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Some classes of p-summing type operators

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Dedication

To my dear Father **Abde alaziz**, who is currently in the holy land of Mecca,And to my beloved mother. **Fatema** I dedicate this graduation to both of you in recognition of your kindness and support, may Allah keep you and grant you good health and a long life.
I love you and am proud of you.

To all my siblings, May our bond continue to strengthen, and may success and happiness always accompany us on our journey together..

To my beloved little niece **Ranim**, I devote this work to you with all my love and pride. I wish you a bright and happy future..

To my friends and my loved ones in this life.

To you, dear reader.

Contents

Notaions

$\mathbb{K}$	The field of real or complex numbers.
$\mathbb{R}_+$	The field of non negative real numbers.
p'	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p'} = 1$
E^*	The topological dual of E.
B_E	The closed unit ball of E
$L(E;F)$	The set of all linear operators.
$\mathcal{L}(E;F)$	The sets of all continuous linear operators.
w	The weak topology.
w^*	The weak * topology.
$\mathcal{I}$	The ideal of all linear operator.
$\mathcal{I}^{inj}$	The injective of the linear operators ideal $\mathcal{I}$.
T^*	The adjoint operator of T.
$\mathcal{L}_f$	The set of all finite rank linear operators.
$\Pi_p(X,Y)$	The ideal linear p-summing
$\ell_p(\mathcal{L}(X,Y))$	$=\left\{(T_n); T_n \in \mathcal{L}(X,Y), \|(T_n)\| = (\sum_{n=1}^{\infty} \|(T_n)\|^p)^{\frac{1}{p}}\right\}$
$\ell_p^w(\mathcal{L}(X,Y))$	$= \ell_p^w(\mathcal{L}(X,Y) = \left\{(T_n); T_n \in \mathcal{L}(X,Y), \|(T_n)\| = \sup_{\psi \in B_{\mathcal{L}(X,Y)}} (\lvert\langle T_n;\psi\rangle\rvert^p)^{\frac{1}{p}}\right\}$
$\ell_p^s(\mathcal{L}(X,Y))$	The space of strongly p-summable sequences of operators (T_n),
$\Pi_{(\ell_p^s,\ell_p)}$	The space of (ℓ_p^s, ℓ_p)-summing operator

Introduction

In our work we highlighted to the one of the most example of linear operator ideals between normed (or Banach) spaces which have been developed by Albert Pietsch [?]. A linear operator $T : E \longrightarrow F$ is said to be p-summing ($1 \leq p < \infty$), if there exists a constant $C \geq 0$ such that for all finite sequence $(x_i)_{1 \leq i \leq n}$ in E

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*} \leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p \right)^{\frac{1}{p}}. \tag{1}$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p(T)$. The collection of all p-summing operators between E and F is denoted by $\Pi_p(E, F)$.
Let $T : X \longrightarrow Y$ is a bounded linear operator between Banach spaces, the correspondence $\hat{T}$ defined from $\ell_p^{weak}(X)$ to $\ell_p(Y)$ by

$$\hat{T} : (x_n)_n \longmapsto (Tx_n)_n$$

induces a linear operator and it is bounded linear operator in the case precisely when T is p-summing , in this case $\|\hat{T}\| = \pi_p(T)$.

Let X and Y two Banach spaces Oscar Blasco and Teresa Signesin [?] introduced a new sequences spaces $\ell_p^w(X, Y)$, $\ell_p^s(X, Y)$ and $\ell_p(X, Y)$,such that $\sup_{\|x\|=1, \|y^*\|=1} |\langle T_n(x), y^* \rangle| < \infty$, $\sup_{\|x\|=1} \|T_n(x)\| < \infty$,and$\sum \|T_n\| < \infty$, respectively. Let $\Phi : \mathcal{L}(X, Y) \longrightarrow \mathcal{L}(Z, W)$ be a bounded operator. The correspondence

$$\hat{\Phi} : (T_n)_{n=1}^{\infty} \longmapsto (\Phi(T_n))_{n=1}^{\infty}$$

always induces a linear bounded operator from $\ell_p(X, Y)$ to $\ell_p(Z, W)$, as well as from $\ell_p^w(X, Y)$ to $\ell_p^w(Z, W)$. Recall that Φ is p-summing if Φ maps $\ell_p^w(X, Y)$ to $\ell_p(Z, W)$
The main goal of our work is to study several questions concerning the class of operators Φ such

that this vector-valued extension $\hat{\Phi}$ produces a bounded linear operator either from $\ell_p^s(X, Y)$ to $\ell_p(Z, W)$,from $\ell_p^s(X, Y)$ to $\ell_p^s(Z, W)$, from $\ell_p^w(X, Y)$ to $\ell_p^s(Z, W)$

This memory is divided into three chapters as follows ,

The first chapter is a preliminaries which contains some basic concepts and fundamentals of p-Summing Operators

In chapter two we present the spaces $\ell_p(X, Y)$, $\ell_p^w(X, Y)$ and $\ell_p^s(X, Y)$, and we study some properties related them.

The last chapter is devoted to study some classes of p-summing type operators and the relations between these classes

Chapter 1

Preliminaries

In this Chapter we give some concepts we need it in the sequel of this memory and we fixed some notations, based on the famous book of J. Diestel, H. Jarchow and A. Tonge,([**?**])

1.1 Vector valued sequence space

Let $1 \leq p < \infty$. let

$$\mathbf{S} = \{x = (x_n)_n \in \mathbb{N} : x_n \in \mathbb{K}, \forall n \in \mathbb{N}\}$$

S armed with law (+)

$$(x + y) = (x_n)_n + (y_n)_n = (x_n + y_n)_n$$

and the law (.)

$$\lambda x = \lambda(x_n)_n = (\lambda x_n)_n, \lambda \in \mathbb{K}$$

is a vector space.
let the following subspace be

$$\ell_p = \ell_p(\mathbb{K}) = \left\{(x_n)_n \subset \mathbb{K} : \|(x_n)_n\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty\right\}$$
$$\ell_\infty = \ell_\infty(\mathbb{K}) = \left\{(x_n)_n \subset \mathbb{K} : \|(x_n)_n\|_\infty = \sup_n |x_n| < \infty\right\}$$
$$c_0 = c_0(\mathbb{K}) = \left\{(x_n)_n \subset \mathbb{K} : \lim_{n\to\infty} |x_n| = 0\right\}$$

ℓ_p: the space of scalar sequences

ℓ_∞: the space of bounded sequences

c_0: the space of sequences convergent towards 0

Definition 1.1 *The vector sequence* (x_n) *in X is absolutely* p*-summable if the corresponding scalar sequence* $(\|x_n\|)$ *is in* ℓ_p*. The set of all such sequences in X denoted by* $\ell_p(X)$*, this is clearly a vector space under pointwise operation, and a natural norm is given by*

$$\|(x_n)\|_p := \left(\sum_n \|x_n\|^p\right)^{\frac{1}{p}}$$

Example 1.1 *The easiest exemples of members of* $\ell_p(X)$ *are the finite sequences* $(x_1, .., x_m)$*-identified with* $(x_1, ..., x_m, 0, 0, ..)$ *these are readily seen to form a dense subspace of* $\ell_p(X)$*.*

We denote by $\ell_p^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\left\|(x_i)_{i=1}^n\right\|_p = (\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}},$$

Definition 1.2 *The space* $\ell_\infty(X)$ *of bounded sequences in X, and the space* $c_0(X)$ *of norm null sequences in X, are Banach spaces with the norm given by*

$$\|(x_n)_n\|_\infty = \sup_n \|x_n\|.$$

Strong p-summability makes reference to the strong (or norm) tobology on X. what about the natural analogue for the weak topology?

Definition 1.3 *Let* $1 \leq p < \infty$*. A sequence* $(x_n)_n$ *in X is said to be weakly* p*-summable if*

$$\sum_{n=1}^\infty |x^*(x_n)|^p < \infty,$$

for every $x^* \in B_{X^*}$*.*

We denote by $\ell_p^w(X)$ *the Banach space of weakly* p*-summable sequences in X becomes a Banach* *space when equipped with the norm given by*

$$\|(x_n)_n\|_p^w = \sup\left\{ (\sum_{n=1}^{\infty} |x^*(x_n)|^p)^{\frac{1}{p}} : x^* \in B_{X^*} \right\}.$$

We denote by $\ell_{p,\omega}^n(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\left\|(x_i)_{i=1}^n\right\|_{p,\omega} = \sup_{\|\xi\|_{X^*}\leq 1} (\sum_{i=1}^n |\langle x_i, \xi\rangle|^p)^{\frac{1}{p}}.$$

Remark 1.1 *In the case $p = \infty$. Then the space $\ell_\infty^w(X)$ of weakly bounded sequences coincide with the space $\ell_\infty(X)$ and*

$$\|(x_n)_n\|_\infty^w = \|(x_n)_n\|_\infty.$$

Definition 1.4 *A sequence $(x_n)_n$ in X is said to be weakly null if*

$$\lim_{n\to\infty} |x^*(x_n)| = 0\ ,$$

for every $x^* \in X^*$.
We denote by $c_0^w(X)$ the Banach space of weakly null sequences in X is a closed subspace of $\ell_\infty(X)$, therefore, it is a Banach space with the supremum norm of $\ell_\infty(X)$.

Let $\ell_p(X)$ be the Banach space of all absolutely p -summable sequences $(x_i)_{i=1}^\infty$ in X with the norm

$$\left\|(x_i)_{i=1}^\infty\right\|_p = (\sum_{i=1}^\infty \|x_i\|^p)^{\frac{1}{p}}.$$

Remark 1.2 *For $(x_n)_n \in \ell_p^{weak}(X)$ the quantity*

$$\|(x_n)\|_p^{weak} := \sup\left\{ (\sum_n |\langle x^*, x_n\rangle|^p)^{\frac{1}{p}} : x^* \in B_{X^*} \right\}.$$

is finite, and for this we call on the closed graph theorem. take (x_n) in $\ell_p^{weak}(X)$ and associate with it the map $T : X^* \to \ell_p$ given by $u(x^*) = (\langle x^*, x_n\rangle)$. certainly, T is well-defined and linear

.Moreover, if (x_k^*)*converge to* x_0^* *in* X^**, then for each* n *the scalar sequence* $(\langle x_k^*, x_n\rangle)_k$ *converges to* $\langle x_0^*, x_n\rangle)$*; As a consequence,* T *has a closed graph and so is bounded in other words*

$$\|T\| := \sup\left\{(\sum_n |\langle x^*, x_n\rangle|^p)^{\frac{1}{p}} : x^* \in B_{X^*}\right\} < \infty$$

which is what we wanted.

