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Existence and uniqueness of the solutions for a nonlinear time-conformable fractional reaction-diffusion equations with delay

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As the Prophet **Muhammad** (Salla Allahu Alayhi Wa Sallam) said: He who does not thank the people is not thankful to Allah.(Haddith Sahih-Al-Albani)

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Dedication

I dedicate this humble work to my father ***Mohammed*** (My God have mercy on him), To the source of happiness, the symbol of love, To my mother ***Ladmia*** who encourage me, gives me help, who sacrificed by all what he has for my happiness and my success.

To my brothers: Abdelghani, Abdelhamid, and Nooreddine.

To my sisters: Safiya, Sabrina, and Fatima.

To the little ones: Farah and Iyad.

To my family, To all my friends.

To my second family of department of

Mathematics

And To my colleagues.

I dedicate this humble work.

Ilham

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General Introduction

In recent decades, the field of fractional calculus has attracted the interest of researches and it has been frequently used to model many fundamental problems in various branches of sciences and engineering [27, 6], [24].

In natural science, there are numerous nonlinear systems that are heredatury when the quantities under consideration depend not only on the present state of a system at a given moment of time, but also on the history of process evolution. In some cases, the state of a system may be determined by a certain moment in the past, but not by its whole history, which referred to as system with delay. This delay systems are occasionally modeled by reaction-diffusion equation of the form

$$\frac{\partial \nu}{\partial t} = a\frac{\partial^2 \nu}{\partial x^2} + f(u, w), \quad w = \nu(x, t - \tau) \tag{1}$$

where a is the transfer coefficient, $\tau > 0$ is the delay time and f is the kinetic function. The special case, $f(u, w) = F(w)$ have a simple physical interpretation: the transfer of substance in a locally non-equilibrium medium is characterized by inertial properties,i. e, that the reaction of the system is not instantaneous but delayed for τ.

In 2014, Khalil Roshdi and Mohammed Al Horani [15] introduce a new definition of fractional derivative named "*conformable fractional derivative*". This new type of fractional derivative is compatible with the classical derivative and it is excellent for studying non regular solutions. The object(subject) of the conformable fractional derivative has attracted the attention of many researches, and it has played an important role in several domain such as population theory [8, 29, 13, 16, 9, 10, 31], medicine [11, 12, 7, 34, 23, 14], biology [32, 18, 22], chemistry [33, 19, 17, 30, 35], electronic, mechanics and anomalous diffusion. We are interested in studying in this paper the reaction diffusion model (1) in framework of the conformable time fractional derivative (this studying based on the article of Z.Ouyang in [21]).
Precisely, we will consider the following transformation

$$\begin{cases} \frac{\partial}{\partial t} \longrightarrow \mathcal{D}_t^{\alpha}, & \frac{\partial^2}{\partial x^2} \longrightarrow \Delta \\ a = \mathbf{c}, & 0 < \alpha \leq 1, \end{cases} \tag{2}$$

where $\mathcal{D}_t^{\alpha}$ denote the conformable time fractional derivative operator [15]. Then, we get the fractional order delay partial differential equation with the transformation (2) as follows:

$$\mathcal{D}_t^{\alpha}\nu(x,t) = \mathbf{c}\Delta\nu(x,t) + f(t, \nu(x, t - \tau)) \tag{3}$$

where $t \in [0, T_0], x \in \Omega$ is M dimension space.
subject to the initial data

$$\nu(x, 0) = \psi(x), \qquad for\ x \in \Omega \tag{4}$$

and the boundary conditions

$$\nu(x, t) = 0, \qquad for\ (x, t) \in \partial\Omega \times [0, T_0] \tag{5}$$

and

$$\frac{\partial \nu}{\partial N} = 0, \qquad for\ (x, t) \in \partial\Omega \times [0, T_0], \tag{6}$$

where N is the exterior unit normal vector to $\partial\Omega$, and $\psi \in C^2(\Omega)$ is a given function.

We will investigate the existence and uniqueness of the solution of two problems (3)-(5) and (3)(4)(6)(we denoted respectively by (P1) and (P2)) by using Leary-Schauder fixed point theorem and the Banach contraction mapping theorem. Before starting and proving the main theorems, we are introduced some hypotheses.

List of Symbols and Notations

We introduce here some necessary symbols/notations which used throughout this paper.

