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Pseudo ordered sets and trellises

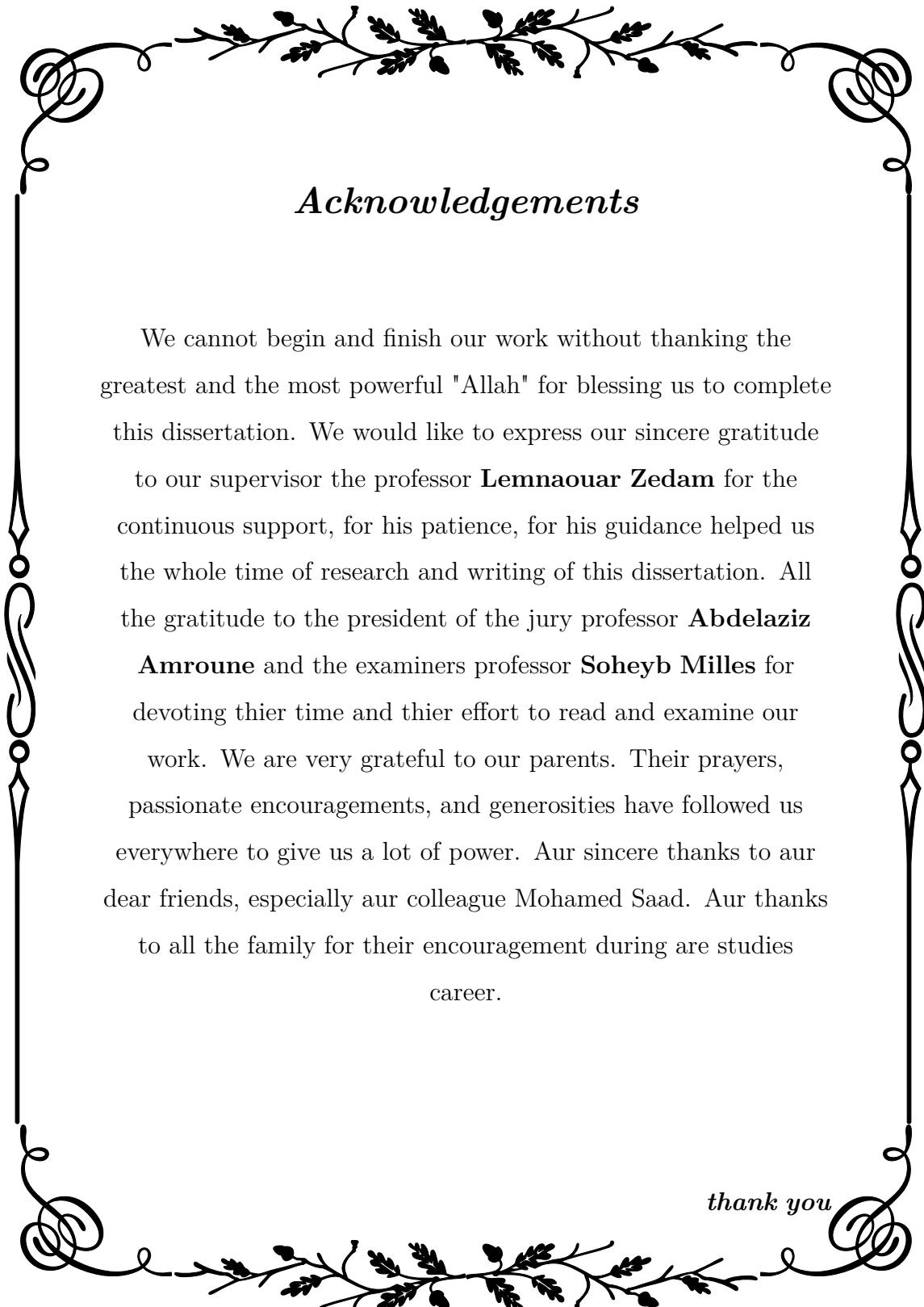
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Introduction

One of the most properties of binary relations is the transitivity property. It is a crucial necessity for numerous curiously areas of unadulterated and applied mathematics. In cross section hypothesis, the transitivity is essential for the associativity of operations of slightest upper and most prominent lower bounds. Nevertheless, the non transitivity connection is additionally critical and omnipresent in our genuine life as well as in mathematics.

One of the curiously scientific structure that consider the non-transitive relations is the non-associative lattice. To facilitate the meaning, it could be a structure like a lattice $(L, \leq, \wedge, \vee)$, but without the property of transitivity of $\leq$, which also means, without the associativity of the cross section meet ($\wedge$) and connect ($\vee$) operations. Fried [8] in 1970 was among the first authors who discussed the idea of non-associative lattice using the title tournament-lattice (T-lattice), and he examined its various properties and characterizations. Within the same setting, Skala in 1972 [19, 20] discussed the same notion, but she called it trellis.

She defined the notion of pseudo order as a reflexive and antisymmetric but not necessarily transitive relation. By stipulation of least upper bounds and greatest lower bounds of finite subsets on a given pseudo ordered set, the result is a structure called trellis (weakly associative lattice as it is called by several authors). In the same setting, she extended to this structure of trellis several known notions and results on lattices. Since then a lot of studies used these notions of pseudo ordered sets and trellises. For instance, Freid and Gratzner in 1973 [9] studied the non-associative extension of the class of distributive lattices. Later in 1975, Fried and Sós [10] studied the connection between trellis and projective planes. In 1973, Gladstien [11] characterized the complete trellises of finite length. In 2004 Bhatta and Shashirekha [1] generalized this characterization in terms of joins of cycles, and their connections with congruence relations.

This dissertation is divided into two chapters :

In the first chapter, we give the fundamental concepts of the ordered sets and lat-

Introduction

tices, and give some examples and properties of these concepts.

In the second chapter, we study some generalities of the above notations, it is the notion of pseudo-ordered sets and the notion of trellises.

These notions are almost similar, but without the property of transitivity (without the associativity properties in trellises).

Chapter 1

Ordered sets and lattices

In this chapter, we recall some basic notions and definitions of ordered sets, lattices, some of their characteristics and related concepts. More information on these notions can be found in [3, 4, 17, 18].

1.1 Ordered sets

In this section, We introduced the basic concepts of binary relations, order relations and types of relations.

1.1.1 Binary relations

Definition 1.1 (Binary relations).

Let A and B be a sets. A binary relation or, simply, relation from A to B is a subset of $A \times B$.

Suppose $\mathcal{R}$ is a relation from A to B . Then $\mathcal{R}$ is a set of ordered pairs where each first element comes from A and each second element comes from B . That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true :

- (i) $(a, b) \in \mathcal{R}$; we then say " a is $\mathcal{R}$ -related to b ", written $a\mathcal{R}b$;
- (ii) $(a, b) \notin \mathcal{R}$; we then say " a is not $\mathcal{R}$ -related to b ", written $a\not\mathcal{R}b$.

