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Thesis

For obtaining the doctorate degree "LMD" in Mathematics

Option: Analyse Numérique, EDPs et Applications

Theme

Problèmes inverses régis par des équations aux dérivées fractionnaires locales

Presented by:

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A. Qaddour

Dedications

I dedicate this thesis:

To my caring mother, who prioritized my happiness and comfort over her own, this achievement is the fruit of her hard work and effort.

To my beloved mother, who sacrificed everything for me. Thanks to her encouragement and financial and moral support, this achievement was made possible.

To my dear mother, whose constant prayers were the secret to every achievement.

To my dear father, who has been my primary support throughout this journey. Thanks to his support and encouragement, this achievement was made possible.

To my beloved wife, the miracle woman who makes everything possible with her patience and moral support.

To the apple of my eye and the light of my eyes, my beloved daughter DJOWEYRIA.

To my beloved sister and my dear brothers.

ملخص: في هذه الأطروحة قمنا بدراسة مسألتين عكسيتين جديدتين للمصدر في الزمن وفي المكان لمعادلة التلغراف الزمني الكسري أحادية البعد باستخدام المشتقات الكسرية التو افقية اليسرى من رتبة $0 < \alpha \leq 1$. إن معادلة التلغراف تمثل النمذجة الرياضية للجهد أو التيار في أسلاك النقل، مع دمج معاملاتها عبر الصيغ التالية $\omega = \frac{1}{LC}$ و $2a = G/C + R/L$ و $b^2 = GR/LC$.

أولا: أثبتنا الوجود والوحدانية والارتباط المستمر للحل مع معطيات المسألة العكسية للمصدر المتغير زمنياً مع شروط حدودية مختلطة وشرط إضافي موضعي وذلك باستخدام طريقة فوريه ومبدأ التقلص لبناخ.

ثانياً: قمنا بإثبات الوجود والوحدانية والارتباط المستمر للحل بالنسبة للمعطيات الخاصة بالمسألة العكسية للمصدر المتغير مكانياً تحت شروط حدودية دورية وشرط إضافي عند الوقت النهائي، اين استخدمنا طريقة فوريه وخصائص المشتقات التو افقية مع أمثلة توضيحية.

أخيراً، اقترحنا خوارزمية فعالة تعتمد على طريقة الفروق المنتهية من أجل الحل التقريبي للمسألة العكسية للمصدر المتعلق بالزمن. تعزز هذه النتائج نظرية المشكلات العكسية للنماذج الز ائدية الكسرية، مع تطبيقاتها في انتشار الإشارات والظواهر الموجية.

كلمات مفتاحية: مسألة عكسية، معادلة التلغراف، طريقة فوريه، الحساب الكسري التو افقي، مبدأ التقلص لبناخ، طريقة الفروق المنتهية.

Abstract

In this thesis, we consider an inverse time-dependent source problem for a time-fractional telegraph equation with mixed boundary conditions and an additional measurement at a fixed point. The fractional derivative is described in the conformable sense. Under some assumptions on the input data, the well-posedness of this inverse problem is shown by using Fourier's method and Banach's contraction mapping principle. An inverse space-dependent source problem for a time-fractional telegraph equation with periodic boundary conditions and an additional condition at final time is considered. The fractional derivative is also in the conformable sense. The well-posedness of this inverse problem is shown under some assumptions on the input data by using Fourier's method. The algorithm using the finite difference method is presented to solve the first inverse problem.

Keywords and phrases: Inverse problem, Telegraph equation, Fourier's method, Conformable fractional calculus, Banach's contraction mapping principle, Finite difference method.

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Introduction

Let $\ell, T > 0$ be some fixed numbers and let Ω_T be a rectangular region defined by:

$$\Omega_T := \{(x,t) : 0 \leq x \leq \ell,\ 0 \leq t \leq T\}.$$

We consider the one-dimensional time-fractional telegraph equation

$$\mathcal{D}_t^{(2\alpha)} u(x,t) + 2a\mathcal{D}_t^{(\alpha)} u(x,t) + b^2 u(x,t) = \omega u_{xx}(x,t) + F(x,t),\ (x,t) \in \Omega_T, \tag{1}$$

where $\mathcal{D}_t^{(\alpha)}$ represent the left-conformable fractional derivative of order $0 < \alpha \leq 1$ with respect to t such that $\mathcal{D}_t^{(2\alpha)} := \mathcal{D}_t^{(\alpha)}\left(\mathcal{D}_t^{(\alpha)}\right)$, $F(x,t)$ is the source term, $a \geq b \geq 0$, $\omega > 0$ and $u(x,t)$ represent the voltage or the current inside a piece of telegraph or transmission wire, whose electrical properties per unit length are: resistance R, inductance L, capacitance C, and conductance of leakage current G where $\omega = \frac{1}{LC}$, $2a = \frac{G}{C} + \frac{R}{L}$, and $b^2 = \frac{GR}{LC}$.

For $\alpha = 1$, equation (1) is the classical telegraph equation developed by Oliver Heaviside in last decades of 19^{th} century [11]. This equation is a second-order linear hyperbolic equation and it models several phenomena in many different fields such as signal analysis [15], wave propagation [24], random walk theory [9].

Suppose the unknown function u satisfies the following initial conditions

$$u(x,0) = \varphi(x),\ \mathcal{D}_t^{(\alpha)} u(x,0) = \psi(x),\ 0 \leq x \leq \ell, \tag{2}$$

and the homogeneous mixed boundary conditions

$$u_x(0,t) = u(\ell,t) = 0,\ 0 \leq t \leq T, \tag{3}$$

where φ and ψ are given functions. If all functions $F(x,t), \varphi(t), \psi(t)$ are given appropriately, the problem (1)-(3) is a direct problem. It should be noted that the direct problem (1)-(3) has been investigated in the works [2,3,20], and the references therein. In this thesis, we will study the following two inverse problems:

- *Inverse time-dependent source problem:*

  When the source term $F(x,t) = r(t) f(x,t)$ with $f(x,t)$ is a given function. The problem of finding the solution pair $\{u(x,t), r(t)\}$ of the problem (1)-(3) with additional measurement condition

  $$u(x_*,t) = h(t), \ 0 \le t \le T, \tag{4}$$

  is called the inverse problem where $x_* \in [\,0,\ell\,[$ is a fixed point and $h(t)$ is a given function.

- *Inverse space-dependent source problem:*

  When the source term $F(x,t) = f(x)$. The problem of finding the solution pair $\{u(x,t), f(x)\}$ of the problem (1)-(2) with periodic boundary conditions

  $$u(0,t) = u(\ell,t), \ u_x(0,t) = u_x(\ell,t), \ 0 \le t \le T, \tag{5}$$

  and additional measurement condition

  $$u(x,T) = h(x), \ 0 \le x \le \ell, \tag{6}$$

  is called the inverse problem where $h(x)$ is a given function.

We note that inverse source problems for fractional diffusion and wave equations were investigated in [5, 12, 13, 22, 23, 25] and inverse coefficient problems for a semilinear time fractional telegraph equation in [18,19]. It should also be noted that all the articles on the inverse problems mentioned used the fractional derivative of the Caputo sense. However, the inverse source problems for fractional telegraph equations have not yet been studied. As far as we know, the well-posedness of the inverse time-dependent source problem (1)-(4) is studied in our article [4] for the first time.

The works [7,17] investigate inverse space-dependent source problems for time-fractional diffusion and heat equations and inverse space-dependent source problem for space-time fractional differential equation is studied in [6].

The rest of this thesis is structured as follows: in Chapter 1, some definitions and properties of the conformable fractional calculus theory and two theorems of stability theory of the finite difference schemes are given.

In Chapter 2, under some natural regularity and consistency conditions on the input data, the well-posedness of the inverse time-dependent source problem (1)-(4) is shown by using eigenfunction expansion of a self-adjoint spectral problem along the Fourier's method and Banach's contraction mapping principle. This chapter is the subject of an article titled "*The well-posedness of an inverse source problem for a time-fractional telegraph equation*" published in an international journal "Palest. J. Math.", see [4].

In Chapter 3, under some assumptions on the input data, the existence, uniqueness and continuous dependence upon the data of the solution of the inverse space-dependent source problem (1), (2), (5), (6) are shown by using the Fourier's method and some properties of the conformable fractional derivative. Finally, we provide two examples of this inverse problem.

In Chapter 4, we proposed an algorithm to find the numerical solution of the inverse time-dependent source problem (1)-(4) by using the finite difference method. Two numerical examples are presented to confirm the effectiveness of this algorithm.

CHAPTER 1

PRELIMINARIES

In this chapter, we give some definitions and properties of the conformable fractional calculus theory and two theorems of stability theory of the finite difference schemes.

1.1 The conformable fractional calculus

In this section, we provide some definitions and properties of the conformable fractional calculus theory.

Definition 1.1 ([16]). Given a function $f : [\,0, \infty\,[\longrightarrow \mathbb{R}$. Then, the conformable fractional derivative of f of order α is defined by

$$\mathcal{D}^{(\alpha)} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in \,]\,0, 1\,]$. If $\mathcal{D}^{(\alpha)} f(t)$ exists in some $]\,0, a\,[$, $a > 0$, and $\lim_{t \to 0^+} \mathcal{D}^{(\alpha)} f(t)$ exists, then define

$$\mathcal{D}^{(\alpha)} f(0) = \lim_{t \to 0^+} \mathcal{D}^{(\alpha)} f(t).$$

If the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

Definition 1.2 ([16]). The conformable fractional integral of a function f starting from $a \geq 0$ of order α is defined by

$$I_{\alpha}^{a}(f)(t) = \int_{a}^{t} \frac{f(x)}{x^{1-\alpha}} dx,$$

where $\alpha \in \,]\,0, 1\,]$.

Theorem 1.1 ([16]). *Let $\alpha \in \,]\,0, 1\,]$ and f, g be α-differentiable at a point $t > 0$. Then*

(1) $\mathcal{D}^{(\alpha)}(af + bg) = a\mathcal{D}^{(\alpha)}(f) + b\mathcal{D}^{(\alpha)}(g)$, *for all $a, b \in \mathbb{R}$.*

(2) *$\mathcal{D}^{(\alpha)}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.*

(3) *$\mathcal{D}^{(\alpha)}(fg) = f\mathcal{D}^{(\alpha)}(g) + g\mathcal{D}^{(\alpha)}(f)$.*

(4) *If f is differentiable, then $\mathcal{D}^{(\alpha)}(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.*

Theorem 1.2 ([16]). *Let $\alpha \in \,]\,0,1\,]$, $a \geq 0$ and $f : [\,a,\infty\,[\, \longrightarrow \mathbb{R}$ be continuous function. Then, for all $t > a$ we have*

$$\mathcal{D}^{(\alpha)} I_\alpha^a(f)(t) = f(t).$$

Definition 1.3 ([1]). Let $0 < \alpha \leq 1$ and $f : [\,0,\infty\,[\, \longrightarrow \mathbb{R}$ be function. Then the **fractional Laplace transform** of order α of f is defined by

$$\mathcal{L}_\alpha\{f(t)\}(s) = \mathcal{F}_\alpha(s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t)\, t^{\alpha-1} dt.$$

Theorem 1.3 ([1]). *Let $0 < \alpha \leq 1$ and $f : \,]\,0,\infty\,[\, \longrightarrow \mathbb{R}$ be differentiable function. Then*

$$\mathcal{L}_\alpha\left\{\mathcal{D}^{(\alpha)} f(t)\right\}(s) = s\mathcal{F}_\alpha(s) - f(0).$$

Property 1.1 ([1, 8, 14]). Let $0 < \alpha \leq 1$, $a, b \in \mathbb{R}$ and $f, g : [\,0,\infty\,[\, \longrightarrow \mathbb{R}$ be a functions such that $\mathcal{L}_\alpha\{f(t)\}(s) = \mathcal{F}_\alpha(s)$ and $\mathcal{L}_\alpha\{g(t)\}(s) = \mathcal{G}_\alpha(s)$ exists. Then

1. The fractional Laplace transform is linear operator:

$$\mathcal{L}_\alpha\{af(t) + bg(t)\}(s) = a\mathcal{F}_\alpha(s) + b\mathcal{G}_\alpha(s). \tag{1.1}$$

2. We have:

$$\mathcal{L}_\alpha\{f(t)\}(s) = \mathcal{L}\left\{f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right\}(s), \tag{1.2}$$

where $\mathcal{L}$ is the usual Laplace transform such that $\mathcal{L}\{g(t)\}(s) = \displaystyle\int_0^\infty e^{-st} g(t)\, dt$.

3. We have $\mathcal{L}_\alpha\left\{e^{-a\frac{t^\alpha}{\alpha}}f(t)\right\}(s)=\mathcal{L}\left\{e^{-at}f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right\}(s)=\mathcal{L}\left\{f\left((\alpha t)^{\frac{1}{\alpha}}\right)\right\}(s+a)$. For example:

$$\mathcal{L}_\alpha\left\{e^{-a\frac{t^\alpha}{\alpha}}\cosh\left(b\frac{t^\alpha}{\alpha}\right)\right\}(s)=\mathcal{L}\left\{e^{-at}\cosh(bt)\right\}(s)=\frac{s+a}{(s+a)^2-b^2}, \tag{1.3}$$

$$\mathcal{L}_\alpha\left\{e^{-a\frac{t^\alpha}{\alpha}}\sinh\left(b\frac{t^\alpha}{\alpha}\right)\right\}(s)=\mathcal{L}\left\{e^{-at}\sinh(bt)\right\}(s)=\frac{b}{(s+a)^2-b^2}. \tag{1.4}$$

4. The derivative of the fractional Laplace transform satisfies:

$$\frac{d\mathcal{L}_\alpha\{f(t)\}(s)}{ds}=-\mathcal{L}_\alpha\left\{\frac{t^\alpha}{\alpha}f(t)\right\}(s). \tag{1.5}$$

5. The fractional Laplace transform of the α-convolution of $f(t)$ and $g(t)$ is:

$$\mathcal{L}_\alpha\{(f*g)(t)\}(s)=\mathcal{F}_\alpha(s)\cdot\mathcal{G}_\alpha(s), \tag{1.6}$$

where $(f*g)(t)=\int_0^t f\left((t^\alpha-\tau^\alpha)^{1/\alpha}\right)g(\tau)\tau^{\alpha-1}d\tau$.

Theorem 1.4. *Let $\eta,\gamma>0$ and $g:[\,0,\infty\,[\longrightarrow\mathbb{R}$ be continuous function. For all $0<\alpha\leq 1$, the following Cauchy problem:*

$$\begin{cases}\mathcal{D}^{(2\alpha)}y(t)+2\eta\mathcal{D}^{(\alpha)}y(t)+\gamma^2y(t)=g(t), \quad 0\leq t,\\ y(0)=y_1,\ \mathcal{D}^{(\alpha)}y(0)=y_2,\end{cases} \tag{1.7}$$

admits a unique solution given in the following three cases:

1. *If $\eta<\gamma$ (by Theorem 2.1 in [2])*

$$\begin{aligned}y(t)&=y_1e^{-\eta\frac{t^\alpha}{\alpha}}\cos\left(\sqrt{\gamma^2-\eta^2}\frac{t^\alpha}{\alpha}\right)+\frac{\eta y_1+y_2}{\sqrt{\gamma^2-\eta^2}}e^{-\eta\frac{t^\alpha}{\alpha}}\sin\left(\sqrt{\gamma^2-\eta^2}\frac{t^\alpha}{\alpha}\right)\\&+\frac{1}{\sqrt{\gamma^2-\eta^2}}\int_0^t g(\tau)e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}}\sin\left(\sqrt{\gamma^2-\eta^2}\frac{t^\alpha-\tau^\alpha}{\alpha}\right)\tau^{\alpha-1}d\tau.\end{aligned} \tag{1.8}$$

2. *If $\eta=\gamma$*

$$y(t)=y_1e^{-\eta\frac{t^\alpha}{\alpha}}+(\eta y_1+y_2)\frac{t^\alpha}{\alpha}e^{-\eta\frac{t^\alpha}{\alpha}}+\int_0^t g(\tau)\frac{t^\alpha-\tau^\alpha}{\alpha}e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}}\tau^{\alpha-1}d\tau. \tag{1.9}$$

3. *If $\eta > \gamma$*

$$\begin{aligned} y(t) &= y_1 e^{-\eta \frac{t^\alpha}{\alpha}} \cosh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha}{\alpha}\right) + \frac{\eta y_1 + y_2}{\sqrt{\eta^2-\gamma^2}} e^{-\eta\frac{t^\alpha}{\alpha}} \sinh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha}{\alpha}\right) \\ &+ \frac{1}{\sqrt{\eta^2-\gamma^2}} \int_0^t g(\tau) e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}} \sinh\left(\sqrt{\eta^2-\gamma^2}\frac{t^\alpha-\tau^\alpha}{\alpha}\right)\tau^{\alpha-1} d\tau. \end{aligned} \tag{1.10}$$

Proof. According to Theorem 1.3 and from (1.1) and (1.7), we get:

$$\begin{aligned} \mathcal{L}_\alpha \{y(t)\}(s) &= \frac{y_1(s+\eta)}{(s+\eta)^2 - (\eta^2-\gamma^2)} \\ &+ \frac{y_1\eta + y_2}{(s+\eta)^2-(\eta^2-\gamma^2)} + \frac{1}{(s+\eta)^2-(\eta^2-\gamma^2)} \cdot \mathcal{L}_\alpha\{g(t)\}(s). \end{aligned} \tag{1.11}$$

1. If $\eta < \gamma$, the proof in [2, page 30].

2. If $\eta = \gamma$, using (1.2) we have

$$\mathcal{L}_\alpha\left\{e^{-\eta\frac{t^\alpha}{\alpha}}\right\}(s) = \mathcal{L}\left\{e^{-\eta t}\right\}(s) = \frac{1}{s+\eta}, \tag{1.12}$$

using (1.5), from (1.12) we obtain:

$$\frac{d\mathcal{L}_\alpha\left\{-e^{-\eta\frac{t^\alpha}{\alpha}}\right\}(s)}{ds} = \mathcal{L}_\alpha\left\{\frac{t^\alpha}{\alpha}e^{-\eta\frac{t^\alpha}{\alpha}}\right\}(s) = \frac{1}{(s+\eta)^2}, \tag{1.13}$$

by using (1.6), from (1.13) we obtain:

$$\mathcal{L}_\alpha\left\{\int_0^t g(\tau)\frac{t^\alpha-\tau^\alpha}{\alpha}e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}}\tau^{\alpha-1}d\tau\right\}(s) = \mathcal{L}_\alpha\left\{\frac{t^\alpha}{\alpha}e^{-\eta\frac{t^\alpha}{\alpha}}\right\}(s)\cdot\mathcal{L}_\alpha\{g(t)\}(s), \tag{1.14}$$

after substituting (1.12)-(1.14) in equation (1.11), we find

$$\mathcal{L}_\alpha\{y(t)\}(s) = \mathcal{L}_\alpha\left\{y_1 e^{-\eta\frac{t^\alpha}{\alpha}} + (\eta y_1 + y_2)\frac{t^\alpha}{\alpha}e^{-\eta\frac{t^\alpha}{\alpha}} + \int_0^t g(\tau)\frac{t^\alpha-\tau^\alpha}{\alpha}e^{-\eta\frac{t^\alpha-\tau^\alpha}{\alpha}}\tau^{\alpha-1}d\tau\right\}(s),$$

hence, by using the inverse fractional Laplace transform, we get (1.9).

3. The last case (if $\eta > \gamma$): using (1.4) and (1.6) with putting $b = \sqrt{\eta^2 - \gamma^2}$ in equation (1.4), we obtain:

$$\begin{aligned} &\mathcal{L}_\alpha \left\{ \int_0^t g(\tau) \frac{e^{-\eta \frac{t^\alpha - \tau^\alpha}{\alpha}}}{\sqrt{\eta^2 - \gamma^2}} \sinh\left(\sqrt{\eta^2 - \gamma^2} \frac{t^\alpha - \tau^\alpha}{\alpha} \right) \tau^{\alpha - 1} d\tau \right\}(s) \\ &\qquad = \mathcal{L}_\alpha \left\{ \frac{e^{-\eta \frac{t^\alpha}{\alpha}}}{\sqrt{\eta^2 - \gamma^2}} \sinh\left(\sqrt{\eta^2 - \gamma^2} \frac{t^\alpha}{\alpha} \right) \right\}(s) \cdot \mathcal{L}_\alpha \{ g(t) \}(s), \end{aligned} \tag{1.15}$$

after rearranging (1.3), (1.4) and (1.15) in equation (1.11), we find

$$\begin{aligned} \mathcal{L}_\alpha \{ y(t) \}(s) = \mathcal{L}_\alpha \Bigg\{ & y_1 e^{-\eta \frac{t^\alpha}{\alpha}} \cosh\left(\sqrt{\eta^2 - \gamma^2} \frac{t^\alpha}{\alpha} \right) + \frac{\eta y_1 + y_2}{\sqrt{\eta^2 - \gamma^2}} e^{-\eta \frac{t^\alpha}{\alpha}} \sinh\left(\sqrt{\eta^2 - \gamma^2} \frac{t^\alpha}{\alpha} \right) \\ & + \frac{1}{\sqrt{\eta^2 - \gamma^2}} \int_0^t g(\tau) e^{-\eta \frac{t^\alpha - \tau^\alpha}{\alpha}} \sinh\left(\sqrt{\eta^2 - \gamma^2} \frac{t^\alpha - \tau^\alpha}{\alpha} \right) \tau^{\alpha - 1} d\tau \Bigg\}(s), \end{aligned}$$

thus, by using the inverse fractional Laplace transform, we get (1.10).

□

Definition 1.4 ([10]). Let $\alpha \in \,]\,0,1\,]$. Define a function space

$$\mathcal{C}^\alpha[0,1] = \left\{ u : u(t) = I_\alpha^0 x(t) + c, c \in \mathbb{R}, x \in \mathcal{C}[0,1] \right\}.$$

Define

$$\|u\|_\alpha = \|u\|_0 + \left\| \mathcal{D}^{(\alpha)} u \right\|_0,$$

where $\|u\|_0 = \max_{t \in [0,1]} |u(t)|$.