Remark 1.3 1. *Notice that* $\ell_p(X)$ *is a linear subspace of* $\ell_p^\omega(X)$ *and*

$$\left\|(x_i)_{i=1}^\infty\right\|_{p,\omega} \le \left\|(x_i)_{i=1}^\infty\right\|_p \text{ for all } (x_i)_{i=1}^\infty \in \ell_p(X).$$

2. *If* X *is finite dimensional with* $\dim X = n$*, then* $\ell_p(X) = \ell_p^\omega(X)$ *and*

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} \le \|(x_i)_{i=1}^\infty\|_p \le n^{\frac{1}{p}} \|(x_i)_{i=1}^\infty\|_{p,\omega} \quad \text{for all } (x_i)_{i=1}^\infty \in \ell_p(X). \tag{1.1}$$

3. *If we take* $n = 1$ *in (??), or* $X = \mathbb{K}$*, then the spaces* $\ell_p(\mathbb{K})$ *and* $\ell_p^\omega(\mathbb{K})$ *coincide and we denote* $\ell_p(\mathbb{K})$ *by* ℓ_p*. In this case we have*

$$\|(x_i)_{i=1}^\infty\|_{p,\omega} = \|(x_i)_{i=1}^\infty\|_p \quad \text{for all } (x_i)_{i=1}^\infty \in \ell_p. \tag{1.2}$$

4. *Let* $1 \le p < \infty$ *and write* p^* *for the conjugate index.Take* $x_1, ..., x_n$ *in the Banach space and set* $K = B_X^*$*. The following interchange of suprema argument exploits the usual duality between*ℓ_p^m *and* $\ell_{p^*}^m$ *:*

$$\begin{aligned}\sup_{x^*\in B_X^*} \|\langle x^*, x\rangle\|_{\ell_p} = \sup_{x^*\in k} (\sum_{i\le m} |\langle x^*, x_i\rangle|^p)^{\frac{1}{p}} &= \sup_{x^*\in k} \sup_{a\in B_{\ell_{p^*}^m}} \left|(\sum_{i\le m} a_i \langle x^*, x_i\rangle\right| \\ &= \sup_{a\in B_{\ell_{p^*}^m}} \sup_{x^*\in k} \left|\langle x^*, \sum_{i\le m} a_i x_i\rangle\right| \\ &= \sup_{a\in B_{\ell_{p^*}^m}} \|\sum_{i\le m} a_i x_i\|\end{aligned}$$

5. *We know (see [?, Theorem 2.1]) that* $\left(\ell_p^n(X)\right)^* = \ell_{p^*}^n(X^*)$ *isometrically i.e.,*

$$\left\|(x_i)_{i=1}^n\right\|_p = \sup\left\{\left|\sum_{i=1}^n \langle x_i, x_i^*\rangle\right| : (x_i^*)_{i=1}^n \subset X^*, \left\|(x_i^*)_{i=1}^n\right\|_{p^*} \leq 1\right\}. \tag{1.3}$$

For the particular case $p = 1$ *and* $X == \mathbb{K}$ *we have*

$$\left\|(x_i)_{i=1}^n\right\|_1 = \sup\left\{\left|\sum_{i=1}^n \lambda_i x_i\right| : (\lambda_i)_{i=1}^n \subset \mathbb{K}, \left\|(\lambda_i)_{i=1}^n\right\|_\infty \leq 1\right\}. \tag{1.4}$$

Let $\left(x_i^*\right)_{i=1}^n \subset X^*$. *Then it is also known that*

$$\left\|(x_i^*)_{i=1}^n\right\|_{p,\omega} = \sup_{\beta \in B_{X^{**}}} (\sum_{i=1}^n |\beta(x_i^*)|^p)^{\frac{1}{p}} = \sup_{x \in B_X} \left\|(x_i^*(x))_{i=1}^n\right\|_p. \tag{1.5}$$

1.2 Fundamentals of p-Summing Operators

Definition 1.5 *Let* $1 \leq p < \infty$. *A linear operator* $T : E \longrightarrow F$ *is said to be* p*-summing if there exists a constant* $C \geq 0$ *such that for all finite sequence* $(x_i)_{1 \leq i \leq n}$ *in* E

$$\left(\sum_{i=1}^n \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*} \leq 1} \left(\sum_{i=1}^n |\xi(x_i)|^p\right)^{\frac{1}{p}}. \tag{1.6}$$

The infimum of all such constants $C \geq 0$ *is denoted by* $\pi_p(T)$. *The collection of all* p*-summing operators between* E *and* F *is denoted by* $\Pi_p(E, F)$.

We mention that (Π_p^L, π_p^L) is an injective Banach Lipschitz operator ideal.

Theorem 1.1 *[?, Page 39] If* $1 \leq p \leq q < \infty$, *then* $\Pi_p(E, F) \subset \Pi_q(E, F)$. *Moreover,* $\pi_q(T) \leq \pi_p(T)$ *for every* $u \in \Pi_p(E, F)$.

The following basic result about p-summing operators is due to A. Pietsch, and it characterizes the p-summability by means of a domination theorem.

Theorem 1.2 *(Pietsch Domination Theorem) [?, page 44]*
Let $1 \leq p < \infty$ *and* $T \in \mathcal{L}(E, F)$. *Then* T *is* p*-summing if and only if there exist a constant* C

and a regular Borel probability measure μ on B_{E^*} (with the weak star topology) so that

$$\|T(x)\| \leq C \int_{B_{E^*}} |\langle x, x^* \rangle|^p \, d\mu(x^*), \qquad x \in E. \tag{1.7}$$

In this case, $\pi_p(T)$ is the least of all the constants C such that (??) holds.

In order to adapt the previous result into a factorization theorem, we present basic examples of p-summing linear operators.

Example 1.2 *see [?, Example 2.9 (b),(d)]*
(1) Let K be a compact Hausdorff space, let μ be a positive regular Borel measure on K, and let $1 \leq p < \infty$. The canonical inclusion

$$J_p : C(K) \longrightarrow L_p(\mu),$$

is p-summing with $\pi_p(J_p) = \|J_p\| = \mu(K)^{\frac{1}{p}}$.
(2) Let (Ω, Σ, μ) be a finite measure space and let $1 \leq p < \infty$. The formal inclusion map

$$I_{\infty,p} : L_\infty(\mu) \longrightarrow L_p(\mu),$$

is p-summing, with $\pi_p(I_{\infty,p}) = \mu(\Omega)^{\frac{1}{p}}$.

We denote by i_E the isometric embedding $E \longrightarrow C(B_{E^*})$ given by $i_E(x) = \langle x, . \rangle$.

Corollary 1.1 *[?, page 45] (Pietsch Factorization Theorem)*
Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$. The following are equivalent
(i) T is p-summing.
(ii) There exist a regular Borel probability measure μ on B_{E^*} (with the weak star topology), a closed subspace E_p of $L_p(\mu)$ and a linear continuous operator $\widetilde{u} : E_p \longrightarrow F$ such that $J_p \circ i_E(E) \subset E_p$ and $\widetilde{u} \circ J_p \circ i_E(x) = T(x)$ for all $x \in E$.

In other words, if $\overline{J_p}$ is the map $i_E(E) \longrightarrow E_p$ induced by J_p, then the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ i_E \downarrow & & \uparrow \widetilde{T} \\ i_E(E) & \xrightarrow{\overline{J_p}} & E_p \\ \cap & & \cap \\ C\left(B_{E^*}\right) & \xrightarrow{J_p} & L_p(\mu). \end{array}$$

In addition, we may choose μ and $\widetilde{T}$ so that $\left\|\widetilde{T}\right\| = \pi_p(T)$.

1.2.1 The correspondence of a linear operator

Let $T : X \longrightarrow Y$ be a bounded linear operator between Banach spaces, the correspondence

$$\hat{T} : (x_n)_n \longmapsto (T(x_n))_n$$

always induces a bounded linear operator $\ell_p^{weak}(X) \longrightarrow \ell_p^{weak}(Y)$, as well as a bounded linear operator $\ell_p(X) \longrightarrow \ell_p(Y)$. in both cases, the norm is clearly $\|T\|$. sometimes, this process even produces a linear operator form $\ell_p^{weak}(X)$ to $\ell_p(Y)$; such is the case precisely when T is p-summing.

Proposition 1.1 *T is p-summing if and only if $\hat{T}(\ell_p^{weak}(X))$is contained in $\ell_p(Y)$ in this case $\|\hat{T} : \ell_p^{weak}(X) \longrightarrow \ell_p(Y)\| = \pi_p(T)$.*

Proof 1.1 *Suppose first that T is p-summing. then, for any finite collection of vectors $x_1, ..x_m \in X$, we have*

$$\left(\sum_{n<m} \|Tx_n\|^p\right)^{\frac{1}{p}} \leq \pi_p(T) \cdot sup\left\{(\sum_{n<m} |\langle x^*, x_n\rangle|^p)^{\frac{1}{p}} : x^* \in B_{X^*}\right\}$$

Let $(x_n) \in \ell_p^{weak}(X)$. then

$$\begin{aligned} \|\hat{T}((x_n))_n\|_{\ell_p} &= \sup_m \left(\sum_{n<m} \|Tx_n\|^p\right)^{\frac{1}{p}} \leq \pi_p(T) \cdot \sup_m \sup_{x^* \in B_{X^*}} \left(\sum_{n<m} |\langle x^*, x_n\rangle|^p\right)^{\frac{1}{p}} \\ &= \pi_p(T) \sup_{x^* \in B_{X^*}} \sup_m \left(\sum_{n<m} |\langle x^*, x_n\rangle|^p\right)^{\frac{1}{p}} = \pi_p(T) \cdot \|(x_n)\|_p^{weak}. \end{aligned}$$

Consequently, $\hat{T}$ *maps* $\ell_p^{weak}(X)$ *continuously into* $\ell_p(Y)$*, and* $\|\hat{T}\| \leq \pi_p(T)$
To prove that T *is* p*-summing with* $\pi_p(T) \leq \|\hat{T}\|$ *see [**?**, page(34)].*

1.3 Linear operator ideals

1.3.1 Finite rank operator

Recall that a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite dimensional subspace of F. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only it has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$$

i.e. if $u \in \mathcal{L}_f(E, F)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ (see [**?**, Page 25]).
It is not hard to establish that $T = x^* \otimes y$ is in $\Pi_p(X, Y)$,with $\pi_p(u) = \|x^*\| \cdot \|y\|$. clearly $\|x^*\| \cdot \|y\| = \|T\| \leq \pi_p(T)$, so we only need to check that $\pi_p \leq \|x^*\| \cdot \|y\|$. but this follows from

$$(\sum_{k=1}^{m} \|T(x)_k\|^p)^{\frac{1}{p}} = \|x^*\| \cdot \|y\| \cdot (\sum_{k=1}^{m} |\langle \frac{x^*}{\|x^*\|}, x_k \rangle|^p)^{\frac{1}{p}} \leq \|x^*\| \cdot \|y\| \cdot \|(x^k)_1^m\|_p^{weak}$$

which is valide for all choices of finitely many vectors $x_1, ..., x_m$ from X

Remark 1.4 *Let* $T \in \mathcal{L}(X, Y)$ *have finite rank. then* T *is* p*-summing for every* $1 \leq p < \infty$
To see why this is so, take $y_1, ...y_n$ *to be a basis for* $T(x)$*. then we can find* $x_1^*, ...x_n^* \in X$ *with* $T(x) = \sum_{k=1}^{n} \langle x_k^*, x \rangle y_k$ *for all* $x \in X$ *this exhibits* T *as a sum of rank one operators and so as a member of the vector space* $\Pi_p(X, Y)$

1.3.2 Linear operator ideals

Definition 1.6 *An operator ideal* $\mathcal{I}$ *is a subclass of the class* $\mathcal{L}$ *of all continuous linear operators between Banach spaces such that for all Banach spaces* E *and* F *its components* $\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$ *satisfy:*

(i) $\mathcal{I}(E,F)$ *is a linear subspace of* $\mathcal{L}(E,F)$ *which contains the finite rank operators.*

(ii) The ideal property: if $v \in \mathcal{L}(G,E)$, $u \in \mathcal{I}(E,F)$ *and* $w \in \mathcal{L}(F,H)$, *then the composition* $w \circ v \circ u$ *is in* $\mathcal{I}(G,H)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ *satisfies:*

(i') $(\mathcal{I}(E,F), \|.\|_{\mathcal{I}})$ *is a normed (Banach) space for all Banach spaces* E *and* F,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') If $v \in \mathcal{L}(G,E)$, $u \in \mathcal{I}(E,F)$ *and* $w \in \mathcal{L}(F,H)$,

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\| \,,$$

then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ *is called a normed (Banach) operator ideal.*

The operator ideal $\mathcal{I}$ is said to be closed if each $\mathcal{I}(E,F)$ is a closed subspace of $\mathcal{L}(E,F)$ for the sup norm.