$L^q(\Omega)$	The Lebesgue spaces where $(1 \leq q < \infty)$ on Ω.
$L^q(\Omega)$	$= \{v : \Omega \longrightarrow \mathbb{R}\text{measurable} : \lvert v\rvert^q \in L^1\}$ with $1 \leq q < \infty$.
$\lVert v\rVert_{L^q}$	$= \left(\int_\Omega \lvert v(x)\rvert^q \, dx\right)^{\frac{1}{q}}$.
$\mathcal{C}(\Omega)$	Space of continuous functions on Ω.
$L^1(\Omega)$	Space of integrable functions on Ω.
$\lVert g(x)\rVert_{L^1}$	$= \int_\Omega \lvert g(x)\rvert \, dx$, or $\lVert g\rVert_{L^1} = \int_\Omega \lvert g\rvert$.
DCT	Dominated Convergence Theorem.
$u = u(x,t)$	unknown function (for instant, concentration, temperature, rate component,etc).
$w = u(x, t-\tau)$	unknown function at a moment of time $t - \tau$.
Δ, ∇	Laplace and Nabla operators respectively.
ΔU	$= \sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_j^2}, \quad n \in \mathbb{N}$.
∇U	$= \left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \ldots, \frac{\partial U}{\partial x_n}\right) = gradU$.
F or f	kinetic functions.
CFD	Conformable fractional derivative.
CFI	Conformable fractional integral.
N	is the exterior unit normal vector to $\partial\Omega$.

CHAPTER 1

PRELIMINARIES

In this chapter, we collect various definitions and theorems which are key tools for proving our main theorems. Some proofs are omitted and may be found in [26] or [3, 21].

1.1 Some Basic definition and Tools

Definition 1.1. [2](***The 'left' and 'right' conformable fractional derivative (CFD)***)

✓ Lets $g : [a, +\infty[\longrightarrow \mathbb{R}$ be a given function and $\alpha \in]0, 1]$. The left CFD of order α is defined by:

$$\mathcal{D}_a^{(\alpha)}(g)(t) := \lim_{\kappa \longrightarrow 0} \frac{g\left(t + \kappa(t-a)^{1-\alpha}\right) - g(t)}{\kappa}, \tag{1.1}$$

•If $\mathcal{D}^{(\alpha)}(g)(t)$ exist on $]a, +\infty[$, then $\mathcal{D}_a^{(\alpha)}(g)(a) = \lim_{t \longrightarrow a^+} \mathcal{D}_a^{(\alpha)}(g)(t)$.

• When $a = 0$,

$$\mathcal{D}_0^{\alpha} = \lim_{\kappa \longrightarrow 0} \frac{g\left(t + \kappa(t)^{1-\alpha}\right) - g(t)}{\kappa},$$

we write $\mathcal{D}^{(\alpha)}$ instead of $\mathcal{D}_0^{\alpha}$, which is the original definition introduced by Khalil et al (see more in [2])

✓ Lets $g :] - \infty, b] \longrightarrow \mathbb{R}$ be a given function and $\alpha \in]0, 1]$.
The right CFD of order α is defined by:

$$\mathcal{D}_b^{(\alpha)}(g)(t) := -\lim_{\kappa \longrightarrow 0} \frac{g\left(t + \kappa(b-t)^{1-\alpha}\right) - g(t)}{\kappa}, \tag{1.2}$$

• If $\mathcal{D}^{(\alpha}(g)(t)$ exist on $] - \infty, b[$, then $\mathcal{D}_b^{(\alpha)}(g)(b) = \lim_{t \to b^-} \mathcal{D}_b^{(\alpha)}(g)(t)$.

Properties 1.1. (see[25])For all $f, g : [0, +\infty[\longrightarrow \mathbb{R}$ and $\alpha \in]0, 1]$, we have the following properties:

1. If g is α-differentiable, then g is continuous.

2. Linearity:
$$\mathcal{D}^{(\alpha)}\left(\mathbf{c}f+\mathbf{d}g\right)=\mathbf{c}\mathcal{D}^{(\alpha)}(f)+\mathbf{d}\mathcal{D}^{(\alpha)}(g),\quad \forall \mathbf{c},\mathbf{d}\in\mathbb{R}. \tag{1.3}$$

3. $\mathcal{D}^{(\alpha)}(C_1)=0$, where C_1 is a constant.

4. $\mathcal{D}^{(\alpha)}t^q=\begin{cases}\frac{\Gamma(q+1)}{\Gamma(q-n)}t^{q-\alpha} & if\ \ q\in\mathbb{N}\ \ and\ \ q\succ\alpha,\\ 0 & if\ \ q\in\mathbb{N}\ \ and\ \ q<\alpha,\end{cases}$
where $\Gamma(.)$ denote the Euler Gamma function and $n<\alpha\leq n+1$.

5. (Leibniz rule)
$$\mathcal{D}^{(\alpha)}(gf)=g\mathcal{D}^{(\alpha)}(f)+f\mathcal{D}^{(\alpha)}(g). \tag{1.4}$$

6. If g is n-times differentiable on $[a,+\infty[$, then
$$\mathcal{D}^{(\alpha)}(g)(t)=(t-a)^{n+1-\alpha}g^{n+1}(t),\ \ \forall n<\alpha\leq n+1.$$

7. $\mathcal{D}^{(\alpha)}(\frac{g}{f})(t)=\frac{f\mathcal{D}^{(\alpha)}(g)-g\mathcal{D}^{(\alpha)}(f)}{f^2}$ with $f\neq 0$.

