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Mohamed Boudiaf university of Msila
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Department of Mathematics

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Theme

On bounded linear operators between quasi-normed linear spaces

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Presented by
LAGUEAGUE Ilham

In front of the jury composed of

M^r *HAMIDI* kHALED	M.C.B,	University of BBA	**President**
M^r *TALLAB* Abdelhamid	M.C.A,	University of M'sila	**Supervisor**
M^r *MAZOUZ* Ahmed	M.A.A,	University of M'sila	**Examiner**

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Dedications

In the name of ALLAH the most gracious the most merciful

I dedicate this work to :

The deare

their virtue

My dear sister and my brothers

All my friends and family

All the student

2021/2022.

Acknowledgments

First, I am so grateful to “ **Allah** ” for giving us protection and ability to start and finish this work.

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Contents

Introduction

The past fifty years. The theory of the geometry of Banach spaces has developed very quickly. In other hand the study of quasi-Banach spaces has lagged far behind, even though the first articles on the subject appeared at the beginning 1940's in [7, 8]. In this way, several authors introduced some topological notions and properties in quasi-normed linear spaces, for example, Rano in [2] introduce the concepts of Cauchy sequence, Convergent sequence, Open set, Closed set, etc. in a quasi-metric space and established some basic theorems such as Cantor's intersection theorem and Baire's category theorem in complete quasi-metric spaces. In this memory we detail the work introduced by G. Rano and T. Bag in [**?**] such that they studied the continuity and boundedness of linear operators in quasi-normed linear spaces. Quasi-norm linear space of bounded linear operators is deduced. Concept of dual space is developed. This memory is divided as follows.

The first chapter of quasi-normed linear spaces we shall see the concepts Preliminairs on quasi-normed linear spaces, topological properties, the product quasi-normed linear spaces, quasi-Banach spaces and some examples.

The second chapter called of bounded linear operators in quasi-normed linear spaces, we introduce and study the relationships between the continuity and boundedness of a linear operators between quasi-normed linear spaces, all this in more details.

The third chapter is denoted to Hahn-Banach extension theorem in quasi-normed linear spaces

CHAPTER 1

QUASI-NORMED SPACES

1.1 Quasi-Normed

Definition 1.1.1. Let X be a linear space over the field $\mathbb{K}$ and 0 the origin of X Let $\|.\|_q$: X $\to [0, \infty)$ satisfying the following conditions:

(1) $\|x\|_q = 0$ iff $x = 0$;

(2) $\|\alpha x\|_q$ = $|\alpha|\|x\|_q$ for $x \in X$ and $\alpha \in \mathbb{K}$;

(3) There exists a $K \geq 1$ such that

$$\|x + y\|_q \leq K\{\|x\|_q + \|y\|_q\}$$

for $x, y \in X$.

Then $(X, \|.\|_q)$ is called a quasi-normed linear space (qnls) and the least value of the constant $K \geq 1$ is called the index of the quasi-norm $\|.\|_q$.

Definition 1.1.2. The quasi-normed linear space $(X, \|.\|_q)$ is called a strong quasi-normed linear space (sqnls) if it satisfies the following additional condition:

There exists $K \geqslant 1$ such that:

$$\|\Sigma_{i=1}^n x_i\|_q \leqslant K\{\Sigma_{i=1}^n \|x_i\|_q\}$$

for all $x_i \in X$ and for all $n \in \mathbb{N}$.

Note 1.1.1. *In a quasi-normed linear space $(X, \|.\|_q)$ with quasi index K,*

$$\|\Sigma_{i=1}^n x_i\|_q \leqslant K^{n-1}\{\Sigma_{i=1}^n \|x_i\|_q\}$$

for all $x_i \in X$ and for all $n \in \mathbb{N}$.

Note 1.1.2. *If $K = 1$, then the quasi-norm $\|.\|_q$ is reduced to a norm on X and $(X, \|.\|_q)$ a normed linear space.*

Th following example show that every normed linear space is a quasi-normed linear space but the converse is not true.

Example 1.1.1. Let $X = \mathbb{R}^2$ be a linear space. For $x = (x_1, x_2) \in X$ we define

$$\|x\|_q = (\sqrt{|x_1|} + \sqrt{|x_2|})^2$$

.

Then $(X, \|.\|_q)$is a quasi-normed linear space but not a normed linear space.

Lemma 1.1.1. *Let $x_1, x_2, x_3, ..., x_n$ be a linearly independent set of vectors in a quasi-normed linear space $(X, \|.\|_q)$. Then $\exists C \geq 0$ such that for any choice of scalars $\lambda_1, \lambda_2, ..., \lambda_n$ we have*

$$\|\lambda_1 x_1, \lambda_2 x_2, ..., \lambda_n x_n\|_q \geqslant c(|\lambda_1| + |\lambda_2| + ... + |\lambda_n|).$$

Definition 1.1.3. Let$(X, \|.\|_q)$ be a quasi-normed linear space, we have.

(i) A sequence $\{x_n\}_{n=1^\infty} \subset X$ is said

(a) to converge to$x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{m,n\to\infty} \|x_n - x_m\|_q = 0$

(b) to be a Cauchy sequence if $\lim_{n\to\infty} \|x_n - x\|_q = 0$

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.

(iii) A subset A of X is said to be bounded if there exists a real number $M > 0$ such that $\|x\|_q \leq M$ forall $x \in A$.

(iv) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim_{n\to\infty} x_n = x$ implies $x \in A$.

(v) A subset A of X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A.

Proposition 1.1.1. *Let $(X, \|.\|_q)$ be a quasi-normed linear space. Then*

(a) the limit of a sequence x_n in X if exists is unique;

(b) every subsequence of a convergent sequence converges to the same limit;

(c) every convergent sequence in X is a Cauchy sequence.

1.2 Topological properties and metrizability

Definition 1.2.1. A b-metric on a nonempty set X is a function $d : X \times X \to [0, \infty)$ satisfying the conditions

$(i) d(x, y) = 0 \Leftrightarrow x = y$;

$(ii) d(x, y) = d(y, x)$;

$(iii) d(x, y) \leqslant s[d(x, z) + d(z, y)]$,

for all $x, y, z \in X$, and for some fixed number $s \geq 1$. The pair (X, d) is called a b-metric space. Obviously, for $s = 1$ one obtains a metric on X.

Definition 1.2.2. Let (X, d) be a quasi-metric or b-metric space. As in the case of metric spaces, we have a topology $\tau(d)$ can be defined on (X, d) by considering the family $V_p(x)$ of neighbourhoods from a point $x \in X$:

$V \in V_p(x) \Leftrightarrow \exists r > 0$ such that $B_X, d(x_0, r) \subset V$.

A set $G \subset X$ will be $\tau(d)$-open if for all $x \in G$,there exists $r = r_x > 0$ such that

$$B_X, d(x_0, r) \subset G.$$

Thus, if we consider the family F of subsets U of E, empty or not, such that for all $x \in U$, there exists an open ball centred at x which is contained in U, it is clear that F satisfies the properties of a topology. (E, F) is therefore a topological space. We denote by $\tau(d)$ the topology generated by F on X.

