PEOPLE's DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Computer Sciences
Department of Mathematics

Master memory

Field : Mathematics and Computer Sciences
Branch : Mathematics
Option : Functional Analysis

Theme

Locally uniformly of some functional spaces

Presented by : Rahmani Abdeldjalal

Publicly supprted on: 07/06/2026

In front of the jury composed of :

Douadi Drihem, Prof.	Mohamed Boudiaf University of M'sila	**President**
Madani Moussai, Prof.	Mohamed Boudiaf University of M'sila	**Supervisor**
Aissa Lakhal, MCB.	Mohamed Boudiaf University of M'sila	**Examiner**

University year 2025/2026

Acknowledgements

First and foremost, I would like to thank Almighty God for granting us the strength, determination, and patience to complete this work.

I would like to express my sincere gratitude to our supervisor, Prof. Madani Moussai, for all the support, guidance, and means he provided us to prepare this dissertation. I am also deeply grateful to all the members of the jury for accepting to preside and examine our work, Prof. Douadi Drihem and Prof. Aissa Lakhal .

Finally, a big thank you to everyone who contributed, whether directly or indirectly, in helping us achieve and finalize this project.

Dedication

...To my parents, my wife, Israa, my brothers and sisters.

Table of Contents

Contents

Introduction

The study of function spaces and their localization is a central field in modern mathematical analysis, where the concept of "localization" aims to understand the local behavior of functions, which is crucial for solving complex problems in functional analysis and differential equations. These foundational spaces include Lebesgue spaces $L_p(\mathbb{R}^n)$, which laid the groundwork for integration, and Sobolev spaces $W_p^m(\mathbb{R}^n)$, which provided the framework for studying weak solutions to differential equations, in addition to Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$, which emerged to meet the need for more precise classifications with the development of harmonic analysis.

The historical roots of this field date back to the early 20th century with the work of Henri Lebesgue, followed in the 1930s and 1940s by the contributions of Sergei Sobolev, which brought about a paradigm shift in the study of differential equations. By the 1970s, research focus shifted toward a deeper understanding of the local properties of functions, leading to the development of Besov spaces (named after Oleg Besov) and Triebel-Lizorkin spaces (named after Hans Triebel and Pyotr Lizorkin). This historical evolution has contributed to building a comprehensive mathematical system that today allows for the high-precision treatment of complex mathematical and physical challenges.

In this work, we investigate several advanced functional spaces and present illustrative examples of their applications. Our work is based on some references, in particular [2], [4], [6], [9], [10], [11].

The dissertation is structured into three main chapters:

Chapter 1: We provide a comprehensive review of the fundamental concepts and preliminary results concerning Lebesgue, Sobolev, Triebel-Lizorkin, and Besov spaces.

Chapter 2: We introduce and defines the local homogeneous versions of the aforementioned functional spaces. Furthermore, we establish and prove several key properties that clarify the relationships between these spaces and their general definitions.

Chapter 3: This chapter is dedicated to studying specific perturbations, properties, and direct applications related to these functional spaces.

Finally, we note that the following:

This work is in the sense to understand the locally uniformly function spaces theory, in the goal of hoping, in the future, to continue with scientific research in this file.

Notations

$\mathcal{D}(\mathbb{R}^n)$ Space of infinitely differentiable functions with compact support (test functions).

$\mathcal{D}'(\mathbb{R}^n)$ Space of distributions (generalized functions).

$\mathcal{S}'(\mathbb{R}^n)$ Space of tempered distributions.

$\mathcal{S}(\mathbb{R}^n)$ Schwartz space of rapidly decreasing functions.

$L_p(\mathbb{R}^n)$ Lebesgue space of p-integrable functions.

$W_p^m(\mathbb{R}^n)$ Sobolev space of order m based on L_p.

$F_{p,q}^s(\mathbb{R}^n)$ Triebel-Lizorkin space of smoothness s and parameters p, q.

$B_{p,q}^s(\mathbb{R}^n)$ Besov space of smoothness s and parameters p, q.

E_{loc} Local version of the function space E.

E_{lu} Local uniform version of the function space E.

$|\alpha|$ Length of the multi-index α, defined by $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $\alpha_i \in \mathbb{N}_0$ for $i = 1, \ldots, n$.

$B(x,r)$ Open ball centered at x with radius r, defined as $B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$.

D^α Differential operator corresponding to the multi-index α, defined by $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$.

If $f \in \mathcal{S}(\mathbb{R}^n)$, then its Fourier transform is defined as:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx.$$

Chapter 1

Preliminaries

1.1 Taylor's Formula

Let $f \in C^k(\mathbb{R}^n)$, we recall Taylor's formula with integral remainder, that is :

$$f(x) = \sum_{|\alpha| \leq k} (x-a)^\alpha \frac{\partial^\alpha f(a)}{\alpha!} + \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(x-a)^\alpha \int_0^1 (1-t)^k \partial^\alpha f(a + t(x-a))\, dt.$$

Remark 1.1.1 This formula plays an important role in the proof of Nikol'skii estimates. Indeed, the polynomial part vanishes when we work on spaces modulo polynomials.

Example 1.1.1 Consider the function f defined by:

$$f(x) = \frac{e^x(2x+3)}{x+1}.$$

Our objective is to compute the Taylor's Formula of f around the center $a = 0$ up to the order $k = 3$ with its explicit integral remainder. By successive differentiation of the function, we obtain the following derivatives along with their respective evaluations at $x = 0$:

$$f(0) = 3.$$

$$f^{(1)}(x) = 2e^x + \frac{xe^x}{(1+x)^2}, \qquad f^{(1)}(0) = 2.$$

$$f^{(2)}(x) = 2e^x + \frac{e^x(1+x^2)}{(1+x)^3}, \qquad f^{(2)}(0) = 3.$$

$$f^{(3)}(x) = 2e^x + \frac{e^x(x^3+3x-2)}{(1+x)^4}, \qquad f^{(3)}(0) = 0.$$

The general Taylor's formula with the integral remainder of order $k = 3$ is given by:

$$f(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{x^4}{3!}\int_0^1 (1-t)^3 f^{(4)}(tx)\,dt.$$

Therefore :

$$f(x) = 3 + 2x + \frac{3}{2}x^2 + \frac{x^4}{6}\int_0^1 \frac{(1-t)^3 e^{tx}\left[2(1+tx)^5 + t^4x^4 + 6t^2x^2 - 8tx + 9\right]}{(1+tx)^5}\,dt.$$

1.2 The Lebesgue spaces

Definition 1.1.1 Let $1 \leq p \leq \infty$. We define $L_p(\mathbb{R}^n)$ the set of all measurable functions f such that $||f||_p < \infty$, where :

$$\|f\|_p = \begin{cases} \left(\int\limits_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}^n} |f(x)| & \text{for } p = \infty. \end{cases}$$

Clearly, the $L_p(\mathbb{R}^n)$ are normed vector spaces and have the following property :

Theorem 1.2.1 For $1 \leq p \leq \infty$, the $L_p(\mathbb{R}^n)$ spaces are complete normed vector spaces, i.e., they are Banach spaces.

Proof See [10].

Theorem 1.2.2 (Hölder's inequality) Let $f \in L_p(\mathbb{R}^n)$ and $g \in L_q(\mathbb{R}^n)$ with $1 \leq p, q \leq \infty$, then:

$$\|fg\|_r \leq \|f\|_p \|g\|_q \quad \text{for} \quad \frac{1}{r} := \frac{1}{p} + \frac{1}{q}.$$

Proof See [7].

Theorem 1.2.3 (Young's inequality) Let $p, q, r \in [1, +\infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, then for any $f \in L_p(\mathbb{R}^n)$ and $g \in L_q(\mathbb{R}^n)$ we have:

$$f * g \in L^r(\mathbb{R}^n) \text{ and } \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof See [7].

Theorem 1.2.4 (Bernstein's inequality) Let $1 \leq p \leq r \leq \infty$, there exists $C > 0$, such that for all $R > 0$, and all $f \in L_p(\mathbb{R}^n)$ with supp $\hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq R\}$, we have:

$$\|f^{(\alpha)}\|_r \leq CR^{n(\frac{1}{p}-\frac{1}{r})+|\alpha|}\|f\|_p, \quad \forall \alpha \in \mathbb{N}^n.$$

Proof See [6].

Theorem 1.2.5 (Minkowski's inequality) Let $p \in [1, +\infty[$, and let f and g be two measurable functions on Ω with values in $[0, +\infty]$, then $f + g \in L_p(\mathbb{R}^n)$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof See [10].

