جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

N°d'ordre : 2021/01/TH/P

الجمهورية الجزائرية الديمقراطية الشعبية

Ministry of Higher Education and Scientific Research

Mohamed Boudiaf University - M'sila

Faculty of Sciences

Department of Physics

MEMORY

Presented to obtained the graduated from:

MASTER

Domaine : **Sciences de la matière**

Filière : **Physique**

Option : **Physique théorique**

Par **Metah Amal and Zitouni Amel**

THEME

The nonrelativistic study of the energy spectrum producing from a central potential in the extended quantum mechanics symmetries: the case of Hulthén potential as a model

---

It **was discussed on: 24/06/2021**

In front of the jury composed of:

Salim MEJBAR	**MC-A Univ. of M'sila**	**President**
Abdelmadjid MAIRECHE	**Prof. Univ. of M'sila**	**Rapporter**
Abboud METATLA	**MC-A Univ. of M'sila**	**Examinator**

Promotion Juin 2021

بسم الله الرحمن الرحيم

تشكرات

بعد حمد المولى عز و جل على توفيقه لنا في انجاز المذكرة

نتوجه بالشكر و العرفان لمؤطرنا الاستاذ الدكتور عبدالمجيد معيرش اضافة لاساتذتنا الافاضل و نخص بالذكر الاستاذين مجبر سليم و عبود مطاطلة اللذان قبلا تقييم مذكرتنا

كما لا يفوتنا ان نشكر كل من ساعدنا من قريب او بعيد على انجاز هاته المذكرة

After the praise of God Almighty for His success for us in completing the memory of master in theortical physics.

We extend our thanks and gratitude to our mentor, Prof. Dr. Abdelmadjid Maireche, in addition to our distinguished professors, especially the two professors, Salim MEJBAR and Abboud METATLA, who accepted the evaluation of our master memory

We also do not forget to thank everyone who helped us from near or far to accomplish this memory.

The contents

General introduction

The beginning of the last century heralded the emergence of a multipolar scientific revolution. One of its poles represents Einstein's special and general relativity. Which was a successor to classical mechanics in the field of speeds comparable to the speed of light in a vacuum, which takes into account the effect of gravity. But their influence is in the macroscopic and astronomical fields.

While the second pole of the contemporary scientific revolution lies in the microscopic field, where the distances between objects are compared to Planck's constant known in the physical literature. In other words, quantum mechanics is concerned with physical phenomena at both the atomic and nuclear levels.

It is possible to know the various physical information about the studied systems by addressing the solutions of the basic equations known in the literature.

The Schrödinger equation describes the state of atomic particles with small energies of the order of the electron volt, this is according to the value of the spin.

While at the level of high energies of the order of Mev, the Schrödinger equation must be replaced by the Klein Gordon equation or by the Dirac equation. The Schrödinger equation has so far attracted researchers to it. Among the potentials that occupied the researchers' thought is the so-called Hulthén potential [1,2 and (1-11 chapter2)]. The Hulthén potential is used in nuclear and particle physics, atomic physics, solid-state physics, this potential is one of the important short-range potentials in physics, this potential has been used in nuclear and particle physics and atomic physics [1-11_chapter 2].

Since noncommutative quantum mechanics includes large physical symmetries than the symmetries of quantum mechanics known in the literature, we will reserve this study to obtain a master's degree in theoretical physics from Mohamed Boudiaf

University in M'sila to study this potential in noncommutative quantum mechanics promotion 2020-2021 intituled by the nonrelativistic model study of the energy spectrum resulting from a central potential in the extended quantum mechanics symmetries: The case of Hulthén potential as a model :

.

This master memory is organized as follows. In chapter one, the noncommutative quantum mechanics is represented. In chapter two, the Schrödinger equation is revised under the Hulthén potential. In chapter three, we study the effect of noncommutativity properties on the Hulthén potential.

CHAPTER I

The noncommutative space-phase formalism

I-Introduction

We reserve this chapter to a review of physical and mathematical concepts that are useful for a better understanding of noncommutative quantum mechanics.

I-Reminds about the structure of ordinary quantum mechanics:

The beginning of quantum physics is known in 1900 when Planck quantifies the energy of light ($E_\gamma = h\nu$) where ($h \approx 6{,}6262.10^{-34}$ js). Currently, ordinary quantum mechanics is formulated on the commutative space of the coordinates of variable and the canonical moment of hermetic operators (x_i, p_i), as following [1-2]**:**

$$\begin{cases} [x_i, p_j] = i\hbar\delta_{ij} \\ [x_i, x_j] = 0 \\ [p_i, p_j] = 0 \end{cases} \tag{I-1}$$

Where $\hbar = \frac{h}{2\pi}$ Is the reduced Planck constant and δ_{ij} Is Kronecker's ordinary symbol. The quantization procedure satisfies both relations principles concerning the energy E and impulsion P:

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \qquad \text{and} \qquad p_i \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x^i} \tag{I-2}$$

We know that the energy of a particle of mass m_0 subjected to the forces produced by a potential $V(\vec{r}, t)$, in a classical mechanic is given by

$$E = \frac{\vec{P}^2}{2m_0} + V(\vec{r}, t) \tag{I-3}$$

Now apply the two principles of canonical quantization presented in Eq. (0–1), we obtain the following Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m_0}\nabla + V(\vec{r}, t)\right]\Psi(\vec{r}, t) = i\hbar\frac{\partial\Psi(\vec{r}, t)}{\partial t} \tag{I-4}$$

Where the Laplacian operator in Cartesian coordinates is given by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(I-5)}$$

While in the spherical coordinate is determined from the following relation:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial^2}{\partial \varphi^2} \qquad \text{(I-6)}$$

Eq. (0–2) known by the Schrödinger equation in ordinary space-time, based on the postulates presented, $\Psi(\vec{r}, t)$ is the complex wave function, which determines the probability of finding the particle at time t in an elementary volume d^3r, rounding the point r.

$$dP = |\Psi(\vec{r}, t)|^3 d^3r \qquad \text{(I-7)}$$

The complex wave functions in the space of the impulsion allow us to determine the probability of impulsion

$$dP(\vec{p}) = |\Psi(\vec{p}, t)|^3 d^3p \qquad \text{(I-8)}$$

The incertitude relation of Heisenberg in ordinary quantum mechanics:

$$\begin{cases} \Delta x \Delta p_x \geq \frac{\hbar}{2} \\ \Delta y \Delta p_y \geq \frac{\hbar}{2} \\ \Delta z \Delta p_z \geq \frac{\hbar}{2} \end{cases} \qquad \text{(I-9)}$$

Furthermore, in ordinary quantum mechanics, the mean value is very important which denoted by $\langle A \rangle$, it takes the form:

$$\begin{cases} \langle A \rangle = \int \Psi(\vec{r}, t)^* \hat{A}\, \Psi(\vec{r}, t) d^2r \quad 2D \\ \langle A \rangle = \int \Psi(\vec{r}, t)^* \hat{A} \Psi(\vec{r}, t) d^3r \quad 3D \end{cases} \qquad \text{(I-10)}$$

We know that

$$\begin{cases} d^2r = r dr d\varphi \\ d^3r = r^2 \sin(\theta) dr d\theta d\varphi \end{cases} \qquad \text{(I-11)}$$

The probability current density vector $\vec{J}(\vec{r}, t)$ given by

$$\vec{J}(\vec{r},t) = \frac{\hbar}{2mi}(\Psi^* \Delta\Psi - \Psi\Delta\Psi^*) \quad \text{(I-12)}$$

Satisfies the continuity equation:

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \vec{\nabla}\vec{J}(\vec{r},t) = 0 \quad \text{(I-13)}$$

Where the term $\rho(\vec{r},t)$ represent the probability density, it's perfectly similar to the equation of charge conservation.

In the symmetries of traditional quantum mechanics QM, the global angular momentum $\vec{J}$ is the sum of the two moments, angular momentum $\vec{L}$ and the spin momentum $\vec{S}$ as follows:

$$\vec{J} = \vec{L} + \vec{S} \quad \text{(I-14)}$$

The spin-orbit coupling takes the form

$$\vec{L}\vec{S} = \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2) \quad \text{(I-15)}$$

The eigenvalues of the operators $\vec{J}$, $\vec{L}$ and $\vec{S}$

$$\begin{cases} \vec{J}^2\Psi = j(j+1)\Psi \\ \vec{L}^2\Psi = l(l+1)\Psi \\ \vec{S}^2\Psi = s(s+1) \end{cases} \quad \text{(I-16)}$$

The previous two equations (I-15) and (I-16) allow deducing the eigenvalue of the coupling $\vec{L}\vec{S}$ as follows:

$$\vec{L}\vec{S}\,\Psi = \frac{1}{2}\{\, j(j+1) - l(l+1) - s(s+1)\,\}\,\Psi \quad \text{(I-17)}$$

For a fermion particle as the electron, the possible value of 'j'

$$\begin{cases} j = l + \frac{1}{2} \\ j = l - \frac{1}{2} \end{cases} \xrightarrow{\text{With}} \begin{cases} j \leq \left|l + \frac{1}{2}\right| \\ j \geq \left|l - \frac{1}{2}\right| \end{cases} \quad \text{(I-18)}$$

III_ Noncommutative space-phase:

The idea of studying noncommutative space in particle physics is very old, the purpose was to be able to eliminate ultraviolet divergences from the field of quantum theory. In quantum mechanics, the phase space is defined by replacing the variables and the canonic momentum $(\hat{x}_i, \hat{p}_j)$ by hermetic operators. The idea of the noncommutativity of space is introduced by W. Heisenberg in 1930 [3] and then developed by H. Syndre in 1947 [4], satisfied by the new algebraic stricter followed by the rule of noncommutative commutations relations [1-2]:

$$\begin{cases} [x_i, p_j] = i\,\hbar\delta_{ij} \\ [x_i, x_j] = 0 \\ [p_i, p_j] = 0 \end{cases} \qquad \text{(In ordinary QM)}$$

It will be replaced by the following new algebra [5-20]:

$$\begin{cases} [\hat{x}_i, \hat{p}_j] = i\hbar_{eff}\delta_{ij} \\ [\hat{x}_i, \hat{x}_j] = i\hbar\theta_{ij} \\ [\hat{p}_i, \hat{p}_j] = i\hbar\bar{\theta}_{ij} \end{cases} \qquad \text{(In the extended QM)} \tag{I-19}$$

Where $i, j = \overline{1,3}$ and $\hbar_{eff} = \hbar(1 + \frac{\theta\bar{\theta}}{4\hbar^2})$ is the effective constant of Planck. The noncommutative coordinate taking

$$\begin{cases} x_i \to \hat{x}_i = f(x_i, p_i) \\ p_i \to \hat{p}_i = g(x_i, p_i) \end{cases} \tag{I-20}$$

