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Tensor Representations of Summing Polynomials via Finitely Supported Sequences

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Bouadi Khalil
(Abu Akram al-Bashir)

Dedication

بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

« وَمَا تَوْفِيقِي إِلَّا بِاللَّهِ ۚ عَلَيْهِ تَوَكَّلْتُ وَإِلَيْهِ أُنِيبُ »

«And my success is not but through Allah. Upon Him I have relied, and to Him I return.»

I dedicate the fruit of this humble effort...

To the source of boundless love and blessing, whose prayers were the secret to my success, and whose sacrifices were the bridge I crossed toward my ambitions... **My beloved parents**, may Allah protect them and grant them a long, healthy life.

To my lifelong companion and greatest support, who shared the hardships of this journey with me and provided me with every means of comfort and success... **My wonderful wife**.

To the joy of my eyes and the beat of my heart, the blossoms of my present and the hope of my future... **My precious children: Akram Al-Bashir and Rudaina Zineb**.

To those who strengthen my resolve, and with whom I share my finest memories and joys... **My brothers and sisters, their spouses, their children, and my entire extended family**.

To the beacons of light from whose knowledge I have drawn... **My honorable teachers and professors**.

To the promise of tomorrow, whose paths I always strive to illuminate... **All my dear students**.

To my partners in effort... **My colleagues at work**.

To my companions in charitable and educational endeavors, with a special mention to the esteemed members of **The "Al-Itqan" Center for Quran Memorization**.

To all of you... I dedicate this work.

List of Notations

Throughout this thesis, we adopt the standard notations of functional analysis. Symbols that are introduced and defined locally within a chapter are omitted from this list.

$\mathbb{K}$	The scalar field (either $\mathbb{R}$ or $\mathbb{C}$).
X, Y	Banach spaces over the field $\mathbb{K}$.
X^*	The continuous topological dual space of X.
B_X	The closed unit ball of the Banach space X.
p^*	The conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$.
ℓ_p	The Banach space of absolutely p-summable scalar sequences.
c_0	The Banach space of scalar sequences converging to zero.
c_{00}	The space of finitely supported scalar sequences.
$(e_n)_{n\geq 1}$	The canonical unit vectors, $e_n = (0, \ldots, 0, 1, 0, \ldots)$ with the entry 1 in the nth coordinate; the same notation $(e_j)_{j\geq 1}$ is used for the coordinates of ℓ_{mp}^0 and ℓ_p^0.
ℓ_p^0	The subspace of finitely supported sequences of ℓ_p; $\ell_p^0 = c_{00}$.
$\ell_p(X)$	The space of absolutely p-summable X-valued sequences.
$\ell_p^w(X)$	The space of weakly p-summable X-valued sequences.
$\ell_p\langle X\rangle$	The space of Cohen strongly p-summable X-valued sequences.
$c_{00}(X)$	The space of finitely supported X-valued sequences.
$S(X)$	A class of X-valued sequences (sequence class).
$X \otimes Y$	The algebraic tensor product of X and Y.
$\epsilon,\ \pi$	The injective and the projective tensor norms.
Δ_p	The Chevet–Saphar tensor norm.
$X \widehat{\otimes}_\alpha Y$	The completed tensor product with respect to the norm α.

$\mathcal{L}(X;Y)$	The Banach space of bounded linear operators from X to Y.
$\mathcal{L}(^mX;Y)$	The Banach space of continuous m-linear mappings.
$\mathcal{L}_s(^mX;Y)$	The space of continuous symmetric m-linear mappings.
$\mathcal{P}(^mX;Y)$	The Banach space of continuous m-homogeneous polynomials.
$\check{P}$	The symmetric m-linear mapping associated with P.
$\widehat{P}$	The sequence polynomial associated with P.
$\overline{P}$	The tensor polynomial associated with P.
$\mathcal{N}_p(X;Y)$	The ideal of Cohen p-nuclear linear operators.
$n_p(T)$	The Cohen p-nuclear norm of the operator T.
$\mathcal{P}^c_{p,N}(^mX;Y)$	The ideal of Cohen p-nuclear m-homogeneous polynomials.

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Introduction

The theory of operator ideals, initiated by Pietsch [32, 33], provides a unified framework for the study of classes of bounded linear operators sharing common structural and geometric properties. Although the classical theory was developed without explicitly using the language of tensor products, tensor product techniques have subsequently proved to be one of the most powerful tools in the study of operator ideals. In particular, the monograph of Defant and Floret [18] systematically develops the tensor product approach to operator ideals and demonstrates its effectiveness in describing the structure of many classical ideals and the Banach spaces associated with them.

Another important viewpoint is provided by vector-valued sequence spaces. In his seminal work [17], Cohen established a close connection between tensor products and vector-valued sequence spaces, leading to tensor characterizations of several operator ideals. Since then, many important operator ideals have been characterized in terms of tensor products, vector-valued sequence spaces, and factorization properties. These characterizations reveal deep connections between Banach space geometry, tensor norms, and the behavior of linear operators, and have become a fundamental tool in the modern theory of operator ideals.

A remarkable feature of this theory is that several classical operator ideals can be characterized through the continuity of suitable tensor operators. More precisely, appropriate tensor norms on algebraic tensor products admit canonical identifications with vector-valued sequence spaces, allowing operator ideals to be described through associated coordinatewise mappings. Classical examples include absolutely p-summing operators, Cohen p-nuclear operators, and related operator ideals, all of which admit equivalent tensor and sequence-space descriptions (see, for example, [24, 20, 38]).

In recent years, considerable effort has been devoted to extending the theory of operator ideals to nonlinear settings, particularly to multilinear mappings and homogeneous polynomials. Many fundamental concepts from the linear theory admit natural nonlinear counterparts, although their study requires new techniques reflecting the richer algebraic structure of multilinear and polynomial mappings. Significant progress in this direction has been achieved through the theory of polynomial and multilinear ideals (see, for example, [28, 26, 5, 13, 31, 11, 12, 35, 14, 29, 30]), and more recently through tensor and sequence-space methods for nonlinear mappings [37, 23, 25, 21, 15, 10, 36]. In particular,

the work of Achour, Alouani, Rueda, and Sánchez Pérez [4] provides a unified framework for obtaining tensor characterizations of polynomial ideals via vector-valued sequence spaces and associated tensor polynomials. Their approach extends the classical linear correspondence between tensor products and sequence spaces to the polynomial setting and serves as a foundation for several tensor characterizations of polynomial ideals.

The aim of this work is to develop a general framework for tensor characterizations of polynomial ideals, see [4]. Starting from suitable sequence classes and tensor norms, we introduce the associated sequence polynomial and the associated tensor polynomial and establish the fundamental relationship between them. This approach extends the classical correspondence between tensor operators and vector-valued sequence spaces to the polynomial setting. As a consequence, a wide class of polynomial ideals can be characterized through the continuity of their associated sequence polynomials or, equivalently, through the continuity of their associated tensor polynomials. The abstract results obtained here provide a unified framework that will be applied in the subsequent sections to obtain tensor characterizations of several polynomial ideals, including Cohen p-nuclear m-homogeneous polynomials.

The present work is organized into three chapters.

Chapter 1 contains the notation, terminology, and background material required throughout the thesis. We review the basic concepts of Banach spaces, tensor products, sequence spaces, and operator ideals, together with the fundamental results that will be used in the subsequent chapters.

Chapter 2 develops the general framework of the thesis. We introduce a technique for defining associated tensor polynomials by means of suitable finitely determined sequence spaces. We establish the relationship between associated sequence polynomials and associated tensor polynomials and derive abstract continuity and tensor characterization results that provide a unified approach to polynomial ideals.

Chapter 3 is devoted to applications of the abstract theory developed in Chapter 2. As an illustration of the general framework, we apply the obtained results to Cohen p-nuclear homogeneous polynomials, obtaining tensor and sequence-space characterizations together with corresponding norm identities.

Chapter 1

Preliminaries

In this chapter, we establish the foundational notation, sequence spaces, and tensor products required for the remainder of this thesis. We also present an example illustrating how these sequence spaces can be characterized through tensor products in the theory of linear operator ideals. Throughout this work, $\mathbb{K}$ denotes the scalar field ($\mathbb{R}$ or $\mathbb{C}$), and X, Y represent Banach spaces over $\mathbb{K}$. The continuous dual space of X is denoted by X^*, and the closed unit balls of X and X^* are represented by B_X and B_{X^*}, respectively. The Banach space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X;Y)$. For $1 \leq p \leq \infty$, p^* denotes the conjugate exponent satisfying $1/p + 1/p^* = 1$.

1.1 Classical Sequence Spaces

Sequence spaces provide some of the most important examples of Banach spaces. They serve throughout this work as fundamental models of infinite-dimensional normed spaces (for a comprehensive treatment, see, e.g., [20, 33]).

Classical Banach Sequence Spaces

- For $1 \leq p < \infty$, the space ℓ_p is defined by

$$\ell_p = \left\{ x = (x_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\},$$

  equipped with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

- The space ℓ_∞ consists of all bounded scalar sequences is defined by

$$\ell_\infty = \left\{ x = (x_i)_{i=1}^\infty : \sup_{i\in\mathbb{N}} |x_i| < \infty \right\},$$

equipped with the norm

$$\|x\|_\infty = \sup_{i\in\mathbb{N}} |x_i|.$$

- The space c_0 is the closed subspace of ℓ_∞ consisting of all sequences converging to zero:

$$c_0 = \left\{ x = (x_i)_{i=1}^\infty \in \ell_\infty : \lim_{i\to\infty} x_i = 0 \right\}.$$

Definition 1.1. The vector space of finitely supported scalar sequences, denoted by c_{00}, is defined by

$$c_{00} = \left\{ x = (x_i)_{i=1}^\infty \in \mathbb{K}^\mathbb{N} : x_i = 0 \text{ for all sufficiently large } n \right\}.$$

Equivalently,

$$c_{00} = \left\{ x = (x_i)_{i=1}^\infty \in \mathbb{K}^\mathbb{N} : \exists\, k \in \mathbb{N} \text{ such that } x_i = 0 \text{ for every } i > k \right\}.$$

The space c_{00} plays a fundamental role in the theory of sequence spaces, serving as a common dense subspace of many classical Banach sequence spaces [10].

Let $(e_n)_{i=1}^\infty$ denote the canonical unit vectors, where

$$e_n = (0, \ldots, 0, 1, 0, \ldots),$$

with the entry 1 in the nth coordinate. Every element $x \in c_{00}$ admits a unique finite representation

$$x = \sum_{i=1}^{k} x_i e_i,$$

for some $k \in \mathbb{N}$. Consequently,

$$c_{00} = \operatorname{span}\{e_n : n \in \mathbb{N}\}.$$

The space c_{00} may be equipped with any of the norms $\|\cdot\|_p$ $(1 \le p < \infty)$ or $\|\cdot\|_\infty$. With any of these norms, c_{00} is a normed space but is never complete. Its completion with respect to these norms yields the classical Banach sequence spaces [20, 33]:

$$\overline{c_{00}}^{\|\cdot\|_p} = \ell_p, \qquad 1 \le p < \infty,$$

and

$$\overline{c_{00}}^{\|\cdot\|_\infty} = c_0.$$

Proposition 1.2. *For every $1 \leq p < \infty$, the following strict continuous embeddings hold:*

$$c_{00} \subsetneq \ell_p \subsetneq c_0 \subsetneq \ell_\infty.$$

Proof. We prove each inclusion separately.

(i) We begin by showing that $c_{00} \subsetneq \ell_p$, for every $1 \leq p < \infty$

Let $x \in c_{00}$. Then x has finite support, so there exists $k \in \mathbb{N}$ such that $x_i = 0$ for every $i > k$. Hence,

$$\sum_{i=1}^{\infty} |x_i|^p = \sum_{i=1}^{k} |x_i|^p < \infty,$$

which shows that $x \in \ell_p$. Moreover,

$$\|x\|_{\ell_p} = \|x\|_p,$$

so the canonical embedding $c_{00} \hookrightarrow \ell_p$ is continuous.

To prove that the inclusion is strict, consider the sequence $x = (2^{-n})_{i=1}^{\infty}$. Since

$$\sum_{i=1}^{\infty} 2^{-np} < \infty,$$

we have $x \in \ell_p$ for every $1 \leq p < \infty$. However, x has infinite support, so $x \notin c_{00}$.