1.3 The Sobolev spaces

Definition 1.3.1 Let $m \in \mathbb{N}$ and $1 \leq p < \infty$. We define $W_p^m(\mathbb{R}^n)$, the Sobolev space, as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the distributional derivatives $f^{(\alpha)}$, $|\alpha| \leq m$, belong to $L_p(\mathbb{R}^n)$.
The $W_p^m(\mathbb{R}^n)$ are equipped with the norm :

$$\|f\|_{W_p^m(\mathbb{R}^n)} := \sum_{|\alpha| \leq m} \|f^{(\alpha)}\|_p.$$

$W_p^m(\mathbb{R}^n)$ is a Banach space, see [8].

Remark 1.3.1 If $p = 2$, we have $W_2^m(\mathbb{R}^n)$ coincides with the usual Sobolev spaces $H^m(\mathbb{R}^n)$.

Remark 1.3.2 We can define $W_p^s(\mathbb{R}^n)$, for s a real number, but here we are with Slobodeckij spaces, see [11] and [12].

Proposition 1.3.1 Let $m \geq 1$ be an integer and $1 \leq p < \infty$, we have:

1. If $\frac{1}{p} - \frac{m}{n} > 0$, then $W_p^m(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n)$ where $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$.

2. If $\frac{1}{p} - \frac{m}{n} = 0$, then $W_p^m(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n) \quad \forall q \in [p, +\infty[$.

3. If $\frac{1}{p} - \frac{m}{n} < 0$, then $W_p^m(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$.

Proof See [5].

Theorem 1.3.1 For each $k \in \mathbb{N}_0$ and $1 < p \leq \infty$, the space $W_p^m(\mathbb{R}^n)$ is a Banach space.

Proof See [7] .

Proposition 1.3.2 Let $p_1 \leq p_2$ and m_1, m_2 be two integers such that :

$$m_1 - \frac{n}{p_1} \geq m_2 - \frac{n}{p_2},$$

then it holds

$$W_{p_1}^{m_1}(\mathbb{R}^n) \hookrightarrow W_{p_2}^{m_2}(\mathbb{R}^n).$$

Proof See [7] .

In connection with the space of bounded continuous functions on $\mathbb{R}^n$, denoted by $C_b(\mathbb{R}^n)$, we have the following statement:

Proposition 1.3.3 If $m > n/p$, or $p = 1$ and $m = n$, it holds that :

$$W_p^m(\mathbb{R}^n) \subset C_b(\mathbb{R}^n).$$

On the contrary, $W_p^m(\mathbb{R}^n)$ is not embedded in $L_\infty(\mathbb{R}^n)$ in case

$$m < n/p \textit{ or } m = n/p \textit{ and } p > 1.$$

Proof See [12].

1.4 Besov spaces

1.4.1 The Littlewood-Paley decomposition

Littlewood-Paley series play an important role in the definition of inhomogeneous and homogeneous Besov spaces, for this, We recall the definition of the this decomposition .

We begin by a cut-off function .

A function ρ is called a *cut-off* function if ρ is a $D(\mathbb{R}^n)$, $0 \leq \rho \leq 1$, radial and even ; and $\rho(\xi) = 1$ if $|\xi| \leq 1$ and $\rho(\xi) = 0$ if $|\xi| \geq \frac{3}{2}$.

We define $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$, $\forall \xi \in \mathbb{R}^n$.
The function $\gamma \in \mathcal{S}(\mathbb{R}^n)$ is such that:

1. $\operatorname{supp} \gamma \subset \{\xi : \frac{1}{2} \leq |\xi| \leq \frac{3}{2}\}$.

2. $\gamma(\xi) \geq 0$ for $\frac{1}{2} \leq |\xi| \leq \frac{3}{2}$.

3. $\gamma(\xi) = 1$ if $\frac{3}{4} \leq |\xi| \leq 1$.

An easy calculation gives finally

$$\rho(\xi) + \sum_{j=1}^{\infty} \gamma(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n, \tag{1.1}$$

and in the second

$$\sum_{j\in\mathbb{Z}} \gamma(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.2}$$

These two partitions of unity present the crucial role in Littlewood-Paley decomposition.

For ρ and γ, we associate a sequence of convolution operators denoted by:

- $Q_j : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ for $j = 1, 2, 3, \ldots.$
  $Q_j f(x) = \mathcal{F}^{-1}(\gamma(2^{-j}\cdot)\mathcal{F}f)(x) = \gamma_j * f(x)$

- $S_k : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ for $k = 0, 1, 2, \ldots.$
  $S_k f(x) = \mathcal{F}^{-1}(\rho(2^{-k}\cdot)\mathcal{F}f)(x) = \rho_k * f(x)$

Where $\gamma_j(x) := 2^{jn}\mathcal{F}^{-1}\gamma(2^j x)$ and ρ is defined similarly. We also put $Q_0 = S_0$.

Definition 1.4.1 (Besov spaces) Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$. The inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that:

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\sum_{j\geq 0} (2^{js}\|Q_j f\|_p)^q \right)^{1/q} < \infty & \text{for } q \neq \infty, \\ \sup_{j\geq 0} 2^{js}\|Q_j f\|_p < \infty & \text{for } q = \infty. \end{cases}$$

Note. Q_0 here corresponds to the operator associated with ρ, i.e. $Q_0 = S_0$.

For the following property, we define:

(i) The Bessel Potential space $H_p^s(\mathbb{R}^n)$ by :

$$\|f\|_{H_p^s} := \|(1-\Delta)^{s/2} f\|_p < +\infty, \tag{1.3}$$

where Δ is the Laplacian operator and $1 < p < \infty$.

(ii) The Hölder space $C^s(\mathbb{R}^n)$ by :

$$\|f\|_{C^s} := \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^s} < +\infty, \tag{1.4}$$

if $s \in]0,1[$, in case $s > 1$ ($s \notin \mathbb{N}$), we define :

$$\|f\|_{C^s} = \sum_{|\alpha| \leq [s]} \|\partial^\alpha f\|_{C^{s-[s]}}. \tag{1.5}$$

Proposition 1.4.1 The following properties hold:

- $(B^s_{p,q}(\mathbb{R}^n), \|\cdot\|_{B^s_{p,q}(\mathbb{R}^n)})$ is a Banach space.
- $B^s_{\infty,\infty}(\mathbb{R}^n) = C^s(\mathbb{R}^n)$ if $s \in \mathbb{R}^+ \setminus \mathbb{N}$.

Proof See [11].

We have a several embeddings associated with Besov space .We will recall some of them .

Proposition 1.4.2 Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$; then:

- $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.
- $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n)$ withe $s > \frac{n}{p} - \frac{n}{r}$.

Proof See [12].

Proposition 1.4.3 We have :

1. If $s, t \in \mathbb{R}$, $s > t$ and $1 \leq p, q \leq \infty$, then: $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^t_{p,q}(\mathbb{R}^n)$.
2. If $s \in \mathbb{R}$, and $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q \leq \infty$,$s - \frac{n}{p_1} = t - \frac{n}{p_2}$ then: $B^s_{p_1,q}(\mathbb{R}^n) \hookrightarrow B^t_{p_2,q}(\mathbb{R}^n)$.
3. If $s_1, s_2 \in \mathbb{R}$, $s_1 < s_2$, $1 \leq q_1 \leq q_2 \leq \infty$, $1 \leq p_1 \leq p_2 \leq \infty$,if $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$ then: $B^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$.

4. If $s \in \mathbb{R}$ and $1 \leq p, r, q \leq \infty$, and $s > \frac{n}{p} - \frac{n}{r}$ or $s = \frac{n}{p} - \frac{n}{r}$, $q = 1$ then: $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n)$.

Proof

1. Let $f \in B^s_{p,q}(\mathbb{R}^n)$, then: $\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j\geq 0} 2^{jsq}\|Q_j f\|_p^q\right)^{\frac{1}{q}} < \infty$.

   We have $s > t$, then $2^{tj} < 2^{sj}$, therefore :

$$\left(\sum_{j\geq 0} 2^{jtq}\|Q_j f\|_p^q\right)^{\frac{1}{q}} \leq \left(\sum_{j\geq 0} 2^{jsq}\|Q_j f\|_p^q\right)^{\frac{1}{q}},$$

   hence :

$$\|f\|_{B^t_{p,q}(\mathbb{R}^n)} \leq \|f\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty,$$

   which gives $f \in B^t_{p,q}(\mathbb{R}^n)$.

