamics, spread of epidemics, and semi-conductor devices... etc.

♡ The equation

$$1+\lambda\int_a^b \frac{t\varphi(t)\varphi(x)}{t+x}dx=\varphi(t),\;\;\lambda\in\mathbb{R}$$

is called a nonlinear integral equation of *Chandrasekhar* (see, [54]), where $\varphi(x)$ is the unknown function.

♡ The nonlinear singular integral equation of *Cauchy* with the unknown function $\varphi(x)$ is the following

$$a(t)\varphi(t)+b(t)\int_\Gamma \frac{\varphi(x)}{t-x}dx+\int_\Gamma k\left(t,x,\varphi(x)\right)dx=f(t)$$

where $a,b$ and $k$ are functions given, and $\Gamma$ is an arc open or closed in $\mathbb{R}^2$.

♡ An interesting nonlinear integral equation which does not fall into the above categories is *Nekrasov's* equation:

$$\varphi\left(t\right)=\lambda\int_0^\pi L\left(t,x\right)\frac{\sin\varphi(x)}{1+3\lambda\int_0^x\sin\varphi\left(s\right)ds}dx,\quad 0\le t\le\pi,$$

where

$$L\left(t,x\right)=\frac{1}{\pi}\log\frac{\sin\left(\left(t+x\right)/2\right)}{\sin\left(\left(t-x\right)/2\right)}.$$

One solution is $\varphi(x)\equiv 0$, and it is the nonzero solutions that are of interest. This arises in the study of the profile of water waves on liquids of infinite depth, see, [5].

♡ The nonlinear integral equation of *Ambarzumian*[1]

$$\varphi\left(\eta\right)=1+\frac{1}{2}\lambda\eta\varphi\left(\eta\right)\int_0^1\frac{\varphi\left(\xi\right)}{\eta+\xi}d\xi,\quad 0\le\eta\le 1 \tag{2.5}$$

which arises in the theory of radiative transfer. In (2.5), $\varphi\left(\eta\right)$ is the unknown function, which will be sought in $C[0,1]$. The quantity $\lambda$ is a parameter called the *albedo* that will have a value in the range $0\le\lambda\le 1$ (see, for example [27, 45]).

[1] This equation was developed by Ambartsumian and Chandrasekhar to solve the problem of determination of the angular distribution of the radiant flux emerging from a semi-infinite, plane-parallel, isotropic atmosphere, see [45].

♡ The nonlinear integral equation of Bratu is the following

$$\varphi(t) = \lambda \int_a^b G(t, x)\, e^{\varphi(x)} dx,$$

where the kernel is a Green's function defined as follows:

$$G(t, x) = \begin{cases} \frac{(b-t)(x-a)}{b-a}, & x \leq t \\ \frac{(b-x)(t-a)}{b-a}, & t \leq x. \end{cases}$$

♡ The nonlinear integral equation of Laplace is the following

$$\begin{aligned} \varphi(P) &= \frac{1}{\pi} \int_\Gamma \varphi(Q) \frac{\partial}{\partial n_Q} (\log |P - Q|)\, d\sigma(Q) \\ &\quad + \frac{1}{\pi} \int_\Gamma (g(Q, \varphi(Q)) - f(Q)) \log |P - Q|\, d\sigma(Q), \end{aligned}$$

where $P \in \Gamma$. This nonlinear integral operator is the sum of a linear operator and a Hammerstein operator, see [6].

## 2.3 Nonlinear differential and integral equations

Integral equations appear in most applied areas and are as important as differential equations. In fact, as we will see, many problems can be formulated (equivalently) as either a differential or an integral equation because a large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations, see [44, 2, 58].

Let $f(t, \varphi)$ be a continuous real-valued function on $[a, b] \times [c, d]$. The Cauchy initial value problem is to find a continuous differentiable function $\varphi$ on $[a, b]$ satisfying the differential equation

$$\frac{d\varphi}{dt} = f(t, \varphi), \;\; \varphi(t_0) = \varphi_0. \tag{2.6}$$

Consider the Banach space $C[a, b]$ of continuous real-valued functions with supremum norm defined by $\|\varphi\| = sup\{|\varphi(t)| : t \in [a, b]\}$. Integrating (2.6), we obtain an integral equation

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} f(x, \varphi(x))\, dx. \tag{2.7}$$

The problem (2.6) is equivalent the problem solving the integral equation (2.7).

**Example 2.4.** *[15, 32] We consider the mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction, the model can be reduced to the nonlinear ordinary differential equation*

$$\varphi'' - \lambda\varphi' + F(\lambda, \mu, \beta, \varphi) = 0, \tag{2.8}$$

*with boundary conditions*

$$\varphi'(0) = \lambda\varphi(0), \varphi'(1) = 0, \tag{2.9}$$

*where*

$$F(\lambda, \mu, \beta, \varphi) = \lambda\mu(\beta - \varphi)\exp(\varphi).$$

*The unknown* $\varphi$ *represents the steady-state temperature of the reaction, and the parameters* $\lambda, \mu$, *and* $\beta$ *represent the Peclet number, the Damkohler number and the dimensionless adiabatic temperature rise, respectively.*

*The problem* (2.8) − (2.9) *can be converted by Green's function*[2] *techniques into a Hammerstein integral equation*

$$\varphi(x) = \mu\int_0^1 k(x, y) f(y, \varphi(y))\,dy,\ \ 0 \le x \le 1$$

*where*

$$k(x, y) = \begin{cases} e^{\lambda(x-y)} & \text{if } 0 \le x \le y \\ 1 & \text{if } y \le x \le 1, \end{cases}$$

*and*

$$f(y, \varphi) = (\beta - \varphi)\exp(\varphi),$$

*which we consider in the space* $C[0, 1]$ *of continuous functions on the closed interval* $[0, 1]$ *with the usual sup norm. Throughout, we assume* $\lambda$ *and* $\mu$, *are positive, and* $\beta$ *is nonnegative.*

**Remark 2.5.** *We presented a analytical method which it transform a linear differential equations with initial conditions to a linear Volterra equations of second kind, efficient some methods (trapezoidal, Euler and finite differences methods) for approximate numerical solution of these equations. Our results have been published in [37] with* Nadir.

[2] The Green function is a tool use for solving the inhomogeneous linear integral or differential equation $L\varphi = f$. It is an integral kernel representing the inverse operator $L^{-1}$. Green functions play an important role in many areas of physics, for more information, see [49].

For further studies on nonlinear integral and differential equations, we refer to Agarwal et al. [2], Constanda et al. [58], Brown [12], and Rahman [44].

In the next section, we will present existence theorems for the solutions of nonlinear Volterra and Fredholm integral equations.

## 2.4 Existence and uniqueness theorems

In the previous papers [28, 11, 8, 42] and in [18], the main tool, in order to find solutions of the nonlinear integral equations, was to use the Schaefer and Schauder fixed point theorems. Some times proofs could be simplified using the Banach fixed point theorem. In this section we will present the using of most fixed point theorems.

### 2.4.1 Nonlinear Fredholm integral equation

Banach's fixed point theorem can be used to prove the following result.

**Theorem 2.6.** *[58] Suppose that $k(t,x,\varphi)$ is defined and continuous on the set $[a,b]\times[a,b]\times\mathbb{R}$ and that it satisfies a Lipschitz condition of the form*

$$|k(t,x,\varphi_1)-k(t,x,\varphi_2)| < C\,|\varphi_1-\varphi_2|\,,$$

*suppose further that $f\in C[a,b]$. Then the nonlinear Urysohn integral equation*

$$\varphi(t)=\lambda\int_a^b k(t,x,\varphi(x))dx+f(t)$$

*has a unique solution on the interval $[a,b]$ whenever $|\lambda|<1/(C(b-a))$.*

*Proof.* See, [58]. □

**Theorem 2.7.** *[28] Consider the nonlinear Urysohn integral equation*

$$\varphi(t)=\lambda\int_a^b k(t,x,\varphi(x))dx+f(t),\qquad -\infty<a\le t\le b<+\infty \tag{2.10}$$

*where $f(.)\in C\,[a,b]$. Assume that the function $g\,(t,s,x)$ satisfies the following conditions:*

$$\sup\left(|g\,(t,s,x)|\,,\left|\frac{\partial g}{\partial t}\,(t,s,x)\right|\right)\le V_1\,(t)\,V_2\,(s)\,\phi\,(|x|)\,,\quad\left|\frac{\partial g}{\partial t}\,(t,s,x)\right|\le V_1\,(t)\,V_2\,(s)\,\psi\,(|x|)\,,$$

*where $V_1\,(.)\in C\,[a,b]\,,\ V_2\,(.)\in L^1\,[a,b]\,,\ \phi\,(.)$ is positive and bounded over $[0,+\infty[$ and $\psi\,(.)$ is positive and continuous over $[0,+\infty[$. Under the above conditions, the equation (2.10) has a solution in $C\,[a,b]$.*

*Proof.* By using Schaefer's fixed-point theorem, see for example [28]. □

**Theorem 2.8.** *[28] Consider the nonlinear integral equation*

$$\varphi(t) = \lambda \int_a^b k(t, x, \varphi(x))dx + f(t), \quad -\infty < a \le t \le b < +\infty \tag{2.11}$$

*assume that* $f(.)$ *is bounded and* $g(t, s, x)$ *satisfies the following conditions:*

$$|g(t, s, x)| \le V_1(t) V_2(s) \phi(|x|), \quad \left|\frac{\partial g}{\partial t}(t, s, x)\right| \le V_1(t) V_2(s) \psi(|x|),$$

*where* $V_1(.)$ *is a measurable and bounded positive function,* $\varphi(.)$ *is a positive and measurable function satisfying the condition*

$$\sup_{x \ge 0} \frac{\phi(x)}{x} = L < +\infty,$$

*and* $\psi(.)$ *is positive and continuous function over* $[0, +\infty[$*. Moreover, assume that there exists a continuous and strictly positive function* $\mu(.)$ *satisfying the following condition:*

$$\|V_1.\mu\|_\infty \left\|\frac{V_2}{\mu}\right\|_1 < \frac{1}{L}.$$

*Under these conditions, the equation* (2.11) *has a solution in* $C[a, b]$.

*Proof.* By using Schauder's fixed-point theorem see, for example [28]. □

**Theorem 2.9.** *Let* $h : G^2 \times B \longrightarrow \mathbb{R}^n$ *be continuous. Assume that*

$$|\varphi|_\infty < R$$

*for any solution* $\varphi \in C(G; B)$ *to*

$$\varphi(t) = \lambda \int_G h(t, x, \varphi(x))\, dx, \quad t \in G \tag{2.12}$$

*for each* $\lambda \in (0, 1)$*. Then* (2.12) *has a solution in* $(G; B)$*.*

*Proof.* By using Leray-schauder's fixed-point theorem see, for example [42]. □

### 2.4.2 Nonlinear Volterra integral equation

Fixed point theorems can also be used to prove the existence and uniqueness of solutions to nonlinear Volterra integral equations.

**Theorem 2.10.** *[58] Suppose that $k(t, x, \varphi)$ is defined and continuous on the set $[a, b] \times [a, b] \times \mathbb{R}$ and that it satisfies a Lipschitz condition of the form*

$$|k(t, x, \varphi_1) - k(t, x, \varphi_2)| < C\,|\varphi_1 - \varphi_2|\,.$$

*Suppose further that $f \in C[a, b]$. Then the nonlinear Volterra integral equation*

$$\varphi(t) = \lambda \int_a^t k(t, x, \varphi(x))dx + f(t)$$

*has a unique solution on the interval $[a, b]$ for every value of $\lambda$, where $a \le t \le b$.*

*Proof.* See, [58]. □

**Theorem 2.11.** *[28] Consider the nonlinear Volterra integral equation*

$$\varphi(t) = \lambda \int_a^t k(t, x, \varphi(x))dx + f(t), \qquad -\infty < a \le t \le b < +\infty \tag{2.13}$$

*where $f$ is continuous over $[a, b]$. Assume that the function $g\,(t, s, x)$ satisfies the following conditions:*

$$|g\,(t, s, x)| \le V_1\,(t)\,V_2\,(s)\,\phi\,(|x|)\,, \qquad \left|\frac{\partial g}{\partial t}\,(t, s, x)\right| \le V_1\,(t)\,V_2\,(s)\,\psi\,(|x|)\,,$$

*where $V_1\,(.)$ is a positive and continuous function over $[a, b]\,,$ $V_2\,(.)$ is a positive and integrable function over $[a, b]\,,$ $\psi\,(.)$ is positive and continuous function over $[0, +\infty[$. Finally, assume that the function $\phi\,(.)$ is positive, continuous and satisfies the condition*

$$\lim_{\varphi \longrightarrow +\infty} \frac{\phi\,(\varphi)}{\varphi} = L < +\infty.$$

*Under the above conditions, the equation (2.13) has a solution in $C\,[a, b]$.*

The proof of this theorem can be found in [28].

In what follows we will present a brief summary of the method of regularization that will be used to handle the nonlinear integral equations of the first kind.

## 2.5 Introduction to regularization methods for some nonlinear ill–posed problems

Nonlinear integral equation of the first kind is considered *ill-posed problem* because it does not satisfy the following three properties:

1. Existence of a solution.

2. Uniqueness of a solution.

3. Continuous dependence of the solution $\varphi(t)$ on the data $f(t)$. This property means that small errors in the data $f(t)$ should cause small errors in the solution $\varphi(t)$.

Any problem that satisfies the three pervious properties is called *well-posed problem*[3]. For any ill-posed problem, a very small change on the data $f(t)$ can give a large change in the solution $\varphi(t)$. Methods for obtaining a stable approximate solution of an ill-posed problem are called *regularization methods*, some of these methods are reviewed in this section:

We consider the nonlinear operator equation

$$T(\varphi) = f, \tag{2.14}$$

where $T(\varphi) : X \longrightarrow Y$ is a nonlinear, weakly closed and continuous operator, $X$ and $Y$ are the Hilbert spaces, $f \in Y$ is the given function. We assume that the equation (2.14) has the unique solution $\overline{\varphi}^0 \in X$.

*Tikhonov regularization*[4] consists in approximation of the desired solution (2.14) by the minimizer of the functional

$$\mathcal{F}_N(\varphi) = \left\|T(\varphi) - f^\delta\right\|_Y^2 + \alpha \left\|\varphi - \varphi_0\right\|_X^2.$$

In the functional $f^\delta$ denotes perturbation of $f$, $\alpha > 0$, $\|.\|_Y$ and $\|.\|_X$ denote the norm on the Hilbert spaces $Y$ and $X$, respectively. If the operator $T$ is Fréchet-differentiable, a minimizer $\varphi_\alpha^\delta$ satisfies the optimality equation

$$T'(\varphi)^* \left(T(\varphi) - f^\delta\right) + \alpha(\varphi - \varphi_0) = 0.$$

Here $T'(\varphi)^*$ denotes the adjoint of the Fréchet-derivative $T'(\varphi)$ (see, for example, [30, 53, 48]).

**Theorem 2.12.** *[30]. Assume there exists a minimum norm solution of (2.14), denoted by $\overline{\varphi}^0 \in X$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence where $f_k \longrightarrow f^\delta$ and let $\varphi_k$ be a minimizer of $\mathcal{F}_N$ where $f^\delta$ is replaced by $f_k$. Then there exists a convergent subsequence of $\{\varphi_k\}_{k\in\mathbb{N}}$ and the limit of every convergent subsequence is a minimizer of $\mathcal{F}_N$.*

[3] The concept of a well-posedness was introduced by J. Hadamard at the beginning of the 20*th* century.

[4] Name of the Russian mathematician Andrey Nikolayevich Tikhonov (30 October 1906– 7 October 1993). And the method of regularization was established independently by Phillips[41] and Tikhonov[51].

### 2.5.1 Hammerstein integral equation of the first kind

Hammerstein integral equation of the first kind is

$$f(t) = \int_a^b k(t,x)\,F(\varphi(x))\,dx, \tag{2.15}$$

to determine a solution for this integral equation (2.15), we first convert it to a linear Fredholm integral equation of the first kind of the form

$$f(t) = \int_a^b k(t,x)\,\upsilon(x)\,dx,\;\; t \in [a,b] \tag{2.16}$$

by using the transformation

$$\upsilon(t) = F(\varphi(t)),$$

we assume that $F(\varphi(t))$ is invertible, then we can set

$$\varphi(t) = F^{-1}(\upsilon(t)).$$

The method of regularization transforms the linear Fredholm integral equation of the first kind (2.16) to the approximation Fredholm integral equation

$$\mu\upsilon_\mu(t) = f(t) - \int_a^b k(t,x)\,\upsilon_\mu(x)\,dx,\;\; t \in [a,b] \tag{2.17}$$

where $\mu$ is a small positive parameter. It is clear that (2.17) is a Fredholm integral equation of the second kind that can be rewritten

$$\upsilon_\mu(t) = \frac{1}{\mu} f(t) - \frac{1}{\mu}\int_a^b k(t,x)\,\upsilon_\mu(x)\,dx,\;\; t \in [a,b] \tag{2.18}$$

the solution $\upsilon_\mu$ of equation (2.18) converges to the solution $\upsilon(x)$ of (2.16) as $\mu \longrightarrow 0$ according to the following Lemma:

**Lemma 2.13.** *Suppose that the integral operator of (2.16) is continuous and coercive in the Hilbert space where $f(t)$, $\varphi(t)$ and $\upsilon_\mu(t)$ are defined, then:*

1. *$|\upsilon_\mu|$ is bounded independently of $\mu$, and*

2. *$|\upsilon_\mu(t) - \upsilon(t)| \longrightarrow 0$ when $\mu \longrightarrow 0$.*

The proof of this lemma can be found in [54].

The resulting integral equation (2.18) can be solved by any method that was presented for linear Fredholm integral equation. The exact solution $\upsilon(t)$ of (2.16) can be obtained by

$$\upsilon(t) = \lim_{\mu\longrightarrow 0} \upsilon_\mu(t).$$

It is important to note that the Fredholm integral equation of the first kind is an ill-posed problem. The solution for an ill-posed problem may not exist, and if it does exists it may not be unique, for more information see [55, 56].