(ii) $\ell_p \subsetneq c_0$**.**

Let $x = (x_i) \in \ell_p$. Since

$$\sum_{i=1}^{\infty} |x_i|^p < \infty,$$

it follows that

$$|x_i|^p \longrightarrow 0,$$

and therefore

$$x_i \longrightarrow 0.$$

Hence $x \in c_0$.

Furthermore,

$$|x_i|^p \leq \sum_{k=1}^{\infty} |x_k|^p,$$

for every $n \in \mathbb{N}$, and consequently,

$$\|x\|_\infty = \sup_i |x_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} = \|x\|_p.$$

Thus the embedding $\ell_p \hookrightarrow c_0$ is continuous.

To see that the inclusion is strict, consider the sequence $x = (n^{-1/p})_{i=1}^{\infty}$. Since

$$n^{-1/p} \longrightarrow 0,$$

we have $x \in c_0$. On the other hand,

$$\sum_{i=1}^{\infty} |x_i|^p = \sum_{i=1}^{\infty} \frac{1}{n} = \infty,$$

so $x \notin \ell_p$.

(iii) Now we prove that $c_0 \subsetneq \ell_\infty$.

Every convergent sequence is bounded. Since every element of c_0 converges to 0, it follows that $c_0 \subset \ell_\infty$. Moreover,

$$\|x\|_{c_0} = \|x\|_\infty,$$

so the canonical embedding $c_0 \hookrightarrow \ell_\infty$ is an isometry, and hence continuous.

Finally, the constant sequence

$$\mathbf{1} = (1, 1, \ldots)$$

belongs to ℓ_∞ but does not converge to zero. Therefore, $\mathbf{1} \notin c_0$, and the inclusion is strict. □

Vector-Valued Sequence Spaces

In what follows, let X be a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). We denote by $X^{\mathbb{N}}$ the vector space of all X-valued sequences. In particular, $\mathbb{K}^{\mathbb{N}}$ denotes the space of all scalar-valued sequences.

Many classical scalar sequence spaces admit natural vector-valued analogues. We briefly recall the vector-valued versions of the spaces ℓ_p, ℓ_p^w, and $\ell_p\langle\cdot\rangle$, which will be used throughout this work (see [18, 20]).

- **Absolutely p-summable sequences.** For $1 \leq p < \infty$, the space $\ell_p(X)$ is defined

by

$$\ell_p(X) = \left\{ (x_i)_{i=1}^{\infty} \subset X : \sum_{i=1}^{\infty} \|x_i\|^p < \infty \right\},$$

equipped with the norm

$$\|x\|_{\ell_p(X)} = \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p}.$$

It is a Banach space.

- **Weakly p-summable sequences.** For $1 \leq p < \infty$, the space $\ell_p^w(X)$ consists of all sequences $x = (x_i)_{i=1}^{\infty}$ in X such that

$$(x^*(x_i))_{i=1}^{\infty} \in \ell_p$$

for every $x^* \in X^*$. It becomes a Banach space when equipped with the norm

$$\|x\|_{\ell_p^w(X)} = \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{\infty} |x^*(x_i)|^p \right)^{1/p}.$$

- **Cohen strongly p-summable sequences.**

Let $1 < p < \infty$ and let p^* denote the conjugate exponent of p, that is $\frac{1}{p} + \frac{1}{p^*} = 1$.

The space $\ell_p\langle X \rangle$ consists of all sequences $x = (x_i)_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty} |x_i^*(x_i)| < \infty$$

for every sequence $(x_i^*)_{i=1}^{\infty} \in \ell_{p^*}^w(X^*)$.

It is a Banach space under the norm

$$\|x\|_{\ell_p\langle X \rangle} = \sup \left\{ \sum_{i=1}^{\infty} |x_i^*(x_i)| : (x_i^*) \in B_{\ell_{p^*}^w(X^*)} \right\}.$$

Proposition 1.3 ([19, Chapter III])**.** *Let $1 < p < \infty$, and let p^* be the conjugate exponent of p. Then the dual space of ℓ_p is isometrically isomorphic to ℓ_{p^*}. More precisely, the mapping*

$$\Phi : \ell_{p^*} \longrightarrow (\ell_p)^*$$

defined by

$$\Phi(y)(x) = \sum_{i=1}^{\infty} x_i y_i, \qquad x \in \ell_p,\ y \in \ell_{p^*},$$

is an isometric isomorphism. Consequently,

$$(\ell_p)^* \cong \ell_{p^*}. \tag{1.1}$$

For $p = 1$, one has

$$(\ell_1)^* \cong \ell_\infty.$$

Proof. By Hölder's inequality, for every $x = (x_i) \in \ell_p$ and $y = (y_i) \in \ell_{p^*}$,

$$|\Phi(y)(x)| \le \sum_{i=1}^{\infty} |x_i||y_i| \le \|x\|_p \, \|y\|_{p^*},$$

so $\Phi(y) \in (\ell_p)^*$ and $\|\Phi(y)\| \le \|y\|_{p^*}$.

For the reverse inequality, fix $y \in \ell_{p^*}$ and $N \in \mathbb{N}$, and define $x^{(N)} = (x_i^{(N)})$ by

$$x_i^{(N)} = \begin{cases} |y_i|^{p^*-1} \operatorname{sgn}(y_i), & i \le N, \\ 0, & i > N, \end{cases}$$

with the convention $\operatorname{sgn}(y_i) = 1$ when $\mathbb{K} = \mathbb{R}$ and $y_i \ge 0$, and the obvious analogue for $\mathbb{K} = \mathbb{C}$. Then $\|x^{(N)}\|_p^p = \sum_{i=1}^N |y_i|^p$, using $p(p^* - 1) = p^*$, and

$$\Phi(y)\big(x^{(N)}\big) = \sum_{i=1}^{N} |y_i|^{p^*} = \|x^{(N)}\|_p \left(\sum_{i=1}^{N} |y_i|^{p^*} \right)^{1/p^*}.$$

Hence

$$\|\Phi(y)\| \ge \frac{\Phi(y)(x^{(N)})}{\|x^{(N)}\|_p} = \left(\sum_{i=1}^{N} |y_i|^{p^*} \right)^{1/p^*} \xrightarrow[N\to\infty]{} \|y\|_{p^*},$$

so $\|\Phi(y)\| \ge \|y\|_{p^*}$, and therefore $\|\Phi(y)\| = \|y\|_{p^*}$. Thus Φ is a linear isometry; in particular it is injective.

It remains to prove that Φ is surjective. Let $\psi \in (\ell_p)^*$ and set $y_i = \psi(e_i)$, where $(e_i)_{i\ge1}$ are the canonical unit vectors. For every $N \in \mathbb{N}$, applying ψ to the vector $x^{(N)}$ constructed above (with y replaced by $(y_1, \dots, y_N, 0, \dots)$) gives, exactly as above,

$$\left(\sum_{i=1}^{N} |y_i|^{p^*} \right)^{1/p^*} \le \|\psi\|.$$

Letting $N \to \infty$ shows that $y = (y_i) \in \ell_{p^*}$ with $\|y\|_{p^*} \le \|\psi\|$. Since ψ and $\Phi(y)$ agree on the dense subspace c_{00} of ℓ_p and are both continuous, $\psi = \Phi(y)$. Hence Φ is onto, which completes the proof. □

Finitely Determined Sequence Classes

We recall from Botelho and Campos [10] the following definition, which will play a fundamental role in our study.

Definition 1.4. A class of vector-valued sequences S, or simply a sequence class S, is a rule that assigns to each Banach space X a Banach space $S(X)$ of X-valued sequences such that

$$c_{00}(X) \subset S(X) \hookrightarrow \ell_\infty(X) \quad \text{and} \quad \|e_j\|_{S(\mathbb{K})} = 1 \text{ for every } j.$$

A sequence class S is said to be *finitely determined* if for every sequence $(x_j)_{j=1}^\infty \in X^{\mathbb{N}}$,

$$(x_j)_{j=1}^\infty \in S(X)$$

if and only if

$$\sup_k \|(x_j)_{j=1}^k\|_{S(X)} < \infty,$$

and, in this case,

$$\|(x_j)_{j=1}^\infty\|_{S(X)} = \sup_k \|(x_j)_{j=1}^k\|_{S(X)}.$$

We first isolate an elementary fact that will let us verify finite determination for all four classes below in a uniform way.

Lemma 1.5. *Let Θ be a non-empty set and, for each $\theta \in \Theta$, let $(a_k(\theta))_{k\geq 1}$ be a non-decreasing sequence in $[0, +\infty]$. Then*

$$\sup_{k\geq 1} \sup_{\theta\in\Theta} a_k(\theta) = \sup_{\theta\in\Theta} \lim_{k\to\infty} a_k(\theta).$$

Proof. Both sides equal $\sup\{a_k(\theta) : k \geq 1,\ \theta \in \Theta\}$: this is immediate for the left-hand side, and for the right-hand side it follows because $\lim_k a_k(\theta) = \sup_k a_k(\theta)$ for each fixed θ, by monotonicity. □

Example 1.6. The sequence classes $\ell_p(\cdot)$ and $\ell_\infty(\cdot)$, for $1 \leq p < \infty$, together with $\ell_p^w(\cdot)$ and $\ell_p\langle\cdot\rangle$, are finitely determined. We verify all four cases; compare [10].

Let X be a Banach space and let $(x_j)_{j=1}^\infty \in X^{\mathbb{N}}$.

(a) $\ell_p(X)$**,** $1 \leq p < \infty$**.** Here $\|(x_j)_{j=1}^k\|_{\ell_p(X)} = \left(\sum_{j=1}^k \|x_j\|^p\right)^{1/p}$ is non-decreasing in k, with limit $\left(\sum_{j=1}^\infty \|x_j\|^p\right)^{1/p} = \|(x_j)_j\|_{\ell_p(X)}$. Lemma 1.5 (with Θ a singleton) gives at once $\sup_k \|(x_j)_{j=1}^k\|_{\ell_p(X)} = \|(x_j)_j\|_{\ell_p(X)}$, which is exactly finite determination.

(b) $\ell_\infty(X)$**.** Similarly, $\|(x_j)_{j=1}^k\|_{\ell_\infty(X)} = \max_{j\leq k} \|x_j\|$ is non-decreasing in k with limit $\sup_j \|x_j\| = \|(x_j)_j\|_{\ell_\infty(X)}$, and Lemma 1.5 again gives finite determination.

(c) $\ell_p^w(X)$**,** $1 \le p \le \infty$**.** Take $\Theta = B_{X^*}$ and, for $x^* \in \Theta$,

$$a_k(x^*) = \Big(\sum_{j=1}^{k} |x^*(x_j)|^p\Big)^{1/p},$$

which is non-decreasing in k with limit $\big(\sum_{j=1}^{\infty} |x^*(x_j)|^p\big)^{1/p}$. By Lemma 1.5,

$$\sup_k \|(x_j)_{j=1}^k\|_{\ell_p^w(X)} = \sup_k \sup_{x^*\in\Theta} a_k(x^*) = \sup_{x^*\in\Theta} \lim_{k\to\infty} a_k(x^*) = \|(x_j)_j\|_{\ell_p^w(X)},$$

so $\ell_p^w(\cdot)$ is finitely determined, and, in particular, $(x_j)_j \in \ell_p^w(X)$ as soon as the left-hand side is finite.

(d) $\ell_p\langle X\rangle$**,** $1 < p < \infty$**.** Take $\Theta = B_{\ell_{p^*}^w(X^*)}$ and, for $(x_i^*)_{i\ge1} \in \Theta$,

$$a_k\big((x_i^*)_i\big) = \sum_{j=1}^{k} |x_j^*(x_j)|,$$

again non-decreasing in k with limit $\sum_{j=1}^{\infty} |x_j^*(x_j)|$. Lemma 1.5 gives, exactly as in (c),

$$\sup_k \|(x_j)_{j=1}^k\|_{\ell_p\langle X\rangle} = \|(x_j)_j\|_{\ell_p\langle X\rangle},$$

so $\ell_p\langle\cdot\rangle$ is finitely determined.

