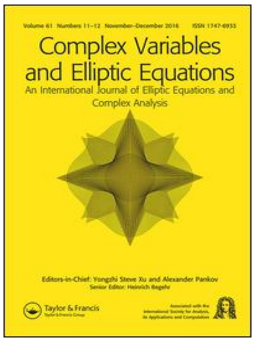

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Anisotropic non-local problems: asymptotic behaviour and existence results

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ABSTRACT

We deal with anisotropic singular perturbation problems in some weighted spaces of Sobolev type. The perturbed problems are elliptic, semi-linear and non-local. Using a variational approach, we establish the existence of their solutions as critical points of some C^1-functionals. Besides, we study the asymptotic behaviour of these solutions with respect to the parameter of perturbation ε and, as $\varepsilon \to 0$, we obtain existence results for some non-standard integro-differential problems. A non-local eigenvalue problem related to the considered problems is also investigated and used to carry out this study.

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1. Introduction

As a model problem in the unit square $\Omega_1 = (0,1) \times (0,1)$, we consider the following problem:

$$\begin{cases} -\partial_{x_2}\left(\mathbf{a}(x)\,\partial_{x_2} u\right) = g\left(\int_0^1 h(x_1)\, u \,\mathrm{d}x_1\right) + f(x), & \text{in } \Omega_1, \\ u(x_1, 0) = u(x_1, 1) = 0, & \text{for a.e. } x_1 \in (0,1) \end{cases} \tag{1}$$

where $x := (x_1, x_2)$, g is a nonlinear function, h and $\mathbf{a}$ are positive functions, with $\mathbf{a} > \alpha$ for some constant $\alpha > 0$. First, we note that due to the presence of the non-local term that involves the values of u on each segment $(0,1) \times \{x_2\}$, this problem cannot be considered as a family of elliptic problems parametrized by x_1. The above problem, as well as the one arising from the time-dependent diffusion equation

$$\frac{\partial u}{\partial t} - \partial_{x_2}\left(\mathbf{a}(x)\,\partial_{x_2} u\right) = g\left(\int_0^1 h(x_1)\, u\,\mathrm{d}x_1\right) + f(x), \quad t \geq 0, \tag{2}$$

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involve a non-standard type of equations with second-order differentiation only in some of the variables (like $\partial_{x_2}(\mathbf{a}(x)\partial_{x_2}u)$ in (2)) and a non-local term given by a partial integral on the other variables. Such problems appear in certain areas of physics and engineering. For instance, an equation similar to (2) appears in the context of time evolution of some populations of nonlinearly coupled random oscillators, described by a model proposed by Kuramoto and Sakaguchi, see [1, Equations (11) and (21)] and [2–4]. In neutron transport theory, the authors of [5] applied the so-called Vladimiriov method (see also [6,7]) to Boltzmann's transport equation. The produced model is an integro-differential problem that is close to (1).

To illustrate the type of problems dealt with in this work, let us take

$$g(s) = -|s|^{r-1}s, \quad r > 1.$$

Then, a weak solution for Problem (1) is a function u satisfying

$$\int_{\Omega_1} \mathbf{a}\partial_{x_2}u\partial_{x_2}v\,\mathrm{d}x + \int_{\Omega_1}\left(\left|\int_0^1 hu\,\mathrm{d}x_1\right|^{r-1}\int_0^1 hu\,\mathrm{d}x_1\right)v\,\mathrm{d}x = \int_{\Omega_1} fv\,\mathrm{d}x, \tag{3}$$

for every $v \in \mathcal{D}(\Omega_1)$, the set of C^∞ function with compact support in Ω_1. To establish some a-priori estimates for u, and due to the presence of the non-local term $\int_0^1 hu\,\mathrm{d}x_1$, it will be more convenient to take hv as a test function (rather than v) in the weak formulation (3). By analogy to Kufner and Rákosník [8], we say that u is an 'h-weak solution' to (1) whenever u satisfies

$$\int_{\Omega_1} \mathbf{a}h\partial_{x_2}u\partial_{x_2}v\,\mathrm{d}x + \int_0^1\left(\left|\int_0^1 hu\,\mathrm{d}x_1\right|^{r-1}\int_0^1 hu\,\mathrm{d}x_1\int_0^1 hv\,\mathrm{d}x_1\right)\mathrm{d}x_2 = \int_{\Omega_1} fvh\,\mathrm{d}x, \tag{4}$$

for every $v \in \mathcal{D}(\Omega_1)$. The fact that $\int_0^1 hu\,\mathrm{d}x_1$ depends only on x_2 is taken into account.

The lack of differentiation in the direction x_1 in the differential operator made Problem (1) less familiar. To overcome this inconvenience and profit from the theory of elliptic operators, we employ an anisotropic perturbation in the x_1-direction. Thus we consider

$$\begin{cases} -\frac{\varepsilon^2}{h}\partial_{x_1}^2 u_\varepsilon - \partial_{x_2}\left(\mathbf{a}\partial_{x_2}u_\varepsilon\right) + \left|\int_0^1 hu_\varepsilon\,\mathrm{d}x_1\right|^{r-1}\int_0^1 hu_\varepsilon\,\mathrm{d}x_1 = f & \text{in } \Omega_1, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_1, \end{cases} \tag{5}$$

where $\varepsilon > 0$ is a small parameter. Of course, the produced problem is elliptic and the function u_ε should have more regularity in the x_1-direction. The corresponding integral equation is given by

$$\begin{aligned} &\int_{\Omega_1} \varepsilon^2\partial_{x_1}u_\varepsilon\partial_{x_1}v + \mathbf{a}h\partial_{x_2}u_\varepsilon\partial_{x_2}v\,\mathrm{d}x \\ &\quad + \int_0^1\left(\left|\int_0^1 hu_\varepsilon\,\mathrm{d}x_1\right|^{r-1}\int_0^1 hu_\varepsilon\,\mathrm{d}x_1\int_0^1 hv\,\mathrm{d}x_1\right)\mathrm{d}x_2 = \int_{\Omega_1} hfv\,\mathrm{d}x, \quad \forall\, v \in \mathcal{D}(\Omega_1). \end{aligned} \tag{6}$$

Dividing by h in the term of perturbation $-\frac{\varepsilon^2}{h}\partial_{x_1}^2 u_\varepsilon$ ensures the symmetry of the bilinear form in the h-weak formulation of (5). This allows us to use a variational approach which is more adapted to the nonlinearity considered here (see for instance [9–11]). Therefore, we examine the existence of solutions u_ε for (6) as critical points for the functional

$$J^\varepsilon(v) := \frac{1}{2}\int_{\Omega_1} \varepsilon^2 \left|\partial_{x_1} v\right|^2 + \mathbf{a}h\left|\partial_{x_2} v\right|^2 \, \mathrm{d}x + \frac{1}{r+1}\int_0^1 \left|\int_0^1 hv\, \mathrm{d}x_1\right|^{r+1} \mathrm{d}x_2 - \int_{\Omega_1} fvh\, \mathrm{d}x.$$

Besides the existence and uniqueness of the solution, our main interest is to study the asymptotic behaviour for anisotropic singular perturbed problems where the solution u_ε is a critical point of the functional J^ε, as $\varepsilon \to 0$. Roughly speaking, we show the convergences

$$u_\varepsilon \to u_0 \quad \text{and} \quad J^\varepsilon(u_\varepsilon) \to J^0(u_0), \quad \text{as } \varepsilon \to 0$$

where u_0 is an h-weak solution to Problem (1) that is also a critical point to the functional

$$J^0(v) := \frac{1}{2}\int_{\Omega_1} \mathbf{a}h\left|\partial_{x_2} v\right|^2 \, \mathrm{d}x + \frac{1}{r+1}\int_0^1 \left|\int_0^1 hv\, \mathrm{d}x_1\right|^{r+1} \mathrm{d}x_2 - \int_{\Omega_1} fvh\, \mathrm{d}x.$$

The problems treated in this work are multidimensional and involve more general nonlinearities, as we will see in the rest of this section. Let ω_1 and ω_2 be two open bounded subsets of $\mathbb{R}^{n_1}$ (resp. $\mathbb{R}^{n_2}$), where n_1 and n_2 are two positive integers. We set

$$\Omega = \omega_1 \times \omega_2$$

and split the components of a point $x \in \mathbb{R}^{n_1+n_2}$ into the n_1 first components and the n_2 last ones, i.e.

$$x := (X_1, X_2) \quad \text{where } X_1 := \left(x_1, \ldots, x_{n_1}\right) \quad \text{and} \quad X_2 := \left(x_1^{'}, \ldots, x_{n_2}^{'}\right).$$

With this notation, we set

$$\nabla_{X_1} u := \left(\partial_{x_1} u, \ldots, \partial_{x_{n_1}} u_2\right)^T, \quad \nabla_{X_2} u := \left(\partial_{x_1^{'}} u, \ldots, \partial_{x_{n_2}^{'}} u\right)^T \quad \text{and} \quad \triangle_{X_1} := \sum_{i=1}^{n_1} \partial_{x_i}^2.$$

Let us denote by $A = (\mathbf{a}_{ij}(x))$ an $n_2 \times n_2$ matrix that is symmetric, positive definite and satisfying

$$\mathbf{a}_{ij} \in L^\infty(\Omega), \quad \forall\, i,j = 1 \cdots n_2. \tag{7}$$

$$\alpha\,|\zeta|^2 \le A(x)\,\zeta \cdot \zeta \le \beta\,|\zeta|^2, \quad \forall\, \zeta \in \mathbb{R}^{n_2}, \quad \text{a.e. } x \in \Omega, \tag{8}$$

for some constants $\alpha, \beta > 0$. Then, we consider the integro-differential problem to find u satisfying

$$\begin{cases} -\nabla_{X_2} \cdot \left(A(x)\,\nabla_{X_2} u\right) = g(X_2, l(u)) + f_1(x) + f_2(X_2) & \text{in } \Omega, \\ u(X_1, \cdot) = 0, \text{ for a.e. } X_1 \in \omega_1 & \text{on } \partial\omega_2, \end{cases} \tag{9}$$

where $g : \omega_2 \times \mathbb{R} \to \mathbb{R}$ is a (nonlinear) function involving the non-local term l defined as

$$l(u) := \int_{\omega_1} h(X_1) u(X_1, X_2)\, \mathrm{d}X_1. \tag{10}$$

Although the function h does not appear in the main operator in (9), it will play a weight role. This forces the study to be established in the framework of weighted Sobolev spaces.

Furthermore, the function h may have zeros on the boundary of ω_1, which makes the functional setting more delicate.

As it is illustrated in the model problem (5), the method of anisotropic singular perturbations will be applied by approximating the first equation in (9) by the following one:

$$-\frac{\varepsilon^2}{h}\triangle_{X_1}u_\varepsilon - \nabla_{X_2}\cdot\left(A\left(x\right)\nabla_{X_2}u_\varepsilon\right) = g(X_2, l(u_\varepsilon)) + f_1\left(x\right) + f_2\left(X_2\right) \quad \text{in } \Omega, \tag{11}$$

coupled with some boundary conditions. The functional setting and the assumptions on the real-valued functions g, h, f_1 and f_2 will be further stipulated in the next section.

In a series of recent works, the authors in [12–17] considered anisotropic singular perturbations for different types of equations (elliptic, parabolic, hyperbolic, ...). This technique was reused in [14,18,19] as a tool to show the existence of solutions for some degenerate non-local semilinear problems. By adding perturbation terms involving a small parameter ε, the problem becomes non-degenerate and the existence of weak solutions, denoted by u_ε, can be obtained by the Schauder fixed point theorem. Then, some a-priori estimates independent of ε are established for the solutions u_ε and, letting ε tend to 0, it is shown that the limit is a weak solution for the unperturbed problem. This technique of anisotropic singular perturbation was applied earlier in [20–22] to show the existence of solutions for certain nonlinear integro-parabolic equations (of the Fokker–Planck type).

In the paper at hands, we deal with Problem (11) when g is superlinear and should be handled by a different method as the variational approach. This type of nonlinearity was not considered in the previously cited works. We also appeal to different techniques as convexity and compactness arguments to perform the passage to the limit. Since Problem (11) is strongly related to the eigenvalue problem in the framework of Fredholm theory, we also investigate the associated non-local eigenvalue problem. This was indispensable since the principal eigenvalue (called here l-eigenvalue) is involved in the assumptions that ensure the existence of solutions for the problem considered in the last section.

After the present section, we introduce some weighted spaces of Sobolev type, as well as some C^1-functional involving the non-local term $l(u)$. In Section 3, we show the existence of solutions (denoted by u_ε) for the perturbed problem (11) under some upper boundedness assumption on the nonlinear term. Then, we investigate the asymptotic behaviour of solutions, as well as of the related functionals, when $\varepsilon \to 0$. By passage to the limit, we obtain the existence of solutions for the original problem (9) as limits of the sequences of solutions u_ε. Section 4 is devoted to the non-local eigenvalue problem. We also investigate the asymptotic behaviour of the smallest l-eigenvalue and its associated l-eigenfunctions. In Section 5, we review the same issues considered in Section 3 for Problem (11) under a weaker upper boundedness on the nonlinear term combined with some compactness assumption.

2. Assumptions and functional setting

In this section, we state the assumptions and define the functional spaces and functionals related to the considered problems.

2.1. Anisotropic weighted Sobolev-type spaces

Let h be a weight function on ω_1 that satisfies

$$h \in L^1(\omega_1), \quad h^{-1} \in L^1_{loc}(\omega_1), \quad h > 0 \text{ a.e. in } \omega_1. \tag{12}$$

For $1 \le p < +\infty$, we denote by $L^p(\Omega, h)$ the set of all measurable functions on Ω for which the norm

$$v \mapsto \|v\|_{L^p(\Omega,h)} := \left(\int_\Omega h\,|v|^p \; \mathrm{d}x \right)^{1/p}$$

is finite. Then, we consider the following weighted anisotropic Sobolev space

$$\mathcal{V}(\Omega, h) := \left\{ v \in L^2(\Omega, h) \mid \left|\nabla_{X_2} v\right| \in L^2(\Omega, h) \right\}.$$

Throughout this paper, the derivatives are taken in the sense of distributions. The above space is equipped with the norm

$$v \mapsto \left(\|v\|^2_{L^2(\Omega,h)} + \left\|\nabla_{X_2} v\right\|^2_{L^2(\Omega,h)} \right)^{1/2}. \tag{13}$$

For a weight function h satisfying (12), $\mathcal{V}(\Omega, h)$ is a Hilbert space (see [23]). We will denote

$$\mathcal{V}_0(\Omega, h) := \overline{\mathcal{D}(\Omega)}, \text{ the closure with respect to the norm (13).} \tag{1}$$

For convenience, this space will be equipped by the norm

$$v \mapsto \left\|\nabla_{X_2} v\right\|_{L^2(\Omega,h)},$$

which is equivalent to the norm (13), thanks to Poincaré's inequality in the X_2-direction.