2. By Bernstein's inequality, we have:

$$\|Q_j f\|_{p_2} \leq c \cdot 2^{jn\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}\|Q_j f\|_{p_1}$$

   Multiplying both sides of the inequality by 2^{jt}:

$$2^{jt}\|Q_j f\|_{p_2} \leq c \cdot 2^{j\left(t + \frac{n}{p_1} - \frac{n}{p_2}\right)}\|Q_j f\|_{p_1}$$

   Given the condition $s = t + \frac{n}{p_1} - \frac{n}{p_2}$, we get :

$$2^{jt}\|Q_j f\|_{p_2} \leq c \cdot 2^{js}\|Q_j f\|_{p_1}$$

   Hence :

$$\left(\sum_{j\geq 0} \left(2^{jt}\|Q_j f\|_{p_2}\right)^q\right)^{\frac{1}{q}} \leq c\left(\sum_{j\geq 0} \left(2^{js}\|Q_j f\|_{p_1}\right)^q\right)^{\frac{1}{q}}$$

   Consequently :

$$\|f\|_{B^t_{p_2,q}(\mathbb{R}^n)} \leq c\|f\|_{B^s_{p_1,q}(\mathbb{R}^n)}$$

3. By Bernstein's inequality, for each frequency block $Q_j f$, we have:

$$\|Q_j f\|_{p_2} \leq c \cdot 2^{jn\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|Q_j f\|_{p_1}$$

Multiplying both sides by 2^{js_2} and incorporating the embedding condition $s_1 = s_2 + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, we simplify the expression to:

$$2^{js_2}\|Q_j f\|_{p_2} \leq c \cdot 2^{js_1}\|Q_j f\|_{p_1}$$

by using Hölder's inequality:

$$\left(\sum_{j\geq 0} \left(2^{js_2}\|Q_j f\|_{p_2}\right)^{q_2}\right)^{\frac{1}{q_2}} \leq c\left(\sum_{j\geq 0} \left(2^{js_1}\|Q_j f\|_{p_1}\right)^{q_1}\right)^{\frac{1}{q_1}}$$

Consequently :

$$\|f\|_{B^{s_2}_{p_2,q_2}} \leq c\|f\|_{B^{s_1}_{p_1,q_1}}$$

4. We have $L_r(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$,
   for $f \in L_r(\mathbb{R}^n)$ then $f \in \mathcal{S}'(\mathbb{R}^n)$, therefore :

$$f = \sum_{j\geq 0} Q_j f,$$

   we have :

$$\|f\|_r = \left\|\sum_{j\geq 0} Q_j f\right\|_r \leq \sum_{j\geq 0} \|Q_j f\|_r, \quad \text{for } r \geq 1.$$

   By Bernstein's inequality, we obtain :

$$\sum_{j\geq 0} \|Q_j f\|_r \leq c\sum_{j\geq 0} 2^{j\left(\frac{n}{p}-\frac{n}{r}\right)}\|Q_j f\|_p,$$

   after simplification, we have :

$$c\sum_{j\geq 0} 2^{j\left(\frac{n}{p}-\frac{n}{r}\right)}\|Q_j f\|_p \leq c\sum_{j\geq 0} 2^{j\left(\frac{n}{p}-\frac{n}{r}-s\right)}2^{js}\|Q_j f\|_p.$$

For $1 \leq q \leq \infty$, such that $\frac{1}{q} + \frac{1}{q'} = 1$, by using Hölder's inequality:

$$c \sum_{j \geq 0} 2^{j\left(\frac{n}{p} - \frac{n}{r} - s\right)} 2^{js} \|Q_j f\|_p \leq c \left(\sum_{j \geq 0} 2^{j\left(\frac{n}{p} - \frac{n}{r} - s\right)q'} \right)^{\frac{1}{q'}} \left(\sum_{j \geq 0} 2^{jsq} \|Q_j f\|_p^q \right)^{\frac{1}{q}}.$$

Consequently :

$$\|f\|_r \leq c \left(\sum_{j \geq 0} 2^{j\left(\frac{n}{p} - \frac{n}{r} - s\right)q'} \right)^{\frac{1}{q'}} \|f\|_{B^s_{p,q}(\mathbb{R}^n)},$$

since $c \left(\sum_{j \geq 0} 2^{j\left(\frac{n}{p} - \frac{n}{r} - s\right)q'} \right)^{\frac{1}{q'}}$ converges if:

- $\frac{n}{p} - \frac{n}{r} - s < 0$ implies $\frac{n}{p} - \frac{n}{r} < s$ therefore: $\|f\|_r \leq c\|f\|_{B^s_{p,q}(\mathbb{R}^n)}$,
  or
- $\frac{n}{p} - \frac{n}{r} - s = 0$ implies $\frac{n}{p} - \frac{n}{r} = s$ *and* q=1 therefore: $\|f\|_r \leq c\|f\|_{B^s_{p,q}(\mathbb{R}^n)}$.

Now we define the homogeneous Besov spaces on $\mathbb{R}^n$. We do not go into details, but we refer to the following references: [6], [11], [12], [13].

Definition 1.4.2 (Homogeneous Besov spaces) Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$. The homogeneous Besov space, denoted by $\dot{B}^s_{p,q}(\mathbb{R}^n)$, is the set of functions all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that:

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|Q_j f\|_p^q \right)^{\frac{1}{q}} < \infty & \text{for } q \neq \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|Q_j f\|_p < \infty & \text{for } q = \infty. \end{cases}$$

Proposition 1.4.4 $(\dot{B}^s_{p,q}(\mathbb{R}^n), \|\cdot\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)})$ is a Banach space.

Proof See [12].

Proposition 1.4.5 Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, then:

$$\mathcal{S}_\infty(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^n).$$

Proof See [4] .

Proposition 1.4.6 We have :

i. Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $q_1 \leq q_2 \leq \infty$, then: $\dot{B}^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p,q_2}(\mathbb{R}^n)$

ii. Let $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$ and $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q \leq \infty$ and $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$ then: $\dot{B}^{s_2}_{p_1,q}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s_1}_{p_2,q}(\mathbb{R}^n)$.

Proof See [11] .

Theorem 1.4.1 (Nikol'skiĭ) Let $0 < a < b$ and $(u_j)_{j\in\mathbb{Z}} \subset \mathcal{S}'_\infty$ such that:

- supp $\hat{u}_j \subset \{\xi : a2^j \leq |\xi| \leq b2^j\}$.
- $\left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|u_j\|_p^q\right)^{\frac{1}{q}} < \infty$.

Then $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'_\infty$ and

$$\|\sum_{j\in\mathbb{Z}} u_j\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \leq C \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|u_j\|_p^q\right)^{\frac{1}{q}} .$$

Proof See [4] .

The relation between inhomogeneous and homogeneous Besov spaces is given by the following assertion:

Proposition 1.4.7 : Let $s > 0$ and $1 \leq p, q \leq \infty$. For all $f \in B^s_{p,q}(\mathbb{R}^n)$ one has $f \in L_p(\mathbb{R}^n)$ and $[f]_p \in \dot{B}^s_{p,q}(\mathbb{R}^n)$. We write :

$$B^s_{p,q}(\mathbb{R}^n) = L_p(\mathbb{R}^n) \cap \dot{B}^s_{p,q}(\mathbb{R}^n).$$

Proof See [13].

1.5 Triebel-Lizorkin spaces

Definition 1.5.1 (Triebel-Lizorkin spaces) Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. The Triebel-Lizorkin space is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that:

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left\| \left(\sum_{j\geq 0} 2^{sjq} |\varphi_j f|^q \right)^{\frac{1}{q}} \right\|_p < \infty & \text{for } q \neq \infty, \\ \| \sup_{j\in\mathbb{N}} (2^{sj} |\varphi_j f|) \|_p < \infty & \text{for } q = \infty. \end{cases}$$

Proposition 1.5.1 Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ then:

- $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$
- $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n)$ withe $s \geq \frac{n}{p} - \frac{n}{r}$

Proof See [12] and [13].

Proposition 1.5.2 The following properties hold:

- $(F^s_{p,q}(\mathbb{R}^n), \|\cdot\|_{F^s_{p,q}(\mathbb{R}^n)})$ is a Banach space if $s \in \mathbb{R}$ and $1 \leq p, q < \infty$.
- $F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$ if $s \in \mathbb{R}$ and $1 < p < \infty$.