In each of the four cases, finiteness of $\sup_k \|(x_j)_{j=1}^k\|$ is equivalent, by the same equality, to membership of $(x_j)_j$ in the corresponding sequence space, which is precisely Definition 1.4.

Example 1.7. The sequence class $c_0(\cdot)$ is not finitely determined Indeed, consider the sequence class $c_0(\cdot)$ over the scalar field $\mathbb{K}$. Let

$$x = (1, 1, 1, \ldots) \in \mathbb{K}^{\mathbb{N}}.$$

For each $k \in \mathbb{N}$, its k-th finite truncation is

$$x^{(k)} = (\underbrace{1, \ldots, 1}_{k \text{ terms}}, 0, 0, \ldots).$$

Clearly, $x^{(k)} \in c_0(\mathbb{K})$ and

$$\|x^{(k)}\|_{c_0} = 1.$$

Hence,

$$\sup_{k\in\mathbb{N}} \|x^{(k)}\|_{c_0} = 1 < \infty.$$

However,

$$x = (1, 1, 1, \ldots) \notin c_0(\mathbb{K}),$$

since x does not converge to 0. Therefore,

$$\sup_{k\in\mathbb{N}} \|x^{(k)}\|_{c_0} < \infty \quad \text{but} \quad x \notin c_0(\mathbb{K}),$$

showing that $c_0(\cdot)$ is not a finitely determined sequence class.

The following theorem, due to Botelho and Campos [10], plays a fundamental role in the theory of finitely determined sequence classes. It shows that the operator norm can be computed by considering only finitely supported sequences.

Theorem 1.8 ([10])**.** *Let $S_1(X)$ and $S_2(Y)$ be two finitely determined sequence classes. Let*

$$T : S_1(X) \longrightarrow S_2(Y)$$

be a bounded linear operator. Then

$$\|T\| = \sup\left\{\|T(x)\|_{S_2(Y)} : x \in c_{00}(X),\ \|x\|_{S_1(X)} \leq 1\right\}.$$

Proof. Let

$$N = \sup\Big\{\|T(x)\|_{S_2(Y)} :\ x \in B_{S_1(X)} \cap c_{00}(X)\Big\}.$$

Clearly,

$$N \leq \|T\|,$$

since $B_{S_1(X)} \cap c_{00}(X) \subseteq B_{S_1(X)}$.

Suppose, by contradiction, that $N < \|T\|$. Then there exists $x = (x_j)_{j=1}^{\infty} \in B_{S_1(X)}$ such that

$$\|T(x)\|_{S_2(Y)} > N.$$

For each $k \in \mathbb{N}$, let

$$x^{(k)} = (x_1, \ldots, x_k, 0, 0, \ldots).$$

Since $S_1(X)$ is finitely determined,

$$\|x^{(k)}\|_{S_1(X)} \leq \|x\|_{S_1(X)} \leq 1,$$

and therefore $x^{(k)} \in B_{S_1(X)} \cap c_{00}(X)$.

Because T maps $S_1(X)$ into the finitely determined sequence class $S_2(Y)$,

$$\|T(x)\|_{S_2(Y)} = \sup_{k\in\mathbb{N}} \|T(x^{(k)})\|_{S_2(Y)}.$$

On the other hand, by the definition of N,

$$\|T(x^{(k)})\|_{S_2(Y)} \leq N \qquad \text{for every } k \in \mathbb{N}.$$

Hence,

$$\|T(x)\|_{S_2(Y)} = \sup_{k\in\mathbb{N}} \|T(x^{(k)})\|_{S_2(Y)} \leq N,$$

which contradicts $\|T(x)\|_{S_2(Y)} > N$.

Therefore,

$$\|T\| = N,$$

which completes the proof. □

Corollary 1.9. *Let $S_1(X)$ and $S_2(Y)$ be two finitely determined sequence classes. A mapping*

$$T : S_1(X) \longrightarrow S_2(Y)$$

is bounded if and only if there exists a constant $C > 0$ such that

$$\|T(u)\|_{S_2(Y)} \leq C\|u\|_{S_1(X)}$$

for every $u \in c_{00}(X)$. Moreover,

$$\|T\| = \inf\left\{C > 0 : \ \|T(u)\|_{S_2(Y)} \leq C\|u\|_{S_1(X)} \text{ for all } u \in c_{00}(X)\right\}.$$

1.2 Tensor Products of Banach Spaces

The preceding section showed that vector-valued sequence spaces provide a natural framework for describing summability properties of sequences. A fundamental fact, established by Cohen [17] and later developed systematically by Defant and Floret [18], is that these sequence spaces are isometrically isomorphic to suitable tensor product spaces endowed with appropriate tensor norms. Consequently, tensor products provide the natural bridge between sequence spaces and operator ideals, a connection that will be exploited throughout the remainder of this chapter.

Let X and Y be Banach spaces over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and let X^* and Y^* denote their continuous dual spaces. The closed unit ball of a Banach space E is denoted by B_E.

Definition 1.10. The *algebraic tensor product* of X and Y, denoted by $X \otimes Y$, is the vector space generated by all elementary tensors $x \otimes y$, where $x \in X$ and $y \in Y$. Every

element $u \in X \otimes Y$ admits a finite (not necessarily unique) representation of the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i,$$

where $x_i \in X$, $y_i \in Y$, and $n \in \mathbb{N}$.

Definition 1.11. A norm α on $X \otimes Y$ is called a *reasonable cross-norm* if it satisfies the following conditions:

1. For every $x \in X$ and $y \in Y$,

$$\alpha(x \otimes y) = \|x\| \, \|y\|.$$

2. For every $x^* \in X^*$ and $y^* \in Y^*$, the functional $x^* \otimes y^*$ belongs to $(X \otimes_\alpha Y)^*$ and satisfies

$$\|x^* \otimes y^*\| \leq \|x^*\| \, \|y^*\|.$$

We now introduce the specific cross-norms. We begin with the two extremal reasonable cross-norms, namely the injective and projective norms, and then define the intermediate Chevet–Saphar norm.

Definition 1.12. Let X and Y be Banach spaces.

1. The *injective norm* $\|\cdot\|_\epsilon$ is the smallest reasonable cross-norm on $X \otimes Y$. For

$$u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y,$$

it is defined by

$$\|u\|_\epsilon = \sup \left\{ \left| \sum_{i=1}^{n} \langle x^*, x_i \rangle \langle y^*, y_i \rangle \right| : x^* \in B_{X^*},\ y^* \in B_{Y^*} \right\}.$$

2. The *projective norm* $\|\cdot\|_\pi$ is the largest reasonable cross-norm on $X \otimes Y$. For $u \in X \otimes Y$,

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

where the infimum is taken over all finite representations of u.

3. Let $1 \leq p \leq \infty$, and let p^* denote the conjugate exponent of p. The *Chevet–Saphar*

norm Δ_p is defined by

$$\|u\|_{\Delta_p} = \inf\left\{ \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} \sup_{y^*\in B_{Y^*}} \left(\sum_{i=1}^n |\langle y^*, y_i\rangle|^{p^*}\right)^{1/p^*} : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

Proposition 1.13 ([18])**.** *Let* $1 \le p \le \infty$*. Then, for every* $u \in X \otimes Y$*, the Chevet–Saphar norm* Δ_p *satisfies*

$$\|u\|_\epsilon \le \|u\|_{\Delta_p} \le \|u\|_\pi.$$

Consequently, the identity map on $X \otimes Y$ *extends by continuity to bounded linear operators between the corresponding completed tensor products.*

Proof. Let $u = \sum_{i=1}^n x_i \otimes y_i$ be an arbitrary representation of u.

For every $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$, Hölder's inequality yields

$$\left|\sum_{i=1}^n \langle x^*, x_i\rangle\langle y^*, y_i\rangle\right| \le \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p\right)^{1/p} \left(\sum_{i=1}^n |\langle y^*, y_i\rangle|^{p^*}\right)^{1/p^*}.$$

Since $\|x^*\| \le 1$,

$$|\langle x^*, x_i\rangle| \le \|x_i\|,$$

and therefore

$$\left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p\right)^{1/p} \le \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}.$$

Taking the supremum over $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$ gives

$$\|u\|_\epsilon \le \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p} \sup_{y^*\in B_{Y^*}} \left(\sum_{i=1}^n |\langle y^*, y_i\rangle|^{p^*}\right)^{1/p^*}.$$

Finally, taking the infimum over all representations of u yields

$$\|u\|_\epsilon \le \|u\|_{\Delta_p}.$$

To prove the second inequality, observe that

$$|\langle y^*, y_i\rangle| \le \|y_i\| \qquad (y^* \in B_{Y^*}),$$

so that

$$\sup_{y^*\in B_{Y^*}} \left(\sum_{i=1}^n |\langle y^*, y_i\rangle|^{p^*}\right)^{1/p^*} \le \left(\sum_{i=1}^n \|y_i\|^{p^*}\right)^{1/p^*}.$$

Hence,

$$\|u\|_{\Delta_p} \leq \inf\left\{ \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \left(\sum_{i=1}^{n} \|y_i\|^{p^*} \right)^{1/p^*} \right\},$$

where the infimum is taken over all representations of u.

Now, replacing each $x_i \otimes y_i$ by $\tilde{x}_i \otimes \tilde{y}_i$, where $\tilde{x}_i = \lambda_i x_i$ and $\tilde{y}_i = \lambda_i^{-1} y_i$ with

$$\lambda_i = \left(\frac{\|y_i\|^{p^*}}{\|x_i\|^p} \right)^{1/(pp^*)},$$

one obtains

$$\|\tilde{x}_i\|^p = \|\tilde{y}_i\|^{p^*} = \|x_i\| \, \|y_i\|.$$

Therefore,

$$\left(\sum_{i=1}^{n} \|\tilde{x}_i\|^p \right)^{1/p} \left(\sum_{i=1}^{n} \|\tilde{y}_i\|^{p^*} \right)^{1/p^*} = \sum_{i=1}^{n} \|x_i\| \, \|y_i\|.$$

Taking the infimum over all representations gives

$$\|u\|_{\Delta_p} \leq \|u\|_{\pi}.$$

This completes the proof. □

The algebraic tensor product $X \otimes Y$, endowed with a reasonable cross-norm α, is generally not complete. Its completion with respect to the norm α is denoted by

$$X \widehat{\otimes}_{\alpha} Y,$$

and is called the *completed tensor product* of X and Y with respect to α. In particular, the completions associated with the injective, Chevet–Saphar, and projective norms are denoted by $X \widehat{\otimes}_{\epsilon} Y, X \widehat{\otimes}_{\Delta_p} Y,$ and $X \widehat{\otimes}_{\pi} Y$, respectively.

Proposition 1.14 ([18, 38])**.** *Let X and Y be Banach spaces. Then the dual of the projective tensor product $X \hat{\otimes}_{\pi} Y$ is isometrically isomorphic to the space $\mathcal{L}(X, Y^*)$ of bounded linear operators from X into Y^*. More precisely,*

$$(X \hat{\otimes}_{\pi} Y)^* \cong \mathcal{L}(X, Y^*) \tag{1.2}$$

isometrically.