We also consider the weighted Sobolev space

$$\mathcal{W}(\Omega, h) := \left\{ v \in L^2(\Omega, h) \mid \left|\nabla_{X_1} v\right| \in L^2(\Omega) \text{ and } \left|\nabla_{X_2} v\right| \in L^2(\Omega, h) \right\},$$

equipped with the norm (without weight for ∇_{X_1})

$$v \mapsto \left(\|v\|^2_{L^2(\Omega,h)} + \left\|\varepsilon \nabla_{X_1} v\right\|^2_{L^2(\Omega)} + \left\|\nabla_{X_2} v\right\|^2_{L^2(\Omega,h)} \right)^{1/2}. \tag{15}$$

The parameter ε is introduced in the norm for convenience. We can easily see that if u is in $\mathcal{W}(\Omega, h)$, it is also in $L^2(\Omega)$. Under the assumption (12), $\mathcal{W}(\Omega, h)$ is also a Hilbert space and clearly $\mathcal{W}(\Omega, h) \subset \mathcal{V}(\Omega, h)$. Then, we define

$$\mathcal{W}_0(\Omega, h) := \overline{\mathcal{D}(\Omega)}, \text{ the closure with respect to the norm(15).} \tag{2}$$

Using Poincaré's inequality, the mapping

$$v \mapsto \left(\varepsilon^2 \left\|\nabla_{X_1} v\right\|^2_{L^2(\Omega)} + \left\|\nabla_{X_2} v\right\|^2_{L^2(\Omega,h)} \right)^{1/2}, \tag{17}$$

is a norm for $\mathcal{W}_0(\Omega, h)$ equivalent to the norm (15).

The two source terms f_1 and f_2 are supposed to satisfy

$$f_1 \in \mathcal{V}_0' (\Omega, h), \quad f_2 \in L^{\frac{p}{p-1}} (\omega_2), \tag{18}$$

where $\mathcal{V}_0'(\Omega, h)$ is the dual space of $\mathcal{V}_0(\Omega, h)$. Of course it holds that $\mathcal{V}_0'(\Omega, h) \subset \mathcal{W}_0'(\Omega, h)$.

To characterize the elements of $\mathcal{V}_0'(\Omega, h)$, consider $f \in \mathcal{V}_0'(\Omega, h)$. Then, by Riesz's representation theorem there exists $\phi \in \mathcal{V}_0(\Omega, h)$, such that

$$\langle f, v\rangle_{\mathcal{V}_0(\Omega,h)} = \int_\Omega h\nabla_{X_2}\phi \cdot \nabla_{X_2} v \,\mathrm{d}x, \quad \forall\, v \in \mathcal{V}_0 (\Omega, h).$$

Here and in the sequel, the brackets $\langle \cdot, \cdot\rangle$ denotes the dual product between a given Banach space and its dual. Setting $\phi_i = -\partial_{x_i'}\phi \in L^2(\Omega, h)$, $i = 1, \dots, n_2$, we can write

$$f = h\sum_{i=1}^{n_2} \partial_{x_i'}\phi_i.$$

This defines a one-to-one correspondence between $\mathcal{V}_0'(\Omega, h)$ and the set

$$\left\{\sum_{i=1}^{n_2} \partial_{x_i'}\phi_i, \ \phi_i \in L^2 (\Omega, h)\right\}.$$

From now on, we shall mean by $f \in \mathcal{V}_0'(\Omega, h)$ its corresponding element $\sum_{i=1}^{n_2} \partial_{x_i'}\phi_i$ in the above set. May be we have to point out that such element should be tested by hv to recover an element of $\mathcal{V}_0'(\Omega, h)$, i.e.

$$\left\langle -\sum_{i=1}^{n_2} \partial_{x_i'}\phi_i, hv\right\rangle := \left\langle -h\sum_{i=1}^{n_2} \partial_{x_i'}\phi_i, v\right\rangle_{\mathcal{D}(\Omega)} = \int_\Omega h\sum_{i=1}^{n_2} \phi_i\partial_{x_i'} v \,\mathrm{d}x = \langle f, v\rangle_{\mathcal{V}_0(\Omega,h)},$$
$$\times\ \forall\, v \in \mathcal{D}(\Omega).$$

Due to the presence of the non-local term, we also consider the space

$$\mathcal{L}^p (\Omega) := \left\{v \text{ measurable in } \Omega \mid l (v) \in L^p (\omega_2)\right\},$$

equipped with the semi-norm $v \mapsto \|l(v)\|_{L^p(\omega_2)}$.

Remark 2.1: By Hölder's inequality, we have

$$|l (u)|^p = \left|\int_{\omega_1} hu \,\mathrm{d}X_1\right|^p \le \|h\|_{L^1(\omega_1)}^{p-1} \int_{\omega_1} h\, |u|^p \,\mathrm{d}X_1$$

for $p \ge 1$. Then, integrating on ω_2, we get

$$\|l (u)\|_{L^p(\omega_2)} \le \|h\|_{L^1(\omega_1)}^{\frac{p-1}{p}} \|u\|_{L^p(\Omega,h)} \tag{19}$$

and thus $L^p(\Omega, h) \subset \mathcal{L}^p(\Omega)$.

Following the usual notation, we set

$$2^* := \begin{cases} 2n_2/(n_2-2) & \text{if } n_2 > 2 \\ +\infty & \text{if } n_2 \le 2. \end{cases}$$

Note that 2^* is related to n_2, the dimension of ω_2, and not to $n_1 + n_2$. The next compactness lemma is very useful to carry out some passages to the limit in the sequel.

Lemma 2.1: *Let w_n be a sequence such that $w_n \rightharpoonup w$ in $\mathcal{V}_0(\Omega, h)$. Then, we have*

$$l(w_n) \rightharpoonup l(w) \quad \text{in } H_0^1(\omega_2). \tag{20}$$

If $1 \le p < 2^*$, we moreover have the strong convergence

$$l(w_n) \to l(w) \quad \text{in } L^p(\omega_2). \tag{21}$$

Proof: Using Remark 2.1 and the fact that h is independent of X_2, we derive for $p = 2$ and $v \in \mathcal{V}_0(\Omega, h)$,

$$\|\nabla_{X_2} l(v)\|^2_{L^2(\omega_2)} = \sum_{i=1}^{n_2} \int_{\omega_2} \left|l(\partial_{x_i} v)\right|^2 \, \mathrm{d}X_2 \le \|h\|_{L^1(\omega_1)} \int_\Omega h \left|\nabla_{X_2} v(x)\right|^2 \, \mathrm{d}x.$$

Recalling that $v \mapsto \|\nabla_{X_2} v\|_{L^2(\omega_2)}$ defines a norm on $H_0^1(\omega_2)$, then

$$\|l(v)\|_{H_0^1(\omega_2)} \le \|h\|^{\frac{1}{2}}_{L^1(\omega_1)} \left\|\nabla_{X_2} v\right\|_{L^2(\Omega,h)}. \tag{22}$$

Consequently, the linear operator $v \mapsto l(v)$ from $\mathcal{V}_0(\Omega, h)$ into $H_0^1(\omega_2)$ is continuous and the image of the weakly converging sequence $(w_n)_n$ remains weakly converging. This shows (20).

If $1 \le p < 2^*$, the injection $H_0^1(\omega_2) \subset L^p(\omega_2)$ is compact and (21) follows. ∎

Remark 2.2: (i) We can replace $\mathcal{V}_0(\Omega, h)$ in Lemma 2.1 by a Banach space V such that $V \subset \mathcal{V}_0(\Omega, h)$ with continuous injection. In the sequel, we often take $V = \mathcal{W}_0(\Omega, h)$.

(ii) Since $h \in L^1(\omega_1)$, if a function $v \in H_0^1(\omega_2)$ then we have also $v \in \mathcal{V}_0(\Omega, h)$ and

$$\left\|\nabla_{X_2} v\right\|_{L^2(\Omega,h)} = \|h\|^{1/2}_{L^1(\omega_1)} \left\|\nabla_{X_2} v\right\|_{L^2(\omega_2)}.$$

2.2. The unperturbed problem

Let $r \geq 1$ and consider the space $\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ equipped with the norm

$$v \mapsto \left\| \nabla_{X_2} v \right\|_{L^2(\Omega,h)} + \| l(v) \|_{L^{r+1}(\omega_2)} .$$

We can show that $\mathcal{V}_0'(\Omega, h) \subset (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$ and $L^{\frac{r+1}{r}}(\omega_2) \subset (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$ with continuous injections, hence

$$\mathcal{V}_0' (\Omega, h) + L^{\frac{r+1}{r}} (\omega_2) \subset \left(\mathcal{V}_0 (\Omega, h) \cap \mathcal{L}^{r+1} (\Omega)\right)' . \tag{23}$$

The space $\mathcal{V}_0'(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2)$ is equipped with the norm

$$v \mapsto \inf_{v=v_1+v_2} \left\{ \|v_1\|_{\mathcal{V}_0'(\Omega,h)} + \|v_2\|_{L^{\frac{r+1}{r}}(\omega_2)} , \text{ for } v_1 \in \mathcal{V}_0' (\Omega, h) , \ v_2 \in L^{\frac{r+1}{r}} (\omega_2) \right\} . \tag{24}$$

Arguing as in [16, Corollary 1], we can show that the elements of $\mathcal{V}_0(\Omega, h)$ are the functions with zero trace on $\omega_1 \times \partial\omega_2$. More precisely,

$$\mathcal{V}_0 (\Omega, h) = \left\{ v \in \mathcal{V} (\Omega, h) \mid v (X_1, \cdot) \in H_0^1 (\omega_2) \text{ a.e. } X_1 \in \omega_1 \right\} . \tag{25}$$

This allows us to state Problem (9) as follows:

$$\begin{cases} -\nabla_{X_2} \cdot \left(A \nabla_{X_2} u\right) = g(X_2, l(u)) + f_1 (x) + f_2 (X_2) & \text{in } \Omega, \\ u \in \mathcal{V}_0 (\Omega, h) \cap \mathcal{L}^{r+1} (\Omega) . \end{cases} \tag{26}$$

As mentioned in the introduction, we shall investigate the existence of solutions in the following sense. We say that a function $u \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ is an h-weak solution to Problem (26) if the integral identity

$$\int_\Omega hA\nabla_{X_2} u \cdot \nabla_{X_2} v \,\mathrm{d}x = \int_{\omega_2} g(X_2, l(u)) l (v) \,\mathrm{d}X_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega,h)} + \int_{\omega_2} f_2 l (v) \,\mathrm{d}X_2. \tag{$\mathbf{P_0}$}$$

holds for every $v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

2.3. The perturbed problem

We deal with the following anisotropic perturbation of Problem (26):

$$\begin{cases} -\frac{\varepsilon^2}{h} \Delta_{X_1} u_\varepsilon - \nabla_{X_2} \cdot \left(A \nabla_{X_2} u_\varepsilon\right) = g(X_2, l(u_\varepsilon)) + f_1 (x) + f_2 (X_2) & \text{in } \Omega, \\ u_\varepsilon \in \mathcal{W}_0 (\Omega, h) \cap \mathcal{L}^{r+1} (\Omega) , \end{cases} \tag{27}$$

where the space $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ is equipped with the norm

$$v \mapsto \left(\varepsilon^2 \left\| \nabla_{X_1} v \right\|_{L^2(\Omega)}^2 + \left\| \nabla_{X_2} v \right\|_{L^2(\Omega,h)}^2 \right)^{1/2} + \| l (v) \|_{L^{r+1}(\omega_2)} . \tag{28}$$

We can show as for (23) that

$$\mathcal{W}_0' (\Omega, h) + L^{\frac{r+1}{r}} (\omega_2) \subset \left(\mathcal{W}_0 (\Omega, h) \cap \mathcal{L}^{r+1} (\Omega)\right)' ,$$

where the norm of $\mathcal{W}_0'(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2)$ is defined as in (24), with $\mathcal{V}_0'(\Omega, h)$ is replaced by $\mathcal{W}_0'(\Omega, h)$.

We say that a function $u_\varepsilon \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ is an h-weak solution to Problem (27) if the integral identity

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + hA\nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \, \mathrm{d}x = \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(v) \, \mathrm{d}X_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega,h)} + \int_{\omega_2} f_2 l(v) \, \mathrm{d}X_2 \qquad (\mathbf{P}_\varepsilon)$$

holds for every $v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Remark 2.3: Let $v \in \mathcal{W}_0(\Omega, h)$. On the boundary $\omega_1 \times \partial\omega_2$, we have $v(X_1, \cdot) \in H_0^1(\omega_2)$ for a.e. $X_1 \in \omega_1$ since $\mathcal{W}_0(\Omega, h) \subset \mathcal{V}_0(\Omega, h)$. There is no issue to define the trace of v on the boundary $\partial\omega_1 \times \omega_2$ since the L^2-norm of the gradient ∇_{X_1} is independent of h. In general, a definition of the boundary values depends on the speed at which the weight degenerates on the boundary, see for instance [24,25].

2.4. C^1-functionals involving the non-local term $l(u)$

Concerning the nonlinearity, we assume that $g : \omega_2 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e.

$$\begin{aligned} &\text{(i) } s \mapsto g(X_2, s) \text{ is continuous for a.e. } X_2 \in \omega_2, \\ &\text{(ii) } X_2 \mapsto g(X_2, s) \text{ is measurable for every } s \in \mathbb{R}, \end{aligned} \qquad (29)$$

and satisfies

$$\left| g\left(X_2, s\right) \right| \le \gamma_1 \left(|s| + |s|^r \right), \quad \text{for a.e. } X_2 \in \omega_2 \quad \text{and} \quad \forall\, s \in \mathbb{R}, \qquad (30)$$

where $1 \le r < +\infty$ and $\gamma_1 > 0$ are constants. Then, define

$$G\left(X_2, t\right) := \int_0^t g\left(X_2, s\right) \mathrm{d}s, \quad \text{for } X_2 \in \omega_2 \quad \text{and} \quad t \in \mathbb{R}. \qquad (31)$$

As a consequence of (30), it follows that

$$\left| G\left(X_2, t\right) \right| \le \gamma_1 \left(\frac{|t|^2}{2} + \frac{|t|^{r+1}}{r+1} \right), \quad \text{for a.e. } X_2 \in \omega_2 \quad \text{and} \quad t \in \mathbb{R}. \qquad (32)$$

This allows us to define the functional

$$J_g\left(\bar{v}\right) := \int_{\omega_2} G\left(X_2, \bar{v}\right) \mathrm{d}X_2, \quad \text{for every } \bar{v} \in L^{r+1}\left(\omega_2\right). \qquad (33)$$

To determine the derivative of J_g, we first need the following lemma.