**Example 2.14.** *[55] Combine the method of regularization and the direct computation method to solve the nonlinear Hammerstein integral equation of the first kind*

$$e^t = \int_0^{\frac{1}{2}} 2e^{t-4x}\varphi^2(x)\,dx, \tag{2.19}$$

*we set*

$$\upsilon(t) = \varphi^2(t), \ \ \varphi(t) = \pm\sqrt{\upsilon(t)}, \tag{2.20}$$

*to carry out (2.19) into*

$$e^t = \int_0^{\frac{1}{2}} 2e^{t-4x}\upsilon(x)\,dx, \tag{2.21}$$

*using the method of regularization, equation (2.21) can be transformed to*

$$\upsilon_\mu(t) = \frac{1}{\mu}e^t - \frac{1}{\mu}\int_0^{\frac{1}{2}} 2e^{t-4x}\upsilon_\mu(x)\,dx. \tag{2.22}$$

*To use the direct computation method, Equation (2.22) can be written as*

$$\upsilon_\mu(t) = \left(\frac{1}{\mu} - \frac{\alpha}{\mu}\right)e^t, \tag{2.23}$$

*where*

$$\alpha = \int_0^{\frac{1}{2}} 2e^{t-4x}\upsilon_\mu(x)\,dx. \tag{2.24}$$

*To determine $\alpha$, we substitute (2.23) into (2.24) to find*

$$\alpha = \left(\frac{1}{\mu} - \frac{\alpha}{\mu}\right)\int_0^{\frac{1}{2}} 2e^{-3x}dx.$$

*Integrating the right side and solve to find that*

$$\alpha = \frac{2\left(1 - e^{\frac{1}{2}}\right)}{2 - (2+\mu)\,e^{\frac{1}{2}}}.$$

*This in turn gives*

$$\upsilon_{\mu}(t) = \frac{1}{\mu}\left(1 - \frac{2\left(1 - e^{\frac{1}{2}}\right)}{2 - (2+\mu)\, e^{\frac{1}{2}}}\right) e^{t}.$$

*The exact solution* $\upsilon(t)$ *of* (2.22) *can be obtained by*

$$\upsilon(t) = \lim_{\mu \longrightarrow 0} \upsilon_{\mu}(t) = \frac{e^{t+\frac{1}{2}}}{2\left(e^{\frac{1}{2}} - 1\right)}.$$

*Using* (2.20) *gives the exact solution of* (2.19) *by*

$$\varphi(t) = \pm\sqrt{\frac{e^{t+\frac{1}{2}}}{2\left(e^{\frac{1}{2}} - 1\right)}}.$$

*Two more solutions to equation* (2.19) *are given by*

$$\varphi(t) = \pm e^{2t}.$$

### 2.5.2 Hammerstein-Volterra integral equation of the first kind

Consider Hammerstein-Volterra integral equation of the first kind of this form

$$f(t) = \int_0^t k(t,x)\, F(\varphi(x))\, dx, \quad t, x \in [0,1] \tag{2.25}$$

we set $\upsilon(x) = F(\varphi(x))$, so the integral equation (2.25) can be written as

$$\int_0^t k(t,x)\, \upsilon(x)\, dx = f(t),$$

taking the derivative with respect to $t$ in both sides of the above equation, leads to

$$k(t,t)\, \upsilon(t) + \int_0^t \frac{\partial k(t,x)}{\partial t} \upsilon(x)\, dx = f'(t). \tag{2.26}$$

With assumption $k(t,t) \neq 0$, Equation (2.26) is converted to

$$\upsilon(t) + \int_0^t \left(\frac{\partial k(t,x)}{\partial t} / k(t,t)\right) \upsilon(x)\, dx = f'(t) / k(t,t), \tag{2.27}$$

by setting $K(t,x) = \frac{-\partial k(t,x)}{\partial t} / k(t,t)$ and $H(t) = f'(t)/k(t,t)$, equation (2.27) can be written in the following form

$$\upsilon(t) + \int_0^t K(t,x)\, \upsilon(x)\, dx = H(t) \Longleftrightarrow F(\varphi(t)) + \int_0^t K(t,x)\, F(\varphi(x))\, dx = H(t),$$

which is a nonlinear Volterra integral equation of the third kind in the unknown $\varphi(t)$, for example see [47].

# Chapter 3

# Numerical methods to solve nonlinear integral equations

We will define and analyze numerical methods for the discretization of fixed point problems

$$\varphi = T(\varphi), \tag{3.1}$$

with $T : X \longrightarrow X$ a completely continuous nonlinear operator. The space $X$ is a Banach space, and $G$ is an closed bounded region in $\mathbb{R}^n$, some $n \geq 1$. The prototype example of $T$ is the nonlinear integral equation of the form

$$\varphi(t) = \int_G k(t, x, \varphi(x))\, dx + f(t), \tag{3.2}$$

assume $k \in C(G^2 \times \mathbb{R}^n; \mathbb{R}^n)$ is continuously differentiable with respect to its third argument. The space $X = C(G; \mathbb{R}^n)$ with the maximum norm is a Banach space. Occasional use is made of some other spaces, in particular, $L^2(G; \mathbb{R}^n)$ and the Sobolev spaces $H^r(G; \mathbb{R}^n)$. But most of the analysis of the numerical methods can be done in $C(G; \mathbb{R}^n)$, and uniform error bounds are usually considered superior to those in the norms of $L^2(G; \mathbb{R}^n)$ and $H^r(G; \mathbb{R}^n)$. In all the following chapters in this thesis we will take $G = [a, b]$.

In this chapter, we look at the most important classes of numerical methods for the nonlinear integral equations: *Nyström method, projection method*, *Newton-Kantorovich method*, and others.

In general, it is not possible to solve a integral equation analytically. However there are some special cases for relatively simple equations where certain tricks can be used to arrive at a solution:

## 3.1 Degenerate kernel method

The degenerate kernel method will be applied to solve the some nonlinear integral equations with separable kernels. The method approaches integral equation in a direct manner and gives the solution in an exact form, this method will be applied for the *degenerate* or *separable* kernels of the form

$$k(x,t,\varphi(t)) = \sum_{i=1}^{n} \alpha_i(x)\,\beta_i(t,\varphi(t)),$$

where each $\alpha_i$ is continuous and each $\beta_i(t,\varphi(t))$ is integrable on $G=[a,b]$.

**Lemma 3.1.** *[24]. If we consider the nonlinear Volterra integral equation*

$$\varphi(t) = \int_a^t k(t,x,\varphi(x))\,dx + f(t), \quad a \le t, x \le b \tag{3.3}$$

*has a unique solution on $[a,b]$ and $k$ has the form*

$$k(x,t,\varphi) = \sum_{i=1}^{n} \alpha_i(x)\,\beta_i(t,\varphi), \quad |\varphi| < +\infty,$$

*if $f$ is continuous, then the solution of (3.3) is given by*

$$\varphi(t) = f(t) + \sum_{i=1}^{n} \alpha_i(t)\,x_i(t), \quad a \le t \le b,$$

*where*

$$\begin{cases} x_i'(t) = \beta_i\left(t, f(t) + \sum\limits_{i=1}^{n} \alpha_i(t)\,x_i(t)\right), & a < t \le b, \\ x_i(a) = 0, \quad i = 1,2,\ldots,n. \end{cases}$$

*Proof.* Let $\varphi(t)$ be the unique solution of (3.3) and set

$$x_i(t) = \int_a^t \beta_i(x,\varphi(x))\,dx, \quad i = 1,2,\ldots,n$$

then, since

$$\varphi(t) = f(t) + \int_a^t \sum_{i=1}^{n} \alpha_i(t)\,\beta_i(x,\varphi(x))\,dx,$$

it is clear that

$$\varphi(t) = f(t) + \sum_{i=1}^{n} \alpha_i(t)\,x_i(t),$$

and

$$\begin{cases} x_i'(t) = \beta_i\left(t, f(t) + \sum\limits_{i=1}^{n} \alpha_i(t)\,x_i(t)\right), & a < t \le b, \\ x_i(a) = 0, \quad i = 1,2,\ldots,n. \end{cases}$$

□

For $G = [0, 1]$ see [13]. Now, we discuss Hammerstein integral equation of the second kind of the form

$$\varphi(t) = f(t) + \lambda \int_a^b k(t,x) F(\varphi(x))\, dx, \quad t, x \in [a, b] \tag{3.4}$$

with a degenerate kernel

$$k(t,x) = \sum_{i=1}^{n} \alpha_i(t) \beta_i(x), \tag{3.5}$$

where $F(\varphi(x))$ is a nonlinear function of $\varphi(x)$. The direct computation method can be applied as follows:

- We first substitute (3.5) into the nonlinear integral equation (3.4).
- This substitution gives

$$\begin{aligned} \varphi(t) &= f(t) \\ &+ \lambda\alpha_1(t) \int_a^b \beta_1(x) F(\varphi(x))\, dx + \lambda\alpha_2(t) \int_a^b \beta_2(x) F(\varphi(x))\, dx + \cdots \\ &+ \lambda\alpha_n(t) \int_a^b \beta_n(x) F(\varphi(x))\, dx. \end{aligned} \tag{3.6}$$

- Each integral at the right side of (3.6) depends only on the variable $x$ with constant limits of integration for $x$. This means that each integral is equivalent to a constant, the equation (3.6) becomes

$$\varphi(t) = f(t) + \lambda c_1 \alpha_1(t) + \lambda c_2 \alpha_2(t) + \cdots + \lambda c_n \alpha_n(t), \tag{3.7}$$

  where

$$c_i = \int_a^b \beta_i(x) F(\varphi(x))\, dx, \quad 1 \leq i \leq n. \tag{3.8}$$

- Substituting (3.7) into (3.8) gives a system of $n$ algebraic equations that can be solved to determine the constants $c_i$, $1 \leq i \leq n$. Using the obtained numerical values of $c_i$ into (3.7), the solution $\varphi(t)$ of Hammerstein integral equation (3.4) follows immediately.

For further studies on degenerate kernel method, we refer to Wazwaz [55], Brunner [13] and Golberg [24].

## 3.2 Successive approximation method

This section is devoted to the discussion of a simple but powerful method of attacking nonlinear equations. This method is sometimes called the method of *successive substitutions* or the method of *Picard iteration*[1]. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen later, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations. In many cases, the successive approximation method can be successfully applied to solve various types of nonlinear integral equations (see, [55, 33, 5, 45]). For NVIE of the second kind in the Urysohn form

$$\varphi(t) = f(t) + \int_a^t k(t, x, \varphi(x))\, dx, \quad a \le t \le b \tag{3.9}$$

the corresponding recursive expression has the form

$$\varphi_{n+1}(t) = f(t) + \int_a^t k(t, x, \varphi_n(x))\, dx, \quad n = 0, 1, 2, \ldots \tag{3.10}$$

It is customary to take the initial approximation either in the form $\varphi_0(t) = 0$ or in the form $\varphi_0(t) = f(t)$.

**Theorem 3.2.** *[5]. Assume $k(t, x, \varphi)$ is continuous for $a \le t \le x \le b$, and let $f \in C[a, b]$. Furthermore, assume*

$$|k(t, x, \varphi_1) - k(t, x, \varphi_2)| \le M |\varphi_1 - \varphi_2|, \; a \le t \le x \le b, \quad \varphi_1, \varphi_2 \in \mathbb{R}$$

*for some constant $M$. Then the integral equation (3.9) has a unique solution $\varphi \in C[a, b]$. Moreover, the iterative method (3.10) converges for any initial function $\varphi_0 \in C[a, b]$.*

*Proof.* We define the nonlinear integral operator

$$T : C[a, b] \longrightarrow C[a, b], \; T\varphi(t) = \int_a^t k(t, x, \varphi(x))\, dx + f(t).$$

Let us show that for $m$ sufficiently large, the operator $T^m$ is a contraction on $C[a, b]$. For $\varphi, \phi \in C[a, b]$,

$$T\varphi(t) - T\phi(t) = \int_a^t [k(t, x, \varphi(x)) - k(t, x, \phi(x))]\, dx.$$

[1] This procedure originated in antiquity, appearing, for example, in the writings of Heron of Alexandria in the second century B.C. in connection with the extraction of roots. In modern times Cauchy and Picard used this technique to establish the existence of solutions of differential equations, (See, for more information [45]).

Then

$$|T\varphi(t) - T\phi(t)| \leq M \int_a^t |\varphi(x) - \phi(x)|\, dx,$$

and

$$|T\varphi(t) - T\phi(t)| \leq M \|\varphi - \phi\|_\infty (t-a).$$

Since

$$T^2\varphi(t) - T^2\phi(t) = \int_a^t [k(t,x,T\varphi(x)) - k(t,x,T\phi(x))]\, dx,$$

we get

$$\begin{aligned} \left|T^2\varphi(t) - T^2\phi(t)\right| &\leq M \int_a^t |T\varphi(x) - T\phi(x)|\, dx, \\ &\leq \frac{(M(t-a))^2}{2!} \|\varphi - \phi\|_\infty . \end{aligned}$$

By a mathematical induction, we obtain

$$|T^m\varphi(t) - T^m\phi(t)| \leq \frac{(M(t-a))^m}{m!} \|\varphi - \phi\|_\infty .$$

Thus

$$\|T^m\varphi(t) - T^m\phi(t)\|_\infty \leq \frac{(M(t-a))^m}{m!} \|\varphi - \phi\|_\infty .$$

Since

$$\frac{(M(t-a))^m}{m!} \longrightarrow 0 \text{ as } m \longrightarrow 0,$$

the operator $T^m$ is a contraction on $C[a,b]$ when $m$ is chosen sufficiently large. Than the operator $T$ has a unique fixed point in $C[a,b]$ and the iteration sequence converges to the solution. □

**Example 3.3.** *[55] Use the successive approximations method to solve the nonlinear Volterra integral equation*

$$\varphi(t) = 1 + \frac{1}{2}t^2 - \frac{1}{6}t^4 - \frac{1}{30}t^6 + \int_0^t (t-x)\varphi^2(x)\, dx. \tag{3.11}$$

*For the zeroth approximation $\varphi_0(t)$, we can select*

$$\varphi_0(t) = 1. \tag{3.12}$$

*The method of successive approximations admits the use of the iteration formula*

$$\varphi_{n+1}(t) = 1 + \frac{1}{2}t^2 - \frac{1}{6}t^4 - \frac{1}{30}t^6 + \int_0^t (t-x)\varphi_n^2(x)\, dx, \quad n \geq 0 \tag{3.13}$$

*Substituting (3.12) into (3.13) we obtain these approximations*

$$\begin{aligned}
\varphi_0(t) &= 1,\\
\varphi_1(t) &= 1+t^2-\frac{1}{6}t^4-\frac{1}{30}t^6,\\
\varphi_2(t) &= 1+t^2+\left(\frac{1}{6}t^4-\frac{1}{6}t^4\right)-\frac{1}{90}t^6+\cdots,\\
\varphi_3(t) &= 1+t^2+\left(\frac{1}{6}t^4-\frac{1}{6}t^4\right)+\left(\frac{1}{90}t^6-\frac{1}{90}t^6\right)+\cdots,
\end{aligned}$$

*and so on. By canceling the noise terms, the solution $\varphi(t)$ of (3.11) is given by*

$$\varphi(t)=\lim_{n\longrightarrow\infty}\varphi_n(t)=1+t^2.$$

New, for nonlinear Fredholm integral equation of the second kind of the form

$$\varphi(t)=\lambda\int_a^b k(t,x,\varphi(x))\,dx+f(t),\quad a\le t\le b \tag{3.14}$$

we assume that

$$f\in C[a,b]\ \text{and}\ k\in C([a,b]\times[a,b]\times\mathbb{R}). \tag{3.15}$$

Moreover, we assume $k$ satisfies a uniform Lipschitz condition with respect to its third argument:

$$|k(t,x,\varphi_1)-k(t,x,\varphi_2)|\le M|\varphi_1-\varphi_2|,\ a\le x,t\le b,\ \varphi_1,\varphi_2\in\mathbb{R}. \tag{3.16}$$

Since (3.14) is of the form $\varphi=T\varphi$, we can introduce the fixed point iteration

$$\varphi_n(t)=\lambda\int_a^b k\left(t,x,\varphi_{n-1}(x)\right)dx+f(t),\ a\le t\le b,\ n\ge 1. \tag{3.17}$$

**Theorem 3.4.** *[5]. Assume $f$ and $k$ satisfy the conditions (3.15), (3.16). Moreover, assume*

$$|\lambda|M(b-a)<1.$$

*Then the integral equation (3.14) has a unique solution $\varphi\in C[a,b]$, and it can be approximated by the iteration method of (3.17).*

**Example 3.5.** *[55] Use the successive approximations method to solve the nonlinear Fredholm integral equation*

$$\varphi(t)=\sin t+1-\frac{\pi}{12}-\frac{5\pi^2}{144}+\frac{1}{36}\int_0^\pi x\left(\varphi(x)+\varphi^2(x)\right)dx. \tag{3.18}$$

*For the zeroth approximation $\varphi_0(t)$, we can select*

$$\varphi_0(t) = 1. \tag{3.19}$$

*The method of successive approximations admits the use of the iteration formula*

$$\varphi_{n+1}(t) = \sin t + 1 - \frac{\pi}{12} - \frac{5\pi^2}{144} + \frac{1}{36}\int_0^{\pi} x\left(\varphi_n(x) + \varphi_n^2(x)\right) dx, \quad n \geq 0 \tag{3.20}$$

*substituting (3.19) into (3.20), and proceeding as before we obtain the approximations*

$$\begin{aligned} \varphi_1(t) &\simeq \sin t + 0.6696616927, \\ \varphi_2(t) &\simeq \sin t + 0.8214573046, \\ \varphi_3(t) &\simeq \sin t + 0.8997853785, \\ \varphi_4(t) &\simeq \sin t + 0.9426743063, \\ \varphi_5(t) &\simeq \sin t + 0.9668710030, \end{aligned}$$

*and so on. Consequently, the solution $\varphi(t)$ of (3.18) is given by*

$$\varphi(t) = \lim_{n \longrightarrow \infty} \varphi_{n+1}(t) = 1 + \sin t.$$

*The direct computation method gives an additional solution to this equation given by*

$$\varphi(t) = \sin t - 2 - \frac{4}{\pi}\left(1 - \frac{18}{\pi}\right).$$

The remaining nonlinear integral equations which cannot be solved analytically must be solved numerically. The main methods we will be using are Nyström method, projection methods and Newton-Kantorovich method.

## 3.3 Nyström method

In this section, we shall describe the quadrature or Nyström method for the approximate solution of nonlinear integral equations of the second kind with continuous kernels.

The goal of the quadrature method[2] is to approximate the definite integral of $f(t)$ over the interval $G = [a, b]$ by evaluating $f(t)$ at a finite number of sample points.