Then we have

$$(\theta^{\mu\nu}, \theta^{-\mu\nu}) \equiv -(\theta^{\nu\mu}, \theta^{-\nu\mu}) = \mathcal{E}^{\mu\nu}(\theta, \bar{\theta}) \tag{I-21}$$

Whereas the first parameter is an antisymmetric tensor induced by noncommutativity space-space, and the second is an antisymmetric tensor induced by the noncommutativity phase-phase. It is very important to note the dimensions:

$$((\text{Lenght})^2 = (\text{impulsion})^2) \to (\theta^{\mu\nu} =, \theta^{-\mu\nu})$$

In two-dimensional noncommutative phase space, the canonical switch relation is:

$$\begin{cases} [\hat{x}_1, \hat{p}_1] = [\hat{x}_2, \hat{p}_2] = [\hat{x}_3, \hat{P}_3] = i\hbar \\ [\hat{x}_1, \hat{p}_2] = [\hat{x}_2, \hat{p}_1] = 0 \\ [\hat{x}_1, \hat{x}_2] = i\theta_{12} \\ [\hat{p}_1, \hat{p}_2] = i\bar{\theta}_{12} \end{cases} \quad \text{(I-22)}$$

Or

$$\begin{cases} [\hat{x}\,, \hat{p}_x] = [\hat{y}\,, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i \\ [\hat{x}\,, \hat{p}_y] = [\hat{y}\,, \hat{p}_x] = 0 \\ [\hat{x}\,, \hat{y}\,] = i\theta_{12} \\ [\hat{p}_x, \hat{p}_y] = i\bar{\theta}_{12} \end{cases} \quad \text{(I-23)}$$

I- 4 Product star

We will see the star product properties and the Moyal-Weyl formula

I-4-1 Quantification of Moyal-Weyl

Within the framework of the canonical quantification of quantum mechanics, Weyl gives a prescription that makes it possible to associate operators with the classical function of canonical variables. The transform of each function f(x) or g(x), is denoted by $\tilde{f}(k)$ and $\tilde{g}(k)$ [21-26]:

$$\begin{cases} \mathbf{f}(\mathbf{x}) = \int d^4k\, e^{ikx}\, \tilde{f}(k) \\ \mathbf{g}(\mathbf{x}) = \int d^4k\, e^{ikx}\, \tilde{g}(k) \end{cases} \rightarrow \begin{cases} \tilde{\mathbf{f}}(\mathbf{k}) = \int d^4x\, e^{-ikx}\, f(x) \\ \tilde{\mathbf{g}}(\mathbf{k}) = \int d^4x\, e^{-ikx}\, g(x) \end{cases} \quad \text{(I-24)}$$

can be defined as follow, with associates f and g by the operators $w(f)$ and $w(g)$:

$$\begin{cases} \mathbf{w}(\mathbf{f}) = (2\pi)^{-4} \int d^4 e^{ik\hat{x}} \tilde{f}(k) \\ \mathbf{w}(\mathbf{g}) = (2\pi)^{-4} \int d^4 e^{ik\hat{x}} \tilde{g}(k) \end{cases} \quad \text{(I-25)}$$

I-4-2 Products of Moyal

The product $w(f)w(g)$ is defined[21-26]:

$$w(f)w(g) = (2\pi)^{-4}(2\pi)^{-4} \int \mathbf{d^4}\, k\;\; d^4p\; e^{ik\hat{x}} e^{ip\hat{x}} \tilde{f}(k)\tilde{g}(p) \quad \text{(I-26)}$$

Used the formula of Campbl-Backer-Hausdorff:

$$e^{A}e^{B} = e^{A+B+\frac{1}{2}h[A,B]+\frac{1}{2}h^{2}[[A,B],B]-\frac{1}{2}h^{3}[[A,B],A]+\cdots}. \tag{I-27}$$

With $[A,[A,B]] = [B,[A,B]]$. A new product noted product star (*) then the product of w(f)w(g)takes the form:

$$w(f)w(g) = (2\pi)^{-4}(2\pi)^{-4}\int d^{4}k\ d^{4}p\ e^{ik\hat{x}+ip\hat{x}-h\frac{i}{2}k_{i}p_{j}\theta^{ij}}\tilde{f}(k)\tilde{g}(p)$$

$$= w(f * g) \tag{I-28}$$

Where (f∗g) is a new function defined

$$(f * g) = e^{\frac{i}{2}h\theta^{ij}\frac{\partial}{\partial x_{i}}\frac{\partial}{\partial y^{j}}}[f(x)g(x)]_{y\rightarrow x} \tag{I-29}$$

This relation is developed as follows

$$(f * g) = f(x)g(x) + \frac{i}{2}h\theta^{ij}\frac{\partial}{\partial x^{i}}f(x)\frac{\partial}{\partial x^{j}} + o(\theta^{2}) \tag{I-30}$$

The new products $(f * g)$ represent the product in noncommutative phase space and the f(x)g(x) represents the ordinary product in the commutative space

$$f(x) * g(x) = f(x)g(x) + \sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^{n}\frac{1}{n!}\theta^{\mu_{1}\nu_{1}}\ldots.\theta^{\mu_{n}\nu_{n}}\partial_{\mu_{1}}\partial_{\mu_{n}}f(x)\partial_{\nu}\partial_{\nu_{n}}g(x) \tag{I-31}$$

I-4-3 Product star properties

There are deferent properties of the star product, we have [21-26]:

- Noncommutative

$$f(x) * g(x) \neq g(x) * f(x) \tag{I-32}$$

- The star product is an association

$$(f(x) * g(x)) * h(x) = f(x) * (g(x) * f(x)) \tag{I-33}$$

- The conjugate complex relation

$$(f(x) * g(x))^* = f(x)^* * g(x)^* \qquad \text{(I-34)}$$

- The integral relation

$$\int d^4x\, (f * g)(x) = \int d^4x\, f(x)g(x) \qquad \text{(I-35)}$$

Cyclic permutation

$$\int d^Dx\, (f * g * h)(x) = \int d^Dx\, (h * f * g)(x) = \int d^Dx\, (g * h * f)(x) \qquad \text{(I-36)}$$

- Satisfies Leibniz's rule

$$\partial_\mu (f * g) = (\partial_\mu f) * g + f * (\partial_\mu g) \qquad \text{(I-37)}$$

I-5-The Boop's shift method

In this Master memory, we apply Boop's shift method instead of solving the Schrödinger equation will be treated by using directly the two commutators [26-29]:

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} \qquad \text{(I-38)}$$

$$[\hat{p}_i, \hat{p}_j] = i\bar{\theta}_{ij} \qquad \text{(I-39)}$$

To write the Schrödinger equation in noncommutative phase space, we apply the following steps [30-36]:

- We replace the ordinary wave function with a new wave function $\Psi(\vec{r}, t) \longrightarrow \widehat{\Psi}(\vec{\hat{r}}, t)$
- We replace the ordinary Hamiltonian operator with a new operator $H(p_i, x_i) \rightarrow \widehat{H}(\hat{p}_i, \hat{x}_i)$
- The ordinary energy E replaced by a new valueE_{nc}.
- The ordinary product is replaced by the star product (*)

The four steps lead to obtains Schrödinger equation in the noncommutative phase space

$$\widehat{H}(\hat{p}_i, \hat{x}_i) * \widehat{\Psi}(\vec{\hat{r}}, t) = E_{nc}\widehat{\Psi}(\vec{\hat{r}}, t) \qquad \text{(I-40)}$$

We can write the wave function as:

$$\hat{\Psi}(\vec{\hat{r}},t)=\hat{\Psi}(\vec{\hat{r}})f(t) \tag{I-41}$$

So, the equation (I-40) becomes:

$$\hat{H}(\hat{p}_i,\hat{x}_i)*\hat{\Psi}(\vec{\hat{r}})=E_{nc}\hat{\Psi}(\vec{\hat{r}}) \tag{I-42}$$

The Hamiltonian operator takes the forms [34-40]:

$$\begin{cases}\hat{H}(\hat{p}_i,\hat{x}_i)=H\left(\hat{p}_i=p_i+\frac{\bar{\theta}^{ij}}{2}x_j,\hat{x}_i=x_i-\frac{\theta^{ij}}{2}p_j\right)\\ \qquad\to\quad (\text{NC}-3\text{D}:\text{RSP})\\ \hat{H}(\hat{p}_i,\hat{x}_i)=H\left(\hat{p}_i=p_i\quad,\quad\hat{x}_i=x_i-\frac{\theta^{ij}}{2}p_j\right)\\ \qquad\to\quad (\text{NC}-3\text{D}:\text{RS})\\ \hat{H}(\hat{p}_i,\hat{x}_i)=H\left(\hat{p}_i=p_i+\frac{\bar{\theta}^{ij}}{2}x_j\quad,\quad\hat{x}_i=x_i\right)\\ \qquad\to\quad (\text{Nc}-3\text{D}:\text{RS})\end{cases} \tag{I-43}$$

The variety (I-43) corresponds:

$$\begin{cases}p_i\to\hat{p}_i=p_i+\frac{\bar{\theta}^{ij}}{2}x_j\\ x_i\to\hat{x}_i=x_i-\frac{\theta^{ij}}{2}p_j\end{cases} \tag{I-44}$$

With

$$\begin{cases}p_i\to\hat{p}_i=p_i\\ x_i\to\hat{x}_i=x_i-\frac{\theta^{ij}}{2}p_j\end{cases} \tag{I-45}$$

and

$$\begin{cases}p_i\to\hat{p}_i=p_i+\frac{\bar{\theta}^{ij}}{2}x_j\\ x_i\to\hat{x}_i=x_i\end{cases} \tag{I-46}$$

We are studying the noncommutative phase space in three dimensions, so the commutation relations in Eq. (I.44) are replaced by a new commutation’s relation

$$\begin{cases} \hat{x}_1 = \hat{x} \\ \hat{x}_2 = \hat{y} \\ \hat{x}_3 = \hat{z} \end{cases} \text{ and } \begin{cases} \hat{p}_1 = \hat{p}_x \\ \hat{p}_2 = \hat{p}_y \\ \hat{p}_3 = \hat{p}_z \end{cases} \tag{I-47}$$