Proof See [12] .

Definition 1.5.2 (Homogeneous Triebel-Lizorkin spaces) Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. The homogeneous Triebel–Lizorkin space, denoted by $\dot{F}^s_{p,q}(\mathbb{R}^n)$, is the set of functions all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that:

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} := \begin{cases} \left\| \left(\sum_{j\in\mathbb{Z}} 2^{jsq} |Q_j f|^q \right)^{\frac{1}{q}} \right\|_p < \infty & \text{for } q \neq \infty \\ \left\| \sup_{j\in\mathbb{Z}} (2^{js} |Q_j f|) \right\|_p < \infty & \text{for } q = \infty. \end{cases}$$

Proposition 1.5.3 Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, then:

$$\mathcal{S}_\infty(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^n).$$

Proof See [12] .

Proposition 1.5.4 We have :

i. Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $q_1 \leq q_2 \leq \infty$, then: $\dot{F}^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p,q_2}(\mathbb{R}^n)$.

ii. Let $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$ and $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q \leq \infty$ and $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$ then: $\dot{F}^{s_2}_{p_1,q}(\mathbb{R}^n) \hookrightarrow \dot{F}^{s_1}_{p_2,q}(\mathbb{R}^n)$.

Proof See [12] and [13].

Chapter 2

Locally uniform spaces

2.1 The local uniform space

To define the local uniform space, we first introduce the following notion :

Definition 2.1.1 Let E be a subspace of $\mathcal{D}'(\mathbb{R}^n)$. We say that E is a Banach distributions space if it is embedded into $\mathcal{D}'(\mathbb{R}^n)$ continuously. We use the abbreviation E.B.D.

Definition 2.1.2 (The Multiplier space) Let E be a Banach space of functions (or distributions) defined on $\mathbb{R}^n$. The multiplier space $M(E)$ is defined as follows :

$$M(E) = \{g : g \cdot f \in E, \quad \forall f \in E\}.$$

The norm on this space, denoted by : $\|g\|_{M(E)}$, is typically the operator norm defined by

$$\|g\|_{M(E)} = \sup\left\{\|g \cdot f\|_E : \forall f \in E, \quad \|f\|_E \leq 1\right\}.$$

Proposition 2.1.1 The space $M(E)$ satisfies the following properties:

i. $(M(E), \|\cdot\|_{M(E)})$ is a Banach space.

ii. If $\mathcal{D}(\mathbb{R}^n) \hookrightarrow M(E)$, then $M(E)$ is an algebra.

Proof See [6].

Example of such spaces. We consider $E = L_p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, then $M(L_p(\mathbb{R}^n)) = L_\infty(\mathbb{R}^n)$.

2.2 Uniform localization

A Banach space of distributions ($E.B.D$) is said to be isometrically translation-invariant if :

$$\tau_a f \in E \quad \text{and} \quad \|\tau_a f\|_E = \|f\|_E, \quad \forall f \in E, \quad \forall a \in \mathbb{R}^n.$$

Furthermore, if E is a $\mathcal{D}(\mathbb{R}^n)$-module, the following property holds :

$$\|\tau_a \psi\|_{M(E)} = \|\psi\|_{M(E)}, \quad \forall \psi \in \mathcal{D}(\mathbb{R}^n), \quad \forall a \in \mathbb{R}^n.$$

If any function space E satisfies the above conditions, we say that:
f belongs to E locally uniformly if :

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^n) \quad \text{and} \quad \forall a \in \mathbb{R}^n, \quad \varphi(\cdot - a) f \in E.$$

E endowed with the norm :

$$\|f\|_{E_{lu}} := \sup_{a \in \mathbb{R}^n} \|\varphi(\cdot - a) f\|_E,$$

is a Banach space .

Definition 2.2.1 We say that a function (or distribution) f belongs to E_{lu} (locally uniformly function in space (E) if the following two conditions are satisfied :

(i) There exists a positive function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, not identically zero, such that :

$$\|f\|_{E_{lu}} = \sup_{a \in \mathbb{R}^n} \|(\tau_a \varphi) f\|_E < +\infty.$$

(ii) For every function $\psi \in \mathcal{D}(\mathbb{R}^n)$, we have $(\tau_a \psi) f \in E$ for all $a \in \mathbb{R}^n$ and :

$$\sup_{a \in \mathbb{R}^n} \|(\tau_a \psi) f\|_E < +\infty.$$

In the Definition 2.1.2, the two conditions (i) and (ii) are equivalent. For the proof of this assertion, see [6]. In our work, we take these properties as a definition.

For the following statement we need to define the algebra property of any function space. To do this, any function space E is said "has the algebra property" if for any $g_1 \in E$ and $g_2 \in E$, the product $g_1 g_2$ belongs to E. An easy example, that $E = L_\infty(\mathbb{R}^n)$, we have :

$$g_1, g_2 \in L_\infty(\mathbb{R}^n) \Rightarrow g_1 g_2 \in L_\infty(\mathbb{R}^n).$$

Proposition 2.1.1 If E is an algebra spaces, then:

$$M(E) = E_{lu}.$$

proof

($\subset$) Let $g \in M(E)$. For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have :

$$\begin{aligned} \|(\tau_a \varphi) \cdot g\|_E &\leq \|g\|_{M(E)} \|\tau_a \varphi\|_E \\ &= \|g\|_{M(E)} \|\varphi\|_E \\ &= C_\varphi. \end{aligned}$$

Where C_φ is a constant independent of a. Hence, $g \in E_{lu}$.

($\supset$) Let $f \in E_{lu}$ and let $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ be such that $\varphi\psi = \varphi$. Then:

$$\begin{aligned} \|f\varphi\|_E &= \|(f\psi)\varphi\|_E \\ &\leq \|f\psi\|_E \|\varphi\|_E. \end{aligned}$$

Since $f\psi = f\tau_0\psi$ and E is an algebra, moreover:

$$\|f\psi\|_E = \|f\tau_0\psi\|_E \leq C_\psi \quad \text{(by definition)}.$$

Hence, we obtain:

$$\|f\varphi\|_E \leq C\|\varphi\|_E.$$

Consequently, $f \in M(E)$.

2.3 The local uniform Lebesgue space $L_{p,lu}(\mathbb{R}^n)$

Definition 2.3.1 We say that a function (or distribution) f belongs to the space $L_{p,lu}(\mathbb{R}^n)$ if it satisfies the following condition:

$$\sup_{a\in\mathbb{R}^n} \|(\tau_a\varphi)\cdot f\|_p < \infty, \quad \forall\varphi\in\mathcal{D}(\mathbb{R}^n).$$

Proposition 2.3.1 Alternatively, we can say that a function (or distribution) f belongs to $L_{p,lu}(\mathbb{R}^n)$ if it satisfies:

$$\|f\|_{L_{p,lu}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n}\left(\int_{B(a,1)} |f(x)|^p\,dx\right)^{1/p} < \infty.$$

Proof

($\Longleftarrow$) Suppose . By choosing the function $\varphi_0\in\mathcal{D}(\mathbb{R}^n)$ with support in the ball B, we then have :

$$\begin{aligned}
\|(\tau_a\varphi_0)f\|_p &= \left(\int_{\mathbb{R}^n} |\varphi_0(x-a)|^p|f(x)|^p\,dx\right)^{\frac{1}{p}}\\
&= \left(\int_{B+a} |\varphi_0(x-a)|^p|f(x)|^p\,dx\right)^{\frac{1}{p}}, \text{ since } \operatorname{supp}\varphi_0\subset B\\
&\leq \|\varphi_0\|_\infty\left(\int_{B+a}|f(x)|^p\,dx\right)^{\frac{1}{p}}, \quad \forall a\in\mathbb{R}^n.
\end{aligned}$$

Which implies that :

$$\begin{aligned}
\sup_{a\in\mathbb{R}^n}\|(\tau_a\varphi_0)f\|_p \leq \|\varphi_0\|_\infty \sup_{a\in\mathbb{R}^n}\left(\int_{B+a}|f(x)|^p\,dx\right)^{\frac{1}{p}}\\
< +\infty,
\end{aligned}$$

this shows that $f\in L^p(\mathbb{R}^n)_{lu}$.