If $\mathcal{R}$ is a relation from a set A to itself, that is, if $\mathcal{R}$ is a subset of $A^2 = A \times A$, then we say that $\mathcal{R}$ is a relation on A .

The domain of a relation $\mathcal{R}$ is the set of all first elements of the ordered pairs which belong to $\mathcal{R}$, and the range is the set of second elements.

Example 1.1.

(a) $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $\mathcal{R} = \{(1, y), (1, z), (3, y)\}$. Then $\mathcal{R}$ is a relation from A to B since $\mathcal{R}$ is a subset of $A \times B$. With respect to this relation,

$$1\mathcal{R}y, 1\mathcal{R}z, 3\mathcal{R}y, \text{ but } 1\not\mathcal{R}x, 2\not\mathcal{R}x, 2\not\mathcal{R}z, 3\not\mathcal{R}x, 3\not\mathcal{R}z.$$

The domain of $\mathcal{R}$ is $\{1, 3\}$ and the range is y, z .

(b) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of set A and B , either $A \subseteq B$ or $A \not\subseteq B$.

(c) A familiar relation on the set $\mathbb{Z}$ of integers is "m divides n". A common notation for this relation is to write $m \mid n$ when m divides n . Thus $6 \mid 30$ but $7 \nmid 25$.

(d) Consider the set L of lines in the plane. Perpendicularity, written " $\perp$ ", is a relation on L . That is, given any pair of lines a and b , either $a \perp b$ or $a \not\perp b$. Similarly, "is parallel to," written " $\parallel$ ", is a relation on L since either $a \parallel b$ or $a \not\parallel b$.

(e) Let A be any set. An important relation on A is that of equality,

$$\{(a, b) / a \in A\}$$

which is usually denoted by " $=$ ". This relation is also called the identity or diagonal relation on δ_A and it will also be denoted by δ or simply.

(f) Let A be any set. Then $A \times A$ and ϕ are a subsets of $A \times A$ and hence are relations on A called the universal relation and empty relation, respectively.

Definition 1.2 (Inverse Relations).

Let $\mathcal{R}$ be a relation from a set A to a set B . The inverse of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to $\mathcal{R}$; that is,

$$\mathcal{R}^{-1} = \{(b, a) | (a, b) \in \mathcal{R}\}.$$

Example 1.2. Let $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $\mathcal{R} = \{(1, y), (1, z), (3, y)\}$. Then $\mathcal{R}^{-1} = \{(y, 1), (y, 3), (z, 1)\}$ is a inverse of $\mathcal{R}$.

Clearly, if $\mathcal{R}$ is any relation, then $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$. Also, the domain and range of $\mathcal{R}^{-1}$ are equal, respectively, to the range and domain of $\mathcal{R}$. Moreover, if $\mathcal{R}$ is a relation on A , then $\mathcal{R}^{-1}$ is also a relation on A .

Definition 1.3 (Composition of Relations).

Let A, B and C be sets, and let $\mathcal{R}$ be a relation from A to B and let $\mathcal{S}$ be a relation from B to C . That is, $\mathcal{R}$ is a subset of $A \times B$ and $\mathcal{S}$ is a subset of $B \times C$. Then $\mathcal{R}$ and $\mathcal{S}$ give rise to a relation from A to C denoted by $\mathcal{R} \circ \mathcal{S}$ and defined by :

$$a(\mathcal{R} \circ \mathcal{S})c \text{ if for some } b \in B \text{ we have } a\mathcal{R}b \text{ and } b\mathcal{S}c$$

That is ,

$$\mathcal{R} \circ \mathcal{S} = \{(a, c) ; \text{there exists } b \in B \text{ for which } (a, b) \in \mathcal{R} \text{ and } (b, c) \in \mathcal{S}\}.$$

The relation $\mathcal{R} \circ \mathcal{S}$ is called the composition of $\mathcal{R}$ and $\mathcal{S}$, it is sometimes denoted simply by $\mathcal{RS}$.

Suppose $\mathcal{R}$ is a relation on a set A , that is $\mathcal{R}$ is a relation from a set A to itself. Then $\mathcal{R} \circ \mathcal{R}$, the composition of $\mathcal{R}$ with itself, is always defined. Also, $\mathcal{R} \circ \mathcal{R}$ is sometimes denoted by $\mathcal{R}^2$. Similarly, $\mathcal{R}^3 = \mathcal{R}^2 \circ \mathcal{R} = \mathcal{R} \circ \mathcal{R} \circ \mathcal{R}$, and so on. Thus $\mathcal{R}^n$ is defined for all positive n . In the following Theorem, the composition of associative.

Theorem 1.1.

Let A, B, C and D be sets. Suppose $\mathcal{R}$ is a relation from A to B , $\mathcal{S}$ is a relation from B to C , and $\mathcal{T}$ is a relation from C to D . Then

$$(\mathcal{R} \circ \mathcal{S}) \circ \mathcal{T} = \mathcal{R} \circ (\mathcal{S} \circ \mathcal{T}).$$

1.1.2 Types of relations

This section discusses several of important types of relations defined on a set A .

Definition 1.4 (Reflexive Relation).

A relation $\mathcal{R}$ on a set A is reflexive if $a\mathcal{R}a$ for every $a \in A$, that is, if $(a, a) \in \mathcal{R}$ for every $a \in A$. Thus $\mathcal{R}$ is not reflexive if there exists $a \in A$ such that $(a, a) \notin \mathcal{R}$.

Example 1.3.

Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$$\mathcal{R}_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$$

$$\mathcal{R}_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$$

$$\mathcal{R}_3 = \{(1, 3), (2, 1)\}$$

$$\mathcal{R}_4 = \phi, \text{ the empty relation}$$

$$\mathcal{R}_5 = A \times A, \text{ the universal relation}$$

Determine which of the relations are reflexive.

Since A contains the four elements 1, 2, 3, and 4, a relation $\mathcal{R}$ on A is reflexive if it contains the four pairs $(1, 1), (2, 2), (3, 3)$, and $(4, 4)$. Thus only $\mathcal{R}_2$ and the universal relation $\mathcal{R}_5 = A \times A$ are reflexive. Note that $\mathcal{R}_1, \mathcal{R}_3$, and $\mathcal{R}_4$ are not reflexive since, for example, $(2, 2)$ does not belong to any of them.

Example 1.4.