[2] The word "quadrature" stems from a technique used by ancient Greeks to measure areas in the plane by transforming them with straight edge and compass into a square using area-preserving transformations. Finding an area-preserving transformation that maps a circle into a square (quadrature of the circle) became a famous unsolved problem until the $19th$ century, when it was proved using Galois theory that the problem cannot be solved using straight edge and compass. Since the definite integral measures the area between the graph of a function and the $x-$axis, it is natural to use the word "quadrature" to denote the approximation of the integral, see, [21].

**Definition 3.6.** *Suppose that* $a = t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} = b$*. A formula of the form*

$$Q_n[f] = \sum_{i=1}^{n} w_i^{(n)} f\left(t_i^{(n)}\right),$$

*with a property that*

$$\int_a^b f(t)\,dt = Q_n[f] + E[f],$$

*is called a numerical integration or quadrature formula. The term* $E[f]$ *is called the truncation error for integration. The values* $\left\{t_i^{(n)}\right\}_{i=1}^{n}$ *are called the quadrature nodes*[3] *and* $\left\{w_i^{(n)}\right\}_{i=1}^{n}$ *are called the weights.*

**Definition 3.7.** *A sequence* $Q_n[f]$ *of quadrature formulas is called convergent if* $Q_n[f] \longrightarrow Q[f]$, $n \longrightarrow \infty$*, for all* $f \in C(G)$*, i.e., if the sequence of linear functionals* $(Q_n)$ *converges pointwise*[4] *to the integral* $Q$*.*

Newton-Cotes[5] methods describes a group of numerical methods which the nodes are equally spaced in the integration interval, with some step-size, $h$. There are two main types of Newton-Cotes methods. Closed Newton-Cotes are such methods which use the end points of the integration interval, examples of such methods are the trapezoidal rule and Simpson's rule. The other type is open Newton-Cotes. This type does not use the end points of the interval, i.e. the nodes are inside the interval. An example of open Newton-Cotes is the midpoint rule. In this thesis we will consider closed Newton-Cotes methods only.

The significance of the next two results is to understand that the error terms for the composite the trapezium rule (also known as trapezoidal rule) and composite Simpson rule are of the order $O(h^2)$ and $O(h^4)$, respectively. This shows that the error for Simpson's rule converges to zero faster than the error for the trapezoidal rule as the step size $h$ decreases to zero.

---

[3] The nodes $\{t_i\}_{i=0}^{n}$ are chosen in various ways. For the trapezoidal rule and Simpson's rule, the nodes are chosen to be equally space. For Gauss-Legendre quadrature, the nodes are chosen to be zeros of certain Legendre polynomials.

[4] A sequence $Q_n$ of functions defined on a set $C(G)$ *converges pointwise* to a function $Q$ defined on $C(G)$ if

$$\lim_{n\longrightarrow\infty} Q_n[f] = Q[f],$$

written in logical notation, $\forall f \in C(G)$ and $\forall \epsilon > 0, \exists N \in \mathbb{N}$ so that $\forall n \geq N$ we have $|Q_n[f] - Q[f]| < \epsilon$.

[5] The open *1–point Newton–Cotes* rule is the midpoint rule, the closed *2–point Newton–Cotes* rule is the trapezoidal rule, the closed *3–point Newton–Cotes* rule is Simpson's rule and the closed *4–point Newton–Cotes* rule is Weddle's rule.

**Corollary 3.8.** ***(Trapezoidal rule: Error analysis)***. *Suppose that* $[a, b]$ *is subdivided into* $n$ *subintervals* $[x_i, x_{i+1}]$ *of width* $h = (b - a)/n$. *The composite trapezoidal rule*

$$T(f, h) = \frac{h}{2}(f(a) + f(b)) + h\sum_{i=1}^{n-1} f(x_i) \tag{3.21}$$

*is an approximation to the integral*

$$\int_a^b f(x)\, dx = T(f, h) + E_T(f, h).$$

*Furthermore, if* $f \in C^2[a, b]$, *there exists a value* $c$ *with* $a < c < b$ *so that the error term* $E_T(f, h)$ *has the form*

$$E_T(f, h) = \frac{-(b - a) f^{(2)}(c) h^2}{12} = O\left(h^2\right).$$

**Corollary 3.9.** ***(Simpson's rule: Error analysis)***. *Suppose that* $[a, b]$ *is subdivided into* $2n$ *subintervals* $[x_i, x_{i+1}]$ *of width* $h = (b - a)/(2n)$. *The composite Simpson rule*

$$S(f, h) = \frac{h}{3}(f(a) + f(b)) + \frac{2h}{3}\sum_{i=1}^{n-1} f(x_{2i}) + \frac{4h}{3}\sum_{i=1}^{n} f(x_{2i-1}) \tag{3.22}$$

*is an approximation to the integral*

$$\int_a^b f(x)\, dx = S(f, h) + E_S(f, h).$$

*Furthermore, if* $f \in C^4[a, b]$, *there exists a value* $c$ *with* $a < c < b$ *so that the error term* $E_S(f, h)$ *has the form*

$$E_S(f, h) = \frac{-(b - a) f^{(4)}(c) h^4}{180} = O\left(h^4\right).$$

**Theorem 3.10.** ***(Szegö)[29].*** *The quadrature formulas* $(Q_n)$ *converge if and only if* $Q_n[f] \longrightarrow Q[f]$, $n \longrightarrow \infty$, *for all* $f$ *in some dense subset* $U \subset C(G)$ *and*

$$\sup_{n\in\mathbb{N}} \sum_{j=1}^{n} \left|w_j^{(n)}\right| < \infty.$$

Using any quadrature rule, it is possible to numerically approximate the nonlinear integral equation of the second kind in the following section:

For Urysohn integral equations with continuous kernels, perhaps the most natural and well-known approximation technique is the Nyström method. That is, if $\varphi(t)$ satisfies

$$\varphi(t) + \int_G k(t, x, \varphi(x))\, dx = f(t), \tag{3.23}$$

then one can obtain an approximation to $\varphi(t)$ by replacing $\int_G k(t, x, \varphi(x))\, dx$ with a numerical integration

$$\int_G k(t, x, \varphi(x))\, dx \approx \sum_{j=1}^{n+1} w_j k(t, x_j, \varphi(x_j)), \tag{3.24}$$

with quadrature points (nodes) $\{x_j\}_{j=1}^{n+1}$ contained in $G$ and real quadrature weights $\{w_j\}_{j=1}^{n+1}$. Substituting the right-hand side of (3.24) for the integral in (3.23) generates the functional equation

$$\varphi_n(t) + \sum_{j=1}^{n+1} w_j k(t, x_j, \varphi_n(x_j)) = f(t), \tag{3.25}$$

where $\varphi_n(t)$ is an approximation to $\varphi(t)$. All the quadrature rules can be written in the form of equation (3.24). The trapezoidal and Simpson rules are special cases of (3.24).
A solution to a functional equation (3.25) may be obtained if we assign $t_i$'s to $t$ in which $i = 1, 2, \ldots, n+1$ and $t_i \in G$. In this way, (3.25) is reduced to the system of nonlinear equations

$$\varphi_n(t_i) + \sum_{j=1}^{n+1} w_j k(t_i, x_j, \varphi_n(x_j)) = f(t_i), \quad i = 1, 2, \ldots, n+1$$

**Theorem 3.11.** *[31] Consider the approximate solution of*

$$\varphi(t) + \int_0^t k(t, x, \varphi(x))\, dx = f(t), \tag{3.26}$$

*by (3.25) and assume that:*

*i) The solution $\varphi(t)$ of (3.26) and the kernel $k(t, x, \varphi)$ are such that the approximation method is consistent of order $p$ [6] with (3.26),*

*ii) The weights satisfy*

$$\sup_i |w_i| \le W < \infty,$$

[6] If for every nonlinear Volterra integral equation, there exists a constant $c$

$$\max_{0 \le n \le N} \left| \int_0^{t_n} k(t_n, x, \varphi(x))\, dx - \sum_{i=1}^{n+1} w_i k(t_n, x_i, \varphi(x_i)) \right| \le ch^p,$$

then the method is said to be consistent of order $p$.

*iii) The starting errors* $\varphi_n - \varphi(t_i)$, $i = 0, 1, \ldots, r-1$ *go to zero as* $h \longrightarrow 0$. *Since* $r$ *is fixed, this implies that*

$$\lim_{h \longrightarrow 0} \sum_{i=0}^{r-1} |\varphi_i - \varphi(t_i)| = 0.$$

*Then the method is a convergent approximation method. Also, in the absence of starting errors, the order of convergence is at least* $p$.

*Proof.* [31] □

For a more detailed analysis of quadrature method, we refer to [5, 6, 29] and others.

## 3.4 Method of Newton-Kantorovich

**Definition 3.12.** *[3] A proper functional* $F : U \longrightarrow V$ *is said to be Fréchet differentiable (or strongly differentiable) at a point* $\varphi \in D$, *where* $D$ *is an open set in* $U$, *if there is a linear operator* $L : U \longrightarrow U^*$ *such that, for any* $\varphi + h \in D$,

$$F(\varphi + h) = F(\varphi) + \langle L(\varphi), h \rangle + \omega(\varphi, h)$$

*and*

$$\lim_{\|h\| \longrightarrow 0} \frac{\omega(\varphi, h)}{\|h\|} = 0.$$

*The quantity* $\langle L(\varphi), h \rangle$ *is called the Fréchet differential (or strong differential) and* $L(\varphi) = F'(\varphi)$ *is called the Fréchet derivative (or strong derivative) of the functional* $F$ *at a point* $\varphi$.

Now, we will consider Newton's method for solving the equation

$$F(\varphi) = 0, \tag{3.27}$$

where $F : U \longrightarrow V$ is Fréchet differentiable and $U$, $V$ be two Banach spaces. This method provides a powerful tool for the theoretical as well as the numerical investigation of nonlinear operator equations. In this section will present the theory of Newton's method, together with some of its applications to important types of nonlinear equations. Kantorovich proposed to solve a functional equation (3.27) where $F$ is defined and Fréchet differentiable on some open convex set of a Banach space $X$, with range in a Banach space $Y$.
One way to approach this problem (3.27) would be to find a linear equation

$$L\varphi = \phi, \tag{3.28}$$

which approximates (3.27) in some sense, at least in a neighborhood of exact solution $\varphi^*$. It is easy to construct a linear equation (3.28) which approximates the nonlinear equation (3.27) if the operator $F$ is differentiable. By the first two terms of Taylor's formula

$$\begin{aligned} 0 &= F(\varphi^*) \\ &= F(\varphi_n) + F'(\varphi_n)(\varphi^* - \varphi_n) + O(|\varphi^* - \varphi_n|) \\ &\approx F(\varphi_n) + F'(\varphi_n)(\varphi^* - \varphi_n). \end{aligned}$$

This approximate solution is said to be obtained from (3.27) by the process of *linearization by differentiation*, sometimes called *quasilinearization* or *the tangent method*. Thus,

$$\varphi^* \approx \varphi_n - [F'(\varphi_n)]^{-1} F(\varphi_n).$$

This leads to the well-known Newton method for solving the equation (3.27) :

$$\varphi_{n+1} = \varphi_n - [F'(\varphi_n)]^{-1} F(\varphi_n),$$

choose an initial guess $\varphi_0 \in U$; for $n = 0, 1, \ldots$.

**Theorem 3.13.** *[5]* ***(Local convergence)*** *Assume $\varphi^*$ is a solution of the equation (3.27) such that $[F'(\varphi_n)]^{-1}$ exists and is a continuous linear map from $V$ to $U$. Assume further that $F'(\varphi)$ is locally Lipschitz continuous at $\varphi^*$,*

$$\|F'(\varphi) - F'(\upsilon)\| \le L\|\varphi - \upsilon\|, \forall\varphi, \upsilon \in N(\varphi^*),$$

*where $N(\varphi^*)$ is a neighborhood of $\varphi^*$, and $L > 0$ is a constant. Then there exists a $\delta > 0$ such that if $\|\varphi_0 - \varphi^*\| \le \delta$, the Newton's sequence $\{\varphi_n\}$ is well-defined and converges to $\varphi^*$. Furthermore, for some constant $M$ with $M\delta < 1$, we have the error bounds*

$$\left\|\varphi_{n+1} - \varphi^*\right\| \le M\|\varphi_n - \varphi^*\|^2 \tag{3.29}$$

*and*

$$\|\varphi_n - \varphi^*\| \le (M\delta)^{2^n}/M. \tag{3.30}$$

*Proof.* Upon redefining the neighborhood $N(\varphi^*)$ if necessary, we may assume $[F'(\varphi)]^{-1}$ exists on $N(\varphi^*)$ and

$$c_0 = \sup_{\varphi\in N(\varphi^*)} \left\|[F'(\varphi)]^{-1}\right\| < \infty.$$

Let us define

$$T(\varphi) = \varphi - [F'(\varphi)]^{-1} F(\varphi), \quad \varphi \in N(\varphi^*).$$

Notice that $T(\varphi^*) = \varphi^*$. For $\varphi \in N(\varphi^*)$, we have

$$\begin{aligned} T(\varphi) - T(\varphi^*) &= \varphi - \varphi^* - [F'(\varphi)]^{-1} F(\varphi) \\ &= [F'(\varphi)]^{-1} [F(\varphi^*) - F(\varphi) - F'(\varphi)(\varphi^* - \varphi)] \\ &= [F'(\varphi)]^{-1} \int_0^1 [F'(\varphi + t(\varphi^* - \varphi)) - F'(\varphi)]\, dt\, (\varphi^* - \varphi), \end{aligned}$$

and by taking the norm,

$$\begin{aligned} \|T(\varphi) - T(\varphi^*)\| &\leq \left\|[F'(\varphi)]^{-1}\right\| \int_0^1 \|[F'(\varphi + t(\varphi^* - \varphi)) - F'(\varphi)]\|\, dt\, \|\varphi^* - \varphi\| \\ &\leq \left\|[F'(\varphi)]^{-1}\right\| \int_0^1 Lt \|\varphi^* - \varphi\|\, dt\, \|\varphi^* - \varphi\|. \end{aligned}$$

Hence,

$$\|T(\varphi) - T(\varphi^*)\| \leq \frac{c_0 L}{2} \|\varphi^* - \varphi\|^2. \tag{3.31}$$

Choose $\delta < \frac{2}{c_0 L}$ with the property $\overline{B}(\varphi^*, \delta) \subset N(\varphi^*)$; and note that

$$\alpha \equiv \frac{c_0 L\ delta}{2} < 1.$$

Then (3.31) implies

$$\begin{aligned} \|T(\varphi) - \varphi^*\| &= \|T(\varphi) - T(\varphi^*)\| \\ &\leq \alpha \|\varphi - \varphi^*\|, \quad \varphi \in \overline{B}(\varphi^*, \delta). \end{aligned} \tag{3.32}$$

Assume an initial guess $\varphi_0 \in \overline{B}(\varphi^*, \delta)$. Then (3.32) implies

$$\begin{aligned} \|\varphi_1 - \varphi^*\| &= \|T(\varphi) - \varphi^*\| \\ &\leq \alpha \|\varphi_0 - \varphi^*\| \leq \alpha\delta < \delta. \end{aligned}$$

Thus $\varphi_1 \in \overline{B}(\varphi^*, \delta)$. Repeating this argument inductively, we have $\varphi_n \in \overline{B}(\varphi^*, \delta)$ for all $n \geq 0$.

To obtain the convergence of $\{\varphi_n\}$, we begin with

$$\begin{aligned} \left\|\varphi_{n+1} - \varphi^*\right\| &= \|T(\varphi_n) - \varphi^*\| \\ &\leq \alpha \|\varphi_n - \varphi^*\|, \quad n \geq 0. \end{aligned}$$

By induction,

$$\left\|\varphi_{n+1}-\varphi^*\right\| \leq \alpha^n \left\|\varphi_0-\varphi^*\right\|, \quad n \geq 0.$$

and $\varphi_n \longrightarrow \varphi^*$ as $n \longrightarrow \infty$. Returning to (3.31), denote $M\delta = \alpha < 1$. Then we get the estimate

$$\left\|\varphi_{n+1}-\varphi^*\right\| = \left\|T\left(\varphi_n\right)-\varphi^*\right\| \leq M\left\|\varphi_n-\varphi^*\right\|^2,$$

proving (3.29). Multiply both sides by $M$, obtaining

$$M\left\|\varphi_{n+1}-\varphi^*\right\| \leq \left(M\left\|\varphi_n-\varphi^*\right\|\right)^2.$$

An inductive application of this inequality leads to

$$M\left\|\varphi_n-\varphi^*\right\| \leq \left(M\left\|\varphi_n-\varphi^*\right\|\right)^{2n},$$

thus proving (3.30). □

Even though we shall assume enough to guarantee $F'(\varphi)$ invertible, it may not be desirable to have to pay the price of computing $F'(\varphi^n)$ at each stage of the iteration and then solving for $\varphi^{n+1}$. At a sacrifice in convergence rates, one may fix $F'(\varphi) = F'(\varphi_0)$, $\varphi_0 \in U$. Successive approximations in these cases are known as *modified Newton's methods*. We commence with Kantorovich's main theorem:

**Theorem 3.14.** *[57, 5]**(Kantorovich)** Suppose that:*

1. *$F : D(F) \subset U \longrightarrow V$ is differentiable on an open convex set $D(F)$, and the derivative is Lipschitz continuous:*

$$\left\|F'(\varphi)-F'(v)\right\| \leq L\left\|\varphi-v\right\| \qquad \forall\ \varphi, v \in D(F).$$

2. *For some $\varphi_0 \in D(F)$, $\left[F'(\varphi_0)\right]^{-1}$ exists and is a continuous operator from $V$ to $U$, and such that $h = abL \leq 1/2$ for some $a \geq \left\|\left[F'(\varphi_0)\right]^{-1}\right\|$ and $b \geq \left\|\left[F'(\varphi_0)\right]^{-1} F'(\varphi_0)\right\|$. Denote*

$$t^* = \frac{1-(1-2h)^{1/2}}{aL}, \quad t^{**} = \frac{1+(1-2h)^{1/2}}{aL}.$$

3. *$\varphi_0$ is chosen so that $\overline{B}(\varphi_1, r) \subset D(F)$, where $r = t^* - b$.*

*Then the equation (3.27) has a solution* $\varphi^* \in B(\varphi_1, r)$ *and the solution is unique in* $\overline{B}(\varphi_0, t^{**}) \subset D(F)$*; the sequence* $\{\varphi_n\}$ *converges to* $\varphi^*$*, and we have the error estimate*

$$\|\varphi_n - \varphi^*\| \leq \frac{\left[1-(1-2h)^{1/2}\right]^{2n}}{2^n a L}, \quad n = 0, 1, \ldots.$$

*Proof.* [57] □

**Theorem 3.15.** ***(Modified Newton's method)*** *[57] Under the same conditions as in* ***Theorem 3.14****, if* $h < \frac{1}{2}$*, then the iterations*

$$\hat{\varphi}^{n+1} = \hat{\varphi}^n - [F_0']^{-1} F(\hat{\varphi}^n)$$

*are defined for all n, for any* $\hat{\varphi}^0 \in \Omega_{t^*} = \{\varphi \,|\, \|\varphi - \varphi^0\| \leq t^*\} \subset D(F)$*, and converge to a root* $\hat{\varphi}^0 \in \Omega_{t^*}$*. Further,* $F(\varphi) = 0$ *has a unique root in* $\Omega_{t^*}$*. Also,*

$$\|\hat{\varphi}^n - \varphi^*\| \leq 2\left(\frac{\eta}{h}\right)\left[1-\sqrt{1-2h}\right]^{n+1}; \quad n = 1, 2, 3, \ldots$$

*Proof.* [57] □

The modified Newton's method has proven to be robust under perturbation of the initial point $\hat{\varphi}^0$. In the next theorem we establish robustness for Newton's method:

**Theorem 3.16.** *[57] Let* $F: U \longrightarrow V$ *be a Fréchet differentiable function for* $\varphi \in D(F)$*, an open convex set. Let* $\Gamma = [F'(\varphi^0)]^{-1} \in [V \longrightarrow U], \Gamma^{-1} \in [U \longrightarrow V]$ *and* $\varphi^0 \in D(F)$ *satisfy*

1. $\|\Gamma\| \leq B^1$,
2. $\|\Gamma F(\varphi^0)\| \leq \eta^1$,
3. $\|\Gamma F'(\varphi^0) - I\| \leq \delta < 1$,
4. $\|F'(\varphi) - F'(v)\| \leq K^1 \|\varphi - v\|, \quad \varphi, v \in D(F)$.