We can obtain the forms:

$$\begin{cases} \hat{x}_1 = x_1 - \frac{\theta_{12}}{2} p_2 - \frac{\theta_{13}}{2} p_3 \\ \hat{x}_2 = x_2 - \frac{\theta_{21}}{2} x_2 - \frac{\theta_{23}}{2} x_3 \\ \hat{x}_3 = x_3 - \frac{\theta_{31}}{2} x_1 - \frac{\theta_{32}}{2} x_2 \end{cases} \tag{I-48}$$

and

$$\begin{cases} \hat{p}_1 = p_1 + \frac{\bar{\theta}_{12}}{2} x_2 + \frac{\bar{\theta}_{13}}{2} x_3 \\ \hat{p}_2 = p_2 + \frac{\bar{\theta}_{21}}{2} x_2 + \frac{\bar{\theta}_{23}}{2} x_3 \\ \hat{p}_3 = p_3 + \frac{\bar{\theta}_{31}}{2} x_1 + \frac{\bar{\theta}_{32}}{2} x_2 \end{cases} \tag{I-49}$$

With the square of $(\vec{\hat{r}})$and $(\vec{\hat{p}})$ are given by:

$$\begin{cases} \vec{\hat{r}}^2 = \vec{\hat{r}}^2 + \vec{\hat{r}}^2 + \vec{\hat{r}}^2 \\ \vec{\hat{p}}^2 = \vec{\hat{p}}^2 + \vec{\hat{p}}^2 + \vec{\hat{p}}^2 \end{cases} \tag{I-50}$$

For getting the solution of the noncommutative Schrödinger equation we used the product star by Bopp's shift method that it is a consequence of the product star between the potential operator $\hat{V}(\hat{x})$ and the complex wave function $\hat{\Psi}(\hat{r})$ [26-29]:

$$\left[\frac{\vec{\hat{p}}^2}{2m} + \hat{V}(\hat{x})\right] * \hat{\Psi}(\hat{x}) \quad \rightarrow \quad \left[\frac{\vec{p}^2}{2m} + V(x)\right] \Psi(x) \tag{I-51}$$

In the noncommutative three-dimensional phase space, the tow operator $(\vec{\hat{r}})$and $(\vec{\hat{p}})$, takes the form [35-40];

$$\begin{cases} \hat{r}^2 = r^2 - \overrightarrow{L\Theta} \\ \frac{\hat{p}^2}{2m_0} = \frac{p^2}{2m_0} + \frac{\vec{L}\vec{\theta}}{2m_0} \end{cases} \quad \text{(I-52)}$$

With $\Theta = \theta/2$

$$\begin{cases} L\Theta = L_x\Theta_1 + L_y\Theta_2 + L_z\Theta_3 \\ \vec{L}\vec{\bar{\theta}} = L_x\bar{\theta}_{12} + L_x\bar{\theta}_{23} + L_z\bar{\theta}_{31} \end{cases} \quad \text{(I-53)}$$

I-7-Application on Hulthén potential

Now we ready to apply the noncommutative space-phase properties on the Hulthén potential which has the following traditional expression [1-11 chapter 2]:

$$V(\acute{r}) = \frac{-ze^2\delta e^{-\delta\acute{r}}}{1-e^{-\delta\acute{r}}} \quad \text{(I-54)}$$

From Eq. (I-52), we can write $\acute{r}$ as follows:

$$\acute{r} = r\left(1 - \frac{\vec{L}\vec{\theta}}{r^2}\right)^{1/2} \quad \text{(I-55)}$$

We use the following formula:

$$\begin{cases} (1+x)^n \simeq 1 + nx \\ \text{if } \ x \ll 1 \end{cases} \quad \text{(I-56)}$$

Eq. (I-55) take the form

$$\acute{r} \simeq r\left(1 - \frac{\vec{L}\vec{\theta}}{2r^2}\right) \simeq r - \frac{\vec{L}\vec{\theta}}{2r} \quad \text{(I-57)}$$

Now we begin to apply the Hulthén potential, the expression of $e^{-\delta\acute{r}}$ can be written in the symmetries of extended quantum mechanics symmetries as follows:

$$e^{-\delta\acute{r}} = e^{-\delta r}e^{\delta\vec{L}\vec{\theta}/2r} \simeq e^{-\delta r}\left(1 + \frac{\delta\vec{L}\vec{\theta}}{2r}\right) \quad \text{(I-58)}$$

Deducing that

$$e^{-\delta \acute{r}} \simeq e^{-\delta r} + \frac{\delta \vec{L}\vec{\theta}}{2r} e^{-\delta r} \qquad \text{(I-59)}$$

Therefore, we can get the following approximation:

$$1 - e^{-\delta \acute{r}} = 1 - e^{-\delta r} + \frac{\delta \vec{L}\vec{\theta}}{2r} e^{-\delta r} \qquad \text{(I-60)}$$

$$= (1 - e^{-\delta r}) \left[1 + \frac{\delta \vec{L}\vec{\theta}}{2r} \; e^{-\delta r} \left(1 - e^{-\delta r} \right)^{-1} \right] \qquad \text{(I-61)}$$

Allows us to obtain the following:

$$\frac{e^{-\delta \acute{r}}}{1 - e^{-\delta \acute{r}}} = \frac{e^{-\delta r} + \frac{\delta \vec{L}\vec{\theta}}{2r} e^{-\delta r}}{(1 - r^{-\delta r}) \left[1 + \frac{\delta \vec{L}\vec{\theta}}{2r} \; e^{-\delta r} \left(1 - e^{-\delta r} \right)^{-1} \right]} \qquad \text{(I-61)}$$

If we put:

$$A = \frac{\delta \vec{L}\vec{\theta}}{2r} \; e^{-\delta r} \left(1 - e^{-\delta r} \right)^{-1} \qquad \text{(I-62)}$$

And rewrite the expression $\frac{e^{-\delta \acute{r}}}{1 - e^{-\delta \acute{r}}}$ as follows:

$$\underbrace{\frac{e^{-\delta \acute{r}}}{1 - e^{-\delta \acute{r}}} = \frac{e^{-\delta r}}{(1 - r^{-\delta r})[1 + A]}}_{L} + \underbrace{\frac{\frac{\delta \vec{L}\vec{\theta}}{2r} e^{-\delta r}}{(1 - r^{-\delta r})[1 + A]}}_{M}$$

(I-63)

Then we simplify the term L and M

$$L \simeq \frac{e^{-\delta r}}{1 - e^{-\delta r}} \left[1 - \frac{\delta \vec{L}\vec{\theta}}{2r} \frac{e^{-\delta r}}{1 - e^{-\delta r}} \right] \qquad \text{(I-64)}$$

And

$$M = \frac{\frac{\delta \vec{L}\vec{\theta}}{2r} e^{-\delta r}}{(1 - r^{-\delta r}) \left[1 + \frac{\delta \vec{L}\vec{\theta}}{2r} \; e^{-\delta r} \left(1 - e^{-\delta r} \right)^{-1} \right]} \qquad \text{(I-65)}$$

After straightforward simplification, we obtain:

$$L \simeq \frac{e^{-\delta r}}{1-e^{-\delta r}} - \frac{\delta \vec{L}\vec{\theta}}{2r}\left(\frac{e^{-\delta r}}{1-e^{-\delta r}}\right)^2 \tag{I-66}$$

And

$$M \simeq \frac{\delta \vec{L}\vec{\theta}}{2r}\frac{e^{-\delta r}}{1-e^{-\delta r}} + O(\theta^2) \tag{I-67}$$

Now,

$$L + M = \frac{e^{-\delta r}}{1-e^{-\delta r}} - \frac{\delta \vec{L}\vec{\theta}}{2r}\left(\frac{e^{-\delta r}}{1-e^{-\delta r}}\right)^2 + \frac{\delta \vec{L}\vec{\theta}}{2r}\frac{e^{-\delta r}}{1-e^{-\delta r}} \tag{I-68}$$

Finally, we get the Hulthén potential in the noncommutative space as follows:

$$V(\acute{r}) = \frac{-Ze^2 e^{-\delta r}}{1-e^{-\delta r}} - \frac{\delta \vec{L}\vec{\theta}}{2r}\left[\left(\frac{e^{-\delta r}}{1-e^{-\delta r}}\right)^2 - \frac{e^{-\delta r}}{1-e^{-\delta r}}\right] \tag{I-69}$$

First chapter references

[1] J. L. Basidevant, Mécanique Quantique, ellipses, ISBN 2-7298-8614-1 (1986), Paris, France.

[2] E. Elbaz, Quantum, The quantum theory of particles, Fields, and Cosmology, Springere, ISBN 3-540-62093-1 (1995), New York, USA.

[3] W. Heisenberg, ”Letter to R. Peierls (1930), in ’Wolfgang Pauli, Scienti- fic Correspondence’, Vol. III, p.15, Ed. K. von Meyenn“, (Springer Verlag 1985)

[4] H. Snyder, The Quantization of space-time, Phys. Rev. 71 (1946) 38-41.

[5] R. J. Szabo, “Quantum field theory on noncommutative spaces”, Phys. Rept. 378 207 (2003) hep-th/0109162.

[6] F. A. Schaposnik “Three lectures on noncommutative field theories”, hep-th/0408132.

[7] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, “On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and Its Implications on Noncommutative QFT,” Phys. Lett. B 604 98 (2004) hep-th/0408069.

[8] J. Wess, “Deformed Coordinate Spaces: Derivatives,” hep-th/0408080.

[9] A. Connes and M. A. Rieffel, “Yang-Mills for Noncommutative Two-Tori”, Contemp. Math. 62 237 (1987).

[10] A. Connes and J. Lott, “Particle Models and Noncommutative Geometry”, Nucl. Phys. (Proc. Suppl.) 18 B 29 (1991).

[11] J. Gamboa, M. Loewe, F. Mendez and J. C. Rojas, Noncommutative Quantum Mechanics: The Twodimensional central Field, International Journal of Modern Physics A, Vol. 17 Issue: 19 Pages: 2555- 2565 (2002).

[12] Gamboa, J., Loewe, M., & Rojas, J. C. (2001). Noncommutative quantum mechanics. Physical Review D, 64(6). doi:10.1103/physrevd.64.067901

[13] Curtright, T., Fairlie, D., & Zachos, C. (1998). Features of time-independent Wigner functions. Physical Review D, 58(2). doi:10.1103/physrevd.58.025002

[14] A. E. F. Djemei and H. Smail, On Quantum Mechanics on Noncommutative Quantum Phase Space, Commun. Theor. Phys. (Beijinig, China). 41 (2004) pp.837-844.

[15] Justin Gabriel encadré par François Gieres, Diverses Approches de la Mécanique Quantique sur Espace Non-Commutatif, Master Science de la matière, Université Claude Bernard Lyon I (2013-2014).

[16] Jumakari-Mamat; Sayipjamal Dulat and Hekim Mamatabdulla, Landau-like Atomic Proplem on a Non-commutative Phase Space, Int J Theor Phys; DIO 10.1007/s10773-016-2922-1 (2016).