Lemma 2.2: *Assume that h satisfies* (12) *and let $g_0 : \omega_2 \times \mathbb{R} \to \mathbb{R}$ be a function satisfying* (29) *like g and*

$$\left| g_0 (X_2, s) \right| \leq \gamma_0 \left| s \right|^r, \quad \textit{for } s \in \mathbb{R} \quad \textit{and a.e. } X_2 \in \omega_2, \tag{34}$$

for $1 \leq r < +\infty$ and some constant $\gamma_0 \geq 0$. Set

$$J_{g_0} (u) := \int_{\omega_2} G_0 (X_2, l(u))\, \mathrm{d}X_2, \quad \textit{for } u \in \mathcal{W}_0 (\Omega, h) \cap \mathcal{L}^{r+1} (\Omega),$$

where G_0 is defined by g_0 as in (31). Then, the mapping

$$u \mapsto g_0 (\cdot, l(u)) \textit{ is continuous from } \mathcal{W}_0 (\Omega, h) \cap \mathcal{L}^{r+1} (\Omega) \textit{ into } L^{\frac{r+1}{r}} (\omega_2),$$

$J_{g_0} \in C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and its derivative is given by

$$\left\langle J'_{g_0} (u), v \right\rangle_{\mathcal{W}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} = \int_{\omega_2} g_0 (X_2, l(u))\, l(v)\, \mathrm{d}X_2, \tag{35}$$

for every $u, v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Proof: Let us consider the functional

$$j_{g_0} (\bar{v}) = \int_{\omega_2} G_0 (X_2, \bar{v})\, \mathrm{d}X_2, \quad \text{for } \bar{v} \in L^{r+1} (\omega_2). \tag{36}$$

Thanks to (34), we can argue as in [9, p.37], to show that

$$\bar{u} \mapsto g_0 (\cdot, \bar{u}) \text{ is continuous from } L^{r+1} (\omega_2) \text{ into } L^{\frac{r+1}{r}} (\omega_2),$$

$j_{g_0} \in C^1(L^{r+1}(\omega_2), \mathbb{R})$ and its derivative is given by

$$\left\langle j'_{g_0} (\bar{u}), \bar{v} \right\rangle_{L^{r+1}(\omega_2)} = \int_{\omega_2} g_0 (X_2, \bar{u})\, \bar{v}\, \mathrm{d}X_2, \quad \forall\, \bar{u}, \bar{v} \in L^{r+1} (\omega_2).$$

From (19) with $p = r + 1$, we infer in particular that $u \mapsto l(u)$, from $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ into $L^{r+1}(\omega_2)$ is a continuous linear mapping. Consequently, the composite mapping

$$u \mapsto \left(g_0 \circ l \right) (u) = g_0 (\cdot, l(u))$$

is continuous from $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ into $L^{\frac{r+1}{r}}(\omega_2)$, and as we have $J_{g_0} = j_{g_0} \circ l(\cdot) \in C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$, then the derivative of J_{g_0} is given by

$$\left\langle J'_{g_0} (u), v \right\rangle_{\mathcal{W}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} = \left\langle j'_{g_0} (l(u)), l(v) \right\rangle_{L^{r+1}(\omega_2)}, \quad \text{for } u, v \in \mathcal{W}_0 (\Omega, h) \cap \mathcal{L}^{r+1} (\Omega).$$

This shows (35) and the lemma follows. ■

In the next section, we investigate the existence of solutions for $(\mathbf{P}_\epsilon)$ as critical points of the functional

$$\mathbf{J}^\varepsilon (u) := \frac{1}{2} \int_\Omega \varepsilon^2 \left| \nabla_{X_1} u \right|^2 + hA\nabla_{X_2} u \cdot \nabla_{X_2} u\, \mathrm{d}x$$

$$-\int_{\omega_2} G(X_2, l(u))\,\mathrm{d}X_2 - \langle f_1, u\rangle_{\mathcal{V}_0(\Omega,h)} - \int_{\omega_2} f_2 l(u)\,\mathrm{d}X_2. \tag{37}$$

Thus, we need to compute the derivative of $\mathbf{J}^\varepsilon$.

Theorem 2.1: *Let* $1 \le r < +\infty$. *Assume that* g *satisfies* (29) *and* (30). *Then, the mapping*

$$u \mapsto g(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \text{ into } \mathcal{W}_0'(\Omega, h) + L^{\frac{r+1}{r}}(\Omega),$$

$\mathbf{J}^\varepsilon \in C^1(\mathcal{W}_0(\Omega,h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ *and its derivative is given by*

$$\left\langle \left(\mathbf{J}^\varepsilon\right)'(u), v\right\rangle_{\mathcal{W}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} = \int_\Omega \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + hA\nabla_{X_2} u \cdot \nabla_{X_2} v\,\mathrm{d}x$$
$$-\int_{\omega_2} g(X_2, l(u))\, l(v)\,\mathrm{d}X_2 - \langle f_1, v\rangle_{\mathcal{V}_0(\Omega,h)} - \int_{\omega_2} f_2 l(v)\,\mathrm{d}X_2, \tag{38}$$

for all $u, v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Proof: First, to show the continuity of $u \mapsto g(X_2, l(u))$ we split g into two parts g_1 and g_2

$$g(X_2, s) = g_1(X_2, s) + g_2(X_2, s),$$

where $g_1(X_2, s) = g(X_2, s)$ for $|s| \le 1$ and $g_1(X_2, s) = 0$ for $|s| \ge 2$, such that

$$\left|g_1(X_2, s)\right| \le C|s|, \quad \left|g_2(X_2, s)\right| \le C|s|^r,$$

for a.e. $X_2 \in \omega_2$, and some positive constant C.

Thanks to Lemma 2.2 and Inequality (19), for $p = 2$, we see that the mapping

$$u \mapsto g_1(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^2(\Omega) \text{ into } L^2(\omega_2).$$

Since $\mathcal{W}_0(\Omega, h) \subset L^2(\Omega, h) \subset \mathcal{L}^2(\Omega)$ and $L^2(\omega_2) \subset L^2(\Omega, h) \subset \mathcal{W}_0'(\Omega, h)$, we infer that

$$u \mapsto g_1(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \text{ into } \mathcal{W}_0'(\Omega, h).$$

As well, due to Lemma 2.2, the mapping

$$u \mapsto g_2(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \text{ into } L^{\frac{r+1}{r}}(\omega_2).$$

Recalling that $\mathcal{W}_0'(\Omega, h) + L^{\frac{r+1}{r}}(\Omega)$ is equipped with the norm defined in (24), then the mapping

$$u \mapsto g(X_2, l(u)) = \left(g_1 + g_2\right)(X_2, l(u))$$

is continuous from $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ into $\mathcal{W}_0'(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2)$ and

$$\left\langle J_g'(u), v\right\rangle_{\mathcal{W}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} = \int_{\omega_2} g(X_2, l(u))\, l(v)\,\mathrm{d}X_2, \quad \text{for } u, v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega). \tag{39}$$

To give the derivatives for the remaining terms in the definition of $\mathbf{J}^\varepsilon$, we set

$$J_\nabla^\varepsilon : \mathcal{W}_0(\Omega, h) \longmapsto \mathbb{R}, J_\nabla^\varepsilon(u) := \frac{1}{2}\int_\Omega \varepsilon^2 \left|\nabla_{X_1} u\right|^2 + hA\nabla_{X_2} u \cdot \nabla_{X_2} u\,\mathrm{d}x,$$

$$J_{f_1} : \mathcal{V}_0(\Omega, h) \mapsto \mathbb{R}, J_{f_1}(u) := \langle f_1, u \rangle_{\mathcal{V}_0(\Omega,h)},$$

$$J_{f_2} : \mathcal{L}^{r+1}(\Omega) \cap \mathcal{W}_0(\Omega, h) \mapsto \mathbb{R}, J_{f_2}(u) := \int_{\omega_2} f_2 l(u) \, dX_2.$$

First, $J^\varepsilon_\nabla \in C^1(\mathcal{W}_0(\Omega, h), \mathbb{R}) \subset C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and $(J^\varepsilon_\nabla)'$ is given by

$$\begin{aligned}\left\langle \left(J^\varepsilon_\nabla\right)'(u), v\right\rangle_{\mathcal{W}_0(\Omega,h)} &:= \lim_{t\to 0} \frac{J^\varepsilon_\nabla(u+tv) - J^\varepsilon_\nabla(u)}{t} \\ &= \lim_{t\to 0} \int_\Omega \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + t\varepsilon^2 \left|\nabla_{X_1} v\right|^2 + hA\nabla_{X_2} u \cdot \nabla_{X_2} v \\ &\quad + thA\nabla_{X_2} v \cdot \nabla_{X_2} v \, dx,\end{aligned}$$

hence

$$\left\langle \left(J^\varepsilon_\nabla\right)'(u), v\right\rangle_{\mathcal{W}_0(\Omega,h)} = \int_\Omega \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + hA\nabla_{X_2} u \cdot \nabla_{X_2} v \, dx. \tag{40}$$

As the functionals J_{f_1} and J_{f_2} are linear, this immediately implies that $J_{f_1} \in C^1(\mathcal{V}_0(\Omega, h), \mathbb{R}) \subset C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ with

$$\left\langle J'_{f_1}(u), v\right\rangle_{\mathcal{W}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} = \langle f_1, v\rangle_{\mathcal{V}_0(\Omega,h)} \tag{41}$$

and $J_{f_2} \in C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ with

$$\left\langle J'_{f_2}(u), v\right\rangle_{\mathcal{W}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} = \int_{\omega_2} f_2 l(v) \, dX_2. \tag{42}$$

Taking into account (39)–(42), we obtain (38) since $\mathbf{J}^\varepsilon = J^\varepsilon_\nabla - J_g - J_{f_1} - J_{f_2}$. ■

Corollary 2.1: *Let* $1 \le r < 2^* - 1$. *Assume that g satisfies* (29) *and* (30). *Then,*

$$u \to g(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \text{ into } \mathcal{W}'_0(\Omega, h),$$

$\mathbf{J}^\varepsilon \in C^1(\mathcal{W}_0(\Omega, h), \mathbb{R})$ *and* (38) *holds for all* $u, v \in \mathcal{W}_0(\Omega, h)$.

Proof: By Sobolev's embedding $H^1_0(\omega_2) \subset L^{r+1}(\omega_2)$ and (22), we get

$$\begin{aligned}\|l(v)\|_{L^{r+1}(\omega_2)} &\le C_S \|h\|^{\frac{1}{2}}_{L^1(\omega_1)} \left\|\nabla_{X_2} v\right\|_{L^2(\Omega,h)} \\ &\le C_S \|h\|^{\frac{1}{2}}_{L^1(\omega_1)} \left(\varepsilon^2 \left\|\nabla_{X_1} v\right\|^2_{L^2(\Omega)} + \left\|\nabla_{X_2} v\right\|^2_{L^2(\Omega,h)}\right)^{\frac{1}{2}}\end{aligned} \tag{43}$$

for all $v \in \mathcal{W}_0(\Omega, h)$. Here, C_S denotes the Sobolev constant. Thus, $\mathcal{W}_0(\Omega, h) \subset \mathcal{L}^{r+1}(\Omega)$ and the corollary results follow from Theorem 2.1. ■

Remark 2.4: In a similar manner, we can also show that Theorem 2.1 and Corollary 2.1 still hold if we replace $\mathcal{W}_0(\Omega, h)$ by $\mathcal{V}_0(\Omega, h)$ and $\mathbf{J}^\varepsilon$ by the functional

$$\mathbf{J}^0(v) := \frac{1}{2}\int_\Omega hA\nabla_{X_2} v \cdot \nabla_{X_2} v \, dx - \int_{\omega_2} G(X_2, l(v)) \, dX_2 - \langle f_1, v\rangle_{\mathcal{V}_0(\Omega,h)} - \int_{\omega_2} f_2 l(u) \, dX_2. \tag{44}$$

3. A problem with superlinear growth of non-local terms

In this section, we aim to establish the existence of solutions u_ε for $(\mathbf{P}_\epsilon)$ as global minimizers for $\mathbf{J}^\varepsilon$, defined by (37), provided that G satisfies some upper boundedness assumption. Then, we study the asymptotic behaviour of u_ε and $\mathbf{J}^\varepsilon(u_\varepsilon)$, as $\varepsilon \to 0$.

3.1. The perturbed problem

We have the following existence result for Problem $(\mathbf{P}_\epsilon)$.

Theorem 3.1: *Let $\varepsilon > 0$ and assume that g satisfies* (29) *and* (30) *for $r \geq 1$. Suppose further that G, defined by* (31) *satisfies*

$$G(X_2, t) \leq C_1 - a\,|t|^{r+1}, \quad \forall\, t \in \mathbb{R} \tag{45}$$

for a.e. $X_2 \in \omega_2$ and some constants $C_1 \geq 0, a > 0$. Then, there exists $u \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ such that

$$\mathbf{J}^\varepsilon(u) = \inf_{v \in \mathcal{W}_0(\Omega,h) \cap \mathcal{L}^{r+1}(\Omega)} \mathbf{J}^\varepsilon(v). \tag{46}$$

In particular, u is a solution of $(\mathbf{P}_\epsilon)$.