Consider the following five relations :

- (1) Relation $\leq$ (less than or equal) on the set $\mathbb{Z}$ of integers.
- (2) Set inclusion $\subseteq$ on a collection $\mathbb{C}$ of sets.
- (3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.
- (4) Relation (parallel) on the set L of lines in the plan.
- (5) Relation $|$ of divisibility on the set $\mathbb{N}$ of positive integers. (Recall $x | y$ if there exists z such that $xz = y$).

Determine which of the relations are reflexive. The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every $x \in \mathbb{Z}$, $A \subseteq A$ for any set $A \in \mathbb{C}$, and $n \mid n$ for every positive integer $n \in \mathbb{N}$.

Definition 1.5 (Symmetric and Antisymmetric Relation).

A relation $\mathcal{R}$ on a set A is symmetric if whenever $a\mathcal{R}b$ then $b\mathcal{R}a$, that is, if whenever $(a, b) \in \mathcal{R}$ then $(b, a) \in \mathcal{R}$. Thus $\mathcal{R}$ is not symmetric if there exists $(a, b) \in A$ such that $(a, b) \in \mathcal{R}$ but $(b, a) \notin \mathcal{R}$.

Example 1.5.

(a) Determine which of the relations in Example 1.3 are symmetric; $\mathcal{R}_1$ is not symmetric since $(1, 2) \in \mathcal{R}_1$ but $(2, 1) \notin \mathcal{R}_1$. $\mathcal{R}_3$ is not symmetric since $(1, 3) \in \mathcal{R}_3$ but $(3, 1) \notin \mathcal{R}_3$. The other relations are symmetric ;

(b) Determine which of the relations in Example 1.4 are symmetric. The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a . Also, $\parallel$ is symmetric since if line a is parallel to line b then b is parallel to line a . The other relations are not symmetric. For example :

$$3 \leq 4 \text{ but } 4 \not\leq 3, \quad \{1, 2\} \subseteq \{1, 2, 3\} \text{ but } \{1, 2, 3\} \not\subseteq \{1, 2\}, \text{ and } 2 \mid 6 \text{ but } 6 \nmid 2.$$

A relation $\mathcal{R}$ on a set A is antisymmetric if whenever $a\mathcal{R}b$ and $b\mathcal{R}a$ then $a = b$, that is, if $a = b$ and $a\mathcal{R}b$ then $b\mathcal{R}a$. Thus $\mathcal{R}$ is not antisymmetric if there exist elements a and b in A such that $a\mathcal{R}b$ and $b\mathcal{R}a$.

Example 1.6.

(1) Determine which of the relations in Example 1.3 are antisymmetric.

$\mathcal{R}_2$ is not antisymmetric since $(1, 2)$ and $(2, 1)$ belong to $\mathcal{R}_2$, but $1 \neq 2$. Similarly, the universal relation $\mathcal{R}_3$ is not antisymmetric. All the other relations are antisymmetric.

(2) Determine which of the relations in Example 1.4 are antisymmetric.

The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$.

Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$. Also, divisibility on $\mathbb{N}$ is antisymmetric since whenever $m \mid n$ and $n \mid m$ then $m = n$.

(Note that divisibility on $\mathbb{Z}$ is not antisymmetric since $3 \mid -3$ and $-3 \mid 3$ but $3 \neq -3$.) The relations $\perp$ and $\parallel$ are not antisymmetric.

Remark 1.1.

The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation $\mathcal{R} = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $\mathcal{R} = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

Definition 1.6 (Transitive Relation).

A relation $\mathcal{R}$ on a set A is transitive if whenever $a\mathcal{R}b$ and $b\mathcal{R}c$ then $a\mathcal{R}c$, that is, if whenever $(a, b), (b, c) \in \mathcal{R}$ then $(a, c) \in \mathcal{R}$. Thus $\mathcal{R}$ is not transitive if there exist $a, b, c \in A$ such that $(a, b), (b, c) \in \mathcal{R}$ but $(a, c) \notin \mathcal{R}$.

Example 1.7.

(a) Determine which of the relations in Example 1.3 are transitive. The relation $\mathcal{R}_3$ is not transitive since $(2, 1), (1, 3) \in \mathcal{R}_3$ but $(2, 3) \notin \mathcal{R}_3$. All the other relations are transitive;

(b) Determine which of the relations in Example 1.4 are transitive. The relations $\leq$, $\subseteq$, and $\mid$ are transitive, but certainly not $\perp$. Also, since no line is parallel to itself, we can have ab and ba , but not aa . Thus $\parallel$ is not transitive. (We note that the relation "is parallel or equal to" is a transitive relation on the set L of lines in the plane). The property of transitivity can also be expressed in terms of the composition of relations. For a relation $\mathcal{R}$ on A , we define $\mathcal{R}^2 = \mathcal{R} \circ \mathcal{R}$ and, more generally, $\mathcal{R}^n = \mathcal{R}^{n-1} \circ \mathcal{R}$. Then we have the following result.

In the following Theorem, we present an equivalent definition of transitive.

Theorem 1.2.

A relation $\mathcal{R}$ is transitive if and only if, for every $n \geq 1$, $\mathcal{R}^n \subseteq \mathcal{R}$.

1.1.3 Order relations

Definition 1.7 (Order relation).

A binary relation $\mathcal{O}$ on a set X is an order if it satisfies the following three properties:

- (1) *Reflexivity: for each $x \in X$, $x\mathcal{O}x$.*
- (2) *Antisymmetry: for all $x, y \in X$, $x\mathcal{O}y$ and $y\mathcal{O}x$ imply $x = y$.*
- (3) *Transitivity: for all $x, y, z \in X$, $x\mathcal{O}y$ and $y\mathcal{O}z$ imply $x\mathcal{O}z$.*

The order $\mathcal{O}$ is linear (or total) if, for all $x, y \in X$, $x\mathcal{O}y$ or $y\mathcal{O}x$.

Example 1.8.

- (a) *Let S be any collection of sets. The relation $\subseteq$ of set inclusion is an ordering of S . Specifically, $A \subseteq A$ for any set A ; if $A \subseteq B$ and $B \subseteq A$ then $A = B$; and if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$;*
- (b) *Consider the set $\mathbb{N}$ of positive integers. We say "a divides b," written $a \setminus b$, if there exists an integer c such that $ac = b$. For example, $2 \setminus 4, 3 \setminus 12, 7 \setminus 21$, and so on. This relation of divisibility is an ordering of $\mathbb{N}$;*
- (c) *The relation " $\mid$ " of divisibility is not an ordering of the set $\mathbb{Z}$ of integers. Specifically, the relation is not antisymmetric. For instance, $2 \mid -2$ and $-2 \setminus 2$, but $2 \neq -2$;*
- (d) *Consider the set $\mathbb{Z}$ of integers. Define $a\mathcal{R}b$ if there is a positive integer r such that $b = ar$. For instance, $2\mathcal{R}8$ since $8 = 2 \times 4$. Then $\mathcal{R}$ is an ordering of $\mathbb{Z}$.*

Definition 1.8 (Ordered set).