($\Longrightarrow$) Conversely, suppose that $f\in L^p(\mathbb{R}^n)_{lu}$ (i.e., $\sup_{a\in\mathbb{R}^n}\|(\tau_a\varphi)f\|_p < +\infty$). By choosing the function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi(x) = 1$ on B, we then obtain :

$$\begin{aligned}
\left(\int_{B+a} |f(x)|^p \, dx\right)^{\frac{1}{p}} &= \left(\int_{B+a} |f(x)\varphi(x-a)|^p \, dx\right)^{\frac{1}{p}}, \text{ since } \varphi(x) = 1 \text{ on } B \\
&\leq \left(\int_{\mathbb{R}^n} |f(x)\varphi(x-a)|^p \, dx\right)^{\frac{1}{p}}, \text{ since } B + a \subset \mathbb{R}^n \\
&\leq \|(\tau_a\varphi)f\|_p, \quad \forall a \in \mathbb{R}^n.
\end{aligned}$$

Which implies that :

$$\sup_{a\in\mathbb{R}^n} \left(\int_{B+a} |f(x)|^p \, dx\right)^{\frac{1}{p}} \leq \sup_{a\in\mathbb{R}^n} \|(\tau_a\varphi)f\|_p < +\infty,$$

Proposition 2.3.2

(1) If $1 \leq q < p < \infty$, then $L_{p,lu}(\mathbb{R}^n) \hookrightarrow L_{q,lu}(\mathbb{R}^n)$.

(2) $L_p(\mathbb{R}^n) \hookrightarrow L_{p,lu}(\mathbb{R}^n) \hookrightarrow L_{p,loc}(\mathbb{R}^n)$.

Proof

1- Assume $L_{p,lu}(\mathbb{R}^n) \hookrightarrow L_{q,lu}(\mathbb{R}^n)$ for $1 \leq q < p < \infty$. We have :

$$\|f\|_{L_{q,lu}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n} \|f\|_{L_q(B(a,1))}.$$

By using Hölder's inequality on the bounded set $B(a, 1)$:

$$\left(\int_{B(a,1)} |f(x)|^q \, dx\right)^{1/q} \leq \left(\int_{B(a,1)} |f(x)|^p \, dx\right)^{1/p} \cdot |B(a,1)|^{\frac{1}{q}-\frac{1}{p}}.$$

Since $|B(a, 1)| = V_n$ (where V_n is a constant representing the volume of the unit ball in $\mathbb{R}^n$), we have :

$$\left(\int_{B(a,1)} |f(x)|^q \, dx\right)^{1/q} \leq C \cdot \left(\int_{B(a,1)} |f(x)|^p \, dx\right)^{1/p}.$$

Where $C = V_n^{\frac{1}{q}-\frac{1}{p}}$. Hence:

$$\sup_{a\in\mathbb{R}^n} \left(\int_{B(a,1)} |f(x)|^q \, dx\right)^{1/q} \leq C \sup_{a\in\mathbb{R}^n} \left(\int_{B(a,1)} |f(x)|^p \, dx\right)^{1/p}.$$

Therefore:

$$\|f\|_{L_{q,lu}(\mathbb{R}^n)} \leq C \cdot \|f\|_{L_{p,lu}(\mathbb{R}^n)}.$$

Which completes the proof of the continuous inclusion $L_{p,lu}(\mathbb{R}^n) \hookrightarrow L_{q,lu}(\mathbb{R}^n)$.

2- $L_p(\mathbb{R}^n) \hookrightarrow L_{p,lu}(\mathbb{R}^n)$

Let $f \in L_p(\mathbb{R}^n)$. For any $a \in \mathbb{R}^n$, we have $B(a,1) \subset \mathbb{R}^n$, therefore :

$$\int_{B(a,1)} |f(x)|^p \, dx \leq \int_{\mathbb{R}^n} |f(x)|^p \, dx.$$

It follows that :

$$\left(\int_{B(a,1)} |f(x)|^p \, dx \right)^{1/p} \leq \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.$$

By taking the supremum over all $a \in \mathbb{R}^n$, we obtain :

$$\sup_{a \in \mathbb{R}^n} \|f\|_{Lp(B(a,1))} \leq \|f\|_{Lp(\mathbb{R}^n)}.$$

Thus :

$$L_p(\mathbb{R}^n) \subset L_{p,lu}(\mathbb{R}^n).$$

3- $L_{p,lu}(\mathbb{R}^n) \subset L_{p,loc}(\mathbb{R}^n)$

Recall that $f \in L_{p,loc}(\mathbb{R}^n)$ if for every compact set $K \subset \mathbb{R}^n$, we have:

$$\left(\int_K |f(x)|^p \, dx \right)^{1/p} < \infty.$$

Since $K \subset \mathbb{R}^n$ is compact, it can be covered by a finite number of balls with radius 1 :

$$K \subset \bigcup_{i=1}^{m} B(y_i, 1).$$

Let $f \in L_{p,lu}(\mathbb{R}^n)$. This implies that for all $a \in \mathbb{R}^n$:

$$\int_{B(a,1)} |f(x)|^p \, dx \leq \|f\|^p_{L_{p,lu}(\mathbb{R}^n)} < \infty.$$

Therefore, we have the following estimation :

$$\begin{aligned}\int_K |f(x)|^p \, dx &\leq \sum_{i=1}^{m} \int_{B(a_i,1)} |f(x)|^p \, dx \\ &\leq \sum_{i=1}^{m} \|f\|^p_{L_{p,lu}(\mathbb{R}^n)} \\ &= m \cdot \|f\|^p_{L_{p,lu}(\mathbb{R}^n)} \\ &< \infty.\end{aligned}$$

From this, it follows that :

$$\int_K |f(x)|^p \, dx < \infty.$$

Thus, we conclude that :

$$f \in L_{p,loc}(\mathbb{R}^n).$$

2.4 The local uniform Sobolev space $W^m_{p,lu}(\mathbb{R}^n)$

Definition 2.4.1 We say that a function (or distribution) f belongs to $W^m_{p,lu}(\mathbb{R}^n)$ if it satisfies :

$$\sup_{a \in \mathbb{R}^n} \|(\tau_a \varphi) \cdot f\|_{W^m_p(\mathbb{R}^n)} < \infty, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Definition 2.4.2 We say that a function (or distribution) f belongs to $W^m_{p,lu}(\mathbb{R}^n)$ if it satisfies :

$$\|f\|_{W^m_{p,lu}(\mathbb{R}^n)} := \sup_{a \in \mathbb{R}^n} \left(\sum_{|\alpha| \leq m} \int_{B(a,1)} |D^{(\alpha)} f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Proposition 2.4.1 we have :

(1) $W^m_{q,lu}(\mathbb{R}^n) \hookrightarrow W^m_{p,lu}(\mathbb{R}^n) \quad (1 \leq p < q < \infty)$

(2) $W^m_p(\mathbb{R}^n) \hookrightarrow W^m_{p,lu}(\mathbb{R}^n) \hookrightarrow W^m_{p,loc}(\mathbb{R}^n)$

(3) $W^k_{p,lu}(\mathbb{R}^n) \hookrightarrow W^m_{p,lu}(\mathbb{R}^n) \quad (\text{if } k \geq m)$

Proof

1- $W^m_{q,lu}(\mathbb{R}^n) \hookrightarrow W^m_{p,lu}(\mathbb{R}^n)$.

Let $f \in W^m_{q,lu}(\mathbb{R}^n)$, since $D^{(\alpha)}f$ is defined for every $|\alpha| \leq k$:

$$\int_{B(a,1)} |D^{(\alpha)}f(x)|^p \cdot 1\, dx.$$

by applying Hölder's inequality with exponents $\frac{q}{p}$ and $\frac{q}{q-p}$:

$$\int_{B(a,1)} |D^{(\alpha)}f(x)|^p\, dx \leq \left(\int_{B(a,1)} (|D^{(\alpha)}f(x)|^p)^{\frac{q}{p}}\, dx\right)^{\frac{p}{q}} \cdot \left(\int_{B(a,1)} 1^{\frac{q}{q-p}}\, dx\right)^{\frac{q-p}{q}}.$$

Hence :

$$\int_{B(a,1)} |D^{(\alpha)}f(x)|^p\, dx \leq \left(\int_{B(a,1)} |D^{(\alpha)}f(x)|^q\, dx\right)^{\frac{p}{q}} \cdot |B(a,1)|^{\frac{q-p}{q}}.$$

Let $c = |B(a,1)|^{\frac{q-p}{q}}$, then :

$$\int_{B(a,1)} |D^{(\alpha)}f(x)|^p\, dx \leq c \cdot \left(\int_{B(a,1)} |D^{(\alpha)}f(x)|^q\, dx\right)^{\frac{p}{q}}.$$

Therefore :

$$\left(\int_{B(a,1)} |D^{(\alpha)}f(x)|^p\, dx\right)^{\frac{1}{p}} \leq c^{\frac{1}{p}} \cdot \left(\int_{B(a,1)} |D^{(\alpha)}f(x)|^q\, dx\right)^{\frac{1}{q}}.$$

By taking the sup over $a \in \mathbb{R}^n$:

$$\sup_{a\in\mathbb{R}^n} \left(\int_{B(a,1)} |D^{(\alpha)}f(x)|^p\, dx\right)^{\frac{1}{p}} \leq c' \cdot \sup_{a\in\mathbb{R}^n} \left(\int_{B(a,1)} |D^{(\alpha)}f(x)|^q\, dx\right)^{\frac{1}{q}}.$$

Therefore : $\|f\|_{W^m_{p,lu}(\mathbb{R}^n)} \leq c' \cdot \|f\|_{W^m_{q,lu}(\mathbb{R}^n)}$.