*If* $h^1 = \dfrac{B^1 K^1 \eta^1}{(1-\delta)^2} \leq \dfrac{1}{2}$*,* $t^* = \left[1-\sqrt{1-2h^1}\right]\left[\eta^1/h^1(1-\delta)\right]$*, and* $\Omega_{t^*} \subset D(F)$ *then Newton's method starting at* $\varphi^0$ *produces a sequence* $\varphi^n$ *converging to a root* $\varphi^*$ *of* $F(\varphi) = 0$ *with*

$$\|\varphi^* - \varphi^n\| \leq \frac{1}{2^n}\left[1-\sqrt{1-2h^1}\right]^{2n}\left[\eta^1/h^1(1-\delta)\right].$$

*The solution* $\varphi^*$ *is unique in the open sphere*

$$\|\varphi - \varphi^0\| \leq \left[1+\sqrt{1-2h^1}\right]\left[\eta^1/h^1(1-\delta)\right], \quad \text{if } h^1 < \tfrac{1}{2}$$

*and is unique in* $\Omega_{t^*}$ *if* $h^1 = \frac{1}{2}$.

*Proof.* [57] □

We will consider Urysohn's equation of the second kind

$$\varphi(t) = f(t) + \int_G k(t, x, \varphi(x))dx, \quad t \in G \tag{3.33}$$

where $G$ a closed bounded interval in $\mathbb{R}$, $f(t)$ and $k(t, x, \varphi)$ are given and continuous functions in $G$, and $\varphi(t)$ is to be determined. If we take $G = [a, b]$ and the value $b$ varies then we have a nonlinear Volterra integral equation, and if the values $a, b$ are fixed, then the equation (3.33) describes a Fredholm nonlinear integral equation.

Let us apply the Newton-Kantorovich method to solve a nonlinear integral equations of the second kind in the Urysohn form (3.33) . We obtain the following iteration process

$$\begin{cases} \varphi_n(t) = \varphi_{n-1}(t) + \phi_{n-1}(t), \\ \phi_{n-1}(t) = \varepsilon_{n-1}(t) + \int_G \frac{\partial k}{\partial \varphi}(t, x, \varphi_{n-1}(x))\phi_{n-1}(x)dx, \ n = 1, 2, \dots \\ \varepsilon_{n-1}(t) = f(t) + \int_G k(t, x, \varphi_{n-1}(x))dx - \varphi_{n-1}(t). \end{cases} \tag{3.34}$$

We can to write (3.34) as

$$\begin{cases} \varphi_n(t) = \varphi_{n-1}(t) + \phi_{n-1}(t), \ n = 1, 2, \dots, \\ \phi_{n-1}(t) = f(t) + \int_G k(t, x, \varphi_{n-1}(x))dx - \varphi_{n-1}(t) + \int_G \frac{\partial k}{\partial \varphi}(t, x, \varphi_{n-1}(x))\phi_{n-1}(x)dx. \end{cases} \tag{3.35}$$

The last algorithm (3.35) based on the solution of the linear integral equation for the correction $\phi_{n-1}(t)$ with the kernel and right-hand side that vary from step to step. This process has a high rate of convergence, but it is rather complicated because we must solve a new equation at each step of iteration. To simplify the problem, we can replace the second equation of the algorithm (3.35) by the equation

$$\phi_{n-1}(t) = f(t) + \int_G k(t, x, \varphi_{n-1}(x))dx - \varphi_{n-1}(t) + \int_G \frac{\partial k}{\partial \varphi}(t, x, \varphi_0(x))\phi_{n-1}(x)dx. \tag{3.36}$$

Or by the equation

$$\phi_{n-1}(t) = f(t) + \int_G k(t, x, \varphi_{n-1}(x))dx - \varphi_{n-1}(t) + \int_G \frac{\partial k}{\partial \varphi}(t, x, \varphi_m(x))\phi_{n-1}(x)dx. \tag{3.37}$$

Whose kernels do not vary. In the equation (3.36), $\varphi_0$ is the initial solution, and in the equation (3.37) $m$ is fixed and satisfies the condition $m < n - 1$.

For further studies on the method of Newton-Kantorovich, we refer to [33, 55, 57, 5, 45, 42, 3].

## 3.5 Projection method

The projection method [7][8] is a method for the approximating to the solution of problems by using finite expansions in terms of orthogonal functions with the trial functions (also called the expansion or approximating functions) for this method equal to some global smooth basis functions, an example of which would be the Fourier series. These types of methods, were originally used for the solution of boundary value problems, for more information about the history of projection method, see [43]. However as explained by a number of researchers, among these are [23, 5, 6, 24, 13, 34] and [7], projection or spectral method can be used as a numerical method for the solution of nonlinear integral equations of the Urysohn form, which is superior to other more conventional methods, some of which are described in the chapter.

Projection operators are useful in discussing many approximation methods. In this section, we consider projection operators on subspaces of more general linear spaces or normed spaces.

**Definition 3.17.** *Let $X$ be a normed space and $U \subset X$ a nontrivial subspace. A bounded linear operator $P : X \longrightarrow U$ with the property $P\varphi = \varphi$ for all $\varphi \in U$ is called a projection operator from $X$ onto $U$.*

**Theorem 3.18.** *A nontrivial bounded linear operator $P$ mapping a normed space $X$ into itself is a projection operator if and only if $P^2 = P$. Projection operators satisfy $\|P\| \geq 1$.*

*Proof.* [29] □

[7] This method is due to Torsten Carleman. It was first published in 1922. See "*Sur la résolution de certaines équations integrates*", Arkiv For Matematik, Astronomi Och Fysik, Band 16, No. 26. It was reprinted in *Edition Complete des Articles de* Torsten Carleman, Malmo 1960, pp. 141-159. (For more information, see [26])

[8] Sometimes the types of projection methods are referred to as *spectral methods*, especially when trigonometric polynomials are used.

Let $X$ be a Banach space, usually $C(G)$ or $L^2(G)$; and let $X_n, n \geq 1$, be a sequence of finite dimensional subspaces being used to approximate $\varphi^*$. Denote the dimension of $X_n$ by $\kappa_n$, and we assume $\kappa_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

We define the bounded projection operator $P_n : X \overset{onto}{\longrightarrow} X_n$([9]) for solving (3.1) as follows

$$P_n\varphi \longrightarrow \varphi \quad \text{as} \quad n \longrightarrow \infty, \quad \varphi \in X$$

for a given discretization parameter $n$, find $\varphi_n \in X_n$ satisfying the operator equation

$$\varphi_n = P_n T(\varphi_n), \tag{3.38}$$

is called an approximate fixed point or simply an approximate solution of the fixed point problem. And (3.38) is equivalently

$$P_n(\varphi_n - T(\varphi_n)) = 0, \quad \varphi_n \in X_n.$$

In the literature, the name "*Galerkin method*" is used in the case $P_n$ is an orthogonal projection. In case $P_n$ is an interpolation operator (3.38) is called a "*collocation method*". But these methods both start out the same way, one seeks an approximation to $\varphi$ of the form $\varphi_n \in X_n$.

### 3.5.1 Collocation method

Consider the Urysohn nonlinear equation (3.2). Let

$$\varphi_n(t) = \sum_{j=1}^{\kappa_n} c_j \phi_j(t), \quad t \in G. \tag{3.39}$$

Let $X$ be a space of continuous functions, and let $P_n$ be an interpolatory projection operator from $X$ to $X_n$, where $\{\phi_1, \phi_2, \ldots, \phi_{\kappa_n}\}$ forms an orthonormal basis for $X_n$. Let us rewrite the Urysohn integral equation of the second kind in the form

$$\varepsilon[\varphi_n(t)] \equiv \varphi_n(t) - \int_G k(t, x, \varphi_n(x))\, dx - f(t) = 0. \tag{3.40}$$

If $\varphi(t)$ is an exact solution, then, clearly the residual $\varepsilon[\varphi(t)]$ is zero. Therefore, one tries to choose the parameters $\{c_j\}$ so that, in a sense, the residual $\varepsilon[\varphi_n(t)]$ is as small as possible.

[9]The operator is said to map $X$ *onto* $V$ or is called *surjective.*

On substituting the expression (3.39) into the left-hand side of Eq. (3.40), we obtain the residual

$$\varepsilon\left[\varphi_n(t)\right]=\sum_{j=1}^{\kappa_n} c_j\phi_j(t)-f(t)-\int_G k\left(t,x,\sum_{j=1}^{\kappa_n} c_j\phi_j(x)\right)dx,\quad i=1,\ldots,\kappa_n$$

according to the collocation method, we require that the residual $\varepsilon\left[\varphi_n(t)\right]$ be zero at the given system of the collocation points $t_1,t_2,\ldots,t_n$ on the interval $G=[a,b]$, i.e., we set

$$\varepsilon\left[\varphi_n(t_i)\right]=0,\quad i=1,2,\ldots,n.$$

We obtain the following system for the coefficients $\{c_j\}$ :

$$\sum_{j=1}^{\kappa_n} c_j\phi_j(t_i)-f(t_i)-\int_G k\left(t_i,x,\sum_{j=1}^{\kappa_n} c_j\phi_j(x)\right)dx=0,\quad i=1,\ldots,\kappa_n$$

this is a nontrivial system to solve, and usually some variant of Newton's method is used to find an approximating solution. For more details, see [33, 6, 24, 34] and [5].

### 3.5.2 Galerkin method

The solution of equation (3.38) may be considered an approximate equation of the Urysohn nonlinear equation (3.2) , where $P_n$ is the unique orthogonal projection from a *Hilbert space* containing $X$ into $X_n$, in this case the equation (3.38) is called the Galerkin equation.

Applying the Galerkin method to the equation (3.2), let $\left\{\phi_1,\phi_2,\ldots,\phi_{\kappa_n}\right\}$ be a basis of $X_n$. Assume

$$\varphi_n(t)=\sum_{j=1}^{\kappa_n} c_j\phi_j(t),\quad t\in G.$$

Then equation (3.38) is equivalent to solving for $\{c_j\}$ in the nonlinear system

$$\sum_{j=1}^{\kappa_n} c_j\left\langle\phi_j,\phi_i\right\rangle=\left\langle\int_G k\left(t,x,\sum_{j=1}^{\kappa_n} c_j\phi_j(x)\right)dx,\phi_i\right\rangle,\quad i=1,\ldots,\kappa_n \tag{3.41}$$

where $\langle .,.\rangle$ denotes the usual $L^2(G)$ inner product, that is

$$\langle\varphi,\phi\rangle=\int_G\varphi(t)\,\phi(t)\,dt,\quad \varphi,\phi\in L^2(G)\,.$$

To solve (3.41), we need the numerical integration scheme.

### 3.5.3 Iterated projection method

The iterated projection[10] solution is defined by

$$\tilde{\varphi}_n = T(\varphi_n), \tag{3.42}$$

where $X_n$ is a Hilbert space and $P$ is an orthogonal projection, the sequence $\{\tilde{\varphi}_n\}$ always converges more rapidly then does $\{\varphi_n\}$, from (3.42), it is immediate that

$$P\tilde{\varphi}_n = \varphi_n,$$

and hence $\tilde{\varphi}_n$ satisfies

$$\tilde{\varphi}_n = T(P\tilde{\varphi}_n).$$

It is known that the orders of convergence for Galerkin and collocation solutions are $O(h^r)$ and for the iterated Galerkin and iterated collocation solutions are $O(h^{2r})$, such that, $X_n$ is a space of piecewise polynomial functions of degree $\leq r$ (see, for more informations [34]).

### 3.5.4 The convergence of the projection method

There are two major approaches to the error analysis of (3.38), see [5]:

Linearize the problem and apply the Banach fixed point theorem: The linearization procedure is a commonly used approach for the convergence analysis of approximation methods for solving (3.38). This generally means we replace the nonlinear function by a linear Taylor series approximation

$$T(\varphi) \approx T(\varphi_0) + T'(\varphi_0)(\varphi - \varphi_0)$$

We begin the linearization process by discussing the error in the linearization of $T(\varphi)$ about a point $\varphi_0$

$$R(\varphi; \varphi_0) \equiv T(\varphi) - [T(\varphi_0) + T'(\varphi_0)(\varphi - \varphi_0)]$$

**Lemma 3.19.** *Let $X$ be a Banach space, and let $H$ be an open subset of $X$. Let $T : H \subset X \longrightarrow X$ be twice continuously differentiable with $T''(\varphi)$ bounded over any bounded subset of $H$. Let $B \subset H$ be a closed, bounded, and convex set with a non-empty interior. Let $\varphi_0$ belong to the interior of $B$, and define $R(\varphi; \varphi_0)$, as above. Then for all $\varphi_1, \varphi_2 \in B$,*

$$\|R(\varphi_1; \varphi_2)\| \leq \frac{1}{2} M \|\varphi_1 - \varphi_2\|^2,$$

[10] This method was first introduced by Ian Sloan (1976) for linear integral equation, (For more information, see [7]).

*with* $M = \sup_{\varphi \in B} \|T''(\varphi)\|$. *Moreover*

$$\|T'(\varphi_2) - T'(\varphi_1)\| \leq M \|\varphi_2 - \varphi_1\|,$$

*implying* $T'(\varphi)$ *is Lipschitz continuous; and*

$$\|R(\varphi_1; \varphi_0) - R(\varphi_2; \varphi_0)\| \leq M \left[ \|\varphi_1 - \varphi_0\| - \frac{1}{2} \|\varphi_1 - \varphi_2\| \right] \|\varphi_1 - \varphi_2\|.$$

*Proof.* [5] □

**Proposition 3.20.** *[5] Let* $X$ *be a Banach space and let* $H \subset X$ *be an open set. Assume* $T : H \longrightarrow X$ *is a completely continuous operator which is differentiable at* $\varphi_0 \in H$. *Then* $T'(\varphi_0)$ *is a compact operator from* $X$ *to* $X$.

As a consequence, we can apply the Fredholm alternative theorem to the operator $I - T'(\varphi_0)$. Assume (3.2) has an isolated solution $\varphi^* \in H$, and assume it is unique within the ball $B(\varphi^*, \epsilon) = \{\varphi | \|\varphi - \varphi^*\| \leq \epsilon\}$ for some $\epsilon > 0$ and with $B(\varphi^*, \epsilon) \subset H$. We assume $T$ is twice continuously differentiable over $H$.

**Lemma 3.21.** *[5] Let* $X$ *be a Banach space, and let* $\{P_n\}$ *be a family of bounded projections on* $X$ *with*

$$P_n \varphi \longrightarrow \varphi \quad as \quad n \longrightarrow \infty, \ \varphi \in X.$$

*If* $L : X \longrightarrow X$ *is compact operator, then*

$$\|L - P_n L\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

*Proof.* [5] □

Then from last proposition and lemma $\|(I - P_n) L\| \longrightarrow 0$ as $n \longrightarrow \infty$, $L = T'(\varphi^*)$.

**Theorem 3.22.** *[5] Assume* $K : X \longrightarrow X$ *is bounded, with* $X$ *a Banach space; and assume* $\lambda - K : X \xrightarrow[onto]{1-1} X$. *Further assume*

$$\|K - P_n K\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

*Then for all sufficiently large* $n$, *say* $n \geq N$, *the operator* $(\lambda - P_n K)^{-1}$ *exists as a bounded operator from* $X$ *to* $X$.

*Proof.* [5]. □

From above theorem, $(I - P_n K)^{-1}$ exists for all sufficiently large $n$ and is uniformly bounded with respect to all such $n$.

Now, we want to show that for all sufficiently large $n$, (3.38) has a unique solution within $B(\varphi^*, \epsilon)$ for some $0 < \epsilon_1 < \epsilon$.

Equation (3.38) can be rewritten as the equivalent equation

$$(I - P_n K)(\varphi_n - \varphi^*) = P_n \varphi^* - \varphi^* + P_n R(\varphi_n; \varphi^*)$$

Introduce a new unknown $\delta_n = \varphi_n - \varphi^*$, and then write

$$\begin{aligned} \delta_n &= (I - P_n K)^{-1} (P_n \varphi^* - \varphi^*) + (I - P_n K)^{-1} R(\delta_n + \varphi^*; \varphi^*) \\ &= F_n(\delta_n). \end{aligned}$$

We can to show that this fixed point equation has a unique solution $\delta_n$ for all sufficiently large $n$ and apply the Banach contractive mapping theorem (for more details, see [5]). This proves that the approximating equation (3.38) has a unique solution un in some ball of fixed radius about $\varphi^*$.