An ordered set (or a partially ordered set or a poset) is an ordered pair $P = (X, \mathcal{O})$ where X is a set and $\mathcal{O}$ an order on X . If $\mathcal{O}$ is a linear order, $P = (X, \mathcal{O})$ is then called a linearly ordered set (or a totally ordered set or a chain).

Example 1.9.

The set $\mathbb{R}$ of real numbers with the usual order relation $\leq$ is a poset. In particular, the poset $(\mathbb{R}, \leq)$ is called a chain or, equivalently, totally ordered set

Definition 1.9.

Suppose a and b are elements in a partially ordered set S . We say that a and b are comparable if

$$a \lesssim b \text{ or } b \lesssim a$$

that is, if one of them precedes the other. Thus a and b are noncomparable, written

$$a || b$$

if neither $a \lesssim b$ nor $b \lesssim a$.

Definition 1.10.

Let Q be a subset of an ordered set P . If the elements of Q are pairwise incomparable, we say that Q is an antichain of P . An antichain of P is maximal if it is not contained in any other antichain of P .

Definition 1.11.

The diagram (or Hasse diagram) of an ordered set $P = (X, \leq)$ is a representation of its covering graph in which the elements x of P are represented by points $p(x)$ of the plane, with the following two rules :

- If $x < y$ (the horizontal line going through) $p(x)$ is below (the horizontal line going through) $p(y)$;
- $p(x)$ and $p(y)$ are linked by a line segment if and only if $x < y$.

Example 1.10.

- (a) Let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by the relation " x divides y ". The diagram of A is given in Fig. 1.1 (a).
- (b) Let $B = \{a, b, c, d, e\}$. The diagram in Fig. 1.1 (b) defines a partial order on B in the natural way. That is, $d \leq b, d \leq a, e \leq c$ and so on.
- (c) The diagram of a finite linearly ordered set, i.e., a finite chain, consists simply of one path. For example, Fig. 1.1 (c) shows the diagram of a chain with five elements.

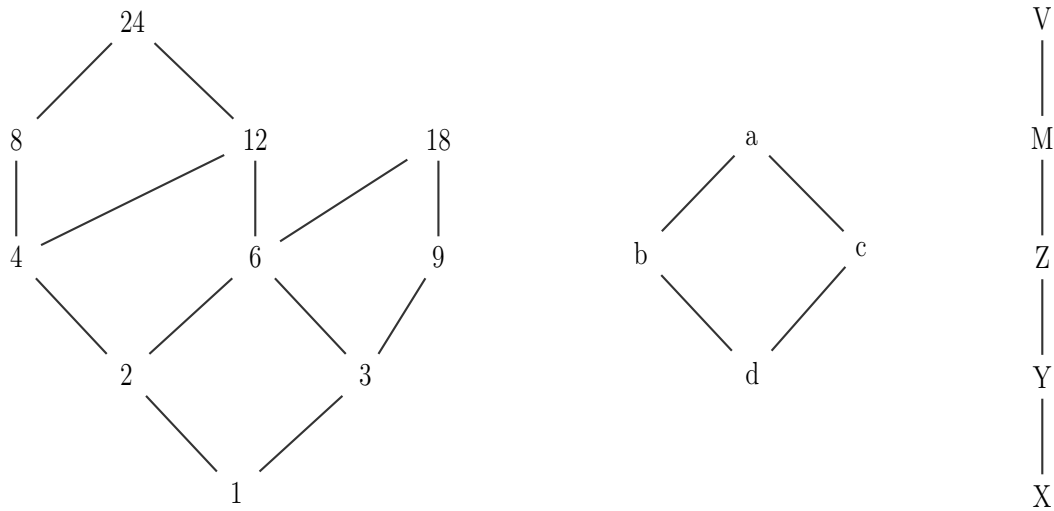


Figure 1.1: Fig. 1.1

Definition 1.12.

Let $P = (X, \leq)$ be an ordered set.

- The minimum of P if $x \leq y$ for any $y \in P$.
- The maximum of P if $y \leq x$ for any $y \in P$.
- A minimal element of P if there does not exist $y \in P$ such that $y < x$. (The set of minimal elements of P is denoted by $\text{Min}P$).

- *A maximal element of P if there does not exist $y \in P$ such that $x < y$. (The set of maximal elements of P is denoted by $MaxP$).*
- *$Sup(A) = \min(MaxP)$.*
- *$Inf(A) = \max(MinP)$.*

Rather than "minimum" (respectively, "maximum"), we also say the smallest or the least-element (respectively, the greatest element) of P . These elements are necessarily unique when they exist and are often respectively denoted by $0P$ and $1P$ (or, simply, 0 and 1) and we then say that P is bounded.

1.2 Lattices

In this section, We provided the concepts and examples of lattices, distributive lattices and complemented lattices.

1.2.1 Concepts and examples

Definition 1.13.

An ordered set P is a meet-semilattice if any pair $\{x, y\}$ of its elements has a meet $x \wedge y$. It is a join-semilattice if any pair $\{x, y\}$ of its elements has a join $x \vee y$. It is a lattice if any pair of its elements has a meet and a join, so if it is a meet and a join-semilattice.

Definition 1.14.

A lattice is an ordered set $(L, \leq)$ any two of whose elements have both a greatest lower bound (g.l.b.), denoted by $x \wedge y$, and a least upper bound (l.u.b.), denoted by $x \vee y$.

Notation 1.1.

$$Sup\{x, y\} = x \vee y$$

$$Inf\{x, y\} = x \wedge y$$

We will sometimes denote the lattice by $(L, \wedge, \vee)$ when we want to show which operations are involved.

Example 1.11.

(1) Let C be a collection of sets closed under intersection and union. Then $(C, \cap, \cup)$ is a lattice. In this lattice, the partial order relation is the same as the set inclusion relation ;

(2) Every chain C is a lattice such that : $\forall x, y \in C$:

$$x \vee y = \max(x, y);$$

$$x \wedge y = \min(x, y) .$$

(3) $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ with the division is a lattice such that :

$$x \vee y = \text{lcm}(x, y). \quad x \wedge y = \text{gcd}(x, y).$$

Now, we will study properties for lattice.

Properties 1.