2-1 $W^m_p(\mathbb{R}^n) \hookrightarrow W^m_{p,lu}(\mathbb{R}^n)$.

Given that $B(a,1) \subset \mathbb{R}^n$ for every $a \in \mathbb{R}^n$ then :

$$\int_{B(a,1)} |D^{(\alpha)}f(x)|^p\, dx \leq \int_{\mathbb{R}^n} |D^{(\alpha)}f(x)|^p\, dx.$$

Hence :

$$\sum_{|\alpha|\le k}\int_{B(a,1)}|D^{(\alpha)}f(x)|^p\,dx \le \sum_{|\alpha|\le k}\int_{\mathbb{R}^n}|D^{(\alpha)}f(x)|^p\,dx.$$

By taking the sup we obtain :

$$\sup_{a\in\mathbb{R}^n}\left(\sum_{|\alpha|\le k}\int_{B(a,1)}|D^{(\alpha)}f(x)|^p\,dx\right)^{\frac{1}{p}} \le \left(\sum_{|\alpha|\le k}\int_{\mathbb{R}^n}|D^{(\alpha)}f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Therefore :

$$\|f\|_{W^m_{p,lu}(\mathbb{R}^n)} \le \|f\|_{W^m_p(\mathbb{R}^n)}.$$

2-2 $W^m_{p,lu}(\mathbb{R}^n) \hookrightarrow W^m_{p,loc}(\mathbb{R}^n)$

We have :

$$f\in W^m_{p,loc}(\mathbb{R}^n) \implies \|f\|_{W^m_p(L)}<\infty.$$

Let L be a compact subset of $\mathbb{R}^n$, since L is compact, it is bounded, thus : $L\subset B(0,R)\subset\bigcup_{i=1}^n B(a_i,1)$.

Let $f\in W^m_{p,lu}(\mathbb{R}^n)$, this implies :

$$\forall a\in\mathbb{R}^n : \sum_{|\alpha|\le k}\int_{B(a,1)}|D^{(\alpha)}f(x)|^p\,dx \le \|f\|^p_{W^m_{p,lu}(\mathbb{R}^n)}.$$

We have :

$$\begin{aligned}\sum_{|\alpha|\le k}\int_L|D^{(\alpha)}f(x)|^p\,dx &\le \sum_{|\alpha|\le k}\int_{\bigcup_{i=1}^n B(a_i,1)}|D^{(\alpha)}f(x)|^p\,dx\\ &\le \sum_{i=1}^n\sum_{|\alpha|\le k}\int_{B(a_i,1)}|D^{(\alpha)}f(x)|^p\,dx\\ &\le n\cdot\|f\|^p_{W^m_{p,lu}(\mathbb{R}^n)}<\infty.\end{aligned}$$

From this, it follows that :

$$\sum_{|\alpha|\le k}\int_L|D^{(\alpha)}f(x)|^p\,dx<\infty.$$

Thus :

$$f \in W^m_{p,loc}(\mathbb{R}^n).$$

3- $W^k_{p,lu}(\mathbb{R}^n) \hookrightarrow W^m_{p,lu}(\mathbb{R}^n) \quad (k \geq m)$.

Since $k \geq m$, then :

$$\sum_{|\alpha|\leq m} \int_{B(a,1)} |D^{(\alpha)} f(x)|^p \, dx \leq \sum_{|\alpha|\leq k} \int_{B(a,1)} |D^{(\alpha)} f(x)|^p \, dx.$$

And thus :

$$\left(\sum_{|\alpha|\leq m} \int_{B(a,1)} |D^{(\alpha)} f(x)|^p \, dx \right)^{\frac{1}{p}} \leq \left(\sum_{|\alpha|\leq k} \int_{B(a,1)} |D^{(\alpha)} f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

By taking the sup of both sides with respect to a :

$$\sup_{a\in\mathbb{R}^n} \left(\sum_{|\alpha|\leq m} \int_{B(a,1)} |D^{(\alpha)} f(x)|^p \, dx \right)^{\frac{1}{p}} \leq \sup_{a\in\mathbb{R}^n} \left(\sum_{|\alpha|\leq k} \int_{B(a,1)} |D^{(\alpha)} f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Therefore :

$$\|f\|_{W^m_{p,lu}(\mathbb{R}^n)} \leq \|f\|_{W^k_{p,lu}(\mathbb{R}^n)}.$$

2.5 The local uniform Besov space $B^s_{p,q,lu}(\mathbb{R}^n)$

Definition 2.5.1 We say that a distribution f belongs to $B^s_{p,q,lu}(\mathbb{R}^n)$ if it satisfies :

$$\sup_{a\in\mathbb{R}^n} \|(\tau_a\varphi)f\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Definition 2.5.2 We say that a distribution f belongs to $B^s_{p,q,lu}(\mathbb{R}^n)$ if it satisfies :

$$\|f\|_{B^s_{p,q,lu}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n} \|f\|_{B^s_{p,q}(B(a,1))} < \infty.$$

Proposition 2.5.1

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,q,lu}(\mathbb{R}^n).$$

Proof

Let $f \in B^s_{p,q}(\mathbb{R}^n)$, we aim to prove that:

$$\sup_{a\in\mathbb{R}^n} \|(\tau_a\varphi)f\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty \quad \forall\varphi \in \mathcal{S}(\mathbb{R}^n).$$

To achieve this, it is sufficient to find a constant $c > 0$ such that:

$$\forall a \in \mathbb{R}^n \quad \|(\tau_a\varphi)f\|_{B^s_{p,q}(\mathbb{R}^n)} \leq c\|f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Given that $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and we know that $\mathcal{S}(\mathbb{R}^n) \subset M(B^s_{p,q})$ it follows that :

$$\|(\tau_a\varphi)f\|_{B^s_{p,q}(\mathbb{R}^n)} \leq \|\tau_a\varphi\|_{M(B^s_{p,q})} \cdot \|f\|_{B^s_{p,q}(\mathbb{R}^n)}$$

From the properties of Besov spaces, we know that the translation operator $\tau_a : f(\cdot) \mapsto f(\cdot - a)$ is an since we have :

$$\|\tau_a\varphi\|_{M(B^s_{p,q})} = \|\varphi\|_{M(B^s_{p,q})} = C.$$

Then :

$$\|(\tau_a\varphi)f\|_{B^s_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Taking the sup over $a \in \mathbb{R}^n$:

$$\sup_{a\in\mathbb{R}^n} \|(\tau_a\varphi)f\|_{B^s_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Therefore :

$$\|f\|_{B^s_{p,q,lu}(\mathbb{R}^n)} \leq C\|f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Which concludes that $f \in B^s_{p,q,lu}(\mathbb{R}^n)$.

2.6 The local uniform Triebl-Lizorkin space $F^s_{p,q,lu}(\mathbb{R}^n)$

Definition 2.6.1 We say that a distribution f belongs to $F^s_{p,q,lu}(\mathbb{R}^n)$ if it satisfies:

$$\sup_{a\in\mathbb{R}^n} \|(T_a\varphi)f\|_{F^s_{p,q}(\mathbb{R}^n)} < \infty, \quad \forall\varphi \in D(\mathbb{R}^n).$$

Definition 2.6.2 We say that a distribution f belongs to $F^s_{p,q}(\mathbb{R}^n)$ if it satisfies:

$$\|f\|_{F^s_{p,q,lu}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n} \|f\|_{F^s_{p,q}(B(a,1))} < \infty.$$

Proposition 2.6.1

$$F^s_{p,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,q,lu}(\mathbb{R}^n) \hookrightarrow F^s_{p,q,loc}(\mathbb{R}^n).$$

Proof See [12] and See [13].