Proof. Let $\varphi \in (X \hat{\otimes}_{\pi} Y)^*$. Define

$$T_{\varphi} : X \longrightarrow Y^*$$

by

$$(T_\varphi x)(y) = \varphi(x \otimes y), \qquad x \in X,\ y \in Y.$$

Since φ is linear, T_φ is linear. Moreover,

$$|(T_\varphi x)(y)| = |\varphi(x \otimes y)| \leq \|\varphi\|\, \|x \otimes y\|_\pi = \|\varphi\|\, \|x\|\, \|y\|,$$

and hence

$$\|T_\varphi x\| \leq \|\varphi\|\, \|x\|,$$

which shows that $T_\varphi \in \mathcal{L}(X, Y^*)$ and

$$\|T_\varphi\| \leq \|\varphi\|.$$

Conversely, let $T \in \mathcal{L}(X, Y^*)$. Define

$$\varphi_T : X \otimes Y \longrightarrow \mathbb{K}$$

by

$$\varphi_T\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sum_{i=1}^n \langle Tx_i, y_i\rangle.$$

For every representation $u = \sum_{i=1}^n x_i \otimes y_i$,

$$|\varphi_T(u)| \leq \sum_{i=1}^n \|Tx_i\|\, \|y_i\| \leq \|T\| \sum_{i=1}^n \|x_i\|\, \|y_i\|.$$

Taking the infimum over all representations of u yields

$$|\varphi_T(u)| \leq \|T\|\, \|u\|_\pi.$$

Thus φ_T is continuous on $X \otimes_\pi Y$ and extends uniquely to a continuous linear functional on $X \hat{\otimes}_\pi Y$ satisfying

$$\|\varphi_T\| \leq \|T\|.$$

Finally, for $x \in X$ and $y \in Y$,

$$(T_{\varphi_T} x)(y) = \varphi_T(x \otimes y) = \langle Tx, y\rangle,$$

and similarly $\varphi_{T_\varphi} = \varphi$. Therefore the correspondences $\varphi \mapsto T_\varphi$ and $T \mapsto \varphi_T$ are inverse isometries, proving the result. $\square$

Characterizations of Vector-Valued Sequence Spaces

We now describe how the tensor product construction is related to vector-valued sequence spaces. It is well known that the algebraic tensor product $\ell_p \otimes X$ can be identified with a subspace of the corresponding sequence space $S(X)$ (see [17]). More precisely, the canonical map

$$\Theta : \ell_p \otimes X \longrightarrow S(X), \qquad \Theta\left(\sum_{j=1}^{n} a_j \otimes x_j\right) = \left(\sum_{j=1}^{n} a_j(k) x_j\right)_{k\geq 1},$$

is an algebraic isomorphism onto its image.

For later purposes, we restrict this identification to finite sequences. We denote by

$$\ell_p^0 = \{(a_j)_{j\geq 1} \in \ell_p : a_j = 0 \text{ for all but finitely many } j\}$$

the subspace of finitely supported sequences. Equivalently,

$$\ell_p^0 = c_{00}.$$

The canonical vectors $(e_j)_{j\geq 1}$ form an algebraic basis of ℓ_p^0.

Under the canonical algebraic identification

$$\Phi : \ell_p^0 \otimes X \longrightarrow c_{00}(X) \tag{1.3}$$

defined by

$$\Phi\left(\sum_{j=1}^{m} e_j \otimes x_j\right) = (x_1, \ldots, x_m, 0, \ldots),$$

we shall identify $\ell_p^0 \otimes X$ with $c_{00}(X)$ throughout this work.

Indeed, every element of $\ell_p^0 \otimes X$ admits a unique representation

$$u = \sum_{j=1}^{m} e_j \otimes x_j,$$

because $(e_j)_{j\geq 1}$ is an algebraic basis of ℓ_p^0. Consequently, the sequence

$$(x_1, \ldots, x_m, 0, \ldots)$$

depends only on u, and hence Φ is well defined. Linearity is immediate from the definition. Moreover, if

$$\Phi\left(\sum_{j=1}^{m} e_j \otimes x_j\right) = 0,$$

then

$$(x_1, \ldots, x_m, 0, \ldots) = 0,$$

which implies $x_1 = \cdots = x_m = 0$; therefore Φ is injective. Finally, for every

$$(x_1, \ldots, x_m, 0, \ldots) \in c_{00}(X),$$

we have

$$(x_1, \ldots, x_m, 0, \ldots) = \Phi\left(\sum_{j=1}^{m} e_j \otimes x_j\right),$$

so Φ is surjective. Hence Φ is a canonical algebraic isomorphism.

The following classical results of Cohen [17] show that the norms of these sequence spaces induce the injective, Chevet–Saphar, and projective tensor norms on $\ell_p \otimes X$.

Theorem 1.15 ([17, Theorem 1.2.2])**.** *Let $1 \leq p \leq \infty$. The space $\ell_p^w(X)$ induces the injective cross-norm ϵ on the algebraic tensor product $\ell_p \otimes X$.*

Proof. Let

$$u = \sum_{j=1}^{m} e_j \otimes x_j \in \ell_p \otimes X.$$

By the definition of the injective tensor norm,

$$\|u\|_\epsilon = \sup\left\{\left|\sum_{j=1}^{m} \langle \phi, e_j\rangle x^*(x_j)\right| : \phi \in B_{(\ell_p)^*},\, x^* \in B_{X^*}\right\}.$$

From (1.1), since $(\ell_p)^* \cong \ell_{p^*}$, every $\phi \in B_{(\ell_p)^*}$ is represented by a sequence $\alpha = (\alpha_j) \in B_{\ell_{p^*}}$, and therefore

$$\|u\|_\epsilon = \sup_{x^* \in B_{X^*}} \sup_{\alpha \in B_{\ell_{p^*}}} \left|\sum_{j=1}^{m} \alpha_j x^*(x_j)\right|.$$

By the duality between ℓ_p and ℓ_{p^*},

$$\sup_{\alpha \in B_{\ell_{p^*}}} \left|\sum_{j=1}^{m} \alpha_j x^*(x_j)\right| = \left(\sum_{j=1}^{m} |x^*(x_j)|^p\right)^{1/p},$$

(with the obvious modification when $p = \infty$). Hence,

$$\|u\|_\epsilon = \sup_{x^* \in B_{X^*}} \left(\sum_{j=1}^{m} |x^*(x_j)|^p\right)^{1/p} = \|(x_j)_{j=1}^m\|_{\ell_p^w(X)},$$

which is precisely the norm induced on $\ell_p \otimes X$ by $\ell_p^w(X)$. □

Theorem 1.16 ([17, Theorem 1.2.3])**.** *Let $1 < p < \infty$. The space $\ell_p\langle X\rangle$ induces the projective cross-norm π on the algebraic tensor product $\ell_p \otimes X$.*

Proof. Let

$$u = \sum_{j=1}^{m} e_j \otimes x_j \in \ell_p \otimes X.$$

By definition,

$$\|u\|_{\ell_p\langle X\rangle} = \sup\left\{ \sum_{j=1}^{m} |x_j^*(x_j)| : (x_j^*)_{j=1}^m \in B_{\ell_{p^*}^w(X^*)} \right\}.$$

On the other hand, by the duality (1.2)

$$(\ell_p \hat{\otimes}_\pi X)^* \cong \mathcal{L}(\ell_p, X^*),$$

the projective norm admits the representation

$$\|u\|_\pi = \sup\left\{ \left| \sum_{j=1}^{m} T(e_j)(x_j) \right| : T \in B_{\mathcal{L}(\ell_p, X^*)} \right\}.$$

Since every operator $T \in \mathcal{L}(\ell_p, X^*)$ is uniquely determined by the sequence $(T(e_j))$, and

$$\|T\| = \|(T(e_j))\|_{\ell_{p^*}^w(X^*)},$$

we obtain

$$\|u\|_\pi = \sup\left\{ \sum_{j=1}^{m} |x_j^*(x_j)| : (x_j^*)_{j=1}^m \in B_{\ell_{p^*}^w(X^*)} \right\} = \|u\|_{\ell_p\langle X\rangle}.$$

Hence, the norm induced by $\ell_p\langle X\rangle$ on $\ell_p \otimes X$ is precisely the projective cross-norm π. □

Theorem 1.17 ([16, 34, 17])**.** *Let $1 \le p \le \infty$. The space $\ell_p(X)$ induces the Chevet–Saphar cross-norm Δ_p on the algebraic tensor product $\ell_p \otimes X$.*

The previous results identify the classical sequence spaces with the injective, projective, and Chevet–Saphar tensor norms. These identifications make it possible to characterize important classes of linear operators either through their action on sequence spaces or, equivalently, through induced operators between tensor products. We illustrate this principle with classical example.

1.3 Tensor Characterization of the Ideal of Cohen p-Nuclear Linear Operators

The theory developed in the previous sections provides a natural framework for the study of operator ideals. As an illustration, we consider the class of Cohen p-nuclear operators.

Definition 1.18 ([17])**.** Let $1 < p < \infty$. A bounded linear operator $T \in \mathcal{L}(X, Y)$ is called *Cohen p-nuclear* if there exists $C > 0$ such that, for every finite families $(x_i)_{i=1}^n \subset X$ and

$(y_i^*)_{i=1}^n \subset Y^*$,

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| \leq C \|(x_i)\|_{\ell_p^w(X)} \, \|(y_i^*)\|_{\ell_{p^*}^w(Y^*)}.$$

The least such constant is denoted by $n_p(T)$.

The collection of all Cohen p-nuclear operators from X to Y, denoted by $\mathcal{N}_p(X,Y)$, forms a Banach operator ideal.

Proposition 1.19. *Let $1 \leq p \leq \infty$ and let $T \in \mathcal{L}(X,Y)$. The following assertions are equivalent:*

1. *T is Cohen p-nuclear;*
2. *The canonical mapping*

$$\widehat{T} : \ell_p^w(X) \longrightarrow \ell_p\langle Y \rangle, \qquad \widehat{T}((x_i)_i) = (T(x_i))_i,$$

is well-defined and bounded.

Moreover,

$$n_p(T) = \|\widehat{T}\|.$$

Proof. Assume first that T is Cohen p-nuclear. For a sequence $(x_i) \in \ell_p^w(X)$, consider the mapping

$$\widehat{T}((x_i)_i) = (T(x_i))_i.$$

We prove that $(T(x_i))_i \in \ell_p\langle Y \rangle$ and that this mapping is bounded.

By the definition of Cohen p-nuclearity, for every finite sequences $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$,

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| \leq n_p(T) \|(x_i)\|_{w,p} \|(y_i^*)\|_{w,p^*}.$$

Now let $(x_i) \in \ell_p^w(X)$ and $(y_i^*) \in \ell_{p^*}^w(Y^*)$. Applying the above inequality to the first n terms and passing to the limit gives

$$\sum_{i=1}^{\infty} |\langle T(x_i), y_i^* \rangle| \leq n_p(T) \|(x_i)\|_{w,p} \|(y_i^*)\|_{w,p^*}.$$

By the definition of $\ell_p\langle Y \rangle$, we conclude that $(T(x_i))_i \in \ell_p\langle Y \rangle$ and

$$\|\widehat{T}((x_i))\|_{\ell_p\langle Y \rangle} \leq n_p(T) \|(x_i)\|_{w,p}.$$

Therefore $\widehat{T}$ is well-defined and bounded, with

$$\|\widehat{T}\| \leq n_p(T).$$

Conversely, suppose that $\widehat{T}$ is bounded. Let $x_1, \dots, x_n \in X$ and $y_1^*, \dots, y_n^* \in Y^*$. Extending these families by zero, we may regard them as elements of $\ell_p^w(X)$ and $\ell_{p^*}^w(Y^*)$, respectively. Then

$$\begin{aligned}\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| &\leq \|(T(x_i))\|_{\ell_p\langle Y\rangle} \|(y_i^*)\|_{w,p^*} \\ &= \|\widehat{T}(x_i)\|_{\ell_p\langle Y\rangle} \|(y_i^*)\|_{w,p^*} \\ &\leq \|\widehat{T}\| \|(x_i)\|_{w,p} \|(y_i^*)\|_{w,p^*}.\end{aligned}$$

Therefore T is Cohen p-nuclear and

$$n_p(T) \leq \|\widehat{T}\|.$$

The equality follows by combining both inequalities. □

Before stating the tensor characterization, we recall how a bounded linear operator induces a mapping between tensor products, and we deliberately denote it by a symbol different from $\widehat{T}$, so as not to confuse it with the sequence-side map of Proposition 1.19.

Definition 1.20. Let $T \in \mathcal{L}(X, Y)$. The *associated tensor operator*

$$I \otimes T : \ell_p \otimes X \longrightarrow \ell_p \otimes Y$$

is defined on elementary tensors by

$$(I \otimes T)(a \otimes x) := a \otimes T(x), \qquad a \in \ell_p,\ x \in X,$$

and is extended by linearity to $\ell_p \otimes X$.