Theorem 1.5 ([10]). *$(\mathcal{C}^\alpha[0,1], \|\cdot\|_\alpha)$ is a Banach space.*

Definition 1.5. Let $\alpha \in \,]\,0,1\,]$ and $T > 0$. We define the set of functions as:

$$\mathcal{C}^{2\alpha}[0,T] = \left\{ u : \mathcal{D}^{(\alpha)} u(t) = I_\alpha^0 x(t) + c, c \in \mathbb{R}, x \in \mathcal{C}[0,T] \right\}.$$

Define

$$\|u\|_{\mathcal{C}^{2\alpha}[0,T]} = \|u\|_{\mathcal{C}[0,T]} + \left\| \mathcal{D}^{(\alpha)} u \right\|_{\mathcal{C}[0,T]} + \left\| \mathcal{D}^{(2\alpha)} u \right\|_{\mathcal{C}[0,T]},$$

where $\|u\|_{\mathcal{C}[0,T]} = \max_{t\in[0,T]} |u(t)|$.

Theorem 1.6. $\left(\mathcal{C}^{2\alpha}[0,T], \|\cdot\|_{\mathcal{C}^{2\alpha}[0,T]}\right)$ *is a Banach space.*

Proof. To prove this theorem, we follow the same steps as the proof of Theorem 1.5. It is easy to verify that $\|\cdot\|_{\mathcal{C}^{2\alpha}[0,T]}$ satisfies the norm axioms.

The following proof is the completeness of $\mathcal{C}^{2\alpha}[0,T]$. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{C}^{2\alpha}[0,T]$:

$$\mathcal{D}^{(\alpha)}u_n(t) = I_\alpha^0 x_n(t) + c_n,$$

where $x_n \in \mathcal{C}[0,T]$, and $c_n \in \mathbb{R}$. Then

$$\mathcal{D}^{(\alpha)}u_n(t) - \mathcal{D}^{(\alpha)}u_m(t) = I_\alpha^0(x_n(t) - x_m(t)) + c_n - c_m,$$

by using parts **(1)** and **(2)** of Theorem 1.1 and using Theorem 1.2, we find

$$\mathcal{D}^{(2\alpha)}(u_n(t) - u_m(t)) = x_n(t) - x_m(t).$$

Because $\{u_n\}_{n=1}^{\infty}$ is Cauchy sequence in $\mathcal{C}^{2\alpha}[0,T]$, we have

$$\begin{aligned}\|u_n - u_m\|_{\mathcal{C}^{2\alpha}[0,T]} = \|u_n - u_m\|_{\mathcal{C}[0,T]} &+ \left\|\mathcal{D}^{(\alpha)}(u_n - u_m)\right\|_{\mathcal{C}[0,T]} \\ &+ \left\|\mathcal{D}^{(2\alpha)}(u_n - u_m)\right\|_{\mathcal{C}[0,T]} \xrightarrow[n,m\to+\infty]{} 0.\end{aligned}$$

Thus every term of the above formula converges to 0. By

$$\|u_n - u_m\|_{\mathcal{C}[0,T]} \xrightarrow[n,m\to+\infty]{} 0 \text{ and } \left\|\mathcal{D}^{(2\alpha)}(u_n - u_m)\right\|_{\mathcal{C}[0,T]} = \|x_n - x_m\|_{\mathcal{C}[0,T]} \xrightarrow[n,m\to+\infty]{} 0$$

we know $\{u_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathcal{C}[0,T]$. By the completeness of $\mathcal{C}[0,T]$, there exists $u, x \in \mathcal{C}[0,T]$ such that $u_n \xrightarrow[n\to+\infty]{} u$ and $x_n \xrightarrow[n\to+\infty]{} x$. The second term is

$$\left\|\mathcal{D}^{(\alpha)}(u_n - u_m)\right\|_{\mathcal{C}[0,T]} = \left\|I_\alpha^0(x_n - x_m) + (c_n - c_m)\right\|_{\mathcal{C}[0,T]} \xrightarrow[n,m\to+\infty]{} 0.$$

We have

$$\|c_n - c_m\|_{\mathcal{C}[0,T]} \leq \left\|I_\alpha^0 (x_n - x_m) + (c_n - c_m)\right\|_{\mathcal{C}[0,T]} + \left\|I_\alpha^0 (x_n - x_m)\right\|_{\mathcal{C}[0,T]} \xrightarrow[n,m\to+\infty]{} 0,$$

that is to say, $\{c_n\}_{n=1}^{\infty}$ is Cauchy sequence in $\mathbb{R}$. By the completeness of $\mathbb{R}$, there exists $c \in \mathbb{R}$ such that $c_n \xrightarrow[n\to+\infty]{} c$. Let $\mathcal{D}^{(\alpha)}u(t) = I_\alpha^0 x(t) + c$, then $u \in \mathcal{C}^{2\alpha}[0,T]$ and

$$\|u_n - u\|_{\mathcal{C}^{2\alpha}[0,T]} \xrightarrow[n\to+\infty]{} 0.$$

The completeness of $\mathcal{C}^{2\alpha}[0,T]$ is proved. □

1.2 Stability theory of the finite difference schemes

In this section, we mention the two theorems we need in chapter four.

Let $0 < \alpha \leq 1, T > 0$ and $N_y \in \mathbb{N}^*$, we consider the following scheme:

$$\begin{cases} BV_y^j + k^2 PV_{yy}^j + AV^j = \bar{F}^j, \ j = \overline{1, N_y - 1}, \ k = \frac{T^\alpha}{\alpha N_y}, \\ V^0 = \Phi, \ V^1 = \Psi, \end{cases} \tag{1.16}$$

where Φ and Ψ are given vectors of a finite-dimensional real space H_h, for all $j = \overline{1, N_y - 1}$ the vector $\bar{F}^j \in H_h$ is given; A, B and P are linear operators in the space H_h. The notations V_{yy}^j and V_y^j are simplifications of formulas $\dfrac{V^{j+1} - 2V^j + V^{j-1}}{k^2}$ and $\dfrac{V^{j+1} - V^{j-1}}{2k}$, respectively, with $V^j \in H_h$ for all $j = \overline{0, N_y}$. For the stability of scheme (1.16) with respect to the initial data and right-hand side, we have the following two theorems:

Theorem 1.7. *[21, page 433] Let $A = A^* > 0$ and $P = P^* > 0$ be positive operators. Then the conditions*

$$B \geq 0 \textit{ and } P > \frac{1}{4}A$$

are sufficient for the stability of scheme (1.16) with respect to the initial data.

Theorem 1.8. *[21, page 436] Let the conditions of Theorem 1.7 be satisfied. Then scheme (1.16) is stable with respect to the right-hand side.*

CHAPTER 2

THE WELL-POSEDNESS OF AN INVERSE SOURCE PROBLEM FOR A TIME-FRACTIONAL TELEGRAPH EQUATION

In this chapter, under some natural regularity and consistency conditions on the input data, we show the well-posedness of the inverse time-dependent source problem (1)-(4) by using eigenfunction expansion of a self-adjoint spectral problem along the Fourier's method and Banach's contraction mapping principle. This chapter is the subject of an article titled "*The well-posedness of an inverse source problem for a time-fractional telegraph equation*" published in an international journal "Palest. J. Math.", see [4].

The following lemma is obtained with the help of integration by parts and the Cauchy–Schwarz inequality.

Lemma 2.1. *Let* $\mu_n = \frac{\pi(2n+1)}{2\ell}$ *with* $n \in \mathbb{N}$. *We have:*

1. *If* $g \in \mathcal{C}^3[0,\ell]$ *satisfies the conditions* $g'(0) = g(\ell) = g''(\ell) = 0$, *then the inequality*

$$\left| \frac{2}{\ell} \int_0^\ell g(x) \cos(\mu_n x)\, dx \right| \leq \frac{\sqrt{2}}{\mu_n^3} \|g\|_{\mathcal{C}^3[0,\ell]},\ \textit{holds.}$$

2. *If* $g \in \mathcal{C}^4[0,\ell]$ *satisfies the conditions* $g'(0) = g(\ell) = g''(\ell) = g'''(0) = 0$, *then the inequality*

$$\left| \frac{2}{\ell} \int_0^\ell g(x) \cos(\mu_n x)\, dx \right| \leq \frac{\sqrt{2}}{\mu_n^4} \|g\|_{\mathcal{C}^4[0,\ell]},\ \textit{holds.}$$

Lemma 2.2. *The numerical series* $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{\mu_n \sqrt{\omega \mu_n^2 + b^2 - a^2}}$ *converges to* $S > 0$, *where* $\mu_n = \frac{\pi}{2\ell}(2n+1)$ *and* $\frac{a^2-b^2}{\omega} < \mu_0^2$.

Proof. Using Riemann's rule, we get:

$$\lim_{n\to+\infty} \frac{n^2}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}} = \frac{\ell^2}{\pi^2\sqrt{\omega}},$$

then, the numerical series $\sum_{n=0}^{+\infty} \frac{1}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}}$ converges to $S>0$. □

Let $\alpha \in \,]\,0,1\,]$, $T>0$ and $L>0$. For $f\in\mathcal{C}\,[0,T]$, we define in $\mathcal{C}\,[0,T]$ the norm

$$\|f\|_{L,\alpha} := \max_{0\le t\le T} e^{-Lt^\alpha/\alpha}\,|f\,(t)|\,.$$

Lemma 2.3. *The norms* $\|\cdot\|_{L,\alpha}$ *and* $\|\cdot\|_{\mathcal{C}[0,T]}$ *are equivalent.*

Proof. For $f\in\mathcal{C}\,[0,T]$, we have

$$e^{-LT^\alpha/\alpha}\,|f\,(t)| \le e^{-Lt^\alpha/\alpha}\,|f\,(t)| \le |f\,(t)|\,.$$

For all $t\in[0,T]$, we obtain

$$e^{-LT^\alpha/\alpha}\,\|f\|_{\mathcal{C}[0,T]} \le \|f\|_{L,\alpha} \le \|f\|_{\mathcal{C}[0,T]}\,,$$

where, $\|f\|_{\mathcal{C}[0,T]} = \max\limits_{0\le t\le T}|f\,(t)|$. □

Consider the following equation:

$$\mathcal{D}_t^{(2\alpha)}h\,(t) + 2a\mathcal{D}_t^{(\alpha)}h\,(t) + b^2h\,(t) = \omega u_{xx}\,(x_*,t) + r\,(t)\,f\,(x_*,t)\,,\ 0\le t\le T. \tag{2.1}$$

The following lemma is valid.

Lemma 2.4. *Let* $\varphi,\psi\in\mathcal{C}\,[0,\ell]\,,\ f\,(x_*,\cdot)\in\mathcal{C}\,[0,T]\,,\ h\in\mathcal{C}^{2\alpha}\,[0,T]$ *and the consistency conditions*

$$\varphi\,(x_*) = h\,(0)\,,\ \ \psi\,(x_*) = \mathcal{D}_t^{(\alpha)}h\,(0)\,, \tag{2.2}$$

be satisfied. Then, we have:

1. *Each solution* $\{u\,(x,t)\,,r\,(t)\}$ *of problem* (1)-(4) *is a solution of* (1)-(3), (2.1).

2. *Each solution* $\{u(x,t), r(t)\}$ *of problem* (1)-(3), (2.1) *is a solution of* (1)-(4).

Proof. 1. Let $\{u(x,t), r(t)\}$ be a solution of the problem (1)-(4), from (4) we have

$$u(x_*,t) = h(t), \ \mathcal{D}_t^{(\alpha)} u(x_*,t) = \mathcal{D}_t^{(\alpha)} h(t) \text{ and } \mathcal{D}_t^{(2\alpha)} u(x_*,t) = \mathcal{D}_t^{(2\alpha)} h(t), \tag{2.3}$$

we put $x = x_*$ in (1), we find

$$\mathcal{D}_t^{(2\alpha)} u(x_*,t) + 2a\mathcal{D}_t^{(\alpha)} u(x_*,t) + b^2 u(x_*,t) = \omega u_{xx}(x_*,t) + r(t) f(x_*,t), \ 0 \leq t \leq T, \tag{2.4}$$

after substituting (2.3) in equation (2.4), we get (2.1).

2. Let $\{u(x,t), r(t)\}$ be a solution of the problem (1)-(3), (2.1), then from (2.1) and (2.4), we obtain:

$$\mathcal{D}_t^{(2\alpha)} [u(x_*,t) - h(t)] + 2a\mathcal{D}_t^{(\alpha)} [u(x_*,t) - h(t)] + b^2 [u(x_*,t) - h(t)] = 0, \ 0 \leq t \leq T, \tag{2.5}$$

by using (2) and (2.2), we get

$$\begin{cases} u(x_*,0) - h(0) = \varphi(x_*) - h(0) = 0, \\ \mathcal{D}_t^{(\alpha)} [u(x_*,0) - h(0)] = \psi(x_*) - \mathcal{D}_t^{(\alpha)} h(0) = 0. \end{cases} \tag{2.6}$$

According to Theorem 1.4, the solution of (2.5)-(2.6) is:

$$u(x_*,t) - h(t) = 0, \ 0 \leq t \leq T.$$

Hence, the condition (4) is obtained.

□

2.1 Existence and uniqueness of the solution

In this section, the first main result on existence and uniqueness of the solution of the inverse time-dependent source problem (1)-(4) is presented as follows.

Theorem 2.1. *Suppose that the following assumptions holds:*

$$(A_1) : \varphi \in \mathcal{C}^4 [0, \ell] ; \ \varphi' (0) = \varphi (\ell) = \varphi'' (\ell) = \varphi''' (0) = 0;$$

$$(A_2) : \psi \in \mathcal{C}^3 [0, \ell] ; \ \psi' (0) = \psi (\ell) = \psi'' (\ell) = 0;$$

$$(A_3) : f \in \mathcal{C} (\Omega_T) ; \ f (\cdot, t) \in \mathcal{C}^3 [0, \ell] , f (x_*, t) \neq 0 \text{ for all } t \in [0, T] ;$$
$$\frac{\partial f}{\partial x} (0, t) = f (\ell, t) = \frac{\partial^2 f}{\partial x^2} (\ell, t) = 0, \text{ for all } t \in [0, T] ;$$

$$(A_4) : h \in \mathcal{C}^{2\alpha} [0, T] ; \varphi (x_*) = h (0) \text{ and } \psi (x_*) = \mathcal{D}_t^{(\alpha)} h (0) .$$

Then, the inverse problem (1)-(4) *has a unique solution* $\{u (x, t) , r (t)\}$.

Proof. The proof of this theorem takes place in three steps:

Step 1: Construction of solution

By using the Fourier's method (separation of variables), the associated spectral problem of the direct problem (1)-(3) is given by:

$$\begin{cases} X'' (x) + \lambda X (x) = 0, \ 0 \leq x \leq \ell, \\ X' (0) = X (\ell) = 0. \end{cases} \tag{2.7}$$

Eigenvalues and eigenfunctions of the spectral problem (2.7) are

$$\lambda_n = \mu_n^2 \text{ where } \mu_n = \frac{\pi (2n + 1)}{2\ell} \text{ and } X_n (x) = \cos (\mu_n x) , \ n \in \mathbb{N}. \tag{2.8}$$

We can easily show that problem (2.7) is self-adjoint, then the eigenfunctions (2.8) forms a complete system in the space $L^2 [0, \ell]$.

By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of the direct problem (1)-(3) for arbitrary $r \in \mathcal{C} [0, T]$,

$$u (x, t) = \sum_{n=0}^{\infty} u_n (t) X_n (x) .$$

where the functions $u_n(t)$ with $n \in \mathbb{N}$ satisfies the following sequence of Cauchy problems:

$$\begin{cases} \mathcal{D}^{(2\alpha)} u_n(t) + 2a\mathcal{D}^{(\alpha)} u_n(t) + \left(b^2 + \omega\mu_n^2\right) u_n(t) = r(t) f_n(t), \; 0 \le t \le T, \\ u_n(0) = \varphi_n, \; \mathcal{D}^{(\alpha)} u_n(0) = \psi_n, \end{cases} \tag{2.9}$$

where

$$\begin{aligned} f_n(t) &= \frac{2}{\ell}\int_0^\ell f(x,t)\cos(\mu_n x)\,dx, \\ \varphi_n &= \frac{2}{\ell}\int_0^\ell \varphi(x)\cos(\mu_n x)\,dx, \\ \psi_n &= \frac{2}{\ell}\int_0^\ell \psi(x)\cos(\mu_n x)\,dx. \end{aligned}$$

According to Theorem 1.4, the solutions of (2.9) are given in the following three cases:
Case 1: If $\dfrac{a^2-b^2}{\omega} < \mu_0^2$, then the solutions of (2.9) are:

$$\begin{aligned} u_n(t) = {} & e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right) \\ & + \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds, \end{aligned} \tag{2.10}$$

where $\delta_n = \sqrt{\omega\mu_n^2 + b^2 - a^2}$.
Case 2: If exist $n_0 \in \mathbb{N}$ such that $\mu_{n_0}^2 < \dfrac{a^2-b^2}{\omega} < \mu_{n_0+1}^2$, then the solutions of (2.9) are:

$$\begin{aligned} \text{for } n \le n_0,\;\; u_n(t) = {} & e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cosh\left(\Delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\Delta_n}\sinh\left(\Delta_n\frac{t^\alpha}{\alpha}\right)\right) \\ & + \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\Delta_n}\int_0^t \sinh\left(\Delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds \\ \text{for } n > n_0,\; u_n(t) = {} & e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right) \\ & + \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds, \end{aligned} \tag{2.11}$$

where $\Delta_n = \sqrt{a^2 - b^2 - \omega\mu_n^2}$.

Case 3: If exist $n_0 \in \mathbb{N}$ such that $\dfrac{a^2-b^2}{\omega} = \mu_{n_0}^2$, then the solutions of (2.9) are:

$$\begin{aligned}
&\text{for } n < n_0,\ \ u_n(t) = e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cosh\left(\Delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\Delta_n}\sinh\left(\Delta_n \frac{t^\alpha}{\alpha}\right)\right)\\
&\qquad + \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\Delta_n}\int_0^t \sinh\left(\Delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds\\
&\text{if } n = n_0,\ u_{n_0}(t) = e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_{n_0} + (\psi_{n_0} + a\varphi_{n_0})\frac{t^\alpha}{\alpha} + \int_0^t \frac{t^\alpha - s^\alpha}{\alpha} e^{a\frac{s^\alpha}{\alpha}} s^{\alpha-1} r(s) f_{n_0}(s)\,ds\right)\\
&\text{for } n > n_0,\ u_n(t) = e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)\right)\\
&\qquad + \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds.
\end{aligned} \tag{2.12}$$

Hence, the representation of the first component of solution pair $\{u(x,t), r(t)\}$ is given in the following three cases:

☞ If $\dfrac{a^2-b^2}{\omega} < \mu_0^2$, then

$$\begin{aligned}
u(x,t) = \sum_{n=0}^{\infty}\Bigg[& e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_n \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n}\sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)\right)\\
&+ \frac{e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{a\frac{s^\alpha}{\alpha}} r(s) f_n(s)\,ds\Bigg]\cos(\mu_n x).
\end{aligned} \tag{2.13}$$

☞ If $\mu_{n_0}^2 < \dfrac{a^2-b^2}{\omega} < \mu_{n_0+1}^2$, then

$$u(x,t) = \sum_{n=0}^{n_0} u_n(t)\cos(\mu_n x) + \sum_{n=n_0+1}^{\infty} u_n(t)\cos(\mu_n x), \tag{2.14}$$

where $u_n(t)$ is defined by (2.11).

☞ If $\dfrac{a^2-b^2}{\omega} = \mu_{n_0}^2$, then

$$u(x,t) = \sum_{n=0}^{n_0-1} u_n(t)\cos(\mu_n x) + u_{n_0}(t)\cos(\mu_{n_0} x) + \sum_{n=n_0+1}^{\infty} u_n(t)\cos(\mu_n x), \tag{2.15}$$

where $u_n(t)$ is defined by (2.12). In this case, if $n_0 = 0$, we delete the first series from the representation (2.15).