Proof: *A priori estimates.* Thanks to (45), we deduce that

$$\begin{aligned}\mathbf{J}^\varepsilon(v) \geq \frac{1}{2}\int_\Omega \varepsilon^2 \left|\nabla_{X_1} v\right|^2 + hA\nabla_{X_2} v \cdot \nabla_{X_2} u \,\mathrm{d}x + a\int_{\omega_2} |l(v)|^{r+1}\,\mathrm{d}X_2 \\ - \langle f_1, v\rangle_{\mathcal{V}_0(\Omega,h)} - \int_{\omega_2} f_2 l(v)\,\mathrm{d}X_2 - C_1\,|\omega_2|,\end{aligned} \tag{47}$$

where $|\omega_2|$ denotes the Lebesgue measure of the set ω_2. Since $\mathcal{V}_0(\Omega, h)$ is equipped with the norm $v \mapsto \|\nabla_{X_2} v\|_{L^2(\Omega,h)}$, we infer that

$$\begin{aligned}\langle f_1, v\rangle_{\mathcal{V}_0(\Omega,h)} &\leq \|f_1\|_{\mathcal{V}_0'(\Omega,h)} \left(\int_\Omega h\left|\nabla_{X_2} v\right|^2\,\mathrm{d}x\right)^{1/2} \\ &\leq \|f_1\|_{\mathcal{V}_0'(\Omega,h)} \left(\int_\Omega \varepsilon^2\left|\nabla_{X_1} v\right|^2 + h\left|\nabla_{X_2} v\right|^2\,\mathrm{d}x\right)^{1/2}.\end{aligned} \tag{48}$$

Then, by (8) and Hölder's inequality it comes that

$$\begin{aligned}\mathbf{J}^\varepsilon(v) \geq \frac{1}{2}\int_\Omega \varepsilon^2 \left|\nabla_{X_1} v\right|^2 + h\alpha\left|\nabla_{X_2} v\right|^2\,\mathrm{d}x + a\int_{\omega_2} |l(v)|^{r+1}\,\mathrm{d}X_2 \\ - \|f_1\|_{\mathcal{V}_0'(\Omega,h)} \left(\int_\Omega \varepsilon^2\left|\nabla_{X_1} v\right|^2 + h\left|\nabla_{X_2} v\right|^2\,\mathrm{d}x\right)^{1/2} \\ - \|f_2\|_{L^{\frac{r+1}{r}}(\omega_2)} \|l(v)\|_{L^{r+1}(\omega_2)} - C_1\,|\omega_2|.\end{aligned}$$

Applying Young's inequality, we get

$$\mathbf{J}^\varepsilon(v) \geq \frac{\min\{1,\alpha\}}{4}\int_\Omega \varepsilon^2\left|\nabla_{X_1} v\right|^2 + h\left|\nabla_{X_2} v\right|^2\,\mathrm{d}x + \frac{a}{2}\int_{\omega_2} |l(v)|^{r+1}\,\mathrm{d}X_2$$

$$- \frac{1}{\min\{1,\alpha\}} \|f_1\|^2_{\mathcal{V}_0'(\Omega,h)} - \frac{2^{\frac{1}{r}} r}{a^{\frac{1}{r}} (r+1)^{\frac{r+1}{r}}} \|f_2\|^{\frac{r+1}{r}}_{L^{\frac{r+1}{r}}(\omega_2)} - C_1 |\omega_2|, \tag{49}$$

for all $v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. Thus $\mathbf{J}^\varepsilon$ is bounded below in $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Passage to the limit. We set

$$d^\varepsilon := \inf_{v \in \mathcal{W}_0(\Omega,h) \cap \mathcal{L}^{r+1}(\Omega)} \mathbf{J}^\varepsilon(v)$$

and let $(u_n)_{n\geq 0}$ be a minimizing sequence. By (49), such a sequence is necessarily bounded in $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$, i.e.

$$\begin{aligned} u_n, \left|\nabla_{X_2} u_n\right| &\text{ are bounded in } L^2(\Omega, h), \\ \left|\nabla_{X_1} u_n\right| &\text{ is bounded in } L^2(\Omega), \\ l(u_n) &\text{ is bounded in } L^{r+1}(\omega_2). \end{aligned}$$

The boundedness of u_n is a consequence of Poincaré's inequality in the X_2-direction. Then, there exist $u \in \mathcal{W}_0(\Omega, h)$, $U_1 \in (L^2(\Omega))^{n_1}$ and $U_2 \in (L^2(\Omega, h))^{n_2}$ such that – up to a subsequence –

$$\begin{aligned} &u_n \rightharpoonup u, \nabla_{X_2} u_n \rightharpoonup U_2 \text{ in } L^2(\Omega, h) \text{ and } \nabla_{X_1} u_n \rightharpoonup U_1, \text{ in } L^2(\Omega), \\ &l(u_n) \rightharpoonup l(u) \text{ in } L^{r+1}(\omega_2), \end{aligned}$$

as $n \to +\infty$. (These convergences meant component by component convergences). Since the above convergences imply the convergence in the distributional sense, we can easily see that

$$u_n \rightharpoonup u, \quad \nabla_{X_2} u_n \rightharpoonup \nabla_{X_2} u \text{ in } L^2(\Omega, h) \tag{50}$$

$$\nabla_{X_1} u_n \rightharpoonup \nabla_{X_1} u \text{ in } L^2(\Omega). \tag{51}$$

Using Lemma 2.1, with $p = 2$, we obtain the strong convergence $l(u_n) \to l(u)$ in $L^2(\omega_2)$. Then, up to a subsequence

$$l(u_n) \to l(u) \quad \text{a.e. in } \omega_2 \tag{52}$$

and the continuity of G implies that

$$G(X_2, l(u_n)) \to G(X_2, l(u)) \quad \text{a.e. in } \omega_2, \quad \text{as } n \to +\infty. \tag{53}$$

To perform the passage to the limit in $\mathbf{J}^\varepsilon$, we write

$$\begin{aligned} \mathbf{J}^\varepsilon(v) = &\frac{1}{2} \int_\Omega \varepsilon^2 \left|\nabla_{X_1} u_n\right|^2 + hA\nabla_{X_2} u_n \cdot \nabla_{X_2} u_n \, dx \\ &+ \int_{\omega_2} (C_1 - G(X_2, l(u_n))) \, dX_2 - C_1 |\omega_2| - \langle f_1, u_n \rangle_{\mathcal{V}_0(\Omega,h)} - \int_{\omega_2} f_2 l(u_n) \, dX_2, \end{aligned} \tag{54}$$

where $C_1 - G(X_2, v) \geq 0$, for a.e. $X_2 \in \omega_2$. Then, Fatou's lemma and the convergences (52) and (53) imply

$$\int_{\omega_2} C_1 - G(X_2, l(u))\,\mathrm{d}X_2 \leq \liminf_{n\to\infty} \int_{\omega_2} C_1 - G(X_2, l(u_n))\,\mathrm{d}X_2.$$

Besides, since A is symmetric and satisfies (8), then the mapping

$$v \mapsto \left(\int_\Omega \varepsilon^2 \left|\nabla_{X_1} v\right|^2 + hA\nabla_{X_2} v \cdot \nabla_{X_2} v\,\mathrm{d}x\right)^{1/2}$$

defines a norm on $\mathcal{W}_0(\Omega, h)$. Thanks to (50), (51) and the lower semi-continuity of norms, we deduce that

$$\int_\Omega \varepsilon^2 \left|\nabla_{X_1} u\right|^2 + hA\nabla_{X_2} u \cdot \nabla_{X_2} u\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega \varepsilon^2 \left|\nabla_{X_1} u_n\right|^2 + hA\nabla_{X_2} u_n \cdot \nabla_{X_2} u_n\,\mathrm{d}x.$$

Passing to the limit in (54), we get

$$\mathbf{J}^\varepsilon(u) \leq \liminf_{n\to\infty} \mathbf{J}^\varepsilon(u_n) = d^\varepsilon.$$

Therefore $\mathbf{J}^\varepsilon(u) = d^\varepsilon$.

Finally, by Theorem 2.1 we have $(J^\varepsilon)'(u) = 0$, i.e. u is a solution to $(\mathbf{P}_\epsilon)$. This ends the proof of the theorem. ■

Remark 3.1: Problems $(\mathbf{P}_\epsilon)$ may have one or several solutions, depending on the assumptions on g, f_1 and f_2. If the mapping $t \mapsto g(\cdot, t)$ is nonincreasing, then the operator

$$u \mapsto -\frac{\varepsilon^2}{h}\Delta_{X_1} u - \nabla_{X_2} \cdot \left(A\nabla_{X_2} u\right) - g(\cdot, u) \tag{55}$$

is strictly monotone. In this case, the solution of Problem $(\mathbf{P}_\epsilon)$ given by Theorem 3.1 is unique.

Example 3.1: Theorem 3.1 applies to the problem

$$\begin{cases} -\frac{\varepsilon^2}{h}\Delta_{X_1} u - \nabla_{X_2} \cdot \left(A\nabla_{X_2} u\right) + a_1 \left|l(u)\right|^{r-1} l(u) \\ \quad -a_2 \left|l(u)\right|^{q-1} l(u) = f_1 + f_2 \text{ in } \mathcal{W}_0'(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2), \\ u \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega). \end{cases}$$

Here we have taken

$$g(X_2, t) = -a_1 |t|^{r-1} t + a_2 |t|^{q-1} t,$$

where $a_1, a_2 \in \mathbb{R}, a_1 > 0$ and $1 \leq q < r$. The assumptions (30) and (45) hold, since

$$|t|^q < |t| + |t|^r \quad \text{and} \quad a_2 |t|^{q+1} \leq C + \frac{a_1(q+1)}{2(r+1)} |t|^{r+1},$$

for some constant $C > 0$.

3.2. Passage to the limit

Letting now $\varepsilon \to 0$, we have the following convergence results that can be improved in the next theorem.

Lemma 3.1: *Let u_ε be a solution of Problem $(\mathbf{P}_\epsilon)$. Under the assumptions of Theorem 3.1, we have – up to a subsequence –*

$$u_\varepsilon \rightharpoonup u_0, \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h), \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \text{ in } L^2(\Omega), \tag{56}$$

$$l(u_\varepsilon) \to l(u_0) \text{ in } L^2(\omega_2), \tag{57}$$

$$l(u_\varepsilon) \rightharpoonup l(u_0) \text{ in } L^{r+1}(\omega_2), \tag{58}$$

as $\varepsilon \to 0$, where u_0 is a solution of Problem $(\mathbf{P}_\epsilon)$.

Moreover, the above convergences hold for the whole sequence if the solution of Problem $(\mathbf{P}_\epsilon)$ is unique.

Proof: Without loss of generality, we assume that $0 < \varepsilon < 1$. Then, we know that

$$\mathbf{J}^\varepsilon(u_\varepsilon) \leq \mathbf{J}^\varepsilon(v) \leq \mathbf{J}^1(v), \quad \forall\, v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega). \tag{59}$$

Thanks to (49), we deduce that

$$\begin{aligned} &\frac{\min\{1,\alpha\}}{4} \int_\Omega \varepsilon^2 \left|\nabla_{X_1} u_\varepsilon\right|^2 \mathrm{d}x + h\left|\nabla_{X_2}(u_\varepsilon)\right|^2 \mathrm{d}x + \frac{a}{2}\|l(u_\varepsilon)\|_{L^{r+1}(\omega_2)}^{r+1} \\ &\leq \mathbf{J}^1(v) + \|f_1\|_{\mathcal{V}_0'(\Omega,h)}^2 + C_1|\omega_2| + \frac{2^{\frac{1}{r}} r}{a^{\frac{1}{r}}(r+1)^{\frac{r+1}{r}}} \|f_2\|_{L^{\frac{r+1}{r}}(\omega_2)}^{\frac{r+1}{r}}. \end{aligned} \tag{60}$$

and it follows that

$$\begin{aligned} u_\varepsilon, \left|\nabla_{X_2} u_\varepsilon\right| &\text{ are bounded in } L^2(\Omega, h), \\ \left|\varepsilon \nabla_{X_1} u_\varepsilon\right| &\text{ is bounded in } L^2(\Omega) \end{aligned}$$

and

$$l(u_\varepsilon) \text{ is bounded in } L^{r+1}(\omega_2). \tag{61}$$

Then, there exist $u_0 \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$, $\tilde{U}_1 \in (L^2(\Omega))^{n_1}$, $\tilde{U}_2 \in (L^2(\Omega, h))^{n_2}$ such that – up to a subsequence –

$$u_\varepsilon \rightharpoonup u_0, \nabla_{X_2} u_\varepsilon \rightharpoonup \tilde{U}_2 \text{ in } L^2(\Omega, h) \quad \text{and} \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup \tilde{U}_1 \quad \text{in } L^2(\Omega),$$

as $\varepsilon \to 0$. Since the above convergences imply the convergence in the distributional sense, we can check that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{and} \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \quad \text{in } L^2(\Omega, h). \tag{62}$$

As well, since $|\varepsilon \nabla_{X_1} u_\varepsilon|$ is bounded in $L^2(\Omega)$ and $\frac{div_{X_1}\Phi}{\sqrt{h}} \in L^2(\Omega, h), \forall \Phi \in (D(\Omega))^{n_1}$, then

$$\int_\Omega \varepsilon \nabla_{X_1} u_\varepsilon \cdot \Phi \,\mathrm{d}x = \int_\Omega \varepsilon \sqrt{h} u_\varepsilon \left(-\frac{div_{X_1}\Phi}{\sqrt{h}}\right) \mathrm{d}x \to 0, \quad \forall\, \Phi \in (D(\Omega))^{n_1}.$$

It follows that

$$\varepsilon\nabla_{X_1}u_\varepsilon \rightharpoonup 0, \quad \text{in } L^2(\Omega). \tag{63}$$

Besides, we have

$$l(u_\varepsilon) \rightharpoonup l(u_0) \quad \text{in } L^{r+1}(\omega_2) \tag{64}$$

and also the strong convergence $l(u_\varepsilon) \to l(u_0)$ in $L^2(\omega_2)$ from Lemma 2.1 with $p = 2$. Then, we have – up to a new subsequence –

$$l(u_\varepsilon) \to l(u_0), \quad \text{a.e. in } \omega_2. \tag{65}$$

Assumption (30) and Young's inequality $t \leq (r - 1 + 2|t|^{r+1})/r + 1$ imply that

$$\left|g(X_2, l(u_\varepsilon))\right|^{\frac{r+1}{r}} \leq C\left(1 + |l(u_\varepsilon)|^{r+1}\right). \tag{66}$$

Thus $g(X_2, l(u_\varepsilon))$ is bounded in $L^{\frac{r+1}{r}}(\omega_2)$ since $l(u_\varepsilon)$ is bounded in $L^{r+1}(\omega_2)$. Moreover, (65) and the continuity of g ensure that

$$g(X_2, l(u_\varepsilon)) \to g(X_2, l(u_0)), \quad \text{a.e. in } \omega_2.$$

Then, due to Lemma 1.3 in [26, p.12], we infer that

$$g(X_2, l(u_\varepsilon)) \rightharpoonup g(X_2, l(u_0)) \quad \text{in } L^{\frac{r+1}{r}}(\omega_2). \tag{67}$$

The convergences (62)–(64) and (67) allow us to pass to the limit in ($\mathbf{P}_\epsilon$). It comes that u_0 satisfies

$$\int_\Omega hA\nabla_{X_2}u_0 \cdot \nabla_{X_2}v\,\mathrm{d}x = \int_{\omega_2} g(X_2, l(u_0))l(v)\,\mathrm{d}X_2 + \langle f_1, v\rangle_{\mathcal{V}_0(\Omega,h)} + \int_{\omega_2} f_2 l(v)\,\mathrm{d}X_2, \tag{68}$$

for all $v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. By density of $\mathcal{W}_0(\Omega, h)$ in $\mathcal{V}_0(\Omega, h)$, this means that the limit u_0 is a solution to Problem ($\mathbf{P_0}$). ■

Remark 3.2: Under the assumption (30) and (45), the function $g(X_2, t)$ behaves like $-|t|^{r-1}t$ for large t, but it is not necessarily monotone. This prevents the use of the monotonicity argument to get the strong convergence as in [15,16]. An example illustrating this situation can be given by $g(t) = -t - (2 + \cos(e^t))|t|^{r-1}t$.