Using the conjugate quasi-metric d^{-1} , we get another topology $\tau(d^{-1})$. A third topology can be obtained by considering the quasi-pseudo-metric d^s, we will denote it $\tau(d^s)$.

Definition 1.2.3. Let (X, d) be quasi-metric space, $x_0 \in X$ et $r > 0$. We define the balls in X by the formulas :

1. The open ball of the center $x_0 \in X$ and of radius r denoted by $B_{x,d}(x_0, r)$ is defined by

$$B_{x,d}(x_0, r) = \{x \in X / d(x_0, x) < r\}.$$

2. The closed ball of a center x_0 and of radius r denoted $B^{'}_{X,d}(x_0, r)$ of (X, d) is defined by :

$$B^{'}_{X,d}(x0, r) = \{x \in X / d(x_0, x) \leq r\}.$$

3. The sphere with center x_0 and radius r denoted by $S_{X,d}(x_0, r)$ is defined by:

$$S_{X,d}(x_0, r) = \{x \in X / d(x_0, x) = r\}.$$

Proposition 1.2.1. *Let (X, d) be a quasi-metric or a b-metric space. Any ball $B_{X,d}(x, r)$ is open for $\tau(d)$ and any ball $B'{}_{X,d}(x, r)$ is closed for $\tau(d^{-1})$. The ball $B^{'}_{X,d}(x, r)$ is not necessarily $\tau(d) - closed.$*

Demonstration:

(a)For all $x \in X$ and $r > 0$, we have for all $y \in B_{X,d}(x, r), r^{'} = r - d(x, y) > 0$et $\forall z \in B_{X,d}(y, r^{'})$:

$$\begin{aligned} d(y, z) &< r^{'} \\ d(x, z) &\leq d(x, y) + d(y, z) \\ &< d(x, y) + r^{'} \\ &< d(x, y) + r - d(x, y) \\ &< r \end{aligned}$$

Then $z \in B_{X,d}(x, r)$. So $B_{X,d}(x, r)$ is $\tau(d)$-open.

(b)$\forall y \in X \ B^{'}_{X,d}(x, r), d(x, y) > r$

Let's pose $r^{'} = d(x, y) - r > 0$ let $z \in B_{X,d^{-1}}(y, r^{'}) so d^{-1}(y, z) < r^{'}$

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) \\ &\leq d(x, z) + d^{-1}(y, z) \\ &< d(x, z) + r^{'} \\ d(x, y) &< d(x, z) + d(x, y) - r. \end{aligned}$$

So $d(x,z) > r$, hence $z \in X/B'_{X,d}(x,r)$; hence $B'_{X,d^{-1}}(y,r') \subset X/B'_{X,d}(x,r)$.

As a result $B'_{X,d^{-1}}(x,r)$ is $\tau(d^{-1})$-closed.

1.3 The quasi product-normed linear space

Theorem 1.3.1. *Let $T : X \to X$ be a quasi-normed linear space from X onto X, where $X \in [0,2]$. T is a quasi-product-normed linear space if, X is a quasi normed space with index $K = (\alpha+\beta)\frac{1}{n}$, in addition to, the following axiom*

$$\|x_1\|\|x_2\| \leq \|x_1\| + \|x_2\|, \forall x_1, x_2 \in X.$$

Proof. We have

$(x+y)^n \geq 0$

$\Rightarrow \langle x,x\rangle^{\frac{n}{2}} +^n C_1\langle x,y\rangle\langle x,x\rangle^{\frac{n-2}{2}} +^n C_2\langle x,x\rangle^{\frac{n-2}{2}}\langle y,y\rangle +^n C_3\langle x,x\rangle^{\frac{n-4}{2}}\langle y,y\rangle\langle x,y\rangle +^n C_4\langle x,x\rangle^{\frac{n-4}{2}}\langle y,y\rangle^2 +^n C_5\langle x,x\rangle^{\frac{n-6}{2}}\langle y,y\rangle^2\langle x,y\rangle +^n C_6\langle x,x\rangle^{\frac{n-6}{2}}\langle y,y\rangle^3 + +^n C_7\langle x,x\rangle^{\frac{n-8}{2}}\langle y,y\rangle^3\langle x,y\rangle +^n C_{\frac{n}{2}}\langle x,x\rangle^{\frac{n}{4}}\langle y,y\rangle^{\frac{n}{4}} + ... + \langle x,x\rangle^{\frac{n}{2}} \geq 0$

$\Rightarrow -^n C_{\frac{n}{2}}\langle x,x\rangle^{\frac{n}{4}}\langle y,y\rangle^{\frac{n}{4}} \leq \{\Rightarrow \langle x,x\rangle^{\frac{n}{2}} +^n C_1\langle x,y\rangle\langle x,x\rangle^{\frac{n-2}{2}} +^n C_2\langle x,x\rangle^{\frac{n-2}{2}}\langle y,y\rangle +^n C_3\langle x,x\rangle^{\frac{n-4}{2}}\langle y,y\rangle\langle x,y\rangle +^n C_4\langle x,x\rangle^{\frac{n-4}{2}}\langle y,y\rangle^2 +^n C_5\langle x,x\rangle^{\frac{n-6}{2}}\langle y,y\rangle^2\langle x,y\rangle +^n C_6\langle x,x\rangle^{\frac{n-6}{2}}\langle y,y\rangle^3 +^n C_7\langle x,x\rangle^{\frac{n-8}{2}}\langle y,y\rangle^3\langle x,y\rangle +^n C_{\frac{n}{2}}\langle x,x\rangle^{\frac{n}{4}}\langle y,y\rangle^{\frac{n}{4}} + ... + \langle x,x\rangle^{\frac{n}{2}}\}$

$\Rightarrow -^n C_{\frac{n}{2}}\langle x,x\rangle^{\frac{n}{4}}\langle y,y\rangle^{\frac{n}{4}} \leq \{\Rightarrow \langle x,x\rangle^{\frac{n}{2}} +^n C_1\langle x,y\rangle\langle x,x\rangle^{\frac{n-2}{2}} +^n C_2\langle x,x\rangle^{\frac{n-2}{2}}\langle y,y\rangle +^n C_3\langle x,x\rangle^{\frac{n-4}{2}}\langle y,y\rangle\langle x,y\rangle +^n C_4\langle x,x\rangle^{\frac{n-4}{2}}\langle y,y\rangle^2 +^n C_5\langle x,x\rangle^{\frac{n-6}{2}}\langle y,y\rangle^2\langle x,y\rangle +^n C_6\langle x,x\rangle^{\frac{n-6}{2}}\langle y,y\rangle^3 +^n C_7\langle x,x\rangle^{\frac{n-8}{2}}\langle y,y\rangle^3\langle x,y\rangle + ... +^n C_{\frac{n}{2}}\langle x,x\rangle^{\frac{n}{4}}\langle y,y\rangle^{\frac{n}{4}} + ... + \langle x,x\rangle^{\frac{n}{2}}\} \Rightarrow$
$-^n C_{\frac{n}{2}}\langle x,x\rangle^{\frac{n}{4}}\langle y,y\rangle^{\frac{n}{4}}\frac{1}{^n C_{\frac{n}{2}}}(x,x)^{\frac{n}{4}}(y,y)^{\frac{n}{4}} \leq (x+y)^n$