Let L be a lattice, and let a, b, c are an elements in L :

$$\begin{aligned} 1. \quad a \leq b &\Leftrightarrow a = a \wedge b \\ &\Leftrightarrow b = a \vee b. \end{aligned}$$

2. Commutative law :

$$(1a) \quad a \wedge b = b \wedge a$$

$$(1b) \quad a \vee b = b \vee a$$

3. Associative law :

$$(2a) \quad (a \wedge b) \wedge c = a \wedge (b \wedge c)$$

$$(2b) \quad (a \vee b) \vee c = a \vee (b \vee c)$$

4. Absorption law :

$$(3a) \quad a \wedge (a \vee b) = a$$

$$(3b) \quad a \vee (a \wedge b) = a$$

Definition 1.15 (Duality).

The dual of any statement in a lattice $(L, \wedge, \vee)$ is defined to be the statement that is obtained by interchanging $\wedge$ and $\vee$. For example, the dual of

$$a \wedge (b \vee a) = a \vee a \quad \text{is} \quad a \vee (b \wedge a) = a \wedge a$$

In the following Theorem, we show that idempotent Law.

Theorem 1.3 (Idempotent Law).

(i) $a \wedge a = a$, (ii) $a \vee a = a$. The proof of (i) requires only two lines :

$$\begin{aligned} a \wedge a &= a \wedge (a \vee (a \wedge b)) \quad (\text{using}(3b)) \\ &= a \quad (\text{using}(3a)). \end{aligned}$$

The proof of (ii) follows from the above principle of duality (or can be proved in a similar manner).

In this Theorem, We'll apply some properties lattice.

Theorem 1.4.

Let L be a lattice. Then:

- (i) $a \wedge b = a$ if and only if $a \vee b = b$;
- (ii) The relation $a \leq b$ (defined by $a \wedge b = a$ or $a \vee b = b$) is a partial order on L .

Proof. (i) Suppose $a \wedge b = a$. Using the absorption law in the first step we have:

$$b = b \vee (b \wedge a) = b \vee (a \wedge b) = b \vee a = a \vee b$$

Now suppose $a \vee b = b$. Again using the absorption law in the first step we have:

$$a = a \wedge (a \vee b) = a \wedge b$$

Thus. $a \wedge b = a$ if and only if $a \vee b = b$.

- (ii) For any a in L , we have $a \wedge a = a$ by idempotency. Hence $a \leq a$, and so $\leq$ is reflexive. Suppose $a \leq b$ and $b \leq a$. Then $a \wedge b = a$ and $b \wedge a = b$. Therefore, $a = a \wedge b = b \wedge a = b$, and so $\leq$ is antisymmetric. Lastly, suppose $a \leq b$ and $b \leq c$. Then $a \wedge b = a$ and $b \wedge c = b$. Thus

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a$$

Therefor. $a \leq c$, and so $\leq$ is transitive. Accordingly, $\leq$ is a partial order on L . □

Definition 1.16 (Bounded lattice).

A lattice L is said to have a lower bound 0 if for any element x in L we have $0 \leq x$. Analogously, L is said to have an upper bound I if for any x in L we have $x \leq I$. We say L is bounded if L has both a lower bound 0 and an upper bound I . In such a lattice we have the identities

$$a \vee I = I, a \wedge I = a, a \vee 0 = a, a \wedge 0 = 0.$$

for any element a in L .

Theorem 1.5.

Every finite lattice L is bounded.

1.2.2 Distributive lattices

Definition 1.17.

A lattice L is said to be distributive if for any elements a, b, c in L we have the following:

$$(1) \ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$(2) \ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Example 1.12.

(1) Every chain is a distributive lattice

$$\min(x, \max(y, z)) = \max(\min(x, y), \min(x, z)) ;$$

(2) $(P(E), \subseteq)$ is a distributive lattice.

In this Theorem, We'll apply some properties lattice.

Theorem 1.6.

A lattice L is distributive if and only if it satisfies any of the following properties:

(a) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z);$

(b) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z);$

(c) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x);$

(d) $[x \wedge y = x \wedge z \text{ and } x \vee y = x \vee z] \implies (y = z).$

1.2.3 Complemented lattices**Definition 1.18** (Complements).

Let L be a bounded lattice with lower bound 0 and upper bound I . Let a be an element of L . An element x in L is called a complement of a if

$$a \vee x = I \quad \text{and} \quad a \wedge x = 0.$$

Complements need not exist and need not be unique.

In the following Theorem, we show that oneness complement in bounded distributive lattice.

Theorem 1.7.

Let L be a bounded distributive lattice. Then complements are unique if they exist.

Proof. Suppose x and y are complements of any element a in L . Then

$$a \vee x = I, a \vee y = I, a \wedge x = 0, a \wedge y = 0$$

Using distributivity,

$$x = x \vee 0 = x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y) = I \wedge (x \vee y) = x \vee y$$

Similarly,

$$y = y \vee 0 = y \vee (a \wedge x) = (y \vee a) \wedge (y \vee x) = I \wedge (y \vee x) = y \vee x$$

Thus.

$$x = x \vee y = y \vee x = y$$

and the theorem is proved. □

Definition 1.19 (Complemented Lattices).

A lattice L is said to be complemented if L is bounded and every element in L has a complement.

Example 1.13.

(1) $(P(E), \subseteq)$ is complemented indeed $A \cap \mathcal{C}(A) = \phi$, $A \cup \mathcal{C}(A) = E$

(2) $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ is complemented :

$$2' = 15, 15' = 2, 3' = 10, 10' = 3, 5' = 6, 6' = 5, 1' = 30, 30' = 1.$$

Chapter 2

Pseudo ordered sets and trellises

This chapter contains the basic definitions, examples and properties of pseudo-ordered sets and trellises. More information can be found in [20, 8].

2.1 Pseudo ordered sets

In this section, we provided a definition of pseudo order relations and contraction sets and the examples.

2.1.1 Pseudo order relations

Definition 2.1. [20]

Let X be a non-empty set, a pseudo order on X is a binary relation $\trianglelefteq$ on X such that, for all $x, y \in X$, the conditions hold

(1) $x \trianglelefteq x$ (reflexivity) ;

(2) $x \trianglelefteq y$ and $y \trianglelefteq x$ implies $x = y$ (antisymmetry).

A set X equipped with a pseudo order relation $\trianglelefteq$ is called a pseudo ordered set (or psoet, for short). For $x, y \in X$, if $x \trianglelefteq y$ and $x \neq y$, then we write $x \triangleleft y$. For a subset $A \subseteq X$, the notions of a lower bound, an upper bound, greatest lower bound (g.l.b), least upper bound (l.u.b) and other notions are defined analogously to the

corresponding notion in a partially ordered set (poset, for short). Now, we will only introduce the new concepts related to the pseudo-order relation.

Remark 2.1.

It is easily seen that any order relation on a set X is a pseudo order relation on X .

Example 2.1.