[17] A. Maireche, New nonrelativistic bound state solutions of the NGECSC potential involving GECSC potential and new central terms for hydrogen atom, Afr. Rev. Phys. 13: 0023,142-149 (2019). http://lamp.ictp.it/index.php/aphysrev/article/view/1635/590.

[18] A. Maireche, A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential, Sri Lankan Journal of Physics21, 11-35(2020). DOI: http://DOI.org/ 10.4038/sljp.v20i0

[19] A. Maireche, Modified Schrödinger Equation with Modified QM Gravitational and Harmonic Oscillator Potentials in Symmetries of Non-commutative QM, Afr. Rev. Phys. 14: 0018,130-1398 (2019). http://lamp.ictp.it/index.php/aphysrev/article/view/1717/613.

[20] A. Maireche, A Complete Analytical Solution of the Mie-Type Potentials in Non-commutative 3-Dimensional Spaces and Phases Symmetries, Afr. Rev. Phys. 11:0015, 111-117 (2016). Available at: http://lamp.ictp.it/index.php/aphysrev/article/view/1259/490.

[21] A. Maireche, Nonrelativistic treatment of Hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative Quantum mechanics, International Journal of Geometric Methods in Modern Physics17(5) (2020) 2050067. DOI: 10.1142/S021988782050067X

[22] A. Connes, M. Douglas, A. Schwarz. Noncommutative geometry and matrix theory: Compactification on tori. Journal of High Energy Physics 1998, 1998.

[23] Thomas Curtright, Features of time-independent Wigner functions, PHYSICAL REVIEW D, VOLUME 58, 025002.

[24] Ilyas Haouam, ANALYTICAL SOLUTION OF (2+1) DIMENSIONAL DIRAC EQUATION IN TIME-DEPENDENT NONCOMMUTATIVE PHASE-SPACE, Acta Polytechnica 60(2):111–121, 2020 DOI:10.14311/AP.2020.60.0111
[25] B. Mirza, M. Zarei. Non-commutative quantum mechanics and the Aharonov-Casher effect. European Physical Journal C 32:583 – 586, 2004. doi:10.1140/epjc/s2003-01522-8.
[26] KHELILI FARID, ASPECTS MATHEMATIQUES ET PHYSIQUES DE LA GEOMETRIE NON COMMUTATIVE, The University of Arizona (2007).
[27] Gouba, L. (2016). *A comparative review of four formulations of noncommutative quantum mechanics. International Journal of Modern Physics A, 31(19), 1630025.* doi:10.1142/s0217751x16300258
[28] L. Mezincescu, Star operation in quantum mechanics, e-print (2000). arXiv: hep-th/0007046v2.
[29] F. Bopp, La mécanique quantique est-elle une mécanique statistique classique particulière ?, Ann. Inst. Henri Poincar´e 15, 81 (1956).
[30] L. Mezincescu, Star operation in quantum mechanics, e-print (2000). arXiv: hep-th/0007046v2.
[31] A. Maireche, Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles, International Frontier Science Letters11, 29-44 (2017). DOI: https://doi.org/10.18052/www.scipress.com/IFSL.11.29
[32] A. Maireche, New Relativistic Bound States for Modified Pseudoharmonic Potential of Dirac Equation with Spin and Pseudo-Spin Symmetry in One-electron Atoms, Afr. Rev. Phys. 12: 0018,130-143 (2018). Available at: http://lamp.ictp.it/index.php/aphysrev/article/view/1533/564.
[33] A. Maireche, New Relativistic Atomic Mass Spectra of Quark (u, d and s) for Extended Modified Cornell Potential in Nano and Plank's Scales, J. Nano- Electron. Phys. 8(1), 01020 (2016). DOI: 10.21272/jnep.8(1).01020
[34] A. Maireche, A New Relativistic Study for Interactions in One-electron atoms (Spin ½Particles) with Modified Mie-type Potential, J. Nano- Electron. Phys. 8(4), 04027(2016). DOI: 10.21272/jnep.8(4(1)).04027
[35] Abdelmadjid Maireche, Deformed Quantum Energy Spectra with Mixed Harmonic Potential for Nonrelativistic Schrödinger equation, J. Nano- Electron. Phys. 7 No 2, (2015) 02003.
[36] Abdelmadjid Maireche, A Study of Schrödinger Equation with Inverse Sextic Potential in 2-dimensional Non-commutative Space, The African Rev. Phys. 9:0025, (2014) 185-193.
[37] Abdelmadjid Maireche Deformed Bound States for Central Fraction Power Potential: Non Relativistic Schrödinger Equation, The African Rev. Phys. 10:0014, (2015) 97-103.
[38] Abdelmadjid. Maireche, Nonrelativistic Atomic Spectrum for Companied Harmonic Oscillator Potential and its Inverse in both NC-2D: RSP, International Letters of Chemistry, Physics and Astronomy, Vol. 56, pp. 1-9, Jul. 2015.
[39] Abdelmadjid Maireche, New exact bound states solutions for (C.F.P.S.) potential in the case of Noncommutative three dimensional non relativistic quantum mechanics, Med. J. Model. Simul. 04 (2015) 060-072.
[40] Abdelmadjid Maireche A (2017) New Exact Nonrelativistic Energy Eigen Values for Modified Inversely Quadratic Hellmann Plus Inversely Quadratic Potential. J Nanosci Curr Res 2:115. DOI: 10.4172/2572-0813.1000115.
[41] Abdelmadjid Maireche, New Bound State Energies for Spherical Quantum Dots in Presence of a Confining Potential Model at Nano and Plank's Scales, NanoWorld J, 1(4): (2016) 120-127.

CHAPTER II

The Schrödinger equation with Hulthén potential in ordinary Quantum mechanics

II-1-Introduction

The study of the Hulthén potential in the extended noncommutative space-phase requires a necessary review of this potential within the framework of quantum mechanics known in the literature.

II-2-The Hulthén potential

The Hulthén potential is used in nuclear and particle physics, atomic physics, solid-state physics, this potential is one of the important short-range potentials in physics, this potential has been used in nuclear and particle physics, atomic physics, and solid-and solid- state physics [1,2]. Ahmadovet al. studied the bound state solutions of the D-dimensional Schrödinger equation for the Hulthén potential within SUSY quantum mechanics [3]. In 2009, Agboola studied the Hulthén potential in D -dimensions with an exponential approximation of the centrifugal term using the Nikiforov–Uvarov method [4]. By using the Pekeris approximation type, Onate solved the Schrodinger equation for the interaction of Coulomb and Hulthén potentials within the framework of supersymmetric approach and Nikiforov-Uvarov method and obtained energy levels are obtained with the corresponding wave functions in terms of hypergeometric functions [5]. The bound state approximate solution of the Schrodinger equation is obtained by C.O. Edeta and P.O. Okoi for the q-deformed Hulthén plus generalized inverse quadratic Yukawa potential in generalized D-dimensions using the Nikiforov-Uvarov method and the corresponding eigenfunctions are expressed in Jacobi polynomials [6]. Qiang *et al*. using the exact quantization rule presented the approximate analytical solutions of the radial Schrödinger equation with nonzero values for the Hulthén potential in the frame of approximation to the centrifugal potential for any *l* state [7]. E. P. Inyang *et al.* obtained analytical solutions of the N-dimensional Schrodinger equation for the Varshni-Hulthén potential within the framework of the Nikiforov-Uvarov method by using the Greene-Aldrich approximation scheme to the centrifugal barrier [8]. Bayrak et al. Were presented the analytical solution of the radial Schrodinger equation for the Hulthén potential within the framework of the asymptotic iteration method by using an approximation to the centrifugal potential for any *l* state and obtained the energy eigenvalues and corresponding eigenfunctions for different screening parameters [9]. The researchers Eser Olgar, Ramazan Koç, and Hayriye Tütüncüler studied the bound state solution of the (1+1)-dimensional Klein–Gordon for the generalized Hulthén potential in the framework of the asymptotic iteration method [10]. Qlang *et al*. developed a new and simple approximation scheme for centrifugal terms using the new approximate formula for $1/r^2$, and they derived approximately analytical solutions to the radial Schrodinger equation of the Hulthén potential with arbitrary *l*-states [11].

II-3-Revised of the Schrödinger equation for the Hulthénén potential

In order to find the energy spectrum of the **Hulthénén potential** in the extended quantum symmetries, it is necessary to review its energy spectrum within the framework of ordinary quantum mechanics by referring to the references we have already mentioned, especially references 1 and 2 of this chapter:

$$V(r) = -Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}} \quad \text{(II- 1)}$$

Where Z is the atomic number and δ is the screening parameter determining the range for the Hulthén potential. As mentioned in Ref 11, this potential behaves like the Coulomb potential near the origin of coordinates, but in the asymptotic region at a large distance similar to the potential decreases exponentially, so its capacity for bound states is smaller than the Coulomb potential.

In the Hilbert space, the Schrödinger equation gives

$$\hat{H}\Psi(\vec{r},t) = E\Psi(\vec{r},t) \quad \text{(II-2)}$$

E represents the energy of the system, and the Hamiltonian obtains by

$$\hat{H} = \frac{\vec{P}^2}{2\mu} - Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}} \quad \text{(II-3)}$$

For a stationary state, the wave function takes the form

$$\Psi(\vec{r},t) = e^{-iE/\hbar}\Psi(\vec{r}) \quad \text{(II-4)}$$

We obtain the ordinary Schrödinger equation by applying the procedure of quantization which we have seen in the first chapter:

$$\left(-\frac{\hbar^2}{2M}\Delta - Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}}\right)\Psi(\vec{r}) = i\hbar\frac{\partial\Psi(\vec{r})}{\partial t} \quad \text{(II-5)}$$

In the three-dimensions with the spherical coordinate, we know that

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\, sin(\theta)}\frac{\partial}{\partial\theta}(sin(\theta)\frac{\partial}{\partial\theta}) + \frac{1}{r^2\, sin(\theta)}\frac{\partial^2}{\partial\varphi^2} \quad \text{(II-6)}$$

In quantum mechanics, the classical momentum obtains the forms

- $\vec{L} \longrightarrow$ The orbital angular momentum
- $\vec{S} \longrightarrow$ The spin operator
- $\vec{J} \longrightarrow$ The total moment $\quad \vec{J} = \vec{L} + \vec{S}$

The orbital angular momentum gives

$$\overrightarrow{L} = \vec{r} \wedge \vec{p} \tag{II-7}$$

The Cartesian components are given

$$\vec{L}=\begin{cases} L_x = yp_x - zp_y \\ L_y = zp_x - xp_z \\ L_z = xp_y - yp_x \end{cases} \tag{II-8}$$

We can write also

$$\begin{cases} L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ L_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{cases} \tag{II-9}$$