$\|-(x,x)^{\frac{n}{2}}(y,y)^{\frac{n}{2}}\| = \|(x+y)^n\|$

$\{\|x\|\|y\|\}^n = \|(x+y)\|^n$

$|x\|\|y\| \leq \|x\| + \|y\|, \forall x, y \in [0.2].$ ☐

Theorem 1.3.2. *The quasi product quasi-complete linear space $(X, \|.\|_{qn})$ is complete.*

Proof. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence in $(X, \|.\|qn)$ and $x \in X$. For every $\epsilon > 0$, there exists an integer $n_0 > N$ such that $\|x_n - x\|_{qn} < \epsilon$, for all $n > n_0$, then $\{X_n\}_{n=1}^{\infty}$ converges x. Thus, $\lim_{n\to\infty} \|x_n - x\|_{qn} = 0$. Also, we see that for every $\epsilon > 0$, there exists integers $m, n > N$ such that $\lim_{m,n\to\infty} \|x_m - x_n\|_{qn} = 0$. In this case, we see that every Cauchy sequence converges to a point $(X, \|.\|qn)$, then $(X, \|.\|qn)$ is a complete quasi-product-normed linear space. □

Remark 1.3.1. The quasi-product normed linear space is a subspace of the quasi-normed linear space, which is also a subspace of the Banach space. In each of these spaces a sequence of points in that space converges to a point in the space. Thus, every Cauchy sequence converges to a point in each of these spaces.

Theorem 1.3.3 (A Unique Fixed Point in Quasi-product normed linear space)**.** *Let*$(X, \|.\|)_{qn}$ *be a quasi-product-normed linear space and* $T : X \to X$ *be a contractive operator, where* $X \in [0, 2]$*. Then* T *has a unique fixed point* $\hat{x} \in X$*.*

Proof. Setting $x_0 \in X$, then

$$X_n = T(X_n) \Rightarrow X_n = T_n(x_0), \forall n \in N$$

Setting $\{X\}_n^{\infty}$ a Cauchy sequence in $(X, \|.\|)_{qn}$. Since $(X, \|.\|)_{qn}$ is a complete, which implies that$x \in X$ such that

$$\lim_{n\to\infty} X_n = x.$$

Again, we see that T is a contractive mapping, then it is continuous. Thus

$$\lim_{n\to\infty} T(X_n) = T(x)$$

We can see that

$$T(x) = \lim_{n\to\infty} T(X_n)$$

$$\Rightarrow T(x) = \lim_{n\to\infty} X_{n+1}$$

$$T(x) = x.$$

Hence, x is a fixed point of the operator T.

In addition, we show that the fixed point is unique $(X, \|?\|)_{qn}$. By contradiction, we set $x_1, x_2 \in X$ be the fixed points of T , then

$$\|T(x_1) - T(x_2)\|_{qn} = \|x_1 - x_2\|_{qn}$$

Also, we see that:

$$\|T(x_1) - T(x_2)\|_{qn} \leq k\|x_1 - x_2\|_{qn}, \forall k = 1.$$

so

$$(1 - k)\|x_1 - x_2\|_{qn} \leq 0$$

$$\Rightarrow \|x_1 - x_2\|_{qn} \leq 0$$

The above equation holds if $x_1 = x_2$. Hence, the fixed point x is unique in a quasi-product-normed linear space. □

1.4 Quasi-banach spaces

In this section we based on p-Banach spaces. Where$K \geq 1$ is a constant independent of $x, y \in X$. The smallest possible constant $K = K_X \geq 1$ is called the quasi-triangle constant of $X = (X, \|.\|_q)$ A quasi-norm induces a locally bounded topology on X and conversely any locally bounded topology is given by a quasi-norm. If, in addition, we have for some $0 < p \leqslant 1$,

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$, then a functional$\|.\|$ is called a p-norm. A complete quasi-normed (or p-normed) space is called a quasi-Banach space (or p-Banach space).

Definition 1.4.1. Let E be a quasi-normed vector space, it is said to be a quasi-Banach space if it is complete. That is, E is complete if any Cauchy sequence in E is convergent.

Theorem 1.4.1. *Let$X$ be a quasi-Banach space of type p where $0 < p \leq 2$. Then:*

(1) If $p < 1$ then X is p-normable.

(2) If $p > 1$ then X is normable (i.e., a Banach space).

Theorem 1.4.2. *(completeness of quasi-normed spaces). A quasi-normed space $X = (X, \|.\|_q)$ with a quasi-triangle constant $C \geq 1$ is complete (quasi-Banach space) if and only if for every series such that :*

$$\Sigma_{k=1}^{\infty} C^k \|x_k\| < \infty$$

we have

$$\Sigma_{k=1}^{\infty} x_k \in X$$

and

$$\|\Sigma_{k=1}^{\infty} x_k\| \leq C \Sigma_{k=1}^{\infty} C^k \|x_k\|.$$

Proof. Note that in a quasi-normed space X for natural numbers $n > m$ we have:

$$\|\Sigma_{k=m}^{n} x_k\| \leqslant c(\|x_m\| + \|\Sigma_{k=m+1}^{n} x_k\|)$$

$$\leq c[\|x_m\| + c(\|x_{m+1}\| + \|\Sigma_{k=m+2}^{n} x_k\|)]$$

$$\leq ... \leq \Sigma_{k=m}^{n} c^{k-m+1} \|x_k\|.$$

Let X be a quasi-Banach space and assume that

$$\Sigma_{k=1}^{n} c^k \|x_k\| = S < \infty$$

. For each $n \in N$ let $S_n = \Sigma_{k=1}^{n} x_k$ be the sequence of partial sums. Then:

$$\|s_{n+m} - s_n\| = \|\Sigma_{k=n+1}^{n+m} x_k\| \leq \Sigma_{k=n+1}^{n+m} c^{k-n} \|x_k\|$$

$$\leq \Sigma_{k=n+1}^{n+m} c^k \|x_k\| = \Sigma_{k=1}^{n+m} c^k \|x_k\| - \Sigma_{k=1}^{n} c^k \|x_k\|$$

$$s_n - s_m \to 0$$

where $n, m \to \infty$, imply that $\{S_n\}$ is a Cauchy sequence, and so, convergent in X.