Definition 1.7 *(injective operator ideal)*
A normed operator ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ *is said to be injective if for every metric injection* $i : F \hookrightarrow G$ *and every* $u \in \mathcal{L}(E,F)$ *it follows from* $i \circ u \in \mathcal{I}(E,G)$ *that* $u \in \mathcal{I}(E,F)$. *Moreover*

$$\|i \circ u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}} \,,$$

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one [**?**, Theorem 1.2.2].

1.3.3 Ideal proprty of p-summing operators

Let $1 \leq p < \infty$ and let $v \in \Pi_p(X,Y)$. then the composition of v with any bounded linear operator is p-summing. More spesifically, if X_0 and Y_0 are banach space then, regardless of how we choose $u \in \mathcal{L}(X,Y)$ and $w \in \mathcal{L}(X_0,X)$, we always have $uvw \in \Pi_p(X_0,Y_0)$ with $\pi_p(uvw) \leq \|u\| \cdot \pi_p(v) \cdot \|w\|$.

Proof 1.2 . *The operators u,v,w give rise to canonical operators* $\hat{u} : \ell_p(Y) \to \ell_p(Y_0), \hat{v} : \ell_p^{weak}(X) \to \ell_p^{strong}(Y)$ *and* $\hat{w} : \ell_p^{weak}(X_0) \to \ell_p^{weak}(X)$ *with* $\|u\| = \|\hat{u}\|, \pi_p(v) = \|\hat{v}\|$ *and* $\|w\| = \|\hat{w}\|$. *the operator* $u\hat{v}w$ *corresponding to* $uvw : X_0 \to Y_0$*coincides with* $\hat{u}\hat{v}\hat{w}$ *and thus* *induces a mape* $\ell_p^{weak}(X_0) \to \ell_p^{strong}(Y_0)$. *Hence* uvw *is* p*-summing, with*

$$\pi_p(uvw) = \|u\hat{v}w\| \leq \|\hat{u}\| \cdot \|\hat{v}\| \cdot \|\hat{w}\| = \|u\| \cdot \pi_p(v) \cdot \|w\|$$

Remark 1.5 *If* X_0 *is a subspace of* X *and* $v : X \to Y$ *is p-summing, then the restriction map* $v|_{X_0} : X \to Y$ *is also p-summing, with* $\pi_p(v|_{X_0}) \leq \pi_p(v)$. *This follows from the ideal property when we take* $u : X_0 \to X$ *to be the inclusion map and set* w *to be the identity operator on* Y.

Proposition 1.2 ***(Injectivity of*** Π_p***:)*** *If* $i : Y \to Y_0$ *is isometric, then* $v \in \Pi_p(X, Y)$ *if and only if* $iv \in \Pi_p(X, Y_0)$ *in this case, we even have* $\pi_p(iv) = \pi_p(v)$

Proposition 1.3 (Π_p, π_p) *is an injective Banach operator ideal.*

Chapter 2

The spaces $\ell_p(X,Y)$, $\ell_p^w(X,Y)$ and $\ell_p^s(X,Y)$

In the sequel of this we based on the article of Oscar Blasco and Teresa Signesin [?]

2.1 The spaces $\ell_p(X,Y)$, $\ell_p^w(X,Y)$

Given tow Banach spaces X and Y we shall be denoting by $\ell_p(X,Y)$ and $\ell_p^w(X,Y)$ the space $\ell_p(\mathcal{L}(X,Y))$ and $\ell_p^w(\mathcal{L}(X,Y))$ respectively.
$\ell_p(X,Y) = \ell_p(\mathcal{L}(X,Y)) = \left\{(T_n); T_n \in \mathcal{L}(X,Y), \|(T_n)\| = (\sum_{n=1}^{\infty} \|(T_n)\|^p)^{\frac{1}{p}}\right\}$ and the space
$\ell_p^w(X,Y) = \ell_p^w(\mathcal{L}(X,Y) = \left\{(T_n); T_n \in \mathcal{L}(X,Y), \|(T_n)\| = \sup_{\psi \in B_{\mathcal{L}(X,Y)}} (|\langle T_n; \psi\rangle|^p)^{\frac{1}{p}}\right\}$

Lemma 2.1 *Let (e_n) be a sequence in a Banach space X then*

$$\|(e_n)\|_{\ell_p^w(X)} = \sup_{(a_n)\in B_{\ell_{p'}}} \|\sum_{n=1}^{\infty} \alpha_n e_n\|_X.$$

Proof 2.1 $1 \leq p < \infty$ *,let $e_n \in \ell_p^w$ for every $x^* \in X^*$*

$$\begin{aligned}
\|(e_n)\|_{\ell_p^w} &= \sup_{x^*\in B_{X^*}} \left(\sum_{n=1}^{\infty} |\langle x^*, e_n\rangle|^p\right)^{\frac{1}{p}} \\
&= \sup_{x^*\in B_{X^*}} \|x^*(e_n)\|_{\ell_p} \\
&= \sup_{x^*\in B_{X^*}} \sup_{\alpha_n\in B_{\ell p'}} |\langle x^*(e_n), \alpha_n\rangle| \\
&= \sup_{x^*\in B_{X^*}} \sup_{\alpha_n\in B_{\ell p'}} |\sum_{n=1} x^*(e_n)\alpha_n| \\
&= \sup_{\alpha_n\in B_{\ell p'}} \sup_{x^*\in B_{X^*}} \|x^*(\sum_{n=1} \alpha_n e_n)\| \\
&= \sup_{\alpha_n\in B_{\ell p'}} \|\sum_{n=1}^{\infty} \alpha_n e_n\|_E.
\end{aligned}$$

Hence one easily get that

Corollary 2.1 *Let* $1 \leq p < \infty$, *and,* $(T_n)_n \in \ell_p^w(X,Y)$

$$\|(T_n)\|_{\ell_p^w(X,Y)} = \sup_{x\in B_X} \sup_{Y^*\in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle y^*, T_n x\rangle|^p\right)^{\frac{1}{p}}$$

Proof 2.2 *Using Lemma (??), so, we get*

$$\begin{aligned}
\|(T_n)\|_{\ell_p^w(X,Y)} &= \|(T_n)\|_{\ell_p^w[\mathcal{L}(X,Y)]} \\
&= \sup_{(a_n)\in B_{\ell_{p'}}} \|\sum_{n=1}^{\infty} \alpha_n T_n\|_{\mathcal{L}(X,Y)} \\
&= \sup_{(a_n)\in B_{\ell_{p'}}} \sup_{x\in B_X} \|\sum_{n=1}^{\infty} \alpha_n T_n(x)\|_Y \\
&= \sup_{x\in B_X} = \sup_{(a_n)\in B_{\ell_{p'}}} \|\sum_{n=1}^{\infty} \alpha_n T_n(x)\|_Y \\
&= \sup_{x\in B_X} \|(T_n(x))_n\|_{p,w} \\
&= \sup_{x\in B_X} \sup_{Y^*\in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle y^*, T_n x\rangle|^p\right)^{\frac{1}{p}}
\end{aligned}$$

Let $\Phi : \mathcal{L}(X,Y) \longrightarrow \mathcal{L}(Z,W)$ be a bounded operator. The correspondence

$$\Phi : (T_n)_{n=1}^{\infty} \longmapsto (\Phi(T_n))_{n=1}^{\infty}$$

always induces a linear bounded operator from $\ell_p(X,Y)$ to $\ell_p(Z,W)$, as well as from $\ell_p^w(X,Y)$ to $\ell_p^w(Z,W)$. Recall that Φ is p-summing if Φ maps $\ell_p^w(X,Y)$ to $\ell_p(Z,W)$

2.2 The space $\ell_p^s(X,Y)$

The fact that we have the strong operator topology at our disposal allows to consider the following intermediate space of sequences of operators.

Definition 2.1 *Let* $1 \leq p \leq \infty$ *and* X, Y *be Banach spaces. A sequence of operators* $(T_n)_{n=1}^{\infty} \subseteq \mathcal{L}(X,Y)$ *is said to be strongly* p*-summable in* $\mathcal{L}(X,Y)$ *if the vector-valued sequence* $(T_n x)_{n=1}^{\infty}$

belongs to $\ell_p(Y)$ for every $x \in X$.

We shall use the notation $\ell_p^s(X,Y)$ for the space of all strongly p-summable sequences. The norm in $\ell_p^s(X,Y)$ is given by

$$\|(T_n)\|_{\ell_p^s(X,Y)} = \sup_{x\in B_X} \left(\sum_{n=1}^{\infty} \|T_n x\|_Y^p \right)^{1/p}.$$

It is rather easy to see that $(\ell_p^s(X,Y), \|\cdot\|_{\ell_p^s(X,Y)})$ is a Banach space for any $1 \le p \le \infty$.

Easy examples of sequences in $\ell_p^s(X,Y)$ are given in the next examples.