- (1) Let $X = \mathbb{R}$ be the set of real numbers. The relation on X defined, for any $x, y \in X$

$$X \trianglelefteq Y \quad \text{if and only if} \quad 0 \leq y - x \leq a \quad \text{where} \quad a \in \mathbb{R}^+$$

is a pseudo order relation on X .

- (2) Let $X = \{a, b, c, d, e\}$ and the binary relation defined on X as follows

$$\trianglelefteq = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e), (b, c), (b, e), (c, d), (c, e), (d, e)\},$$

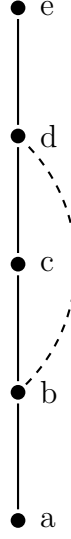
the couple $(X, \trianglelefteq)$ is a pseudo ordered set. We note that $\trianglelefteq$ is not transitive because (b, c) and $(c, d) \in \trianglelefteq$, while $(b, d) \notin \trianglelefteq$.

In the sequel we will represent finite pseudo ordered sets by Hasse diagram with the convention that if x is below y and is joined to y by a line if $x \trianglelefteq y$.

If x is below y , and x and y are not related, then x and y will be joined by a dashed curve.

Example 2.2.

Let $A = \{a, b, c, d, e\}$ be a set and $\trianglelefteq$ be a pseudo order relation on A , where $a \trianglelefteq b \trianglelefteq c \trianglelefteq d \trianglelefteq e$ (i.e.; $a \trianglelefteq x \trianglelefteq e$, for any $x \in A$). The Hasse diagram of A is depicted by the following figure:

Figure 2.1: Hasse diagram of the psoset A

According to Skala[20], we have the following definitions

Definition 2.2. [20]

Let $(X, \trianglelefteq)$ be a psoset and B be a subset of X . We define a relation $\trianglelefteq_B$ on B as follows : for any $\hat{b}, y \in B$ there is a finite sequence $(b_1, \dots, b_n) \in B$ such that $b \trianglelefteq b_1 \trianglelefteq \dots \trianglelefteq b_n \trianglelefteq \hat{b}$. $(B, \trianglelefteq_B)$

is called *pseudo chain* or (*P-chain*, for short) if for each pair of elements $b, \hat{b}$ of B either of the relations $b \trianglelefteq_B \hat{b}$ or $\hat{b} \trianglelefteq_B b$ holds, ($\trianglerighteq_B$ can be defined dually). A subset B is called a *cycle* if, for each pair of elements b and $\hat{b}$ of B both the relations $b \trianglelefteq_B \hat{b}$ and $\hat{b} \trianglelefteq_B b$ hold.

Empty set is a cycle, any single element set on psoset is also a cycle. A non-trivial cycle is a cycle having more than one element and it contains at least three elements.

A psoset is called *acyclic* if it does not contain a non-trivial cycle.

A pseudo chain (*P-chain*) $C = \{a_i | i = 1, 2, \dots\}$ of elements of psoset $(X, \trianglelefteq)$ is said to be an *infinite ascending p-chain* in X if $a_1 \trianglelefteq a_2 \trianglelefteq a_3, \dots$, where C is infinite. A psoset $(X, \trianglelefteq)$ is said to satisfy the *ascending p-chain condition* if there is no infinite ascending p-chain in X .

Definition 2.3. [20]

A preorder relation (preorder, for short) is a binary relation $\lesssim$ over a set X which is reflexive ($a \lesssim a$ for all $a \in X$), and transitive ($a \lesssim b$ and $b \lesssim c$ implies $a \lesssim c$, for any $a, b, c \in X$) but not necessarily antisymmetric. A set with a preorder relation is called a preordered set.

In the following Proposition, we will study properties pseudo ordered sets.

Proposition 2.1. [20]

Let $(X, \trianglelefteq)$ be a pseudo-ordered set and A be a subset on X . A binary relation $\lesssim$ can be defined on A by setting $x \lesssim y$ for two elements of A if and only if there exists a finite sequence $(a_1, \dots, a_n)$ of elements from X such that $x \trianglelefteq a_1 \trianglelefteq \dots \trianglelefteq a_n \trianglelefteq y$, is a preorder relation on A .

Proof. Let $(X, \trianglelefteq)$ be a pseudo ordered set and A be a non-empty set on X .

- (i) For any $x \in A$, we have $x \trianglelefteq x$ this implies $x \lesssim x$. Hence, $\lesssim$ is reflexive;
- (ii) Let $x, y, z \in A$. Suppose that $x \lesssim y$ and $y \lesssim z$ and we show that $x \lesssim z$.
From the supposition, it follows that there exist two sequences $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n) \in X$ such that $x \trianglelefteq a_1 \trianglelefteq \dots \trianglelefteq a_n \trianglelefteq y$ and $y \trianglelefteq b_1 \trianglelefteq \dots \trianglelefteq b_n \trianglelefteq z$, it is easy to see that, there exists another sequence $(a_1, \dots, b_n) \in X$ such that $x \trianglelefteq a_1 \trianglelefteq \dots \trianglelefteq b_n \trianglelefteq z$. That is, $x \lesssim z$. Hence, $\lesssim$ is transitive. Therefore, $\lesssim$ is a preorder relation on A .

□

Remark 2.2.

If for each pair of elements x, y of A at least one of the relations $x \lesssim y$ or $y \lesssim x$ holds then A will be called a pseudo-chain or (p -chain, for short).

A pseudo order on a set X is said to be linear if X itself is a pseudo-chain.

It can easily shown that any pseudo order can be extended to a linear which leads the following theorem.

Theorem 2.1. [20]

Any pseudo order relation $\trianglelefteq$ on a set X can be extended to a linear pseudo order $\trianglelefteq^*$ on X such that any cycle which respects $\trianglelefteq$ also respects $\trianglelefteq^*$.

The following theorem is immediately obtained from Theorem 2.1.

Theorem 2.2. [20]

Any pseudo order is the intersection of all its linear extensions.

Definition 2.4. [20]

Let $(X, \trianglelefteq)$ be a psoet. We can define the dual psoet of X , $(X^d, \trianglerighteq)$ by: $x \trianglerighteq y$ to holds in X^d if and only if $x \trianglelefteq y$ holds in X . For X finite, we obtain a diagram for X^d simply by "tuning upside down" a diagram of X . Given a statement ϕ about pseudo ordered sets which is true in all pseudo ordered sets, the dual statement ϕ^d is also true in all pseudo ordered sets. The notion of consistent family and contraction set of a pseudo ordered set $(X, \trianglelefteq)$ was introduced by Skala [20].

Definition 2.5. [20]

Let $(X, \trianglelefteq)$ be a psoet and C be a family of subsets of X , (i.e., $C \subseteq \wp(X)$, where $\wp(X)$ is the power set of X). The family C is called a consistent family of $(X, \trianglelefteq)$ if it satisfies the following condition: For any A and B two distinct members of C , there do not exist elements $x_1, x_2 \in A$ and $y_1, y_2 \in B$ such that $x_1 \trianglelefteq y_1$ and $y_2 \trianglelefteq x_2$.