We have the following result about the convergence for the minimum of $\mathbf{J}^\varepsilon$ as $\varepsilon \to 0$.

Theorem 3.2: *Under the assumptions of Lemma* 3.1, *the converging subsequences give a smooth minimizing sequences for the functional* $\mathbf{J}^0$. *That is*

$$\mathbf{J}^\varepsilon(u_\varepsilon) \to \mathbf{J}^0(u_0) = \inf_{v\in\mathcal{V}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} \mathbf{J}^0(v), \tag{69}$$

as $\varepsilon \to 0$, *where the functional* $\mathbf{J}^0$ *is defined by* (44). *Moreover, we have the strong convergences*

$$u_\varepsilon \to u_0, \nabla_{X_2}u_\varepsilon \to \nabla_{X_2}u_0 \text{ in } L^2(\Omega, h), \quad \varepsilon\nabla_{X_1}u_\varepsilon \to 0 \text{ in } L^2(\Omega), \tag{70}$$

$$G(\cdot, l(u_\varepsilon)) \to G(\cdot, l(u_0)) \ \ in\ L^1(\omega_2), \tag{71}$$

$$l(u_\varepsilon) \to l(u_0) \ \ in\ L^{r+1}(\omega_2). \tag{72}$$

Proof: We begin by establishing (69). Since u_ε realizes the minimum of $\mathbf{J}^\varepsilon$, then

$$\mathbf{J}^\varepsilon(u_\varepsilon) \le \mathbf{J}^\varepsilon(v), \quad \forall\, v \in \mathcal{D}(\Omega).$$

Passing to the lim sup, we get

$$\limsup_{\varepsilon\to 0} \mathbf{J}^\varepsilon(u_\varepsilon) \le \mathbf{J}^0(v), \quad \forall\, v \in \mathcal{D}(\Omega).$$

By density and the fact that $\mathbf{J}^0 \in C^1(\mathcal{V}_0(\Omega,h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ (see Remark 2.4), the precedent inequality also holds for every $v \in \mathcal{V}_0(\Omega,h) \cap \mathcal{L}^{r+1}(\Omega)$. Thus

$$\limsup_{\varepsilon\to 0} \mathbf{J}^\varepsilon(u_\varepsilon) \le \inf_{v\in\mathcal{V}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} \mathbf{J}^0(v). \tag{73}$$

Since G satisfies (32), then we have

$$|G(X_2, l(u_\varepsilon))| \le C\left(1 + |l(u_\varepsilon)|^{r+1}\right), \tag{74}$$

for some constant $C > 0$. Thanks to (61), this means that $G(\cdot, l(u_\varepsilon))$ is a bounded sequence in $L^1(\omega_2)$. Due to (65) and the continuity of $t \mapsto G(X_2, t)$, for a.e. $X_2 \in \omega_2$, we get – up to a subsequence –

$$G(X_2, l(u_\varepsilon)) \to G(X_2, l(u_0)), \quad \text{a.e. in } \omega_2, \quad \text{as } \varepsilon \to 0. \tag{75}$$

Writing $\mathbf{J}^\varepsilon$ as in (54), then the convergences (56), (58) and (75) allow us to pass to the infimum limit in $\mathbf{J}^\varepsilon(u_\varepsilon)$. We deduce that

$$\mathbf{J}^0(u_0) \le \liminf_{\varepsilon\to 0} \mathbf{J}^\varepsilon(u_\varepsilon).$$

Reporting this to (73), we obtain $\mathbf{J}^0(u_0) \le \inf_{v\in\mathcal{V}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} \mathbf{J}^0(v)$ and thus (69) is proved.

To show the strong convergences, we consider

$$\begin{aligned} I_\varepsilon := \frac{1}{2}\int_\Omega \varepsilon^2 \left|\nabla_{X_1} u_\varepsilon\right|^2 + hA\nabla_{X_2}(u_\varepsilon - u_0)\cdot\nabla_{X_2}(u_\varepsilon - u_0)\,\mathrm{d}x \\ + \int_{\omega_2} |G(X_2, l(u_\varepsilon)) - G(X_2, l(u_0))|\,\mathrm{d}X_2. \end{aligned} \tag{76}$$

Then, expending I_ε, we get

$$\begin{aligned} I_\varepsilon = \frac{1}{2}\int_\Omega \varepsilon^2 \left|\nabla_{X_1} u_\varepsilon\right|^2 + hA\nabla_{X_2} u_\varepsilon\cdot\nabla_{X_2} u_\varepsilon\,\mathrm{d}x - \int_\Omega hA\nabla_{X_2} u_\varepsilon\cdot\nabla_{X_2} u_0\,\mathrm{d}x \\ + \frac{1}{2}\int_\Omega hA\nabla_{X_2} u_0\cdot\nabla_{X_2} u_0\,\mathrm{d}x + \int_{\omega_2} |G(X_2, l(u_\varepsilon)) - G(X_2, l(u_0))|\,\mathrm{d}X_2. \end{aligned}$$

Going back to the definition of $\mathbf{J}^{\varepsilon}$, given in (37), we can write

$$\begin{aligned} I_\varepsilon = \mathbf{J}^{\varepsilon}(u_\varepsilon) + \langle f_1, u_\varepsilon\rangle_{\mathcal{V}_0(\Omega,h)} + \int_{\omega_2} f_2 l(u_\varepsilon)\,\mathrm{d}X_2 \\ - \int_\Omega hA\nabla_{X_2}u_\varepsilon \cdot \nabla_{X_2}u_0\,\mathrm{d}x + \frac{1}{2}\int_\Omega hA\nabla_{X_2}u_0\cdot\nabla_{X_2}u_0\,\mathrm{d}x \\ - \int_{\omega_2} C_1 - G(X_2, l(u_\varepsilon)) - |G(X_2,l(u_\varepsilon)) - G(X_2,l(u_0))|\,\mathrm{d}X_2 + C_1|\omega_2|. \end{aligned} \quad (77)$$

(Recall that $C_1 - G(X_2,t) \geq 0$, for $t \in \mathbb{R}$). Thanks to the pointwise convergence (75), we can apply Brezis–Lieb's Lemma [27] to the sequence $C_1 - G(X_2, l(u_\varepsilon))$, we get

$$\begin{aligned} \int_{\omega_2} C_1 - G(X_2,l(u_\varepsilon)) - |G(X_2,l(u_\varepsilon)) - G(X_2,l(u_0))|\,\mathrm{d}X_2 \\ \to \int_{\omega_2} C_1 - G(X_2,l(u_0))\,\mathrm{d}X_2, \end{aligned} \quad (78)$$

as $\varepsilon \to 0$. Using (69), (78) and the convergences (56), (58) from Lemma 3.1, we can pass to the limit in (77). It comes that

$$\begin{aligned} \lim_{\varepsilon\to 0} I_\varepsilon = \mathbf{J}^0(u_0) + \langle f_1, u_0\rangle_{\mathcal{V}_0(\Omega,h)} + \int_{\omega_2} f_2 l(u_0)\,\mathrm{d}X_2 \\ - \frac{1}{2}\int_\Omega hA\nabla_{X_2}u_0\cdot\nabla_{X_2}u_0\,\mathrm{d}x + \int_{\omega_2} G(X_2,l(u_0))\,\mathrm{d}X_2 = 0, \end{aligned}$$

since $\mathbf{J}^0$ is defined by (44). The convergences (70) and (71) are consequences of the inequality

$$\begin{aligned} I_\varepsilon \geq \min\{1,\alpha\}\int_\Omega \varepsilon^2\left|\nabla_{X_1}u_\varepsilon\right|^2 + h\left|\nabla_{X_2}(u_\varepsilon - u_0)\right|^2\,\mathrm{d}x \\ + \int_{\omega_2} |G(X_2,l(u_\varepsilon)) - G(X_2,l(u_0))|\,\mathrm{d}X_2. \end{aligned}$$

Finally, due to the convergences (65), (71) and the fact that $a|l(u_\varepsilon)|^{r+1} \leq C_1 - G(X_2, l(u_\varepsilon))$, we deduce (72) by applying Lebesgue's theorem. This ends the proof. ■

Remark 3.3: Taking $\varepsilon = 0$, replacing $\mathcal{W}_0(\Omega,h)$ by $\mathcal{V}_0(\Omega,h)$ and arguing as in Theorem 3.1, we can show directly that under the assumptions of Theorem 3.1 there exists $u \in \mathcal{V}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)$ such that

$$\mathbf{J}^0(u) = \inf_{v\in\mathcal{V}_0(\Omega,h)\cap\mathcal{L}^{r+1}(\Omega)} \mathbf{J}^0(v), \quad (79)$$

where u is a solution of ($\mathbf{P_0}$). Moreover, this solution is unique when $t \mapsto g(\cdot,t)$ is nonincreasing.

4. A non-local eigenvalue problem

To go further and study problems ($\mathbf{P_0}$) and ($\mathbf{P}_\epsilon$) under some assumptions weaker than (45), we need to consider the following non-local eigenvalue problem.

4.1. l-eigenvalues and l-eigenfunctions

We say that λ^ε is an l-eigenvalue of the linear problem

$$\begin{cases} -\frac{\varepsilon^2}{h}\triangle_{X_1}\varphi - \nabla_{X_2}\cdot\left(A\nabla_{X_2}\varphi\right) = \lambda^\varepsilon l(\varphi) & \text{in } \mathcal{W}_0'(\Omega,h) \\ \varphi\in\mathcal{W}_0(\Omega,h), \end{cases} \tag{80}$$

provided that there exists a solution φ such that $l(\varphi)$ is not identically null. A solution φ to (80), called an l-eigenfunction associated with λ^ε, is taken in the h-weak sense, i.e. φ satisfies

$$\int_\Omega \varepsilon^2\nabla_{X_1}\varphi\cdot\nabla_{X_1}v + hA\nabla_{X_2}\varphi\cdot\nabla_{X_2}v\,\mathrm{d}x = \lambda^\varepsilon\int_{\omega_2} l(\varphi)l(v)\,\mathrm{d}X_2,\quad \forall\, v\in\mathcal{W}_0(\Omega,h). \tag{$\mathbf{EVP}_\varepsilon$}$$

Theorem 4.1: *The smallest l-eigenvalue of Problem* ($\mathbf{EVP}_\varepsilon$) *exists, is positive and defined by*

$$\lambda_1^\varepsilon := \inf_{v\in\mathcal{W}_0(\Omega,h),\quad l(v)\not\equiv 0} \frac{\displaystyle\int_\Omega \varepsilon^2\left|\nabla_{X_1}v\right|^2 + hA\nabla_{X_2}v\cdot\nabla_{X_2}v\,\mathrm{d}x}{\|l(v)\|^2_{L^2(\omega_2)}}. \tag{81}$$

The associated l-eigenfunctions are the minimizers of the above functional.

Proof: Let λ^ε be an l-eigenvalue of Problem ($\mathbf{EVP}_\varepsilon$). Taking $v=\varphi\in\mathcal{W}_0(\Omega,h)$ in ($\mathbf{EVP}_\varepsilon$), we get

$$\int_\Omega \varepsilon^2\left|\nabla_{X_1}\varphi\right|^2 + hA\nabla_{X_2}\varphi\cdot\nabla_{X_2}\varphi\,\mathrm{d}x = \lambda^\varepsilon\int_{\omega_2}|l(\varphi)|^2\,\mathrm{d}X_2$$

and we have clearly $\lambda_1^\varepsilon\le\lambda^\varepsilon$ and $\lambda^\varepsilon>0$ since $l(\varphi)\not\equiv 0$.

Let us show that the infimum in (81) is indeed achieved for a function in $\mathcal{W}_0(\Omega,h)$. Thanks to the density of $\mathcal{D}(\Omega)$ in $\mathcal{W}_0(\Omega,h)$, we can take a minimizing sequence $\varphi_n\in\mathcal{D}(\Omega)$, i.e.

$$\frac{\displaystyle\int_\Omega \varepsilon^2\left|\nabla_{X_1}\varphi_n\right|^2 + hA\nabla_{X_2}\varphi_n\cdot\nabla_{X_2}\varphi_n\,\mathrm{d}x}{\|l(\varphi_n)\|^2_{L^2(\omega_2)}} \longrightarrow \inf_{v\in\mathcal{W}_0(\Omega,h),l(v)\not\equiv 0}\frac{\displaystyle\int_\Omega \varepsilon^2\left|\nabla_{X_1}v\right|^2 + hA\nabla_{X_2}v\cdot\nabla_{X_2}v\,\mathrm{d}x}{\|l(v)\|^2_{L^2(\omega_2)}} := Inf, \tag{82}$$

as $n\to+\infty$. Replacing $h\varphi_n$ by $h\varphi_n/\|l(\varphi_n)\|_{L^2(\omega_2)}$, we can assume that $\|l(\varphi_n)\|_{L^2(\omega_2)}=1$, $\forall n\in\mathbb{N}$. By (8), (82) and for n large enough, we derive

$$\alpha\int_\Omega h\left|\nabla_{X_2}\varphi_n\right|^2\,\mathrm{d}x \le \int_\Omega \varepsilon^2\left|\nabla_{X_1}\varphi_n\right|^2 + hA\nabla_{X_2}\varphi_n\cdot\nabla_{X_2}\varphi_n\,\mathrm{d}x \le Inf+1, \tag{83}$$

which guarantees that

$$\left|\nabla_{X_1}\varphi_n\right| \text{ is bounded in } L^2(\Omega) \quad\text{and}\quad \left|\nabla_{X_2}\varphi_n\right| \text{ is bounded in } L^2(\Omega,h).$$

Then, we have – up to a subsequence –

$$\varphi_n\rightharpoonup\varphi \quad\text{and}\quad \nabla_{X_2}\varphi_n\rightharpoonup\nabla_{X_2}\varphi \text{ in } L^2(\Omega,h),$$

$$\nabla_{X_1}\varphi_n \rightharpoonup \nabla_{X_1}\varphi \text{ in } L^2(\Omega),$$

for some $\varphi \in \mathcal{W}_0(\Omega, h)$. The first convergence $\varphi_n \rightharpoonup \varphi$ is a consequence of Poincaré's inequality in the X_2 direction. Moreover, using Lemma 2.1, with $p = 2$, we obtain

$$l(\varphi_n) \to l(\varphi) \text{ in } L^2(\omega_2) \quad \text{and} \quad \|l(\varphi_n)\|_{L^2(\omega_2)} \to \|l(\varphi)\|_{L^2(\omega_2)} = 1. \tag{84}$$