Now we construction the second component of solution pair $\{u(x,t), r(t)\}$. Under the assumptions

$(A_3)-(A_4)$, by using Lemma 2.4, from (2.1) we obtain:

$$r(t)=\frac{\mathcal{D}_t^{(2\alpha)}h(t)+2a\mathcal{D}_t^{(\alpha)}h(t)+b^2h(t)-\omega u_{xx}(x_*,t)}{f(x_*,t)},\ 0\leq t\leq T, \tag{2.16}$$

by using the series (2.13)-(2.15), from (2.16) we obtain the following Volterra integral equation of the second kind for $r(t)$:

$$r(t)=B(t)+\int_0^t Q(t,s)\,r(s)\,ds,\ t\in[0,T], \tag{2.17}$$

where

✓ If $\dfrac{a^2-b^2}{\omega}<\mu_0^2$, then

$$\begin{aligned}B(t)=&\frac{\mathcal{D}^{(2\alpha)}h(t)+2a\mathcal{D}^{(\alpha)}h(t)+b^2h(t)}{f(x_*,t)}\\&+\frac{\omega e^{-a\frac{t^\alpha}{\alpha}}}{f(x_*,t)}\sum_{n=0}^{\infty}\left[\varphi_n\cos\left(\delta_n\frac{t^\alpha}{\alpha}\right)+\frac{\psi_n+a\varphi_n}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right]\mu_n^2\cos(\mu_n x_*),\end{aligned} \tag{2.18}$$

and

$$Q(t,s)=\frac{\omega e^{a\frac{s^\alpha-t^\alpha}{\alpha}}s^{\alpha-1}}{f(x_*,t)}\sum_{n=0}^{\infty}\frac{\mu_n^2\cos(\mu_n x_*)}{\delta_n}\sin\left(\delta_n\frac{t^\alpha-s^\alpha}{\alpha}\right)f_n(s). \tag{2.19}$$

✓ If $\mu_{n_0}^2<\dfrac{a^2-b^2}{\omega}<\mu_{n_0+1}^2$, then

$$\begin{aligned}B(t)=&\frac{\mathcal{D}^{(2\alpha)}h(t)+2a\mathcal{D}^{(\alpha)}h(t)+b^2h(t)}{f(x_*,t)}\\&+\frac{\omega e^{-a\frac{t^\alpha}{\alpha}}}{f(x_*,t)}\left(\sum_{n=0}^{n_0}\left[\varphi_n\cosh\left(\Delta_n\frac{t^\alpha}{\alpha}\right)+\frac{\psi_n+a\varphi_n}{\Delta_n}\sinh\left(\Delta_n\frac{t^\alpha}{\alpha}\right)\right]\mu_n^2\cos(\mu_n x_*)\right.\\&\left.+\sum_{n=n_0+1}^{\infty}\left[\varphi_n\cos\left(\delta_n\frac{t^\alpha}{\alpha}\right)+\frac{\psi_n+a\varphi_n}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right]\mu_n^2\cos(\mu_n x_*)\right),\end{aligned} \tag{2.20}$$

and

$$\begin{aligned} Q\left(t,s\right) = \frac{\omega e^{a\frac{s^{\alpha}-t^{\alpha}}{\alpha}} s^{\alpha-1}}{f\left(x_{*},t\right)} &\left(\sum_{n=0}^{n_0} \frac{\mu_n^2 \cos\left(\mu_n x_*\right)}{\Delta_n} \sinh\left(\Delta_n \frac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f_n\left(s\right)\right. \\ &\left.+ \sum_{n=n_0+1}^{\infty} \frac{\mu_n^2 \cos\left(\mu_n x_*\right)}{\delta_n} \sin\left(\delta_n \frac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f_n\left(s\right)\right). \end{aligned} \tag{2.21}$$

✓ If $\dfrac{a^2-b^2}{\omega} = \mu_{n_0}^2$, then

$$\begin{aligned} B\left(t\right) = &\frac{\mathcal{D}^{(2\alpha)} h\left(t\right) + 2a\mathcal{D}^{(\alpha)} h\left(t\right) + b^2 h\left(t\right)}{f\left(x_*,t\right)} \\ &+ \frac{\omega e^{-a\frac{t^{\alpha}}{\alpha}}}{f\left(x_*,t\right)} \left(\sum_{n=0}^{n_0-1} \left[\varphi_n \cosh\left(\Delta_n \frac{t^{\alpha}}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\Delta_n} \sinh\left(\Delta_n \frac{t^{\alpha}}{\alpha}\right)\right] \mu_n^2 \cos\left(\mu_n x_*\right)\right. \\ &+ \left[\varphi_{n_0} + \left(\psi_{n_0} + a\varphi_{n_0}\right) \frac{t^{\alpha}}{\alpha}\right] \mu_{n_0}^2 \cos\left(\mu_{n_0} x_*\right) \\ &\left.+ \sum_{n=n_0+1}^{\infty} \left[\varphi_n \cos\left(\delta_n \frac{t^{\alpha}}{\alpha}\right) + \frac{\psi_n + a\varphi_n}{\delta_n} \sin\left(\delta_n \frac{t^{\alpha}}{\alpha}\right)\right] \mu_n^2 \cos\left(\mu_n x_*\right)\right), \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} Q\left(t,s\right) = \frac{\omega e^{a\frac{s^{\alpha}-t^{\alpha}}{\alpha}} s^{\alpha-1}}{f\left(x_{*},t\right)} &\left(\sum_{n=0}^{n_0-1} \frac{\mu_n^2 \cos\left(\mu_n x_*\right)}{\Delta_n} \sinh\left(\Delta_n \frac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f_n\left(s\right)\right. \\ &+\mu_{n_0}^2 \cos\left(\mu_{n_0} x_*\right) f_{n_0}\left(s\right) \frac{t^{\alpha}-s^{\alpha}}{\alpha} \\ &\left.+ \sum_{n=n_0+1}^{\infty} \frac{\mu_n^2 \cos\left(\mu_n x_*\right)}{\delta_n} \sin\left(\delta_n \frac{t^{\alpha}-s^{\alpha}}{\alpha}\right) f_n\left(s\right)\right). \end{aligned} \tag{2.23}$$

In this case, if $n_0 = 0$, we delete the first series from the two representations (2.22) and (2.23).

Step 2: Existence of the solution

To establish the regularity of the first component $u(x,t)$, we need to show $u\left(x,t\right)$, $u_x\left(x,t\right)$, $u_{xx}\left(x,t\right)$, $\mathcal{D}_t^{(\alpha)} u\left(x,t\right)$ and $\mathcal{D}_t^{(2\alpha)} u\left(x,t\right)$ are continuous functions in Ω_T.

Under the assumptions $(A_1)-(A_3)$ and Lemma 2.1, by using the series (2.13), the following inequalities holds for any $(x,t)\in\Omega_T$ such that

$$
\begin{aligned}
|u(x,t)| \leq \sum_{n=0}^{\infty}\Bigg[& \frac{\sqrt{2}\,\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^4}+\frac{\sqrt{2}\,\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n^3}+\frac{\sqrt{2}a\,\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\delta_n\mu_n^4} \\
& +\frac{\sqrt{2}e^{aT^\alpha/\alpha}\,\|r\|_{\mathcal{C}[0,T]}\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{a\alpha\delta_n\mu_n^3}\Bigg].
\end{aligned} \tag{2.24}
$$

$$
\begin{aligned}
|u_x(x,t)| \leq \sum_{n=0}^{\infty}\Bigg[& \frac{\sqrt{2}\,\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^3}+\frac{\sqrt{2}\,\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n^2}+\frac{\sqrt{2}a\,\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\delta_n\mu_n^3} \\
& +\frac{\sqrt{2}e^{aT^\alpha/\alpha}\,\|r\|_{\mathcal{C}[0,T]}\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{a\alpha\delta_n\mu_n^2}\Bigg].
\end{aligned} \tag{2.25}
$$

$$
\begin{aligned}
|u_{xx}(x,t)| \leq \sum_{n=0}^{\infty}\Bigg[& \frac{\sqrt{2}\,\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^2}+\frac{\sqrt{2}\,\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n}+\frac{\sqrt{2}a\,\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\delta_n\mu_n^2} \\
& +\frac{\sqrt{2}e^{aT^\alpha/\alpha}\,\|r\|_{\mathcal{C}[0,T]}\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{a\alpha\delta_n\mu_n}\Bigg].
\end{aligned} \tag{2.26}
$$

$$
\begin{aligned}
\left|\mathcal{D}_t^{(\alpha)}u(x,t)\right| \leq \sum_{n=0}^{\infty}\Bigg[& \sqrt{2}\left(2a+\delta_n+\frac{a^2}{\delta_n}\right)\frac{\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^4}+\left(\sqrt{2}+\frac{a\sqrt{2}}{\delta_n}\right)\frac{\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\mu_n^3} \\
& +\left(\frac{\sqrt{2}}{a}+\frac{\sqrt{2}}{\delta_n}\right)e^{aT^\alpha/\alpha}\,\|r\|_{\mathcal{C}[0,T]}\,\frac{\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{\mu_n^3}\Bigg].
\end{aligned} \tag{2.27}
$$

$$
\begin{aligned}
\left|\mathcal{D}_t^{(2\alpha)}u(x,t)\right| \leq \sum_{n=0}^{\infty}\Bigg[& \left(\delta_n^2+\frac{2a}{\delta_n}+2\right)\left(\sqrt{2}\left(1+\frac{a}{\delta_n}\right)\frac{\|\varphi\|_{\mathcal{C}^4[0,\ell]}}{\mu_n^4}+\frac{\sqrt{2}\,\|\psi\|_{\mathcal{C}^3[0,\ell]}}{\delta_n\mu_n^3}\right) \\
& +\sqrt{2}\left(\frac{a}{\delta_n}+\frac{\delta_n}{a}+2\right)e^{aT^\alpha/\alpha}\,\|r\|_{\mathcal{C}[0,T]}\,\frac{\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{\mu_n^3}\Bigg].
\end{aligned} \tag{2.28}
$$

From (2.24)-(2.28) and by Weierstrass M-test, the series corresponding to $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $\mathcal{D}_t^{(\alpha)}u(x,t)$ and $\mathcal{D}_t^{(2\alpha)}u(x,t)$ are uniformly convergent on Ω_T. Hence, $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $\mathcal{D}_t^{(\alpha)}u(x,t)$ and $\mathcal{D}_t^{(2\alpha)}u(x,t)$ are continuous functions on Ω_T.

Similarly, we show that $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $\mathcal{D}_t^{(\alpha)}u(x,t)$ and $\mathcal{D}_t^{(2\alpha)}u(x,t)$ are continuous functions on Ω_T in the other cases. Thus, $u(x,t)$ satisfies the conditions (1)-(3) for arbitrary $r\in\mathcal{C}[0,T]$.

We define the following operator:

$$\mathcal{P}\left(r\left(t\right)\right) := B\left(t\right) + \int_0^t Q\left(t,s\right) r\left(s\right) ds,$$

on the space $\mathcal{C}\left[0,T\right]$ with $\left\|\phi\right\|_{L,\alpha} := \max\limits_{0\leq t\leq T} e^{-Lt^{\alpha}/\alpha}\left|\phi\left(t\right)\right|$. To show $\mathcal{P}$ is well defined.

Under the assumptions $(A_3)-(A_4)$, the function $t\mapsto \frac{\mathcal{D}^{(2\alpha)}h(t)+2a\mathcal{D}^{(\alpha)}h(t)+b^2h(t)}{f(x_*,t)}$ is continuous on $[0,T]$. Above we proved that the function $u_{xx}\left(x,t\right)$ is continuous in Ω_T for $r\in\mathcal{C}\left[0,T\right]$, then, under the assumption (A_3), the function $t\mapsto\frac{\omega u_{xx}(x_*,t)}{f(x_*,t)}$ is continuous on $[0,T]$. From (2.16), the operator $\mathcal{P}$ is well defined.

For the rest of this proof, we take only the first case (if $\frac{a^2-b^2}{\omega}<\mu_0^2$). Now we prove that $\mathcal{P}$ is a contraction operator in the space $\mathcal{C}\left[0,T\right]$. We choose $L>0$ and let $r_1,r_2\in\mathcal{C}\left[0,T\right]$. Under the assumption (A_3) and using (2.19), we have the following estimates:

$$\begin{aligned}
e^{-Lt^{\alpha}/\alpha}\left|\mathcal{P}\left(r_1\left(t\right)\right)-\mathcal{P}\left(r_2\left(t\right)\right)\right| &\leq e^{-Lt^{\alpha}/\alpha}\int_0^t e^{Ls^{\alpha}/\alpha}\left|Q\left(t,s\right)\right| e^{-Ls^{\alpha}/\alpha}\left|r_1\left(s\right)-r_2\left(s\right)\right| ds \\
&\leq \left\|r_1-r_2\right\|_{L,\alpha}\int_0^t e^{L(s^{\alpha}-t^{\alpha})/\alpha}\left|Q\left(t,s\right)\right| ds \\
&\leq \frac{\sqrt{2}\omega\left\|f\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}\sum\limits_{n=0}^{+\infty}\dfrac{1}{\delta_n\mu_n}}{\left(L+a\right)\min\limits_{0\leq t\leq T}\left|f\left(x_*,t\right)\right|}\left\|r_1-r_2\right\|_{L,\alpha}.
\end{aligned}$$

Consequently, we obtain:

$$\left\|\mathcal{P}\left(r_1\right)-\mathcal{P}\left(r_2\right)\right\|_{L,\alpha} \leq \frac{\sqrt{2}\omega S\left\|f\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{\left(L+a\right)\min\limits_{0\leq t\leq T}\left|f\left(x_*,t\right)\right|}\left\|r_1-r_2\right\|_{L,\alpha}, \tag{2.29}$$

where $S=\sum\limits_{n=0}^{+\infty}\dfrac{1}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}}$. It is easy to choose the real $L>0$ such that,

$$\frac{\sqrt{2}\omega S\left\|f\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{\left(L+a\right)\min\limits_{0\leq t\leq T}\left|f\left(x_*,t\right)\right|} < 1. \tag{2.30}$$

Then the operator $\mathcal{P}$ is a contraction. Consequently, by Banach's contraction mapping principle, $\mathcal{P}$ has a unique fixed point $r\in\mathcal{C}\left[0,T\right]$.

Step 3: Uniqueness of the solution

Let $\{u\left(x,t\right),r\left(t\right)\}$ and $\{\tilde{u}\left(x,t\right),\tilde{r}\left(t\right)\}$ be two solution sets of the inverse problem (1)-(4). From (2.13) and (2.17) , we have

$$u(x,t)-\tilde{u}(x,t)=e^{-a\frac{t^{\alpha}}{\alpha}}\sum_{n=0}^{\infty}\frac{\cos(\mu_n x)}{\delta_n}\int_0^t \sin\left(\delta_n\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)s^{\alpha-1}e^{a\frac{s^{\alpha}}{\alpha}}(r(s)-\tilde{r}(s))\,f_k(s)\,ds, \tag{2.31}$$

and

$$r(t)-\tilde{r}(t)=\mathcal{P}(r(t))-\mathcal{P}(\tilde{r}(t)). \tag{2.32}$$

From (2.29) and (2.32) we get:

$$\|r-\tilde{r}\|_{L,\alpha}\leq\frac{\sqrt{2}\omega S\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\min\limits_{0\leq t\leq T}|f(x_*,t)|}\|r-\tilde{r}\|_{L,\alpha}, \tag{2.33}$$

which implies that $r=\tilde{r}$. After inserting $r=\tilde{r}$ in (2.31), we have $u=\tilde{u}$. □

Remark 2.1. We can prove Theorem 2.1 so that the function $Q(t,s)$ given by (2.21) and (2.23). In both cases, the contraction constant in (2.30) will be given as follows:

✓ If $\mu_{n_0}^2<\dfrac{a^2-b^2}{\omega}<\mu_{n_0+1}^2$, the contraction constant in (2.30) is replaced by

$$\begin{aligned}\frac{\sqrt{2}\omega S'\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\min\limits_{0\leq t\leq T}|f(x_*,t)|}\text{ where }S'=&\sum_{n=0}^{n_0}\frac{e^{a\frac{T^{\alpha}}{\alpha}}}{2\mu_n\sqrt{a^2-b^2-\omega\mu_n^2}}\\&+\sum_{n=n_0+1}^{+\infty}\frac{1}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}}.\end{aligned} \tag{2.34}$$

✓ If $\dfrac{a^2-b^2}{\omega}=\mu_{n_0}^2$, the contraction constant in (2.30) is replaced by

$$\begin{aligned}\frac{\sqrt{2}\omega S''\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\min\limits_{0\leq t\leq T}|f(x_*,t)|}\text{ where }S''=&\frac{T^{\alpha}}{\alpha\mu_{n_0}}+\sum_{n=0}^{n_0-1}\frac{e^{a\frac{T^{\alpha}}{\alpha}}}{2\mu_n\sqrt{a^2-b^2-\omega\mu_n^2}}\\&+\sum_{n=n_0+1}^{+\infty}\frac{1}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}}.\end{aligned} \tag{2.35}$$

2.2 Continuous dependence upon the data of the solution

In this section, we give the second main result on continuous dependence upon the data of the solution pair $\{u(x,t),r(t)\}$ of the inverse problem (1)-(4).

Let $\Im$ be the set of quartiles $\{\varphi, \psi, f, h\}$ where the functions φ, ψ, f and h satisfies the assumptions $(A_1) - (A_4)$ of Theorem 2.1 and

$$\begin{aligned} &\|\varphi\|_{\mathcal{C}^4[0,\ell]} \leq M_1,\ \|\psi\|_{\mathcal{C}^3[0,\ell]} \leq M_2,\ \|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]} \leq M_3,\ \|h\|_{\mathcal{C}^{2\alpha}[0,T]} \leq M_4, \\ &M_5 = \min_{0\leq t\leq T} |f(x_*, t)|\,,\ M_6 = \max\left\{1, 2a, b^2\right\}. \end{aligned} \tag{2.36}$$

For $\phi \in \Im$, we define the norm

$$\|\phi\|_{\Im} := \|\varphi\|_{\mathcal{C}^4[0,\ell]} + \|\psi\|_{\mathcal{C}^3[0,\ell]} + \|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]} + \|h\|_{\mathcal{C}^{2\alpha}[0,T]}\,. \tag{2.37}$$

Theorem 2.2. *The solution $\{u(x,t), r(t)\}$ of the inverse problem (1)-(4) under the assumptions of Theorem 2.1, depends continuously upon the data.*

Proof. Let $\{u(x,t), r(t)\}$ and $\{\tilde{u}(x,t), \tilde{r}(t)\}$ be two solution sets of the inverse problem (1)-(4), corresponding to the data $\phi = \{\varphi, \psi, f, h\}$ and $\tilde{\phi} = \left\{\tilde{\varphi}, \tilde{\psi}, \tilde{f}, \tilde{h}\right\}$, respectively. We prove only the first case $\left(\text{if } \frac{a^2-b^2}{\omega} < \mu_0^2\right)$ and similarly, we prove the other cases.

From (2.18), we have

$$\begin{aligned} B(t) - \tilde{B}(t) &= \frac{1}{f(x_*, t)}\left[\mathcal{D}^{(2\alpha)}\left(h(t) - \tilde{h}(t)\right) + 2a\mathcal{D}^{(\alpha)}\left(h(t) - \tilde{h}(t)\right) + b^2\left(h(t) - \tilde{h}(t)\right)\right] \\ &+ \frac{\tilde{f}(x_*, t) - f(x_*, t)}{f(x_*, t)\tilde{f}(x_*, t)}\left[\mathcal{D}^{(2\alpha)}\tilde{h}(t) + 2a\mathcal{D}^{(\alpha)}\tilde{h}(t) + b^2\tilde{h}(t)\right] \\ &+ \frac{\omega e^{-at^\alpha/\alpha}}{f(x_*, t)}\sum_{n=0}^{+\infty}\left[(\varphi_n - \tilde{\varphi}_n)\cos(\delta_n t^\alpha/\alpha) + \frac{1}{\delta_n}\left(\psi_n - \tilde{\psi}_n + a(\varphi_n - \tilde{\varphi}_n)\right)\sin(\delta_n t^\alpha/\alpha)\right]\mu_n^2\cos(\mu_n x_*) \\ &+ \frac{\omega e^{-at^\alpha/\alpha}\left(\tilde{f}(x_*, t) - f(x_*, t)\right)}{f(x_*, t)\tilde{f}(x_*, t)}\sum_{n=0}^{+\infty}\left(\tilde{\varphi}_n\cos(\delta_n t^\alpha/\alpha) + \frac{\tilde{\psi}_n + a\tilde{\varphi}_n}{\delta_n}\sin(\delta_n t^\alpha/\alpha)\right)\mu_n^2\cos(\mu_n x_*)\,. \end{aligned} \tag{2.38}$$

Under the assumptions $(A_1) - (A_4)$ and using Lemma 2.1, (2.36) and (2.38) we obtain:

$$\begin{aligned} \left\|B - \tilde{B}\right\|_{\mathcal{C}[0,T]} \leq M_7\|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + M_8\left\|\psi - \tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} &+ M_9\left\|f - \tilde{f}\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]} \\ &+ M_{10}\left\|h - \tilde{h}\right\|_{\mathcal{C}^{2\alpha}[0,T]}, \end{aligned} \tag{2.39}$$

where

$$\begin{aligned}
M_7 &= \frac{\sqrt{2}\omega}{M_5}\sum_{n=0}^{+\infty}\left(\frac{1}{\mu_n^2}+\frac{a}{\delta_n\mu_n^2}\right),\\
M_8 &= \frac{\sqrt{2}\omega}{M_5}\sum_{n=0}^{+\infty}\frac{1}{\delta_n\mu_n},\\
M_9 &= \frac{M_4M_6}{M_5^2}+\frac{\sqrt{2}\omega}{M_5^2}\sum_{n=0}^{+\infty}\left(\left(1+\frac{a}{\delta_n}\right)\frac{M_1}{\mu_n^2}+\frac{M_2}{\delta_n\mu_n}\right),\\
M_{10} &= M_6/M_5.
\end{aligned}$$

From (2.17) and for all $L>0$, we have

$$\begin{aligned}
e^{-Lt^\alpha/\alpha}\left[r(t)-\tilde{r}(t)\right] = e^{-Lt^\alpha/\alpha}&\left[B(t)-\tilde{B}(t)\right]\\
&+\int_0^t e^{-Lt^\alpha/\alpha}Q(t,s)\left[r(s)-\tilde{r}(s)\right]ds\\
&+\int_0^t e^{-Lt^\alpha/\alpha}\left[Q(t,s)-\tilde{Q}(t,s)\right]\tilde{r}(s)\,ds.
\end{aligned}\tag{2.40}$$

Under the assumption (A_3), using Lemma 2.1, Lemma 2.2 and (2.19), we obtain

$$\int_0^t e^{-Lt^\alpha/\alpha}Q(t,s)\left[r(s)-\tilde{r}(s)\right]ds \le \frac{\sqrt{2}\omega S\left\|f\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\,M_5}\left\|r-\tilde{r}\right\|_{L,\alpha},\tag{2.41}$$

and

$$\int_0^t e^{-Lt^\alpha/\alpha}\left[Q(t,s)-\tilde{Q}(t,s)\right]\tilde{r}(s)\,ds \le \frac{2\sqrt{2}W\omega M_3S}{(L+a)\,M_5^2}\left\|f-\tilde{f}\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]},\tag{2.42}$$

where $W=\left\|\tilde{r}\right\|_{L,\alpha}$.