Example 2.1 *Let X and Y be Banach spaces and $1 \le r_1, r_2, p \le \infty$ with $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{p}$. If $(x_n^*)_{n=1}^{\infty} \in \ell_{r_1}^w(X^*)$ and $(y_n)_{n=1}^{\infty} \in \ell_{r_2}(Y)$, then $(x_n^* \otimes y_n)_{n=1}^{\infty} \in \ell_p^s(X,Y)$.*

Example 2.2 *Let X, Y, and Z be Banach spaces and $1 \le p \le \infty$. If $(T_n)_{n=1}^{\infty} \in \ell_p^w(X,Y)$ and $S \in \Pi_p(Y,Z)$, then $(ST_n)_{n=1}^{\infty} \in \ell_p^s(X,Z)$.*

Proof 2.3 *Since $(T_n)_{n=1}^{\infty} \in \ell_p^w(X,Y)$there exists a constant C such that for all$x \in X$ and $y^* \in Y$*

$$\left(\sum_{n=1}^{\infty} \|y^*(T_n x)\|^p \right)^{1/p} \le C\|x\|$$

$S \in \pi_p(Y,Z)$ there exists a constant C_1such that for any finite sequence

$$\left(\sum_i \|Sy_i\|^p \right)^{1/p} \le C_1 \sup_{\|y^*\|\le 1} \left(\sum_i \|y^*(y_i)\|^{p'} \right)^{1/p'}.$$

so To show that $(ST_n)_{=1}^{\infty} \in \ell_p^s(X,Z)$ We Applying the $p-summing$ property of S to the sequence $(T_n x_i)_n$ in Y

$$\left(\sum_i \|S(T_n x_i)\|^p \right)^{1/p} \le C_1 \sup_{\|y^*\|\le 1} \left(\sum_i \|y^*(T_n x_i)\|^{p'} \right)^{1/p'}.$$

From the weak $p-summability$ of (T_n) we know for all $y^* \in Y^*$:

$$\left(\sum_{n=1}^{\infty} \|y^*(T_n x_i)\|^p \right)^{1/p} \le C_2\|x_i\|$$

Hence

$$\sup_{\|y^*\|\le 1} \left(\sum_i \|y^*(T_n x_i)\|^{p'} \right)^{1/p'} \le C_2 \sup_{\|y^*\|\le 1} \left(\sum_i \|x_i\|^{p'} \right)^{\frac{1}{p'}}$$

Thus

$$\left(\sum_i \|ST_n x_i\|^p\right)^{1/p} \leq C_1 C_2 \sup_{\|y^*\|\leq 1} \left(\sum_i \|x_i\|^{p'}\right)^{1/p'}$$

$$\left(\sum_i \|ST_n x_i\|^p\right)^{1/p} \leq K \sup_{\|y^*\|\leq 1} \left(\sum_i \|x_i\|^{p'}\right)^{1/p'}$$

For all $(x_i) \subset X$ we have shown that (ST_n) is absolutely $p-summing$.
Therefore, $(ST_n)_{n=1}^{\infty} \in \ell_p^s(X,Z)$

As mentioned in the introduction, $\ell_p(X,Y)$ stands for $\ell_p(\mathcal{L}(X,Y))$ and $\ell_p^w(X,Y)$ for $\ell_p^w(\mathcal{L}(X,Y))$. Let us first notice the inclusions appearing between these spaces and the new one. Note that $\ell_p^s(\mathbb{K},Y) = \ell_p(\mathbb{K},Y) = \ell_p(Y)$ and $\ell_p^s(X,\mathbb{K}) = \ell_p^w(X,\mathbb{K}) = \ell_p^w(X^*)$. Observe also that for every bounded sequence (T_n) in $\mathcal{L}(X,Y)$ we have

$$\sup_n \|T_n\| = \sup_{x\in B_X} \sup_n \|T_n x\|_Y = \sup_{x\in B_X} \sup_{y^*\in B_{Y^*}} \sup_n |\langle y^*, T_n x\rangle|.$$

Hence for $p=\infty$, $\ell_\infty(X,Y) = \ell_\infty^s(X,Y) = \ell_\infty^w(X,Y)$.

Proposition 2.1 *Let X,Y be Banach spaces and $1 \leq p < \infty$ .then*

$$\ell_p(X,Y) \subseteq \ell_p^s(X,Y) \subseteq \ell_p^w(X,Y)$$

Moreover, each of the inclusions is strict in general.

Proof 2.4 *We shall only show the last statement, since the inclusions are immediate by definition. Let $(e_n)_{n=1}^{\infty}$ be the usual basis in $\ell_{p\prime}$ (or c_0 for $p=1$) and consider $(e_n \otimes e_n)_{n=1}^{\infty}$ as operators from ℓ_p into ℓ_∞. Since $(e_n) \in \ell_p^w(\ell_{p\prime})$ and $(e_n) \in \ell_\infty(\ell_\infty)$, by example ?? we get that $(e_n \otimes e_n) \in \ell_p^s(\ell_p, \ell_\infty)$. On the other hand, $\|e_n \otimes e_n\|_{(\ell_{p\prime}),\ell_\infty} = \|e_n\|_{\ell p\prime}\|e_n\|_{\ell_\infty} = 1$ thus $(e_n \otimes e_n) \notin \ell_p(\ell_p, \ell_\infty)$. Let us now find a weakly p-summable sequence which is not strongly p-summable. Simply choose $p < r < \infty$ and take s so that $1/r + 1/p\prime = 1/s$. A direct computation, using (1.1), shows that $(e_n \otimes e_n) \in \ell_p^w(\ell_r, \ell_s)$. However $(e_n \otimes e_n) \notin \ell_p^s(\ell_r, \ell_s)$ as it is shown by selecting any $\lambda = (\lambda_n)_{n=1}^{\infty} \in \ell_r \backslash \ell_p$, since $\|\langle e_n, \lambda e_n\rangle\|_{\ell_s} = |\lambda_n| \notin \ell_p$.*

Observe that if $(T_n)_{n=1}^{\infty}$ in $\mathcal{L}(X,Y)$ is a strongly p-summable sequence, using the closed graph theorem, then one can associate a linear and bounded operator $S : X \to \ell_p(Y)$ defined by $S(x) =$

$(T_n x)_{n=1}^{\infty}$. In this way we get $\ell_p^s(X,Y)$ as a closed subspace of $\mathcal{L}(X,\ell_p(Y))$. Let us see that they are actually isometrically isomorphi.

Proposition 2.2 *Let X,Y be Banach spaces and $1 \le p \le \infty$. For each $n \in \mathbb{N}$, let $Q_n : \ell_p(Y) \to Y$ be the operator $Q_n((y_i)_{i=1}^{\infty}) = y_n$. The correspondence $T \mapsto (Q_n T)_{n=1}^{\infty}$ is an isometric isomorphism from $\mathcal{L}(X,\ell_p(Y))$ onto $\ell_p^s(X,Y)$.*

Proof 2.5 . *Given $T \in \mathcal{L}(X,\ell_p(Y))$, the sequence $T_n = Q_n T$, $n = 1,2,\ldots$, belongs to $\ell_p^s(X,Y)$ and*

$$\|(T_n)\|_{\ell_p^s(X,Y)} = \sup_{x\in B_X}\left(\sum_{cn=1}^{\infty}\|Q_n Tx\|_Y^p\right)^{1/p} = \sup_{x\in B_X}\|Tx\|_{\ell_p(Y)} = \|T\|_{(X,\ell_p(Y))}.$$

Therefore, the correspondence $T \mapsto (Q_n T)_{n=1}^{\infty}$ induces an isometry. To see the isomorphism, let us take $(S_n) \in \ell_p^s(X,Y)$ and note that $S : X \to \ell_p(Y)$ defined by $S(x) = (S_n x)_{n=1}^{\infty}$ gives a bounded operator and $Q_n S = S_n$.

*As a consequence, we get that $\ell_p^w(X^*)$ is isometrically isomorphic to $\mathcal{L}(X,\ell_p)$ for any $1 \le p \le \infty$ and any Banach space X (see [**?**], Prop. 19.4.3).*

Let us also recall that, for every Banach space Z, $\ell_p^w(Z)$ is also isometrically isomorphic to $\mathcal{L}(\ell_{p'},Z)$ if $1 < p < \infty$ and $\mathcal{L}(c_0,Z)$ for $p = 1$. In these last cases, the isomorphisms are given by associating to each operator T the sequence $x_n = T(e_n)$. Of course, the connection between both results goes through the adjoint operator.

Combining both identifications, we get that the strongly p-summable sequences in $\mathcal{L}(\ell_{q'},X)$ are precisely the weakly q-summable sequences in $\ell_p(X)$.

Corollary 2.2 *Let X be a Banach space, $1 \le p \le \infty$, and $1 < q < \infty$. Then the map Ψ : $\ell_q^w(\ell_p(X)) \to \ell_p^s(\ell_{q'},X)$, defined by $\Psi((x_k)_k) = (T_n)_n$ where T_n are defined by $T_n(e_k) = Q_n(x_k)$ for all $n,k \in \mathbb{N}$, is an isometric isomorphism. A similar result is true for $q = 1$ replacing ℓ_∞ by c_0.*

Chapter 3

(ℓ_p^s, ℓ_p)-summing operators

The concept of strongly p-summing sequence can now be used to define new classes of p-summing type operators.

Definition 3.1 *Let $1 \leq p < \infty$. Let X, Y, Z, and W be Banach spaces.*
An operator $\Phi : \mathcal{L}(X, Y) \to Z$ is said to be (ℓ_p^s, ℓ_p)-summing if there exists a constant $C > 0$ such that

$$\left(\sum_{i=1}^{n} \|\Phi(T_i)\|_Z^p\right)^{\frac{1}{p}} \leq C \sup_{x \in B_X} \left(\sum_{i=1}^{n} \|T_i x\|_Y^p\right)^{\frac{1}{p}} \tag{3.1}$$

for any finite family of operators $T_1, \ldots, T_n$ in $\mathcal{L}(X, Y)$.
An operator $\Phi : X \to \mathcal{L}(Y, Z)$ is said to be (ℓ_p^w, ℓ_p^s)-summing if there exists a constant $C > 0$ such that

$$\sup_{y \in B_Y} \left(\sum_{i=1}^{n} \|\Phi(x_i)(y)\|_Z^p\right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x^*, x_i \rangle|^p\right)^{\frac{1}{p}} \tag{3.2}$$

for every choice of elements $x_1, \ldots, x_n$ in X.
An operator $\Phi : \mathcal{L}(X, Y) \to \mathcal{L}(Z, W)$ is said to be (ℓ_p^s, ℓ_p^s)-summing if there exists a constant $C > 0$ such that

$$\sup_{z \in B_Z} \left(\sum_{i=1}^{n} \|\Phi(T_i)(z)\|_W^p\right)^{\frac{1}{p}} \leq C \sup_{x \in B_X} \left(\sum_{i=1}^{n} \|T_i x\|_Y^p\right)^{\frac{1}{p}} \tag{3.3}$$

for every choice of operators $T_1, \ldots, T_n$ in $\mathcal{L}(X, Y)$.
*The least constants in **??**, **??**,and **??** are denoted by $\pi_{(\ell_p^s, \ell_p)}(\Phi)$, $\pi_{(\ell_p^w, \ell_p^s)}(\Phi)$, and $\pi_{(\ell_p^s, \ell_p^s)}(\Phi)$ respectively.*

We shall denote by $\Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X, Y), Z)$ the space of all (ℓ_p^s, ℓ_p)-summing operators from $\mathcal{L}(X, Y)$ to Z, by $\Pi_{(\ell_p^w, \ell_p^s)}(X, \mathcal{L}(Y, Z))$ the space of all (ℓ_p^w, ℓ_p^s)-summing operators from X to $\mathcal{L}(Y, Z)$, and by $\Pi_{(\ell_p^s, \ell_p^s)}(\mathcal{L}(X, Y), \mathcal{L}(Z, W))$ the space of all strongly (ℓ_p^s, ℓ_p)-summing operators from $\mathcal{L}(X, Y)$ to $\mathcal{L}(Z, W)$.