By the spherical coordinate (r, θ, φ) the Eq. (II-9) becomes:

$$\begin{cases} L_x = \frac{\hbar}{i}\left(-\cos\varphi \cot\theta\frac{\partial}{\partial\varphi} - \sin\varphi\frac{\partial}{\partial\theta}\right) \\ L_y = \frac{\hbar}{i}\left(-\sin\varphi \cot\theta\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\theta}\right) \\ L_z = \frac{\hbar}{i}\left(\frac{\partial}{\partial\varphi}\right) \end{cases} \tag{II-10}$$

We have

$$\begin{cases} L_z\, Y_m^l(\theta,\varphi) = m\hbar\, Y_m^l \\ \\ L^2\, Y_m^l(\theta,\varphi) = l(l+1)\hbar^2\, Y_m^l(\theta,\varphi) \end{cases} \tag{II-11}$$

With $\quad l = \overline{0, n-1}$, whereas the total momentum takes the following forms

$$\vec{J}^2 = \overrightarrow{J_x}^2 + \overrightarrow{J_y}^2 + \overrightarrow{J_z}^2 \tag{II-12}$$

and

$$\begin{cases} \vec{J}^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle \\ \overrightarrow{J_z} |j,m\rangle = m |j,m\rangle \end{cases} \tag{II-13}$$

The coupling operator spin-orbit $\vec{\hat{L}}\vec{\hat{S}}$ can be written

$$\vec{\hat{L}}\vec{\hat{S}} = \frac{1}{2}\left(\left(\vec{\hat{L}} + \vec{\hat{S}}\right)^2 - \vec{\hat{L}}^2 - \vec{\hat{S}}^2\right)$$

$$= \frac{1}{2}\left(\vec{\hat{J}}^2 + \vec{\hat{L}}^2 + \vec{\hat{S}}^2\right) \tag{II-14}$$

Interesting, the complex wave function $\Psi(\vec{r}) = R(r)Y(\theta,\varphi)$ into Eq. (II-5), the radial part of the Schrödinger equation for the Hulthén potential can be expressed as [3,11]:

$$\left[\frac{\boldsymbol{d}}{\boldsymbol{dr^3}} + \frac{\mathbf{2}}{\boldsymbol{r}}\frac{\boldsymbol{d}}{\boldsymbol{dr}} - \frac{\boldsymbol{l(l+1)}}{\boldsymbol{r^2}} + \mathbf{2}\mu\left(\boldsymbol{E_{nl}} + \boldsymbol{Ze^2\delta}\frac{\boldsymbol{e^{-\delta r}}}{\boldsymbol{1-e^{-\delta r}}}\right)\right]\boldsymbol{R(r)} = \mathbf{0} \tag{II-15}$$

If we used the short-hand notations:

$$V_{eff}(r) \;\; = \frac{l(l+1)\hbar^2}{2\mu r^2} - Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}} \tag{II-16}$$

Here $V_{eff}(r)$ is the effective potential. The radial part of the Schrödinger equation for the Hulthén potential can be expressed as [3,11]:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)R(r) = \frac{2\mu}{\hbar^2}\left[E_{nl} - V_{eff}(r)\right]R(r) \tag{II-17}$$

If we assume $R_{nl}(r) = \frac{\chi_{nl}}{r}$, the hyper-radial Schrodinger equation (II-17) becomes:

$$\chi''(r) + \left[\frac{2\mu}{\hbar^2}\left(E + Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}}\right) - \frac{l(l+1)}{2r^2}\right]\chi(r) = 0 \tag{II-18}$$

To solve Eq. (II-18) H. I. Ahmadov et al. for $l\neq 0$, were applying the approximation for the centrifugal term [3,12,13]:

$$\frac{1}{r^2} \approx \left(c_0 + \frac{e^{-\delta r}}{1-e^{-\delta r}}\right) \tag{II-19}$$

The parameter $C_0 = 1/21$ is a dimensionless constant, when $C_0 = 0$, the approximation scheme in Eq. (II-20) becomes the convectional approximation scheme suggested by Greene and Aldrich [3,12,13,14]:

$$\tilde{V}_{eff}(r) = -Ze^2 \frac{\delta e^{-\delta r}}{1-e^{-\delta r}} + \frac{l(l+1)\hbar^2}{2\mu} \frac{e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2} \tag{II-20}$$

The radial Schrödinger equation (II-18) after applying approximation for the centrifugal term is written as follows:

$$\chi''(r) + \frac{\chi'(s)}{s}[-\varepsilon^2(1-s)^2 - \lambda((1-s)^2 + s) + \alpha^2 s(s-1)]\chi(s) = 0 \tag{II-21}$$

Where $s = e^{-\delta r}$, $-\varepsilon^2 = \frac{2\mu E}{\hbar^2 \delta^2}$, $\alpha^2 = \frac{2\mu Z e^2}{\hbar^2 \delta}$ and $\lambda = l(l+1)$. The solutions of Eq. (II-21) in terms of hypergeometric polynomials ${}_2F_1\left(-n, n+2\sqrt{C_{nl}}+K_{nl}; 1+2\sqrt{C_{nl}}, s\right)$ as [3,11]:

$$\chi(r) = \frac{C_{nl}\Gamma\left(n+2\sqrt{C_{nl}}+1\right)}{n!\Gamma\left(2\sqrt{C_{nl}}+1\right)} s^{\sqrt{C_{nl}}} (1-s)^{K_{nl}} {}_2F_1\left(-n, n+2\sqrt{C_{nl}}+K_{nl}; 1+2\sqrt{C_{nl}}, s\right) \quad \text{(II-21)}$$

The complete wave function $\Psi_{n,l,m}(r,\theta,\varphi)$, and the energy of the system $E_{n,l}$ for the Hulthén potential are given in this subsection where [3,11]:

$$\Psi_{nlm}(r,\theta,\varphi) = \frac{C_{nl}\Gamma\left(n+2\sqrt{C_{nl}}+1\right)}{n!\Gamma\left(2\sqrt{C_{nl}}+1\right)} \frac{s^{\sqrt{C_{nl}}}}{r} (1-s)^{K_{nl}} {}_2F_1\left(-n, n+2\sqrt{C_{nl}}+K_{nl}; 1+2\sqrt{C_{nl}}, s\right) Y_l^m(\theta,\varphi) \quad \text{(II-22)}$$

Where

$$\begin{cases} C_{nl} = -\dfrac{2\mu}{\delta^2} E_{nl} + l(l+1)C_0 \\ K_l = l+1 \end{cases} \tag{II-23}$$

The normalization constant C_{nl} can be found from normalization condition [3]:

$$\int_0^\infty |R(r)|^2 r^2 dr = \int_0^\infty |\chi(r)|^2 dr = b\int_0^1 \frac{1}{s}|\chi(s)|^2 ds = 1 \qquad \text{(II-24)}$$

Which gives the following values of normalization constant:

$$C_{nl} = \sqrt{\frac{n!\,2\sqrt{c}\left(n+k+\sqrt{c}\right)\Gamma\left(2\left(k+\sqrt{c}\right)+n\right)}{b(n+k)\Gamma\left(n+2\sqrt{c}+1\right)\Gamma(n+2k)}} \qquad \text{(II- 25)}$$

While the energy eigenvalues E_{nl} is given also from [3] when *D=3* as follows:

$$E_{nl} = \frac{\delta^2}{2\mu}\left[\frac{\sqrt{1/4+l(l+1)}+n+1/2}{2} - \frac{\left(2\mu Ze^2/\delta\right)^2}{2\sqrt{1/4+l(l+1)}+n+1/2}\right]^2 + \frac{\delta^2}{2\mu}C_0 l(l+1) \qquad \text{(II-26)}$$

For the ground state, we have $n = 0$, the hypergeometric polynomials function

$${}_2F_1\left(-0, 2\sqrt{C_{0l}} + K_{nl}; 1+2\sqrt{C_{0l}}, s\right) = 1$$

This allows us to simplify the complex wave function and the corresponding energy eigenvalues E_{0l} to the following two formulas as follows:

$$\Psi_{0lm}(r,\theta,\varphi) = C_{0l}\frac{s^{\sqrt{C_{0l}}}}{r}(1-s)^{K_l}Y_l^m(\theta,\varphi) \qquad \text{(II-27)}$$

And

$$E_{Ol} = \frac{\delta^2}{2\mu}\left[\frac{\sqrt{1/4+l(l+1)}+1/2}{2} - \frac{\left(2\mu Ze^2/\delta\right)^2}{2\sqrt{1/4+l(l+1)}+1/2}\right]^2 + \frac{\delta^2}{2\mu}C_0 l(l+1) \qquad \text{(II-28)}$$

References for the second chapter

1. L. Hulthén, On the characteristic solutions of the Schrödinger deuteron equation, Ark. Mat. Astron. Fys. A 28 (1942) 5.
2. L. Hulthén, Über die Eigenlösungen der Schrödinger-Gleichung des Deuterons Ark. Mat. Astron. Fys. B 29 (1942) 1.
3. Ahmadov, H. I., Qocayeva, M. V., & Huseynova, N. S. (2017). The bound state solutions of the D-dimensional Schrödinger equation for the Hulthén potential within SUSY quantum mechanics. International Journal of Modern Physics E, 26(05), 1750028. doi:10.1142/s0218301317500288
4. Agboola, D (2009). The Hulthén potential in D -dimensions. Physica Scripta, 80(6), 065304–. doi:10.1088/0031-8949/80/06/065304
5. Onate, C.A (2014) Approximate Solutions of the Non-Relativistic Schrodinger Equation with An Interaction of Coulomb and Hulthen Potentials. SOP TRANSACTIONS ON THEORETICAL PHYSICS, 1 (2). pp. 118-127.
6. C.O. Edeta and P.O. Okoi, Any l-state solutions of the Schrodinger equation for q-deformed Hulthen plus generalized inverse quadratic Yukawa potential in arbitrary dimensions, Revista Mexicana de Fısica 65 (2019) 333-344
7. Qiang, W.-C., Gao, Y., & Zhou, R.-S. (2008). Arbitrary /-state approximate solutions of the Hulthén potential through the exact quantization rule. Open Physics, 6(2). doi:10.2478/s11534-008-0041-1
8. E. P. Inyang, E. S. William and J. A. Obu, Eigensolutions of the N-dimensional Schrodinger equation interacting with Varshni-Hulthen potential model, Revista Mexicana de Fısica 67 (2) 193–205
9. Bayrak, O., Kocak, G., & Boztosun, I. (2006). Any *l*-state solutions of the Hulthén potential by the asymptotic iteration method. Journal of Physics A: Mathematical and General, 39(37), 11521–11529. doi:10.1088/0305-4470/39/37/012
10. Olğar, E., Koç, R., & Tütüncüler, H. (2008). The exact solution of the s-wave Klein–Gordon equation for the generalized Hulthén potential by the asymptotic iteration method. Physica Scripta, 78(1), 015011. doi:10.1088/0031-8949/78/01/015011