Theorem 1.21 (Cohen [17])**.** *Let $T \in \mathcal{L}(X, Y)$. The following are equivalent:*

1. *T is Cohen p-nuclear;*
2. *$I \otimes T : \ell_p \widehat{\otimes}_\epsilon X \longrightarrow \ell_p \widehat{\otimes}_\pi Y$ is well-defined and bounded.*

Moreover,

$$n_p(T) = \|I \otimes T\|.$$

Proof. By Proposition 1.19,

$$T \text{ is Cohen } p\text{-nuclear} \iff \widehat{T} : \ell_p^w(X) \to \ell_p\langle Y\rangle \text{ is bounded.}$$

We show that, under the canonical isometric isomorphisms of Theorems 1.15 and 1.16,

$$\ell_p^w(X) \cong \ell_p \widehat{\otimes}_\epsilon X, \qquad \ell_p\langle Y\rangle \cong \ell_p \widehat{\otimes}_\pi Y,$$

the sequence-side map $\widehat{T}$ corresponds exactly to the tensor-side map $I \otimes T$ of Definition 1.20.

Indeed, let $u = \sum_{j=1}^{m} e_j \otimes x_j \in \ell_p \otimes X$. By the proof of Theorem 1.15, under the identification $\ell_p \otimes X \hookrightarrow \ell_p^w(X)$, the tensor u corresponds to the finitely supported sequence $(x_j)_{j=1}^m$. By Definition 1.20,

$$(I \otimes T)(u) = \sum_{j=1}^{m} e_j \otimes T(x_j),$$

which, by the proof of Theorem 1.16, corresponds under $\ell_p \otimes Y \hookrightarrow \ell_p\langle Y \rangle$ to the sequence $(T(x_j))_{j=1}^m = \widehat{T}\big((x_j)_{j=1}^m\big)$.

Hence $I \otimes T$ and $\widehat{T}$ agree on the dense subspace $\ell_p^0 \otimes X \cong c_{00}(X)$. Consequently, $I \otimes T$ extends to a bounded operator between the completed tensor products $\ell_p \widehat{\otimes}_\epsilon X$ and $\ell_p \widehat{\otimes}_\pi Y$ if and only if $\widehat{T}$ is bounded from $\ell_p^w(X)$ to $\ell_p\langle Y \rangle$, and in that case the two operators have the same norm. This proves the result. □

Remark 1.22. The previous theorem illustrates a general principle. Once a sequence space induces a tensor norm on an algebraic tensor product, operator ideals can be characterized by the continuity of the associated linearized tensor operator between the corresponding completed tensor products. The characterization of Cohen p-nuclear operators presented above is one instance of this general scheme. Similar tensor characterizations hold for other classes of summing operators, such as absolutely p-summing operators, where the injective and Chevet–Saphar tensor norms naturally appear. This viewpoint will serve as the guiding principle for the nonlinear theory developed in the next chapter.

Chapter 2

Homogeneous Polynomials and Tensor Products

Chapter 1 ended with a linear statement: a bounded operator is Cohen p-nuclear exactly when the associated tensor operator is bounded between the corresponding completed tensor products. The aim of this chapter is to construct the machinery that makes such a statement meaningful when the linear operator is replaced by an m-homogeneous polynomial. No particular summability class is used here: all the results are proved for arbitrary sequence classes.

2.1 m-Homogeneous Polynomials

Let $m \in \mathbb{N}$ and let $X_1, \ldots, X_m$ and Y be Banach spaces over the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We equip the Cartesian product

$$X_1 \times ... \times X_m = \left\{ \left(x^1, ..., x^m\right) : x^j \in X_j,\ \forall 1 \leq j \leq m \right\},$$

which is a normed space equipped with the norm

$$\left\| \left(x^1, ..., x^m\right) \right\| := \max \left\{ \left\| x^j \right\| ;\ x^j \in X_j, \forall 1 \leq j \leq m \right\}. \tag{2.1}$$

Definition 2.1. A mapping

$$T : X_1 \times \cdots \times X_m \longrightarrow Y$$

is called *multilinear* (or *m-linear*) if it is linear in each variable separately, in other words

$$T\left(x^1, ..., \lambda x^j + y^j, ..., x^m\right) = \lambda T\left(x^1, ..., x^j, ..., x^m\right) + T\left(x^1, ..., y^j, ..., x^m\right);$$

for all $\lambda \in \mathbb{K}$ and x^j, $y^j \in X_j$ $(1 \leq j \leq m)$. The vector space of all multilinear mappings

from $X_1 \times \cdots \times X_m$ into Y is denoted by

$L(X_1, \ldots, X_m; Y)$. When $Y = \mathbb{K}$, we simply write $L(X_1, \ldots, X_m)$.

A multilinear mapping

$$T : X_1 \times \cdots \times X_m \longrightarrow Y$$

is said to be *continuous* if it is continuous with respect to the product topology on $X_1 \times \cdots \times X_m$. We denote by $\mathcal{L}(X_1, \ldots, X_m; Y)$ the vector space of all continuous multilinear mappings from $X_1 \times \cdots \times X_m$ into Y.

When $X_1 = \cdots = X_m = X$, we write $\mathcal{L}({}^m X; Y)$, and, if $Y = \mathbb{K}$, we simply write $\mathcal{L}(X_1, \ldots, X_m)$.

In particular,

$$\mathcal{L}(X; \mathbb{K}) = X^*.$$

The space $\mathcal{L}(X_1, \ldots, X_m; Y)$ is endowed with the operator norm

$$\|T\| = \sup_{\|x_j\| \leq 1} \|T(x_1, \ldots, x_m)\|,$$

which is equivalently given by

$$\begin{aligned} \|T\| \quad & \sup_{\substack{x_j \neq 0 \\ 1 \leq j \leq m}} \frac{\|T(x_1, \ldots, x_m)\|}{\|x_1\| \cdots \|x_m\|} \\ & = \inf \{C > 0 : \|T(x_1, \ldots, x_m)\| \leq C\|x_1\| \cdots \|x_m\|, \ \forall\, x_j \in X_j\}. \end{aligned}$$

With this norm, $\mathcal{L}(X_1, \ldots, X_m; Y)$ is a Banach space whenever Y is Banach.

Let $\mathcal{S}_m$ denote the symmetric group on $\{1, \ldots, m\}$. A multilinear mapping

$$T : X^m \longrightarrow Y$$

is called *symmetric* if

$$T(x_1, \ldots, x_m) = T(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$$

for every $\sigma \in \mathcal{S}_m$. The space of all continuous symmetric multilinear mappings is denoted by $\mathcal{L}_s({}^m X; Y)$.

The concept of a homogeneous polynomial is closely related to that of a symmetric multilinear mapping.

Definition 2.2. Let X and Y be Banach spaces and let $m \in \mathbb{N}$. A mapping

$$P : X \longrightarrow Y$$

is called an *m-homogeneous polynomial* if there exists a symmetric m-linear mapping

$$\widetilde{P} : X^m \longrightarrow Y$$

such that

$$P(x) = \widetilde{P}(x, \ldots, x), \qquad x \in X.$$

The mapping $\widetilde{P}$ is called the *symmetric multilinear mapping associated with* P.

The correspondence between homogeneous polynomials and symmetric multilinear mappings is one-to-one. In particular, the associated multilinear mapping can be recovered from P by the classical polarization formula (see [27]),

$$\widetilde{P}(x_1, \ldots, x_m) = \frac{1}{m!\, 2^m} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_m P\left(\sum_{j=1}^{m} \varepsilon_j x_j\right).$$

A homogeneous polynomial $P : X \longrightarrow Y$ is said to be *continuous* if there exists a constant $C \geq 0$ such that

$$\|P(x)\| \leq C\|x\|^m, \qquad x \in X.$$

We denote by $\mathcal{P}({}^m X; Y)$ the Banach space of all continuous m-homogeneous polynomials from X into Y, endowed with the norm

$$\|P\| = \sup_{x \in B_X} \|P(x)\|.$$

When $Y = \mathbb{K}$, we simply write $\mathcal{P}({}^m X)$. In the particular case $m = 1$, the space $\mathcal{P}({}^1 X; Y)$ coincides isometrically with $\mathcal{L}(X; Y)$, the Banach space of all bounded linear operators from X into Y.

If $P \in \mathcal{P}({}^m X; Y)$ and $\widetilde{P}$ is its associated symmetric multilinear mapping, then

$$\|P\| \leq \|\widetilde{P}\| \leq \frac{m^m}{m!}\|P\|.$$

The relationship between a homogeneous polynomial and its associated symmetric multilinear mapping is summarized by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Delta_m} & X^m \\ & \searrow^{P} & \downarrow{\scriptstyle \widetilde{P}} \\ & & Y \end{array}$$

where $\Delta_m : X \to X^m$ is the diagonal mapping defined by

$$\Delta_m(x) = (x, \ldots, x) \qquad x \in X.$$

For a comprehensive treatment of homogeneous polynomials, we refer the reader to [22, 27].

Example 2.3. Let X and Y be Banach spaces, let $\varphi \in X^*$, and let $u \in \mathcal{L}(X;Y)$. For a fixed $m \in \mathbb{N}$, define

$$P : X \longrightarrow Y, \qquad P(x) = \big(\varphi(x)\big)^{m-1} u(x).$$

Then P is a continuous m-homogeneous polynomial.

Indeed, consider the mapping $T : X^m \longrightarrow Y$ defined by

$$T(x_1, \ldots, x_m) = \frac{1}{m} \sum_{j=1}^{m} \varphi(x_1) \cdots \varphi(x_{j-1}) \varphi(x_{j+1}) \cdots \varphi(x_m)\, u(x_j).$$

The mapping T is clearly symmetric. We prove directly that it is bounded. For $x_1, \ldots, x_m \in X$, each of the m terms in the defining sum contains, as factors, $\varphi(x_i)$ for every index $i \neq j$ and $u(x_j)$ for the remaining index j; hence every term is bounded by $\|\varphi\|^{m-1}\|u\| \prod_{i=1}^{m} \|x_i\|$, and

$$\|T(x_1, \ldots, x_m)\| \leq \frac{1}{m} \sum_{j=1}^{m} \|\varphi\|^{m-1}\|u\| \prod_{i=1}^{m} \|x_i\| = \|\varphi\|^{m-1}\|u\| \prod_{i=1}^{m} \|x_i\|.$$

Thus T is a bounded, and so continuous, symmetric m-linear mapping, with $\|T\| \leq \|\varphi\|^{m-1}\|u\|$. Moreover,

$$T(x, \ldots, x) = \big(\varphi(x)\big)^{m-1} u(x) = P(x),$$

so P is an m-homogeneous polynomial. Setting $x_1 = \cdots = x_m = x$ in the estimate above gives, for every $x \in X$,

$$\|P(x)\| = |\varphi(x)|^{m-1}\|u(x)\| \leq \|\varphi\|^{m-1}\|u\|\, \|x\|^m,$$

and therefore

$$\|P\| = \sup_{\|x\| \leq 1} \|P(x)\| \leq \|\varphi\|^{m-1}\|u\|.$$

2.2 Tensor Polynomials on Sequence Spaces

Recall from Definition 1.20 that a bounded linear operator $T : X \to Y$ induces, on elementary tensors, an associated tensor operator

$$I \otimes T : \ell_p \otimes X \longrightarrow \ell_p \otimes Y, \qquad (I \otimes T)\left(\sum_{i=1}^{n} a_i \otimes x_i\right) = \sum_{i=1}^{n} a_i \otimes T(x_i),$$

where $a_i \in \ell_p$ and $x_i \in X$, and that this is precisely the map whose continuity characterizes Cohen p-nuclearity in Theorem 1.21. A similar construction cannot be obtained by simply replacing the linear operator T with an m-homogeneous polynomial $P : X \to Y$, since the resulting mapping fails to be well defined whenever $m \geq 2$. Consequently, a different construction is required in order to define an associated tensor polynomial. Such tensor products of m- homogeneous polynomials have previously been considered in the literature (see, for example, [8, Section 6] and [7]). Following this approach, we introduce the associated tensor polynomial.

Let

$$\ell_{mp}^0 = \ell_{mp} \cap c_{00} = \{(a_j) \in \ell_{mp} : a_j = 0 \text{ for all but finitely many } j\},$$

denote the subspace of finitely supported sequences in ℓ_{mp}.