This means that $l(\varphi) \not\equiv 0$ and a fortiori $\varphi \not\equiv 0$. By lower semi-continuity of norms, we deduce that

$$\begin{aligned} Inf &\le \int_\Omega \varepsilon^2 \left|\nabla_{X_1}\varphi\right|^2 + hA\nabla_{X_2}\varphi \cdot \nabla_{X_2}\varphi \,\mathrm{d}x \le \liminf_{n\to+\infty} \int_\Omega \varepsilon^2 \left|\nabla_{X_1}\varphi_n\right|^2 \\ &\quad + hA\nabla_{X_2}\varphi_n \cdot \nabla_{X_2}\varphi_n \,\mathrm{d}x := Inf. \end{aligned}$$

To summarize, $\varphi \in \mathcal{W}_0(\Omega, h)$ is a function for which the infimum (81) is achieved, i.e.

$$\int_\Omega \varepsilon^2 \left|\nabla_{X_1}\varphi\right|^2 + hA\nabla_{X_2}\varphi \cdot \nabla_{X_2}\varphi \,\mathrm{d}x = \lambda_1^\varepsilon, \quad \|l(\varphi)\|_{L^2(\omega_2)} = 1. \tag{85}$$

Let us show that this φ is an l-eigenfunction. Let $v \in \mathcal{D}(\Omega)$ and consider

$$\begin{aligned} T(t) &:= \int_\Omega \varepsilon^2 \left|\nabla_{X_1}(\varphi + tv)\right|^2 + hA\nabla_{X_2}(\varphi + tv) \cdot \nabla_{X_2}(\varphi + tv) \,\mathrm{d}x \Big/ \\ &\quad \times \int_{\omega_2} |l(\varphi + tv)|^2 \,\mathrm{d}X_2. \end{aligned}$$

Then, we should have $T'(0) = 0$. That is

$$\begin{aligned} &\int_\Omega \varepsilon^2 \nabla_{X_1}\varphi \cdot \nabla_{X_1}v + hA\nabla_{X_2}\varphi \cdot \nabla_{X_2}v \,\mathrm{d}x \cdot \int_{\omega_2} |l(\varphi)|^2 \,\mathrm{d}X_2 \\ &\quad - \int_{\omega_2} l(\varphi)l(v) \,\mathrm{d}X_2 \cdot \int_\Omega \varepsilon^2 \left|\nabla_{X_1}\varphi\right|^2 + hA\nabla_{X_2}\varphi \cdot \nabla_{X_2}\varphi \,\mathrm{d}x = 0. \end{aligned}$$

Using (85), we obtain

$$\int_\Omega \varepsilon^2 \nabla_{X_1}\varphi \cdot \nabla_{X_1}v + hA\nabla_{X_2}\varphi \cdot \nabla_{X_2}v \,\mathrm{d}x = \lambda_1^\varepsilon \int_{\omega_2} l(\varphi)l(v) \,\mathrm{d}X_2, \quad \forall\, v \in \mathcal{D}(\Omega).$$

This shows that φ is an l-eigenfunction associated with the smallest l-eigenvalue λ_1^ε. ■

Concerning the (expected) limit problem, we say that λ^0 is an l-eigenvalue of the linear problem

$$\begin{cases} -\nabla_{X_2} \cdot \left(A\nabla_{X_2}\varphi\right) = \lambda^0 l(\varphi) & \text{in } \mathcal{V}_0'(\Omega, h) \\ \varphi \in \mathcal{V}_0(\Omega, h), \end{cases} \tag{86}$$

provided that there exists a solution φ such that $l(\varphi) \not\equiv 0$. This solution φ is also taken in the h-weak sense, i.e. φ satisfies

$$\int_\Omega hA\nabla_{X_2}\varphi \cdot \nabla_{X_2}v \,\mathrm{d}x = \lambda^0 \int_{\omega_2} l(\varphi)l(v) \,\mathrm{d}X_2, \quad \forall\, v \in \mathcal{V}_0(\Omega, h). \tag{$\mathbf{EVP_0}$}$$

The proof of Theorem 4.1 can be easily reproduced, by taking $\varepsilon = 0$ and replacing $\mathcal{W}_0(\Omega, h)$ by $\mathcal{V}_0(\Omega, h)$, to prove that:

Theorem 4.2: *The smallest l-eigenvalue of Problem* ($\mathbf{EVP}_0$) *exits, is positive and defined by*

$$\lambda_1^0 := \inf_{v \in \mathcal{V}_0(\Omega,h), l(v) \not\equiv 0} \frac{\int_\Omega hA\nabla_{X_2} v \cdot \nabla_{X_2} v \, \mathrm{d}x}{\|l(v)\|^2_{L^2(\omega_2)}}. \tag{87}$$

Remark 4.1: If the matrix A is independent of X_1, it is clear from ($\mathbf{EVP}_0$) that an l-eigenfunction $\tilde{\varphi}_0$, associated with an l-eigenvalue λ^0, is independent of X_1 and

$$-\nabla_{X_2} \cdot \left(A\,(X_2)\,\nabla_{X_2} \tilde{\varphi}_0\right) = \lambda^0 \, \|h\|_{L^1(\omega_1)} \, \tilde{\varphi}_0, \quad \text{with } \tilde{\varphi}_0 \in H_0^1\,(\omega_2)\,.$$

This means that $\lambda^0 \|h\|_{L^1(\omega_1)}$ and $\tilde{\varphi}_0$ are respectively an eigenvalue and eigenfunction (in the classical sense) of the operator $-\nabla_{X_2} \cdot (A(X_2)\nabla_{X_2} \cdot)$ on ω_2, with Dirichlet boundary conditions. Denoting by λ_{1,ω_2} the usual first eigenvalue $-\nabla_{X_2} \cdot (A(X_2)\nabla_{X_2} \cdot)$ on ω_2, with Dirichlet boundary conditions, then by definition $\lambda_{1,\omega_2} \leq \lambda_1^0 \|h\|_{L^1(\omega_1)}$. Besides, if $\tilde{\varphi}_0 \in \mathcal{V}_0(\Omega, h)$ is the eigenvalue corresponding to λ_{1,ω_2}, then (87) yields

$$\lambda_1^0 \leq \frac{\|h\|_{L^1(\omega_1)} \int_{\omega_2} A\nabla_{X_2} \tilde{\varphi}_0 \cdot \nabla_{X_2} \tilde{\varphi}_0 \, \mathrm{d}X_2}{\|l(\tilde{\varphi}_0)\|^2_{L^2(\omega_2)}} = \frac{\lambda_{1,\omega_2}}{\|h\|_{L^1(\omega_1)}}. \tag{88}$$

Thus $\lambda_1^0 = \lambda_{1,\omega_2} / \|h\|_{L^1(\omega_1)}$.

Even if A depends on x, we still have:

Corollary 4.1: *For every* $\varepsilon > 0$, *the l-eigenvalues* λ_1^ε *and* λ_1^0 *satisfy*

$$\lambda_1^\varepsilon \geq \lambda_1^0 \geq \alpha \lambda_{1,\omega_2}^\Delta / \, \|h\|_{L^1(\omega_1)}\,, \tag{89}$$

where α *is the ellipticity constant given in* (8) *and* $\lambda_{1,\omega_2}^\Delta$ *denotes the first eigenvalue of* $-\Delta_{X_2}$ *on* ω_2 *with Dirichlet boundary conditions.*

Proof: By the density of $\mathcal{W}_0(\Omega, h)$ in $\mathcal{V}_0(\Omega, h)$ and from the definitions (81) and (87), it is clear that $\lambda_1^\varepsilon \geq \lambda_1^0$. Let φ_0 be a normalized l-eigenfunction of Problem ($\mathbf{EVP}_0$) associated with λ_1^0, then

$$\lambda_1^0 = \int_\Omega hA\nabla_{X_2}\varphi_0 \cdot \nabla_{X_2}\varphi_0 \, \mathrm{d}x \geq \alpha \int_\Omega h \left|\nabla_{X_2}\varphi_0\right|^2 \, \mathrm{d}x. \tag{90}$$

On the other hand, taking $p = 2$ in (19) then using Poincaré's inequality in the X_2-direction, we get

$$1 = \|l(\varphi_0)\|^2_{L^2(\omega_2)} \leq \|h\|_{L^1(\omega_1)} \int_\Omega h \, |\varphi_0|^2 \, \mathrm{d}x \leq \frac{\|h\|_{L^1(\omega_1)}}{\lambda_{1,\omega_2}^\Delta} \int_\Omega h \left|\nabla_{X_2}\varphi_0\right|^2 \, \mathrm{d}x. \tag{91}$$

Combining this with (90), we deduce (89). ■

4.2. Some proprieties of the first l-eigenvalue and l-eigenfunction

As for the classical eigenvalue problems, we can obtain some proprieties like simplicity of λ_ε^1, the positivity of φ_ε and its uniqueness up to a multiplicative constant.

Theorem 4.3: *Let $\varepsilon > 0$ and φ_ε be an l-eigenfunction for* $(\mathbf{EVP}_\varepsilon)$ *associated with λ_1^ε. Then, λ_1^ε is simple and φ_ε does not change sign in Ω.*

Proof: Let $\varphi_\varepsilon \in \mathcal{W}_0(\Omega, h)$, $l(\varphi_\varepsilon) \not\equiv 0$, be an h-weak solution of $(\mathbf{EVP}_\varepsilon)$ and thus φ_ε realizes the infimum (81). Set $\varphi_\varepsilon^+ := \max\{0, \varphi_\varepsilon\}$. Arguing by density as in [28, Lemma 1.19], we can show that

$$\varphi_\varepsilon^+ \in \mathcal{W}_0(\Omega, h) \quad \text{and} \quad \partial_{x_i}\varphi_\varepsilon^+ = \left(\partial_{x_i}\varphi_\varepsilon\right)\chi_{\{\varphi_\varepsilon > 0\}}, \quad \text{for } i = 1, \dots, n_1 + n_2.$$

In particular, this means that $|\varphi_\varepsilon| = (2\varphi_\varepsilon^+ - \varphi_\varepsilon) \in \mathcal{W}_0(\Omega, h)$. Since $\|l(\varphi_\varepsilon)\|_{L^2(\omega_2)} \leq \|l(|\varphi_\varepsilon|)\|_{L^2(\omega_2)}$, we infer that

$$\frac{\displaystyle\int_\Omega \varepsilon^2 \left|\nabla_{X_1} |\varphi_\varepsilon|\right|^2 + hA\nabla_{X_2} |\varphi_\varepsilon| \cdot \nabla_{X_2} |\varphi_\varepsilon| \, \mathrm{d}x}{\|l(|\varphi_\varepsilon|)\|^2_{L^2(\omega_2)}} \leq \frac{\displaystyle\int_\Omega \varepsilon^2 \left|\nabla_{X_1}\varphi_\varepsilon\right|^2 + hA\nabla_{X_2}\varphi_\varepsilon \cdot \nabla_{X_2}\varphi_\varepsilon \, \mathrm{d}x}{\|l\,(\varphi_\varepsilon)\|_{L^2(\omega_2)}} = \lambda_1^\varepsilon, \tag{92}$$

hence the function $|\varphi_\varepsilon|$ also realizes the infimum (81). Thanks to Theorem 4.1, $|\varphi_\varepsilon|$ is also an l-eigenfunction for (80).

Next, we show that φ_ε does not change sign in Ω. Going back to (80), the non-negative function $|\varphi_\varepsilon|$ satisfies

$$-\frac{\varepsilon^2}{h}\Delta_{X_1} |\varphi_\varepsilon| - \nabla_{X_2} \cdot \left(A\nabla_{X_2} |\varphi_\varepsilon|\right) \geq 0,$$

in the weak sense. According to [29], under the assumption (12) we must have $|\varphi_\varepsilon| > 0$, a.e. in Ω and thus λ_1^ε has a positive l-eigenfunction. Therefore we must have $\varphi_\varepsilon = \pm|\varphi_\varepsilon|$. Otherwise $\varphi_\varepsilon^+ = (\varphi_\varepsilon + |\varphi_\varepsilon|)/2$ would be a non-negative l-eigenfunction for (80) vanishing on a set of positive measure in contradiction with the precedent assertion $|\varphi_\varepsilon^+| = \varphi_\varepsilon^+ > 0$, a.e. in Ω.

Finally, if λ_1^ε is not a simple l-eigenvalue, then there is a second l-eigenfunction ψ_ε corresponding to λ_1^ε and orthogonal to φ_ε. This ψ_ε would not be of one sign (a.e.) which leads to a contradiction. This shows the simplicity of λ_1^ε. ∎

Remark 4.2: Let $\varphi_0 \in \mathcal{V}_0(\Omega, h)$ be an l-eigenfunction associated with λ_1^0. Taking $\varepsilon = 0$ and replacing φ by φ_0 in (92), we deduce that the function $|\varphi_0| \in \mathcal{V}_0(\Omega, h)$ is also an l-eigenvalue. The function φ_0^+ (or $\varphi_0^- := \max\{0, -\varphi_0\}$) is another non-negative l-eigenvalues whenever it is not identically null.

Remark 4.3: Arguing as in [13, p.185] and if $\partial_{x_k} a_{ij} \in L^\infty(\Omega)$ for $k = 1, \dots, n_1$, we can show that a solutions of $(\mathbf{EVP}_0)$ satisfies $\nabla_{X_1}\varphi_0 \in [L^2(\Omega, h)]^{n_1}$.

4.3. Passage to the limit

We are now interested in the behaviour of the l-eigenvalue λ_1^ε and its associated l-eigenfunction φ_ε, as $\varepsilon \to 0$.