11. QIANG, W.-C., CHEN, W. L., LI, K., & ZHANG, H.-P. (2009). IMPROVED ARBITRARY l-STATE SOLUTIONS OF THE HULTHÉN POTENTIAL. International Journal of Modern Physics A, 24(28n29), 5523–5529. doi:10.1142/s0217751x09046370
12. W.-C. Qiang and S. H. Dong, "Analytical Approximation to the Solutions of the Manning-Rosen Potential with Centrifugal Term," Physics Letters A, Vol. 368, No.1-2, 2007, pp. 13-17. doi:10.1016/j.physleta.2007.03.057
13. Hassanabadi, H., Maghsoodi, E., Ikot, A. N., & Zarrinkamar, S. (2013). Approximate arbitrary-state solutions of Dirac equation for modified deformed Hylleraas and Modified Eckart potentials by the NU method. Applied Mathematics and Computation, 219(17), 9388–9398. doi:10.1016/j.amc.2013.03.011
14. Greene, R. L., & Aldrich, C. (1976). Variational wave functions for a screened Coulomb potential. Physical Review A, 14(6), 2363–2366. doi:10.1103/physreva.14.2363

Chapter III

The Effect of Noncommutativity

On the Energy Spectrum produced by the Hulthén potential in three-dimensional noncommutative quantum mechanics

III-1-Introduction

The object of the last chapter is to know the effect produced by proprieties of deformed space-phase on the Hulthén potential using the Boop's Shift method and the perturbation theory to find the corrections of energy.

III-2-The Schrödinger equation in noncommutative phase- space

In this section, we shall give an overview of a brief preliminary for the modified Hulthén potential in three-dimensional nonrelativistic noncommutative quantum mechanics symmetries. To perform this task, the physical form of the modified Schrödinger equation, it is necessary to replace the ordinary three-dimensional Hamiltonian operators $\hat{H}_{hp}(x_\mu, p_\mu)$, the complex wave function $\Psi\left(\vec{r}\right)$, and energy E_{nl} with the new three Hamiltonian operators $\hat{H}_{nc}^{hp}(\hat{x}_\mu, \hat{p}_\mu)$, the new complex wave function $\Psi\left(\vec{\hat{r}}\right)$, and new values E_{nc}^{hp}, respectively. In addition to replacing the ordinary product with the Weyl Moyal star product which we have seen in the first chapter, which allows us to construct the modified Schrödinger equation in three-dimensional nonrelativistic noncommutative quantum mechanics symmetry framework as [1-8]:

$$\hat{H}_{hp}(x_\mu, p_\mu)\Psi\left(\vec{r}\right) = E_{nl}\Psi\left(\vec{r}\right) \Rightarrow \hat{H}_{nc}^{hp}(\hat{x}_\mu, \hat{p}_\mu) * \Psi\left(\vec{\hat{r}}\right) = E_{nc}^{hp}\Psi\left(\vec{\hat{r}}\right) \qquad \text{(III-1)}$$

Bopp's shift method [9-11] have been successfully applied to relativistic and nonrelativistic noncommutative quantum mechanical problems using the modified Dirac equation [12-16], the modified Klein-Gordon equation [17-25], and modified Schrödinger equation [21-25]. This method has produced very promising results for several situations having physical, chemical interest. The method reduces the modified Dirac equation, the modified Klein-Gordon equation, and modified Schrödinger equation to the Dirac equation, Klein-Gordon equation, and the Schrödinger equation, respectively, under two-simultaneously translations in space and phase. It is based on the following new commutators [9-11,26-32]:

$$[\hat{x}_\mu, \hat{p}_\nu] = [\hat{x}_\mu(t), \hat{p}_\nu(t)] = i\delta_{\mu\nu}\hbar_{eff}\,, \qquad \text{(III-.2)}$$

$$[\hat{x}_\mu, \hat{x}_\nu] = [\hat{x}_\mu(t), \hat{x}_\nu(t)] = i\theta_{\mu\nu}\,, \qquad \text{(III-.3)}$$

and

$$\left[\hat{p}_{\mu},\hat{p}_{\nu}\right]=\left[\hat{p}_{\mu}(t),\hat{p}_{\nu}(t)\right]=i\bar{\theta}_{\mu\nu}. \qquad \text{(III-.4)}$$

III-3-The Hamiltonian operator for the Hulthén potential in extended quantum mechanics:

The generalized position and momentum coordinates $(\hat{p}_i,\hat{x}_i)$in the extended quantum mechanics depend on the corresponding usual generalized positions and momentum coordinates (p_i,x_i) in NRQM as follows [9-11]:

$$(x_i,p_i) \longrightarrow (\hat{x}_i\hat{p}_i)=\left(x_i-\frac{\theta_{ij}}{2}p_j,p_i+\frac{\bar{\theta}_{ij}}{2}x_j\right) \qquad \text{(III-.5)}$$

The above equation allows us to obtain the two operators $\hat{r}^2$ and $\hat{p}^2$ in (NC-3D: RSP) [26-32]:

$$(r^2,p^2)\longrightarrow(\hat{r}^2,\hat{p}^2)=(r^2-L\Theta,p^2+L\bar{\theta}) \qquad \text{(III-6)}$$

The reduced Schrödinger equation can be written as [9-11, 26-32]:

$$\hat{H}_{nc}^{hp}\left(\hat{x}_{\mu},\hat{p}_{\mu}\right)*\Psi\left(\vec{\hat{r}}\right)=E_{nc}^{hp}\Psi\left(\vec{\hat{r}}\right)\Rightarrow H_{nc}^{hp}\left(\hat{x}_{\mu},\hat{p}_{\mu}\right)\Psi\left(\vec{r}\right)=E_{nc}^{hp}\Psi\left(\vec{r}\right) \qquad \text{(III-7)}$$

The Hamiltonian operator$\widehat{H}(\hat{p}_i,\hat{x}_i)$ which we have seen in the first chapter, be written as three models:

$$\begin{cases}\widehat{H}(\hat{p}_i,\hat{x}_i)=H\left(\hat{p}_i=p_i+\dfrac{\bar{\theta}^{ij}}{2}x_j,\hat{x}_i=x_i-\dfrac{\theta^{ij}}{2}p_j\right) & \text{(III- 8)}\\ \qquad\qquad \rightarrow\ (NC-3D:RSP) & \\ \widehat{H}(\hat{p}_i,\hat{x}_i)=H\left(\hat{p}_i=p_i\quad,\quad\hat{x}_i=x_i-\dfrac{\theta^{ij}}{2}p_j\right) & \text{(III- 9)}\\ \qquad\qquad \rightarrow\ (NC-3D:RS) & \\ \widehat{H}(\hat{p}_i,\hat{x}_i)=H\left(\hat{p}_i=p_i+\dfrac{\bar{\theta}^{ij}}{2}x_j\quad,\quad\hat{x}_i=x_i\right) & \text{(III- 10)}\\ \qquad\qquad \rightarrow\ (Nc-3D:RS) & \end{cases}$$

⇒ (III- 8) signify that the deformation is applying on the phase-space**.**

⇒(III- 9) signify that the deformation is applying to space-space noncommutativity.

⟹(III- 10) signifies that the deformation is applying to the phase-phase noncommutativity.

In our Memory, we are interested in the first variety. The Hamiltonian operator which corresponds to the first model takes the form

$$H_{nc}(\hat{p}_i, \hat{x}_i) = \frac{\hat{p}^2}{2\mu} + V(\hat{r}) \qquad \text{(III-11)}$$

The Hulthén potential in three-dimensional noncommutative space-phase symmetries gives

$$V\ (\acute{r}) = \frac{-ze^2\delta e^{-\delta\acute{r}}}{1-e^{-\delta\acute{r}}} \qquad \text{(III-12)}$$

We have seen in the first chapter, the Hulthén potential in the three-dimensional noncommutative space-phase symmetries as follows:

$$V(\acute{r}) = \frac{-Ze^2e^{-\delta r}}{1-e^{-\delta r}} - \frac{\delta\vec{L}\vec{\theta}}{2r}\left[\left(\frac{e^{-\delta r}}{1-e^{-\delta r}}\right)^2 - \frac{e^{-\delta r}}{1-e^{-\delta r}}\right] \qquad \text{(I-69)}$$

We can deduce that

$$V(\acute{r}) = V(r) + \frac{Ze^2\delta\vec{L}\vec{\theta}}{2r}\left[\left(\frac{e^{-\delta r}}{1-e^{-\delta r}}\right)^2 - \frac{e^{-\delta r}}{1-e^{-\delta r}}\right] \qquad \text{(III-13)}$$

We have seen in the first chapter, the kinetic term $\frac{\hat{p}^2}{2m_0}$ can be transformed to the following form:

$$\frac{\hat{p}^2}{2m_0} = \frac{p^2}{2m_0} + \frac{\vec{L}\vec{\theta}}{2m_0} \qquad \text{(III-14)}$$

The last two relations (III-13) and (III-14) allow finding the new Hamiltonian expression $H_{nc}^{hp}\left(\hat{x}_\mu, \hat{p}_\mu\right)$ in the three-dimensional noncommutative space-phase symmetries as follows

$$H_{nc}\left(\hat{p}_\mu, \hat{x}_\mu\right) = H\left(p_\mu, x_\mu\right) + H_{per}^{hp}(\hat{p}, \hat{x}) \qquad \text{(III-15)}$$

Where

$$H\left(p_\mu, x_\mu\right) = \frac{P^2}{2\mu} - Ze^2\delta\frac{e^{-\delta r}}{1-e^{-\delta r}} \qquad \text{(III-16)}$$

And

$$H_{per}^{hp}(\hat{p},\hat{x}) \ = \frac{Ze^2\delta\vec{L}\vec{\Theta}}{2r}\left[\left(\frac{e^{-\delta r}}{1-e^{-\delta r}}\right)^2 - \frac{e^{-\delta r}}{1-e^{-\delta r}}\right] + \frac{\vec{L}\vec{\theta}}{2\mu} \qquad \text{(III-17)}$$

Eq. (III-17) describes the product of the two deformations of space and phase.