Let $P \in \mathcal{P}({}^m X; Y)$. We define the associated tensor mapping

$$\overline{P} : \ell_{mp}^0 \otimes X \longrightarrow \ell_p^0 \otimes Y$$

by

$$\overline{P}\left(\sum_{i=1}^{n} (a_{ij})_{j=1}^{k} \otimes x_i\right) = \sum_{j=1}^{k} e_j \otimes P\left(\sum_{i=1}^{n} a_{ij} x_i\right).$$

We first verify that $\overline{P}$ is well defined. Let

$$u = \sum_{i=1}^{n} (a_{ij})_{j=1}^{k} \otimes x_i \in \ell_p^0 \otimes X.$$

Since

$$(a_{ij})_{j=1}^{k} = \sum_{j=1}^{k} a_{ij} e_j,$$

we have

$$\begin{aligned} u &= \sum_{i=1}^{n} \sum_{j=1}^{k} a_{ij} e_j \otimes x_i \\ &= \sum_{j=1}^{k} e_j \otimes \left(\sum_{i=1}^{n} a_{ij} x_i\right). \end{aligned}$$

Setting

$$y_j = \sum_{i=1}^{n} a_{ij} x_i, \qquad j = 1, \ldots, k,$$

we obtain

$$u = \sum_{j=1}^{k} e_j \otimes y_j.$$

Under the canonical identification (1.3)

$$\ell_p^0 \otimes X \cong c_{00}(X),$$

the tensor u corresponds to the finitely supported sequence $(y_j)_{j\geq 1}$. Therefore, the vectors y_j are uniquely determined by u. Consequently, the formula

$$\overline{P}(u) = \sum_{j=1}^{k} e_j \otimes P(y_j)$$

depends only on u and not on the particular representation

$$u = \sum_{i=1}^{n} (a_{ij})_{j=1}^{k} \otimes x_i.$$

Hence, $\overline{P}$ is well defined.

From the definition of $\overline{P}$, we have

$$\overline{P}\left(\sum_{i=1}^{n} (a_{ij})_{j=1}^{k} \otimes x_i\right) = \sum_{j=1}^{k} e_j \otimes P\left(\sum_{i=1}^{n} a_{ij} x_i\right).$$

Choosing $k = n$ and

$$a_{ij} = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

we have

$$(a_{ij})_{j=1}^{n} = (\delta_{i1}, \ldots, \delta_{in}) = e_i.$$

Consequently,

$$\sum_{i=1}^{n} (a_{ij})_{j=1}^{n} \otimes x_i = \sum_{i=1}^{n} e_i \otimes x_i.$$

Moreover, for every $j = 1, \ldots, n$,

$$\sum_{i=1}^{n} a_{ij} x_i = \sum_{i=1}^{n} \delta_{ij} x_i = x_j.$$

Therefore,

$$\overline{P}\left(\sum_{i=1}^{n} e_i \otimes x_i\right) = \sum_{j=1}^{n} e_j \otimes P\left(\sum_{i=1}^{n} \delta_{ij} x_i\right)$$
$$= \sum_{j=1}^{n} e_j \otimes P(x_j).$$

Let

$$\check{P} : X^m \longrightarrow Y$$

be the symmetric m-linear mapping associated with P, that is,

$$P(x) = \check{P}(x, \ldots, x), \qquad x \in X.$$

We define the associated tensor multilinear mapping

$$\overline{\check{P}} : (\ell_{mp}^0 \otimes X)^m \longrightarrow \ell_p^0 \otimes Y$$

by

$$\overline{\check{P}}\left(\sum_{i=1}^{n} e_i \otimes x_i^1, \ldots, \sum_{i=1}^{n} e_i \otimes x_i^m\right) = \sum_{i=1}^{n} e_i \otimes \check{P}(x_i^1, \ldots, x_i^m).$$

For

$$u = \sum_{i=1}^{n} e_i \otimes x_i \in \ell_{mp}^0 \otimes X,$$

we obtain

$$\overline{\check{P}}(u, \ldots, u) = \sum_{i=1}^{n} e_i \otimes \check{P}(x_i, \ldots, x_i)$$
$$= \sum_{i=1}^{n} e_i \otimes P(x_i)$$
$$= \overline{P}(u).$$

Hence,

$$\overline{P}(u) = \overline{\check{P}}(u, \ldots, u),$$

which shows that $\overline{P}$ is an m-homogeneous polynomial.

Moreover, the symmetric multilinear mapping associated with $\overline{P}$ is precisely the tensor multilinear mapping associated with $\check{P}$; that is,

$$\check{\overline{P}} = \overline{\check{P}}.$$

2.3 The Associated Sequence Polynomial

To relate the tensor polynomial $\overline{P}$ with a coordinatewise polynomial acting on sequence spaces, we introduce the associated sequence polynomial, following the approach of Achour, Alouani, Rueda and Sánchez-Pérez [4]. Let $P \in \mathcal{P}({}^m X; Y)$ be an m-homogeneous polynomial. We define the associated sequence polynomial

$$\widehat{P} : X^{\mathbb{N}} \longrightarrow Y^{\mathbb{N}}$$

defined by,

$$\widehat{P}\big((x_i)_{i=1}^{\infty}\big) = \big(P(x_i)\big)_{i=1}^{\infty}.$$

Via the canonical isomorphism (1.3), we identify $\ell_p^0 \otimes X$ with $c_{00}(X)$. Explicitly,

$$\Phi_{p,X} : \ell_p^0 \otimes X \longrightarrow c_{00}(X)$$

defined by

$$\Phi_{p,X}\left(\sum_{i=1}^{n}(a_{ij})_{j=1}^{k} \otimes x_i\right) = \left(\sum_{i=1}^{n} a_{ij}x_i\right)_{j=1}^{k}.$$

As shown previously, $\Phi_{p,X}$ is an algebraic isomorphism.

Proposition 2.4. *Let $P : X \to Y$ be an m-homogeneous polynomial and let*

$$\widehat{P} : X^{\mathbb{N}} \longrightarrow Y^{\mathbb{N}}, \qquad \widehat{P}((x_i)_i) = (P(x_i))_i,$$

be the associated sequence polynomial. Then

$$\widehat{P} \circ \Phi_{mp,X} = \Phi_{p,Y} \circ \overline{P}.$$

Proof. Let

$$u = \sum_{i=1}^{n}(a_{ij})_{j=1}^{k} \otimes x_i \in \ell_{mp}^0 \otimes X.$$

Then

$$\begin{aligned}(\widehat{P}\circ\Phi_{mp,X})(u) &= \widehat{P}\left(\left(\sum_{i=1}^{n} a_{ij}x_i\right)_{j=1}^{k}\right)\\ &= \left(P\left(\sum_{i=1}^{n} a_{ij}x_i\right)\right)_{j=1}^{k}\\ &= \Phi_{p,Y}\left(\sum_{j=1}^{k} e_j \otimes P\left(\sum_{i=1}^{n} a_{ij}x_i\right)\right)\\ &= \Phi_{p,Y}\big(\overline{P}(u)\big)\\ &= (\Phi_{p,Y}\circ\overline{P})(u).\end{aligned}$$

Therefore,

$$\widehat{P}\circ\Phi_{mp,X} = \Phi_{p,Y}\circ\overline{P}.$$

□

The finite determination property of sequence classes allows us to reduce norm estimates for sequence polynomials to finitely supported sequences. The next proposition formalizes this fact by showing that $c_{00}(X)$ is a norming subset for the associated sequence polynomial whenever the corresponding sequence classes are finitely determined.

Throughout this section, whenever the associated sequence polynomial

$$\widehat{P} : S_1(X) \longrightarrow S_2(Y)$$

is well defined, we write

$$\|\widehat{P}\| := \sup_{\|x\|_{S_1(X)}\leq 1} \|\widehat{P}(x)\|_{S_2(Y)},$$

allowing the value $+\infty$. In particular, $\widehat{P}$ is continuous if and only if

$$\|\widehat{P}\| < \infty.$$

Proposition 2.5. *Let S_1 and S_2 be finitely determined sequence classes, and let $P \in \mathcal{P}({}^{m}X;Y)$ be such that*

$$\widehat{P}(S_1(X)) \subset S_2(Y).$$

Then the associated sequence polynomial

$$\widehat{P} : S_1(X) \longrightarrow S_2(Y)$$

is normed on $c_{00}(X)$; more precisely, $c_{00}(X)$ is a norming subset for $\widehat{P}$.

Proof. Let

$$N(\widehat{P}) = \sup\left\{\|\widehat{P}(x)\|_{S_2(Y)} : x \in c_{00}(X),\ \|x\|_{S_1(X)} \leq 1\right\}.$$

Clearly,

$$N(\widehat{P}) \leq \|\widehat{P}\|.$$

If $N(\widehat{P}) = \infty$, then the assertion is immediate. Suppose, therefore, that $N(\widehat{P}) < \infty$. We prove that

$$N(\widehat{P}) = \|\widehat{P}\|.$$

Assume, by contradiction, that

$$N(\widehat{P}) < \|\widehat{P}\|.$$

Then, by the definition of the norm of $\widehat{P}$, there exists

$$x = (x_i)_{i=1}^{\infty} \in S_1(X), \qquad \|x\|_{S_1(X)} \leq 1,$$

such that

$$\|\widehat{P}(x)\|_{S_2(Y)} > N(\widehat{P}).$$

Since S_1 is finitely determined, every finite truncation

$$(x_1, \ldots, x_N, 0, \ldots)$$

belongs to $c_{00}(X)$ and satisfies

$$\|(x_i)_{i=1}^{N}\|_{S_1(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{S_1(X)} \leq 1, \qquad N \in \mathbb{N}.$$

Therefore, by the definition of $N(\widehat{P})$,

$$\|(P(x_i))_{i=1}^{N}\|_{S_2(Y)} \leq N(\widehat{P}), \qquad N \in \mathbb{N}.$$

On the other hand, since S_2 is finitely determined, we have

$$\|\widehat{P}(x)\|_{S_2(Y)} = \|(P(x_i))_{i=1}^{\infty}\|_{S_2(Y)} = \sup_{N\in\mathbb{N}} \|(P(x_i))_{i=1}^{N}\|_{S_2(Y)}.$$

Consequently,

$$\|\widehat{P}(x)\|_{S_2(Y)} \leq N(\widehat{P}),$$

which contradicts the choice of x.

Hence,

$$N(\widehat{P}) = \|\widehat{P}\|,$$

and therefore $c_{00}(X)$ is a norming subset for $\widehat{P}$. □

Let $P \in \mathcal{P}({}^m X;Y)$ be such that $\widehat{P}(S_1(X)) \subset S_2(Y)$.

Since

$$\Phi_{mp,X}(\ell^0_{mp} \otimes X) \subset c_{00}(X) \subset S_1(X)$$

and

$$\Phi_{p,Y}(\ell^0_p \otimes Y) \subset c_{00}(Y) \subset S_2(Y),$$

we obtain the following commutative diagram:

$$\begin{array}{ccc} S_1(X) & \xrightarrow{\widehat{P}} & S_2(Y) \\ {\scriptstyle \Phi_{mp,X}}\uparrow & & \uparrow{\scriptstyle \Phi_{p,Y}} \\ \ell^0_{mp} \otimes X & \xrightarrow{\overline{P}} & \ell^0_p \otimes Y \end{array}$$

The commutativity of the diagram follows from Proposition 2.4.

We now establish the fundamental result relating the continuity of the associated tensor polynomial to the continuity of the induced sequence polynomial. This theorem provides the abstract framework for the tensor characterizations obtained in the following sections.