$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$, $\Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z))$, and $\Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,W))$ become Banach spaces with the norms $\pi_{(\ell_p^s,\ell_p)}(\cdot)$, $\pi_{(\ell_p^w,\ell_p^s)}(\cdot)$, $\pi_{(\ell_p^s,\ell_p^s)}(\cdot)$, respectively.

The corresponding definitions for $p=\infty$ would simply lead to the space of bounded operators in all cases. Alternative definition for (ℓ_p^s,ℓ_p)-summing operators is the following one:

Remark 3.1 *Let $1\le p<\infty$ and $\Phi\in\mathcal{L}(\mathcal{L}(X,Y),Z)$. The following are equivalent:*

1. *Φ is (ℓ_p^s,ℓ_p)-summing.*
2. *Φ maps sequences $(T_n)\in\ell_p^s(X,Y)$ into sequences $(\Phi(T_n))\in\ell_p(Z)$.*
3. *The linear operator $\tilde{\Phi}:\ell_p^s(X,Y)\to\ell_p(Z)$ defined by $\tilde{\Phi}((T_n)_{n=1}^\infty)=(\Phi(T_n))_{n=1}^\infty$ is continuous.*

Similar equivalences are true for (ℓ_p^w,ℓ_p^s)-summing operators and (ℓ_p^s,ℓ_p^s)-summing operators.

Remark 3.2 *It is rather easy to see that, when some of the spaces is finite dimensional, the classes $\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$, $\Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z))$ reduce either to bounded operators or to p-summing operators and that the class $\Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(X,Y),\mathcal{L}(Z,U))$ reduces to one of the previous cases.*

Remark 3.3 *Recall that $\dim X=\infty$ implies $\Pi_p(X,X)\subset\mathcal{L}(X,X)$ and observe that*

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(K,X^*),Y)=\mathcal{L}(X^*,Y)$$

and

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,K),Y)=\Pi_p(X^*,Y).$$

Therefore, for any infinite dimensional X, we have $\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(K,X^*),Y)\neq\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,K),Y)$,for some Banach space Y,

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,K),\mathcal{L}(X,K))\subset\Pi(wp,sp)((\mathcal{L}X,K),\mathcal{L}(X,K)),$$

$$\Pi(wp,sp)(\mathcal{L}(K,X),\mathcal{L}(K,X))\subset\Pi_{(\ell_p^s,\ell_p)}(L(K,X),\mathcal{L}(K,X)).$$

Remark 3.4 *If A and B are spaces of operators, then*

$$\Pi_p(A,B)\subseteq\Pi_{(\ell_p^w,\ell_p^s)}(A,B)\cap\Pi_{(\ell_p^s,\ell_p)}(A,B),$$

$$\Pi_{(\ell_p^w,\ell_p^s)}(A,B) \cup \Pi_{(\ell_p^s,\ell_p)}(A,B) \subseteq \Pi_{(\ell_p^s,\ell_p^s)}(A,B).$$

It is not difficult to show that the inclusions are strict in general.

Remark 3.5 *(i) For each $x \in X$, the evaluation map $e_x : \mathcal{L}(X,Y) \to Y$ given by $e_x(T) = Tx$ is (ℓ_1^s, ℓ_1)-summing.*

(ii) For each $y \in Y$, the operator $\Phi_y : X^* \to \mathcal{L}(X,Y)$ given by $\Phi_y(x^*) = x^* \otimes y$ is (ℓ_1^w, ℓ_1^s)-summing.

(iii) For each $S \in \mathcal{L}(Y,Z)$, the operator $\Phi_S : \mathcal{L}(X,Y) \to \mathcal{L}(X,Z)$ given by $\Phi_S(T) = ST$ is (ℓ_1^s, ℓ_1^s)-summing. Moreover, if $S \in \Pi_p(Y,Z)$, $1 \le p < \infty$, then Φ_S is (ℓ_p^w, ℓ_p^s)-summing.

We now show that actually $\Pi(\ell_p^w, \ell_p^s)(X, \mathcal{L}(Y,Z))$ and $\mathcal{L}(Y, \Pi_p(X,Z))$ can be identified.

Theorem 3.1 *Let $\Phi : X \to \mathcal{L}(Y,Z)$ be a bounded operator, and let us define $\Phi^\# : Y \to \mathcal{L}(X,Z)$ by $\Phi^\#(y)(x) := \Phi(x)(y)$, $x \in X$, $y \in Y$.*

The correspondence $\Phi \mapsto \Phi^\#$ is an isometric isomorphism between $\Pi_{(\ell_p^w,\ell_p^s)}(X, \mathcal{L}(Y,Z))$ and $\mathcal{L}(Y, \Pi_p(X,Z))$.

Proof 3.1 *Take $\Phi \in \Pi_{(\ell_p^w,\ell_p^s)}(X, \mathcal{L}(Y,Z))$ and $y \in Y$, then*

$$\left(\sum_{i=1}^n \left\|\Phi^\#(y)(x_i)\right\|_Z^p\right)^{1/p} \le \|y\|_Y \Pi_{(\ell_p^w,\ell_p^s)}(\Phi) \sup_{x^* \in BX^*} \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p\right)^{1/p}$$

for every choice of elements $\{x_1, ..., x_n\}$ in X. Thus, $\Phi^\#(y) \in \Pi_p(X,Z)$ and $\pi_p(\Phi^\#(y)) \le \|y\|_Y \pi_{(\ell_p^w,\ell_p^s)}(\Phi)$. Hence $\Phi^\# \in \mathcal{L}(Y, \Pi_p(X,Z))$ and $\|\Phi^\#\| \le \pi_{(\ell_p^w,\ell_p^s)}(\Phi)$.

On the other hand, if $\Psi \in \mathcal{L}(Y, \Pi_p(X,Z))$ then

$$\begin{aligned}
\sup_{y \in B_Y} \left(\sum_{i=1}^n \|\Psi^\#(x_i)(y)\|_z^p\right)^{1/p} &= \sup_{y \in B_Y} \left(\sum_{i=1}^n \|\Psi(y)(x_i)\|_z^p\right)^{1/p} \\
&\le \sup_{y \in B_Y} \{\pi_p(\Psi(y))\} \sup_{x^* \in BX^*} \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p\right)^{1/p} \\
&\le \|\Psi\| \sup_{x^* \in BX^*} \left(\sum_{i=1}^n |\langle x^*, x_i\rangle|^p\right)^{1/p}
\end{aligned}$$

for every finite sequence $\{x_1, ..., x_n\}$ *in* X. *Hence* $\Psi^{\#} \in \Pi_{(\ell_p^w, \ell_p^s)}(X, \mathcal{L}(Y, Z))$ *and* $\pi_{(\ell_p^w, \ell_p^s)}(\Psi^{\#}) \leq \|\Psi\|$. *Since* $(\Psi^{\#})^{\#} = \Psi$, *the proof is finished.*

□

Therefore, the class $\Pi_{(\ell_p^w, \ell_p^s)}$ inherits some properties from those in Π_p. For example, Grothendieck's theorem (see [?], Theorem 1.13) implies the following:

Corollary 3.1 *Let* X *be a Banach space. Every operator from* ℓ_1 *into* $\mathcal{L}(X, \ell_2)$ *is* (ℓ_1^w, ℓ_1^s)-*summing, i.e.,* $\Pi_{(\ell_1^w, \ell_1^s)}(\ell_1, \mathcal{L}(X, \ell_2)) = \mathcal{L}(\ell_1, \mathcal{L}(X, \ell_2))$.

Corollary 3.2 *Let* X *be a Banach space,* K *be a compact Hausdorff space, and* (Ω, μ) *be a* σ-*finite measure space. Every operator from* $C(K)$ *into* $\mathcal{L}(X, L_1(\Omega, \mu))$ *is* (ℓ_2^w, ℓ_2^s)-*summing, i.e.,* $\Pi_{(\ell_2^w, \ell_2^s)}(C(K), \mathcal{L}(X, L_1(\mu))) = \mathcal{L}(C(K), \mathcal{L}(X, L_1(\mu)))$.

Furthermore, arguing as in Theorem **??** *we get also the following result whose proof is left to the interested reader*

Theorem 3.2 *The correspondence* $\Phi \longmapsto \Phi^{\#}$ *is an isometric isomorphism between* $\Pi_{(\ell_p^s, \ell_p^s)}(\mathcal{L}(X, Y), \mathcal{L}(Z, W))$ *and* $\mathcal{L}(Z, \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X, Y), W))$.

We would like to point out also certain behaviour of these classes as operator ideals and some composition results. The proof of the following proposition is straightforward

Proposition 3.1 *Let* $1 \leq p < \infty$ *and let* X, Y, Z, W, U, X_0, *and* Z_0 *be Banach spaces.*

1. *If* $\Phi \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X, Y), Z)$ *and* $\Psi \in \mathcal{L}(Z, Z_0)$, *then the composition* $\Psi\Phi$ *belongs to* $\Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X, Y), Z_0)$ *with* $\pi_{(\ell_p^s, \ell_p)}(\Psi\Phi) \leq \|\Psi\|_{Z, Z_0} \cdot \pi_{(\ell_p^s, \ell_p)}(\Phi)$. *That is,* $\mathcal{L} \circ \Pi_{(\ell_p^s, \ell_p)} \subseteq \Pi_{(\ell_p^s, \ell_p)}$.

2. *If* $\Phi \in \mathcal{L}(X_0, X)$ *and* $\Psi \in \Pi_{(\ell_p^w, \ell_p^s)}(X, \mathcal{L}(Y, Z))$, *then the composition* $\Psi\Phi$ *belongs to* $\Pi_{(\ell_p^w, \ell_p^s)}(X_0, \mathcal{L}(Y, Z))$ *with* $\Pi_{(\ell_p^w, \ell_p^s)}(\Psi\Phi) \leq \pi_{(\ell_p^w, \ell_p^s)}(\Psi) \cdot \|\Phi\|_{X_0, X}$. *That is,* $\Pi_{(\ell_p^w, \ell_p^s)} \circ \mathcal{L} \subseteq \Pi_{(\ell_p^w, \ell_p^s)}$.