2.1.2 Contraction sets**Definition 2.6.** [20]

For any consistent family C of $(X, \trianglelefteq)$, the pseudo order relation defined as follows, for any $A, B \in C$, $A \trianglelefteq_C B$ if and only if there exist $x \in A$ and $y \in B$ such that $x \trianglelefteq y$. Any consistent family C of $(X, \trianglelefteq)$ with the pseudo order $\trianglelefteq_C$ (i.e., $(C, \trianglelefteq_C)$) will be called a contraction set of $(X, \trianglelefteq)$.

Example 2.3.

The family of all pairs $\{2k-1, 2k\}$, and $k = 1, 2, 3, \dots$ is consistent family of $(\mathbb{N}, \leq)$.

Theorem 2.3. [20]

Any pseudo ordered set is isomorphic with a contraction set of partially ordered set.

Definition 2.7.

Let $(X, \trianglelefteq)$ be a psoet, A be a subset of X . The set A is called a convex subset of $(X, \trianglelefteq)$, if for all $b, c \in A$ and $a \in X$:

$$b \trianglelefteq a \trianglelefteq c \text{ implies } a \in A.$$

Definition 2.8. [20]

A psoet is called to have the fixed point property if every order-preserving mapping $f : X \rightarrow X$ has a fixed point (i.e., there exists an element x of X such that $f(x) = x$).

2.2 Trellises

In this section, we defined of trellises and give some examples, and meet and join-associative.

2.2.1 Definitions and examples

The notion of trellis was introduced by Fried [8] and Skala [19] [20]. As one of the most important algebraic structures in order theory. Which is considered as an extension of the notion of lattice by dropping the property of transitivity. A trellis is defined as a psoet $(T, \trianglelefteq)$ in which pair of elements has a least upper bound and a greatest lower bound. We use the following notation, we write $x \sqcup y$ in place of $\sup x, y$ when it exists and $x \sqcap y$ in place of $\inf x, y$ when it exists. Similarly, we write $\sqcup S$ and $\sqcap$ instead of $(\sup S)$ and $(\inf S)$ when these exist.

Definition 2.9.

Let $(T, \trianglelefteq)$ be a non-empty pseudo ordered set.

- (1) $(T, \trianglelefteq)$ is called a $\sqcap$ -semi-trellis if $x \sqcap y$ exists for any $x, y \in T$. ;
- (2) $(T, \trianglelefteq)$ is called a $\sqcup$ -semi-trellis if $x \sqcup y$ exists for any $x, y \in T$. ;

(3) $(T, \trianglelefteq)$ is called a trellis if it is both a $\sqcap$ -semi-trellis and $\sqcup$ -semi-trellis.

In other words, a trellis is an algebraic structure $(T, \sqcap, \sqcup)$ where the binary operations $\sqcap$ and $\sqcup$ satisfy the following properties, for any $x, y, z \in T$.

- (i) $x \sqcap x = x \sqcup x = x$ (idempotency) ;
- (ii) $x \sqcap y = y \sqcap x$ and $x \sqcup y = y \sqcup x$ (commutativity) ;
- (iii) $x \sqcap (x \sqcup y) = x = x \sqcup (x \sqcap y)$ (absorption laws) ;
- (iv) $x \sqcap ((x \sqcup y) \sqcap (x \sqcup z)) = x = x \sqcup ((x \sqcap y) \sqcup (x \sqcap z))$ (weak associativity).

For a given trellis $(T, \trianglelefteq, \sqcap, \sqcup)$ and $x, y \in T$, the following statements are equivalent

- (1) $x \trianglelefteq y$;
- (2) $x \sqcup y = y$;
- (3) $x \sqcap y = x$.

The notions of sub-trellis, complete trellis, distributive and modular trellis are defined analogously to the corresponding notions in partially ordered sets.

Remark 2.3.

It is well known that every finite lattice is complete, but this property is not true in the finite trellis. Indeed, let us consider the trellis $(T, \trianglelefteq, \sqcap, \sqcup)$ given in the following table and figure.

$\sqcap \sqcup$	a	b	c	d	e
a		b	c	d	a
b	a		e	e	e
c	a	a		e	e
d	a	a	a		e
e	e	b	c	d	

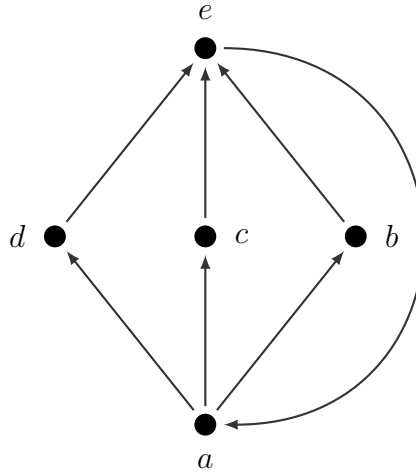


Figure 2.2: Finite trellis not complete.

We can see that T is not complete, since for example $\sqcup a, b, d$ does not exist.

Definition 2.10. [20]

Let T be a trellis. We say that T has a greatest element if there exists $1 \in T$ such that $x \sqcap 1 = x$, for any $x \in T$. Dually, we say that T has a smallest element if there exists $0 \in T$ such that $x \sqcup 0 = x$, for any $x \in T$. A trellis $(T, \sqcap, \sqcup)$ possessing 0 and 1 is called bounded trellis.

Remark 2.4. [20]

If every non-empty subset of a pseudo ordered set T with the smallest element 0 has a least upper bound, then T is a complete trellis.

Example 2.4.

Let $(X, \trianglelefteq)$ be a pset given in Example 2.2 with the l.u.bs and g.l.bs given in the following table (see eg. Figure 2.3)

$\{x, y\}$	$x \sqcap y$	$x \sqcup y$
$\{a, b\}$	b	a
$\{a, c\}$	a	c
$\{a, d\}$	a	d
$\{a, e\}$	a	e
$\{b, c\}$	b	c
$\{b, d\}$	a	e
$\{b, e\}$	b	e
$\{c, d\}$	c	d
$\{c, e\}$	c	e
$\{d, e\}$	d	e

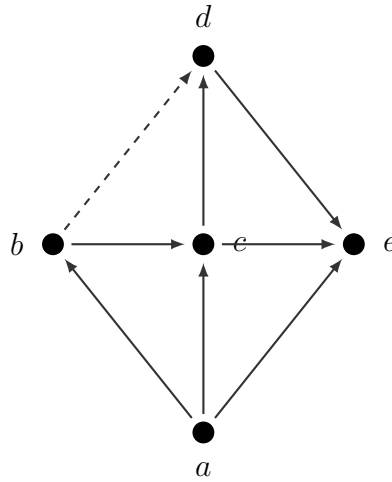


Figure 2.3: Representaiton of a trellis.