Therefore, $\Sigma_{k=1}^{\infty} x_k = x \in X$. Moreover,

$$\|\Sigma_{k=1}^{n} x_k\| \leq \Sigma_{k=1}^{n} c^k \|x_k\| \leq \Sigma_{k=1}^{\infty} c^k \|x_k\|.$$

So

$$\limsup_{n\to\infty} \|s_n\| \leq \Sigma_{k=1}^{\infty} c^k \|x_k\|$$

from which we obtain:

$$\|x\| \leq c \limsup_{n\to\infty} (\|x - s_n\| + \|s_n\|) \leq c\Sigma_{k=1}^{\infty} c^k \|x_k\|.$$

Conversely, assume that $(x_n)_n$ is a Cauchy sequence in X . Then there is a natural number n_1 such that for all $n > n_1$ we have$\|x_n - x_{n_1}\| < (2C)^{-1}$. We take $n_2 \in N$, such that for all $n > n_2$, we have $\|x_n - x_{n_2}\| < (2C)^{-2}$.

Inductively, we find an increasing sequence $n_1 < n_2 < n_3 < ...$ such that for all $n > n_k$, we have $\|x_n - x_{n_k}\| < (2C)^{-k}$. Then (x_{n_k}) is a subsequence of (x_n) which satisfies

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{(2c)^k}.$$

for $k = 1, 2, ...,$

and

$$\|x_{n_1}\| + \Sigma_{k=1}^{\infty} c\|x_{n_{k+1}} - x_{n_k}\| \leq \|x_{n_1}\| + \Sigma_{k=1}^{\infty} 2^{-k} = \|x_{n_1}\| + 1 < \infty$$

which by the assumption is convergent in X, that is

$$S_{N-1} = x_{n_1} + \Sigma_{k=1}^{N-1} (x_{n_{k+1}} - x_{n_k}) = x_{n_N} \to y \in X$$

as $N \to \infty$. Since

$$\|y - x_n\| \leq c(\|y - x_{nN}\| + \|x_{nN} - x_n\|) \to 0,$$

as $n, N \to \infty$ it follows that (x_n) converges to $y \in X$ and so X is complete.

□

1.5 A finite dimensional quasi-normed space

Definition 1.5.1. Let $(X, \|.\|_q)$ be a quasi-normed linear space. If X is a finite dimensional linear space then $(X, \|.\|_q)$ is called a finite dimensional quasi-normed linear space.

As a normed case, we can see the following theorem.

Theorem 1.5.1. *In a finite-dimensional quasi-normed space, all quasi-norms are equivalent.*

Corollary 1.5.1. *A subset A of a finite dimensional quasi-normed space E is compact , if and only if it is closed bounded.*

CHAPTER 2

BOUNDED LINEAR OPERATORS IN QUASI-NORMED LINEAR SPACE

2.1 Linear operators

Definition 2.1.1. let X and Y be quasi-normed spaces over a field $\mathbb{K}$. We say that: $T : X \to Y$ is a linear operator if T is linear that is,

$$T(x + y) = T(x) + T(y)$$

for all $x, y \in X$ and

$$T(\lambda x) = \lambda T(x)$$

for all $x \in X$ and $\lambda \in \mathbb{K}$). or more simply, if and only if: $\forall x, y \in X, \forall \lambda \in \mathbb{K}$

$$T(\lambda x + y) = \lambda T(x) + T(y).$$

Remark 2.1.1. Let X and Y be any two linear space and $T : X \to Y$ be an operator. Then T is said to be a linear operator if for any $\lambda_1, \lambda_2 \in F$ and for any $x_1; x_2 \in X$

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$$

Note 2.1.1. *The linear operator defined from X to Y form a set than this set we note $L(X, Y)$ which has a structure of vector space on when it is provided with both laws of composition. Usual:*

$$L(X, Y) \times L(X, Y) \to L(X, Y)$$

$$(f, g) \to f + g$$

$$(\lambda, f) \to \lambda.f.$$

Definition 2.1.2. We call kernel of an operator $T \in L(X, Y)$, the subset of X of elements x verifying:

$$T(x) = 0_Y.$$

We note this kernel $Ker(T)$ (Ker comes from Kernel). We also write

$$Ker(T) = \{T^{-1}(0_Y)\} = \{x \in X : T(x) = 0Y\}.$$

The kernel of a linear operator from X to Y is a vector subspace of X.

Note 2.1.2. *Let T be a linear operator from X to Y then T is injective if and only if :*

$$KerT = \{0_X\}.$$

Definition 2.1.3. We call image of an element T of $L(X, Y)$, the sub set denoted by $Im(T)$ of Y consisting of element y of Y having an antecedent in X. We put:

$$Im(T) = \{y \in Y : \exists x \in Y, y = f(x)\}.$$

Definition 2.1.4. We call linear form on a vector space, all operators of the definition of X in Y . We indicate the set of linear forms of defined on X by $X^* = L(X, Y)$ and it is called dual algebraic of X , obviously, X^* has a vector space structure.

2.2 Bounded linear operators in quasi-normed linear space

In the following, we recall that the definition of norm of linear operator between normed linear spaces.

Definition 2.2.1 (Bounded Linear Operator)**.** Let $(X, \|.\|), (Y, \|.\|)$ be the normed linear spaces over same field $\mathbb{K}$(real or complex). Let $T : D(T) \to Y$, (domain $D(T) \subset X$) is a linear operator. The operator T is said to be bounded if there is a real number C such that for all$x \in D(T)$,

$$\|Tx\| \leq C\|x\|$$

Note 2.2.1. *A bounded linear operator maps bounded sets in $D(T)$ onto bounded sets in Y. For $x \neq 0$, smallest value of C is given by*

$$\frac{\|Tx\|}{\|x\|} \leq C$$

but C must be at least as big as supremum of the quantity on 2.2.1, which we denote by $\|T\|_{(p,q)}$ such that,

$$\|T\| = \sup_{x \in D(T), x \neq 0} \frac{\|Tx\|}{\|x\|}$$

$\|T\|$ is termed as quasi-norm of T. Clearly,

$$\|Tx\| \leq \|T\| \|x\|.$$

Now, as the same way of normed case, we introduce the definition of quasi-norm of linear operator between qusi-normed linear spaces introduced by Gobardhan Rano and Tarapada Bag in [**?**].

Definition 2.2.2. Let $(X, \|.\|_{q_1})$ and $(Y, \|.\|_{q_2})$ be two quasi-normed linear spaces and $T : X \to Y$ be a linear operator. Then T is said to be continuous at $x \in X$ if for any sequence x_n of X with $(x_n)_n \to x$ i.e.,

$$\lim_{n \to \infty} \|x_n - x\|_{q_1} = 0$$

implies $T(x_n) \to T(x)$. i.e.,

$$\lim_{n \to \infty} \|T(x_n) - T(x)\|_{q_2} = 0.$$

If T is continuous at each point of X, then T is said to be continuous on X.