There are a number of results on the rate of convergence of $\varphi_n$ to $\varphi^*$, and we quote only one of them. With the same hypotheses on $T$ and $\{P_n\}$ as above,

$$\|\varphi^* - \varphi_n\|_X \leq \left\| [I - T'(\varphi^*)]^{-1} \right\| (1 + \gamma_n) \|\varphi^* - P_n \varphi^*\|_X .$$

with $\gamma_n \longrightarrow 0$ as $n \longrightarrow \infty$. A proof of this result is given in [7, Theorem 2.2].

**Lemma 3.23.** *[43]. Let*

$$P_n(\varphi) = \sum_{i=0}^{n} \varphi_i L_i,$$

*denote the truncated Legendre series. The truncation error, denoted as $\varphi - P_n(\varphi)$, can be estimated in a similar fashion to the interpolation error. Therefore for $\varphi \in H^m(-1, 1)$ and $m \geq 0$, the estimate is*

$$\|\varphi - P_n(\varphi)\|_{L^2(-1,1)} \leq C n^{-m} \|\varphi\|_{H^m(-1,1)}$$

For a more detailed analysis of projection method, we refer to [23, 5, 6, 24, 13, 34, 7] and others.

# Chapter 4

# Numerical examples and results

All the numerical methods will be used to see which gives the best solutions and also look at computational time using the *tic toc* functions in MATLAB for different values of iterations, the timings are given in seconds. The test equations that will be used are the examples from NADJAFI et al. [46], HETMANIOK et al. [50] and TANG et al. [25].

## 4.1 Approximation solutions of integral equations by using the Nyström method

Consider the nonlinear Urysohn integral equation of the second kind

$$\varphi(t) = f(t) + \int_G k(t, x, \varphi(x))dx, \ t, x \in G. \tag{4.1}$$

For Fredholm integral equations, we take $G = [a, b]$, let $t_1 = a < t_2 < \cdots < t_n < t_{n+1} = b$ be an equidistant subdivision of a step $h = \dfrac{b-a}{n+1} = t_{j+1} - t_j$, for $1 \leq j \leq n+1$. Our objective, it's to approximate the solution of the NFIE to the nodes $t_j = a + (j-1)h$ by Nyström methods.

Now, we approximate the integral on the right-hand side of the equation (4.1) with $j = 1, \cdots, n+1$

$$\varphi(t_j) = f(t_j) + \int_{t_1}^{t_{n+1}} k(t_j, x, \varphi(x))dx, \quad a \le t_j \le b$$
$$= f(t_j) + \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} k(t_j, x, \varphi(x))dx,$$

then

$$\varphi(t_j) = f(t_j) + \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} k(t_j, x, \varphi(x))dx.$$

By the numerical integration formulas of trapezoidal rule (3.21), so we get

$$\varphi(t_j) = f(t_j) + \frac{h}{2}\left(k(t_j, t_1, \varphi(t_1)) + 2\sum_{i=2}^{n} k(t_j, t_i, \varphi(t_i)) + k(t_j, t_{n+1}, \varphi(t_{n+1}))\right), \tag{4.2}$$

we take $f(t_j) = f_j$ and $k(t_j, t_i, \varphi(t_i)) = k_{j,i,\varphi_i}$, then the equation (4.2) becomes

$$\varphi_j = f_j + \frac{h}{2}\left(k_{j,1,\varphi_1} + 2\sum_{i=2}^{n} k_{j,i,\varphi_i} + k_{j,n+1,\varphi_{n+1}}\right). \tag{4.3}$$

The evaluation of (4.3) on the $t_j$ gives a nonlinear system of algebraic equations of the form

$$\varphi = T\varphi, \quad j = 1, ..., n+1$$

where $\varphi = \left(\varphi_1, ..., \varphi_{n+1}\right)^t$, then we can use the method of fixed point for the resolution the system of nonlinear equations, then we passe to the following diagram of recurrence

$$\varphi^{K+1} = T\varphi^K \Longleftrightarrow \varphi_j^{K+1} = f_j + \frac{h}{2}\left(k_{j,1,\varphi_1^K} + 2\sum_{i=2}^{n} k_{j,i,\varphi_i^K} + k_{j,n+1,\varphi_{n+1}^K}\right). \tag{4.4}$$

If we apply the method of Simpson; we will obtain another discretization compared to the variable of integration $t$ as follows

$$\varphi_j = f_j + \frac{h}{3}\left(k_{j,1,\varphi_1} + 2\sum_{i=1}^{\frac{n}{2}} \left[2k_{j,2i-1,\varphi_{2i-1}} + k_{j,2i,\varphi_{2i}}\right] + k_{j,n+1,\varphi_{n+1}}\right),$$

thus, this formula is a nonlinear system of $n+1$ equations with $n+1$ unknowns $\varphi_1, ..., \varphi_{n+1}$ can also to write in the form

$$\varphi = T\varphi, \quad j = 1, ..., n+1$$

with $\varphi = \left(\varphi_1, ..., \varphi_{n+1}\right)^t$, then we can use the method of fixed point for the resolution the system of nonlinear equations, then we passe to the following diagram of recurrence

$$\varphi^{K+1} = T\varphi^K \Longleftrightarrow \varphi_j^{K+1} = f_j + \frac{h}{3}\left(k_{j,1,\varphi_1^K} + 2\sum_{i=1}^{\frac{n}{2}}\left(2k_{j,2i-1,\varphi_{2i-1}^K} + k_{j,2i,\varphi_{2i}^K}\right) + k_{j,n+1,\varphi_{n+1}^K}\right). \tag{4.5}$$

For (4.4) and (4.5), $K$ is the number of iteration and we can to take the initial value $\varphi_j^0 = f(t_j)$, $j = 1, 2, ..., n+1$. By recurrence, we can to calculate the vector of solutions $\varphi$ in all points $t_j$ for $j = 1, 2, ..., n+1$.

**Example 4.1.** *Consider the nonlinear Fredholm integral equation of the form*

$$\varphi(t) - \frac{1}{2}\int_0^1 tx^2 \sin\left(\varphi(x)\right) dx = t^3 + \frac{t}{6}\left(\cos(1) - 1\right), \quad 0 \leq x, t \leq 1 \tag{4.6}$$

*with the exact solution is given by*

$$\varphi(t) = t^3.$$

*The errors with trapezoidal and Simpson rules are shown in following tables:*

Table 4.1: Comparison of resultants, absolute error, by trapezoidal method for NFIE (4.6) for $K = 5$.

| $t_i$ | Exact solution | Error |
|---|---|---|
| 0.0 | 0.0000 | 0.0000e+000 |
| 0.1 | 0.0010 | 1.5365e-004 |
| 0.2 | 0.0080 | 3.0731e-004 |
| 0.3 | 0.0270 | 4.6096e-004 |
| 0.4 | 0.0640 | 6.1462e-004 |
| 0.5 | 0.1250 | 7.6827e-004 |
| 0.6 | 0.2160 | 9.2193e-004 |
| 0.7 | 0.3430 | 1.0756e-003 |
| 0.8 | 0.5120 | 1.2293e-003 |
| 0.9 | 0.7290 | 1.3829e-003 |
| 1.0 | 1.0000 | 1.5366e-003 |
| C-time (second) | | 0.4 |

**Example 4.2.** *Let be the nonlinear integral equation of Fredholm [46]*

$$\varphi(t) - \frac{1}{5}\int_0^1 \cos(\pi t)\sin\left(\pi x\right)\varphi^3(x)dx = \sin(\pi t), \quad 0 \leq x, t \leq 1 \tag{4.7}$$

Table 4.2: Comparison of resultants, absolute error, by Simpson method for NFIE (4.6) .

| $t_i$ | Exact | $K = 5$ Error | $K = 10$ Error | $K = 100$ Error |
|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0000e+000 | 0.0000e+000 | 0.0000e+000 |
| 0.1 | 0.0010 | 2.1124e-006 | 2.1061e-006 | 2.1061e-006 |
| 0.2 | 0.0080 | 4.2248e-006 | 4.2121e-006 | 4.2121e-006 |
| 0.3 | 0.0270 | 6.3371e-006 | 6.3182e-006 | 6.3182e-006 |
| 0.4 | 0.0640 | 8.4486e-006 | 8.4243e-006 | 8.4243e-006 |
| 0.5 | 0.1250 | 1.0559e-005 | 1.0530e-005 | 1.0530e-005 |
| 0.6 | 0.2160 | 1.2665e-005 | 1.2636e-005 | 1.2636e-005 |
| 0.7 | 0.3430 | 1.4771e-005 | 1.4742e-005 | 1.4742e-005 |
| 0.8 | 0.5120 | 1.6867e-005 | 1.6849e-005 | 1.6849e-005 |
| 0.9 | 0.7290 | 1.8969e-005 | 1.8955e-005 | 1.8955e-005 |
| 1.0 | 1.0000 | 2.1064e-005 | 2.1061e-005 | 2.1061e-005 |
| Computational time | | 0.4 seconds | 0.6 seconds | 5.7 seconds |

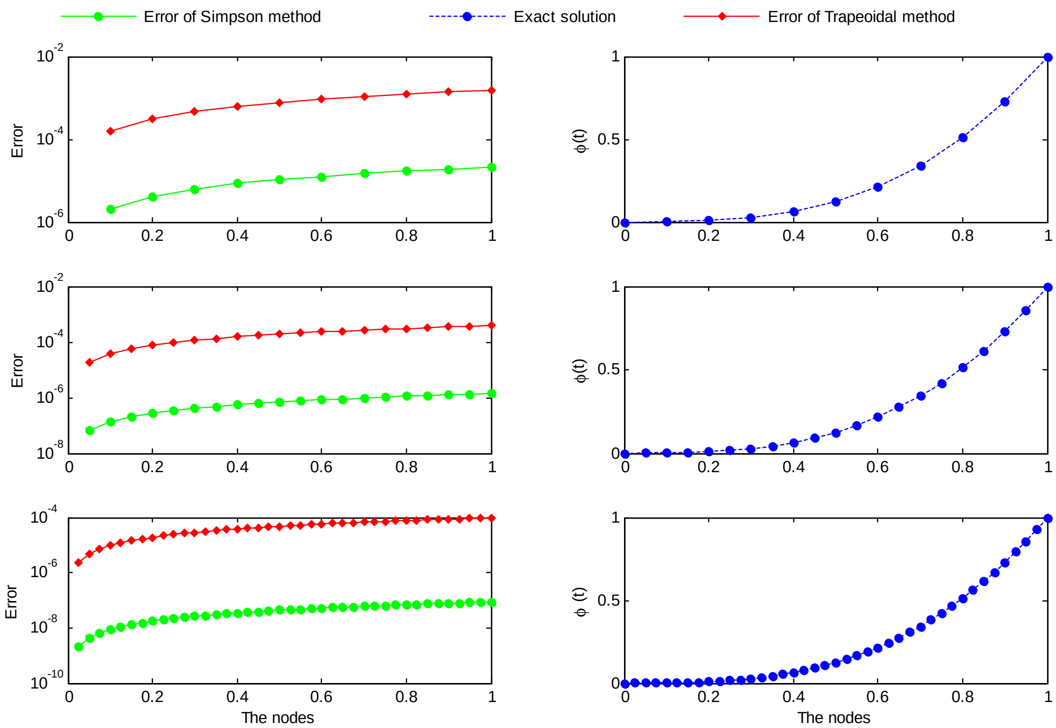


Figure 4.1: Comparison of resultants, absolute error, by trapezoidal and Simpson methods for NFIE (4.6), with $h = 0.1$, $h = 0.05$, $h = 0.025$

*with the exact solution is given by*

$$\varphi(t) = \sin \pi t + \frac{1}{3}\left(20 - \sqrt{391}\right) \cos \pi t.$$

Table 4.3: Comparison of resultants, absolute error, by trapezoidal method for NFIE (4.7), with $h = 0.1,\ K = 15$.

| | | Trapezoidal method | Simpson method |
|---|---|---|---|
| $t_i$ | Exact | Error | Error |
| 0.0 | 0.0754 | 3.1919e-016 | 3.0531e-016 |
| 0.1 | 0.3808 | 3.3307e-016 | 3.3307e-016 |
| 0.2 | 0.6488 | 2.2204e-016 | 2.2204e-16 |
| 0.3 | 0.8534 | 2.2204e-016 | 2.2204e-16 |
| 0.4 | 0.9744 | 1.1102e-016 | 1.1102e-016 |
| 0.5 | 1.0000 | 0.0000e+000 | 0.0000e+000 |
| 0.6 | 0.9277 | 1.1102e-016 | 1.1102e-016 |
| 0.7 | 0.7647 | 2.2204e-016 | 2.2204e-016 |
| 0.8 | 0.5268 | 2.2204e-016 | 2.2204e-016 |
| 0.9 | 0.2373 | 2.7756e-016 | 3.0531e-016 |
| 1.0 | -0.0754 | 3.1919e-016 | 3.0531e-016 |
| C-time (seconds) | | 1 | 1.1 |

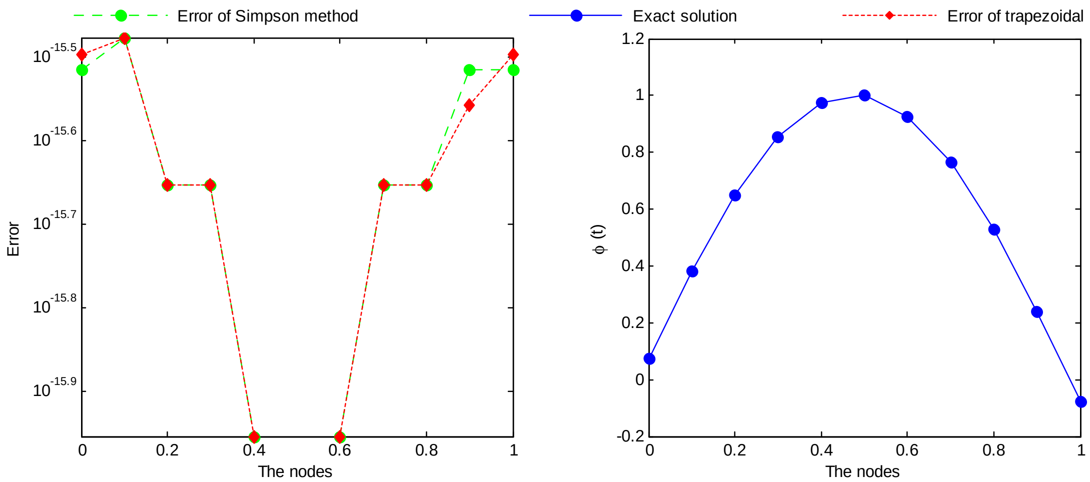


Figure 4.2: Comparison of resultants, absolute error, by trapezoidal method for NFIE (4.7), with $h = 0.1,\ K = 15$.

**Example 4.3.** ***(Pendulum problem)*** *We consider the problem of the pendulum*

$$\begin{cases} \ddot{\theta}(t) + a^2 \sin\theta(t) = z(t), & t \in [0,1] \\ \theta(0) = \theta(1) = 0, \end{cases} \tag{4.8}$$

*where the driving force $z$ is odd. The constant $a$ $\left(a = \frac{g}{l} \neq 0\right)$ depends on the length $l$ of the pendulum and gravity $g$, $\theta$ is the angular displacement and $t$ is the time, see [14, 12]. Since the Green's function of the problem:*

$$\ddot{\theta}(t) = 0, \ \theta(0) = \theta(1) = 0$$

*is given by:*

$$k(t,x) := \begin{cases} t(1-x), & 0 \le t \le x \le 1, \\ x(1-t), & 0 \le x \le t \le 1, \end{cases}$$

*it follows that problem (4.8) is equivalent to the nonlinear integral equation*

$$\theta(t) = -\int_0^1 k(t,x)\left(z(x) - a^2 \sin\theta(x)\right) dx, \quad t \in [0,1].$$

Table 4.4: Approximation solution $\theta$ of pendulum equation (4.8), $\alpha = 2$, $z(t) = \sin(t)$, and $N = 40$, $K = 10$.

| $t$ | Trapezoidal | Simpson |
|---|---|---|
| 0 | 0.010545 | 0.007020 |
| 0.2 | 0.027521 | 0.025244 |
| 0.4 | 0.040933 | 0.039534 |
| 0.6 | 0.045249 | 0.044503 |
| 0.8 | 0.034087 | 0.033877 |
| 1 | -0.000527 | -0.000234 |
| C-time | 10 second | 10.1 second |

Now, for Volterra integral equation we take $G = [a, t]$, this equation of second kind is the following

$$\varphi(t) = f(t) + \int_a^t k(t, x, \varphi(x))dx, \;\; a \leq t \leq b$$

we will use the same subdivision the interval $[a, b]$, and use quadrature rules on each points of subintervals $[t_j,\ t_{j+1}]$. We calculate the solution in the nodes, therefore the nonlinear integral equation of Volterra becomes

$$\begin{aligned}\varphi(t_j) &= f(t_j) + \int_a^{t_j} k(t_j, x, \varphi(x))dx, \quad a \leq t_j \leq b.\\ &= f(t_j) + \sum_{i=1}^{j-1} \int_{t_i}^{t_{i+1}} k(t_j, x, \varphi(x))dx.\end{aligned}$$

By the same last way for NFIE and we use trapezoidal rule, we get

$$\varphi_j = f_j + \frac{h}{2}\left(k_{j,1,\varphi_1} + 2\sum_{i=2}^{j-1} k_{j,i,\varphi_i} + k_{j,j,\varphi_j}\right),$$

with $f(t_j) = f_j$ and $k(t_j, t_i, \varphi(t_i)) = k_{j,i,\varphi i}$. Then

$$\varphi^{K+1} = T\varphi^K \Longleftrightarrow \varphi_j^{K+1} = f_j + \frac{h}{2}\left(k_{j,1,\varphi_1^K} + 2\sum_{i=2}^{j-1} k_{j,i,\varphi_i^K} + k_{j,j,\varphi_j^K}\right),$$

with $K$ the number of iteration and we can to take the initial value $\varphi_j^0 = f(t_j)$, $j = 1, ..., n+1$. By recurrence, we can to calculate the vector of solutions $\varphi$ in all points $t_j$ for $j = 1, ..., n+1$. Now, by using the method of Simpson, we get

$$\varphi_j = f_j + \frac{h}{3}\left(k_{j,1,\varphi_1} + 2\sum_{i=1}^{\frac{j-1}{2}} \left[2k_{j,2i-1,\varphi_{2i-1}} + k_{j,2i,\varphi_{2i}}\right] + k_{j,j,\varphi_j}\right). \tag{4.9}$$

The evaluation of (4.9) on the $t_j$ gives a system of nonlinear equations of the form

$$\varphi^{K+1} = T\varphi^K \Longleftrightarrow \varphi_j^{K+1} = f_j + \frac{h}{3}\left(k_{j,1,\varphi_1^K} + 2\sum_{i=1}^{\frac{j-1}{2}} \left[2k_{j,2i-1,\varphi_{2i-1}^K} + k_{j,2i,\varphi_{2i}^K}\right] + k_{j,j,\varphi_j^K}\right), \tag{4.10}$$

and by recurrence, we can to calculate the vector of solutions $\varphi$ in all points $t_j$ for $j = 1, 2, ..., n+1$, and $K \geq 1$.