Chapter 3

Application on locally uniform spaces

Let E be a functional space on $\mathbb{R}^n$. With real-valued, as Sobolev space $W_p^m(\mathbb{R}^n)$, Besov space $B_{p,q}^s(\mathbb{R}^n)$, Triebel-Lizorkin space $F_{p,q}^s(\mathbb{R}^n)$. In this chapter, we interested to apply some functional calculus on locally uniform space E_{lu}, that is the acting composition of functions in the following sense :

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, we consider the operator (which is not a linear)

$$T_f : g \to f \circ g, \quad \forall g \in E \tag{3.1}$$

If the following assertion

$$T_f(E) \subset E \tag{3.2}$$

Associated to (3.1) is satisfied, we say that f acts by composition on E.

Our application concerns to recall some theorems,given in [6], [11], [12], [13] .
In which the acting function f belongs to E_{lu}, where $E = W_p^m(\mathbb{R}^n)$ or $B_{p,q}^s(\mathbb{R}^n)$ or $F_{p,q}^s(\mathbb{R}^n)$.

Noting that, we will give some examples in it possible.

3.1 A preparation

Let E be a function space on $\mathbb{R}^n$. We will use the following concept:

- E is supercritical if $E \subset L_\infty(\mathbb{R}^n)$.
- E is subcritical if $E \not\subset L_\infty(\mathbb{R}^n)$.

Here are some applications:

- **Sobolev space** $W_p^m(\mathbb{R}^n)$

  $W_p^m(\mathbb{R}^n)$ is *supercritical* if
  $$m > \frac{n}{p} \quad \text{or} \quad m = \frac{n}{p} \text{ with } p = 1.$$
  $W_p^m(\mathbb{R}^n)$ is *subcritical* if
  $$m < \frac{n}{p} \quad \text{or} \quad m = \frac{n}{p} \text{ with } p > 1.$$
  Example $W^m(\mathbb{R}^n) = \{f, f^{(1)}, \ldots, f^{(m)} \in L_2\}$

  (i) if $m > \frac{n}{2}$ then $W_2^m(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$, hence $W_2^m(\mathbb{R}^n)$ is supercritical.

  (ii) if $m < \frac{n}{2}$ then $W_2^m(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$, hence $W^m(\mathbb{R}^n)$ is subcritical.

- **Besov space** $B_{p,q}^s(\mathbb{R}^n)$

  $B_{p,q}^s(\mathbb{R}^n)$ is *supercritical* if
  $$s > \frac{n}{p} \quad \text{or} \quad s = \frac{n}{p} \text{ with } q = 1.$$
  $B_{p,q}^s(\mathbb{R}^n)$ is *subcritical* if
  $$s < \frac{n}{p} \quad \text{or} \quad s = \frac{n}{p} \text{ with } q > 1.$$
  Example $B_{2,2}^m(\mathbb{R}^n) = W_2^m(\mathbb{R}^n) \quad s = m$

  (i) if $m > \frac{n}{2}$ then $B_{2,2}^m(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$, hence $B_{2,2}^m(\mathbb{R}^n)$ is supercritical.

  (ii) if $m < \frac{n}{2}$ then $B_{2,2}^m(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$, hence $B_{2,2}^m(\mathbb{R}^n)$ is subcritical.

- **Triebel-Lizorkin space** $F_{p,q}^s(\mathbb{R}^n)$

  $F_{p,q}^s(\mathbb{R}^n)$ is *supercritical* if
  $$s > \frac{n}{p} \quad \text{or} \quad s = \frac{n}{p} \quad \textit{with} \quad p = 1.$$
  $F_{p,q}^s(\mathbb{R}^n)$is *subcritical* if
  $$s < \frac{n}{p} \quad \textit{or} \quad s = \frac{n}{p} \quad \textit{with} \quad p > 1.$$
  Example

  (i)$F_{p,2}^0(\mathbb{R}^n)$ if $0 < \frac{n}{p}$ then $F_{p,2}^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, hence $F_{p,2}^0(\mathbb{R}^n)$ is subcritical.

  (ii)$F_{2,2}^2(\mathbb{R}^n)$ if $2 > \frac{n}{2}$ then $F_{2,2}^2(\mathbb{R}^n) = W_2^2(\mathbb{R}^n)$, hence $F_{2,2}^2(\mathbb{R}^n)$ is supercritical.

3.2 Theorems on locally uniform spaces

We give some examples related to $L_p(\mathbb{R}^n)$, $W_p^s(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$. All these paragraphs can be found in [2], [4], [9], [10], [11].

3.2.1 Examples related to Lebesgue spaces

Theorem 3.2.1 Suppose $n > m \geq 2$. Then the following conditions are necessary and sufficient for F to operate on $W^m_{n/m}(\mathbb{R}^n)$: $F' \in L_\infty$ and $F^{(m)}$ belongs to $L_{p,lu}(\mathbb{R})$.

Proof See [5]. The prove is based on Taylor's Formula, section 1.1.

Example : See [11] and [3] .

i. Let us take the function :

$$F(t) = \sqrt{1+t^2}.$$

We have :

$$F'(t) = \frac{t}{\sqrt{1+t^2}}.$$

Hence :

$$\|F'\|_{L_\infty(\mathbb{R})} = \sup_{t\in\mathbb{R}} \left|\frac{t}{\sqrt{1+t^2}}\right| = 1 < \infty \implies F' \in L_\infty(\mathbb{R}).$$

Now, For higher-order derivatives, $F^{(m)}(t)$ is a linear combination of terms of the form $\frac{t^k}{(1+t^2)^{j/2}}$. As $t \to \pm\infty$, these fractional terms tend toward zero $(m \geq 2)$, which confirms that they are bounded. Therefore:

$$F^{(m)} \in L_{p,l_u}(\mathbb{R}).$$

Consequently, if $u \in W^m_{n/m}(\mathbb{R}^n)$, then:

$$F(u(x)) \in W^m_{n/m}(\mathbb{R}^n).$$

ii. Let us take the function :

$$F(t) = e^t.$$

We have :

$$F'(t) = e^t.$$

From this, it follows that :

$$\|F'\|_{L_\infty(\mathbb{R})} = \sup_{t\in\mathbb{R}} e^t = +\infty.$$

Hence :

$$F' \notin L_\infty(\mathbb{R}).$$

Therefore, the function does not satisfy the conditions of Theorem 3.2.1 .
Consequently, if $u \in W^m_{n/m}(\mathbb{R}^n)$, then :

$$e^{u(x)} \notin W^m_{n/m}(\mathbb{R}^n).$$

3.2.2 Examples related to Sobolev spaces

Theorem 3.2.2 Let $m = 2, 3, \ldots$, and let $1 < p < \infty$, let W^m_p be subcritical, let $G\colon \mathbb{R} \to \mathbb{R}$ be continuous and let $G(0) = 0$.
If $m = n/p \geq 2$, then $\left(T_G(W^m_p) \subset W^m_p\right)$ holds if and only if $G' \in W^{m-1}_{p,lu}(\mathbb{R})$.

Proof See [11].

Example : See [11] and [3].

i. Let us take the function :

$$G(t) = \sin(t).$$

We have:

- On one hand, the function is continuous on $\mathbb{R}$, and $G(0) = \sin(0) = 0$.
- On the other hand, $G'(t) = \cos(t)$, and since all derivatives of $\cos(t)$ are bounded functions, it follows that for any derivative of order m and for all p, we have:

$$\left|\frac{d^k}{dt^k}\cos(t)\right| \leq 1.$$

Thus, $G' \in W^{m-1}_{p,l_u}(\mathbb{R})$.

Consequently, if $u \in W^m_p(\mathbb{R}^n)$. In this case, the condition is satisfied since $sin(u(x)) \in W^m_p(\mathbb{R}^n)$.

ii. Let us take the function :

$$G(t) = t^m, \quad (m \geq 2).$$

We have :

- $G(0) = 0^m = 0$, and the function is continuous.
- $G'(t) = mt^{m-1}$.
- $G^{(m)}(t) = m!$.