Proposition 2.2.1. *] Let $(X, \|.\|_{q_1})$ and $(Y, \|.\|_{q_2})$be two quasi-normed linear spaces and $T : X \to Y$ be an operator. If T is continuous at a point $x \in X$, then T is continuous everywhere on X.*

Definition 2.2.3. Let $(X, \|.\|_{q_1})$ and $(Y, \|.\|_{q_2})$ be two quasi-normed linear spaces and $T : X \to Y$ be an operator. Then T is said to be bounded if $\exists M > 0$ such that

$$\|T(x)\|_{q_2} \leqslant M \|x\|_{q_1}, \forall x \in X.$$

Example 2.2.1. Let $(X, \|.\|_q)$ be a quasi-normed linear spaces and $T : X \to X$ be a linear operator defined by $T(x) = 2x$. Then T is a bounded linear operator. Indeed,

$$\|T(x)\|_q = \|2x\|_q \leq 2\|x\|_q.$$

Theorem 2.2.1. *Let $(X, \|.\|_{q_1})$ and $(Y, \|.\|_{q_2})$ be two quasi-normed linear spaces and $T : X \to Y$ be an operator. Then T is bounded if and only if T is continuous.*

Proof. Let T be a bounded linear operator. Then $\exists M > 0$ such that:

$$\|T(x)\|_{q_2} \leqslant M\|x\|_{q_2}, \forall x \in X.$$

Let $\{x_n\}$ be any sequence in X with $x_n \to x$ i.e.,

$$\lim_{n\to\infty} \|x_n - x\|_{q_1} = 0.$$

Now

$$\|T(x_n) - T(x)\|_{q_2} = \|T(x_n) - x)\|_{q_2}$$

$$\leqslant M\|x_n - x\|_{q_1}$$

$$\Rightarrow \lim_{n\to\infty} \|T(x_n) - T(x)\|_{q_2} = 0$$

$$\{T(x_n)\} \to T(x).$$

So T is continuous.

Conversely, suppose T is a continuous linear operator. We have to prove that T is bounded. If possible suppose T is not bounded linear operator. Then there exists a sequence $\{x_n\}$ in X such that:

$$x_n^{'} = \frac{x_n}{n\|x_n\|_{q_1}}$$

then

$$\|x_n^{'}\|_{q_1} = \frac{1}{n},$$

this implies that

$$\lim_{n\to\infty} \|x_n^{'}\|_{q_1} = 0,$$

hence

$$\lim_{n\to\infty} x_n^{'} = 0$$

Since T is continuous, it follows that:

$$\lim_{n\to\infty} T(x_n^{'}) = T(0)$$

$$\Rightarrow \lim_{n\to\infty} \|T(x_n^{'})\|_{q2} = 0$$

But $\|T(x_n^{'})\|_{q_2} > 1, \forall n \in \mathbb{N}$, which is a contradiction. Hence T is bounded. □

Theorem 2.2.2. *Let $(X, \|.\|_{q_1})$ and $(Y, \|.\|_{q_2})$ be two quasi-normed linear spaces and $T : X \to Y$ be a linear operator. If X is of finite dimensional, then T is bounded (so continuous).*

Proof. Let dim $X = n$ and $\{x_1; x_2; x_3; ...; x_n\}$ be a basis of X.

Let

$$x = \Sigma_{i=1}^{n} \lambda_i x_i \in X$$

then

$$\|T(x)\|_{q_2} = \|T(\Sigma_{i=1}^{n} \lambda_i x_i)\|_{q_2} = \|\Sigma_{i=1}^{n} \lambda_i T(x_i)\|_{q_2}$$

By Note 1.1.1

$$\|\Sigma_{i=1}^{n} \lambda_i T(x_i)\|_{q_2} \leqslant K^{n-1} \Sigma_{i=1}^{n} \|\lambda_i\| \|T(x_i)\|_{q_2}$$

let $M = k^{n-1} Max\{\|T(x_1)\|_{q_2}, \|T(x_2)\|_{q_2}, ..., \|T(x_n)\|_{q_2}\}$,

then

$$\|T(x)\|_{q_2} \leqslant M \Sigma_{i=1}^{n} \|\lambda_i\|$$

By Lemma 1.1.1 , exists $C > 0$ such that :

$$\|x\|_{q_1} = \|\Sigma_{i=1}^{n} \lambda_i x_i\|_{q1} \geqslant C \Sigma_{i=1}^{n} |\lambda_i|$$

we have

$$\|T(x)\|_{q_2} \leqslant \frac{M}{C} \|x\|_{q_1}$$

Since Lemma 1.1.1 holds for any arbitrary scalars$\lambda_1; \lambda_2; ...; \lambda_n$ we have

$$\|T(x)\|_{q_2} \leqslant \frac{M}{C}\|x\|_{q_1}, \forall x \in X.$$

Hence T is bounded (so continuous). □

Theorem 2.2.3. *Let $(X, \|.\|_{q_1})$ and $(Y, \|.\|_{q_2})$ be two quasi-normed linear spaces. We denote by $B(X, Y)$ the set of all bounded linear operators from $(X, \|.\|_{q1})$ to $(Y, \|.\|_{q2})$. Then $B(X, Y)$ is also a linear space.*

Proof. Let $T_1; T_2 \in B(X, Y)$ and for $x \in X$

$$(T_1 + T_2)(x) = T_1(x) + T_2(x)$$

$$(\lambda T_1)(x) = \lambda T_1(x)$$

Since T_1,T_2 are bounded,$\exists M > 0, N > 0$ Such that $\forall x \in X$

$$\|T_1(x)\|_{q2} \leqslant M\|x\|_{q1}$$

and

$$\|T_2(x)\|_{q2} \leqslant N\|x\|_{q1}$$

Now $\|(k_1T_1 + k_2T_2)(x)\|_{q2} = \|k_1(T_1(x)) + k_2(T_2(x))\|_{q2}$

$$= \|T_1(K_1(x)) + T_2(K_2(x))\|_{q2}$$

$$\leq K\{\|k_1(T_1(x)) + k_2(T_2(x))\|_{q2}\}$$

Thus

$$\|(k_1T_1 + k_2T_2)(x)\|_{q2} \leq MK\|K_1x\|_{q1} + NK\|K_2x\|_{q1}$$

$$= K(M\|K_1| + N|K_2\|)\|x\|_{q1}$$

$\Rightarrow \|(k_1T_1 + k_2T_2)(x)\|_{q2} \leq P\|x\|_{q1}$ where

$$P = K(M\|K_1\| + N\|K_2\|)$$

thus $(k_1T_1 + k_2T_2) \in B(X, Y)$. □

Theorem 2.2.4. *Let* $(X, \|.\|_{q1})$ *and* $(Y, \|.\|_{q2})$ *be two quasi-normed linear spaces. For* $T \in B(X, Y)$ *we define*

$$\|T\|_q = \sup_{x(\neq 0) in X} \|\frac{T(x)\|_{q_2}}{\|x\|_{q_1}}$$

Then $(B(X, Y), \|.\|_q)$ *is a quasi-normed linear space.*

Proof. Clearly $\|T\|_q \geq 0$ and conditions(*). and(**) are directly followed from definition. For(***), let $T_1; T_2 \in B(X, Y)$, then:

$$\|T_1 + T_2\|_q = \sup_{x(\neq 0)\in X} \frac{\|(T_1 + T_2)(x)\|_{q_2}}{\|x\|_{q_1}}$$

$$= \sup_{x(\neq 0)\in X} \frac{\|(T_1(x) + T_2(x)\|_{q_2}}{\|x\|_{q_1}}$$

$$\leq \sup_{x(\neq 0)\in X} k \frac{\|(T_1(x)\|_{q2} + \|T_2(x)\|_{q_2}}{\|x\|_{q_1}}$$

$$\leq \sup_{x(\neq 0)\in X} k \frac{\|(T_1(x)\|_{q_2}}{\|x\|_{q_1}} + \sup_{x(\neq 0)\in X} k \frac{\|(T_2(x)\|_{q_2}}{\|x\|_{q_1}}.$$