**Remark 4.4.** *Simpson's rule can be applied only when $(n+1)$ is even. If $n$ even*

$$\int_{t_1}^{t_{n+1}} f(x)\,dx = \int_{t_1}^{t_3} f(x)\,dx + \int_{t_3}^{t_5} f(x)\,dx + \cdots + \int_{t_n}^{t_{n+1}} f(x)\,dx, \quad a \le t_j \le b,$$
$$= \sum_{i=1}^{n-2} \int_{t_i}^{t_{i+2}} f(x)\,dx + \int_{t_n}^{t_{n+1}} f(x)\,dx.$$

*Then we apply the method of Simpson.*

**Example 4.5.** *Let be the nonlinear integral equation of Volterra*

$$\varphi(t) - \int_0^t e^{(t-x)}\varphi^2(x)dx = \sqrt{t+1} + (t+2) - 2e^t, \quad 0 \le t \le 1 \tag{4.11}$$

*with the exact solution is given by*

$$\varphi(t) = \sqrt{t+1}.$$

Table 4.5: Comparison of resultants, absolute error, for NVIE (4.11), by trapezoidal method.

| | | $h = 0.1$ | | $h = 0.05$ | |
|---|---|---|---|---|---|
| $t_i$ | E solution | A solution | Error | A solution | Error |
| 0 | 1.0000 | 1.0000 | 0.0000e+000 | 1.0000 | 0.0000e+000 |
| 0.1 | 1.0488 | 1.0487 | 9.3049e-005 | 1.0488 | 2.3231e-005 |
| 0.2 | 1.0954 | 1.0952 | 2.1129e-004 | 1.0954 | 5.2724e-005 |
| 0.3 | 1.1402 | 1.1398 | 3.6665e-004 | 1.1401 | 9.1430e-005 |
| 0.4 | 1.1832 | 1.1826 | 5.7662e-004 | 1.1831 | 1.4367e-004 |
| 0.5 | 1.2247 | 1.2239 | 8.6704e-004 | 1.2245 | 2.1583e-004 |
| 0.6 | 1.2649 | 1.2636 | 1.2764e-003 | 1.2646 | 3.1740e-004 |
| 0.7 | 1.3038 | 1.3020 | 1.8623e-003 | 1.3034 | 4.6256e-004 |
| 0.8 | 1.3416 | 1.3389 | 2.7118e-003 | 1.3410 | 6.7269e-004 |
| 0.9 | 1.3784 | 1.3744 | 3.9564e-003 | 1.3774 | 9.8015e-004 |
| 1 | 1.4142 | 1.4084 | 5.7969e-003 | 1.4128 | 1.4342e-003 |
| Computational time | | 1.2 seconds | | 3.9 seconds | |

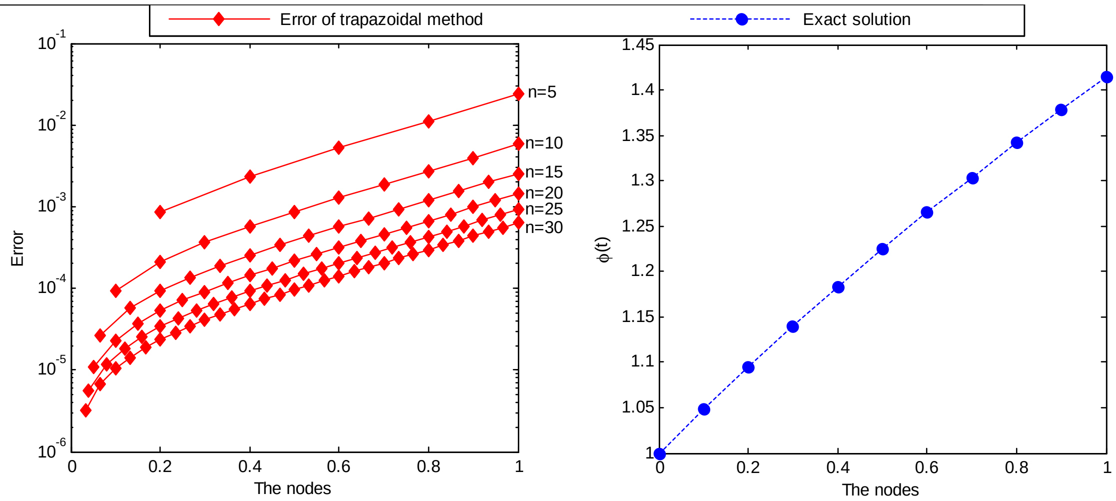


Figure 4.3: Comparison of resultants, absolute error, for NVIE (4.11) by trapezoidal method, with $n = 5, 10, \ldots, 30$.

**Example 4.6.** *The test equation that will be used is the example from Tang et al. [50]*

$$\varphi(t) + \int_{-1}^{t} e^{t-3x}\varphi^2(x)\,dx = f(t)\,, \qquad -1 \le t \le 1 \tag{4.12}$$

*with*

$$f(t) = \frac{-e^t}{2(1+36\pi^2)}\left(e^{-t} + 36\pi^2 e^{-t} - e^{-t}\cos(6\pi t) + 6\pi e^{-t}\sin(6\pi t) - 36e\pi^2\right) + e^t \sin(3\pi t)\,,$$

*and the exact solution is given by* $\varphi(t) = e^t \sin(3\pi t)\,.$

Table 4.6: Comparison of resultants, absolute error, for NVIE (4.12) by trapezoidal method.

| | | $h = 0.1$ | | $h = 0.05$ | |
|---|---|---|---|---|---|
| $t_i$ | E solution | A solution | Error | A solution | Error |
| -1.0 | -0.0000 | -0.0000 | 5.8690e-017 | -0.0000 | 5.8690e-017 |
| -0.8 | -0.4273 | -0.4251 | 2.2469e-003 | -0.4269 | 4.0836e-004 |
| -0.6 | 0.3226 | 0.3125 | 1.0074e-002 | 0.3201 | 2.5201e-003 |
| -0.4 | 0.3940 | 0.4002 | 6.2123e-003 | 0.3954 | 1.3741e-003 |
| -0.2 | -0.7787 | -0.7877 | 9.0070e-003 | -0.7809 | 2.2681e-003 |
| 0.0 | 0.0000 | -0.0070 | 6.9641e-003 | -0.0018 | 1.7989e-003 |
| 0.2 | 1.1616 | 1.1612 | 3.9594e-004 | 1.1614 | 1.9445e-004 |
| 0.4 | -0.8769 | -0.8925 | 1.5671e-002 | -0.8807 | 3.8383e-003 |
| 0.6 | -1.0710 | -1.0747 | 3.6513e-003 | -1.0721 | 1.0688e-003 |
| 0.8 | 2.1166 | 2.0992 | 1.7383e-002 | 2.1122 | 4.3761e-003 |
| 1.0 | 0.0000 | -0.0150 | 1.5044e-002 | -0.0038 | 3.8221e-003 |
| Computational time | | 4.2 seconds | | 14.2 seconds | |

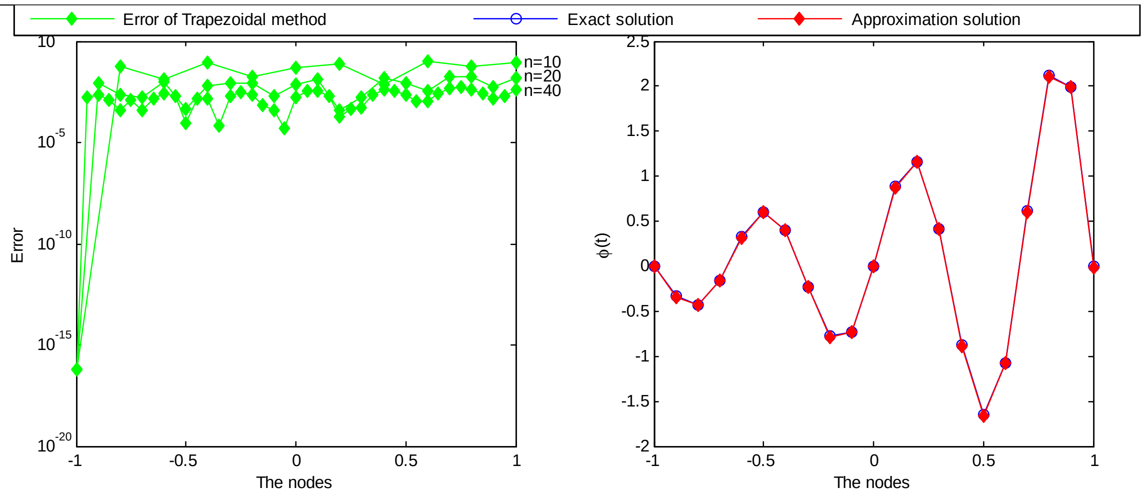


Figure 4.4: Comparison of resultants, absolute error, for NVIE (4.12) by trapezoidal method, with $n = 10$, 20 and 40.

In the previous tables and figures in this section, we compared the solutions obtained by the quadrature methods (trapezoidal and Simpson methods) for NVIEs and NFIEs. Generally, we list pointwise errors with Simpson method for the previous examples. It is found that the convergence order is the best. And it is shown that the amount of computer time for two methods is comparable.

## 4.2 Approximation solutions of integral equations by using the Legendre collocation and Chebyshev collocation methods

Orthogonal polynomials are widely used in applications to a variety of fields in mathematics, mathematical physics, engineering and computer science. The most common set of this kind of polynomials are the set of Trigonometric polynomials, as well as other orthogonal polynomials like Legendre's[1] and Chebyshev's[2], are widely employed in approximation theory.

Let $w = w(x)$ be a weight function on the interval $G = [-1, 1]$. Let us denote by $\{p_k,\ k = 0, 1, \ldots\}$ a system of algebraic polynomials, with $p_k$ of degree equal to $k$ for each

[1]Adrien-Marie Legendre, 1752 – 1833, French/Parisian mathematician.
[2]Pafnuty Lvovich Chebyshev, 1821 – 1894, Russian mathematician.

$k$, mutually orthogonal on the interval $G$ with respect to $w$. This means that

$$\int_{-1}^{1} p_k(x)\, p_m(x)\, w(x)\, dx = 0, \quad \text{if } k \neq m$$

set

$$\langle f, g \rangle_w = \int_{-1}^{1} f(x)\, g(x)\, w(x)\, dx, \quad \text{and} \quad \|f\|_w = \langle f, f \rangle_w^{1/2},$$

$\langle ., . \rangle_w$ and $\|.\|_w$ are respectively the scalar product and the norm for the function space.

**Remark 4.7.** ***(The Jacobi polynomials)*** *The polynomials previously introduced belong to the wider family of Jacobi polynomials* $\left\{J_k^{\alpha\beta}, k = 0, 1, \ldots, n\right\}$, *that are orthogonal with respect to the weight* $w(x) = (1-x)^\alpha (1+x)^\beta$, *for* $\alpha, \beta > -1$. *Indeed, setting* $\alpha = \beta = 0$ *we recover the Legendre polynomials, while choosing* $\alpha = \beta = -1/2$ *gives the Chebyshev polynomials.*

Some of the most useful orthogonal bases for $L^2$ spaces consist of polynomial functions. This section is a brief introduction to the most important of these orthogonal systems of polynomials (see, for more information [4]).

The *Legendre polynomials*, which are written as $L_n(x)$, are an important set of orthogonal functions over the interval $[-1, 1]$, is defined by Rodrigues formula

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n.$$

$L_n$ is a polynomial of degree $n$. And these polynomials satisfy the three-term recurrence relation

$$\begin{cases} L_0(x) = 1, \\ L_1(x) = x, \\ (n+1)\, L_{n+1}(x) = (2n+1)\, x L_n(x) - n L_{n-1}(x), \quad n \geq 1, \end{cases}$$

from which we deduce the special values $L_n(\pm 1) = (\pm 1)^n$.

The *Chebyshev polynomial* of degree $n$ $(n = 0, 1, \ldots)$ on $[-1, 1]$ is defined by formula

$$T_n = \cos\left(n \cos^{-1} x\right).$$

Cleary, $|T_n(x)| \leq 1$ for $x \in [-1, 1]$. $T_n$ are indeed polynomials in $x$. For example,

$$T_0(x) = 1, \quad T_1(x) = x$$

by definition, and using elementary trigonometric identities, we can to obtain the recursion

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \ n \geq 1.$$

Some properties of Chebyshev polynomials are

$$\begin{aligned} |T_n'(x)| &\leq n^2, \\ T_n(\pm 1) &= (\pm 1)^n, \end{aligned}$$

The collocation method shown above has applied to the general form of nonlinear integral equations of the second kind ($t \in G$). However with some alterations, the method can be applied to Fredholm and Volterra integral equations of the second kind.
For the nonlinear Fredholm integral equation of the second kind in the form

$$\varphi(t) - \int_{-1}^{1} k(t, x, \varphi(x))\, dx = f(t), \ t \in [-1, 1]$$

we obtain in the **section 3.5.1**. the residual $\varepsilon[\varphi_n(t_i)]$ of the form

$$\sum_{j=1}^{\kappa_n} c_j L_j(t_i) - f(t_i) - \int_{-1}^{1} k\left(t_i, x, \sum_{j=1}^{\kappa_n} c_j L_j(x)\right) dx = 0, \ i = 1, \ldots, \kappa_n.$$

We use Legendre Gauss integration to compute the integral in the above equation. Denote the set of $(s+1)$ Legendre Gauss-type points, $\{x_p\}_{p=1}^{s}$, and the corresponding weights, $\{w_p\}_{p=1}^{s}$.

$$\int_{-1}^{1} g(x)\, dx = \sum_{p=1}^{s} w_p g(x_p) + R_s(g), \tag{4.13}$$

where the real numbers $w_p$ (called the weights of the formula) are given by

$$w_p = \frac{2}{\left(1 - x_p^2\right)\left[\frac{d}{dx} L_{s+1}(x_p)\right]^2}, \ 1 \leq p \leq s$$

the points $x_p$ (called the nodes of the formula) are the zeros of the Legendre polynomial $L_{s+1}$, and the remainder is $R_s(g)$ is given by

$$R_s(g) = \frac{2^{2s+1}(s!)^4}{(2s+1)\left[(2s)!\right]^3} g^{(2s)}(\zeta), \ \text{for } \zeta \in (-1, 1).$$

The Gauss–Legendre quadrature formula is exact for $g \in L_{2s-1}$, since in such a case the remainder $R_s(g)$ is null.

Using the above numerical integration formula (4.13), we obtain the approximate equation

$$\sum_{j=1}^{\kappa_n} c_j L_j(t_i) - f(t_i) - \sum_{p=1}^{s} w_p k\left(t_i, x_p, \sum_{j=1}^{\kappa_n} c_j L_j(x_p)\right) = 0, \;\; i = 1, \dots, \kappa_n. \tag{4.14}$$

Equation (4.14) can be solved by some methods suitable for solving nonlinear systems. We use the Newton iterative method and initial guess $c_j = 0$.