III-4- The Spin-orbit Hamiltonian operator for the Hulthén potential in (NC: 3DSP)

In this section, we want to derive the physical form of the induced Hamiltonian $H_{per}^{hp}(\hat{p},\hat{x})$ due to the space-phase noncommutativity [26-32]:

$$\begin{cases} \vec{L}\vec{\Theta} \ \ \rightarrow \gamma\Theta\vec{L}\vec{S} \\ \vec{L}\vec{\bar{\theta}} \longrightarrow \gamma\bar{\theta}\vec{L}\vec{S} \end{cases} \qquad \text{(III-18)}$$

Which allowed rewriting Eq. (III-17) as follow:

$$H_{per}^{hp}(\hat{p},\hat{x}) \ = \gamma\left\{\frac{Ze^2\delta\Theta}{2r}\left[\left(\frac{e^{-\delta r}}{1-e^{-\delta r}}\right)^2 - \frac{e^{-\delta r}}{1-e^{-\delta r}}\right] + \bar{\theta}\right\}\vec{L}\vec{S} \qquad \text{(III-19)}$$

Her we have

$$\begin{cases} \Theta = (\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2)^{1/2} \\ \bar{\theta} = (\bar{\theta}_{12}^2 + \bar{\theta}_{23}^2 + \bar{\theta}_{13}^2)^{1/2} \end{cases} \qquad \text{(III-20)}$$

Here $\gamma \approx 1/137$ Is the atomic fine structure constant, and **S** is the spin of the particle which interacted with Hulthén potential. The obtained equation (III-7) cannot be solved analytically for any state *l*≠0, because of the centrifugal term and the studied potential itself. To solve Eq. (III-7), we need to apply the approximation scheme suggested by Greene and Aldrich [3,12,13,14 (chapter 2)]:

$$\frac{1}{r^2} \approx \frac{\delta^2 e^{-\delta r}}{\left(1-e^{-\delta r}\right)^2} \qquad \text{(III-21)}$$

Which allow us to deduce immediately:

$$\frac{1}{r} \approx \frac{\delta e^{(\delta/2)r}}{1-\exp(-\delta r)} = \frac{\delta s^{1/2}}{1-s} \qquad \text{(III-22)}$$

which gives the induced Hamiltonian $H_{per}^{hp}(\hat{p},\hat{x})$ due to the space-phase noncommutativity:

$$H_{pert}^{hp}\left(\hat{x}_{\mu},\hat{p}_{\mu}\right)=\gamma\left\{\frac{Ze^{2}}{2}\frac{\delta s^{1/2}}{1-s}\left[\frac{s^{2}}{(1-s)^{2}}-\frac{s}{1-s}\right]\Theta+\frac{\overline{\theta}}{2m_{0}}\right\}\vec{L}\vec{S} \qquad \text{(III-23)}$$

A direct simplification gives the induced Hamiltonian $\mathrm{H}_{\mathrm{per}}^{\mathrm{hp}}(\hat{\mathrm{p}},\hat{\mathrm{x}})$ due to the space-phase noncommutativity as follows:

$$H_{pert}^{hp}\left(\hat{x}_{\mu},\hat{p}_{\mu}\right)=\gamma\left\{\frac{\delta Ze^{2}}{2}\left[\frac{s^{5/2}}{(1-s)^{3}}-\frac{s^{3/2}}{(1-s)^{2}}\right]\Theta+\frac{\overline{\theta}}{2m_{0}}\right\}\vec{L}\vec{S} \qquad \text{(III-24)}$$

The energy spectrum, by applying the standard perturbation theory as follow**:**

$$E_{nc}^{hp}=E_{nl}+\Delta E_{nl} \qquad \text{(III-25)}$$

Where ΔE_{nl} is the correction of energy which can be determined by using the standard perturbation theory at first order as follows:

$$\Delta E_{nl}=\left\langle \Psi_{nlm}\left|H_{pert}^{hp}\left(\hat{x}_{\mu},\hat{p}_{\mu}\right)\right|\Psi_{nlm}\right\rangle \qquad \text{(III-26)}$$

In this memory of a master, we will confine ourselves to finding the energy correction corresponding to the ground state $n=0$, the hypergeometric polynomials function for this case ${}_2F_1\left(-0,2\sqrt{C_{0l}}+K_{nl};1+2\sqrt{C_{0l}},s\right)=1$, and regarding the corresponding wave function in Eq. (II-27), we apply the formula (III-26) to obtain ΔE_{0l} as follows:

$$\Delta E_{0l}=C_{0l}{}^{2}\int_{0}^{+\infty}s^{2\sqrt{C_{0l}}}(1-s)^{2K_{l}}\,\gamma\left\{\frac{\delta Ze^{2}}{2}\left[\frac{s^{5/2}}{(1-s)^{3}}-\frac{s^{3/2}}{(1-s)^{2}}\right]\Theta+\frac{\overline{\theta}}{2m_{0}}\right\}dr \qquad \text{(III-27)}$$

After a straightforward calculation, we obtain:

$$\Delta E_{0l} = \gamma \frac{\delta Ze^2}{2} \tau(j,l,s) C_{0l}{}^2 \int_0^{+\infty} s^{2\sqrt{C_{0l}}} (1-s)^{2K_l} \frac{s^{5/2}}{(1-s)^3} \Theta dr$$
$$- \gamma \frac{\delta Ze^2}{2} \tau(j,l,s) C_{0l}{}^2 \int_0^{+\infty} s^{2\sqrt{C_{0l}}} (1-s)^{2K_l} \frac{s^{3/2}}{(1-s)^2} \Theta dr \qquad \text{(III-28)}$$
$$+ \gamma C_{0l}{}^2 \tau(j,l,s) \frac{\overline{\theta}}{2m_0} \int_0^{+\infty} s^{2\sqrt{C_{0l}}} (1-s)^{2K_l} dr$$

The value $\tau(j,l,s)$ presents the eigenvalues of the operator $\Xi^2 = \vec{J}^2 - \vec{L}^2 - \vec{S}^2$ is given by:

$$\tau(j,l,s) \equiv \big(j(j+1) - l(l+1) - s(s+1)\big)/2 \qquad \text{(III-29)}$$

A direct simplification of Eq. (III-28) gives:

$$\Delta E_{0l} = \gamma \frac{\delta Ze^2}{2\delta} \tau(j,l,s) C_{0l}{}^2 \Theta \int_0^{+1} s^{2\sqrt{C_{0l}}+3/2} (1-s)^{2K_l-3} ds$$
$$- \gamma \frac{\delta Ze^2}{2\delta} \tau(j,l,s) C_{0l}{}^2 \Theta \int_0^{+1} s^{2\sqrt{C_{0l}}+1/2} (1-s)^{2K_l-2} ds \qquad \text{(III-30)}$$
$$+ \gamma C_{0l}{}^2 \tau(j,l,s) \frac{\overline{\theta}}{2m_0\delta} \int_0^{+1} s^{2\sqrt{C_{0l}}-1} (1-s)^{2K_l} ds$$

We have $s = \exp(-\delta r)$, this allows us to obtain $dr = -\frac{1}{\delta}\frac{ds}{s}$. After introducing a new variable $z = 1-2s$, we have $s = \frac{1-z}{2}$, $ds = \frac{1}{\delta}\frac{dz}{1-z}$ and $1-s = \frac{z+1}{2}$. From the asymptotic behavior of $s = \exp(-\delta r)$ and $z = 1-2s$, when $r \to 0$ $(z \to -1)$ and $r \to +\infty$ $(z \to 1)$, this allows to reformulate Eqs; (III-30) as follows:

$$\Delta E_{0l} = \frac{1}{2^{2\sqrt{C_{0l}}+3/2+2K_l-3}} \gamma \frac{\delta Ze^2}{2\delta^2} \tau(j,l,s) C_{0l}{}^2 \Theta \int_{-1}^{+1} (1-z)^{2\sqrt{C_{0l}}+1/2} (1+z)^{2K_l-3} dz$$
$$- \gamma \frac{1}{2^{2\sqrt{C_{0l}}+1/2+2K_l-2}} \frac{\delta Ze^2}{2\delta^2} \tau(j,l,s) C_{0l}{}^2 \Theta \int_{-1}^{+1} (1-z)^{2\sqrt{C_{0l}}-1/2} (1+z)^{2K_l-2} dz \qquad \text{(III-31)}$$
$$+ \gamma \frac{1}{2^{2\sqrt{C_{0l}}-1+2K_l}} C_{0l}{}^2 \tau(j,l,s) \frac{\overline{\theta}}{2m_0\delta^2} \int_{-1}^{+1} (1-z)^{2\sqrt{C_{0l}}-2} (1+z)^{2K_l} dz$$

Comparing Eqs. (III-31) with the integral of the form [33]:

$$\int_{-1}^{+1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx=\frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)} \tag{III-31}$$

A direct calculation gives the expectation values in Eqs. Eqs. (III-31) As follows:

$$\begin{aligned}\Delta E_{0l}=&\gamma\frac{\delta Ze^2}{2\delta^2}\tau(j,l,s)C_{0l}{}^2\Theta\frac{\Gamma\left(2\sqrt{C_{0l}}+3/2\right)\Gamma(2K_l-2)}{\left(2\sqrt{C_{0l}}+2K_l-3/2\right)\Gamma\left(2\sqrt{C_{0l}}+2K_l-3/2\right)}\\&-\gamma\frac{1}{2}\frac{\delta Ze^2}{2\delta^2}\tau(j,l,s)C_{0l}{}^2\Theta\frac{\Gamma\left(2\sqrt{C_{0l}}+1/2\right)\Gamma(2K_l-1)}{\left(2\sqrt{C_{0l}}+2K_l-3/2\right)\Gamma\left(2\sqrt{C_{0l}}+2K_l-3/2\right)}\\&+\gamma C_{0l}{}^2\tau(j,l,s)\frac{\bar{\theta}}{2m_0\delta^2}\frac{\Gamma\left(2\sqrt{C_{0l}}-1\right)\Gamma(2K_l+1)}{\left(2\sqrt{C_{0l}}+2K_l-1\right)\Gamma\left(2\sqrt{C_{0l}}+2K_l-1\right)}\end{aligned} \tag{III-32}$$

The energy spectrum, by applying the standard perturbation theory as follow**:**

$$E_{0c}^{hp}=E_{0l}+\Delta E_{0l} \tag{III-33}$$

Where E_{0l} is determined from Eq. (II-28) which we have seen in the second chapter:

$$E_{Ol}=\frac{\delta^2}{2\mu}\left[\frac{\sqrt{1/4+l(l+1)}+1/2}{2}-\frac{\left(2\mu Ze^2/\delta\right)^2}{2\sqrt{1/4+l(l+1)}+1/2}\right]^2+\frac{\delta^2}{2\mu}C_0 l(l+1) \tag{II-28}$$

General conclusion

Through this master's memory in physics, theoretical specialty:

Promotion 2020-2021

The nonrelativistic study of the energy spectrum producing from a central potential in the extended quantum mechanics symmetries: the case of Hulthén potential as a model

This memory aims to study physical systems within the framework of the modified Schrödinger equation with the modified Hulthén potential, in three-dimensional non-commutative quantum mechanics.