So

$$\|T_1 + T_2\|_q \leq k\{\|T_1\|_q + \|T_2\|_q\}$$

hence $(B(X, Y), \|.\|_q)$ is a quasi-normed linear space. □

Remark 2.2.1. In Theorem 2.2.4, we can also define

$$\|T\|_q = \sup_{x\in X}\{\|T(x)\|_{q2} : \|x\|_{q1} \leq 1\}$$

then $(B(X, Y), \|.\|_q)$ is a quasi-normed linear space and two quasi-norm in two cases are same. Again

$$\|T(x)\|_{q_2} \leq \|T\|_q \|x\|_{q_1}, \forall x \in X.$$

Lemma 2.2.1. *Let*$(X, \|.\|_q)$ *be a quasi-normed linear space and* $\{x_n\}$ *be a sequence in* X *such that:*

$$\lim_{n\to\infty} x_n = x$$

i.e.,

$$\lim_{n\to\infty} \|x_n - x\|_q = 0$$

then

$$\|x\|_q = \| \lim_{n\to\infty} x_n\|_q \leq k \lim_{n\to\infty} \|x_n\|_q.$$

Proof. Indeed

$$\|x\|_q \leq k\{\|x - x_n\|\}_q + \|x_n\|_q\}$$

this implies that

$$\|x\|_q \leq \lim_{n\to\infty} K\{\|x - x_n\|\}_q + \|x_n\|_q\} = k \lim_{n\to\infty} \|x_n\|_q$$

then

$$\|x\|_q = \| \lim_{n\to\infty} x_n\|_q \leq k \lim_{n\to\infty} \|x_n\|_q.$$

□

Theorem 2.2.5. *Let $(X, |.|_{q_1})$ be a quasi-normed linear space and $(Y, |.|_{q_2})$ be a complete quasi-normed linear space. Then $(B(X,Y), |.|_q)$ is a complete quasi-normed. linear space*

Proof. By Theorem 2.2.4, $(B(X,Y), \|.\|_q)$ is a quasi-normed linear space. Next we shall prove that it is complete. Now we consider an arbitrary Cauchy sequence$\{Tn\}$ in $(B(X,Y), \|.\|_q)$ and show that $\{T_n\}$ converges to an operator in $(B(X,Y), \|.\|_q)$. Since $\{Tn\}$ is Cauchy sequence

$$\lim_{m,n\to\infty} \|T_n - T_m\|_q = 0,$$

so corresponding to any $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that

$$\|T_n - T_m\|_q < \epsilon, \forall m, n \geq N(\epsilon)$$

for all $x \in X$ and $m, n \geqslant N(\epsilon)$

$$\|T_n(x) - T_m(x)\|_{q_2} = \|(T_n - T_m)(x)\|_{q_2}$$

$$\leq \|T_n - T_m\|_q \|x\|_{q_1} < \epsilon \|x\|_{q_1}.$$

Now for any fixed $x \in X$ we see that $\{T_n(x)\}$ is a Cauchy sequence in $(Y, \|.\|_{q2})$ Since $(Y, |.|_{q2})$ is complete, $\{T_n(x)\}$ converges, say, $\lim_{n\to\infty} T_n(x) = y$. Clearly, the limit $y \in Y$ depends on the choice of $x \in X$. This defines an operator

$$T : (X, \|.\|_{q1}) \to (Y, \|.\|_{q2}),$$

where$y = T(x)$. We shall show that the operator T is linear.

Let $\alpha, \beta \in \mathbb{K}$ then,

$$T(\alpha x + \beta z) = \lim_{n\to\infty} T_n(\alpha x + \beta z) = \lim_{n\to\infty} T_n(\alpha x) + T_n(\beta Z).$$

Now

$$\lim_{n\to\infty} \|(T_n(\alpha x) + T_n(\beta z)) - (\alpha T(x) + \beta T(z)\|_{q_2}$$

$$\leq \lim_{n\to\infty} k\{|\alpha|\|T_n(x) - T(x)\|_{q_2} + |\beta|\|T_n(z) - T(z)\|_{q_2}\} = 0$$

hence $T(\alpha x) + T(\beta z) = \alpha T(x) + \beta T(z)$.

Next we have to show that T is bounded and

$$\lim_{n\to\infty} T_n = T\|T_n(x) - T(x)\|_{q_2} = \|T_n(x) - \lim_{m\to\infty} T_m(x)\|_{q_2}$$

$$k \lim_{m\to\infty} \|T_n(x) - T_m(x)\|_{q_2}$$

$$k \lim_{m\to\infty} \|T_n - T_m\|_{q_2}$$

$$k\epsilon\|x\|_{q_1} \forall n \geqslant N(\epsilon) \forall x \in X$$

since $T_{N(\epsilon)}$is bounded, there exists$M_{N(\epsilon)} > 0$ such that:

$$\|T_{N(\epsilon)}(x)\|_{q_2} \leq M_{N(\epsilon)}\|x\|_{q_1} \forall x \in X$$

thus

$$\|T(x)\|_{q_2} = |T(x) - T_{N(\epsilon)}(x) + T_{N(\epsilon)}(x)\|_{q_2}$$

$$\leq K\|T(x) - T_{N(\epsilon)}(x)\|_{q_2} + K\|_{N(\epsilon)}(x)\|_{q_2} \forall x \in X$$

$$< K^2\epsilon\|x\|_{q_1} + KM_{N(\epsilon)}\|x\|_{q_1}, \forall x \in X$$

$$= (k^2\epsilon + kM_{N(\epsilon)})\|x\|_{q_1}, \forall x \in X$$

so T is bounded.

Now

$$\|T_n - T\|_q = \sup_{x(\neq 0)\in X} \frac{\|T_n(x) - T(x)\|_{q2}}{\|x\|_{q_1}}$$

then

$$\lim_{n\to\infty} \|T_n - T\|_q = 0,$$

so

$$\lim_{n\to\infty} T_n = T.$$

□

Definition 2.2.4. Let $(X, \|.\|_{q_1})$ and $(Y, \|.\|_{q_2})$ be two quasi-normed linear spaces. For $T \in B(X,Y)$ we define

$$\|T\|_q = \sup_{x(\neq 0)\in X} \frac{\|T(x)\|_{q_2}}{\|x\|_{q_1}}$$

then $(B(X,Y), \|.\|_q)$ is a quasi-normed linear space. The space $(B(X,Y), \|.\|_q)$ is called the Dual space of $(X, \|.\|_{q1})$ if $Y = \mathbb{R}$ and $\|.\|_{q_2} = |.|$. We denote the set of all bounded linear functional defined on $(X, \|.\|_q)$ by $B(X, \|.\|_q)$ which is the Dual space of $(X, \|.\|_q)$.