Theorem 4.4: *Let λ_1^ε be defined by* (81) *and φ_ε an associated l-eigenfunction such that $\|l(\varphi_\varepsilon)\|_{L^2(\omega_2)} = 1$. Then, we have*

$$\lim_{\varepsilon\to 0} \lambda_1^\varepsilon = \lambda_1^0, \tag{93}$$

where λ_1^0 is defined by (87). *Moreover, – up to a subsequence –*

$$\varphi_\varepsilon \to \varphi_0, \quad \nabla_{X_2}\varphi_\varepsilon \to \nabla_{X_2}\varphi_0 \text{ in } L^2(\Omega, h), \quad \varepsilon\nabla_{X_1}\varphi_\varepsilon \to 0 \text{ in } L^2(\Omega) \tag{94}$$

$$l(\varphi_\varepsilon) \to l(\varphi_0) \quad \text{in } L^2(\omega_2), \tag{95}$$

where φ_0 is a normalized l-eigenfunction associated with λ_1^0, i.e. a solution to ($\mathbf{EVP}_0$)*, satisfying $\|l(\varphi_0)\|_{L^2(\omega_2)} = 1$.*

Proof: The sequence λ_1^ε is decreasing as $\varepsilon \to 0$ and clearly $\lambda_1^\varepsilon \geq \lambda_1^0 > 0, \forall \varepsilon > 0$. Thus λ_1^ε converges to some limit

$$\lambda_1^\varepsilon \to \bar{\lambda}_1 := \inf_{\varepsilon>0} \lambda_1^\varepsilon \geq \lambda_1^0. \tag{96}$$

Let φ_ε be an l-eigenfunction associated with λ_1^ε such that $\|l(\varphi_\varepsilon)\|_{L^2(\omega_2)} = 1$, then

$$\int_\Omega \varepsilon^2 \left|\nabla_{X_1}\varphi_\varepsilon\right|^2 + \alpha \left|\nabla_{X_2}\varphi_\varepsilon\right|^2 h \, dx \leq \int_\Omega \varepsilon^2 \left|\nabla_{X_1}\varphi_\varepsilon\right|^2 + hA\nabla_{X_2}\varphi_\varepsilon \cdot \nabla_{X_2}\varphi_\varepsilon \, dx = \lambda_1^\varepsilon \leq \lambda_1^1, \tag{97}$$

for $0 < \varepsilon \leq 1$. It follows that

$$\varphi_\varepsilon, \left|\nabla_{X_2}\varphi_\varepsilon\right| \text{ are bounded in } L^2(\Omega, h),$$
$$\left|\varepsilon\nabla_{X_1}\varphi_\varepsilon\right| \text{ is bounded in } L^2(\Omega).$$

Of course, the first estimate follows from Poincaré's inequality in the X_2-direction. Then, there exists $\varphi_0 \in \mathcal{V}_0(\Omega, h)$ such that – up to a subsequence –

$$\varphi_\varepsilon \rightharpoonup \varphi_0, \nabla_{X_2}\varphi_\varepsilon \rightharpoonup \nabla_{X_2}\varphi_0 \text{ in } L^2(\Omega, h) \quad \text{and} \quad \varepsilon\nabla_{X_1}\varphi_\varepsilon \rightharpoonup 0, \text{ in } L^2(\Omega). \tag{98}$$

In addition, Lemma 2.1 with $p = 2$ ensures that

$$l(\varphi_\varepsilon) \to l(\varphi_0) \quad \text{in } L^2(\omega_2). \tag{99}$$

Let us now prove that $\bar{\lambda}_1 = \lambda_1^0$ and φ_0 is its associated l-eigenvalue. On one hand, taking $\varphi = \varphi_\varepsilon$ in ($\mathbf{EVP}_\varepsilon$), then using the convergences (98) and (99) to pass to the limit yields

$$\int_\Omega hA\nabla_{X_2}\varphi_0 \cdot \nabla_{X_2} v \, dx = \bar{\lambda}_1 \int_{\omega_2} l(\varphi_0) l(v) \, dX_2, \quad \forall\, v \in \mathcal{D}(\Omega). \tag{100}$$

Besides, the strong convergence (99) means that $\|l(\varphi_0)\|_{L^2(\omega_2)} = 1$, hence $l(\varphi_0) \not\equiv 0$ and the identity (100) shows that φ_0 is an l-eigenvalue associated with $\bar{\lambda}_1$.

On the other hand, by definition we have

$$\lambda_1^\varepsilon \leq \int_\Omega \varepsilon^2 \left|\nabla_{X_1} v\right|^2 + hA\nabla_{X_2} v \cdot \nabla_{X_2} v \,\mathrm{d}x, \quad \forall\, v \in \mathcal{D}(\Omega), \quad \|l(v)\|_{L^2(\omega_2)} = 1,$$

and passing to the limit, when $\varepsilon \to 0$, we get

$$\bar{\lambda}_1 \leq \int_\Omega hA\nabla_{X_2} v \cdot \nabla_{X_2} v \,\mathrm{d}x, \quad \forall\, v \in \mathcal{D}(\Omega), \ \|l(v)\|_{L^2(\omega_2)} = 1.$$

By density of $\mathcal{D}(\Omega)$ in $\mathcal{V}_0(\Omega, h)$ we infer that $\bar{\lambda}_1 \leq \lambda_1^0$. Taking (96) into account, we obtain $\bar{\lambda}_1 = \lambda_1^0$.

It remains to show the strong convergences (94). Set

$$\Lambda_\varepsilon := \int_\Omega \varepsilon^2 \left|\nabla_{X_1} \varphi_\varepsilon\right|^2 + hA\nabla_{X_2}(\varphi_\varepsilon - \varphi_0) \cdot \nabla_{X_2}(\varphi_\varepsilon - \varphi_0) \,\mathrm{d}x.$$

Developing Λ_ε and using the convergences (93) and (98), we get

$$\begin{aligned}\Lambda_\varepsilon &= \lambda_1^\varepsilon - 2\int_\Omega hA\nabla_{X_2}\varphi_0 \cdot \nabla_{X_2}\varphi_\varepsilon \,\mathrm{d}x + \int_\Omega hA\nabla_{X_2}\varphi_0 \cdot \nabla_{X_2}\varphi_0 \,\mathrm{d}x \\ &\to \lambda_1^0 - \int_\Omega hA\nabla_{X_2}\varphi_0 \cdot \nabla_{X_2}\varphi_0 \,\mathrm{d}x = 0.\end{aligned}$$

The strong convergences (94) follow from the fact that

$$\Lambda_\varepsilon \geq \int_\Omega \varepsilon^2 \left|\nabla_{X_1}\varphi_\varepsilon\right|^2 + \alpha h \left|\nabla_{X_2}(\varphi_\varepsilon - \varphi_0)\right|^2 \,\mathrm{d}x, \quad \forall\, \varepsilon > 0.$$

This ends the proof of the theorem. ■

5. A problem with subcritical growth of non-local terms

In this section, we relax the assumption (45) by considering an upper quadratic bound of $G(\cdot, t)$ for t large, and assume that

$$1 \leq r < 2^* - 1.$$

We recall that in this case $\mathcal{W}_0(\Omega, h) \subset \mathcal{L}^{r+1}(\Omega)$ and $L^{\frac{r+1}{r}}(\omega_2) \subset \mathcal{W}_0'(\Omega, h)$. So, without loss of generality, we assume that $f_2 = 0$ and the h-weak formulation of Problem (27) reads

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + hA\nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \,\mathrm{d}x = \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(v) \,\mathrm{d}X_2 + \langle f_1, v\rangle_{\mathcal{V}_0(\Omega,h)}, \tag{$\tilde{\mathbf{P}}_\varepsilon$}$$

for every $v \in \mathcal{W}_0(\Omega, h)$. The functional associated to $(\tilde{\mathbf{P}}_\varepsilon)$ is given here by

$$\tilde{\mathbf{J}}^\varepsilon(u) := \frac{1}{2}\int_\Omega \varepsilon^2 \left|\nabla_{X_1} u\right|^2 + hA\nabla_{X_2} u \cdot \nabla_{X_2} u \,\mathrm{d}x - \int_{\omega_2} G(X_2, l(u)) \,\mathrm{d}X_2 - \langle f_1, u\rangle_{\mathcal{V}_0(\Omega,h)}. \tag{101}$$

We aim to establish the existence of solutions u_ε for $(\tilde{\mathbf{P}}_\varepsilon)$ as global minimizers for $\tilde{\mathbf{J}}^\varepsilon$, then we study the asymptotic behaviour of u_ε and $\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon)$, as $\varepsilon \to 0$.

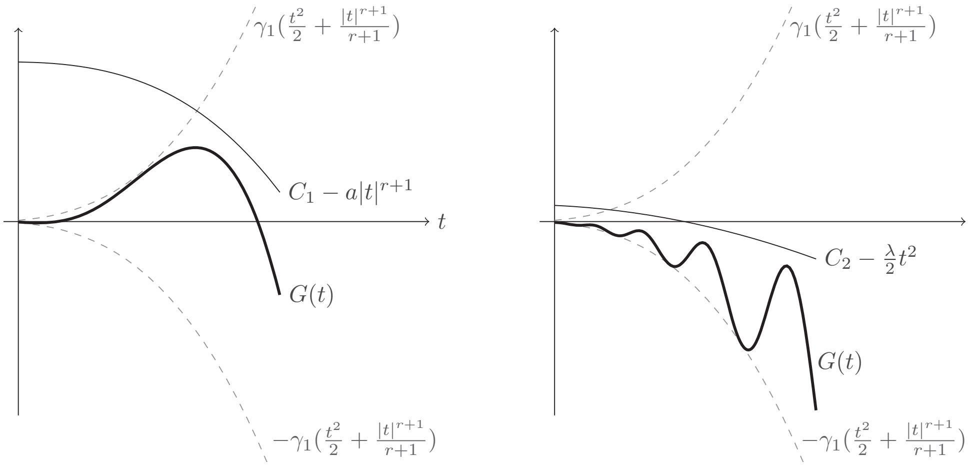

Figure 1. An illustration of the assumptions (45) and (102), as well as their compatibility with (32).

5.1. The perturbed problem

First, we establish the following existence result.

Theorem 5.1: *Let $\varepsilon > 0$. Assume that g satisfies* (29) *and* (30) *with* $1 \leq r < 2^* - 1$. *In addition, assume that G defined by* (31), *satisfies*

$$G(X_2, t) \leq \frac{-\lambda}{2} t^2 \quad \text{a.e. } X_2 \in \omega_2, \quad \text{for } |t| \text{ large enough} \tag{102}$$

and some $\lambda > -\lambda_1^{\varepsilon}$. Then, there exists $u \in \mathcal{W}_0(\Omega, h)$ such that

$$\tilde{\mathbf{J}}^{\varepsilon}(u) = \inf_{v \in \mathcal{W}_0(\Omega,h)} \tilde{\mathbf{J}}^{\varepsilon}(v), \tag{103}$$

where $\tilde{\mathbf{J}}^{\varepsilon}$ is defined by 101. *In particular, u is a solution of* $(\tilde{\mathbf{P}}_{\varepsilon})$.

Proof: *A priori estimates.* Due to (102) we have

$$G(X_2, t) \leq C_2 - \frac{\lambda}{2} t^2, \quad \forall\, t \in \mathbb{R} \tag{104}$$

for some constant $C_2 \geq 0$. Reporting this to (101), we have the lower boundedness

$$\tilde{\mathbf{J}}^{\varepsilon}(v) \geq \Phi_{\lambda}^{\varepsilon}(v) - \langle f_1, v \rangle_{\mathcal{V}_0(\Omega,h)} - C_2 |\omega_2|, \tag{105}$$

where

$$\Phi_{\lambda}^{\varepsilon}(v) := \frac{1}{2} \int_{\Omega} \varepsilon^2 \left|\nabla_{X_1} v\right|^2 + hA\nabla_{X_2} v \cdot \nabla_{X_2} v \, dx + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 \, dX_2.$$

Thanks to (81) we have, for any given $0 < \eta < 1$,

$$\Phi_{\lambda}^{\varepsilon}(v) = \frac{\eta + (1-\eta)}{2} \int_{\Omega} \varepsilon^2 \left|\nabla_{X_1} v\right|^2 + hA\nabla_{X_2} v \cdot \nabla_{X_2} v \, dx + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 \, dX_2$$

$$\geq \frac{\eta}{2}\min\{1,\alpha\}\int_{\Omega}\varepsilon^2\left|\nabla_{X_1}v\right|^2+h\left|\nabla_{X_2}v\right|^2\,\mathrm{d}x+\frac{(1-\eta)\,\lambda_1^{\varepsilon}+\lambda}{2}\int_{\omega_2}|l(v)|^2\,\mathrm{d}X_2.$$

Since $\lambda+\lambda_1^{\varepsilon}>0$, then for $0<\eta\leq\eta_{\varepsilon}:=\min\{1,(\lambda+\lambda_1^{\varepsilon})/\lambda_1^{\varepsilon}\}$, the last term above is nonnegative. Denoting $\delta_{\varepsilon}:=\eta_{\varepsilon}\min\{1,\alpha\}$, we have

$$\Phi_{\lambda}^{\varepsilon}(v)\geq\frac{\delta_{\varepsilon}}{2}\int_{\Omega}\varepsilon^2\left|\nabla_{X_1}v\right|^2+h\left|\nabla_{X_2}v\right|^2\,\mathrm{d}x,\quad\forall\,\varepsilon>0. \tag{106}$$

Going back to (105) and using (48), we obtain

$$\begin{aligned}\tilde{\mathbf{J}}^{\varepsilon}(v)\geq{}&\frac{\delta_{\varepsilon}}{2}\int_{\Omega}\varepsilon^2\left|\nabla_{X_1}v\right|^2+h\left|\nabla_{X_2}v\right|^2\,\mathrm{d}x\\&-\|f_1\|_{\mathcal{V}_0'(\Omega,h)}\left(\int_{\Omega}\varepsilon^2\left|\nabla_{X_1}v\right|^2+h\left|\nabla_{X_2}v\right|^2\,\mathrm{d}x\right)^{1/2}-C_2\,|\omega_2|\,.\end{aligned}$$

Applying Young's inequality $ab\leq\delta_{\varepsilon}^{-1}a^2+\delta_{\varepsilon}b^2/4$, we get

$$\tilde{\mathbf{J}}^{\varepsilon}(v)\geq\frac{\delta_{\varepsilon}}{4}\int_{\Omega}\varepsilon^2\left|\nabla_{X_1}v\right|^2+h\left|\nabla_{X_2}v\right|^2\,\mathrm{d}x-\frac{1}{\delta_{\varepsilon}}\|f_1\|^2_{\mathcal{V}_0'(\Omega,h)}-C_2\,|\omega_2|\,. \tag{107}$$

Thus $\tilde{\mathbf{J}}^{\varepsilon}$ is bounded below, i.e.

$$d_{\varepsilon}:=\inf_{v\in\mathcal{W}_0(\Omega,h)}\tilde{\mathbf{J}}^{\varepsilon}(v)>-\infty.$$

Passage to the limit. The estimation (32) implies that

$$G(X_2,t)\leq\gamma_2t^2,\ \text{ for small } t \text{ and some constant } \gamma_2\geq 0.$$

Combining this with (102), which holds for t large, we deduce that for some constant $\mu\geq 0$ it holds that

$$G(X_2,t)\leq\mu t^2,\quad\text{for all } t\in\mathbb{R}. \tag{108}$$

Then, we rewrite $\tilde{\mathbf{J}}^{\varepsilon}$ as

$$\tilde{\mathbf{J}}^{\varepsilon}(v)=\Phi_{\lambda}^{\varepsilon}(v)+J_1(v)-\left(\frac{\lambda}{2}+\mu\right)\int_{\omega_2}|l(v)|^2\,\mathrm{d}X_2-\langle f_1,v\rangle_{\mathcal{V}_0(\Omega,h)}, \tag{109}$$

where

$$J_1(v):=\int_{\omega_2}-G(X_2,l(v))+\mu\,|l(v)|^2\,\mathrm{d}X_2. \tag{110}$$

Let $(u_n)_{n\geq0}$ be a minimizing sequence. By (107), we deduce that $(u_n)_{n\geq0}$ is necessarily bounded in $\mathcal{W}_0(\Omega,h)$. Then, there exist $u\in\mathcal{W}_0(\Omega,h)$ and a weakly converging subsequence $(u_{n_k})_{n_k\geq0}$ in $\mathcal{W}_0(\Omega,h)$ such that

$$\begin{aligned}&\nabla_{X_1}u_{n_k}\rightharpoonup\nabla_{X_1}u\text{ in }L^2(\Omega),\quad\nabla_{X_2}u_{n_k}\rightharpoonup\nabla_{X_2}u,\ \text{in } L^2(\Omega,h),\\&l\left(u_{n_k}\right)\to l(u)\ \text{ in } L^2(\omega_2),\end{aligned}$$

as $n_k \to +\infty$. The last strong convergence is a consequence of Lemma 2.1 with $p = 2$. Up to a new subsequence, we have $l(u_{n_k}) \to l(u)$ a.e. in ω_2 and since the function $-G(X_2, t) + \mu t^2$ is nonnegative, Fatou's lemma implies that

$$J_1(u) \leq \liminf_{n_k \to \infty} J_1(u_{n_k}).$$

Using the weak lower semi-continuity of norms, we infer that

$$\Phi_\lambda^\varepsilon(u) \leq \liminf_{n_k \to \infty} \Phi_\lambda^\varepsilon(u_{n_k}).$$

Passing to the limit in (109) yields

$$\tilde{\mathbf{J}}^\varepsilon(u) \leq \liminf_{n_k \to \infty} \tilde{\mathbf{J}}^\varepsilon(u_{n_k}) = d_\varepsilon.$$

Therefore, the limit u realizes the minimum of $\tilde{\mathbf{J}}^\varepsilon$ on $\mathcal{W}_0(\Omega, h)$.