From (2.39)-(2.42), we have the estimate

$$\begin{aligned}
\left(1-\frac{\sqrt{2}\omega S\left\|f\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\,M_5}\right)\left\|r-\tilde{r}\right\|_{L,\alpha} \le M_7\left\|\varphi-\tilde{\varphi}\right\|_{\mathcal{C}^4[0,\ell]}&+M_8\left\|\psi-\tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]}\\
&+\left[M_9+\frac{2\sqrt{2}W\omega M_3S}{(L+a)\,M_5^2}\right]\left\|f-\tilde{f}\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}\\
&+M_{10}\left\|h-\tilde{h}\right\|_{\mathcal{C}^{2\alpha}[0,T]}.
\end{aligned}\tag{2.43}$$

From (2.43) and (2.37), we get

$$\left(1-\frac{\sqrt{2}\omega S\,\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)\,M_5}\right)\|r-\tilde{r}\|_{L,\alpha}\leq M_{11}\left\|\phi-\tilde{\phi}\right\|_{\Im}, \tag{2.44}$$

where

$$M_{11}=\max\left\{M_7,M_8,M_9+\frac{2\sqrt{2}W\omega M_3 S}{(L+a)\,M_5^2},M_{10}\right\}.$$

From (2.30) and (2.44), we have

$$\|r-\tilde{r}\|_{L,\alpha}\leq M_{12}\left\|\phi-\tilde{\phi}\right\|_{\Im}, \tag{2.45}$$

where $M_{12}=\frac{M_{11}}{1-\frac{\sqrt{2}\omega S\|f\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}}{(L+a)M_5}}$.

Under the assumptions $(A_1)-(A_3)$ and from (2.13), we obtain

$$\begin{aligned}\|u-\tilde{u}\|_{\mathcal{C}(\Omega_T)}\leq M_{13}\,\|\varphi-\tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]}+M_{14}\left\|\psi-\tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]}+M_{15}\left\|f-\tilde{f}\right\|_{\mathcal{C}^3[0,\ell]\times\mathcal{C}[0,T]}\\ +M_{16}\,\|r-\tilde{r}\|_{L,\alpha},\end{aligned} \tag{2.46}$$

where

$$M_{13}=\sum_{n=0}^{+\infty}\left(\frac{\sqrt{2}}{\mu_n^4}+\frac{\sqrt{2}a}{\delta_n\mu_n^4}\right),\ M_{14}=\sum_{n=0}^{+\infty}\frac{\sqrt{2}}{\delta_n\mu_n^3},\ M_{15}=\sum_{n=0}^{+\infty}\frac{W\sqrt{2}e^{L\frac{T^\alpha}{\alpha}}}{(L+a)\,\delta_n\mu_n^3},\ M_{16}=\sum_{n=0}^{+\infty}\frac{M_3\sqrt{2}e^{L\frac{T^\alpha}{\alpha}}}{(L+a)\,\delta_n\mu_n^3}$$

From (2.45) and (2.46), we get

$$\|u-\tilde{u}\|_{\mathcal{C}(\Omega_T)}+\|r-\tilde{r}\|_{L,\alpha}\leq M_{17}\left\|\phi-\tilde{\phi}\right\|_{\Im},$$

where $M_{17}=\max\{M_{13},M_{14},M_{15}\}+M_{12}M_{16}+M_{12}$. Then, the solution of the inverse problem (1)-(4) depends continuously upon the data. $\square$

CHAPTER 3

AN INVERSE SPACE-DEPENDENT SOURCE PROBLEM FOR A TIME-FRACTIONAL TELEGRAPH EQUATION

In this chapter, under some assumptions on the input data, we show the existence, uniqueness and continuous dependence upon the data of the solution of the inverse space-dependent source problem (1), (2), (5), (6) by using the Fourier's method and some properties of the conformable fractional derivative. Finally, we give two examples of this inverse problem.

Before presenting the main results, we introduce the lemmas, which are used in this chapter. The following lemma is obtained with the help of integration by parts and the Cauchy–Schwarz inequality.

Lemma 3.1. *Let $\ell > 0$ and $\mu_n = \frac{2n\pi}{\ell}$ with $n \in \mathbb{N}^*$. We have:*

1. *If $g \in \mathcal{C}^3[0,\ell]$ satisfies the conditions $g(0) = g(\ell)$, $g'(0) = g'(\ell)$ and $g''(0) = g''(\ell)$, then the inequality*

$$\left|\frac{2}{\ell}\int_0^\ell g(x)\cos(\mu_n x)\,dx\right| \le \frac{\sqrt{2}}{\mu_n^3}\|g\|_{\mathcal{C}^3[0,\ell]} \text{ and } \left|\frac{2}{\ell}\int_0^\ell g(x)\sin(\mu_n x)\,dx\right| \le \frac{\sqrt{2}}{\mu_n^3}\|g\|_{\mathcal{C}^3[0,\ell]}, \text{ holds.}$$

2. *If $g \in \mathcal{C}^4[0,\ell]$ satisfies the conditions $g(0) = g(\ell)$, $g'(0) = g'(\ell)$, $g''(0) = g''(\ell)$ and $g'''(0) = g'''(\ell)$, then the inequality*

$$\left|\frac{2}{\ell}\int_0^\ell g(x)\cos(\mu_n x)\,dx\right| \le \frac{\sqrt{2}}{\mu_n^4}\|g\|_{\mathcal{C}^4[0,\ell]} \text{ and } \left|\frac{2}{\ell}\int_0^\ell g(x)\sin(\mu_n x)\,dx\right| \le \frac{\sqrt{2}}{\mu_n^4}\|g\|_{\mathcal{C}^4[0,\ell]}, \text{ holds.}$$

Lemma 3.2. *The numerical series $\sum_{n=1}^{+\infty}\frac{1}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}}$ converges, where $\mu_n = \frac{2n\pi}{\ell}$ and $\frac{a^2-b^2}{\omega} < \mu_1^2$.*

Proof. Using Riemann's rule, we get:

$$\lim_{n\to+\infty} \frac{n^2}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}} = \frac{\ell^2}{4\pi^2\sqrt{\omega}},$$

then, the numerical series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{\mu_n\sqrt{\omega\mu_n^2+b^2-a^2}}$ converges. □

Lemma 3.3. *Let* $\alpha \in \left]0,1\right]$ *and* $T, a > 0$. *We have:*

$$\int_0^T \sin\left(\delta_n \frac{T^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{T^\alpha - s^\alpha}{\alpha}} ds > \frac{\delta_n e^{-a\frac{T^\alpha}{\alpha}}}{\delta_n^2 + a^2}\left(e^{a\frac{T^\alpha}{\alpha}} - \left(1 + a\frac{T^\alpha}{\alpha}\right)\right) > 0, \tag{3.1}$$

where $\delta_n = \sqrt{\omega\mu_n^2 + b^2 - a^2}$ *with* $\omega > 0$, $\mu_n = \frac{2n\pi}{\ell}, n \in \mathbb{N}^*$ *and* $\frac{a^2-b^2}{\omega} < \mu_1^2$.

Let $t \in [0, T]$, *we have:*

$$\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds \leq \frac{\delta_n}{\delta_n^2 + a^2}\left(2 + \frac{a}{\delta_n}\right). \tag{3.2}$$

Proof. By using changing of variables $\tau = \dfrac{T^\alpha - s^\alpha}{\alpha}$ and integration by parts twice, we obtain:

$$\begin{aligned} \int_0^T \sin\left(\delta_n \frac{T^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{T^\alpha - s^\alpha}{\alpha}} ds &= \int_0^{\frac{T^\alpha}{\alpha}} \sin(\delta_n\tau)\, e^{-a\tau} d\tau \\ &= \frac{\delta_n e^{-a\frac{T^\alpha}{\alpha}}}{\delta_n^2 + a^2}\left(e^{a\frac{T^\alpha}{\alpha}} - \cos\left(\delta_n\frac{T^\alpha}{\alpha}\right) - \frac{a\dfrac{T^\alpha}{\alpha}\sin\left(\delta_n\dfrac{T^\alpha}{\alpha}\right)}{\delta_n\dfrac{T^\alpha}{\alpha}}\right), \end{aligned} \tag{3.3}$$

we have $\delta_n\dfrac{T^\alpha}{\alpha} > 0$ hold for any $n \in \mathbb{N}^*$, this implies that $\dfrac{\sin\left(\delta_n\frac{T^\alpha}{\alpha}\right)}{\delta_n\frac{T^\alpha}{\alpha}} < 1$, then

$$\int_0^T \sin\left(\delta_n \frac{T^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{T^\alpha - s^\alpha}{\alpha}} ds > \frac{\delta_n e^{-a\frac{T^\alpha}{\alpha}}}{\delta_n^2 + a^2}\underbrace{\left(e^{a\frac{T^\alpha}{\alpha}} - \left(1 + a\frac{T^\alpha}{\alpha}\right)\right)}_{>0} > 0.$$

The constant T in (3.3) is replaced by t, we get

$$\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds = \int_0^{\frac{t^\alpha}{\alpha}} \sin(\delta_n \tau) e^{-a\tau} d\tau$$
$$= \frac{\delta_n e^{-a\frac{t^\alpha}{\alpha}}}{\delta_n^2 + a^2} \left(e^{a\frac{t^\alpha}{\alpha}} - \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) - \frac{a\frac{t^\alpha}{\alpha} \sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)}{\delta_n \frac{t^\alpha}{\alpha}} \right),$$

hence, we obtain:

$$\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds = \frac{\delta_n}{\delta_n^2 + a^2} \left(1 - e^{-a\frac{t^\alpha}{\alpha}} \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) - \frac{a e^{-a\frac{t^\alpha}{\alpha}} \sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)}{\delta_n} \right)$$
$$\leq \frac{\delta_n}{\delta_n^2 + a^2} \left(2 + \frac{a}{\delta_n} \right).$$

$\square$

3.1 Existence and uniqueness of the solution

In this section, the first main result on existence and uniqueness of the solution of the inverse space-dependent source problem (1), (2), (5), (6) is presented as follows.

Theorem 3.1. *Suppose that the following assumptions holds:*

$$(A_1): \varphi \in \mathcal{C}^4[0,\ell];\ \varphi(0) = \varphi(\ell),\ \varphi'(0) = \varphi'(\ell),\ \varphi''(0) = \varphi''(\ell) \text{ and } \varphi'''(0) = \varphi'''(\ell);$$
$$(A_2): \psi \in \mathcal{C}^3[0,\ell];\ \psi(0) = \psi(\ell),\ \psi'(0) = \psi'(\ell) \text{ and } \psi''(0) = \psi''(\ell);$$
$$(A_3): h \in \mathcal{C}^4[0,\ell];\ h(0) = h(\ell),\ h'(0) = h'(\ell),\ h''(0) = h''(\ell) \text{ and } h'''(0) = h'''(\ell).$$

Then, the inverse problem (1), (2), (5), (6) has a unique solution $\{u(x,t), f(x)\}$.

Proof. The proof of this theorem takes place in three steps:

Step 1: Construction of solution

By using the Fourier's method (separation of variables), the associated spectral problem of the direct problem (1), (2), (5) is given by:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \ 0 \leq x \leq \ell, \\ X(0) = X(\ell), \ X'(0) = X'(\ell). \end{cases} \tag{3.4}$$

This problem has the eigenvalues

$$\lambda_n = \mu_n^2 \ \text{ where } \ \mu_n = \frac{2n\pi}{\ell}, \ n = 0, 1, \ldots$$

and the eigenfunctions

$$1, \ \cos(\mu_n x), \ \sin(\mu_n x), \ n = 1, 2, \ldots. \tag{3.5}$$

We can easily show that problem (3.4) is self-adjoint, then the system of functions (3.5) forms an orthogonal basis in the space $L^2[0, \ell]$.

The solution of the inverse problem (1), (2), (5), (6) can be written by using the Fourier method

$$\begin{aligned} u(x,t) &= u_0(t) + \sum_{n=1}^{\infty} u_{1,n}(t)\cos(\mu_n x) + u_{2,n}(t)\sin(\mu_n x), \\ f(x) &= f_0 + \sum_{n=1}^{\infty} f_{1,n}\cos(\mu_n x) + f_{2,n}\sin(\mu_n x), \end{aligned} \tag{3.6}$$

where the functions $u_0(t), u_{1,n}(t), u_{2,n}(t), f_0, f_{1,n}$ and $f_{2,n}$ with $n \in \mathbb{N}^*$ satisfies the following sequence of inverse problems:

$$\begin{cases} \mathcal{D}^{(2\alpha)} u_0(t) + 2a\mathcal{D}^{(\alpha)} u_0(t) + b^2 u_0(t) = f_0, \ 0 \leq t \leq T, \\ u_0(0) = \varphi_0, \ \mathcal{D}^{(\alpha)} u_0(0) = \psi_0, \ u_0(T) = h_0, \end{cases} \tag{3.7}$$

with $f_0 = \frac{1}{\ell}\int_0^\ell f(x)\,dx, \ \varphi_0 = \frac{1}{\ell}\int_0^\ell \varphi(x)\,dx, \ \psi_0 = \frac{1}{\ell}\int_0^\ell \psi(x)\,dx$ and $h_0 = \frac{1}{\ell}\int_0^\ell h(x)\,dx$,

$$\begin{cases} \mathcal{D}^{(2\alpha)} u_{k,n}(t) + 2a\mathcal{D}^{(\alpha)} u_{k,n}(t) + \left(b^2 + \omega\mu_n^2\right) u_{k,n}(t) = f_{k,n}, \ 0 \leq t \leq T, \\ u_{k,n}(0) = \varphi_{k,n}, \ \mathcal{D}^{(\alpha)} u_{k,n}(0) = \psi_{k,n}, \ u_{k,n}(T) = h_{k,n}, \end{cases} \quad k = 1, 2, \tag{3.8}$$

where

$$\begin{aligned}
&f_{1,n}=\frac{2}{\ell}\int_0^\ell f(x)\cos(\mu_n x)\,dx,\ \varphi_{1,n}=\frac{2}{\ell}\int_0^\ell \varphi(x)\cos(\mu_n x)\,dx,\ \psi_{1,n}=\frac{2}{\ell}\int_0^\ell \psi(x)\cos(\mu_n x)\,dx,\\
&h_{1,n}=\frac{2}{\ell}\int_0^\ell h(x)\cos(\mu_n x)\,dx,\\
&f_{2,n}=\frac{2}{\ell}\int_0^\ell f(x)\sin(\mu_n x)\,dx,\ \varphi_{2,n}=\frac{2}{\ell}\int_0^\ell \varphi(x)\sin(\mu_n x)\,dx,\ \psi_{2,n}=\frac{2}{\ell}\int_0^\ell \psi(x)\sin(\mu_n x)\,dx,\\
&h_{2,n}=\frac{2}{\ell}\int_0^\ell h(x)\sin(\mu_n x)\,dx.
\end{aligned}$$

According to Theorem 1.4, the solution pair $\{u_0(t), f_0\}$ of (3.7) given in the following two cases:

Case 1: If $a=b$, then the solution pair $\{u_0(t), f_0\}$ of (3.7) is:

$$\begin{aligned}
u_0(t)&=e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_0+(\psi_0+a\varphi_0)\frac{t^\alpha}{\alpha}\right)+f_0\int_0^t\frac{t^\alpha-s^\alpha}{\alpha}e^{-a\frac{t^\alpha-s^\alpha}{\alpha}}s^{\alpha-1}ds,\\
f_0&=\frac{h_0-e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_0+(\psi_0+a\varphi_0)\dfrac{T^\alpha}{\alpha}\right)}{\displaystyle\int_0^T\frac{T^\alpha-s^\alpha}{\alpha}e^{-a\frac{T^\alpha-s^\alpha}{\alpha}}s^{\alpha-1}ds}.
\end{aligned}\tag{3.9}$$

Case 2: If $a>b$, then the solution pair $\{u_0(t), f_0\}$ of (3.7) is:

$$\begin{aligned}
u_0(t)&=e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_0\cosh\left(\Delta_0\frac{t^\alpha}{\alpha}\right)+\frac{\psi_0+a\varphi_0}{\Delta_0}\sinh\left(\Delta_0\frac{t^\alpha}{\alpha}\right)\right)\\
&\quad+\frac{f_0}{\Delta_0}\int_0^t\sinh\left(\Delta_0\frac{t^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{t^\alpha-s^\alpha}{\alpha}}ds,\\
f_0&=\frac{\Delta_0\left(h_0-e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_0\cosh\left(\Delta_0\dfrac{T^\alpha}{\alpha}\right)+\dfrac{\psi_0+a\varphi_0}{\Delta_0}\sinh\left(\Delta_0\dfrac{T^\alpha}{\alpha}\right)\right)\right)}{\displaystyle\int_0^T\sinh\left(\Delta_0\frac{T^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{T^\alpha-s^\alpha}{\alpha}}ds},
\end{aligned}\tag{3.10}$$

where $\Delta_0=\sqrt{a^2-b^2}$.

According to Theorem 1.4 and Lemma 3.3, the solutions pair $\{u_{k,n}(t), f_{k,n}\}$ of (3.8) are given in the following three cases:

Case 1: If $\dfrac{a^2-b^2}{\omega} < \mu_1^2$, then the solutions of (3.8) are:

$$\begin{aligned}
u_{k,n}(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_{k,n}\cos\left(\delta_n\frac{t^\alpha}{\alpha}\right) + \frac{\psi_{k,n}+a\varphi_{k,n}}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right)\\
&\quad + \frac{f_{k,n}}{\delta_n}\int_0^t \sin\left(\delta_n\frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1}e^{-a\frac{t^\alpha-s^\alpha}{\alpha}}ds,\\
f_{k,n} &= \frac{\delta_n\left(h_{k,n} - e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_{k,n}\cos\left(\delta_n\frac{T^\alpha}{\alpha}\right) + \frac{\psi_{k,n}+a\varphi_{k,n}}{\delta_n}\sin\left(\delta_n\frac{T^\alpha}{\alpha}\right)\right)\right)}{\int_0^T \sin\left(\delta_n\frac{T^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{T^\alpha-s^\alpha}{\alpha}}ds},
\end{aligned} \tag{3.11}$$

where $\delta_n = \sqrt{\omega\mu_n^2 + b^2 - a^2}$.

Case 2: If exist $n_0 \in \mathbb{N}^*$ such that $\mu_{n_0}^2 < \dfrac{a^2-b^2}{\omega} < \mu_{n_0+1}^2$, then the solutions of (3.8) are:

$$\begin{aligned}
\text{for } n \le n_0,\ u_{k,n}(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_{k,n}\cosh\left(\Delta_n\frac{t^\alpha}{\alpha}\right) + \frac{\psi_{k,n}+a\varphi_{k,n}}{\Delta_n}\sinh\left(\Delta_n\frac{t^\alpha}{\alpha}\right)\right)\\
&\quad + \frac{f_{k,n}}{\Delta_n}\int_0^t \sinh\left(\Delta_n\frac{t^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{t^\alpha-s^\alpha}{\alpha}}ds\\
f_{k,n} &= \frac{\Delta_n\left(h_{k,n} - e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_{k,n}\cosh\left(\Delta_n\frac{T^\alpha}{\alpha}\right) + \frac{\psi_{k,n}+a\varphi_{k,n}}{\Delta_n}\sinh\left(\Delta_n\frac{T^\alpha}{\alpha}\right)\right)\right)}{\displaystyle\int_0^T \sinh\left(\Delta_n\frac{T^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{T^\alpha-s^\alpha}{\alpha}}ds}\\
\text{for } n > n_0,\ u_{k,n}(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_{k,n}\cos\left(\delta_n\frac{t^\alpha}{\alpha}\right) + \frac{\psi_{k,n}+a\varphi_{k,n}}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right)\\
&\quad + \frac{f_{k,n}}{\delta_n}\int_0^t \sin\left(\delta_n\frac{t^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{t^\alpha-s^\alpha}{\alpha}}ds\\
f_{k,n} &= \frac{\delta_n\left(h_{k,n} - e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_{k,n}\cos\left(\delta_n\frac{T^\alpha}{\alpha}\right) + \frac{\psi_{k,n}+a\varphi_{k,n}}{\delta_n}\sin\left(\delta_n\frac{T^\alpha}{\alpha}\right)\right)\right)}{\displaystyle\int_0^T \sin\left(\delta_n\frac{T^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{T^\alpha-s^\alpha}{\alpha}}ds},
\end{aligned} \tag{3.12}$$

where $\Delta_n = \sqrt{a^2 - b^2 - \omega\mu_n^2}$.

Case 3: If exist $n_0 \in \mathbb{N}^*$ such that $\dfrac{a^2-b^2}{\omega} = \mu_{n_0}^2$, then the solutions of (3.8) are:

$$
\begin{aligned}
\text{for } n < n_0,\ u_{k,n}(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_{k,n}\cosh\left(\Delta_n\frac{t^\alpha}{\alpha}\right) + \frac{\psi_{k,n}+a\varphi_{k,n}}{\Delta_n}\sinh\left(\Delta_n\frac{t^\alpha}{\alpha}\right)\right)\\
&+\frac{f_{k,n}}{\Delta_n}\int_0^t \sinh\left(\Delta_n\frac{t^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{t^\alpha-s^\alpha}{\alpha}}ds\\
f_{k,n} &= \frac{\Delta_n\left(h_{k,n}-e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_{k,n}\cosh\left(\Delta_n\frac{T^\alpha}{\alpha}\right)+\frac{\psi_{k,n}+a\varphi_{k,n}}{\Delta_n}\sinh\left(\Delta_n\frac{T^\alpha}{\alpha}\right)\right)\right)}{\displaystyle\int_0^T \sinh\left(\Delta_n\frac{T^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{T^\alpha-s^\alpha}{\alpha}}ds}\\
\text{if } n = n_0,\ u_{k,n_0}(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_{k,n_0}+(\psi_{k,n_0}+a\varphi_{k,n_0})\frac{t^\alpha}{\alpha}\right)+f_{k,n_0}\int_0^t \frac{t^\alpha-s^\alpha}{\alpha}e^{-a\frac{t^\alpha-s^\alpha}{\alpha}}s^{\alpha-1}ds\\
f_{k,n_0} &= \frac{h_{k,n_0}-e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_{k,n_0}+(\psi_{k,n_0}+a\varphi_{k,n_0})\frac{T^\alpha}{\alpha}\right)}{\displaystyle\int_0^T \frac{T^\alpha-s^\alpha}{\alpha}e^{-a\frac{T^\alpha-s^\alpha}{\alpha}}s^{\alpha-1}ds}\\
\text{for } n > n_0,\ u_{k,n}(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_{k,n}\cos\left(\delta_n\frac{t^\alpha}{\alpha}\right) + \frac{\psi_{k,n}+a\varphi_{k,n}}{\delta_n}\sin\left(\delta_n\frac{t^\alpha}{\alpha}\right)\right)\\
&+\frac{f_{k,n}}{\delta_n}\int_0^t \sin\left(\delta_n\frac{t^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{t^\alpha-s^\alpha}{\alpha}}ds\\
f_{k,n} &= \frac{\delta_n\left(h_{k,n}-e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_{k,n}\cos\left(\delta_n\frac{T^\alpha}{\alpha}\right)+\frac{\psi_{k,n}+a\varphi_{k,n}}{\delta_n}\sin\left(\delta_n\frac{T^\alpha}{\alpha}\right)\right)\right)}{\displaystyle\int_0^T \sin\left(\delta_n\frac{T^\alpha-s^\alpha}{\alpha}\right)s^{\alpha-1}e^{-a\frac{T^\alpha-s^\alpha}{\alpha}}ds}.
\end{aligned}
\tag{3.13}
$$

Hence, the representation of the solution pair $\{u(x,t), f(x)\}$ of the inverse problem (1), (2), (5), (6) is given in the following four cases:

☞ If $a = b$ this implies that $\dfrac{a^2-b^2}{\omega} < \mu_1^2$, then

$$
\begin{aligned}
u(x,t) &= u_0(t) + \sum_{n=1}^{\infty} u_{1,n}(t)\cos(\mu_n x) + u_{2,n}(t)\sin(\mu_n x),\\
f(x) &= f_0 + \sum_{n=1}^{\infty} f_{1,n}\cos(\mu_n x) + f_{2,n}\sin(\mu_n x),
\end{aligned}
\tag{3.14}
$$

where $\{u_0(t), f_0\}$ and $\{u_{k,n}(t), f_{k,n}\}$ are defined by (3.9) and (3.11) respectively.