According to the $L_p(\mathbb{R})$-norm of the derivative $G'(t) = mt^{m-1}$ on the interval $[x, x+1]$, when $x \to \infty$:

$$\|G'(t)\|^p_{L_p([x,x+1])} = \int_x^{x+1} |m \cdot t^{m-1}|^p dt = m^p \int_x^{x+1} t^{p(m-1)} dt.$$

Since $m \geq 2$ and $p > 1$, we have $p(m-1) > 0$. Therefore, for a sufficiently large $x > 0$:

$$\int_x^{x+1} t^{p(m-1)} dt \geq 1 \cdot x^{p(m-1)} = x^{p(m-1)}.$$

By taking the supremum over all $x \in \mathbb{R}$ (where $x > 0$) :

$$\|G'\|^p_{W^{m-1}_{p,l_u}(\mathbb{R})} \geq \sup_{x>0} \left(m^p \cdot x^{p(m-1)} \right) = +\infty.$$

Thus :

$$G' \notin W^{m-1}_{p,l_u}(\mathbb{R}).$$

Consequently, if $u \in W^m_p(\mathbb{R}^n)$, then :

$$u(x)^m \notin W^m_p(\mathbb{R}^n).$$

3.2.3 Examples related to Besov spaces and Triebel-Lizorkin spaces

The following two theorems are given in [1]. We will write some examples on these results.

Theorem 3.2.3 If $0 < s < 1$, then $B^s_{p,q,lu}(\mathbb{R}^n)$ is the set of functions f such that:

$$\sup_{a \in \mathbb{R}^n} \left(\int_0^1 \left(t^{-s} \omega_{p,\mathbb{B}+a}(f,t) \right)^q \frac{dt}{t} \right)^{1/q} + \|f\|_{L_{p,lu}(\mathbb{R}^n)} < +\infty. \tag{3.3}$$

Moreover, the expression above is an equivalent norm on $B^s_{p,q,lu}(\mathbb{R}^n)$.

Proof See [2].

Remark 3.2.1 For $s = 1$, see [2].

Example : See [12] and [13].

Consider the constant function $f(x) = C$, where $C \neq 0$. Implies $f \in B^s_{p,q,l_u}(\mathbb{R}^n)$. .

Theorem 3.2.4 If $0 < s < 1$, then $F^s_{p,q,lu}(\mathbb{R}^n)$ is the set of functions f such that :

$$\sup_{a\in\mathbb{R}^n} \left\| \left(\int_0^1 \left(t^{-s-n} \int_{|h|\leq t} |\Delta_h f(\cdot)| \, dh \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p(\mathbb{B}+a)} + \|f\|_{L_{p,lu}(\mathbb{R}^n)} < +\infty. \tag{3.4}$$

Moreover, the expression above is an equivalent norm on $F^s_{p,q,lu}(\mathbb{R}^n)$.

Proof See [2].

Remark 3.2.2 For $s = 1$, see [2].

Example : See [12] and [13] .
Let $n = 1$ and $0 < s < 1$. consider the smooth bump function $\psi \in C_c^\infty(\mathbb{R})$, where $\psi(x)$ is defined as follows:

$$\psi(x) = \begin{cases} \exp\left(-\dfrac{1}{1-x^2}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

We can prove:

$$\psi \in \ F^s_{p,q,lu}(\mathbb{R})$$

Theorem 3.2.5 Let $0 < s < 1$ and $p, q \in [1, +\infty]$. We have :

$$M(B^s_{p,q}) = \left(M(B^s_{p,q})\right)_{l_\infty}.$$

Proof See [2].

Example : See [12] and [1].
Consider the function $g(x) = \frac{1}{1+x^2}$ where $n = 1$, $p = q = 2$, and $0 < s < 1$.
We have :

$$B^s_{22}(\mathbb{R}) = H^s(\mathbb{R}) \quad \text{for } 0 < s < 1.$$

We can prove that $g \in M(H^s)$.

We can also prove that $g \in (M(H^s))_{l_\infty}$.

Conclusion

We have clarified by several examples the relationship between locally uniform spaces and ordinary spaces through a set of proofs.
Furthermore, we have presented concepts of some of the relevant theories related to locally uniform spaces, such as: $B^s_{p,q,lu}(\mathbb{R}^n)$, $F^s_{p,q,lu}(\mathbb{R}^n)$, $W^m_{p,lu}(\mathbb{R}^n)$, and $L_{p,lu}(\mathbb{R}^n)$.

Bibliography

[1] R.A. Adams, J.J. Fournier : Sobolev spaces, Academic Press, 2003.

[2] S.E. Allaoui, G. Bourdaud : Localisation uniforme des espaces de Besov et de Lizorkin-Triebel, Arch. Math , 2017.

[3] J. Appell, P.P. Zabrejko : Superposition Operators in Function Spaces, Cambridge University Press, 1990.

[4] J. Bergh, J. Löfström : Interpolation spaces. Springer, Berlin, 1976.

[5] G. Bourdaud : Le calcul fonctionnel dans les espaces de Sobolev, Invent. Math, 1991, 435446.

[6] G. Bourdaud : Analyse fonctionnelle dans l'espace Euclidien, 2 ème édition, Pub. Math. Univ. Paris 7, 1995.

[7] H. Brezis :Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer Science & Business Media, New York, 2011.

[8] M. Moussai : Realization of homogeneous Besov and Trieble-Lizorkin spaces and an application to pointwise multipliers, Analysis and Application, 2015, 149-183.

[9] M. Moussai : Characterization by approximation of homogeneous Besov and Triebel-Lizorkin type spaces, Archiv der Mathematik, 2024, 59-69.

[10] W. Rudin : Real and Complex Analysis. 3rd edition, McGraw-Hill Book Co, New York, 1987.

[11] T. Runst, W. Sickel : Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. Walter de Gruyter, Berlin, 1996.

[12] H. Triebel : Theory of function spaces, Birkhauser, Basel, 1983.

[13] H. Triebel : Theory of function spaces II, Birkhauser, Basel, 1992.

[14] W. Yuan, W. Sickel, and D. Yang : Morrey and Campanato Meet Besov, Lizorkin-Triebel and Triebel-Lizorkin Spaces, Springer-Verlag, Berlin, 2010.

ملخص

.

في هذه المذكرة درسنا الفضاءات المحلية المنتظمة من نوع $F^s_{p,q,lu}(\mathbb{R}^n)$، $L_{p,lu}(\mathbb{R}^n)$، $B^s_{p,q,lu}(\mathbb{R}^n)$، و $W^m_{p,lu}(\mathbb{R}^n)$، حيث قمنا بوضع تعاريف هذه الفضاءات وذكر أهم الخواص والتضمينات الخاصة بها.
بعد ذلك قمنا باختيار بعض النظريات المتعلقة بهذه الفضاءات وإعطاء أمثلة مع شرحها لتسهيل وفهم محتواها.

كلمات مفتاحية

الفضاءات المحلية المنتظمة، فضاءات سوبوليف، فضاءات بيزوف، فضاءات تريبل--ليزوركين، تجزئة لتلوود بيلي، فضاء الضرب .

Abstract

In this memory, we study local uniform spaces of the type $F^s_{p,q,lu}(\mathbb{R}^n)$, $L_{p,lu}(\mathbb{R}^n)$,$B^s_{p,q,lu}(\mathbb{R}^n)$, and $W^m_{p,lu}(\mathbb{R}^n)$, where we provide definitions for these spaces and present their main properties and embeddings.

Afterward, we select some theorems related to these spaces and provide examples along with their explanations to facilitate and understand their content.

Key words:

Local uniform spaces, Sobolev spaces, Besov spaces, Triebel–Lizorkin spaces, Littlewood–Paley decomposition, Multiplier spaces .

Résumé

Dans ce mémoire, nous étudions les espaces localement uniformément de type $F^s_{p,q,lu}(\mathbb{R}^n)$, $L_{p,lu}(\mathbb{R}^n)$,$B^s_{p,q,lu}(\mathbb{R}^n)$ et $W^m_{p,lu}(\mathbb{R}^n)$, où nous donnons les définitions de ces espaces et nous présentons leurs propriétés principales ainsi que leurs injections.

Ensuite, nous sélectionnons certains théorèmes liés à ces espaces et nous donnons des exemples accompagnés de leurs explications pour faciliter la compréhension de leur contenu.

Mots clés :

Espaces localement uniformément, espaces de Sobolev, espaces de Besov, espaces de Triebel–Lizorkin, décomposition de Littlewood–Paley, espaces de multiplicateurs .