**Example 4.8.** *Consider the following Fredholm integral equation*

$$\varphi(t) - \frac{1}{60}\int_{-1}^{1} \sin(x+t)\exp(\varphi(x))\,dx = f(t), \quad -1 \le t \le 1 \tag{4.15}$$

*where*

$$f(t) = \frac{1}{60}\left(3\cos(t+1) - \cos(t-1) - \sin(t+1) + \sin(t-1) + 60\log(t+2)\right),$$

*and the exact solution is given by*

$$\varphi(t) = \log(t+2).$$

*Therefore, the approximation solution of this example is given by*

$$\begin{aligned}
\varphi_{10}(t) &= \sum_{j=1}^{10} c_j \phi_j(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_{10}\phi_{10}(t), \\
c_L &= (0.6479, 0.5281, -0.0937, 0.0200, -0.0046, 0.0011, -0.0003, 0, 0, 0)^t \\
c_T &= (0.6171, 0.4909, -0.0698, 0.0148, -0.0026, 0.0005, -0.0001, 0, 0, 0)^t
\end{aligned}$$

*Numerical results are given in following tables.*

Table 4.7: Comparison of resultants, absolute error, for NFIE (4.15) by LC method.

| $\kappa_n = 10$ |  | $K = 2, 3, 4, \ldots$ |  |
|---|---|---|---|
| $t_i$ | Exact solution | Approximation solution | Error |
| 0.9739 | 1.0899 | 1.0899 | 1.1146e-005 |
| -0.1489 | 0.6158 | 0.6158 | 1.7670e-006 |
| -0.8651 | 0.1265 | 0.1266 | 6.2284e-006 |
| -0.6794 | 0.2781 | 0.2781 | 4.3031e-006 |
| -0.4334 | 0.4489 | 0.4489 | 1.5366e-006 |
| 0.1489 | 0.7650 | 0.7650 | 5.0685e-006 |
| 0.8651 | 1.0526 | 1.0526 | 1.0711e-005 |
| 0.4334 | 0.8893 | 0.8893 | 7.8096e-006 |
| 0.6794 | 0.9856 | 0.9856 | 9.6803e-006 |
| -0.9739 | 0.0258 | 0.0258 | 7.2608e-006 |
| Computational time |  | 9.2 seconds |  |

Table 4.8: Comparison of resultants, absolute error, for NFIE (4.15) by TC method.

| $\kappa_n = 10$ |  | $K = 2, 3, 4, \ldots$ |  |
|---|---|---|---|
| $t_i$ | Exact solution | Approximation solution | Error |
| 0.7071 | 0.9959 | 0.9561 | 3.9812e-002 |
| -0.7071 | 0.2569 | 0.2834 | 2.6561e-002 |
| 0.9877 | 1.0945 | 1.0471 | 4.7442e-002 |
| 0.1564 | 0.7684 | 0.7519 | 1.6566e-002 |
| 0.8910 | 1.0616 | 1.0164 | 4.5207e-002 |
| 0.4540 | 0.8977 | 0.8675 | 3.0236e-002 |
| -0.9877 | 0.0122 | 0.0501 | 3.7845e-002 |
| -0.1564 | 0.6117 | 0.6111 | 6.5174e-004 |
| -0.8910 | 0.1035 | 0.1377 | 3.4250e-002 |
| -0.4540 | 0.4357 | 0.4502 | 1.4572e-002 |
| Computational time |  | 8.9 seconds |  |

Now, we consider the nonlinear Volterra integral equation of the second kind

$$\varphi(t) - \int_0^t k(t, x, \varphi(x))\, dx = f(t), \ t \in [0, T] \tag{4.16}$$

we will transfer the equation (4.16) to an equivalent equation defined in $[-1, 1]$. More specifically, we use the change of variables

$$t = \frac{T(1+s)}{2}, \quad s = \frac{2t}{T} - 1.$$

Consider the general case, where we have an integral of a function, $f(t)$ over some interval $[a, b]$ which is not equal to $[-1, 1]$ and we wish to transform the interval to $[-1, 1]$.

$$\int_a^b f(t)\, dt \tag{4.17}$$

Therefore to make the transformation from the integral on the interval $t \in [a, b]$, introduce a new variable $x$, with $x \in [-1, 1]$ and use the linear transforms below to convert $t$ to $x$

$$\begin{aligned} t &= \frac{(b-a)x + b + a}{2}, \\ x &= \frac{2t - b - a}{b - a}, \end{aligned}$$

then

$$\int_{-1}^{1} f\left(\frac{(b-a)x + b + a}{2}\right) \frac{b-a}{2} dx.$$

To rewrite the nonlinear Volterra equation (4.16) as follows

$$u(s) - \int_0^{T(1+s)/2} k\left(\frac{T(1+s)}{2}, x, \varphi(x)\right) dx = g(s), \tag{4.18}$$

where $s \in [-1, 1]$, and

$$u(s) = \varphi\left(\frac{T}{2}(1+s)\right), \quad g(s) = f\left(\frac{T}{2}(1+s)\right)$$

Furthermore, to transfer the integral interval $\left[0, \frac{T}{2}(1+s)\right]$ to the interval $[-1, s]$, we make a linear transformation: $x = \frac{T}{2}(1+\tau), \tau \in [-1, s]$. Then, Eq (4.18) becomes

$$u(s) - \int_{-1}^{s} K(s, \tau, u(\tau))\, d\tau = g(s), \ s \in [-1, 1] \tag{4.19}$$

where

$$K(s, \tau, u(\tau)) = \frac{T}{2} k\left(\frac{T}{2}(1+s), \frac{T}{2}(1+\tau), u(\tau)\right).$$

Set the collocation points as the set of $(N+1)$ Legendre Gauss points, $\{s_i\}_{i=0}^{N}$. Assume that Eq. (4.19) holds at $s_i$

$$u(s_i) - \int_{-1}^{s_i} K(s_i, \tau, u(\tau))\, d\tau = g(s_i), \; 0 \leq i \leq N. \tag{4.20}$$

Now, we transfer the integral interval $[-1, s_i]$ to a fixed interval $[-1, 1]$ and then make use of some appropriate quadrature rule. More precisely, we first make a simple linear transformation:

$$\tau(s, \theta) = \frac{1+s}{2}\theta + \frac{s-1}{2}, \quad -1 \leq \theta \leq 1.$$

Then (4.20) becomes

$$u(s_i) - \frac{1+s_i}{2} \int_{-1}^{1} k(s_i, \tau(s_i, \theta), u(\tau(s_i, \theta)))\, d\theta = g(s_i), \; 0 \leq i \leq N.$$

For more details, see [50]. Using a $(N+1)$-point Gauss quadrature formula relative to the Legendre weights $\{\omega_p\}$ gives

$$u(s_i) - \frac{1+s_i}{2} \sum_{p=0}^{N} \omega_p k(s_i, \tau(s_i, \theta), u(\tau(s_i, \theta))) = g(s_i), \; 0 \leq i \leq N$$

where the set $\{\theta_i\}_{i=0}^{N}$ coincide with the collocation points $\{s_i\}_{i=0}^{N}$.

Using the above numerical integration formula (4.13), we obtain the approximate equation

$$\sum_{j=1}^{\kappa_n} c_j L_j(s_i) - \frac{1+s_i}{2} \sum_{p=0}^{N} \omega_p k\left(s_i, \tau(s_i, \theta_p), \sum_{j=1}^{\kappa_n} c_j L_j(\tau(s_i, \theta_p))\right) = g(s_i). \; 0 \leq i \leq N \tag{4.21}$$

Equation (4.21) can be solved by some methods suitable for solving nonlinear systems.

Although projection methods are widely used, there are severe practical problems in using them to solve nonlinear integral equation. In recent paper the authors discuss modifications of these methods, to make them into more efficient and practical methods. In [34] the authors discussed the convergence of the approximate solutions of Hammerstein integral equation by the Legendre spectral Galerkin and Legendre spectral collocation methods and they obtained the rates of convergence.

**Example 4.9.** *[34] Let us consider the following Hammerstein integral equation*

$$\varphi(t) - \int_{-1}^{1} \frac{3\sqrt{2}\pi}{16} \cos\left(\frac{\pi\,|x-t|}{4}\right) \varphi^2(x)dx = \frac{-1}{4} \cos\left(\frac{\pi t}{4}\right), \quad -1 \leq t \leq 1 \tag{4.22}$$

*where the exact solution is given by*

$$\varphi(t) = \cos\left(\frac{\pi t}{4}\right).$$

Table 4.9: Absolute error for the Eq. (4.22), $\varphi_n$ be the Legendre Galerkin or LC approximations of $\varphi$ (see, [34]).

| n | $\left\|\varphi - \varphi_n^C\right\|_{L^2}$ | $\left\|\varphi - \varphi_n^C\right\|_{\infty}$ | $\left\|\varphi - \varphi_n^G\right\|_{L^2}$ | $\left\|\varphi - \varphi_n^G\right\|_{\infty}$ |
|---|---|---|---|---|
| 2 | 0.23728226e-002 | 0.63183241e-002 | 0.16612096e-002 | 0.34084469e-002 |
| 4 | 0.20524588e-002 | 0.22438193e-002 | 0.20439952e-002 | 0.22166056e-002 |
| 5 | 0.86760263e-005 | 0.20971481e-004 | 0.86759925e-005 | 0.21021360e-004 |
| 7 | 0.24104407e-007 | 0.64465292e-007 | 0.24104371e-007 | 0.64562011e-007 |
| 8 | 0.58748349e-100 | 0.22963709e-009 | 0.41715829e-010 | 0.11321033e-009 |

## 4.3 Approximation solutions of integral equations by using the Newton-Kantorovich method

In this section we will apply a method, which is a combination of the Newton-Kantorovich and modified Simpson method for obtaining approximate solution of the nonlinear Urysohn integral equations, our results in this section have been published in [38] with NADIR, and are presented as follows:

The modified Simpson method, it is an adaptation of the quadrature formula of Simpson, see [40]. If consider $G = [a, b]$, and let $t_0 = a < t_1 < \cdots < t_{2j-1} < t_{2j} < \cdots < t_{2n} = b$ be an equidistant subdivision of a step $h = t_{2j+1} - t_{2j}$ for $j = 0, 1, \ldots, n$. Our objective then, it's to approximate the solution of the integral to the nodes of even indices (at the point $t_{2j}$), then the modified Simpson have the form

$$\int_{t_{2j}}^{t_{2j+2}} f(t)dt \simeq \frac{t_{2j+1} - t_{2j}}{3}\left[f(t_{_{2j}}) + 4f(t_{_{2j+1}}) + f(t_{_{2j+2}})\right]. \tag{4.23}$$

Now, for Volterra integral equations we take $G = [a, t]$ and approximate the two integrals on the right-hand side of the second equation of the algorithms (3.35) with $j = 0, 1, \ldots, n$

$$\begin{aligned}\phi(t_{2j}) &= f(t_{2j}) - \varphi(t_{2j}) + \int_a^{t_{2j}} k(t_{2j}, x, \varphi(x))dx + \int_a^{t_{2j}} \frac{\partial k}{\partial \varphi}(t_{2j}, x, \varphi(x))\phi(x)dx \\ &= f(t_{2j}) - \varphi(t_{2j}) + \sum_{i=0}^{j-1} \int_{t_{2i}}^{t_{2i+2}} \left( k(t_{2j}, x, \varphi(x)) + \frac{\partial k}{\partial \varphi}(t_{2j}, x, \varphi(x))\phi(x) \right) dx\end{aligned}$$

by the numerical integration formulas of modified Simpson method (4.23), so we get

$$\begin{aligned}\phi_{2j} &= f_{2j} - \varphi_{2j} + \sum_{i=0}^{j-1} \frac{h}{3} \left[k_{2j,2i,2i} + 4k_{2j,2i+1,2i+1} + k_{2j,2i+2,2i+2}\right] \\ &+ \sum_{i=0}^{j-1} \frac{h}{3} \left[k'_{2j,2i,2i}\phi_{2i} + 4k'_{2j,2i+1,2i+1}\phi_{2i+1} + k'_{2j,2i+2,2i+2}\phi_{2i+2}\right], \qquad (4.24)\end{aligned}$$

where $\phi(t_{2j}) = \phi_{2j}, f(t_{2j}) = f_{2j}, \varphi(t_{2j}) = \varphi_{2j}, k(t_{2j}, x_{2i}, \varphi(x_{2i})) = k_{2j,2i,2i}$, and

$$\frac{\partial k}{\partial \varphi}(t_{2j}, x_{2i}, \varphi(x_{2i}))\phi(x_{2i}) = k'_{2j,2i,2i}\phi_{2i}.$$

Since $h$ sufficiently small, we approximate $\phi_{2i+1}$ by $\dfrac{\phi_{2i} + \phi_{2i+2}}{2}$, the equation (4.24) becomes

$$\begin{aligned}\phi_{2j} &= g_{2j} - \varphi_{2j} + \sum_{i=0}^{j-1} \frac{h}{3} \left[k'_{2j,2i,2i}\phi_{2i} + 4k'_{2j,2i+1,2i+1}\phi_{2i+1} + k'_{2j,2i+2,2i+2}\phi_{2i+2}\right] \\ &= g_{2j} - \varphi_{2j} + \sum_{i=0}^{j-1} \frac{h}{3} \left[k'_{2j,2i,2i}\phi_{2i} + 4k'_{2j,2i+1,2i+1}\left(\tfrac{\phi_{2i}+\phi_{2i+2}}{2}\right) + k'_{2j,2i+2,2i+2}\phi_{2i+2}\right] \\ &= g_{2j} - \varphi_{2j} + \frac{h}{3}\left[2k'_{2j,2j-1,2j-1} + k'_{2j,2j,2j}\right]\phi_{2j} + \frac{h}{3}\left[2k'_{2j,0,0} + k'_{2j,1,1}\right]\phi_0 \\ &+ \frac{2h}{3} \sum_{i=1}^{j-1} \left[k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i} + k'_{2j,2i+1,2i+1}\right]\phi_{2i}\end{aligned}$$

Finally

$$\begin{aligned}\left[1 - \frac{h}{3}\left[2k'_{2j,2j-1,2j-1} + k'_{2j,2j,2j}\right]\right]\phi_{2j} &= g_{2j} - \varphi_{2j} + \frac{h}{3}\left[2k'_{2j,0,0} + k'_{2j,1,1}\right]\phi_0 \\ &+ \frac{2h}{3}\sum_{i=1}^{j-1}\left[k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i} + k'_{2j,2i+1,2i+1}\right]\phi_{2i} \qquad (4.25)\end{aligned}$$

with

$$g_{2j} = f_{2j} + \sum_{i=1}^{j-1} \frac{h}{3}\left(k_{2j,2i,2i} + 4k_{2j,2i+1,2i+1} + k_{2j,2i+2,2i+2}\right)$$

The evaluation of (4.25) on the $t_{2j}$ gives a system of algebraic equations of the form

$$\left(I - \frac{h}{3}A\right)\Phi = B,$$

where the vectors $\Phi, A$ and $B$ respectively defined the components of $\phi_{2j}, 2k'_{2j,2j-1,2j-1} + k'_{2j,2j,2j}$, and $g_{2j} - \varphi_{2j} + \frac{2h}{3}\sum_{i=1}^{j-1}\left[k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i} + k'_{2j,2i+1,2i+1}\right]\phi_{2i}$
$+\frac{h}{3}\left[2k'_{2j,0,0} + k'_{2j,1,1}\right]\phi_0$, for $j = 0, 1, \ldots, n$ .
And the initial approximations $\Phi^{(0)} = Y^{(0)} = F$ , and $F$ defined the components of $f_{2j}$

$$\begin{cases} Y^{(K+1)} = Y^{(K)} + \Phi^{(K)} \\ \left(I - \frac{h}{3}A\right)\Phi^{(K+1)} = B, \end{cases} \quad K = 0, 1, 2, \ldots$$

and by recurrence, we can to calculate the vector of solutions $Y$ in all points $t_{2j}$ for $j = 0, 1, \ldots, n$.

Now, for Fredholm integral equations we take $G = [a, b]$ and approximate the two integrals on the right-hand side of the second equation of the algorithms (3.35)

$$\begin{aligned} \phi(t_{2j}) &= f(t_{2j}) + \int_a^b k(t_{2j}, x, \varphi(x))dx - \varphi(t_{2j}) + \int_a^b \frac{\partial k}{\partial \varphi}(t_{2j}, x, \varphi_{n-1}(x))\phi(x)dx \\ &= f(t_{2j}) - \varphi(t_{2j}) + \sum_{i=0}^{n-1}\int_{t_{2i}}^{t_{2i+2}}\left(k(t_{2j}, x, \varphi(x)) + \frac{\partial k}{\partial \varphi}(t_{2j}, x, \varphi(x))\phi(x)\right)dx \end{aligned}$$

by the same last method for Volterra, we get

$$\begin{aligned} \phi_{2j} &= G_{2j} - \varphi_{2j} + \sum_{i=0}^{n-1}\frac{h}{3}\left[k'_{2j,2i,2i}\phi_{2i} + 4k'_{2j,2i+1,2i+1}\phi_{2i+1} + k'_{2j,2i+2,2i+2}\phi_{2i+2}\right] \\ &= G_{2j} - \varphi_{2j} + \frac{h}{3}\left[2k'_{2j,2n-1,2n-1} + k'_{2j,2n,2n}\right]\phi_{2n} + \frac{h}{3}\left[2k'_{2j,0,0} + k'_{2j,1,1}\right]\phi_0 \\ &\quad + \frac{2h}{3}\sum_{i=1}^{n-1}\left[k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i} + k'_{2j,2i+1,2i+1}\right]\phi_{2i} \end{aligned}$$

Finally

$$\begin{aligned} \phi_{2j} &= \frac{h}{3}\left[2k'_{2j,2n-1,2n-1} + k'_{2j,2n,2n}\right]\phi_{2n} + \frac{h}{3}\left[2k'_{2j,0,0} + k'_{2j,1,1}\right]\phi_0 \\ &\quad + \frac{2h}{3}\sum_{i=1}^{n-1}\left[k'_{2j,2i-1,2i-1} + k'_{2j,2i,2i} + k'_{2j,2i+1,2i+1}\right]\phi_{2i} + G_{2j} - \varphi_{2j} \end{aligned} \tag{4.26}$$

with

$$G_{2j} = f_{2j} + \sum_{i=1}^{n} \frac{h}{3} \left[ k_{2j,2i,2i} + 4k_{2j,2i+1,2i+1} + k_{2j,2i+2,2i+2} \right]$$

The evaluation of (4.26) on the $t_{2j}$ gives a system of algebraic equations of the form

$$\Phi = D,$$

where the vectors $\Phi$ and $D$ respectively defined the components of $\phi_{2j}$ and the right-hand side of the equation (4.26) , for $j = 0, 1, \ldots, n$ .
And the initial approximations $Y^{(0)} = 0, \quad \Phi^{(0)} = F - Y^{(0)}$ , and $F$ defined the components of $f_{2j}$

$$\begin{cases} Y^{(K+1)} = Y^{(K)} + \Phi^{(K)} \\ \Phi^{(K+1)} = D, \end{cases} \quad K = 0, 1, 2, \ldots$$

and by recurrence, we can to calculate the vector of solutions $Y$ in all points $t_{2j}$ for $j = 0, 1, \ldots, n$.

**Example 4.10.** ***(Chemical reactor model)*** *In this example we will study the mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction (see, the* ***section 2.3.****). The problem is characterized by a Hammerstein nonlinear integral equation of the second kind (see, [32, 12])*

$$\varphi(t) = \mu \int_0^1 k(t,x) (\beta - \varphi(x)) e^{\varphi(x)} dx, \ 0 \le t \le 1, \quad (4.27)$$

*where*

$$k(t,x) = \begin{cases} e^{\lambda(t-x)} & \text{if } 0 \le t \le x, \\ 1 & \text{if } x \le t \le 1, \end{cases}$$

*which we consider in the space $C[0,1]$ of continuous functions on the closed interval $[0,1]$ with the usual sup norm. Throughout, we assume $\lambda = 10$, $\mu = 0.02$ and $\beta = 3$. To validate the application of pervious methods in the chapter four to the Hammerstein integral equation (4.27), we compare our numerical results with results in [32] by using Adomian's method, and we take $\varphi(0) = 0,0060483735$.*

Table 4.10: Computational results of an adiabatic tubular chemical reactor problem (4.27), for $N = 40$, $K = 10$, by using some numerical methods.

| $t$ | [32] | Trapezoidal | Simpson | NKSM |
|---|---|---|---|---|
| 0 | 0.006048 | 0.006080 | 0.007057 | 0.008706 |
| 0.2 | 0.018192 | 0.018225 | 0.019210 | 0.026177 |
| 0.4 | 0.030424 | 0.030457 | 0.031451 | 0.043721 |
| 0.6 | 0.042669 | 0.042701 | 0.043704 | 0.061236 |
| 0.8 | 0.054371 | 0.054401 | 0.055415 | 0.077934 |
| 1 | 0.061458 | 0.061460 | 0.062507 | 0.088035 |
| C-time | - | 9.7 | 9.9 | 10.1 |

**Example 4.11.** *Consider the nonlinear Volterra integral equation [46] of the form*

$$\varphi(t) = \sin(t) - \frac{t}{2} + \frac{1}{4}\sin(2t) + \int_0^t \varphi^2(x)dx, \; 0 \leq t \leq 1 \tag{4.28}$$

*with the exact solution* $\varphi(t) = \sin(t)$.