☞ If $a > b$ and $\dfrac{a^2 - b^2}{\omega} < \mu_1^2$, then

$$\begin{aligned} u(x,t) &= u_0(t) + \sum_{n=1}^{\infty} u_{1,n}(t)\cos(\mu_n x) + u_{2,n}(t)\sin(\mu_n x), \\ f(x) &= f_0 + \sum_{n=1}^{\infty} f_{1,n}\cos(\mu_n x) + f_{2,n}\sin(\mu_n x), \end{aligned} \tag{3.15}$$

where $\{u_0(t), f_0\}$ and $\{u_{k,n}(t), f_{k,n}\}$ are defined by (3.10) and (3.11) respectively.

☞ If $a > b$ and $\mu_{n_0}^2 < \dfrac{a^2 - b^2}{\omega} < \mu_{n_0+1}^2$, then

$$\begin{aligned} u(x,t) &= u_0(t) + \sum_{n=1}^{n_0} u_{1,n}(t)\cos(\mu_n x) + u_{2,n}(t)\sin(\mu_n x) \\ &\quad + \sum_{n=n_0+1}^{\infty} u_{1,n}(t)\cos(\mu_n x) + u_{2,n}(t)\sin(\mu_n x), \\ f(x) &= f_0 + \sum_{n=1}^{n_0} f_{1,n}\cos(\mu_n x) + f_{2,n}\sin(\mu_n x) + \sum_{n=n_0+1}^{\infty} f_{1,n}\cos(\mu_n x) + f_{2,n}\sin(\mu_n x), \end{aligned} \tag{3.16}$$

where $\{u_0(t), f_0\}$ and $\{u_{k,n}(t), f_{k,n}\}$ are defined by (3.10) and (3.12) respectively.

☞ If $a > b$ and $\dfrac{a^2 - b^2}{\omega} = \mu_{n_0}^2$, then

$$\begin{aligned} u(x,t) &= u_0(t) + \sum_{n=1}^{n_0-1} u_{1,n}(t)\cos(\mu_n x) + u_{2,n}(t)\sin(\mu_n x) + u_{1,n_0}(t)\cos(\mu_{n_0} x) \\ &\quad + u_{2,n_0}(t)\sin(\mu_{n_0} x) + \sum_{n=n_0+1}^{\infty} u_{1,n}(t)\cos(\mu_n x) + u_{2,n}(t)\sin(\mu_n x), \\ f(x) &= f_0 + \sum_{n=1}^{n_0-1} f_{1,n}\cos(\mu_n x) + f_{2,n}\sin(\mu_n x) + f_{1,n_0}\cos(\mu_{n_0} x) + f_{2,n_0}\sin(\mu_{n_0} x) \\ &\quad + \sum_{n=n_0+1}^{\infty} f_{1,n}\cos(\mu_n x) + f_{2,n}\sin(\mu_n x), \end{aligned} \tag{3.17}$$

where $\{u_0(t), f_0\}$ and $\{u_{k,n}(t), f_{k,n}\}$ are defined by (3.10) and (3.13) respectively. In this case, if $n_0 = 1$, we delete the first series from the representations of $u(x,t)$ and $f(x)$.

Step 2: Existence of the solution

To establish the regularity of the solution pair $\{u(x,t), f(x)\}$ of the inverse problem (1), (2), (5), (6), we need to show $f(x)$ is continuous function in $[0,\ell]$ and $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $\mathcal{D}_t^{(\alpha)}u(x,t)$ and $\mathcal{D}_t^{(2\alpha)}u(x,t)$

are continuous functions in Ω_T.

Under the assumptions $(A_1)-(A_3)$, Lemma 3.1 and Lemma 3.3, by using the series (3.14), the following inequalities holds for any $(x,t)\in\Omega_T$ such that

$$\left|f\left(x\right)-f_0\right|\leq\sum_{n=1}^{\infty}\frac{4\left(\delta_n^2+a^2\right)}{\gamma}\left[\frac{\|\varphi\|_{C^4[0,\ell]}}{\mu_n^4}+\frac{\|\psi\|_{C^3[0,\ell]}}{\delta_n\mu_n^3}+\frac{a\,\|\varphi\|_{C^4[0,\ell]}}{\delta_n\mu_n^4}+\frac{\|h\|_{C^4[0,\ell]}}{\mu_n^4}\right], \tag{3.18}$$

where $\gamma=e^{-a\frac{T^\alpha}{\alpha}}\left(e^{a\frac{T^\alpha}{\alpha}}-\left(1+a\frac{T^\alpha}{\alpha}\right)\right)$.

$$\left|u\left(x,t\right)-u_0\left(t\right)\right|\leq\sum_{n=1}^{\infty}\chi_n\left[\frac{\|\varphi\|_{C^4[0,\ell]}}{\delta_n\mu_n^4}+\frac{\|\psi\|_{C^3[0,\ell]}}{\delta_n^2\mu_n^3}+\frac{a\,\|\varphi\|_{C^4[0,\ell]}}{\delta_n^2\mu_n^4}+\frac{\|h\|_{C^4[0,\ell]}}{\delta_n\mu_n^4}\right], \tag{3.19}$$

where $\chi_n=\frac{4}{\gamma}\left(a+2\delta_n+\gamma\delta_n\right)$.

$$\left|u_x\left(x,t\right)\right|\leq\sum_{n=1}^{\infty}\chi_n\left[\frac{\|\varphi\|_{C^4[0,\ell]}}{\delta_n\mu_n^3}+\frac{\|\psi\|_{C^3[0,\ell]}}{\delta_n^2\mu_n^2}+\frac{a\,\|\varphi\|_{C^4[0,\ell]}}{\delta_n^2\mu_n^3}+\frac{\|h\|_{C^4[0,\ell]}}{\delta_n\mu_n^3}\right]. \tag{3.20}$$

$$\left|u_{xx}\left(x,t\right)\right|\leq\sum_{n=1}^{\infty}\chi_n\left[\frac{\|\varphi\|_{C^4[0,\ell]}}{\delta_n\mu_n^2}+\frac{\|\psi\|_{C^3[0,\ell]}}{\delta_n^2\mu_n}+\frac{a\,\|\varphi\|_{C^4[0,\ell]}}{\delta_n^2\mu_n^2}+\frac{\|h\|_{C^4[0,\ell]}}{\delta_n\mu_n^2}\right]. \tag{3.21}$$

$$\left|\mathcal{D}_t^{(\alpha)}u\left(x,t\right)-\mathcal{D}_t^{(\alpha)}u_0\left(t\right)\right|\leq\sum_{n=1}^{\infty}\chi_n\left(a+\delta_n\right)\left[\frac{\|\varphi\|_{C^4[0,\ell]}}{\delta_n\mu_n^4}+\frac{\|\psi\|_{C^3[0,\ell]}}{\delta_n^2\mu_n^3}+\frac{a\,\|\varphi\|_{C^4[0,\ell]}}{\delta_n^2\mu_n^4}+\frac{\|h\|_{C^4[0,\ell]}}{\delta_n\mu_n^4}\right]. \tag{3.22}$$

$$\left|\mathcal{D}_t^{(2\alpha)}u\left(x,t\right)-\mathcal{D}_t^{(2\alpha)}u_0\left(t\right)\right|\leq\sum_{n=1}^{\infty}\chi_n\left(a+\delta_n\right)^2\left[\frac{\|\varphi\|_{C^4[0,\ell]}}{\delta_n\mu_n^4}+\frac{\|\psi\|_{C^3[0,\ell]}}{\delta_n^2\mu_n^3}+\frac{a\,\|\varphi\|_{C^4[0,\ell]}}{\delta_n^2\mu_n^4}+\frac{\|h\|_{C^4[0,\ell]}}{\delta_n\mu_n^4}\right]. \tag{3.23}$$

From (3.18)-(3.23) and by Weierstrass M-test, the series corresponding to $f\left(x\right)$, $u\left(x,t\right)$, $u_x\left(x,t\right)$, $u_{xx}\left(x,t\right)$, $\mathcal{D}_t^{(\alpha)}u\left(x,t\right)$ and $\mathcal{D}_t^{(2\alpha)}u\left(x,t\right)$ are uniformly convergent on $[0,\ell]$ and Ω_T, respectively. Hence, $f\left(x\right)$, $u\left(x,t\right)$, $u_x\left(x,t\right)$, $u_{xx}\left(x,t\right)$, $\mathcal{D}_t^{(\alpha)}u\left(x,t\right)$ and $\mathcal{D}_t^{(2\alpha)}u\left(x,t\right)$ are continuous functions on $[0,\ell]$ and Ω_T, respectively.

Similarly, we show that $f\left(x\right)$ is continuous function on $[0,\ell]$ and $u\left(x,t\right)$, $u_x\left(x,t\right)$, $u_{xx}\left(x,t\right)$, $\mathcal{D}_t^{(\alpha)}u\left(x,t\right)$ and $\mathcal{D}_t^{(2\alpha)}u\left(x,t\right)$ are continuous functions on Ω_T in the other cases. Thus, $\{u\left(x,t\right),f\left(x\right)\}$ satisfies the conditions (1), (2), (5), (6).

Step 3: Uniqueness of the solution

Suppose $\{u_1(x,t), f_1(x)\}$ and $\{u_2(x,t), f_2(x)\}$ are two solution sets of the inverse problem (1), (2), (5), (6). Define $\widetilde{u}(x,t) = u_1(x,t) - u_2(x,t)$ and $\widetilde{f}(x) = f_1(x) - f_2(x)$. The functions $\widetilde{u}(x,t)$ and $\widetilde{f}(x)$ satisfies the following system:

$$\begin{aligned}
&\mathcal{D}_t^{(2\alpha)}\widetilde{u}(x,t) + 2a\mathcal{D}_t^{(\alpha)}\widetilde{u}(x,t) + b^2\widetilde{u}(x,t) = \omega\widetilde{u}_{xx}(x,t) + \widetilde{f}(x), \quad (x,t) \in \Omega_T \\
&\widetilde{u}(x,0) = 0,\ \mathcal{D}_t^{(\alpha)}\widetilde{u}(x,0) = 0,\ 0 \le x \le \ell \\
&\widetilde{u}(0,t) = \widetilde{u}(\ell,t),\ \widetilde{u}_x(0,t) = \widetilde{u}_x(\ell,t),\ 0 \le t \le T \\
&\widetilde{u}(x,T) = 0,\ 0 \le x \le \ell.
\end{aligned} \tag{3.24}$$

By using the orthogonal basis (3.5), we have

$$\begin{aligned}
\widetilde{u}_0(t) &= \int_0^\ell \widetilde{u}(x,t)\,dx, \\
\widetilde{u}_{1,n}(t) &= \int_0^\ell \widetilde{u}(x,t)\cos(\mu_n x)\,dx, \\
\widetilde{u}_{2,n}(t) &= \int_0^\ell \widetilde{u}(x,t)\sin(\mu_n x)\,dx, \\
\widetilde{f}_0 &= \int_0^\ell \widetilde{f}(x)\,dx, \\
\widetilde{f}_{1,n} &= \int_0^\ell \widetilde{f}(x)\cos(\mu_n x)\,dx, \\
\widetilde{f}_{2,n} &= \int_0^\ell \widetilde{f}(x)\sin(\mu_n x)\,dx.
\end{aligned} \tag{3.25}$$

Applying the fractional derivative $\mathcal{D}_t^{(\alpha)}$ and $\mathcal{D}_t^{(2\alpha)}$ to both sides of the first equation in (3.25), we get

$$\mathcal{D}_t^{(\alpha)}\widetilde{u}_0(t) = \int_0^\ell \mathcal{D}_t^{(\alpha)}\widetilde{u}(x,t)\,dx \ \text{ and } \ \mathcal{D}_t^{(2\alpha)}\widetilde{u}_0(t) = \int_0^\ell \mathcal{D}_t^{(2\alpha)}\widetilde{u}(x,t)\,dx,$$

from the boundary conditions of $\widetilde{u}(x,t)$, we have

$$\int_0^\ell \widetilde{u}_{xx}(x,t)\,dx = 0,$$

after using these equations and (3.24), we find that the pair $\left\{\widetilde{u}_0(t), \widetilde{f}_0\right\}$ satisfies the following inverse problem:

$$\begin{cases} \mathcal{D}^{(2\alpha)}\widetilde{u}_0(t) + 2a\mathcal{D}^{(\alpha)}\widetilde{u}_0(t) + b^2\widetilde{u}_0(t) = \widetilde{f}_0,\ 0 \leq t \leq T, \\ \widetilde{u}_0(0) = 0,\ \mathcal{D}^{(\alpha)}\widetilde{u}_0(0) = 0,\ \widetilde{u}_0(T) = 0, \end{cases} \tag{3.26}$$

according to Theorem 1.4, the first component of solution pair $\left\{\widetilde{u}_0(t), \widetilde{f}_0\right\}$ of (3.26) is given in the following two cases:

Case 1: If $a = b$, then

$$\widetilde{u}_0(t) = \widetilde{f}_0 \int_0^t \frac{t^\alpha - s^\alpha}{\alpha} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} s^{\alpha-1} ds.$$

Case 2: If $a > b$, then

$$\widetilde{u}_0(t) = \frac{\widetilde{f}_0}{\Delta_0} \int_0^t \sinh\left(\Delta_0 \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds,$$

where $\Delta_0 = \sqrt{a^2 - b^2}$. By using the condition $\widetilde{u}_0(T) = 0$, we obtain:

$$\widetilde{f}_0 = 0, \text{ this implies that } \widetilde{u}_0(t) = 0.$$

Applying the fractional derivative $\mathcal{D}_t^{(\alpha)}$ and $\mathcal{D}_t^{(2\alpha)}$ to both sides of the second equation in (3.25), we get

$$\mathcal{D}_t^{(\alpha)}\widetilde{u}_{1,n}(t) = \int_0^\ell \mathcal{D}_t^{(\alpha)}\widetilde{u}(x,t)\cos(\mu_n x)\,dx \text{ and } \mathcal{D}_t^{(2\alpha)}\widetilde{u}_{1,n}(t) = \int_0^\ell \mathcal{D}_t^{(2\alpha)}\widetilde{u}(x,t)\cos(\mu_n x)\,dx,$$

the following equation is obtained by using the integration by parts and boundary conditions of $\widetilde{u}(x,t)$

$$\widetilde{u}_{1,n}(t) = -\frac{1}{\mu_n^2}\int_0^\ell \widetilde{u}_{xx}(x,t)\cos(\mu_n x)\,dx,$$

after using these equations and (3.24), we find that the pair $\left\{\widetilde{u}_{1,n}(t), \widetilde{f}_{1,n}\right\}$ satisfies the following inverse problem:

$$\begin{cases} \mathcal{D}^{(2\alpha)}\widetilde{u}_{1,n}(t) + 2a\mathcal{D}^{(\alpha)}\widetilde{u}_{1,n}(t) + \left(b^2 + \omega\mu_n^2\right)\widetilde{u}_{1,n}(t) = \widetilde{f}_{1,n},\ 0 \leq t \leq T, \\ \widetilde{u}_{1,n}(0) = 0,\ \mathcal{D}^{(\alpha)}\widetilde{u}_{1,n}(0) = 0,\ \widetilde{u}_{1,n}(T) = 0, \end{cases} \tag{3.27}$$

according to Theorem 1.4, the first component of solution pair $\left\{\widetilde{u}_{1,n}(t), \widetilde{f}_{1,n}\right\}$ of (3.27) is given in the following three cases:

Case 1: If $\dfrac{a^2-b^2}{\omega} < \mu_1^2$, then

$$\widetilde{u}_{1,n}(t) = \frac{\widetilde{f}_{1,n}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds,$$

where $\delta_n = \sqrt{\omega\mu_n^2 + b^2 - a^2}$.

Case 2: If exist $n_0 \in \mathbb{N}^*$ such that $\mu_{n_0}^2 < \dfrac{a^2-b^2}{\omega} < \mu_{n_0+1}^2$, then

$$\text{for } n \le n_0,\ \widetilde{u}_{1,n}(t) = \frac{\widetilde{f}_{1,n}}{\Delta_n}\int_0^t \sinh\left(\Delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds,$$
$$\text{for } n > n_0,\ \widetilde{u}_{1,n}(t) = \frac{\widetilde{f}_{1,n}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds,$$

where $\Delta_n = \sqrt{a^2 - b^2 - \omega\mu_n^2}$.

Case 3: If exist $n_0 \in \mathbb{N}^*$ such that $\dfrac{a^2-b^2}{\omega} = \mu_{n_0}^2$, then

$$\text{for } n < n_0,\ \widetilde{u}_{1,n}(t) = \frac{\widetilde{f}_{1,n}}{\Delta_n}\int_0^t \sinh\left(\Delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds,$$
$$\text{if } n = n_0,\ \widetilde{u}_{1,n_0}(t) = \widetilde{f}_{1,n_0}\int_0^t \frac{t^\alpha - s^\alpha}{\alpha} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} s^{\alpha-1} ds,$$
$$\text{for } n > n_0,\ \widetilde{u}_{1,n}(t) = \frac{\widetilde{f}_{1,n}}{\delta_n}\int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds.$$

By using the condition $\widetilde{u}_{1,n}(T) = 0$ and Lemma 3.3, we obtain:

$$\widetilde{f}_{1,n} = 0, \ \text{ this implies that } \widetilde{u}_{1,n}(t) = 0, \ \ n = 1, 2, \dots.$$

Similarly, we can show that

$$\widetilde{f}_{2,n} = 0 \ \text{ and } \ \widetilde{u}_{2,n}(t) = 0, \ \ n = 1, 2, \dots.$$

For every $t \in [0, T]$, the functions $\widetilde{u}(x,t)$ and $\widetilde{f}(x)$ are orthogonal to the system of functions (3.5), which forms an orthogonal basis in the space $L^2[0,\ell]$, consequently $\widetilde{u}(x,t) = 0$ and $\widetilde{f}(x) = 0$. □

3.2 Continuous dependence upon the data of the solution

In this section, we give the second main result on continuous dependence upon the data of the solution pair $\{u(x,t), f(x)\}$ of the inverse problem (1), (2), (5), (6).

Let $\Im$ be the set of triples $\{\varphi, \psi, h\}$ where the functions φ, ψ and h satisfies the assumptions $(A_1)-(A_3)$ of Theorem 3.1. For $\phi \in \Im$, we define the norm

$$\|\phi\|_{\Im} := \|\varphi\|_{\mathcal{C}^4[0,\ell]} + \|\psi\|_{\mathcal{C}^3[0,\ell]} + \|h\|_{\mathcal{C}^4[0,\ell]}. \tag{3.28}$$

Theorem 3.2. *The solution $\{u(x,t), f(x)\}$ of the inverse problem (1), (2), (5), (6) under the assumptions of Theorem 3.1, depends continuously upon the data.*

Proof. Let $\{u(x,t), f(x)\}$ and $\left\{\tilde{u}(x,t), \tilde{f}(x)\right\}$ be two solution sets of the inverse problem (1), (2), (5), (6), corresponding to the data $\phi = \{\varphi, \psi, h\}$ and $\tilde{\phi} = \left\{\tilde{\varphi}, \tilde{\psi}, \tilde{h}\right\}$, respectively. We have the representation of the solution of the inverse problem (1), (2), (5), (6) is given in four cases. We prove only the first case ($a = b$ and $\frac{a^2-b^2}{\omega} < \mu_1^2$) and similarly, we prove the other cases.