8. (Chain rule) We pose $h(t)=(f\circ g)(t)$ such that f and g are α-differentiable functions, then
$$\mathcal{D}^{(\alpha)}(h)(t)=[\mathcal{D}^{(\alpha)}(f)(g(t))]\cdot[\mathcal{D}^{(\alpha)}(g)(t)]\cdot g^{\alpha-1}(t). \tag{1.5}$$

Definition 1.2. (***The conformable fractional integral (CFI) 'left' and 'right'***)[2]

1. Lets $g:[a,+\infty[\longrightarrow\mathbb{R}$ be a given function and $\alpha\in]0,1]$.
The left CFI of order α is defined by:
$$\mathcal{I}_a^{\alpha}(g)(t):=\int_a^t(s-a))^{\alpha-1}g(s)ds. \tag{1.6}$$

2. Lets $g:]-\infty,b]\longrightarrow\mathbb{R}$ be a given function and $\alpha\in]0,1]$.
The right CFI of order α is defined by:
$$\mathcal{I}_b^{\alpha}(g)(t):=\int_b^t(b-s))^{\alpha-1}g(s)ds. \tag{1.7}$$

Lemma 1.1. *[25] Lets $g:[a,+\infty[\longrightarrow\mathbb{R}$ be a given function and $\alpha\in]0,1]$. For every $t>a$, the CFD and CFI obey the following relations:*

1. *If g is continuous, then $\mathcal{D}_a^{(\alpha)}\mathcal{I}_a^{\alpha}(g)(t)=g(t)$.*

2. *If g is differentiable, then $\mathcal{I}_a^{\alpha}\mathcal{D}_a^{(\alpha)}(g)(t)=g(t)-g(a)$.*

See more in [2]
Green formula.[28]If $U,V\in H^1(\Omega)$ then we have
$$\int_{\Omega}\frac{\partial U(x)}{\partial x_i}V(x)dx=\int_{\partial\Omega}U(x)V(x)\nu_i d\partial\Omega-\int_{\Omega}U(x)\frac{\partial V(x)}{\partial x_i}dx,\ \ 1\leq i\leq n,$$
where ν_i denote the i^{eme} cosinus director of the normal vector ν_i on $\partial\Omega$ directed to the exterior of Ω.

Theorem 1.1. *[28] If $U \in H^2(\Omega)$ and $V \in H^1(\Omega)$, then*

$$-\int_\Omega \Delta U(x)V(x)dx = \int_\Omega \nabla U(x)\cdot\nabla V(x)dx - \int_{\partial\Omega}\frac{\partial U(x)}{\partial\nu}V(x)d\Gamma,$$

where the symbol $\nabla U = \left(\frac{\partial U}{\partial x_i}\right)_{1\leq i\leq n}$ design the gradient vector of U and $\frac{\partial U}{\partial \nu} = \nabla U\cdot\nu$.

Theorem 1.2. *(Bochner)[28] Let $v:\Omega\longrightarrow E$ be a measurable function. We say that v is summable on Ω if and only if the real function $t\longrightarrow \|v(t)\|_E$ is summable on Ω. Moreover,*

$$\left\|\int_\Omega v(t)dt\right\|_E \leq \int_\Omega \|v(t)\|_E\,dt, \tag{1.8}$$

$$\left\langle g, \int_\Omega v(t)dt\right\rangle_E = \int_\Omega \langle g, v(t)dt\rangle_E\,dt,\ \ \forall g\in E'. \tag{1.9}$$

Theorem 1.3. *(The Lebesgue's DCT) [4, 5] Let (f_m) is a sequence of functions of L^1. We suppose that*
(i) $(f_m(x))\longrightarrow f(x)$ a.e on Ω.
(ii) There exist a function $h\in L^1$, such that for every $m\in\mathbb{N}^*$, $|f_m(x)|\leq h(x)$ $a.e$ on Ω. then,

$$f\in L^1 \; and \lim_{m\longrightarrow\infty}\int_\Omega |f_m| = \int_\Omega |f|\,.$$

Lemma 1.2. *(Leary-Schauder fixed point theorem)[1, 21] Let E be a space of Banach and $X\in E$ is a closed, bounded and convex subset. If $T: X\longrightarrow X$ is completely continuous, then T has at least a fixed point in X.*

Lemma 1.3. *(Theorem of the Banach contraction mapping)[1, 21] Let Y be a complete measurable space and $T: Y\longrightarrow Y$ is a contraction mapping of Y, then T has a unique fixed point (i. e., $Tu = u, \forall u\in Y$.)*

CHAPTER 2

EXISTENCE AND UNIQUENESS OF SOLUTION FOR CONFORMABLE FRACTIONAL REACTION-DIFFUSION EQUATION

In this chapter, we consider two coupled systems of a non linear time-conformable fractional reaction-diffusion equation with delay. Under certain conditions, we study the existence and uniqueness of the solution for both systems.