Theorem 2.6. *Let X and Y be Banach spaces and let S_1 and S_2 be sequence classes. Let $P \in \mathcal{P}({}^m X;Y)$ be such that $\widehat{P}(S_1(X)) \subset S_2(Y)$. Let α and β be norms on $\ell^0_{mp} \otimes X$ and $\ell^0_p \otimes Y$ respectively so that $\Phi_{mp,X} : \ell^0_{mp} \otimes_\alpha X \to S_1(X)$ and $\Phi_{p,Y} : \ell^0_p \otimes_\beta Y \to S_2(Y)$ are continuous.*

1. *If $\Phi_{p,Y}$ is an isometric embedding then $\overline{P} : \ell^0_{mp} \otimes_\alpha X \to \ell^0_p \otimes_\beta Y$ is continuous whenever $\widehat{P} : S_1(X) \to S_2(Y)$ is continuous. In this case $\|\overline{P}\| \leq \|\widehat{P}\| \, \|\Phi_{mp,X}\|^m$.*
2. *If $S_1(X)$ and $S_2(Y)$ are finitely determined and $\Phi_{mp,X}$ is an isometric embedding then $\widehat{P} : S_1(X) \to S_2(Y)$ is continuous whenever $\overline{P} : \ell^0_{mp} \otimes_\alpha X \to \ell^0_p \otimes_\beta Y$ is continuous. In this case $\|\widehat{P}\| \leq \|\overline{P}\| \|\Phi_{p,Y}\|$.*

Proof. (1) Let

$$u \in \ell^0_{mp} \otimes_\alpha X.$$

By Proposition 2.4,

$$\widehat{P} \circ \Phi_{mp,X} = \Phi_{p,Y} \circ \overline{P}.$$

Hence,

$$\Phi_{p,Y}\big(\overline{P}(u)\big) = \widehat{P}\big(\Phi_{mp,X}(u)\big).$$

Since $\Phi_{p,Y}$ is an isometric embedding, we obtain

$$\beta\big(\overline{P}(u)\big) = \Big\|\Phi_{p,Y}\big(\overline{P}(u)\big)\Big\|_{S_2(Y)}.$$

Therefore,

$$\begin{aligned}\beta\big(\overline{P}(u)\big) &= \Big\|\widehat{P}\big(\Phi_{mp,X}(u)\big)\Big\|_{S_2(Y)} \\ &\leq \|\widehat{P}\|\,\|\Phi_{mp,X}(u)\|_{S_1(X)}^m \\ &\leq \|\widehat{P}\|\,\|\Phi_{mp,X}\|^m\,\alpha(u)^m.\end{aligned}$$

Hence $\overline{P}$ is continuous and

$$\|\overline{P}\| \leq \|\widehat{P}\|\,\|\Phi_{mp,X}\|^m.$$

(2) Since

$$\widehat{P}(S_1(X)) \subset S_2(Y)$$

and the sequence classes are finitely determined, Proposition 2.5 implies that $c_{00}(X)$ is a norming subset for $\widehat{P} : S_1(X) \to S_2(Y)$.

Let $(x_i)_{i=1}^n \in c_{00}(X)$ satisfy

$$\|(x_i)_{i=1}^n\|_{S_1(X)} \leq 1.$$

Then

$$\begin{aligned}\Big\|\widehat{P}((x_i)_{i=1}^n)\Big\|_{S_2(Y)} &= \|(P(x_i))_{i=1}^n\|_{S_2(Y)} \\ &= \left\|\Phi_{p,Y}\left(\sum_{i=1}^n e_i \otimes P(x_i)\right)\right\|_{S_2(Y)} \\ &= \left\|\Phi_{p,Y}\left(\overline{P}\left(\sum_{i=1}^n e_i \otimes x_i\right)\right)\right\|_{S_2(Y)} \\ &\leq \|\Phi_{p,Y}\|\,\beta\left(\overline{P}\left(\sum_{i=1}^n e_i \otimes x_i\right)\right) \\ &\leq \|\Phi_{p,Y}\|\,\|\overline{P}\|\,\alpha\left(\sum_{i=1}^n e_i \otimes x_i\right)^m.\end{aligned}$$

Since $\Phi_{mp,X}$ is an isometry,

$$\alpha\left(\sum_{i=1}^n e_i \otimes x_i\right) = \left\|\Phi_{mp,X}\left(\sum_{i=1}^n e_i \otimes x_i\right)\right\|_{S_1(X)} = \|(x_i)_{i=1}^n\|_{S_1(X)} \leq 1.$$

Hence,

$$\Big\|\widehat{P}((x_i)_{i=1}^n)\Big\|_{S_2(Y)} \leq \|\Phi_{p,Y}\|\,\|\overline{P}\|.$$

Since $c_{00}(X)$ is a norming subset for $\widehat{P}$ by Proposition 2.5, taking the supremum over all

$$(x_i) \in c_{00}(X), \qquad \|(x_i)\|_{S_1(X)} \leq 1,$$

yields

$$\|\widehat{P}\| \leq \|\Phi_{p,Y}\| \, \|\overline{P}\|.$$

□

Corollary 2.7. *Let X and Y be Banach spaces, let S_1 and S_2 be sequence classes, and let $P \in \mathcal{P}(^mX;Y)$ be such that*

$$\widehat{P}(S_1(X)) \subset S_2(Y).$$

Suppose that α and β are norms on $\ell_{mp}^0 \otimes X$ and $\ell_p^0 \otimes Y$, respectively, such that

$$\Phi_{mp,X} : \ell_{mp}^0 \otimes_\alpha X \longrightarrow S_1(X) \quad \textit{and} \quad \Phi_{p,Y} : \ell_p^0 \otimes_\beta Y \longrightarrow S_2(Y)$$

are isometric embeddings. If $S_1(X)$ and $S_2(Y)$ are finitely determined, then

$$\widehat{P} : S_1(X) \longrightarrow S_2(Y)$$

is continuous if and only if

$$\overline{P} : \ell_{mp}^0 \otimes_\alpha X \longrightarrow \ell_p^0 \otimes_\beta Y$$

is continuous. Moreover,

$$\|\widehat{P}\| = \|\overline{P}\|.$$

The following theorem provides a polynomial analogue of [10, Proposition 1.4] and characterizes the continuity of the coordinatewise polynomial induced by P between sequence spaces.

Theorem 2.8. *Let $m \in \mathbb{N}$, let $P \in \mathcal{P}(^mX;Y)$, and let S_1 and S_2 be sequence classes. The following assertions are equivalent:*

1. *For every $(x_i)_{i=1}^\infty \in S_1(X)$,*
$$(P(x_i))_{i=1}^\infty \in S_2(Y).$$

2. *The induced mapping*
$$\widehat{P} : S_1(X) \longrightarrow S_2(Y),$$
is a well-defined continuous m-homogeneous polynomial.

Moreover, each of the above conditions implies the following:

(3) There exists a constant $C > 0$ such that

$$\|(P(x_i))_{i=1}^n\|_{S_2(Y)} \leq C \, \|(x_i)_{i=1}^n\|_{S_1(X)}^m,$$

for every $x_1, \ldots, x_n \in X$ *and every* $n \in \mathbb{N}$.

If, in addition, S_1 *and* S_2 *are finitely determined, then* (3) *is also equivalent to* (1) *and* (2).

Furthermore,

$$\|\widehat{P}\| = \inf\left\{C > 0 : \|(P(x_i))_{i=1}^n\|_{S_2(Y)} \leq C\,\|(x_i)_{i=1}^n\|_{S_1(X)}^m,\ \forall\, n \in \mathbb{N}\right\}.$$

Proof. The implication *(2)* $\Rightarrow$ *(1)* is immediate from the definition of $\widehat{P}$.

Assume now that *(1)* holds. Then the mapping

$$\widehat{P} : S_1(X) \longrightarrow S_2(Y), \qquad \widehat{P}((x_i)_{i=1}^\infty) = (P(x_i))_{i=1}^\infty,$$

is well defined. Since P is m-homogeneous, $\widehat{P}$ is also m-homogeneous. It remains to prove that $\widehat{P}$ is continuous.

To this end, it is enough to prove that its associated symmetric m-linear mapping $\check{\widehat{P}}$ is continuous. Consider the coordinatewise multilinear operator induced by $\check{P}$,

$$\widehat{\check{P}} : S_1(X) \times \cdots \times S_1(X) \longrightarrow S_2(Y),$$

defined by

$$\widehat{\check{P}}\big((x_i^1)_{i=1}^\infty, \ldots, (x_i^m)_{i=1}^\infty\big) = \big(\check{P}(x_i^1, \ldots, x_i^m)\big)_{i=1}^\infty.$$

For every $i \in \mathbb{N}$, the polarization formula ([27, Theorem 1.10]) gives

$$\check{P}(x_i^1, \ldots, x_i^m) = \frac{1}{m!\,2^m} \sum_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \varepsilon_1 \cdots \varepsilon_m P(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m). \tag{2.2}$$

Since $S_1(X)$ is a Banach space, it is closed under finite linear combinations. Hence, for every choice of signs,

$$(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m)_{i=1}^\infty \in S_1(X).$$

By assumption *(1)*,

$$\big(P(\varepsilon_1 x_i^1 + \cdots + \varepsilon_m x_i^m)\big)_{i=1}^\infty \in S_2(Y).$$

Therefore, (2.2) implies that

$$\big(\check{P}(x_i^1, \ldots, x_i^m)\big)_{i=1}^\infty \in S_2(Y),$$

showing that $\widehat{\check{P}}$ is well defined.

Applying [10, Proposition 1.4] to the multilinear operator $\check{P}$, we conclude that $\widehat{\check{P}}$ is

continuous. Moreover, for every $(x_i^1)_i, \ldots, (x_i^m)_i \in S_1(X)$,

$$\check{\widehat{P}}((x_i^1)_i, \ldots, (x_i^m)_i) = (\check{P}(x_i^1, \ldots, x_i^m))_i = \widehat{\check{P}}((x_i^1)_i, \ldots, (x_i^m)_i).$$

so that $\check{\widehat{P}} = \widehat{\check{P}}$. It follows that $\check{\widehat{P}}$ is continuous, and consequently $\widehat{P}$ is continuous. This proves *(1)* $\Rightarrow$ *(2)*.

Moreover, the previous argument shows that if P satisfies *(1)*, then $\check{P}$ satisfies condition *(a)* of [10, Proposition 1.4]. Hence part *(c)* of that proposition yields condition *(3)*.

The remainder of the proof follows the arguments of [10, Proposition 1.4]. In particular, condition *(2)* implies *(3)* and

$$\|\widehat{P}\| \geq \inf\{C > 0 : \text{condition } (3) \text{ holds}\}.$$

Finally, assume that S_1 and S_2 are finitely determined and that *(3)* holds. Let $(x_i)_{i=1}^{\infty} \in S_1(X)$. Since S_1 is finitely determined,

$$\|(x_i)_{i=1}^n\|_{S_1(X)} \leq \|(x_i)_{i=1}^{\infty}\|_{S_1(X)}$$

for every $n \in \mathbb{N}$. Hence, by (3),

$$\|(P(x_i))_{i=1}^n\|_{S_2(Y)} \leq C\|(x_i)_{i=1}^{\infty}\|_{S_1(X)}^m.$$

Taking the supremum over n and using the finite determination of S_2, we obtain

$$\|\widehat{P}((x_i)_i)\|_{S_2(Y)} \leq C\|(x_i)_i\|_{S_1(X)}^m.$$

Therefore $\widehat{P} : S_1(X) \to S_2(Y)$ is continuous. Consequently, the least constant C satisfying (3) is exactly the norm of $\widehat{P}$, namely

$$\|\widehat{P}\| = \inf\left\{C > 0 : \|(P(x_i))_{i=1}^n\|_{S_2(Y)} \leq C\|(x_i)_{i=1}^n\|_{S_1(X)}^m,\ \forall n \in \mathbb{N}\right\}.$$

□

Remark 2.9. Theorem 2.8 shows that the continuity of the associated sequence polynomial $\widehat{P}$ is completely determined by the behavior of P on finite sequences. In particular, when the sequence classes are finitely determined, the study of $\widehat{P}$ can be reduced to its restriction to $c_{00}(X)$.

Through the canonical identification between finite sequences and algebraic tensor products,

$$c_{00}(X) \cong \ell_p^0 \otimes X,$$

this observation provides the link between sequence polynomial constructions and tensor polynomial representations. This connection will be used in the next section to establish tensor characterizations of homogeneous polynomials.

Chapter 3

Tensor Characterization of Cohen p-Nuclear Polynomials

The machinery built in Chapter 2 is now applied to a concrete class. We recall the definition of the Cohen p-nuclear m-homogeneous polynomials and obtain for them the tensor characterization announced in the introduction, which is the non-linear counterpart of the linear result of Section 1.3.

3.1 Cohen p-Nuclear Homogeneous Polynomials

We begin by recalling the definition of Cohen p-nuclear homogeneous polynomials, which constitute one of the fundamental classes of polynomial ideals.

Definition 3.1. Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$, and let X and Y be Banach spaces. An m-homogeneous polynomial $P \in \mathcal{P}({}^m X; Y)$

is called *Cohen p-nuclear* if there exists a constant $C \geq 0$ such that

$$\left\| \left(P(x_i)\right)_{i=1}^n \right\|_{\ell_p\langle Y\rangle} \leq C \left\| (x_i)_{i=1}^n \right\|_{\ell_{mp}^w(X)}^m,$$

for every $x_1, \ldots, x_n \in X$ and every $n \in \mathbb{N}$.