From (3.9), we have

$$\begin{aligned} u_0(t) - \tilde{u}_0(t) &= e^{-a\frac{t^\alpha}{\alpha}}\left(\varphi_0 - \tilde{\varphi}_0 + \left(\psi_0 - \tilde{\psi}_0 + a(\varphi_0 - \tilde{\varphi}_0)\right)\frac{t^\alpha}{\alpha}\right) + \left(f_0 - \tilde{f}_0\right)\int_0^t \frac{t^\alpha - s^\alpha}{\alpha} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} s^{\alpha-1} ds, \\ f_0 - \tilde{f}_0 &= \frac{h_0 - \tilde{h}_0 - e^{-a\frac{T^\alpha}{\alpha}}\left(\varphi_0 - \tilde{\varphi}_0 + \left(\psi_0 - \tilde{\psi}_0 + a(\varphi_0 - \tilde{\varphi}_0)\right)\frac{T^\alpha}{\alpha}\right)}{\displaystyle\int_0^T \frac{T^\alpha - s^\alpha}{\alpha} e^{-a\frac{T^\alpha - s^\alpha}{\alpha}} s^{\alpha-1} ds}. \end{aligned} \tag{3.29}$$

From (3.29), we obtain:

$$\begin{aligned} \|u_0 - \tilde{u}_0\|_{\mathcal{C}[0,T]} &\leq M_1 \|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + M_1 \left\|\psi - \tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} + M_2 \left|f_0 - \tilde{f}_0\right|, \\ \left|f_0 - \tilde{f}_0\right| &\leq M_3 \left(\|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + \left\|\psi - \tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} + \left\|h - \tilde{h}\right\|_{\mathcal{C}^4[0,\ell]}\right), \end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
M_1 &= 1 + a\frac{T^\alpha}{\alpha} + \frac{T^\alpha}{\alpha},\\
M_2 &= \frac{T^\alpha}{\alpha a} + \frac{2}{a^2},\\
M_3 &= \frac{M_1}{\displaystyle\int_0^T \frac{T^\alpha - s^\alpha}{\alpha} e^{-a\frac{T^\alpha - s^\alpha}{\alpha}} s^{\alpha-1} ds}.
\end{aligned}$$

From (3.11), we have

$$\begin{aligned}
u_{k,n}(t) - \tilde{u}_{k,n}(t) &= e^{-a\frac{t^\alpha}{\alpha}} \left((\varphi_{k,n} - \tilde{\varphi}_{k,n}) \cos\left(\delta_n \frac{t^\alpha}{\alpha}\right) + \frac{\psi_{k,n} - \tilde{\psi}_{k,n} + a(\varphi_{k,n} - \tilde{\varphi}_{k,n})}{\delta_n} \sin\left(\delta_n \frac{t^\alpha}{\alpha}\right)\right)\\
&\quad + \frac{f_{k,n} - \tilde{f}_{k,n}}{\delta_n} \int_0^t \sin\left(\delta_n \frac{t^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{t^\alpha - s^\alpha}{\alpha}} ds,
\end{aligned} \tag{3.31}$$

and

$$f_{k,n} - \tilde{f}_{k,n} = \frac{\delta_n \left(h_{k,n} - \tilde{h}_{k,n} - e^{-a\frac{T^\alpha}{\alpha}} \left((\varphi_{k,n} - \tilde{\varphi}_{k,n}) \cos\left(\delta_n \frac{T^\alpha}{\alpha}\right) + \frac{\psi_{k,n} - \tilde{\psi}_{k,n} + a(\varphi_{k,n} - \tilde{\varphi}_{k,n})}{\delta_n} \sin\left(\delta_n \frac{T^\alpha}{\alpha}\right)\right)\right)}{\int_0^T \sin\left(\delta_n \frac{T^\alpha - s^\alpha}{\alpha}\right) s^{\alpha-1} e^{-a\frac{T^\alpha - s^\alpha}{\alpha}} ds}. \tag{3.32}$$

Under the assumptions $(A_1) - (A_3)$ and using Lemma 3.1, Lemma 3.3, (3.31) and (3.32) we obtain:

$$\|u_{k,n} - \tilde{u}_{k,n}\|_{\mathcal{C}[0,T]} \leq \frac{2}{\mu_n^4} \|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + \frac{2}{\delta_n \mu_n^3} \left\|\psi - \tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} + \frac{2a}{\delta_n \mu_n^4} \|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + \frac{2 + \frac{a}{\delta_n}}{\delta_n^2 + a^2} \left| f_{k,n} - \tilde{f}_{k,n} \right|, \tag{3.33}$$

and

$$\left| f_{k,n} - \tilde{f}_{k,n} \right| \leq \frac{\delta_n^2 + a^2}{\gamma} \left(\frac{2}{\mu_n^4} \|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + \frac{2}{\delta_n \mu_n^3} \left\|\psi - \tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} + \frac{2a}{\delta_n \mu_n^4} \|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + \frac{2}{\mu_n^4} \left\|h - \tilde{h}\right\|_{\mathcal{C}^4[0,\ell]} \right), \tag{3.34}$$

where $\gamma = e^{-a\frac{T^\alpha}{\alpha}}\left(e^{a\frac{T^\alpha}{\alpha}} - \left(1 + a\frac{T^\alpha}{\alpha}\right)\right)$. From (3.33) and (3.34), we get

$$\begin{aligned} \|u_{k,n} - \tilde{u}_{k,n}\|_{\mathcal{C}[0,T]} &\leq M_{4,n}\left(\|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + \left\|\psi - \tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} + \left\|h - \tilde{h}\right\|_{\mathcal{C}^4[0,\ell]}\right), \\ \left|f_{k,n} - \tilde{f}_{k,n}\right| &\leq M_{5,n}\left(\|\varphi - \tilde{\varphi}\|_{\mathcal{C}^4[0,\ell]} + \left\|\psi - \tilde{\psi}\right\|_{\mathcal{C}^3[0,\ell]} + \left\|h - \tilde{h}\right\|_{\mathcal{C}^4[0,\ell]}\right), \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} M_{4,n} &= 2\left(\frac{1}{\mu_n^4} + \frac{1}{\delta_n\mu_n^3} + \frac{a}{\delta_n\mu_n^4}\right)\left(1 + \frac{2}{\gamma} + \frac{a}{\gamma\delta_n}\right), \\ M_{5,n} &= \frac{\delta_n^2 + a^2}{\gamma}\left(\frac{2}{\mu_n^4} + \frac{2}{\delta_n\mu_n^3} + \frac{2a}{\delta_n\mu_n^4}\right). \end{aligned}$$

By using (3.28), (3.30) and (3.35), from (3.14) we obtain:

$$\begin{aligned} \|u - \tilde{u}\|_{\mathcal{C}(\Omega_T)} &\leq M_4\left\|\phi - \tilde{\phi}\right\|_{\Im}, \\ \left\|f - \tilde{f}\right\|_{\mathcal{C}[0,\ell]} &\leq M_5\left\|\phi - \tilde{\phi}\right\|_{\Im}, \end{aligned}$$

where

$$\begin{aligned} M_4 &= M_1 + M_2M_3 + 2\sum_{n=1}^{\infty} M_{4,n}, \\ M_5 &= M_3 + 2\sum_{n=1}^{\infty} M_{5,n}, \end{aligned}$$

hence, we get

$$\|u - \tilde{u}\|_{\mathcal{C}(\Omega_T)} + \left\|f - \tilde{f}\right\|_{\mathcal{C}[0,\ell]} \leq M\left\|\phi - \tilde{\phi}\right\|_{\Im},$$

where $M = M_4 + M_5$. Then, the solution of the inverse problem (1), (2), (5), (6) depends continuously upon the data. □

3.3 Examples

In this section, we give two examples of the inverse space-dependent source problem (1), (2), (5), (6).

Example 3.1. We consider $\alpha = \frac{1}{2}, a = \sqrt{2}, b = \frac{1}{2}, \omega = \frac{3}{4}, \ell = 2\pi, T = 1$ and the functions φ, ψ, h are given by:

$$\begin{cases} \varphi\left(x\right) = -\sqrt{2}\pi, \\ \psi\left(x\right) = \sqrt{2}\pi\cos\left(x\right), \\ h\left(x\right) = \sqrt{2}\pi e^{-2\sqrt{2}}\sinh\left(2\right)\cos\left(x\right) - \sqrt{2}\pi. \end{cases}$$

For these data, the assumptions of Theorem 3.1 are satisfied and the coefficients φ_0, $\varphi_{1,n}$, $\varphi_{2,n}$, ψ_0, $\psi_{1,n}$, $\psi_{2,n}$, h_0, $h_{1,n}$ and $h_{2,n}$ are given by:

$$\varphi_0 = -\sqrt{2}\pi, \;\; \varphi_{1,n} = 0, \;\; \varphi_{2,n} = 0, \;\; \psi_0 = 0, \;\; \psi_{1,n} = \begin{cases} \sqrt{2}\pi & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \quad \psi_{2,n} = 0,$$

$$h_0 = -\sqrt{2}\pi, \;\; h_{1,n} = \begin{cases} \sqrt{2}\pi e^{-2\sqrt{2}}\sinh\left(2\right) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \quad h_{2,n} = 0 \;\text{ for all } n \in \mathbb{N}^*.$$

By using (3.10) and (3.12), we obtain:

$$u_0\left(t\right) = -\sqrt{2}\pi, \;\; u_{1,n}\left(t\right) = \begin{cases} \sqrt{2}\pi e^{-2\sqrt{2t}}\sinh\left(2\sqrt{t}\right) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \quad u_{2,n}\left(t\right) = 0,$$

and

$$f_0 = \frac{-\sqrt{2}\pi}{4}, \;\; f_{1,n} = 0, \;\; f_{2,n} = 0.$$

Hence, the solution pair $\{u\left(x,t\right), f\left(x\right)\}$ of the inverse space-dependent source problem is given by:

$$u\left(x,t\right) = \sqrt{2}\pi\left(e^{-2\sqrt{2t}}\sinh\left(2\sqrt{t}\right)\cos\left(x\right) - 1\right), \quad f\left(x\right) = \frac{-\pi}{2\sqrt{2}}.$$

Example 3.2. In this example, we consider $a = 2, b = \omega = 1, \ell = \pi, T = 2, \alpha$ is arbitrary and the functions

φ, ψ, h are given by:

$$\begin{cases} \varphi(x) = -\sin(2x), \\ \psi(x) = 2\cos(2x), \\ h(x) = 2e^{-\frac{2^{1+\alpha}}{\alpha}} \sin\left(\frac{2^{\alpha}}{\alpha}\right)\cos(2x) - \sin(2x). \end{cases}$$

We can easily show that the assumptions of Theorem 3.1 are satisfied, we have:

$$\varphi_0 = 0, \quad \varphi_{1,n} = 0, \quad \varphi_{2,n} = \begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \quad \psi_0 = 0, \quad \psi_{1,n} = \begin{cases} 2 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \quad \psi_{2,n} = 0,$$

$$h_0 = 0, \quad h_{1,n} = \begin{cases} 2e^{-\frac{2^{1+\alpha}}{\alpha}} \sin\left(\frac{2^{\alpha}}{\alpha}\right) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \quad h_{2,n} = \begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \quad \text{for all } \ n \in \mathbb{N}^*$$

then, by using (3.10) and (3.11), we get:

$$u_0(t) = 0, \quad u_{1,n}(t) = \begin{cases} 2e^{-2\frac{t^{\alpha}}{\alpha}} \sin\left(\frac{t^{\alpha}}{\alpha}\right) & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \quad u_{2,n}(t) = \begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases}$$

and

$$f_0 = 0, \quad f_{1,n} = 0, \quad f_{2,n} = \begin{cases} -5 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

Consequently, the solution pair $\{u(x,t), f(x)\}$ of the inverse space-dependent source problem is given by:

$$u(x,t) = 2e^{-2\frac{t^{\alpha}}{\alpha}} \sin\left(\frac{t^{\alpha}}{\alpha}\right)\cos(2x) - \sin(2x), \quad f(x) = -5\sin(2x).$$

CHAPTER 4

FINITE DIFFERENCE APPROXIMATION FOR INVERSE SOURCE PROBLEM FOR A TIME-FRACTIONAL TELEGRAPH EQUATION

In this chapter, we proposed an algorithm to solve the inverse time-dependent source problem (1)-(4) by using the finite difference method. We present two numerical examples to confirm the effectiveness of this algorithm.

Definition 4.1. [21, page 49] A number λ such that there exists a vector $Y \neq 0$ with $AY = \lambda Y$ is called an eigenvalue of the operator A. This vector Y is called an eigenvector corresponding to the given eigenvalue λ.

4.1 Finite Difference Scheme

We consider the inverse problem (1)-(4) consisting of finding the solution pair $\{u(x,t), r(t)\}$. According to Lemma 2.4, this problem is equivalent to the problem (1)-(3), (2.1). Hence, we will consider the inverse problem (1)-(3), (2.1), this problem is discretized in two steps:

Step 1: Semi-discretization in space. We sub-divide the segment $[0, \ell]$ with

$$x_i = ih,\ i = 0, 1, 2, \ldots N_x$$

where $h = \dfrac{\ell}{N_x}$ is step size. We denote by $u_i(t)$ the numerical approximation to $u(x_i, t)$ and $f_i(t) = f(x_i, t)$.

1. The approximation of $u_{xx}(x_i, t)$, is given by

$$\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{h^2},\ i = 1, 2, \ldots, N_x - 1.$$

2. The approximation of the initial conditions (2), are givens by

$$u_i(0) = \varphi_i \text{ and } \mathcal{D}_t^{(\alpha)} u_i(0) = \psi_i,\ i = 0, 1, \dots, N_x.$$

3. The mixed boundary conditions (3), are approximated

$$u_0(t) = u_1(t) \text{ and } u_{N_x}(t) = 0,\ 0 \le t \le T.$$

Now, by using these approximations we obtain the following semi-discretization of the direct problem (1)-(3):

$$\begin{cases} \mathcal{D}_t^{(2\alpha)} u_i(t) + 2a\mathcal{D}_t^{(\alpha)} u_i(t) + b^2 u_i(t) = \omega \dfrac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{h^2} + r(t) f_i(t),\ i = \overline{1, N_x - 1},\ 0 \le t \le T, \\ u_i(0) = \varphi_i,\ \mathcal{D}_t^{(\alpha)} u_i(0) = \psi_i,\ i = 0, 1, 2, \dots, N_x, \\ u_0(t) = u_1(t),\ u_{N_x}(t) = 0,\ 0 \le t \le T. \end{cases} \tag{4.1}$$

Step 2: Total discretization in space-time. According to Theorem 1.1, we have:

$$\mathcal{D}_t^{(\alpha)} u_i(t) = \frac{du_i(t)}{dt} t^{1-\alpha}, \tag{4.2}$$

on the other hand, we have:

$$\frac{du_i(t)}{dt} = \frac{du_i(t)}{d\left(\frac{t^\alpha}{\alpha}\right)} \cdot \frac{d\left(\frac{t^\alpha}{\alpha}\right)}{dt} = \frac{du_i(t)}{d\left(\frac{t^\alpha}{\alpha}\right)} t^{\alpha-1}, \tag{4.3}$$

from (4.2) and (4.3), we obtain:

$$\mathcal{D}_t^{(\alpha)} u_i(t) = \frac{du_i(t)}{d\left(\frac{t^\alpha}{\alpha}\right)}. \tag{4.4}$$

On the other hand, there exist a real function v_i define on $\left[0, \frac{T^\alpha}{\alpha}\right]$, such that:

$$u_i(t) = v_i\left(\frac{t^\alpha}{\alpha}\right), \tag{4.5}$$

from (4.5) we get:

$$\mathcal{D}_t^{(\alpha)} u_i(t) = \mathcal{D}_t^{(\alpha)} v_i\left(\frac{t^\alpha}{\alpha}\right) \text{ and } \mathcal{D}_t^{(2\alpha)} u_i(t) = \mathcal{D}_t^{(2\alpha)} v_i\left(\frac{t^\alpha}{\alpha}\right), \tag{4.6}$$

by using (4.4), we obtain:

$$\mathcal{D}_t^{(\alpha)} v_i\left(\frac{t^\alpha}{\alpha}\right) = \frac{dv_i\left(\frac{t^\alpha}{\alpha}\right)}{d\left(\frac{t^\alpha}{\alpha}\right)} \text{ and } \mathcal{D}_t^{(2\alpha)} v_i\left(\frac{t^\alpha}{\alpha}\right) = \frac{d}{d\left(\frac{t^\alpha}{\alpha}\right)}\left[\frac{dv_i\left(\frac{t^\alpha}{\alpha}\right)}{d\left(\frac{t^\alpha}{\alpha}\right)}\right], \tag{4.7}$$

we put $\frac{t^\alpha}{\alpha} = y$ in (4.7), we get:

$$\frac{dv_i\left(\frac{t^\alpha}{\alpha}\right)}{d\left(\frac{t^\alpha}{\alpha}\right)} = v_i'(y) \text{ and } \frac{d}{d\left(\frac{t^\alpha}{\alpha}\right)}\left[\frac{dv_i\left(\frac{t^\alpha}{\alpha}\right)}{d\left(\frac{t^\alpha}{\alpha}\right)}\right] = v_i''(y)\,,\ 0 \le y \le \frac{T^\alpha}{\alpha}. \tag{4.8}$$

Hence, we put $\dfrac{t^\alpha}{\alpha} = y$ in (4.1), we obtain the following problem equivalent to (4.1):

$$\begin{cases} v_i''(y) + 2av_i'(y) + b^2 v_i(y) = \omega \dfrac{v_{i+1}(y) - 2v_i(y) + v_{i-1}(y)}{h^2} + R(y)\,F_i(y)\,,\ i = \overline{1, N_x - 1},\ 0 \le y \le \dfrac{T^\alpha}{\alpha}, \\ v_i(0) = \varphi_i,\ v_i'(0) = \psi_i,\ i = 0, 1, 2, \ldots, N_x, \\ v_0(y) = v_1(y)\,,\ v_{N_x}(y) = 0,\ 0 \le y \le \frac{T^\alpha}{\alpha}, \end{cases} \tag{4.9}$$

where $R(y) = R\left(\frac{t^\alpha}{\alpha}\right) = r(t)$ and $F_i(y) = F_i\left(\frac{t^\alpha}{\alpha}\right) = f_i(t)$. We sub-divide the segment $\left[0, \frac{T^\alpha}{\alpha}\right]$ with

$$y_j = jk,\ j = 0, 1, 2, \ldots, N_y$$

where $k = \dfrac{T^\alpha}{\alpha N_y}$ is step size. We denote by v_i^j the numerical approximation to $v_i(y_j)$, $R^j = R(y_j)$ and $F_i^j = F_i(y_j)$.

1. The approximation of $v_i'(y_j)$ and $v_i''(y_j)$, are given by

$$\frac{v_i^{j+1} - v_i^{j-1}}{2k} \text{ and } \frac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{k^2},\ i = 0, 1, \ldots, N_x,\ j = 1, 2, \ldots, N_y - 1.$$

2. The boundary conditions of the problem (4.9), are approximated

$$v_0^j = v_1^j \text{ and } v_{N_x}^j = 0,\ j = 0, 1, \ldots, N_y.$$

3. The approximation of the initial conditions of the problem (4.9), are givens by

$$v_i^0 = \varphi_i \text{ and } v_i^1 = k\psi_i + \varphi_i,\ i = 0, 1, \ldots, N_x.$$

Hence, by using these approximations, from (4.9) we obtain the following discretization of the direct problem (1)-(3):

$$\begin{cases} \dfrac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{k^2} + 2a\dfrac{v_i^{j+1} - v_i^{j-1}}{2k} + b^2 v_i^j = \omega\dfrac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{h^2} + R^j F_i^j,\ i = \overline{1, N_x - 1},\ j = \overline{1, N_y - 1}, \\ v_i^0 = \varphi_i,\ v_i^1 = k\psi_i + \varphi_i,\ i = 0, 1, 2, \ldots, N_x, \\ v_0^j = v_1^j,\ v_{N_x}^j = 0,\ j = 0, 1, 2, \ldots, N_y, \end{cases} \tag{4.10}$$

where $R^j = R(y_j)$ and $F_i^j = F_i(y_j)$.