Table 4.11: Comparison of results to the NVIE (4.28), for $K = 1$, $h = 0.05$, by NKSM and NK methods.

| Nodes $t_{2j}$ | E solutions | A solutions | Error of NKSM | Error of NK [46] |
|---|---|---|---|---|
| 0 | 0.0000 | 0.0000 | 0.0000e+000 | 0.0000e+000 |
| 0.1 | 0.0998 | 0.0995 | 2.9093e-004 | 3.3267e-004 |
| 0.2 | 0.1987 | 0.1971 | 1.5187e-003 | 2.6454e-003 |
| 0.3 | 0.2955 | 0.2919 | 3.6070e-003 | 8.8394e-003 |
| 0.4 | 0.3894 | 0.3832 | 6.2029e-003 | 2.0661e-002 |
| 0.5 | 0.4794 | 0.4709 | 8.5499e-003 | 3.9632e-002 |
| 0.6 | 0.5646 | 0.5554 | 9.2803e-003 | 6.6990e-002 |
| 0.7 | 0.6442 | 0.6381 | 6.1212e-003 | 1.0364e-001 |
| 0.8 | 0.7174 | 0.7219 | 4.5705e-003 | 1.5011e-001 |
| 0.9 | 0.7833 | 0.8120 | 2.8630e-002 | 2.0654e-001 |
| 1 | 0.8415 | 0.9172 | 7.5679e-002 | 2.7268e-001 |
| Computational time | | | 0.4 second | - |

Table 4.12: Comparison of results to the NVIE (4.28), for $K = 4$, $h = 0.05$, by NKSM and NK methods.

| Nodes $t_{2j}$ | E solutions | A solutions | Error of NKSM | Error of NK [46] |
|---|---|---|---|---|
| 0 | 0.0000 | 0.0000e | 0.0000e+000 | 0.0000e+000 |
| 0.1 | 0.0998 | 0.0995 | 2.9110e-004 | 3.3267e-004 |
| 0.2 | 0.1987 | 0.1971 | 1.5343e-003 | 1.5351e-003 |
| 0.3 | 0.2955 | 0.2918 | 3.7722e-003 | 6.6506e-003 |
| 0.4 | 0.3894 | 0.3824 | 7.0249e-003 | 1.3901e-002 |
| 0.5 | 0.4794 | 0.4681 | 1.1333e-002 | 2.4222e-002 |
| 0.6 | 0.5646 | 0.5479 | 1.6761e-002 | 4.6982e-002 |
| 0.7 | 0.6442 | 0.6208 | 2.3407e-002 | 6.8954e-002 |
| 0.8 | 0.7174 | 0.6859 | 3.1439e-002 | 9.5916e-002 |
| 0.9 | 0.7833 | 0.7422 | 4.1167e-002 | 1.4381e-001 |
| 1 | 0.8415 | 0.7882 | 5.3262e-002 | 1.8539e-001 |
| Computational time | | | 1.7 second | - |

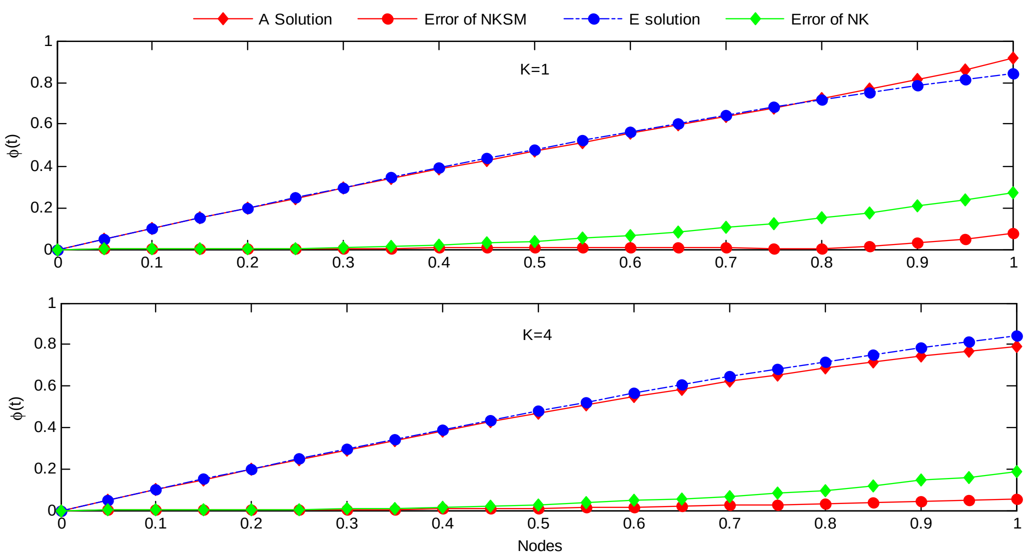


Figure 4.5: Comparison of resultants to the NVIE (4.28), for $K = 1$, and $K = 4$, $h = 0.05$, by NKSM and NK methods.

Table 4.13: Comparison of results to the NFIE (4.7), for $K = 1$, $h = 0.05$, by NKSM and NK methods.

| Nodes $t_{2j}$ | E solutions | A solutions | Error of NKSM | Error of NK [46] |
|---|---|---|---|---|
| 0 | 0.0754 | 0.0000 | 7.5427e-002 | 7.5427e-002 |
| 0.2 | 0.6488 | 0.5898 | 5.9051e-002 | 6.1021e-002 |
| 0.4 | 0.9744 | 0.9596 | 1.4732e-002 | 2.3308e-002 |
| 0.6 | 0.9277 | 0.9303 | 2.5496e-003 | 2.3308e-002 |
| 0.8 | 0.5268 | 0.5097 | 1.7061e-002 | 6.1021e-002 |
| 1 | -0.0754 | -0.1017 | 2.6292e-002 | 7.5427e-002 |
| Computational time | | | 0.5 second | - |

Table 4.14: Comparison of results to the NFIE (4.7), for $K = 4$, $h = 0.05$, by NKSM and NK methods.

| Nodes $t_{2j}$ | E solutions | A solutions | Error of NKSM | Error of NK [46] |
|---|---|---|---|---|
| 0 | 0.0754 | 0.1043 | 2.8855e-002 | 4.9832e-002 |
| 0.2 | 0.6488 | 0.6722 | 2.3344e-002 | 4.0315e-002 |
| 0.4 | 0.9744 | 0.9833 | 8.9168e-003 | 1.5399e-002 |
| 0.6 | 0.9277 | 0.9188 | 8.9168e-003 | 1.5399e-002 |
| 0.8 | 0.5268 | 0.5034 | 2.3344e-002 | 4.0315e-002 |
| 1 | -0.0754 | -0.1043 | 2.8855e-002 | 4.9832e-002 |
| Computational time | | | 1.9 second | - |

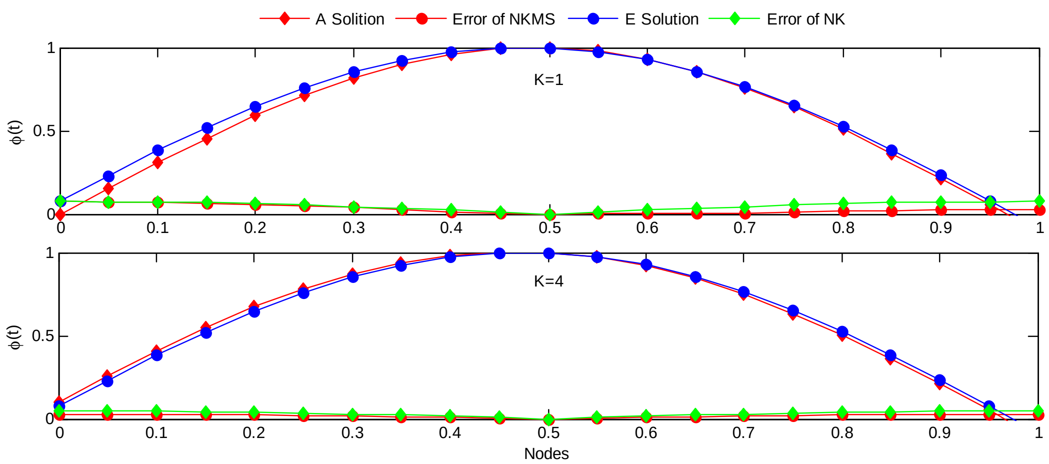


Figure 4.6: Comparison of results to the NFIE (4.7), for $K = 1$, and $K = 4$, $h = 0.05$, by NKSM and NK methods.

## 4.4 Comparison of results

Table 4.15: Absolute errors to the NVIE (4.12). The number of points, $N$, is the total number of discretization points.

| $N$ | Trapezoidal | NKSM | Collocation [50] |
|---|---|---|---|
| 8 | 1.47e-001 | 1.03e+000 | 7.22e-004 |
| 12 | 2.31e-002 | 6.47e-001 | 3.15e-007 |
| 16 | 3.11e-002 | 4.45e-001 | 3.98e-011 |
| 20 | 1.79e-002 | 3.36e-001 | 3.86e-015 |
| 24 | 1.48e-002 | 2.60e-001 | 3.98e-015 |

Table 4.16: Absolute errors to the NFIE (4.7) , for $h = 0.1$ by using the most numerical methods.

| $t_i$ | Trapezoidal | Simpson | NKSM | NK [46] | Collocation[10] |
|---|---|---|---|---|---|
| 0.0 | 3.192e-016 | 3.053e-016 | 2.848e-002 | 4.983e-002 | 9.579e-011 |
| 0.1 | 3.331e-016 | 3.331e-016 | 2.709e-002 | 4.739e-002 | 6.123e-011 |
| 0.2 | 2.220e-016 | 2.220e-016 | 2.304e-002 | 4.031e-002 | 8.451e-011 |
| 0.3 | 2.220e-016 | 2.220e-016 | 1.674e-002 | 2.929e-002 | 1.545e-010 |
| 0.4 | 1.110e-016 | 1.110e-016 | 8.801e-003 | 1.540e-002 | 1.762e-010 |
| 0.5 | 0.000e+000 | 0.000e+000 | 0.000e+000 | 0.000e+000 | 1.421e-014 |
| 0.6 | 1.110e-016 | 1.110e-016 | 8.801e-003 | 1.540e-002 | 3.971e-010 |
| 0.7 | 2.220e-016 | 2.220e-016 | 1.674e-002 | 2.929e-002 | 6.103e-010 |
| 0.8 | 2.220e-016 | 2.220e-016 | 2.304e-002 | 4.031e-002 | 2.156e-010 |
| 0.9 | 2.776e-016 | 3.053e-016 | 2.709e-002 | 4.739e-002 | 2.707e-010 |
| 1.0 | 3.192e-016 | 3.053e-016 | 2.848e-002 | 4.983e-002 | 9.575e-011 |

In this section, we will compare our results:In recent paper the authors discuss modifications of these methods, to make them into more efficient and practical methods. In [10] the authors Bhrawy et al., discussed new spectral collocation technique for solving second kind Fredholm integral equations. They developed a collocation scheme to approximate NFIEs by means of the shifted Legendre–Gauss–Lobatto collocation (SL–GL-C) method. We solved some examples by our presented methods: Nyström method, collocation method and Newton-Kantorovich-modified Simpson method, we solved these equations (4.12) and (4.7) with different values of $h$ to compare our results with results in [10], [46] and in [50] the errors in the solutions are tabulated in tables (4.15) , (4.16) which show that numerical

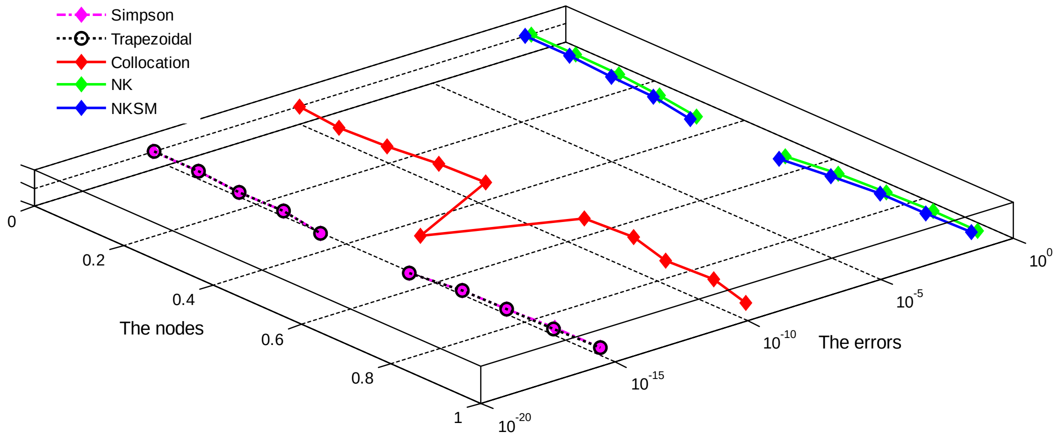


Figure 4.7: Absolute errors to the NFIE (4.7), for $h = 0.1$ by using the most numerical methods.

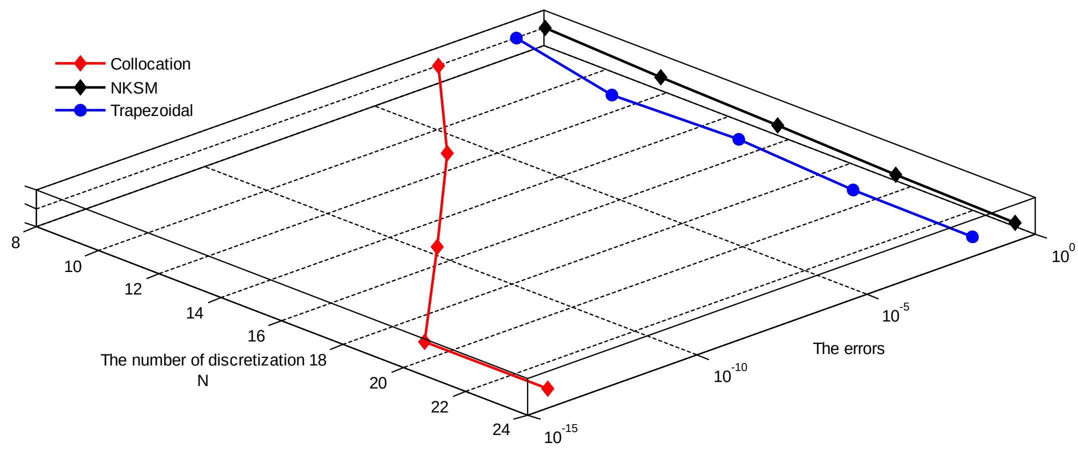


Figure 4.8: Absolute errors to the NVIE (4.12). The number of points, $N$, is the total number of discretization points.

results were acceptable. The above data indicate that the collocation method is faster than Nyström for problems of Urysohn integral equations.

# Conclusion

There are various numerical methods to solve nonlinear integral equations. Most of them transform the integral equation into a system of nonlinear algebraic equations. The solution of a general system of nonlinear equations is a difficult problem. In this work the nonlinear system is solved using the fixed point method or Newton method.

We have presented each numerical method as algorithm and applied these algorithms to both Fredholm and Volterra integral equation using Matlab Software. We tested these methods by using some different examples. It is observed that all methods converge and the absolute error has approached zero which was shown that numerical results were acceptable. Then, the most accurate scheme is the projection method and the least accurate one is Newton-Kantorovich method.

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**Abstract :**

Many problems which arise in mathematical physics, engineering, biology, economics,…etc., lead to mathematical models described by nonlinear integral equations. The aim of this research is to find the solution of nonlinear Volterra and Fredholm integral equation by using analytical and numerical methods such as the degenerate kernel method, the successive approximation method, the projection method, and the Nyström method. Also, we applied the new combination of Newton-Kantorovich method with modified Simpson method. Most of them transform the nonlinear integral equation into a system of linear or nonlinear algebraic equations. Finally, numerical examples are presented which demonstrate the robustness of the expansion numerical methods in determining solutions.

**Keywords**: Nonlinear integral equations, fixed point problem, degenerate kernel method, successive approximation method, projection method, Nyström method, Newton-Kantorovich method.

**Résumé :**

De nombreux problèmes en physique mathématique, ingénierie, biologie, économie,…etc., peuvent être modélisées en tant qu'équations intégrales non linéaire. Le but de la présente recherche est de trouver la solution de l'équation intégrale non linéaire de Volterra ainsi que celle de Fredholm en utilisant les méthodes analytiques et numériques telles que : la méthode du noyau dégénéré, la méthode d'approximation successive, la méthode de projection et la méthode de Nyström. Nous avons appliqué une nouvelle combinaison de la méthode de Newton-Kantorovich avec la méthode de Simpson modifiée. La plupart de ces méthodes transforment l'équation intégrale non linéaire en un système d'équations algébriques linéaires ou non linéaires. Finalement, des exemples numériques ont été présentés pour montrer la robustesse des méthodes d'expansion numérique dans la détermination des solutions.

**الملخص:**

العديد من المسائل التي تنشأ في الفيزياء الرياضية، الهندسة، علم الأحياء، الاقتصاد وغيرها، تؤول إلى حل نماذج رياضية ما يعرف بالمعادلات التكاملية غير الخطية .في هذه الأطروحة تم دراسة وتحليل الحل العددي للمعادلات التكاملية غير الخطية. والهدف من هذا البحث هو إيجاد حل معادلات "فولتيرا" و"فريدولم" التكاملية غير الخطية باستخدام طرق تحليلية وعددية مثل طريقة النواة المنحلة، طريقة التقريب المتتالي، طريقة الإسقاط و طريقة "نيستروم". كما قمنا بتطبيق تركيبة جديدة من طريقة "نيوتن-كانتوروفيتش" مع طريقة "سيمبسون" المعدلة. معظم هذه الطرق تقوم بتحويل المعادلة التكاملية غير الخطية إلى مجموعة معادلات جبرية خطية أو غير الخطية . وأخيرا، تم عرض أمثلة عددية حتى نبرهن على قوة الطرق العددية في تحديد الحلول.