3. *If* $\Phi \in \Pi_{(\ell_p^s, \ell_p^s)}(\mathcal{L}(X, Y), \mathcal{L}(Z, W))$ *and* $\Psi \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(Z, W), U)$, *then* $\Psi\Phi \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(X, Y), U)$ *with* $\pi_{(\ell_p^s, \ell_p)}(\Psi\Phi) \leq \pi_{(\ell_p^s, \ell_p)}(\Psi) \cdot \pi_{(\ell_p^s, \ell_p^s)}(\Phi)$. *That is,* $\Pi_{(\ell_p^s, \ell_p)} \circ \Pi_{(\ell_p^s, \ell_p^s)} \subseteq \Pi_{(\ell_p^s, \ell_p)}$.

4. *If* $\Phi \in \Pi_{(\ell_p^w,\ell_p^s)}(X, \mathcal{L}(Y,Z))$ *and* $\Psi \in \Pi_{(\ell_p^s,\ell_p^s)}(\mathcal{L}(Y,Z), \mathcal{L}(W,U))$, *then* $\Psi\Phi \in \Pi_{(\ell_p^w,\ell_p^s)}(X, \mathcal{L}(W,U))$ *with* $\pi_{(\ell_p^w,\ell_p^s)}(\Psi\Phi) \leq \pi_{(\ell_p^w,\ell_p^s)}(\Psi) \cdot \pi_{(\ell_p^w,\ell_p^s)}(\Phi)$. *That is,* $\Pi_{(\ell_p^s,\ell_p^s)} \circ \Pi_{(\ell_p^w,\ell_p^s)} \subseteq \Pi_{(\ell_p^w,\ell_p^s)}$.

5. *If* $\Phi \in \Pi_{(\ell_p^w,\ell_p^s)}(X, \mathcal{L}(Y,Z))$ *and* $\Psi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(Y,Z), W)$, *then* $\Psi\Phi \in \Pi_p(X,W)$ *with* $\pi_p(\Psi\Phi) \leq \pi_{(\ell_p^s,\ell_p)}(\Psi) \cdot \pi_{(\ell_p^w,\ell_p^s)}(\Phi)$. *That is,* $\Pi_{(\ell_p^s,\ell_p)} \circ \Pi_{(\ell_p^w,\ell_p^s)} \subseteq \Pi_p$.

The classical theorem of Pietsch stated that if Φ is q-summing and Ψ is p-summing then $\Psi\Phi$ is r-summing, with $\frac{1}{r} = \min\left\{1, \frac{1}{p} + \frac{1}{q}\right\}$ (see [?], Theorem 2.22 or [?], Theorem 19.10.3). Next, we establish that when Ψ is (ℓ_p^w, ℓ_p^s)-summing then $\Psi\Phi$ is (ℓ_r^w, ℓ_r^s)-summing. The proof follows along the lines of Theorem 2.22 in [?].

Proposition 3.2 *Let* $\Phi \in \Pi_q(X,Y)$ *and* $\Psi \in \Pi_{(\ell_p^w,\ell_p^s)}(Y, \mathcal{L}(Z,W))$ *with* $1 \leq p, q < \infty$. *Define* $1 \leq r < \infty$ *by* $\frac{1}{r} = \min\left\{1, \frac{1}{p} + \frac{1}{q}\right\}$. *Then* $\Psi\Phi$ *is* (ℓ_r^w, ℓ_r^s)*-summing with* $\pi_r^s(\Psi\Phi) \leq \pi_{(\ell_p^w,\ell_p^s)}(\Psi)\pi_q(\Phi)$. *That is,* $\Pi_{(\ell_p^w,\ell_p^s)} \circ \Pi_q \subseteq \Pi_{(\ell_r^w,\ell_r^s)}$.

Proof 3.2 *We assume first that* $\frac{1}{p} + \frac{1}{q} \leq 1$, *then* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *Let* $(x_n) \in \ell_r^w(X)$ *be given. Applying Lemma 2.23 in [?] we get sequences* $(\sigma_n) \in q$ *and* $(y_n) \in \ell_p^w(Y)$ *such that* $\|(\sigma_n)\|_q \leq \|(\mathbf{x_n})\|_{\ell_r^w(X)}^{r/q}$, $\|(\mathbf{y_n})\|_{\ell_p^w(Y)} \leq \pi_q(\Phi)\|(\mathbf{x_n})\|_{\ell_r^w(X)}^{r/p}$ *and* $\Phi(x_n) = \sigma_n y_n$ *for all* n. *By Hölder's inequality, using that* Ψ *is* (ℓ_p^w, ℓ_p^s)*-summing and that* $[\Psi\Phi(x_n)](z) = \sigma_n[\Psi(y_n)(z)]$ *for every* $z \in Z$, *we get*

$$\begin{aligned}
\sup_{z \in B_Z} \left(\sum_{n=1}^{k} \|[\Psi\Phi(x_n)](z)\|_w^r \right)^{\frac{1}{r}} &\leq \left(\sum_{n=1}^{k} |\sigma_n|^q \right)^{\frac{1}{q}} \sup_{z \in B_Z} \left(\sum_{n=1}^{k} \|\Psi(y_n)(z)\|_w^p \right)^{\frac{1}{p}} \\
&\leq \|(\mathbf{x_n})\|_{\ell_r^w(X)}^{r/q} \pi_{(\ell_p^w,\ell_p^s)}(\Psi) \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{k} |\langle y^*, y_n \rangle|^p \right)^{\frac{1}{p}} \\
&\leq \pi_{(\ell_p^w,\ell_p^s)}(\Psi) \cdot \pi_q(\Phi) \cdot \|(\mathbf{x_n})\|_{\ell_r^w(X)}.
\end{aligned}$$

Then $\Psi\Phi$ *belongs to* $\Pi_{(\ell_p^w,\ell_p^s)}(X, \mathcal{L}(Z,W))$.

If $\frac{1}{p} + \frac{1}{q} > 1$ *and we assume that* $1 < p \leq q$, *then* $\Phi \in \Pi_{p\prime}(X,Y)$ *with* $\pi_{p\prime}(\Phi) \leq \pi_q(\Phi)$ *and applying the first part with* p *and* $p\prime$ *we get* $\Psi\Phi \in \Pi_{(\ell_1^w,\ell_1^s)}(X, L(Z,W))$, *which completes the proof.*

3.1 (ℓ_p^s, ℓ_p)-summing operators

Let us now present several examples of such operators. The first one connects The notion of p-summing and (ℓ_p^s, ℓ_p)-summing operators .

Let $1 \leq p < \infty$, and let $T : X \to Y$ be a bounded operator. Denote $\widetilde{T} : \mathcal{L}(\ell_{p\prime}, X) \to \ell^p(Y)$ the map given by $\widetilde{T}(S) = (TSe_n)_{n=1}^{\infty}$. Recall that T is p-summing if and only if $\widetilde{T}$ is bounded and $\pi_p(T) = \|\widetilde{T}\|$.

We first study when $\widetilde{T}$ is(ℓ_p^s, ℓ_p)-summing operator. The answer is given in the following theorem.

Theorem 3.3 *Let X, Y be Banach spaces, $1 < p < \infty$, and $T \in \mathcal{L}(X, Y)$.*
Then $\widetilde{T} \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(\ell_{p\prime}, X)), \ell_p(Y))$ if and only if $\overline{T} \in \Pi_p(\ell_p(X), \ell_p(Y))$ where $\overline{T} : \ell_p(X) \to \ell_p(Y)$ is given by $\overline{T}((x_j)_{j=1}^{\infty}) = (T(x_j))_{j=1}^{\infty}$. Moreover, $\pi_p(\overline{T}) = \pi_{(\ell_p^s, \ell_p)}(\widetilde{T})$. The same result holds for $p = 1$ with the replacement of ℓ_∞ by c_0.

Proof 3.3 *Assume that $\widetilde{T} \in \Pi_{(\ell_p^s, \ell_p)}(\mathcal{L}(\ell_{p\prime}, X)), \ell_p(Y))$. Given $(x_k)_k \in \ell_p^w(\ell_p(X))$, we define the operators $T_n : \ell_{p\prime} \to X$ such that $T_n(e_k) = Q_n(x_k)$ for all $n, k \in \mathbb{N}$. Since Corollary 2.6 gives that $(T_n) \in \ell_p^s(\ell_{p\prime}, X)$, then $(\widetilde{T}(T_n))_n = ((TT_n(e_k))_k)_n \in \ell^p(\ell_p(Y))$. Now we have*

$$\begin{aligned}\sum_{k=1}^{\infty} \|\overline{T}(x_k)\|_{\ell_p(Y)}^p &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \|T(Q_n(x_k))\|_Y^p = \sum_{n=1}^{\infty} \|\widetilde{T}(T_n)\|_{\ell_p(Y)}^p \\ &\leq (\pi_(\ell_p^s, \ell_p)(\widetilde{T}))^p \|(T_n)\|_{\ell_p^s(\ell_{p\prime}, X)}^p = (\pi_{(\ell_p^s, \ell_p)}(\widetilde{T}))^p \|(x_k)\|_{\ell_p^w(\ell_p(X))}^p.\end{aligned}$$

For the converse assume that $^-$ T is p-summing and, as above, we write

$$\begin{aligned}\sum_{n=1}^{\infty} \|\widetilde{T}(T_n)\|_{\ell_p(Y)}^p &= \sum_{k=1}^{\infty} \|\overline{T}(x_k)\|_{\ell_p(Y)}^p \\ &\leq (\pi_p(\overline{T}))^p \|(x_k)\|_{\ell_p^w(\ell_p(X))}^p = (\pi_p(\overline{T}))^p \|(T_n)\|_{\ell_p^s(\ell_{\acute{p}}, X)}^p\end{aligned}$$

Example 3.1 *Let $1 < p < \infty$, X, Y be Banach spaces. Assume that $T \in \Pi_{\ell_p}(X^*, Y)$, then the operator*

$$\begin{aligned}\Phi_T : \mathcal{L}(X, \ell_p) &\to \ell_p(Y) \\ u &\rightsquigarrow (Tu^*(e_i))_{i=1}^{\infty},\end{aligned}$$

. (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s, \ell_p)}(\Phi_T) = \pi_p(T)$

Proof 3.4 . *For any choice of $u_1, ..., u_N \in \mathcal{L}(X,\ell_p)$, since $T : X^* \to Y$ is p-summing we have*

$$\begin{aligned}\sum_{n=1}^{N} \|\Phi_T(u_n)\|_{\ell_p(Y)}^p &= \sup_m \left\{ \sum_{i=1}^{m} \sum_{n=1}^{N} \|Tu_n^*(e_i)\|_Y^p \right\}, \\ &\leq (\pi_p(T))^p \sup_m \sup_{x \in B_X} \left\{ \sum_{i=1}^{m} \sum_{n=1}^{N} |\langle u_n^*(e_i), x\rangle|^p \right\} \\ &= (\pi_p(T))^p \sup_{x \in B_X} \sup_m \left\{ \sum_{i=1}^{m} \sum_{n=1}^{N} |\langle e_i, u_n x\rangle|^p \right\} \\ &\leq (\pi_p(T))^p \sup_{x \in B_X} \left\{ \sum_{n=1}^{N} \|u_n x\|_{\ell_p}^p \right\}.\end{aligned}$$

Therefore, Φ_T is (ℓ_p^s, ℓ_p)-summing and $\pi_{(\ell_p^s,\ell_p)}(\Phi_T) \leq \pi_p(T)$. It is straightforward to get equality of norms since $\pi_{(\ell_p^s,\ell_p)}(\Phi_T) \geq \|\Phi_T\| = \pi_p(T)$.