The infimum of all constants C satisfying the above inequality defines a norm on the space of all Cohen p-nuclear m-homogeneous polynomials from X to Y, denoted by

$\mathcal{P}_{p,N}^c({}^m X; Y)$. This norm is denoted by

$\|P\|_{p,N}$. Moreover, equipped with this norm, $\mathcal{P}_{p,N}^c({}^m X; Y)$ is a Banach polynomial ideal.

Example 3.2. Let $1 < p < \infty$, $m \in \mathbb{N}$, and let $u : X \to Y$ be a Cohen p-nuclear operator, where X and Y are Banach spaces. Let $\varphi \in X^*$ and define

$$\begin{array}{rccc} P: & X & \longrightarrow & Y, \\ & x & \longmapsto & \varphi(x)^{m-1} u(x). \end{array}$$

Then P is a Cohen p-nuclear m-homogeneous polynomial. Moreover,

$$\|P\|_{p,N} \leq n_p(u)\,\|\varphi\|^{m-1},$$

where $n_p(u)$ denotes the Cohen p-nuclear norm of u.

Proposition 3.3. *An m-homogeneous polynomial P is Cohen p-nuclear if and only if its associated symmetric m-linear mapping $\check{P}$ is Cohen p-nuclear in the sense of [2], or, equivalently, absolutely $(1; mp, \ldots, mp, p^*)$-summing in the sense of [1].*

3.2 Tensor Characterization of Cohen p-Nuclear Polynomials

We now apply the general tensor characterization established in Theorem 2.6 together with Theorem 2.8 to obtain a tensor characterization of Cohen p-nuclear polynomials. The proof is an immediate consequence of these results and the classical tensor identifications for the sequence spaces $\ell_{mp,\,w}(\cdot)$ and $\ell_p\langle\cdot\rangle$.

Theorem 3.4. *Let $1 < p < \infty$, $m \in \mathbb{N}$, and let $P \in \mathcal{P}({}^mX; Y)$. The following assertions are equivalent.*

1. *P is Cohen p-nuclear.*
2. *For every $(x_i)_{i=1}^{\infty} \in \ell_{mp,\,w}(X)$,*
$$(P(x_i))_{i=1}^{\infty} \in \ell_p\langle Y\rangle.$$
3. *The induced polynomial*
$$\widehat{P} : \ell_{mp,\,w}(X) \longrightarrow \ell_p\langle Y\rangle$$
is well defined and continuous.
4. *The associated tensor polynomial*
$$\overline{P} : \ell_{mp}^0 \otimes_\varepsilon X \longrightarrow \ell_p^0 \otimes_\pi Y$$
is continuous.

Moreover,
$$\|P\|_{p,N} = \|\widehat{P}\| = \|\overline{P}\|.$$

Proof. Take
$$S_1(X) = \ell_{mp,\,w}(X) \quad \text{and} \quad S_2(Y) = \ell_p\langle Y\rangle.$$

Since both sequence classes are finitely determined ([10, Remark 1.3]), Theorem 2.8 shows that conditions *(1)*, *(2)* and *(3)* are equivalent, and that

$$\|P\|_{p,N} = \|\widehat{P}\|.$$

Moreover, Cohen's classical tensor characterizations yield the isometric embeddings

$$\phi_{mp,X} : \ell_{mp}^0 \otimes_\varepsilon X \longrightarrow \ell_{mp,\,w}(X)$$

and

$$\phi_{p,Y} : \ell_p^0 \otimes_\pi Y \longrightarrow \ell_p\langle Y\rangle.$$

Therefore, Corollary 2.7 applies and shows that *(3)* and *(4)* are equivalent. Furthermore,

$$\|\widehat{P}\| = \|\overline{P}\|.$$

Combining these equalities yields

$$\|P\|_{p,N} = \|\widehat{P}\| = \|\overline{P}\|,$$

which completes the proof. □

Corollary 3.5. *The correspondence*

$$P \longmapsto \overline{P}$$

defines an isometric embedding of the Banach polynomial ideal
$\mathcal{P}_{p,N}^c({}^m X;Y)$ *into the space of continuous m-homogeneous polynomials from* $\ell_{mp}^0 \otimes_\varepsilon X$ *to* $\ell_p^0 \otimes_\pi Y$.

Remark 3.6. The preceding theorem illustrates a general principle. Whenever appropriate sequence classes give rise to tensor norms on the corresponding algebraic tensor products, polynomial ideals can be characterized by the continuity of their associated sequence polynomials and, equivalently, by the continuity of their associated tensor polynomials. The characterization of Cohen p-nuclear homogeneous polynomials obtained in Theorem 3.4 is an instance of this general framework. Similar tensor characterizations can be established for other classes of homogeneous polynomials by choosing suitable sequence classes, tensor norms, and associated sequence spaces.

Conclusion

What this thesis set out to do was to replace a nonlinear summability condition by a topological one of linear type, and that is what Theorem 3.4 does: an m-homogeneous polynomial $P \in \mathcal{P}({}^m X; Y)$ is Cohen p-nuclear precisely when its associated tensor polynomial $\overline{P} : \ell^0_{mp} \otimes_\epsilon X \to \ell^0_p \otimes_\pi Y$ is continuous, and in that case the ideal norm of P, the norm of $\widehat{P}$ and the norm of $\overline{P}$ all agree. That the proof occupies four lines is the point rather than a weakness: once the machinery of Chapter 2 is in place, the characterization follows from it and from Cohen's old theorems, with no new argument at all.

The last remark of Chapter 3 observed that the result is one instance of something more general, and it seems right to close by saying what. Suppose a class of polynomials is defined, as classes in this theory almost always are, by an inequality $\|(P(x_i))_{i\le n}\|_{S_2(Y)} \le C\|(x_i)_{i\le n}\|^m_{S_1(X)}$ for two classes of vector-valued sequences S_1 and S_2. Then everything we have done applies, provided only that S_1 and S_2 are finitely determined, so that $c_{00}(X)$ is a norming subset, and that their norms induce tensor norms α and β on the corresponding algebraic tensor products. Under those two conditions Theorem 2.6 and Corollary 2.7 give at once that membership in the class, the continuity of $\widehat{P}$ and the continuity of $\overline{P}$ are three ways of saying the same thing, and that the norms coincide.

The particular class never enters the argument; it enters only through the choice of the pair (S_1, S_2), after which Cohen's identifications dictate which tensor norms appear. Taking $S_1 = \ell^w_{mp}(\cdot)$ and $S_2 = \ell_p\langle\cdot\rangle$ gives the class studied here, and it is worth noticing that this is the extremal choice: the smallest reasonable cross-norm on the domain, the largest on the range. Other choices give other ideals. Replacing S_2 by $\ell_p(\cdot)$ leads to the p-dominated polynomials [9], and replacing S_1 by $\ell_{mp}(\cdot)$ to the Cohen strongly p-summing ones [6], the projective norm giving way in each case to the Chevet–Saphar norm Δ_p. The three classes therefore differ only in the pair of tensor norms carried by the source and the range.

There is more to do. The most immediate continuation is to carry out the same scheme for the two classes just mentioned, and for the factorable strongly p-nuclear polynomials of [3]. One might also relate $\overline{P}$ to the linearization of P on the symmetric tensor product, or ask how much of the construction survives outside the Banach setting. We hope, at least, that the mechanism behind these characterizations is now clear enough to be reused.

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Abstract

This thesis studies the tensor characterization of Cohen p-nuclear m-homogeneous polynomials between Banach spaces. We first recall the vector-valued sequence spaces, the notion of a finitely determined sequence class, and Cohen's theorems identifying the norms of $\ell_p^w(X)$, $\ell_p\langle X\rangle$ and $\ell_p(X)$ with the injective, projective and Chevet–Saphar tensor norms on $\ell_p \otimes X$. Since the tensor operator $I \otimes T$ has no analogue for an m-homogeneous polynomial when $m \geq 2$, we construct on finitely supported sequences an associated tensor polynomial $\overline{P}$ and an associated sequence polynomial $\widehat{P}$, and we prove that, for finitely determined sequence classes, their continuity is equivalent, with equality of norms. We deduce that P is Cohen p-nuclear if and only if $\overline{P} : \ell_{mp}^0 \otimes_\epsilon X \to \ell_p^0 \otimes_\pi Y$ is continuous, and then $\|P\|_{p,N} = \|\widehat{P}\| = \|\overline{P}\|$. The method is general: other polynomial ideals are recovered by changing the pair of sequence classes.

Keywords: Banach spaces; tensor products; tensor norms; vector-valued sequence spaces; finitely determined sequence classes; homogeneous polynomials; Cohen p-nuclear polynomials; operator ideals.

Résumé

Ce mémoire étudie la caractérisation tensorielle des polynômes m-homogènes p-nucléaires de Cohen entre espaces de Banach. Nous rappelons les espaces de suites à valeurs vectorielles, la notion de classe de suites finiment déterminée, et les théorèmes de Cohen identifiant les normes de $\ell_p^w(X)$, $\ell_p\langle X\rangle$ et $\ell_p(X)$ aux normes tensorielles injective, projective et de Chevet–Saphar sur $\ell_p \otimes X$. Comme l'opérateur tensoriel $I \otimes T$ n'admet pas d'analogue pour un polynôme m-homogène lorsque $m \geq 2$, nous construisons sur les suites à support fini un polynôme tensoriel associé $\overline{P}$ et un polynôme de suites associé $\widehat{P}$, et nous démontrons que, pour des classes de suites finiment déterminées, leur continuité est équivalente, avec égalité des normes. Nous en déduisons que P est p-nucléaire de Cohen si et seulement si $\overline{P} : \ell_{mp}^0 \otimes_\epsilon X \to \ell_p^0 \otimes_\pi Y$ est continu, et alors $\|P\|_{p,N} = \|\widehat{P}\| = \|\overline{P}\|$. La méthode est générale : on retrouve d'autres idéaux polynomiaux en changeant le couple de classes de suites.

Mots-clés : Espaces de Banach ; produits tensoriels ; normes tensorielles ; espaces de suites à valeurs vectorielles ; classes de suites finiment déterminées ; polynômes homogènes ; polynômes p-nucléaires de Cohen ; idéaux d'opérateurs.

الملخص

تدرس هذه المذكرة التوصيف التنسوري لكثيرات الحدود المتجانسة من الدرجة m النووية لكوهين من النمط p بين فضاءات باناخ. نستعرض أولاً فضاءات المتتاليات ذات القيم الشعاعية ومفهوم صنف المتتاليات المُعيِّن انتهائياً، ثم نبرهن مبرهنات كوهين التي تُطابق معايير الفضاءات $\ell_p^w(X)$ و $\ell_p\langle X\rangle$ و $\ell_p(X)$ مع المعايير التنسورية الحاقنة والجدائية ومعيار شوفيه--سافار على $\ell_p \otimes X$. ولمّا كان المؤثر التنسوري $I \otimes T$ لا يقبل نظيراً في حالة كثير حدود متجانس من الدرجة $m \geq 2$، نبني اعتماداً على المتتاليات ذات السند المنتهي كثير حدود تنسورياً مرافقاً $\overline{P}$ وكثير حدود متتالياتٍ مرافقاً $\widehat{P}$، ونبرهن تكافؤ استمرارهما مع تساوي المعيارين. ونستنتج أن P نووي لكوهين من النمط p إذا وفقط إذا كان $\overline{P} : \ell_{mp}^0 \otimes_\epsilon X \to \ell_p^0 \otimes_\pi Y$ مستمراً، وعندئذٍ $\|P\|_{p,N} = \|\widehat{P}\| = \|\overline{P}\|$. والطريقة عامة، إذ نسترجع مثاليات أخرى بتغيير ثنائية أصناف المتتاليات.

الكلمات المفتاحية: فضاءات باناخ؛ الجداءات التنسورية؛ المعايير التنسورية؛ فضاءات المتتاليات ذات القيم الشعاعية؛ أصناف المتتاليات المعيِّنة انتهائياً؛ كثيرات الحدود المتجانسة؛ كثيرات الحدود النووية لكوهين؛ مثاليات المؤثرات.