We assume that the reader is familiar with the notion of measurable functions and integrable functions $g :\longrightarrow \mathbb{R}$.

2.1 Problem Statement

The object of this chapter is to study the existence and uniqueness of the solution of the following class of nonlinear fractional order partial differential equations with delay

$$\mathcal{D}_t^{\alpha}\nu(x,t) = \mathbf{c}\Delta\nu(x,t) + g(t,\nu(x,\tau)), \qquad t \in [0,T_0] \tag{2.1}$$

where :

- ✓ $\mathcal{D}_t^{\alpha}$: is the conformable fractional derivative of order $\alpha \in [0,1]$,
- ✓ g : is a non linear item defined as $g : \mathbb{R} \longrightarrow \mathbb{R}$,
- ✓ $x \in \Omega$ is M dimension space.
- ✓ $\mathbf{c} > 0$ is the coefficient of diffusion equation.
- ✓ $\tau > 0$ is the parameter of delay.

We consider the initial data (4) and the previous boundary conditions (5)-(6).

We denote (2.1)-(4),(2.1)-(4)-(5) and (2.1)-(4)-(6) respectively by (P0), (P1) and (P2).

2.2 Existence and uniqueness of the solution

To simplify matters throughout this paper, we will need to make some suitable hypotheses to achieve the disered results.

Hypotheses 2.1.

(H_1) $g(t,\nu)$ is :

- ❒ Convex and continuous with respect to ν,
- ❒ Lebesgue measurable with respect to $t \in [0,a]$,

(H_2) $\|g(t,\nu(x,\tau))\| \leq m_1(t)\,\|\nu(t,\tau)\|^{k_1}$,
where $m_1(t) \geq 0$, $k_1 \geq 0$ and there is $\lambda \in (0,\alpha)$ such that

$$m_1(t) \in L^{1/\lambda}([0,a]), \left(\int_0^a [m_1(t)]^{1/\lambda}\,dt\right)^{\lambda} \leq c_1 < 1,$$

(H_3) $0 \leq \tau \leq t$,

(H_4) Suppose that there exists a constant $L \geq 0$ such that

$$\|g(t,\nu_1) - g(t,\nu_2)\| \leq L\,\|\nu_1 - \nu_2\|\,, \;\; \forall t \in [0,T_0].$$

such that

$$L < -\alpha_1 \mathbf{c} + \sigma^{\beta-\alpha}(\frac{1-\beta}{\alpha-\beta})^{\beta-1},$$

where $0 < \beta < \alpha < 1$, and $\sigma = \{\omega, \omega_1\}$
Hint. ω, ω_1 are defined respectively in lemma 2.2 and lemma 2.3.

We shall start by introducing the following lemma.

Lemma 2.1. *Suppose that the previous hypotheses $(H_1)-(H_4)$ holds. Then for every $t \in (0,a)$,*

$$IVP(P_0) \Longleftrightarrow \nu(x,t) = \psi(x) + \int_0^t (t-s)^{\alpha-1}[\boldsymbol{c}\Delta\nu(x,s) + g(s,\nu(x,\tau))]ds, \;\; x \in \Omega \tag{2.2}$$

Proof. $(\Longrightarrow)$: [ν is solution of $(P_0) \Longrightarrow \nu$ satisfy (2.2)]
We suppose that $\nu(x,t)$ is solution of the IVP (P_0),

- ☞ Clearly we have $g(t,\nu(x,\tau))$ is Lebesgue measurable on $(0,a)$ according to (H_1) and (H_2).
- ☞ Simple calculation give that

$$(t-s)^{\alpha-1} \in L^{\frac{1}{1-\lambda}}(0,a), \quad \forall t \in \Omega$$

- ☞ $(t-s)^{\alpha-1}g(s,\nu(x,\tau))$ is Lebesgue measurable with respect to $s \in [0,t]$. In fact, Holder inequality give that

$$\begin{aligned}\int_0^t |(t-s)^{\alpha-1}g(s,\nu(x,\tau))|ds &\leq \left(\int_0^t \left|(t-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}\right)^{1-\lambda} \left(\int_0^t |g(s,\nu(x,\tau))|^{\frac{1}{\lambda}}\right)^{\lambda} \\ &\leq \left\|(t-s)^{\alpha-1}\right\|_{L^{\frac{1}{1-\lambda}}(0,a)} \|g(s,\nu(x,\tau))\|\end{aligned}$$

using (H_2), it becomes

$$\leq \left\|(t-s)^{\alpha-1}\right\|_{L^{\frac{1}{1-\lambda}}(0,a)} m_1(s) \left\|\nu(x,\tau)\right\|^{k_1}$$