Example 3.2 *Let $1 \leq p < \infty$, let X be a reflexive Banach space, Y be a separable Banach space, and $L_p(\mu, Y)$ denote the space of Bochner p-integrable functions. If μ is a finite Borel measure on the compact topological space $(B_X, \sigma(X, X^*))$, then the operator $\Theta_p : \mathcal{L}(X,Y) \to L_p(\mu, Y)$ defined by $\Theta_p(T)(x) = T(x)$, $x \in B_X$, is (ℓ_p^s, ℓ_p)-summing with $\pi_{(\ell_p^s,\ell_p)}(\Theta_p) \leq \mu(B_X)^{\frac{1}{p}}$.*

Proof 3.5 *We first note that $x \to T(x)$ is a weakly continuous function on B_X and hence weakly measurable. Using the separability of Y, we get that it is a bounded measurable function and then in $L_p(\mu, Y)$. This shows that the operator Θ_p is well defined and bounded. If $T_1, \ldots, T_n$ are in $\mathcal{L}(X,Y)$, then*

$$\begin{aligned}\sum_{n=1}^{n} \left\|\Theta_p(T_i)\right\|_{L_p(\mu,Y)}^p &= \sum_{n=1}^{n} \int_{B_X} \|T_i(x)\|_Y^p \, d\mu(x) \\ &\leq \mu(B_X) \sup_{x \in B_X} \left\{ \sum_{n=1}^{n} \|T_i(x)\|_Y^p \right\}.\end{aligned}$$

Hence, we have that Θ_p is (ℓ_p^s, ℓ_p)-summing and $\pi_{(\ell_p^s,\ell_p)}(\Theta_p) \leq \mu(B_X)^{\frac{1}{p}}$ ■

Example 3.3 *Let (Ω, μ) be a σ-finite measure space, X, Y be Banach spaces, $1 \leq p < \infty$, and denote by $L_P(\mu, X)$ the space of Bochner p-integrable functions. For each $f \in L_P(\mu, X)$, the operator $\Phi_f : \mathcal{L}(X,Y) \to L_P(\mu, Y)$ defined by $\Phi_f(T)(w) = T(f(w))$, $w \in \Omega$, is (ℓ_p^s, ℓ_p)-summing and $\pi_{(\ell_p^s,\ell_p)}(\Phi_f) \leq \|f\|_{L_p(\mu,X)}$.*

Proof 3.6 *Observe first that the operator is well defined. Let $T_1, \ldots, T_n$ be operators from X into*

Y. If $E = \{w \in \Omega : f(w) \neq 0\}$, then

$$\begin{aligned}\sum_{i=1}^{n} \|\Phi_f(T_i)\|^p_{L_p(\mu,Y)} &= \sum_{i=1}^{n} \int_E \|T_i(f(w))\|^p_Y \, d\mu(w) \\ &= \sum_{i=1}^{n} \int_E \left\| T_i\Big(\frac{f(w)}{\|f(w)\|}\Big)\right\|^p_Y \|f(w)\|^p d\mu(w) \\ &\leq \|f\|_{L_p(\mu,X)} \sup_{x\in B_X} \left(\sum_{i=1}^{n} \|T_i x\|^p_Y\right). \blacksquare\end{aligned}$$

Example 3.4 *Let $1 \leq p < \infty$ and let X, Y be Banach spaces. Any sequence $(T_n)_{n=1}^{\infty} \in \ell_p(X,Y)$ induces an operator*

$$\begin{aligned}\Delta_T : \mathcal{L}(\ell_1, X) &\longrightarrow \ell_P(Y) \\ S &\rightsquigarrow (T_n S(e_n))_{n=1}^{\infty},\end{aligned}$$

.

which is (ℓ_p^s, ℓ_P)-summing with $\pi_{(\ell_p^s,\ell_P)}(\Delta_T) \leq \|(T_n)\|_{\ell P}(X,Y)$

Proof 3.7 *Take $S_1, \ldots, S_N \in \mathcal{L}(\ell_1, X)$, then*

$$\begin{aligned}\sum_{k=1}^{N} \|\Delta_T(S_k)\|^p &= \sum_{k=1}^{N}\sum_{n=1}^{\infty} \|T_n S_k(e_n)\|^p_Y \\ &\leq \left(\sum_{n=1}^{\infty} \|T_n\|_p\right) \sup_{a\in B_1} \left(\sum_{k=1}^{N} \|S_k(a)\|^p_X\right).\end{aligned}$$

This gives $\pi_{(\ell_p^s,\ell_p)}(\Delta_T) \leq \|(T_n)\|_{\ell_p(X,Y)}$. $\blacksquare$

To finish the section, we give several equivalent formulations for the notion of (ℓ_p^s, ℓ_p)-summing operator.

Note that We can write the fact $(T_n)_{n\in\mathbb{N}} \in \ell_p^s(X,Y)$ using duality:

$$\sup_{x\in B_X} \left(\sum_{i=1}^{n} \|T_i x\|^p_Y\right)^{1/p} = \sup_{(y_i^*)\in B^n_{\ell^p(Y^*)}} \sum_{i=1}^{n} \|T_i^* y_i^*\|_{X^*}.$$

Proposition 3.3 *Let $1 \leq p < \infty$ and $\Phi \in \mathcal{L}(\mathcal{L}(X,Y), Z)$. The following statements are equivalent:*

1. $\Phi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$.

2. *There exists a constant* $C > 0$ *such that*

$$\left(\sum_{i=1}^{n} \|\Phi(T_i)\|_Z^p\right)^{1/p} \leq C \sup_{(y_i^*)\in B^n_{\ell^p(Y^*)}} \sum_{i=1}^{n} \|T_i^* y_i^*\|_{X^*}$$

for every $T_1, \ldots, T_n$ *in* $\mathcal{L}(X,Y)$.

Moreover, $\pi_{(\ell_p^s,\ell_p)}(\Phi) = \inf\{C : C \quad verifying \quad ??\}$.

Proposition 3.4 *Let* $1 \leq p < \infty$ *and* $\Phi \in \mathcal{L}(\mathcal{L}(X,Y),Z)$. *The following statements are equivalent*

1. $\Phi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$.

2. *For each* $u \in \mathcal{L}(X, \ell_P(Y))$, *the operator*

$$\begin{aligned} \Phi_u : \mathcal{L}(\ell_p(Y),Y) &\longrightarrow Z \\ T &\rightsquigarrow \Phi(Tu), \end{aligned}$$

   . *is* (ℓ_p^s, ℓ_p)*-summing.*

3. *There exists a constant* c *such that* $\pi_{(\ell_p^s,\ell_p)}(\Phi_u) \leq c\|u\|$ *for each* $n \in \mathbb{N}$ *and each* $u \in \mathcal{L}(X, \ell_p^n(Y))$.

Moreover,

$$\pi_{(\ell_p^s,\ell_p)}(\Phi) = \sup\left\{\pi_{(\ell_p^s,\ell_p)}(\Phi u) : u \in L(X, p(Y)),\ \|u\| = 1\right\} = \inf\{c : c \text{ verifies (3)}\}.$$

Proof 3.8 ***(i)*** $\Rightarrow$ ***(ii):*** *Observe that* $\Psi_u : (\ell_P(Y), Y) \to (X,Y)$, *defined by* $\Psi_u(T) = Tu$, *is strongly* (ℓ_p^s, ℓ_P)*-summing with* $\pi_{(\ell_p^s,\ell_P)}(\Psi_u) \leq \|u\|$ *and* $\Phi_u = \Phi\Psi_u$. *Then, by (iii) in Proposition 2.17,* Φ_u *is* (ℓ_p^s, ℓ_P)*-summing with* $\pi_{(\ell_p^s,\ell_P)}(\Phi_u) \leq \pi_{(\ell_p^s,\ell_P)}(\Phi)\|u\|$.
(ii) $\Rightarrow$ ***(iii):*** *It follows easily from the closed graph theorem.*
(iii) $\Rightarrow$ ***(i):*** *Let* $T_1, \ldots, T_n$ *be operators in* $\mathcal{L}(X,Y)$, *and let* $S_n : X \to \ell_p^n(Y)$ *be defined by* $S_n(x) = (T_i x)_{i=1}^n$. *Note that* $\|S_n\|_{X,\ell_p^n(Y)} = \|(T_i)_{i=1}^n\|_{\ell_p^s(X,Y)}$. *For* $i = 1, \ldots, n$, *denote by* $Q_{i,n}$ *the projections* $Q_{i,n} : \ell_p^n(Y) \to Y$ *given by* $Q_{i,n}(y_j)_{j=1}^n = y_i$. *Hence, for each* $n \in \mathbb{N}$ *and* $i = 1, \ldots, n$,

$T_i = Q_{i,n}S_n$ and $\Phi(T_i) = \Phi(Q_{i,n}S_n) = \Phi_{S_n}(Q_{i,n})$. By hypothesis, Φ_{S_n} is (ℓ_p^s, ℓ_P)-summing with $\pi_{(\ell_p^s,\ell_P)}(\Phi_{S_n}) \leq B\|S_n\|_{X,\ell_p^n(Y)}$. Thus,

$$\begin{aligned}\left(\sum_{i=1}^{n} \|\Phi(T_i)\|_Z^p\right)^{1/p} &= \left(\sum_{i=1}^{n} \|\Phi_{S_n}(Q_{i,n})\|_Z^p\right)^{1/p} \\ &\leq B\|S_n\|_{X,\ell_p^n(Y)} \sup_{\lambda \in B_{\ell_p^n(Y)}} \left(\sum_{i=1}^{n} \|Q_{i,n}(\lambda)\|_Y^p\right)^{1/p} \\ &\leq B\|(T_i)_{i=1}^n\|_{\ell_p^s(X,Y)}.\end{aligned}$$

Consequently, Φ is (ℓ_p^s, ℓ_P)-summing with $\pi_{(\ell_p^s,\ell_P)} \leq B$. ■

3.2 Relations between the classes

As in the case of p-summing operators, we have the following inclusions.

Proposition 3.5 *Let X, Y, Z, and W be Banach spaces, and $1 \leq p < q \leq \infty$. Then*

(i) $\Pi_{(\ell_p^w,\ell_P^s)}(X, \mathcal{L}(Y,Z)) \subset \Pi_{(\ell_q^w,\ell_q^s)}(X, \mathcal{L}(Y,Z))$.

(ii) $\Pi_{(\ell_p^s,\ell_P)}(\mathcal{L}(X,Y), Z) \subset \Pi_{(\ell_q^s,\ell_q)}(\mathcal{L}(X,Y), Z)$.