Example 2.2.2. Let $X = \mathbb{R}^2$ be a linear space. For $x = (x_1, x_2) \in X$ define

$$\|x\|_q = (\sqrt{\|x_1\|} + \sqrt{\|x_2\|})^2.$$

Then $(X, \|.\|_q)$is a quasi-normed linear space let $f : X \to X$ be an operator defined by

$$f(x) = x.a = x_1a_1 + x_2a_2$$

. Then T is a bounded linear functional.

Proof. Now

$$\frac{\|f(x)\|}{\|x\|_q} = \frac{\|x_1a_1 + x_2a_2\|}{(\sqrt{\|x1\|} + \sqrt{\|x2\|})^2}$$

$$\leq \frac{\|x_1\|\|a_1| + |x_2\|\|a_2\|}{(\|x_1\| + \|x_2 + 2\|\sqrt{\|x_1\|\|x_2\|})}$$
$$= \frac{\|x_1\|\|a_1\|}{(\|x_1\| + \|x_2 + 2\|\sqrt{\|x_1\|\|x_2\|})} + \frac{\|x_2\|\|a_2\|}{(\|x_1\| + \|x_2 + 2\|\sqrt{\|x_1\|\|x_2\|})}$$
$$\leq (\|a_1\| + \|a_2\|) \Rightarrow \|f(x)\|(\|a_1\| + \|a_2\|)\|x\|_q, \forall x \in X.$$

□

Remark 2.2.2. In Example 2.2.2, $(X, \|.\|_q)$is a quasi-normed linear space for quasi index $K = 2$ but not a normed linear space . Therefore there exists such type of function which are quasi-norm but not a norm.

Theorem 2.2.6. *Let $(X, \|.\|_q)$ be a quasi-normed linear spaces. Then the Dual space $B(X, \|.\|_q)$ of $(X, \|.\|_q)$ is a complete normed linear space.*

CHAPTER 3

FUNDAMENTAL THEOREMS

3.1 Hahn-Banach extension theorem in quasi-normed linear spaces

Definition 3.1.1. Let X be a linear space and $P : X \to \mathbb{R}^+$ be a function. Then P is called a quasi sub-linear functional on X if the followings hold :

(i) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{R}, \forall x \in X$;

(ii) There exists $K \geq 1$such that

$$p(x + y) \leqslant Kp(x) + p(y), \forall x, y \in X.$$

Theorem 3.1.1. *Let X be any vector space and $p : X \to \mathbb{R}^+$ a quasi sub-linear functional on X. Let f be a linear functional which is defined on a subspace Z of X satisfying $\|f(x)\| \leq p(x) \forall x \in Z$. Then f has a linear extension $\hat{f}$ from Z to Z_1 having higher dimension of Z satisfying $\|\hat{f}(x)\| \leqslant Kp(x) \forall x \in Z_1$ and $\hat{f}(x) = f(x) \forall, x \in Z$.*

Proof. Let $x_0 \in Z - X$ Clearly $x_0 = 0$ and the space Z_1 generated by $Z \bigcup \{x_0\}$is also a subspace of X and has higher dimension than Z. Let $x, y \in Z$, then:

$$f(x) - f(y) = f(x - y) \leq p(x - y)$$

$$= p(x + x_0 - x_0 - y) \leq k\{p(x + x_0) + p(x_0 + y)\}$$

$$\Rightarrow f(x) - kp(x - x_0) \leq f(y) + kp(y + x_0), \forall x, y \in Z$$

$$sup_{x \in Z}\{f(x) - kp(x + x_0)\} \leq inf_{y \in Z}\{f(y) + kp(y + x_0)\}$$

let$\beta \in R$such that :

$$sup_{x \in Z}\{f(x) - kp(x + x_0)\} \leq \beta \leq inf_{y \in Z}\{f(y) + kp(y + x_0)\}$$

let $z \in Z_1$ then z is of the form $z = x + tx_0$, where $t \in R$and $x \in Z$. Clearly this representation is unique. If we define:

$$\hat{f}(z) = f(x) - t\beta$$

Then clearly $\hat{f}$ is a linear functional defined on Z_1 such that:

$$\hat{f}(x) = f(x), \forall x \in Z$$

if $t > 0$ then

$$\hat{f}(z) = t\{f(\frac{x}{t}) - \beta\} \leq tkp(\frac{x}{t} + x_0) = Kp(x + tx_0) = Kp(z).$$

if $t < 0$ then

$$\hat{f}(\frac{x}{t} - \beta) \geq -kp(\frac{x}{t} + x_0) = -\frac{1}{|t|}Kp(x + tx_0) = \frac{1}{t}Kp(z).$$

hence

$$\hat{f}(z) = t\{f(\frac{x}{t}) - \beta\} \leq kp(z)$$

if $t = 0$ then

$$\hat{f}(z) = f(z) \leq p(z) \leq kp(z), \forall z \in Z$$

Now

$$-\hat{f}(z) = \hat{f}(-z) \leq kp(-z) = |-1|kp(z) = kp(z), \forall z \in Z$$

hence

$$\|\hat{f}(z)\| \leq kp(z), \forall z \in Z_1.$$

□

Corollary 3.1.1. *If X is a finite dimensional vector space and $p : X \to \mathbb{R}^+$ a quasi sub-linear functional on X. Let f be a linear functional which is defined on a subspace Z of X satisfying*

$$\|f(x)\| \leq p(x), \forall x \in Z.$$

Then f has a linear extension $\hat{f}$ from Z to X satisfying

$$\|\hat{f}(x)\| \leq K^n p(x), \forall x \in X.$$

and

$$\hat{f}(x) = f(x), \forall x \in Z.$$

Theorem 3.1.2. *Let $(X, \|.\|q)$ be a quasi-normed linear space and f be a bounded linear functional which is defined on a subspace Z of X. Then f has a linear extension $\hat{f}$ from Z to Z_1 which is a higher dimensional subspace of X and bounded on Z_1 satisfying*

$$\|f\|_q \leq \|\hat{f}\|_q \leq K\|f\|_q$$

Proof. Let $p(x) = \|f\|q\|x\|q \forall x \in X$ Then clearly p is a quasi sub-linear functional on X .$f$has a linear extension $\hat{f}$ from Z to Z_1 satisfying :