From (P_0) and by integrating both sides of (2.1)

$$\mathcal{I}^{\alpha}\mathcal{D}_t^{(\alpha)}\nu(x,t) = \mathcal{I}^{\alpha}\left[\mathbf{c}\Delta\nu(x,t) + g(t,\nu(x,\tau))\right]$$

according to lemma1.1, we get

$$\nu(x,t) - \underbrace{\nu(x,0)}_{\psi(x)} = \mathcal{I}^{\alpha}\left[\underbrace{\mathbf{c}\Delta\nu(x,t) + g(t,\nu(x,\tau))}_{G}\right]$$

using definition.1.2, it becomes

$$\nu(x,t) = \psi(x) + \int_0^t (t-s)^{\alpha-1}\left[\mathbf{c}\Delta\nu(x,s) + g(s,\nu(x,\tau))\right] ds.$$

$(\Longleftarrow)$:[(2.2) $\Longrightarrow$ ν is solution of (P_0)]

Suppose that $\nu(x,t)$ is a solution satisfying 2.2, by deriving both sides of equation (2.2), we get

$$\mathcal{D}^{(\alpha)}\nu(x,t) = \mathcal{D}^{(\alpha)}\left[\psi(x) + \int_0^t (t-s)^{\alpha-1}[\mathbf{c}\Delta\nu(x,s) + g(s,\nu(x,\tau))]ds\right]$$

using the properties 1.1

$$\mathcal{D}^{(\alpha)}\nu(x,t) = \mathcal{D}^{(\alpha)}\psi(x) + \mathcal{D}^{(\alpha)}\underbrace{\int_0^t (t-s)^{\alpha-1}[\underbrace{\mathbf{c}\Delta\nu(x,s) + g(s,\nu(x,\tau))}_{G}]ds}_{\mathcal{I}^{\alpha}G}$$

according lemma1.1 and the properties 1.1, we obtain

$$\mathcal{D}^{(\alpha)}\nu(x,t) = \mathbf{c}\Delta\nu(x,t) + g(t,\nu(x,\tau)),$$

which means that $\nu(x,t)$ is a solution of IVP (P_0). This ends the proof. □

Now, consider the Dirichlet boundary value problem on Ω

$$\Delta\nu + \gamma\nu = 0, \qquad in (x,t) \in \Omega \times [0,T_0], \tag{2.3}$$

$$\nu = 0, \qquad on (x,t) \in \partial\Omega \times [0,T_0], \tag{2.4}$$

where γ is a constant.

It's well known from [21] that:

- ✻ the smallest eigenvalue α_1 of the problem (2.2) is positive,
- ✻ And the corresponding eigenfunction $\varphi(x) \geq 0$ for all $x \in \Omega$.

Now, let $\nu(x,t)$ be a solution of problem (P_i)(i=1,2). We always define throughout this paper, two functions $V(t)$,$U(t)$ as follows:

$$V(t) = \frac{\int_\Omega \nu(x,t)\varphi(x)dx}{\int_\Omega \varphi(x)dx}, \tag{2.5}$$

$$U(t) = \frac{\int_\Omega \nu(x,t)dx}{\int_\Omega dx}, \tag{2.6}$$

In particular

$$V(0) = \frac{\int_\Omega \psi(x)\varphi(x)dx}{\int_\Omega \varphi(x)dx}, \tag{2.7}$$

$$U(0) = \frac{\int_\Omega \psi(x)dx}{\int_\Omega dx}. \tag{2.8}$$

The strategy to prove the existence of the solution is to use the following procedure of Leary-Schauder fixed point theorem.