Definition 2.11. [20]

A mapping f from a trellis $\mathcal{L} = (T, \sqsubseteq)$ into a trellis $(\mathcal{M} = \hat{T}, \hat{\sqsubseteq})$ (i.e., $f : \mathcal{L} \rightarrow \mathcal{M}$) is called

- (1) Homomorphism if $f(x \sqcap_{\mathcal{L}} y) = f(x) \sqcap_{\mathcal{M}} f(y)$ and $f(x \sqcup_{\mathcal{L}} y) = f(x) \sqcup_{\mathcal{M}} f(y)$, for any $x, y \in T$;

- (2) *Embedding if f is a one to one homomorphism ;*
- (3) *Isomorphism if f is both one to one and onto homomorphism.*

A homomorphism f of a trellis T into itself is called an endomorphism.

Next, we introduce some important special elements of a trellis.

Definition 2.12. [20]

An element z of a trellis T is said to be

- (i) *right-transitive if $z \trianglelefteq x \trianglelefteq y$ implies $z \trianglelefteq y$;*
- (ii) *left-transitive if $x \trianglelefteq y \trianglelefteq z$ implies $x \trianglelefteq z$;*
- (iii) *middle-transitive if $x \trianglelefteq z \trianglelefteq y$ implies $x \trianglelefteq y$;*
- (iv) *transitive if z is right, left and middle transitive.*

Example 2.5.

In the trellis T given in Figure 2.3, note that $b \trianglelefteq c \trianglelefteq d$ and $b \not\trianglelefteq d$ then b isn't right-transitive. Otherwise, since if $x \trianglelefteq y \trianglelefteq e$ implies $x \trianglelefteq e$, for any $x, y \in T$, then it holds that e is a left-transitive element.

In the following Theorem, a study left, right-transitive.

Theorem 2.4. [20]

Let T be a trellis and $x, a \in X$. It hold that

- (i) *If a is left-transitive element, then*
 - (a) $x \lesssim a$, *implies $x \trianglelefteq a$;*
 - (b) $x \trianglelefteq a$, *implies that $x \sqcup y \trianglelefteq \trianglelefteq a \sqcup y$, for any $y \in X$.*
- (ii) *Dually, if a is right-transitive element, then ;*
 - (a) $a \lesssim x$, *implies $a \trianglelefteq x$;*
 - (b) $a \trianglelefteq x$, *implies that $a \sqcap y \trianglelefteq x \sqcap y$, for any $y \in X$.*

2.2.2 Meet and join-associative elements

Definition 2.13. [20]

Let $(T, \leq, \sqcap, \sqcup)$ be a trellis and (x, y, z) a sequence of elements, then

(i) (x, y, z) is $\sqcap$ -associative if $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$;

(ii) (x, y, z) is $\sqcup$ -associative if $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$.

Definition 2.14. [20]

An element a is called $\sqcap$ -associative if any sequence of three elements including a is $\sqcap$ -associative. An element $\sqcup$ -associative a is defined dually. If an element is both $\sqcap$ -associative and $\sqcup$ -associative, it is said to be associative.

In the following Theorem, the associative and the transitive elements are equivalent.

Theorem 2.5. [20]

Let T be a trellis and $\mathcal{B}$ be a pseudo chain on T . An element of $\mathcal{B}$ is associative if and only if it is transitive.

In the following Theorem transitive equivalent associative in a trellis.

Theorem 2.6.

Let $(T, \leq, \sqcap, \sqcup)$ be a trellis, then the following statements are equivalent

(i) $\leq$ is transitive ;

(ii) the operations $\sqcup$ and $\sqcap$ are associative.

In the next Theorem, $\sqcup$ -associative and $\sqcap$ -associative elements are equivalent to the transitive element.

Theorem 2.7. [20]

Let $(T, \leq, \sqcap, \sqcup)$ be a trellis and a be an element of T . If a is $\sqcup$ -associative (resp. $\sqcap$ -associative), then a is transitive.

Proof. The proof of this theorem is straightforward. □

Remark 2.5.

The converse of Theorem 2.7 does not hold in general. Indeed, let $(T, \trianglelefteq, \sqcap, \sqcup)$ the trellis given by the Hasse diagram in Figure 2.4. Note that d is transitive but not $\sqcup$ -associative since $(d \sqcup a) \sqcup c = c$ but $d \sqcup (a \sqcup c) = u$.

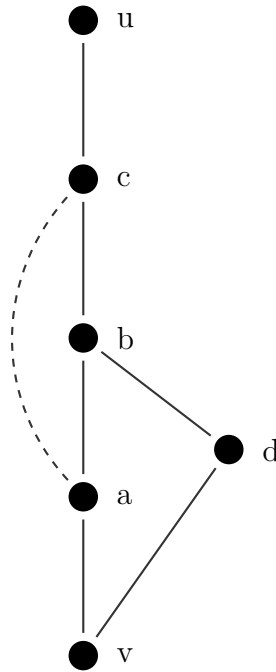


Figure 2.4: Representaiton of a trellis.

Conclusion

In this dissertation, we have studied the notions of pseudo-ordered sets and trellises. We have presented some properties and examples, in particular, those related to the elimination of transitivity.

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ملخص

في هذه المذكرة ، قمنا أولاً بالتذكير بالمفاهيم المتعلقة بالمجموعات المرتبة والشبكات وبعض أنواعها ثم تطرقنا إلى دراسة المجموعات الشبه المرتبة والشبكات واعطاء بعض الأمثلة عنها وتعريف بمجموعات الانكماش والشبكات القابلة وتضم التجميع.

كلمات مفتاحية

مجموعات مرتبة، مجموعات شبه مرتبة، شبكات، مجموعة الانكماش.

Abstract

In this dissertation, we first recall the notions related to ordered sets and lattices. In a similar way, we study the notions of pseudo-ordered sets and trellises, and present some examples. We pay particular attention to the elimination of the property of transitivity.

Key words

Ordered sets, Pseudo ordered sets, Trellises, Contraction set.

Résumé

Dans ce mémoire, nous rappelons d'abord les notions liées aux ensembles ordonnés et aux treillis. De manière similaire, nous étudions des notions d'ensembles pseudo-ordonnés et de treillis et présentons quelques exemples. Nous portons une attention particulière à l'élimination de la propriété de transitivité.

Mot-clés

Ensembles ordonnés, Ensembles pseudo ordonnés, Treillis, Ensemble de contraction.