- For $i = 1$ and $j = 1, 2, \ldots, N_y - 1$,

$$\frac{v_1^{j+1} - 2v_1^j + v_1^{j-1}}{k^2} + a\frac{v_1^{j+1} - v_1^{j-1}}{k} + b^2 v_1^j - \omega\frac{v_2^j - v_1^j}{h^2} = R^j F_1^j,$$

- For $i = 2, 3, \ldots, N_x - 2$ and $j = 1, 2, \ldots, N_y - 1$,

$$\frac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{k^2} + a\frac{v_i^{j+1} - v_i^{j-1}}{k} + b^2 v_i^j - \omega\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{h^2} = R^j F_i^j,$$

- For $i = N_x - 1$ and $j = 1, 2, \ldots, N_y - 1$,

$$\frac{v_{N_x-1}^{j+1} - 2v_{N_x-1}^j + v_{N_x-1}^{j-1}}{k^2} + a\frac{v_{N_x-1}^{j+1} - v_{N_x-1}^{j-1}}{k} + b^2 v_{N_x-1}^j - \omega\frac{v_{N_x-2}^j - 2v_{N_x-1}^j}{h^2} = R^j F_{N_x-1}^j.$$

Hence, we have the following scheme:

$$\begin{cases} V_{yy}^{j} + 2aV_{y}^{j} + AV^{j} = \bar{F}^{j}, \ j = \overline{1, N_y - 1}, \\ V^{0} = \Phi, \ V^{1} = \Psi, \end{cases} \tag{4.11}$$

where

$$V^{j} = \begin{pmatrix} v_1^j \\ v_2^j \\ \vdots \\ v_{N_x-1}^j \end{pmatrix}, \ V_y^j = \begin{pmatrix} \dfrac{v_1^{j+1} - v_1^{j-1}}{2k} \\ \dfrac{v_2^{j+1} - v_2^{j-1}}{2k} \\ \vdots \\ \dfrac{v_{N_x-1}^{j+1} - v_{N_x-1}^{j-1}}{2k} \end{pmatrix}, \ V_{yy}^j = \begin{pmatrix} \dfrac{v_1^{j+1} - 2v_1^j + v_1^{j-1}}{k^2} \\ \dfrac{v_2^{j+1} - 2v_2^j + v_2^{j-1}}{k^2} \\ \vdots \\ \dfrac{v_{N_x-1}^{j+1} - 2v_{N_x-1}^j + v_{N_x-1}^{j-1}}{k^2} \end{pmatrix}, \ \bar{F}^j = \begin{pmatrix} R^j F_1^j \\ R^j F_2^j \\ \vdots \\ R^j F_{N_x-1}^j \end{pmatrix}, \tag{4.12}$$

and

$$AV^{j} = \begin{pmatrix} b^2 v_1^j - \omega \dfrac{v_2^j - v_1^j}{h^2} \\ b^2 v_2^j - \omega \dfrac{v_3^j - 2v_2^j + v_1^j}{h^2} \\ b^2 v_3^j - \omega \dfrac{v_4^j - 2v_3^j + v_2^j}{h^2} \\ \vdots \\ b^2 v_{N_x-2}^j - \omega \dfrac{v_{N_x-1}^j - 2v_{N_x-2}^j + v_{N_x-3}^j}{h^2} \\ b^2 v_{N_x-1}^j - \omega \dfrac{v_{N_x-2}^j - 2v_{N_x-1}^j}{h^2} \end{pmatrix}, \ \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{N_x-1} \end{pmatrix}, \ \Psi = \begin{pmatrix} k\psi_1 + \varphi_1 \\ k\psi_2 + \varphi_2 \\ \vdots \\ k\psi_{N_x-1} + \varphi_{N_x-1} \end{pmatrix}. \tag{4.13}$$

Suppose that scheme (4.11) is stable with respect to the right-hand side $\bar{F}^j$, that is, when we make a small change in the right-hand side $\bar{F}^j$, the solution V^j changes only slightly. Therefore, we can approximate R^j. Now, we consider the inverse problem (1)-(3), (2.1), with r is unknown function and h is a given function, from (2.1) we get

$$r(t) = \frac{\mathcal{D}_t^{(2\alpha)} h(t) + 2a\mathcal{D}_t^{(\alpha)} h(t) + b^2 h(t) - \omega u_{xx}(x_*, t)}{f(x_*, t)}, \ 0 \leq t \leq T, \tag{4.14}$$

where, there exist $I \in \{0, 1, \ldots, N_x - 1\}$, such that $x_* = x_I$. The approximation of $u_{xx}(x_*, t)$, is given by

$$\frac{u_{I+1}(t) - 2u_I(t) + u_{I-1}(t)}{h^2}, \tag{4.15}$$

we put $\dfrac{t^\alpha}{\alpha} = y$ in (4.14) and (4.15), by using (4.5) we obtain:

$$R(y) \approx \frac{H''(y) + 2aH'(y) + b^2 H(y) - \omega \dfrac{v_{I+1}(y) - 2v_I(y) + v_{I-1}(y)}{h^2}}{F_I(y)}, \; 0 \leq y \leq \frac{T^\alpha}{\alpha}, \tag{4.16}$$

where $R(y) = R\left(\frac{t^\alpha}{\alpha}\right) = r(t)$, $H(y) = H\left(\frac{t^\alpha}{\alpha}\right) = h(t)$, $H'(y) = \mathcal{D}_t^{(\alpha)} h(t)$ and $H''(y) = \mathcal{D}_t^{(2\alpha)} h(t)$. From (4.16) we get:

$$R^j \approx \widetilde{R}^j = \frac{1}{F_I^j}\left(H_j'' + 2aH_j' + b^2 H_j - \frac{\omega}{h^2}\left(v_{I+1}^j - 2v_I^j + v_{I-1}^j\right)\right), \tag{4.17}$$

where $j = \overline{0, N_y}$. In scheme (4.11), we replace R^j by $\widetilde{R}^j$, this means we made a small change in $\bar{F}^j$, but we must prove the stability of scheme (4.11). Using (4.17), from (4.11) we obtain the following explicit scheme:

$$\begin{cases} v_1^{j+1} = a_1 v_1^j + a_2 v_1^{j-1} + a_3 v_2^j + a_4 \widetilde{R}^j F_1^j, & \text{for } j = \overline{1, N_y - 1}, \\ v_i^{j+1} = a_5 v_i^j + a_2 v_i^{j-1} + a_3 v_{i+1}^j + a_3 v_{i-1}^j + a_4 \widetilde{R}^j F_i^j & \text{for } i = \overline{2, N_x - 2} \text{ and } j = \overline{1, N_y - 1}, \\ v_{N_x-1}^{j+1} = a_5 v_{N_x-1}^j + a_2 v_{N_x-1}^{j-1} + a_3 v_{N_x-2}^j + a_4 \widetilde{R}^j F_{N_x-1}^j & \text{for } j = \overline{1, N_y - 1}, \\ v_i^0 = \varphi_i, \; v_i^1 = k\psi_i + \varphi_i, \;\; i = \overline{0, N_x}, \end{cases} \tag{4.18}$$

where

$$a_1 = \frac{2 - b^2k^2 - \omega k^2/h^2}{1 + ak}, \; a_2 = \frac{ak - 1}{1 + ak}, \; a_3 = \frac{\omega k^2}{h^2(1 + ak)}, \; a_4 = \frac{k^2}{1 + ak} \text{ and } a_5 = \frac{2 - b^2k^2 - 2\omega k^2/h^2}{1 + ak}.$$

The above scheme (4.18) can be written as the following matrix form:

$$\begin{cases} \mathcal{V}^{j+1} = \mathcal{M}\mathcal{V}^j + a_2 \mathcal{V}^{j-1} + a_4 \widetilde{R}^j \mathcal{F}^j, \quad j = \overline{1, N_y - 1} \\ \mathcal{V}^0 = \Phi, \;\; \mathcal{V}^1 = \Psi, \end{cases} \tag{4.19}$$

where $\mathcal{M}$ is matrix given by

$$\mathcal{M} = \begin{pmatrix} a_1 & a_3 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_5 & a_3 & 0 & 0 & \cdots & 0 \\ 0 & a_3 & a_5 & a_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_3 & a_5 & a_3 & 0 \\ 0 & \cdots & 0 & 0 & a_3 & a_5 & a_3 \\ 0 & \cdots & 0 & 0 & 0 & a_3 & a_5 \end{pmatrix}$$

and

$$\mathcal{V}^j = \begin{pmatrix} v_1^j \\ v_2^j \\ \vdots \\ v_{N_x-1}^j \end{pmatrix}, \mathcal{F}^j = \begin{pmatrix} F_1^j \\ F_2^j \\ \vdots \\ F_{N_x-1}^j \end{pmatrix}, \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{N_x-1} \end{pmatrix}, \Psi = \begin{pmatrix} k\psi_1 + \varphi_1 \\ k\psi_2 + \varphi_2 \\ \vdots \\ k\psi_{N_x-1} + \varphi_{N_x-1} \end{pmatrix}.$$

4.2 Stability analysis

The main objective of this section is to study the stability of the finite difference scheme (4.11). According to Theorem 1.7 and Theorem 1.8, to establish the stability of scheme (4.11) with respect to the initial data and right-hand side, we need to show $A = A^* > 0, P = P^* > 0, B \geq 0$ and $P > \frac{1}{4}A$, where A is defined by (4.13) and

$$P = \frac{1}{k^2}E \text{ and } B = 2aE,$$

where E is the identity operator.

Let H_h be the space of all grid functions defined on the grid $\left\{x_i = ih, i = \overline{1, N_x - 1}, h = \dfrac{\ell}{N_x}\right\}$. In [21, page 99], we have

$$\langle Y, Z \rangle = \sum_{i=1}^{N_x-1} Y_i Z_i h, \tag{4.20}$$

is inner product. We can easily show that the inner product (4.20) satisfies all the axioms of the inner product in the space H_h. Let $Y \in H_h$, from (4.13) we have:

$$AY = \begin{pmatrix} b^2 Y_1 - \omega \dfrac{Y_2 - Y_1}{h^2} \\ b^2 Y_2 - \omega \dfrac{Y_3 - 2Y_2 + Y_1}{h^2} \\ b^2 Y_3 - \omega \dfrac{Y_4 - 2Y_3 + Y_2}{h^2} \\ \vdots \\ b^2 Y_{N_x-2} - \omega \dfrac{Y_{N_x-1} - 2Y_{N_x-2} + Y_{N_x-3}}{h^2} \\ b^2 Y_{N_x-1} - \omega \dfrac{Y_{N_x-2} - 2Y_{N_x-1}}{h^2} \end{pmatrix} \quad \text{with} \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{N_x-1} \end{pmatrix}.$$

- We show that the operator A is self-adjoint. Let $Y, Z \in H_h$, then

$$\begin{aligned}
\langle AY, Z \rangle &= \sum_{i=1}^{N_x-1} (AY)_i Z_i h \\
&= \frac{\omega}{h^2} Y_1 Z_1 h + \sum_{i=1}^{N_x-1} b^2 Y_i Z_i h + \sum_{i=1}^{N_x-2} \frac{-\omega}{h^2} Y_{i+1} Z_i h + \sum_{i=2}^{N_x-1} \frac{2\omega}{h^2} Y_i Z_i h + \sum_{i=2}^{N_x-1} \frac{-\omega}{h^2} Y_{i-1} Z_i h \\
&= \frac{\omega}{h^2} Z_1 Y_1 h + \sum_{i=1}^{N_x-1} b^2 Z_i Y_i h + \sum_{i=2}^{N_x-1} \frac{-\omega}{h^2} Z_{i-1} Y_i h + \sum_{i=2}^{N_x-1} \frac{2\omega}{h^2} Z_i Y_i h + \sum_{i=1}^{N_x-2} \frac{-\omega}{h^2} Z_{i+1} Y_i h \\
&= \langle AZ, Y \rangle \\
&= \langle Y, AZ \rangle ,
\end{aligned}$$

then, A is self-adjoint ($A = A^*$).

- Now, we show that the operator A is positive. Let $Y \in H_h - \{0_{H_h}\}$, then

$$\begin{aligned}
\langle AY, Y \rangle &= b^2 Y_1^2 h + \frac{\omega}{h^2} Y_1^2 h - \frac{\omega}{h^2} Y_2 Y_1 h + \sum_{i=2}^{N_x-2} b^2 Y_i^2 h + \frac{\omega}{h^2} \sum_{i=2}^{N_x-2} \left(-Y_{i+1} Y_i + 2Y_i^2 - Y_{i-1} Y_i \right) h \\
&+ \left(\frac{2\omega}{h^2} + b^2 \right) Y_{N_x-1}^2 h - \frac{\omega}{h^2} Y_{N_x-2} Y_{N_x-1} h \\
&= \sum_{i=1}^{N_x-1} b^2 Y_i^2 h + \frac{\omega}{h^2} \sum_{i=2}^{N_x-1} (Y_i - Y_{i-1})^2 h + \frac{\omega}{h^2} Y_{N_x-1}^2 h \\
&\geq \sum_{i=1}^{N_x-1} b^2 Y_i^2 h > 0,
\end{aligned}$$

then, A is positive $A > 0$.

- We show that the operator P is self-adjoint. Let $Y, Z \in H_h$, then

$$\begin{aligned}\langle PY, Z\rangle &= \frac{1}{k^2}\langle EY, Z\rangle \\ &= \frac{1}{k^2}\sum_{i=1}^{N_x-1} Y_i Z_i h \\ &= \langle Y, PZ\rangle ,\end{aligned}$$

then, P is self-adjoint $(P = P^*)$.

- Now, we show that the operator P is positive. Let $Y \in H_h - \{0_{H_h}\}$, then

$$\begin{aligned}\langle PY, Y\rangle &= \frac{1}{k^2}\langle EY, Y\rangle \\ &= \frac{1}{k^2}\sum_{i=1}^{N_x-1} Y_i^2 h > 0,\end{aligned}$$

then, P is positive $P > 0$. Similarly, we show that the operator B is positive $B > 0$.

- Scheme (4.11) is stable for $P > \frac{1}{4}A$, that is, for

$$\frac{1}{k^2} > \frac{\|A\|}{4}, \tag{4.21}$$

the norm $\|A\|$ of the operator A is equal to its greatest eigenvalue.

Lemma 4.1. *The norm of the operator A satisfies the following inequality*

$$\|A\| < b^2 + \frac{4\omega}{h^2} \tag{4.22}$$

Proof. For $Y \in H_h - \{0_{H_h}\}$ and $\lambda \in \mathbb{R}$, we have

$$AY = \lambda Y \iff \begin{pmatrix} b^2 Y_1 - \omega \dfrac{Y_2 - Y_1}{h^2} \\ b^2 Y_2 - \omega \dfrac{Y_3 - 2Y_2 + Y_1}{h^2} \\ b^2 Y_3 - \omega \dfrac{Y_4 - 2Y_3 + Y_2}{h^2} \\ \vdots \\ b^2 Y_{N_x-2} - \omega \dfrac{Y_{N_x-1} - 2Y_{N_x-2} + Y_{N_x-3}}{h^2} \\ b^2 Y_{N_x-1} - \omega \dfrac{Y_{N_x-2} - 2Y_{N_x-1}}{h^2} \end{pmatrix} = \begin{pmatrix} \lambda Y_1 \\ \lambda Y_2 \\ \lambda Y_3 \\ \vdots \\ \lambda Y_{N_x-2} \\ \lambda Y_{N_x-1} \end{pmatrix},$$

then

$$AY = \lambda Y \iff \begin{cases} Y_{i+1} - 2\left(1 + \dfrac{b^2h^2}{2\omega} - \dfrac{\lambda h^2}{2\omega}\right) Y_i + Y_{i-1} = 0, \quad i = \overline{2, N_x - 2}, \\ Y_2 - \left(1 + \dfrac{b^2h^2}{\omega} - \dfrac{\lambda h^2}{\omega}\right) Y_1 = 0, \\ Y_{N_x-2} - 2\left(1 + \dfrac{b^2h^2}{2\omega} - \dfrac{\lambda h^2}{2\omega}\right) Y_{N_x-1} = 0. \end{cases} \tag{4.23}$$

To solve this problem we rely on pages (24-28) of book [21].

1. If $\lambda < b^2$, then $1 + \frac{b^2h^2}{2\omega} - \frac{\lambda h^2}{2\omega} > 1$. Substituting $Y_i = q^i, q \neq 0$ and $1 + \frac{b^2h^2}{2\omega} - \frac{\lambda h^2}{2\omega} = \cosh(\beta), \beta \neq 0$ into (4.23), we get the quadratic equation for q:

$$q^2 - 2\cosh(\beta)\, q + 1 = 0,$$

with $\Delta = 4\left(\cosh^2(\beta) - 1\right) = 4\sinh^2(\beta) > 0$ and roots

$$q_1 = e^{\beta} \text{ and } q_2 = e^{-\beta},$$

then, particular solutions of (4.23) are given by

$$q_1^i = e^{i\beta} \text{ and } q_2^i = e^{-i\beta}$$

hence, the general solution of (4.23) can be written as follows:

$$Y_i = c_1 e^{i\beta} + c_2 e^{-i\beta}, \text{ with } c_1, c_2 \in \mathbb{R},$$

by using the boundary conditions

$$Y_2 - \left(1 + \frac{b^2h^2}{\omega} - \frac{\lambda h^2}{\omega}\right) Y_1 = 0 \text{ and } Y_{N_x-2} - 2\left(1 + \frac{b^2h^2}{2\omega} - \frac{\lambda h^2}{2\omega}\right) Y_{N_x-1} = 0,$$

of the problem (4.23), we find

$$\begin{cases} c_1\left(e^{2\beta} - (2\cosh(\beta) - 1)\, e^{\beta}\right) + c_2\left(e^{-2\beta} - (2\cosh(\beta) - 1)\, e^{-\beta}\right) = 0, \\ c_1\left(e^{(N_x-2)\beta} - 2\cosh(\beta)\, e^{(N_x-1)\beta}\right) + c_2\left(e^{-(N_x-2)\beta} - 2\cosh(\beta)\, e^{-(N_x-1)\beta}\right) = 0, \end{cases}$$

this problem has only one solution, because

$$\begin{vmatrix} e^{2\beta} - (2\cosh(\beta) - 1)\, e^{\beta} & e^{-2\beta} - (2\cosh(\beta) - 1)\, e^{-\beta} \\ e^{(N_x-2)\beta} - 2\cosh(\beta)\, e^{(N_x-1)\beta} & e^{-(N_x-2)\beta} - 2\cosh(\beta)\, e^{-(N_x-1)\beta} \end{vmatrix} = \left(1 - e^{\beta}\right)\left(e^{-N_x\beta} + e^{(N_x-1)\beta}\right) \neq 0,$$

hence $c_1 = c_2 = 0$, this is implies that $Y_i = 0$ for all $i = \overline{1, N_x - 1}$.

2. If $\lambda > b^2 + \dfrac{4\omega}{h^2}$, then $1 + \frac{b^2h^2}{2\omega} - \frac{\lambda h^2}{2\omega} < -1$. Substituting $Y_i = q^i, q \neq 0$ and $1 + \frac{b^2h^2}{2\omega} - \frac{\lambda h^2}{2\omega} = -\cosh(\beta), \beta \neq 0$ into (4.23), we get the quadratic equation for q:

$$q^2 + 2\cosh(\beta)\, q + 1 = 0,$$

with $\Delta = 4\left(\cosh^2(\beta) - 1\right) = 4\sinh^2(\beta) > 0$ and roots

$$q_1 = -e^{-\beta} \text{ and } q_2 = -e^{\beta},$$

then, particular solutions of (4.23) are given by

$$q_1^i = \begin{cases} e^{-i\beta} & \text{if } i \text{ even} \\ -e^{-i\beta} & \text{if } i \text{ odd} \end{cases} \text{ and } q_2^i = \begin{cases} e^{i\beta} & \text{if } i \text{ even} \\ -e^{i\beta} & \text{if } i \text{ odd} \end{cases}$$

hence, the general solution of (4.23) can be written as follows:

$$Y_i = \begin{cases} c_1 e^{-i\beta} + c_2 e^{i\beta} & \text{if } i \text{ even} \\ -c_1 e^{-i\beta} - c_2 e^{i\beta} & \text{if } i \text{ odd} \end{cases}, \text{ with } c_1, c_2 \in \mathbb{R},$$

by using the boundary conditions of the problem (4.23), we get

$$\text{if } N_x \text{ odd}, \begin{cases} -c_1\left(1 + e^{-\beta}\right) - c_2\left(1 + e^{\beta}\right) = 0, \\ c_1 e^{-N_x\beta} + c_2 e^{N_x\beta} = 0, \end{cases}$$

$$\text{if } N_x \text{ even}, \begin{cases} -c_1\left(1 + e^{-\beta}\right) - c_2\left(1 + e^{\beta}\right) = 0, \\ -c_1 e^{-N_x\beta} - c_2 e^{N_x\beta} = 0, \end{cases}$$

these two problems each have only one solution, because

$$\begin{vmatrix} -\left(1+e^{-\beta}\right) & -\left(1+e^{\beta}\right) \\ e^{-N_x\beta} & e^{N_x\beta} \end{vmatrix} = -\begin{vmatrix} -\left(1+e^{-\beta}\right) & -\left(1+e^{\beta}\right) \\ -e^{-N_x\beta} & -e^{N_x\beta} \end{vmatrix} = -2\left(\sinh\left(N_x\beta\right)+\sinh\left(\left(N_x-1\right)\beta\right)\right) \neq 0,$$

hence $c_1 = c_2 = 0$, this is implies that $Y_i = 0$ for all $i = \overline{1, N_x - 1}$.

3. If $\lambda = b^2 + \dfrac{4\omega}{h^2}$. Substituting $Y_i = q^i, q \neq 0$ into (4.23), we get the quadratic equation for q:

$$q^2 + 2q + 1 = 0,$$

with $\Delta = 0$ and roots

$$q_1 = q_2 = -1 = q_0,$$

then, particular solutions of (4.23) are given by

$$q_0^i = (-1)^i \text{ and } iq_0^i = i\,(-1)^i\,,$$

hence, the general solution of (4.23) can be written as follows:

$$Y_i = (c_1 + ic_2)\,(-1)^i\,, \text{ with } c_1, c_2 \in \mathbb{R},$$

by using the boundary conditions of the problem (4.23), we find

$$\text{if } N_x \text{ odd}, \begin{cases} -2c_1 - c_2 = 0, \\ c_1 + N_x c_2 = 0, \end{cases}$$

$$\text{if } N_x \text{ even}, \begin{cases} -2c_1 - c_2 = 0, \\ -c_1 - N_x c_2 = 0, \end{cases}$$

these two problems each have only one solution, because

$$\begin{vmatrix} -2 & -1 \\ 1 & N_x \end{vmatrix} = -\begin{vmatrix} -2 & -1 \\ -1 & -N_x \end{vmatrix} = -2N_x + 1 \neq 0,$$

hence $c_1 = c_2 = 0$, this is implies that $Y_i = 0$ for all $i = \overline{1, N_x - 1}$.

4. If $\lambda = b^2$. Substituting $Y_i = q^i, q \neq 0$ into (4.23), we get the quadratic equation for q:

$$q^2 - 2q + 1 = 0,$$

with $\Delta = 0$ and roots

$$q_1 = q_2 = 1 = q_0,$$

then, particular solutions of (4.23) are given by

$$q_0^i = 1 \text{ and } iq_0^i = i,$$

hence, the general solution of (4.23) can be written as follows:

$$Y_i = c_1 + ic_2, \text{ with } c_1, c_2 \in \mathbb{R},$$

by using the boundary conditions of the problem (4.23), we get

$$\begin{cases} c_2 = 0, \\ -c_1 - N_x c_2 = 0, \end{cases} \Longrightarrow \begin{cases} c_1 = 0, \\ c_2 = 0, \end{cases}$$

this is implies that $Y_i = 0$ for all $i = \overline{1, N_x - 1}$.