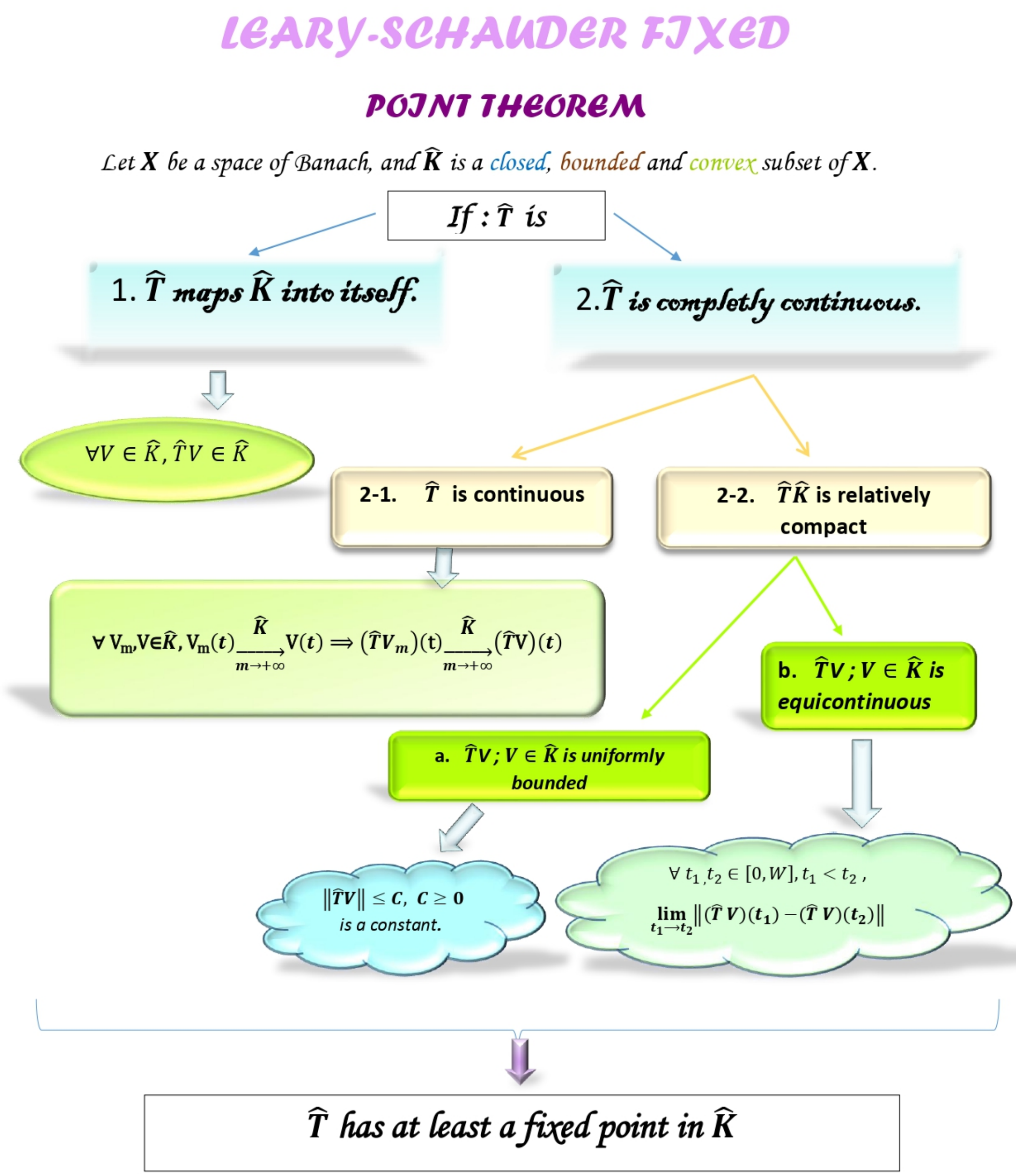

Figure 2.1: Leary-Schauder fixed point theorem.

We start by studying the existence of solution of the problem (P1).

2.2.1 The Existence of Solutions of (P1)

To proof the existence, we need to introduce the following lemma.

Lemma 2.2. *Assume that $(H_1)-(H_4)$ holds, there exists $a\lambda \in (0,\alpha)$. For any constant $\ell > 0$, we suppose that*

$$\omega < min\left\{a, T_0, \left[\frac{\ell}{\alpha_1 \boldsymbol{c}\left(\ell + \|V(0)\|\right)|\Omega|^{\lambda} + \left(\ell + \|V(0)\|\right)^{k_1} c_2}\left(\frac{\alpha-\lambda}{1-\lambda}\right)^{1-\lambda}\right]^{\frac{1}{\alpha-\lambda}}\right\},$$

$k_1 \geq 0$.
Then for every $t \in (0,\omega)$, the problem (P_1) has a solution $\Longleftrightarrow$ the following equation has a solution,

$$V(t) = V(0) - \alpha_1 \boldsymbol{c}\int_0^t (t-s)^{\alpha-1} V(s)ds + \int_0^t (t-s)^{\alpha-1} g(s, V(\tau))ds, \qquad t \in [0,\omega] \tag{2.9}$$

Proof. Necessary($\Longleftarrow$): Suppose that $V(t)$ is solution of(2.9).
Let

$$\nu(x,t) = \exp^{\sqrt{\alpha_1}(x_1+\cdots+x_M)} V(t).$$

Combining (2.1)(2.3)(2.9), it is easy to show that $\nu(x,t)$ is a solution of(2.2). It is obvious that $\nu(x,t)$ is a solution of (P_0)(from lemma2.1)[21].