In the first chapter, we have represented the mathematical and physical formalisms of the noncommutative three-dimensional space-phase, and apply these principles to the atoms of modified Hulthén potential.

In the second chapter, we reviewed the effect of the modified Hulthén potential on a physical system based on many works.

In the third chapter, we studied the effect of the noncommutativity of the three-dimensional phase space, by applying the generalized Bopp shift method and standard perturbation theory at the first order, the modifications on the energy corresponding to the ground state are obtained, we have constructed a noncommutative Hamiltonian operator, we can conclude that the application of the noncommutativity in this work on the modified Hulthén potential, includes the spin-orbit coupling effect internally.

References for the third chapter

1. Gamboa, M. Loewe and J.C. Rojasn, "Noncommutative quantum mechanics," Phys. Rev. D 64, 067901 (2001). DOI: https://doi.org/10.1103/PhysRevD.64.067901.
2. E.F. Djemaï and H. Smail, "On Quantum Mechanics on Noncommutative Quantum Phase Space," Commun. Theor. Phys. (Beijing, China) 41(6), 837-844 (2004). doi:10.1088/0253-6102/41/6/837
3. A.Maireche, "A New Approach to the Approximate Analytic Solution of the Three-Dimensional Schrödinger Equation for Hydrogenic and Neutral Atoms in the Generalized Hellmann Potential Model," Ukr. J. Phys. 65(11), 987 (2020). https://doi.org/10.15407/ujpe65.11.987
4. A. Maireche, "Effects of Two-Dimensional Noncommutative Theories on Bound States Schrödinger Diatomic Molecules under New Modified Kratzer-Type Interactions," International Letters of Chemistry, Physics and Astronomy 76, 1-11(2017). https://doi.org/10.18052/www.scipress.com/ILCPA.76.1
5. O. Bertolami and P. Leal, "Aspects of phase-space noncommutative quantum mechanics," Phys. Lett. B 750, 6--11 (2015). doi: 10.1016/j.physletb.2015.08.024
6. J. Gamboa, M. Loewe and J.C. Rojas, "Noncommutative quantum mechanics," Phys. Rev. D 64(6) (2001). doi:10.1103/physrevd.64.067901
7. K. Li and J. Wang, "The topological AC effect on non-commutative phase space," The European Physical Journal C 50(4), 1007--1011 (2007). doi:10.1140/epjc/s10052-007-0256-0
8. J. Wang and K. Li, "The HMW effect in noncommutative quantum mechanics," Journal of Physics A: Mathematical and Theoretical 40(9), 2197--2202 (2007). doi:10.1088/1751-8113/40/9/021
9. L. Mezincescu, “Star operation in quantum mechanics,” e-print (2000). arXiv: hep-th/0007046v2.
10. F. Bopp, “La mécanique quantique est-elle une mécanique statistique classique particulière ?,” Ann. Inst. Henri Poincar´e 15, 81 (1956).
11. L. Gouba, “A comparative review of four formulations of noncommutative quantum mechanics,” Int. J. Mod. Phys. A31(19), 1630025 (2016).
12. A. Maireche, “Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles,” International Frontier Science Letters11, 29-44 (2017). DOI: https://doi.org/10.18052/www.scipress.com/IFSL.11.29

13. A. Maireche, “New Relativistic Bound States for Modified Pseudoharmonic Potential of Dirac Equation with Spin and Pseudo-Spin Symmetry in One-electron Atoms,” Afr. Rev. Phys. 12: 0018,130-143 (2018). Available at: http://lamp.ictp.it/index.php/aphysrev/article/view/1533/564.
14. A. Maireche, “New Relativistic Atomic Mass Spectra of Quark (u, d and s) for Extended Modified Cornell Potential in Nano and Plank's Scales,” J. Nano- Electron. Phys. 8(1), 01020 (2016). DOI: 10.21272/jnep.8(1).01020
15. A. Maireche, “A New Relativistic Study for Interactions in One-electron atoms (Spin ½Particles) with Modified Mie-type Potential,” J. Nano- Electron. Phys. 8(4), 04027(2016). DOI: 10.21272/jnep.8(4(1)).04027
16. Y. Yi, L. Kang, W. Jian-Hua and C. Chi-Yi, "Spin-1/2 relativistic particle in a magnetic field in NC phase space,». Chinese Physics C 34(5), 543-547 (2010). doi:10.1088/1674-1137/34/5/005
17. A. Maireche, "The Klein-Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space," Mod. Phys. Lett. A 35(5), 052050015 (2020). doi:10.1142/s0217732320500157
18. A. Maireche, “A Theoretical Model of Deformed Klein-Gordon Equation with Generalized Modified Screened Coulomb Plus Inversely Quadratic Yukawa Potential in RNCQM Symmetries,” Few-Body Syst.62, 12 (2021). https://doi.org/10.1007/s00601-021-01596-2
19. A. Maireche, “Bound-state Solutions of Klein-Gordon and Schrödinger Equations for Arbitrary l-state with Linear Combination of Hulthén and Kratzer Potentials,” Afr. Rev Phys.15:003, 19-31 (2020). http://lamp.ictp.it/index.php/aphysrev/article/view/1779/620.
20. H. Motavalli and A.R. Akbarieh, “Klein-Gordon equation for the Coulomb potential in noncommutative space,” Mod. Phys. Lett. A25(29), 2523-2528 (2010). doi:10.1142/s0217732310033529
21. M. Darroodi, H. Mehraban and H. Hassanabadi, “The Klein Gordon equation with the Kratzer potential in the noncommutative space,” Mod. Phys. Lett. A33No. 35, 1850203 (2018). doi:10.1142/s0217732318502036
22. A. Maireche, “Modified Unequal Mixture Scalar Vector Hulthén-Yukawa Potentials Model as a Quark-Antiquark Interaction and Neutral Atoms via Relativistic Treatment Using the Approximation of the Centrifugal Term and Bopp's Shift Method,” Few-Body Syst. 61, 30 (2020). https://doi.org/10.1007/s00601-020-01559-z.
23. A. Maireche, Solutions of Klein-Gordon equation for the modified central complex potential in the symmetries of noncommutative quantum mechanics,” Sri Lankan Journal of Physics. 22(1),1-19 (2021). DOI: http://doi.org/10.4038/sljp.v22i1.8079

24. A.Maireche, “Bound state solutions of the modified Klein-Gordon and Schrödinger equations for arbitrary *l*-state with the modified Morse potential in the symmetries of noncommutative quantum mechanics,” Journal of Physical Studies25(1), 1002 (2021). DOI: https://doi.org/10.30970/jps.25.1002
25. A. Maireche, “Theoretical Investigation of the Modified Screened cosine Kratzer potential via Relativistic and Nonrelativistic treatment in the NCQM symmetries,” Lat. Am. J. Phys. Educ. 14(3), 3310-1 (2020).
26. A. Maireche, "New Bound State Energies for Spherical Quantum Dots in Presence of a Confining Potential Model at Nano and Plank's Scales," NanoWorld J. 1(4), 122-129 (2016). DOI: 10.17756/nwj.2016-016
27. A. Maireche, "The Relativistic and Nonrelativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Noncommutative Quantum Mechanics," Jordan J. Phys. 14 (1), 59-70 (2021). Doi: https://doi.org/10.47011/14.1.6
28. A. Maireche, “New nonrelativistic bound state solutions of the NGECSC potential involving GECSC potential and new central terms for hydrogen atom,” Afr. Rev. Phys. 13: 0023,142-149 (2019). http://lamp.ictp.it/index.php/aphysrev/article/view/1635/590.
29. A. Maireche, “A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential, Sri Lankan Journal of Physics 21, 11-35(2020). DOI: http://DOI.org/ 10.4038/sljp.v20i0
30. A. Maireche, “Modified Schrödinger Equation with Modified QM Gravitational and Harmonic Oscillator Potentials in Symmetries of Non-commutative QM,” Afr. Rev. Phys. 14: 0018,130-1398 (2019). http://lamp.ictp.it/index.php/aphysrev/article/view/1717/613.
31. 26 A. Maireche, “A Complete Analytical Solution of the Mie-Type Potentials in Non-commutative 3-Dimensional Spaces and Phases Symmetries,” Afr. Rev. Phys. 11:0015, 111-117 (2016). Available at: http://lamp.ictp.it/index.php/aphysrev/article/view/1259/490.
32. A. Maireche, “Nonrelativistic treatment of Hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative Quantum mechanics,” International Journal of Geometric Methods in Modern Physics17(5) (2020) 2050067. DOI: 10.1142/S021988782050067X
33. S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series and Products, 7th. ed.: eds. A. Jeffrey D. Zwillinger (Elsevier, 2007).

Abstract

In our work to this master memory, in theoretical physics(2020/2021), we have studied the Schrödinger equation with the Hulthén potential in noncommutative three dimension spaces and phases, by applying the Boop's Shift method to the first order of the parameters $(\Theta, \bar{\theta})$, in addition to the standard perturbation theory, to obtain the spectrum of energy of the system, which is changing radically, and replaced by degenerate new states which depending on the discrete atomic quantum numbers (j, n, l, s) this result was produced from the spin-orbit interaction. we saw that the Hamiltonian operator of the Hulthén potential in the extended symmetries is the sum of two operators, the first part is the usual ordinary Hamiltonian (ordinary kinetic energy and standard Hulthén potential), while the second is the is spin-orbit interaction.

Keywords: Schrödinger equation, Hulthén potential, noncommutative quantum mechanics, star product, Boop's shift method.

ملخص

في عملنا هذا الخاص بمذكرة الماستر هذه، في الفيزياء النظرية (2020-2021)، قمنا بدراسة معادلة شرودينغر لكمون هولثين في الفضاء اللاتبادلي ثلاثي البعد والطور، بتطبيق مبدأ بوب الذي يوافق الحد$(\theta, \bar{\Theta})$، بالإضافة إلى تطبيق نظرية الاضطراب المعيارية، من اجل الحصول على طيف الطاقة للجملة، الذي يتغير بشكل جذري، بحيث يستبدل بحالات منحلة جديدة والتي تتعلق بالأعداد الكمية المتقطعة، (j, n, l, s)هذه النتائج تنتج من تأثير مفعول السبين-مدار المستحدث تلقائيا.

رأينا ان مؤثر الهاميلتوني لكمون هولثين في تناظرات الفضاء ثلاثي الطور والبعد اللاتبادلي هو مجموع لمؤثرين، الأول هو مؤثر الهاميلتونيان لكمون هولثين الطبيعي و الثاني هو مؤثر التفاعل سبين-مدار.

كلمات مفتاحية: معادلة شرودينغر، كمون هولثين، ميكانيك الكم اللاتبادلي، الجداء النجمي، طريقة بوب.