As $1 \leq r < 2^* - 1$, then by Corollary 2.1 we have $\tilde{\mathbf{J}}^\varepsilon \in C^1(\mathcal{W}_0(\Omega, h), \mathbb{R})$. Thus $(\tilde{\mathbf{J}}^\varepsilon)'(u) = 0$ and the limit u is a solution to $(\tilde{\mathbf{P}}_\varepsilon)$. This ends the proof. ∎

Example 5.1: Theorem 5.1 applies to the problem

$$\begin{cases} -\frac{\varepsilon^2}{h}\Delta_{X_1} u - \nabla_{X_2} \cdot \left(A \nabla_{X_2} u\right) + \lambda l(u) + a(l(u))\,|l(u)|^{r-1}\, l(u) \\ \quad -b(l(u))\,|l(u)|^{q-1}\, l(u) = f_1 \text{ in } \mathcal{W}_0'(\Omega, h) \\ u \in \mathcal{W}_0(\Omega, h), \end{cases}$$

where $\lambda > -\lambda_1^\varepsilon$, $1 < q < r < 2^* - 1$ and $a, b : \mathbb{R} \to \mathbb{R}$ are bounded continuous functions satisfying

$$a \geq 0 \quad \text{and} \quad b(t) = O(a(t)), \quad \text{as } |t| \to +\infty.$$

Here we have taken

$$g(\cdot, t) = -\lambda t - a(t)\,|t|^{r-1}\, t + b(t)\,|t|^{q-1}\, t, \quad \text{for } t \in \mathbb{R}.$$

Corollary 5.1: *Let u be a solution of Problem* $(\tilde{\mathbf{P}}_\varepsilon)$ *given by Theorem* 5.1. *If the mapping*

$$s \mapsto g(\cdot, s) + \lambda s \text{ is non increasing,} \tag{111}$$

then u is the unique solution of $(\tilde{\mathbf{P}}_\varepsilon)$.

Proof: We keep the notation of the previous proof. If (111) holds, then the functional

$$v \mapsto -\int_{\omega_2} G(X_2, l(v)) + \frac{\lambda}{2}\,|l(v)|^2 \,\mathrm{d}X_2$$

is convex. In addition, Φ_λ^ε is strictly convex since it satisfies (106). Rewriting $\tilde{\mathbf{J}}^\varepsilon$ as

$$\tilde{\mathbf{J}}^\varepsilon(v) = \Phi_\lambda^\varepsilon(v) - \int_{\omega_2} G(X_2,\, l(v)) + \frac{\lambda}{2}\,|l(v)|^2 \,\mathrm{d}X_2 - \langle f_1, v\rangle_{\mathcal{V}_0(\Omega,h)},$$

we can see that $\tilde{\mathbf{J}}^\varepsilon$ is strictly convex and the uniqueness of the solution realizing the minimum follows. ∎

5.2. Passage to the limit

The next theorem shows the existence of a solution u_0 to the following unperturbed problem

$$\int_\Omega hA\nabla_{X_2}u\cdot\nabla_{X_2}v\,\mathrm{d}x=\int_{\omega_2}g(X_2,l(u))l(v)\,\mathrm{d}X_2+\langle f_1,v\rangle_{\mathcal{V}_0(\Omega,h)},\quad\forall\,v\in\mathcal{V}_0(\Omega,h).\ (\tilde{\mathbf{P}}_0)$$

Theorem 5.2: *Under the assumptions of Theorem 5.1 with $\lambda>-\lambda_1^0$, and if u_ε is a solution of Problem $(\tilde{\mathbf{P}}_\varepsilon)$, then we have – up to a subsequence –*

$$u_\varepsilon\to u_0,\quad\nabla_{X_2}u_\varepsilon\to\nabla_{X_2}u_0\ in\ L^2(\Omega,h),\quad\varepsilon\nabla_{X_1}u_\varepsilon\to 0\ in\ L^2(\Omega),\tag{112}$$

$$l(u_\varepsilon)\to l(u_0)\ \ in\ L^{r+1}(\omega_2),\tag{113}$$

$$g(\cdot,l(u_\varepsilon))\to g(\cdot,l(u_0))\ \ in\ L^{\frac{r+1}{r}}(\omega_2).\tag{114}$$

Moreover, the above convergences hold for the whole sequence if the solution of $(\mathbf{P}_o)$ is unique.

Proof: First, the assumptions of Theorem 5.1 hold since $\lambda>-\lambda_1^0\geq-\lambda_1^\varepsilon$. Thus, for any $\varepsilon>0$, a solution u_ε exists and realizes the minimum of $\tilde{\mathbf{J}}^\varepsilon$ on $\mathcal{W}_0(\Omega,h)$. For $0<\varepsilon<1$, we have

$$\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon)\leq\tilde{\mathbf{J}}^\varepsilon(v)\leq\tilde{\mathbf{J}}^1(v),\quad\forall\,v\in\mathcal{W}_0(\Omega,h)$$

and $\lambda_1^1>\lambda_1^\varepsilon>\lambda_1^0$. In particular,

$$\eta_\varepsilon>\eta_0:=\min\left\{1,\left(\lambda+\lambda_1^0\right)/\lambda_1^1\right\}>0,\quad\text{for }\lambda>-\lambda_1^0,$$

where η_ε is the constant considered in (106). Denoting $\delta_0:=\eta_0\min\{1,\alpha\}$, we have

$$\Phi_\lambda^\varepsilon(v)\geq\frac{\delta_0}{2}\int_\Omega\varepsilon^2\left|\nabla_{X_1}v\right|^2+h\left|\nabla_{X_2}v\right|^2\,\mathrm{d}x,\quad\text{for }0<\varepsilon<1,$$

where this time δ_0 is independent of ε.

The same argument used to derive (107) yields

$$\frac{\delta_0}{4}\int_\Omega\varepsilon^2\left|\nabla_{X_1}u_\varepsilon\right|^2+h\left|\nabla_{X_2}u_\varepsilon\right|^2\,\mathrm{d}x\leq\tilde{\mathbf{J}}^1(v)+\frac{1}{\delta_0}\left\|f_1\right\|^2_{\mathcal{V}_0'(\Omega,h)}+C\left|\omega_2\right|,\quad\forall\,v\in\mathcal{W}_0(\Omega,h).\tag{115}$$

This means that the sequences

$$u_\varepsilon,\left|\nabla_{X_2}u_\varepsilon\right|\ \text{are bounded in }L^2(\Omega,h),$$
$$\left|\varepsilon\nabla_{X_1}u_\varepsilon\right|\ \text{is bounded in }L^2(\Omega).$$

Then, there exists $u_0 \in \mathcal{V}_0(\Omega, h)$ such that – up to a subsequence –

$$u_\varepsilon \rightharpoonup u_0, \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h), \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \quad \text{in } L^2(\Omega), \tag{116}$$

as $\varepsilon \to 0$. Thanks to Lemma 2.1, we have also

$$l(u_\varepsilon) \to l(u_0) \quad \text{in } L^{r+1}(\omega_2) \tag{117}$$

since $1 \leq r < 2^* - 1$. Moreover, up to a new subsequence, $l(u_\varepsilon) \to l(u_0)$ a.e. in ω_2 and there exists $K \in L^{r+1}(\omega_2)$, such that $|l(u_\varepsilon)| \leq K$ a.e. in ω_2. The continuity of g implies that

$$g(X_2, l(u_\varepsilon)) \to g(X_2, l(u_0)) \quad \text{a.e. in } \omega_2. \tag{118}$$

In addition, we have

$$\left| g(X_2, l(u_\varepsilon)) \right|^{\frac{r+1}{r}} \leq C\left(1 + K^{r+1}(X_2)\right) \quad \text{a.e. in } \omega_2,$$

see (66). Applying Lebesgue's theorem, we derive the strong convergence (114).

Going back to $(\tilde{\mathbf{P}}_\varepsilon)$ and passing to the limit, we deduce that the limit u_0 satisfies

$$\int_\Omega hA\nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, \mathrm{d}x = \int_{\omega_2} g(X_2, l(u_0))\, l(v)\, \mathrm{d}X_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega,h)} \tag{119}$$

for every $v \in \mathcal{W}_0(\Omega, h)$. This means that u_0 is a solution to Problem $(\mathbf{P}_o)$ since $\mathcal{W}_0(\Omega, h)$ is dense in $\mathcal{V}_0(\Omega, h)$.

It remains to show the strong convergences (112). To this end, we consider

$$\tilde{I}_\varepsilon := \int_\Omega \varepsilon^2 \left|\nabla_{X_1} u_\varepsilon\right|^2 + hA\nabla_{X_2}(u_\varepsilon - u_0) \cdot \nabla_{X_2}(u_\varepsilon - u_0)\, \mathrm{d}x. \tag{120}$$

Expanding $\tilde{I}_\varepsilon$ and taking $v = u_\varepsilon$ in the h-weak formulation $(\mathbf{P}_o)$, we get

$$\begin{aligned}\tilde{I}_\varepsilon = & \int_{\omega_2} g(X_2, l(u_\varepsilon))\, l(u_\varepsilon)\, \mathrm{d}X_2 + \langle f_1, u_\varepsilon \rangle_{\mathcal{V}_0(\Omega,h)} \\ & - 2\int_\Omega hA\nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_0 \, \mathrm{d}x + \int_\Omega hA\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 \, \mathrm{d}x.\end{aligned}$$

Thanks to (114), (116) and (117), the passage to the limit in $\tilde{I}_\varepsilon$ yields

$$\lim_{\varepsilon \to 0} \tilde{I}_\varepsilon = \int_{\omega_2} g(X_2, l(u_0))\, l(u_0)\, \mathrm{d}X_2 + \langle f_1, u_0 \rangle_{\mathcal{V}_0(\Omega,h)} - \int_\Omega hA\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 \, \mathrm{d}x.$$

Going back to $(\mathbf{P}_\epsilon)$ and taking $v = u_0$, we infer that $\lim_{\varepsilon \to 0} \tilde{I}_\varepsilon = 0$. The convergences (112) follow from the inequality

$$\tilde{I}_\varepsilon \geq \min\{1, \alpha\} \int_\Omega \varepsilon^2 \left|\nabla_{X_1} u_\varepsilon\right|^2 + h\left|\nabla_{X_2}(u_\varepsilon - u_0)\right|^2 \, \mathrm{d}x.$$

This ends the proof of the theorem. ∎

Concerning the convergence of $\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon)$ as $\varepsilon \to 0$, we set

$$\tilde{\mathbf{J}}^0(v) := \frac{1}{2}\int_\Omega hA\nabla_{X_2}v \cdot \nabla_{X_2}v \, dx - \int_{\omega_2} G(X_2, l(v)) \, dX_2 - \langle f_1, v \rangle . \tag{121}$$

Then, we have:

Corollary 5.2: *Under the assumptions of Theorem 5.2, the converging subsequences satisfy*

$$G(\cdot, l(u_\varepsilon)) \to G(\cdot, l(u_0)) \quad \text{in } L^1(\omega_2) \tag{122}$$

and give smooth minimizing sequences for the functional $\tilde{\mathbf{J}}^0$. That is,

$$\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon) \to \tilde{\mathbf{J}}^0(u_0) = \inf_{v \in \mathcal{V}_0(\Omega,h)} \tilde{\mathbf{J}}^0(v) . \tag{123}$$

Proof: Since (113) holds and G still satisfies (32), the convergence (122) follows as in the proof of Theorem 3.2. Combining this with (112), we can pass to the limit in $\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon)$. It comes that

$$\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon) \to \tilde{\mathbf{J}}^0(u_0) \geq \inf_{v \in \mathcal{V}_0(\Omega,h)} \tilde{\mathbf{J}}^0(v) .$$

In addition, $\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon) \leq \tilde{\mathbf{J}}^\varepsilon(v), \forall v \in \mathcal{D}(\Omega)$. Passing to the limit in both sides, we get

$$\tilde{\mathbf{J}}^0(u_0) \leq \tilde{\mathbf{J}}^0(v) , \quad \forall\, v \in \mathcal{D}(\Omega) . \tag{124}$$

Since $\mathcal{D}(\Omega)$ is dense in $\mathcal{V}_0(\Omega, h)$ and $\tilde{\mathbf{J}}^0 \in C^1(\mathcal{V}_0(\Omega, h), \mathbb{R})$ (see Remark 2.4), then (124) still holds for every $v \in \mathcal{V}_0(\Omega, h)$. So u_0 realizes the minimum of $\tilde{\mathbf{J}}^0$ on $\mathcal{V}_0(\Omega, h)$ and the corollary follows. ∎

Remark 5.1: The arguments used in the proof of Theorem 5.1 can be adapted, by setting $\varepsilon = 0$ and replacing $\mathcal{W}_0(\Omega, h)$ by $\mathcal{V}_0(\Omega, h)$, to obtain the existence of a solution for Problem $(\mathbf{P}_O)$ that realizes the infimum of $\tilde{\mathbf{J}}^0$ on $\mathcal{V}_0(\Omega, h)$. Moreover, this solution is unique if (111) holds.

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