Sufficiency: ($\Longrightarrow$)Suppose that(P_1) has a solution $\nu(x,t)$. According to lemma2.1, multiplying by $\varphi(x)$ and integrating on Ω both sides of(2.2), it comes

$$\int_\Omega \varphi(x)\nu(x,t)dx = \int_\Omega \varphi(x)\psi(x)dx + \int_\Omega \varphi(x)\int_0^t (t-s)^{\alpha-1}[\eta\Delta\nu(x,s) + g(s,\nu(x,\tau))]ds$$

$$= \int_\Omega \varphi(x)\psi(x)dx + \int_0^t (t-s)^{\alpha-1}\left[\mathbf{c}\int_\Omega \varphi(x)\Delta\nu(x,s) + \int_\Omega \varphi(x) g(s,\nu(x,\tau))dx\right]ds \tag{2.10}$$

from (2.3) and by using Green's formula, we get

$$\int_\Omega \Delta\nu\varphi dx = -\alpha_1\int_\Omega \nu\varphi dx, \quad t \geq T_0. \tag{2.11}$$

combining (2.10)(2.11) and dividing both sides by $\int_\Omega \psi(x)dx$, it comes

$$\begin{aligned} V(t) &= \frac{\int_\Omega \varphi(x)\nu(x,t)dx}{\int_\Omega \varphi(x)dx} \\ &= \frac{\int_\Omega \varphi(x)\psi(x)dx}{\int_\Omega \varphi(x)dx} - \alpha_1\int_0^t (t-s)^{\alpha-1}\left[\mathbf{c}\frac{\int_\Omega \varphi(x)\nu(x,s)}{\int_\Omega \varphi(x)dx} + \frac{\int_\Omega \varphi(x)g(s,\nu(x,\tau))dx}{\int_\Omega \varphi(x)dx}\right]ds \\ &= V(0) - \alpha_1\mathbf{c}\int_0^t (t-s)^{\alpha-1}V(s)ds + \int_0^t (t-s)^{\alpha-1}\frac{\int_\Omega \varphi(x)g(s,\nu(x,\tau))dx}{\int_\Omega \varphi(x)dx}ds \end{aligned}$$

using Jensen's inequality [21], we obtain

$$V(t) \leq V(0) - \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} V(s) ds + \int_0^t (t-s)^{\alpha-1} g\left(s, \overbrace{\frac{\int_\Omega \varphi(x)\nu(x,\tau)dx}{\int_\Omega \varphi(x)dx}}^{V(\tau)}\right) ds$$
$$= V(0) - \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} V(s) ds + \int_0^t (t-s)^{\alpha-1} g\left(s, V(\tau)\right) ds,$$

which refer to us that $V(t)$ is a solution of

$$V(t) \leq V(0) - \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} V(s) ds + \int_0^t (t-s)^{\alpha-1} g\left(s, V(\tau)\right) ds. \tag{2.12}$$

Let

$$B = \left\{V : V \in \mathcal{C}([0,T_0],\mathbb{R}), \|V(t) - V(0)\| \leq \ell\right\}. \tag{2.13}$$

We can easy show that B is: nonempty(take for instance$V = V(0) \in B,$), closed, and convex set.

We define an operator T as follows

$$(TV)(t) = \begin{cases} V(0), & t = 0, \\ V(0) - \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} V(s) ds \\ + \int_0^t (t-s)^{\alpha-1} g\left(s, V(\tau)\right) ds, & 0 \leq t \leq min\left\{a, T_0\right\}. \end{cases} \tag{2.14}$$

We should prove that T maps B into itself(i. e., $\forall V \in B, TV \in B$), $\forall V \in B$ and using (2.13), we have

$$\|(TV)(t) - V(0)\| = \|(TV)(t) - (TV)(0)\|$$
$$= \left\| V(0) - \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} V(s) ds + \int_0^t (t-s)^{\alpha-1} g\left(s, V(\tau)\right) ds - V(0) \right\|$$

using theorem 1.2, it becomes

$$\|(TV)(t) - V(0)\| \leq \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} \|V(s)\| ds + \int_0^t (t-s)^{\alpha-1} \|g\left(s, V(\tau)\right)\| ds$$

From(2.13), we have

$$\|V(t) - V(0)\| \leq \ell$$

Keeping in mind that

$$|\|V(t)\| - \|V(0)\|| \leq \|V(t) - V(0)\|$$

Thus

$$\Longrightarrow \|V(t)\| - \|V(0)\| \leq \ell \Longrightarrow \|V(t)\| \leq \ell + \|V(0)\|. \tag{2.15}$$

then,

$$\|(TV)(t)-V(0)\| \leq (\ell+\|V(0)\|)\alpha_1\mathbf{c}\int_0^t (t-s)^{\alpha-1}ds + (\ell+\|V(0)\|)^{k_1}\int_0^t (t-s)^{\alpha-1}m_1(s)ds$$