Hence, the eigenvalues λ of the operator A are includes in the interval $]\, b^2, b^2 + \frac{4\omega}{h^2}\, [$, therefor

$$\|A\| < b^2 + \frac{4\omega}{h^2}.$$

□

Now, by using (4.21) and (4.22), we get the following condition:

$$k^2\left(\frac{b^2}{4}+\frac{\omega}{h^2}\right)<1.$$

Hence, we obtain the following Theorem:

Theorem 4.1. *Let the condition*

$$k^2\left(\frac{b^2}{4}+\frac{\omega}{h^2}\right)<1, \tag{4.24}$$

be satisfied. Then scheme (4.11) *is stable with respect to the initial data and right-hand side, thus scheme* (4.11) *is conditionally stable.*

4.3 Algorithm and examples

In this section, an algorithm to solve the inverse problem (1)-(4) and two numerical examples are presented to confirm the effectiveness of this algorithm. The algorithm using the finite difference method is as follows:

Algorithm 1 (The finite difference method)

1: **Initializations:**

1. Give the fractional order $0 < \alpha \leq 1$, the coefficients a, b and ω, the values of T, ℓ and x_*.
2. Give the source term $f(x,t)$, the initial conditions $\varphi(x)$ and $\psi(x)$.
3. Give the additional condition $h(t)$ and its fractional derivatives $\mathcal{D}_t^{(\alpha)} h(t)$, $\mathcal{D}_t^{(2\alpha)} h(t)$.
4. Give the step size of space $h = \dfrac{\ell}{N_x}$ such that there is a $I \in \{0, 1, \ldots, N_x - 1\}$ that satisfies the equality $x_* = Ih$.
5. Give the step size $k = \dfrac{T^\alpha}{\alpha N_y}$ such that it satisfies the condition:
$$k^2 \left(\frac{b^2}{4} + \frac{\omega}{h^2} \right) < 1.$$
6. Give the constants:
$$a_1 = \frac{2 - b^2k^2 - \omega k^2/h^2}{1 + ak},\ a_2 = \frac{ak - 1}{1 + ak},\ a_3 = \frac{\omega k^2}{h^2(1 + ak)},\ a_4 = \frac{k^2}{1 + ak},\ a_5 = \frac{2 - b^2k^2 - 2\omega k^2/h^2}{1 + ak}.$$
7. Compute the matrix $\mathcal{M}$ and the vectors $\mathcal{V}^0$ and $\mathcal{V}^1$.

2: **For** $j = 1, 2, \ldots, N_y - 1$**:**

1. Calculate the source term $\mathcal{F}^j$.
2. Calculate the approximation of r at the point $(\alpha jk)^{\frac{1}{\alpha}}$:
$$\widetilde{R}^j = \frac{1}{F_I^j} \left(H_j'' + 2aH_j' + b^2 H_j - \frac{\omega}{h^2} \left(v_{I+1}^j - 2v_I^j + v_{I-1}^j \right) \right). \tag{4.25}$$
3. Calculate the approximation of u at the points $\left(ih, [\alpha(j+1)k]^{\frac{1}{\alpha}} \right), i = \overline{1, N_x - 1}$:
$$\mathcal{V}^{j+1} = \mathcal{M}\mathcal{V}^j + a_2 \mathcal{V}^{j-1} + a_4 \widetilde{R}^j \mathcal{F}^j.$$

3: **Calculate the approximation of r at the points 0 and T by putting $j = 0$ and $j = N_y$ respectively in (4.25).**

Let L_2, L_∞ and MS be the errors defined by:

$$L_2 = \sqrt{h\sum_{i=0}^{N_x}\left(u\left(x_i,t\right)-\widetilde{u}\left(x_i,t\right)\right)^2},$$

$$L_\infty = \max_{0\le i\le N_x}\left|u\left(x_i,t\right)-\widetilde{u}\left(x_i,t\right)\right|,$$

$$\mathrm{MS} = \frac{\sum_{i=0}^{N_x}\left(u\left(x_i,t\right)-\widetilde{u}\left(x_i,t\right)\right)^2}{N_x+1},$$

where u is the exact solution and $\widetilde{u}$ is the numerical solution.

Example 4.1. We consider the inverse problem (1)-(4) where $\alpha = \frac{1}{2}, a = 2, b = 1, \omega = \frac{3}{\pi^2}, \ell = \frac{5}{2}, T = 5$ and $x_* = 2$, the functions f, φ, ψ and h are given by:

$$\begin{cases} f\left(x,t\right) = \left(4\sqrt{t}+5\right)e^{-\sin(2t)}\cos\left(\pi x\right), \\ \varphi\left(x\right) = -2\cos\left(\pi x\right), \\ \psi\left(x\right) = \cos\left(\pi x\right), \\ h\left(t\right) = 8t+2\sqrt{t}-2, \ \ \mathcal{D}_t^{(\alpha)}h\left(t\right) = 8\sqrt{t}+1, \ \ \mathcal{D}_t^{(2\alpha)}h\left(t\right) = 4. \end{cases}$$

The exact solution pair $\{u\left(x,t\right), r\left(t\right)\}$ is given by:

$$u\left(x,t\right) = \left(8t+2\sqrt{t}-2\right)\cos\left(\pi x\right), \quad r\left(t\right) = 8\sqrt{t}e^{\sin(2t)}.$$

We give the step size of space $h = \dfrac{5}{2N_x}$, according to the data in this example, N_x must be a multiple of 5, we take $N_x = 5000$. Then, the condition (4.24) is satisfied for $\dfrac{20}{N_y^2}\left(\dfrac{1}{4}+\dfrac{3}{\pi^2 h^2}\right) < 1$, that is, for $N_y > 4931$. We apply Algorithm 1 for $N_y = 5000$. The results on the error of the function $r\left(t\right)$ are presented as follows:

$$\mathrm{MS} = \frac{\sum_{j=0}^{N_y}\left|r\left(t_j\right)-\widetilde{r}\left(t_j\right)\right|^2}{N_y+1} = 3.4096\mathrm{E}-08,$$

$$L_\infty = \max_{0\le j\le N_y}\left|r\left(t_j\right)-\widetilde{r}\left(t_j\right)\right| = 3.2110\mathrm{E}-04.$$

where $\widetilde{r}$ is the numerical solution, we present the error graph $e_r = r - \widetilde{r}$ in Figure 4.1. In Table 4.1, we present the MS, L_2 and L_∞ errors of $u(x,t)$ at different values of t. The graphs of the exact and numerical solutions of $r(t)$ and $u(x,t)$ are shown in Figure 4.2 and Figure 4.3, respectively.

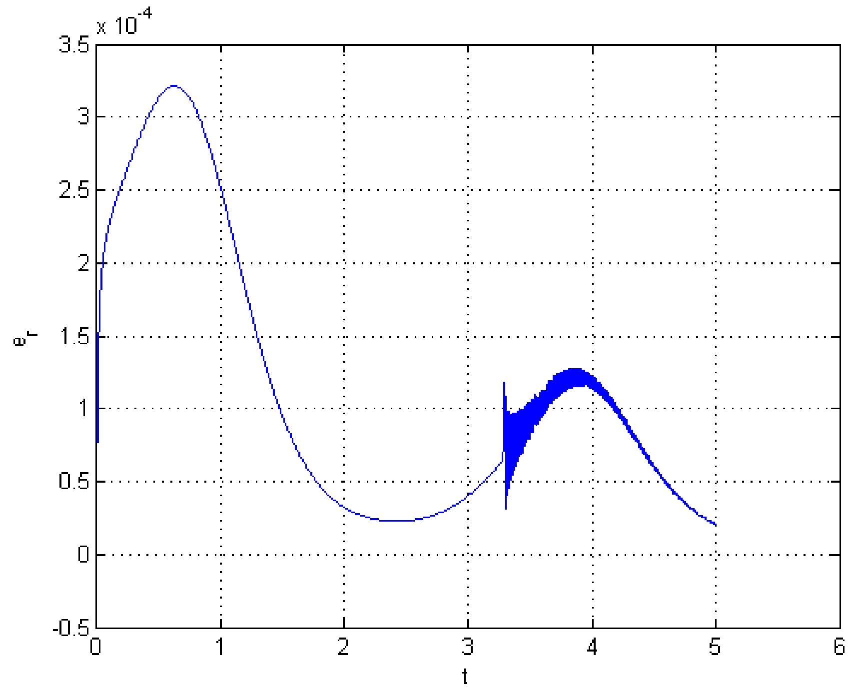

Figure 4.1: The error $e_r = r - \widetilde{r}$

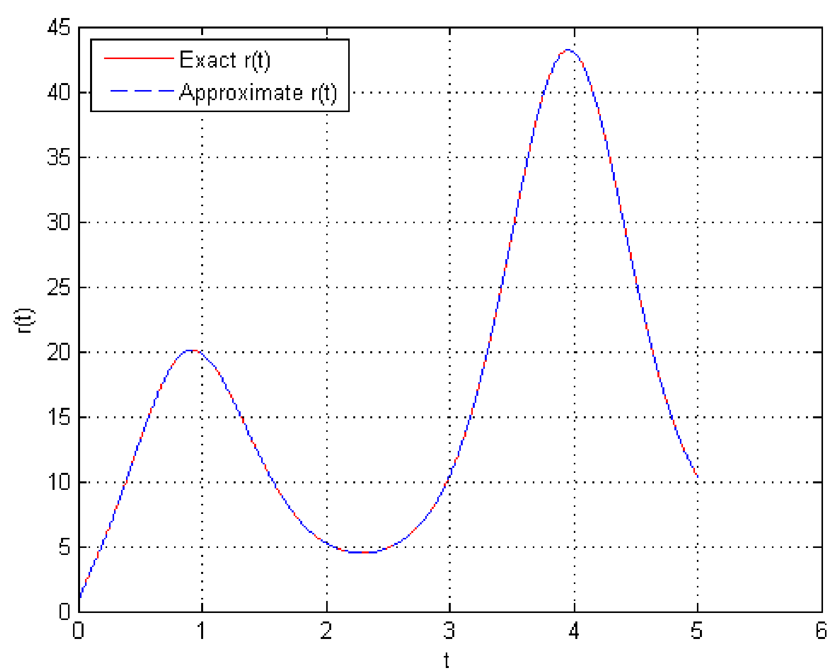

Figure 4.2: The exact and approximate of $r(t)$

Table 4.1: The MS, L_2 and L_∞ errors of the function $u(x,t)$.

t	MS	L_2	L_∞
0.05	5.7460E-08	3.7905E-04	3.6142E-04
0.2	6.1471E-08	3.9206E-04	3.8871E-04
0.45	6.6364E-08	4.0736E-04	8.1058E-04
0.8	2.2287E-07	7.4651E-04	2.4613E-03
1.25	9.0078E-07	1.5008E-03	4.9113E-03
1.8	2.7897E-06	2.6412E-03	8.2018E-03
2.45	6.9750E-06	4.1763E-03	1.2368E-02
3.2	1.5002E-05	6.1248E-03	1.7441E-02
4.05	2.8936E-05	8.5061E-03	2.3448E-02
5	5.1401E-05	1.1337E-02	3.0409E-02

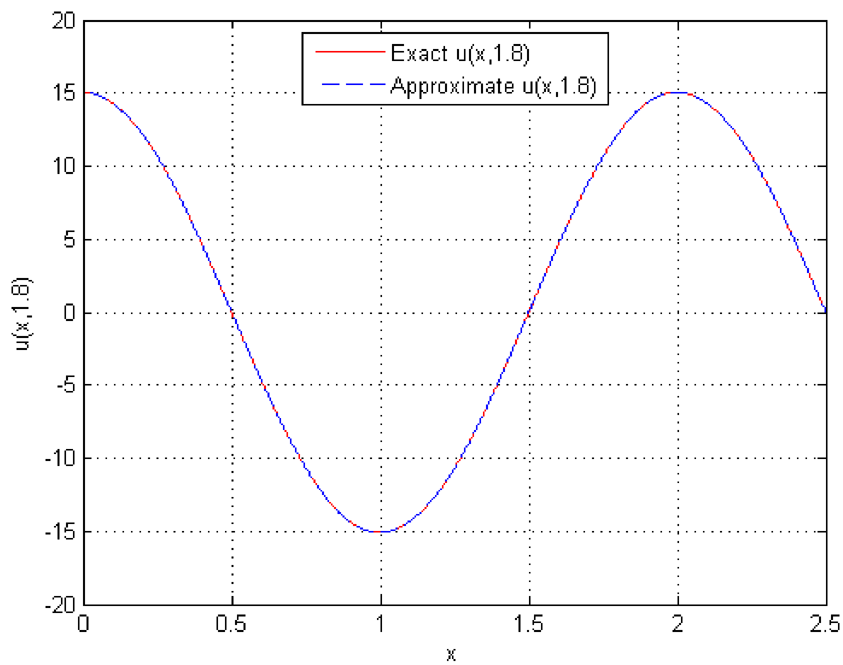

Figure 4.3: The exact and approximate of $u(x,t)$ for $t = 1.8$

Example 4.2. In this example, we consider the inverse problem (1)-(4) with the following data:

$$\begin{cases} \alpha = \frac{3}{4}, \;\; a = 3, \;\; b = 2, \;\; \omega = \frac{4}{\pi^2}, \;\; \ell = 3, \;\; T = 3, \;\; x_* = 2, \\ f\left(x,t\right) = -3e^{-\sin^2\left(4t^{3/4}\right)} \cos\left(\frac{\pi}{2}x\right), \\ \varphi\left(x\right) = \sinh\left(1\right)\cos\left(\frac{\pi}{2}x\right), \\ \psi\left(x\right) = -\cosh\left(1\right)\cos\left(\frac{\pi}{2}x\right), \\ h\left(t\right) = -\sinh\left(1 - \frac{4}{3}t^{3/4}\right), \;\; \mathcal{D}_t^{(\alpha)} h\left(t\right) = \cosh\left(1 - \frac{4}{3}t^{3/4}\right), \;\; \mathcal{D}_t^{(2\alpha)} h\left(t\right) = -\sinh\left(1 - \frac{4}{3}t^{3/4}\right). \end{cases}$$

The exact solution pair $\{u\left(x,t\right), r\left(t\right)\}$ of this problem is:

$$u\left(x,t\right) = \sinh\left(1 - \frac{4}{3}t^{3/4}\right)\cos\left(\frac{\pi}{2}x\right), \;\; r\left(t\right) = 2e^{\frac{4}{3}t^{3/4} - \cos^2\left(4t^{3/4}\right)}.$$

We give the step size of space $h = \dfrac{3}{N_x}$, according to the data in this example, N_x must be a multiple of 3, we take $N_x = 1500$. Hence, the condition (4.24) is satisfied for $\dfrac{16}{\sqrt{3}N_y^2}\left(1 + \dfrac{4}{\pi^2 h^2}\right) < 1$, that is, for $N_y > 967$. We apply Algorithm 1 for $N_y = 1000$. The results on the error of the function $r\left(t\right)$ are presented as follows:

$$\text{MS} = \frac{\sum_{j=0}^{N_y} \left|r\left(t_j\right) - \widetilde{r}\left(t_j\right)\right|^2}{N_y + 1} = 9.1629\text{E} - 09,$$

$$L_\infty = \max_{0 \le j \le N_y} \left|r\left(t_j\right) - \widetilde{r}\left(t_j\right)\right| = 2.2118\text{E} - 04.$$

where $\widetilde{r}$ is the numerical solution, and we present the error graph $e_r = r - \widetilde{r}$ in Figure 4.4. The MS, L_2 and L_∞ errors of $u\left(x,t\right)$ at different values of t are presented in Table 4.2. The graphs of the exact and numerical solutions of $r\left(t\right)$ and $u\left(x,t\right)$ are presented in Figure 4.5 and Figure 4.6, respectively.

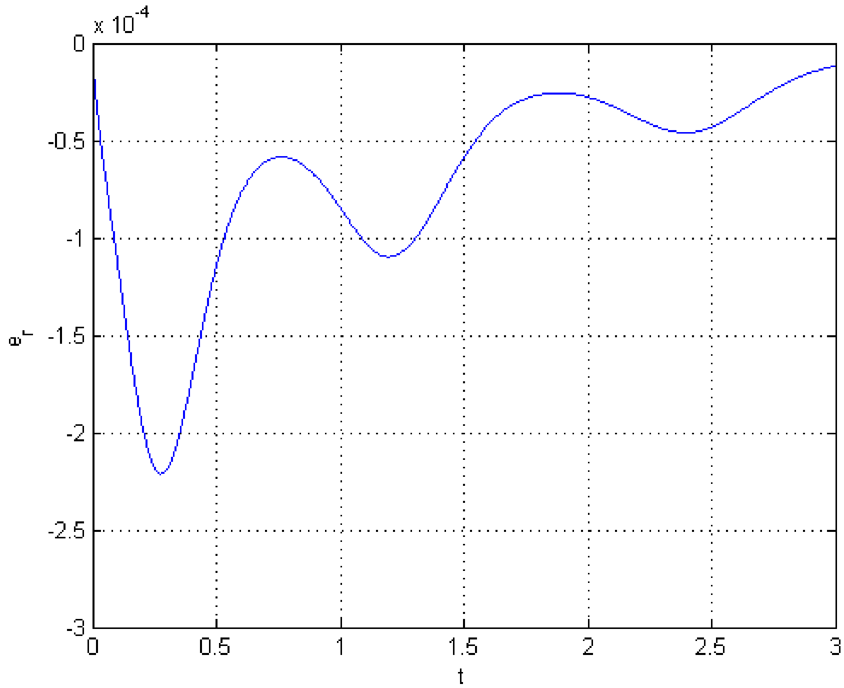

Figure 4.4: The error $e_r = r - \widetilde{r}$

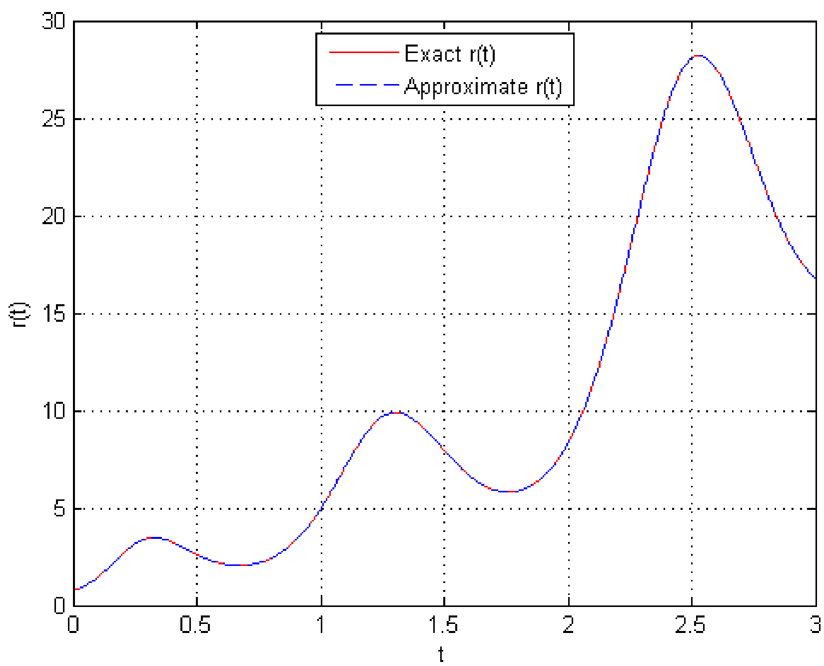

Figure 4.5: The exact and approximate of $r\left(t\right)$

Table 4.2: The MS, L_2 and L_∞ errors of the function $u(x,t)$.

t	MS	L_2	L_∞
0.1875	3.8443E-08	3.3971E-04	5.6840E-04
0.3888	3.9273E-08	3.4336E-04	5.4855E-04
0.7203	2.6193E-08	2.8041E-04	3.7468E-04
1.2288	1.0040E-08	1.7361E-04	1.8565E-04
1.9683	7.2505E-09	1.4753E-04	5.0529E-04
3	8.0810E-08	4.9254E-04	1.6684E-03

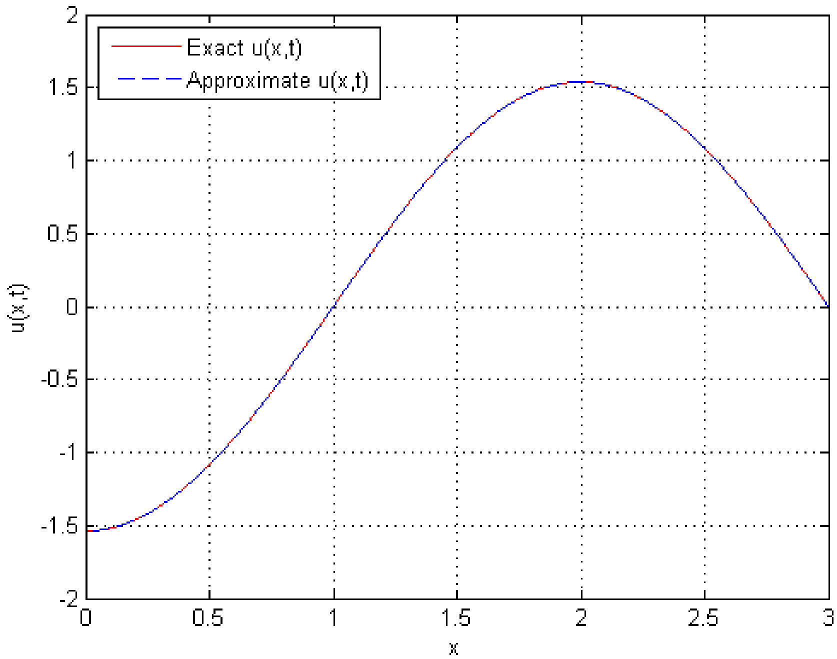

Figure 4.6: The exact and approximate of $u(x,t)$ for $t = 1.9683$

General Conclusion

This thesis investigates two novel inverse source problems for the one-dimensional time-fractional telegraph equation with left-conformable fractional derivatives of order $0 < \alpha \leq 1$. The equation models voltage or current in transmission wires, incorporating electrical parameters via coefficients $\omega = 1/LC$, $2a = G/C + R/L$, and $b^2 = GR/LC$.

We first establish well-posedness of the inverse time-dependent source problem recovering $r(t)$ from a point measurement $u(x_*, t) = h(t)$ using eigenfunctions expansion, Fourier's method, and Banach's contraction principle under mixed boundary conditions. This provides the first such result for conformable fractional telegraph equations.

Second, for the inverse space-dependent source problem recovering $f(x)$ from final-time data $u(x, T) = h(x)$ under periodic boundary conditions, we prove existence, uniqueness, and continuous data dependence via Fourier analysis and conformable derivative properties, with illustrative examples.

Finally, a finite difference algorithm is developed and tested numerically for the time-dependent case. These results advance inverse problem theory for fractional hyperbolic models, with applications in signal propagation and wave phenomena.

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