(iii) $\Pi_{(\ell_p^s,\ell_P^s)}(\mathcal{L}(X,Y), \mathcal{L}(Z,W)) \subset \Pi_{(\ell_q^s,\ell_q^s)}(\mathcal{L}(X,Y), \mathcal{L}(Z,W))$.

Proof 3.9 *(i) Follows from Theorem 2.13 and $\Pi_p(X,Z) \subset \Pi_q(X,Z)$.*

(ii) To see (ii), we take $\Phi \in \Pi_{(\ell_p^s,\ell_P)}(\mathcal{L}(X,Y), Z)$ and $T_k \in \mathcal{L}(X,Y)$ for $k = 1, \ldots, n$. Let us write

$$\sum_{k=1}^{n} \|\Phi(T_k)\|^q = \sum_{k=1}^{n} \|\Phi(\beta_k T_k)\|^p,$$

where $\beta_k = \|\Phi(T_k)\|^{\frac{q-p}{p}}$.

Applying Hölder's inequality with conjugate indices q/p *and* $q/(q-p)$*, we get*

$$\begin{aligned}
\sum_{k=1}^{n} \|\Phi(T_k)\|^q &\leq \left(\pi_{(\ell_p^s,\ell_p)}(\Phi)\right)^p \sup_{x\in B_X} \left(\sum_{k=1}^{n} \|T_k(x)\|^p \beta_k^p\right) \\
&\leq \left(\pi_{(\ell_p^s,\ell_p)}(\Phi)\right)^p \sup_{x\in B_X} \left(\sum_{k=1}^{n} \|T_k(x)\|^q\right)^{\frac{p}{q}} \left(\sum_{k=1}^{n} \beta_k^{\frac{qp}{q-p}}\right)^{\frac{q-p}{q}} \\
&= \left(\pi_{(\ell_p^s,\ell_p)}(\Phi)\right)^p \sup_{x\in B_X} \left(\sum_{k=1}^{n} \|T_K(x)\|^q\right)^{\frac{p}{q}} \left(\sum_{k=1}^{n} \|\Phi(T_k)\|^q\right)^{\frac{q-p}{q}}.
\end{aligned}$$

This gives that

$$\left(\sum_{k=1}^{n} \|\Phi(T_k)\|^q\right)^{\frac{1}{q}} \leq \left(\pi_{(\ell_p^s,\ell_p)}(\Phi)\right) \sup_{x\in B_X} \left(\sum_{k=1}^{n} \|T_k(x)\|^q\right)^{\frac{1}{q}}.$$

(iii) *now follows using Theorem 2.16 and* ***(ii)****.*

Next, we are going to see that, under some assumptions on the Banach spaces, these classes coincide, at least for certain values of p *and* q*. Let us recall that some classical result, due to B. Maurey (see [?] or [?], Theorem 11.13), states that if* Y *has cotype 2 and* $2 < p < \infty$*, then*

$$\Pi_p(X,Y) = \Pi_2(X,Y)$$

.

Using Theorem 2.13, we get the following corollary.

Corollary 3.3 *Let* X*,* Y*,* Z *be Banach spaces, and* $2 < p < \infty$*. Assume that* Z *has cotype 2. Then,*

$$\Pi_{(\ell_p^w,\ell_p^s)}(X,\mathcal{L}(Y,Z)) = \Pi_{(\ell_2^w,\ell_2^s)}(X,\mathcal{L}(Y,Z)).$$

It is natural to ask whether there are generalizations in the framework of $_{(\ell_p^s,\ell_p)}$*-summing operators. The next result is the extension of Theorem 1.2.3 in [?] to our setting.*

Theorem 3.4 *Let* X*,* Y*,* Z*, and* W *be Banach spaces, and* $2 < p < \infty$*. Assume that* Y *has type* 2 *and* Z *has cotype* 2*. Then*

$$\Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z) = \Pi_{(\ell_2^s,\ell_2)}(\mathcal{L}(X,Y),Z).$$

Proof 3.10 *Let $\Phi \in \Pi_{(\ell_p^s,\ell_p)}(\mathcal{L}(X,Y),Z)$, and $T_1,\ldots,T_n$ be a finite sequence of operators in $\mathcal{L}(X,Y)$. Using that Z has cotype 2, one has*

$$\left(\sum_{i=1}^n \|\Phi(T_i)\|_Z^2\right)^{\frac{1}{2}} \le C_2(Z)\int_0^1 \left\|\sum_{i=1}^n r_i(t)\Phi(T_i)\right\|_Z dt$$
$$\le C_2(Z)\left(\int_0^1 \left\|\Phi\left(\sum_{i=1}^n r_i(t)T_i\right)\right\|_z^p dt\right)^{\frac{1}{P}}.$$

Observe that $\sum_{i=1}^n r_i(t)T_i$ is a simple function $\sum_{j=1}^{2n} S_j\chi_{\left[\frac{j-1}{2^n},\frac{j}{2^n}\right)}$for some operators S_j and then

$$\int_0^1 \left\|\Phi\left(\sum_{i=1}^n r_i(t)T_i\right)\right\|_z^p dt = 2^{-n}\sum_{j=1}^{2n}\|\Phi(S_j)\|_Z^p$$
$$\le \left(\pi_{(\ell_p^s,\ell_P)}(\Phi)\right)^P 2^{-n}\sup_{x\in B_X}\sum_{j=1}^{2^n}\|S_j(x)\|_Y^p$$
$$\le \left(\pi_{(\ell_p^s,\ell_P)}(\Phi)\right)^p \sup_{x\in B_X}\int_0^1 \|\sum_{i=1}^n r_i(t)T_i(x)\|_Y^p, dt.$$

Hence, using Kahane's inequality and the type 2 condition on Y, it yields

$$\left(\sum_{i=1}^n \|\Phi(T_i)\|_Z^2\right)^{\frac{1}{2}} \le \pi_{(\ell_p^s,\ell_P)}(\Phi)C_2(Z)\sup_{x\in B_X}\left(\int_0^1 \|\sum_{i=1}^n r_i(t)T_i(x)\|_Y^p dt\right)^{\frac{1}{p}}$$
$$\le \pi_{(\ell_p^s,\ell_P)}(\Phi)C_2(Z)K_p\sup_{x\in B_X}\int_0^1 \|\sum_{i=1}^n r_i(t)T_i(x)\|_Y, dt$$
$$\le \pi_{(\ell_p^s,\ell_P)}(\Phi)C_2(Z)T_2(Y)K_p\sup_{x\in B_X}\left(\sum_{i=1}^n \|T_i(x)\|_Y^2\right)^{\frac{1}{2}},$$

and the proof is finished.

Corollary 3.4 *Let $2 < p < \infty$, X, Y, Z, and W be Banach spaces. If Y has type 2 and W has cotype 2, then*

$$\Pi_{\ell_p^s,\ell_p^s}(\mathcal{L}(X,Y),\mathcal{L}(Z,W)) = \Pi_{\ell_p^s,\ell_p^s}(\mathcal{L}(X,Y),\mathcal{L}(Z,W)).$$

In particular, we get the following applications to operators acting on $\ell_r^w(X) = \mathcal{L}(\ell_{r'},X)$.

Corollary 3.5 *Let X and Y be Banach spaces of type 2 and cotype 2, respectively. If $2 < p < \infty$*

CHAPTER 3. (ℓ_p^s, ℓ_p)-SUMMING OPERATORS

and $1 \leq r, q < \infty$, *then*

1. $\Pi_{\ell_p^s, \ell_p}(\ell_r^w(X), Y) = \Pi_{\ell_2^s, \ell_2}(\ell_r^w(X), Y)$,

2. $\Pi_{\ell_p^s, \ell_p^s}(\ell_r^w(X), \ell_q^w(Y)) = \Pi_{\ell_2^s, \ell_2}(\ell_r^w(X), \ell_q^w(Y))$.

3.3 Pietsch-type domination of(ℓ_p^s, ℓ_p)-summing operators

Throughout this section we fix $1 \leq p < \infty$. *The aforementioned Pietsch-type domination theorem for* (ℓ_p^s, ℓ_p)*-summing operators proved in [[**?**], Theorem 3.1] reads as follows:*

Theorem 3.5 *(Botelho-Santos). Let* U *be a subspace of* $X \epsilon Y$ *and let* $S : U \to Z$ *be an*(ℓ_p^s, ℓ_p) *-summing operator. Then there exist a constant* $K < 0$ *and* $\mu \in P(B_X^*)$ *such that*

$$\|S(T)\|_Z \leq K \left(\int_{B_X^*} \|T(\cdot)\|_Y^p \, d\mu \right)^{\frac{1}{p}}. \tag{3.4}$$

or every $T \in U$

A first comment is that the integral of inequality **??** *is always well-defined for any* $T \in X\epsilon Y$ *and* $\mu \in P(B_X^*)$. *Indeed, the restriction* $T|_{B_X^*}$ *is* $(w^* - to - norm)$ *continuous, so it is universally strongly measurable. Since in addition* $T|_{B_X^*}$ *is bounded, it belongs to the Lebesgue-Bochner space* $L_p(\mu, X)$

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Summary and Keywords

ملخص

في هذه الأطروحة، تناولنا مقالا للباحثين *Blasco O.* و *Signes, T.* حيث قاما بتقديم ودراسة بعض المؤثرات الخطية المحدودة
$\Phi : \mathcal{L}(X,Y) \to \mathcal{L}(Z,W)$ بحيث $(T_n) \to (\Phi(T_n))$ يرفق $l_p^s(X,Y)$ الى $l_p(Z,W)$، $l_p^s(X,Y)$ الى $l_p^s(Z,W)$ و $l_p^w(X,Y)$ الى $l_p^s(Z,W)$. مبرهنة الهيمنة لبيتش والتي قدمت من طرف *Rodríguez José and Enrique A. Sánchez-Pérez* كذلك عولجت في هده المذكرة
الكلمات المفتاحية: مؤثر p جمعي، مؤثر ذي بعد منتهي ، خاصية مثالية المتعلقة بالموثر p الجمعي ،المؤثر الخطي المثالي، المؤثر (ℓ_p^s, ℓ_p)- الجمعي

Abstract

This work based on the article of O. Blasco and T. Signes, where the o authors introduced and studied the classes of bounded linear operators $\Phi : \mathcal{L}(X,Y) \to \mathcal{L}(Z,W)$ *such that* $(T_n) \to (\Phi(T_n))$ *maps* $l_p^s(X,Y)$ *into* $l_p(Z,W)$, $l_p^s(X,Y)$ *into* $l_p^s(Z,W)$ *and* $l_p^w(X,Y)$ *into* $l_p^s(Z,W)$. *The Pietsch-type dominations of* (l_p^s, l_p)*-summing linear operators wich presented nowadays by José Rodríguez and Enrique A. Sánchez-Pérez is also given .*
Keywords *:* $p-$*summing operator, Finite rank operator, ideal property of* $p-$*suming operators , Linear operator ideals,*(ℓ_p^s, ℓ_p)*-summing operators,*

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