$$\|\hat{f}(x)\| \leq Kp(x) \forall x \in Z_1$$

$$\Rightarrow \|f(x)\| \leq K\|f\|_q\|x\||_q \forall x \in Z_1$$

Thus $\hat{f}$ is a bounded linear functional on Z_1

and

$$\|\hat{f}\|_q = sup_{x(\neq 0) \in Z_1} \frac{\|\hat{f}(x)\|}{|x|_q}$$

Since $Z$is a subspace of $Z1$ and $\hat{f}(x) = f(x) \forall x \in Z$

$$sup_{x(\neq 0) \in Z_1} \frac{\|\hat{f}(x)\|}{|x|_q} \geq sup_{x(\neq 0) \in Z_1} \frac{\|f(x)\|}{|x|_q}$$

$$\|\hat{f}\|_q \geq \|f\|_q$$

Hence$\|f\|_q \leq \|\hat{f}\|_q \leq K\|f\|_q$. □

Corollary 3.1.2. *If $(X, \|.\|q)$ is a finite dimensional quasi-normed linear space and f be a bounded linear functional which is defined on a subspace Z of X. Then f has a linear extension $\hat{f}$ from Z to X which is bounded on X satisfying*

$$\|f\|_q \leq \|\hat{f}\|_q \leq K^n\|f\|_q$$

Theorem 3.1.3. *Let $(X, \|.\|q)$ be a n-dimensional quasi-normed linear space and $x_0(\neq 0) \in X$. Then there exists a bounded linear functional $\hat{f}$ on X such that $1 \leq \|\hat{f}\|_q \leq K^n$ and $\hat{f}(x_0) = |x_0|_q$.*

Proof. We consider the subspace Z of X consisting of all elements $x = cx_0$ where c is a scalar. On Z we define a linear functional f by $f(x) = f(cx_0) = c\|x_0\|_q$. Then f is bounded since $\|f(x)\| = |c|\|x_0\|_q = \|cx_0\|_q = \|x\|_q$ and $\|f\|_q = 1$, f has a linear extension $\hat{f}$ from Z to X with $\|f\|_q \leq \|f\|_q \leq K^n\|f\|_q$ i.e. $1 \leq \|f\|_q \leq K^n$ and $\hat{f}(x_0) = f(x_0) = \|x_0\|_q$. □

Corollary 3.1.3. *For every $x \in X$ we have*

$$\|x\|_q = sup_{f\neq 0, f\in B(x,Q)} \frac{\|f(x)\|}{\|f\|_q}$$

Hence if x is such that $f(x) = 0$ for $f \in B(X, Q)$, then $x = 0$

Proof. From the above theorem we have, writing x for x_0

$$sup_{f\neq 0, f\in B(x,Q)} \frac{\|f(x)\|}{\|f\|_q} \geq \frac{\|f(x)\|}{\|f\|_q} = \|x\|_q$$

and from$\|f(x)\| \leq \|f\|_q\|x\|_q$ we have

$$sup_{f\neq 0, f\in B(x,Q)} \frac{\|f(x)\|}{\|f\|_q} \leq \|x\|_q$$

Hence proved. □

Theorem 3.1.4. *Let $(X, \|.\|q)$ be a n-dimensional quasi-normed linear space. Let $y_0 \in X - Z$. Let*

$$d_q = inf_{x\in Z}\|y_0 - x\|_q > 0.$$

then there exists a bounded linear functional f on X such that

$(1) f(x) = 0 \forall x \in Z.$

$(2) f(y_0) = 1.$

$(3) 1 \leq \|f\|_q \leq \dfrac{K_n}{d_q}.$

Proof. The subspace $Z + y_0$ is uniquely representable in the form $y = x + ty_0$ where $x \in Z$ and t is real. Let us define a functional ϕ on$\{Z + y_0\}$ by $\phi(y) = t$for $y = x + ty_0 \in \{Z + y_0\}$. Then ϕ is a linear functional on $\{Z + y_0\}$. Also $\phi(x) = 0, \forall x \in Z$ and $\phi(y_0) = 1$. Now

$$\|\phi(y)\| = \|t\| = \frac{|t|\|y\|_q}{\|Y\|_q}$$

$$\frac{\|ty\|_q}{\|Y\|_q} = \frac{\|ty\|_q}{\|x + ty_0\|_q}$$

$$= \frac{\|y\|_q}{\|\frac{x}{t} + y_0\|_q} = \frac{\|y\|_q}{\|y_0 - (-\frac{x}{t})\|_q} \leq \frac{\|y\|_q}{d_q}$$

So ϕ is a bounded linear functional on$\{Z + y_0\}$ and

$$\|\phi\|_q = sup_{y \in \{Z+y_0\}, \|y\|_q \leq 1}\{\|\phi(y)\|\} \leq \frac{1}{d_q}$$

since

$$d_q = inf_{x \in Z}\|y_0 - x\|_q.$$

there exists a sequence $\{x_n\}$in Z such that

$$\Rightarrow \lim_{n\to\infty} \|x_n - y_0\|_q = d_q.......(i)$$

Now$\|\frac{\phi(x_n - y_0)}{\|x_n - y_0\|_q}\| \leq \|\phi\|_q$

$$\Rightarrow \|\phi(x_n - y_0)\| \leq |\phi|_q\|x_n - y_0\|_q$$

but

$$\|\phi(x_n - y_0)\| = |\phi(x_n) - \phi(y_0)\| = 1$$

$$\Rightarrow |\phi|_q\|x_n - y_0\|_q \geq 1$$

$$\Rightarrow \lim_{n\to\infty} \|\phi\|_q\|x_n - y_0\|_q = d_q\|\phi\|_q \geq 1$$

$$\lim_{n\to\infty} d_q\|\phi\|_q \geq 1$$

$$\Rightarrow \|\phi\|_q \geq \frac{1}{d_q}....(ii)$$

From (i) and (ii) we have, $|\phi|_q = \frac{1}{d_q}$, ϕ has a linear extension f from $Z + y_0$ to X which is a bounded linear functional on X □

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ملخص:

في هذه المذكرة، ندرس استمرارية وحدود المؤثرات الخطية المعرفة بين الفضاءات الشبه نظيمية التي قدمها Gobardhan Rano و Tarapada Bag. أيضًا ، نقدم بعض النظريات الأساسية لهذه الفئة من المؤثرات الخطية المعرفة بين الفضاءات الشبه نظيمية .

الكلمات المفتاحية: المؤثرات الخطية، الفضاءات الشبه نظيمية، الفضاءات الشبه بناخية، نظرية Hahn-Banach

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Résumé:

Dans ce mémoire, nous étudions la continuité et la bornitude des opérateurs linéaires entre les espaces quasi-normés introduits par Gobardhan Rano et Tarapada Bag. Aussi, nous donnons quelques théorémes fondamentaux pour cette classe d'opérateurs dans ce cas.

mots-clés: Opérateur linéaire; espace quasi-normés, espces quasi-Banach, théorèm

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Abstract:

In this memory, we study the continuity and boundedness of linear operators in quasi-normed linear spaces introduced by Gobardhan Rano and Tarapada Bag. Also, we give some fundamental theorems for this class of operators between quasi-normed linear spaces space.

Key-words: Linear operator, quasi-normed spaces, quasi Banach spaces, Hahn- Banch theorem.