Using HÖlder inequality(theoremA.1), then using (H_2), we get

$$\begin{aligned}\|(TV)(t)-V(0)\| \leq &(\ell+\|V(0)\|)\alpha_1\mathbf{c}\left(\int_0^t \left[(t-s)^{\alpha-1}\right]^{\frac{1}{1-\lambda}} ds\right)^{1-\lambda}\left(\int_0^t 1^{\frac{1}{\lambda}}ds\right)^{\lambda}\\ &+(\ell+\|V(0)\|)^{k_1}\left(\int_0^t \left[(t-s)^{\alpha-1}\right]^{\frac{1}{1-\lambda}} ds\right)^{1-\lambda}\left(\int_0^t [m_1(s)]^{\frac{1}{\lambda}} ds\right)^{\lambda}, \lambda\in(0,\alpha)\\ \leq &(\ell+\|V(0)\|)\alpha_1\mathbf{c}\,|\Omega|^{\lambda}\left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda}\omega^{\alpha-\lambda} + c_2(\ell+\|V(0)\|)^{k_1}\left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda}\omega^{\alpha-\lambda}\\ \leq &\left[(\ell+\|V(0)\|)\alpha_1\mathbf{c}\,|\Omega|^{\lambda} + c_2(\ell+\|V(0)\|)^{k_1}\right]\left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda}\omega^{\alpha-\lambda}\\ \leq &\ell,\ \ t\in[0,\omega]\end{aligned} \tag{2.16}$$

So, $TV\in B$, which means that $T(B)\subset B$.

Now, defining a sequence $V_m(t)$ in B as follows

$$\begin{cases} V_0 \equiv V(0), & t\in[0,\omega]\\ V_{m+1}(t) = (TV_m)(t), & t\in[0,\omega], m\in\{0,1,\ldots\}\end{cases}$$

We can show that $V_m(t)$ is a sequence of measurable functions and $V_m(t)\in B$ (by recurrence).

Since B is closed and bounded, then we may extract a subsequence $V_{m_j}(t)$ from $V_m(t)$ which converge towards an element $\mathcal{V}(t)$ of B i. e,

$$\lim_{m_j\to\infty} V_{m_j}(t) = \mathcal{V}(t), \quad 0\leq t\leq\omega.$$

From Lebesgue's dominated convergence theorem it follows that $\mathcal{V}(t)$ satisfies

$$\mathcal{V}(t) = \begin{cases}\mathcal{V}(0), & t=0\\ \mathcal{V}(0) - \alpha_1\mathbf{c}\int_0^t (t-s)^{\alpha-1}\mathcal{V}(s)ds & \\ + \int_0^t (t-s)^{\alpha-1} g\left(s,\mathcal{V}(\tau)\right)ds, & 0\leq t\leq\omega.\end{cases}$$

We deduce that $V(t)$ is a solution of (2.9). This ends the proof. □

We are now ready to produce a theorem which is one of the main results in this chapter leading to the existence of the solution to problem (P1).

Theorem 2.1. *Assume that $(H_1)-(H_4)$ holds. Suppose that ℓ,ω are defined in the previous lemma2.2. Then, the problem (P1)* has at least a solution on $\Omega\times[0,\omega]$.

Proof. According to lemma2.2, we just need to prove that(2.9) has at least a solution. For this we follow the previous procedure (see figure2.1).
For every $t \in [0, \omega]$,(it implies that $t \in [0, a]$, $t \in [0, T_0]$).
Let

$$\overline{B} = \{V : V \in \mathcal{C}([0, \omega], \mathbb{R}), \|V(t) - V(0)\| \leq \ell\}. \tag{2.17}$$

It is easy to show that $\overline{B}$ is:

✎ Nonempty,

✎ Closed,

✎ Convex set (For every $V_1, V_2 \in \overline{B}$ and every $\lambda_1, \lambda_2 \geq 0$ such that$\lambda_1 + \lambda_2 = 1$, we have

$$\|(\lambda_1 V_1 + \lambda_2 V_2) - 1V_0\| = \|(\lambda_1 V_1 + \lambda_2 V_2) - (\lambda_1 + \lambda_2)V_0\|$$

Using triangular inequality, it comes

$$\|(\lambda_1 V_1 + \lambda_2 V_2) - V_0\| \leq \lambda_1 \|V_1 - V_0\| + \lambda_2 \|V_2 - V_0\|$$

We have

$$\begin{cases} V_1 \in \overline{B}, \\ V_2 \in \overline{B}, \end{cases} \Longrightarrow \begin{cases} \|V_1 - V_0\| \leq \ell, \\ \|V_2 - V_0\| \leq \ell. \end{cases}$$

so $\|(\lambda_1 V_1 + \lambda_2 V_2) - V_0\| \leq \lambda_1 \ell + \lambda_2 \ell = (\underbrace{\lambda_1 + \lambda_2}_{=1})\ell = \ell$,

thus,$\lambda_1 V_1 + \lambda_2 V_2 \in \overline{B}$.)

We define an operator $\overline{T}$ with this formula

$$(\overline{T}V)(t) = V(0) - \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} V(s) ds + \int_0^t (t-s)^{\alpha-1} g\left(s, V(\tau)\right) ds. \tag{2.18}$$

In one hand, we show that for all $V \in \overline{B}, \overline{T}V \in \overline{B}$(i. e., $\overline{T}$ maps $\overline{B}$ into itself). In fact, with similar method as in (2.16), it is obvious.

On the other hand, we show that $\overline{T}$ is completely continuous,

☞ $\overline{T}$ is continuous, i. e., $\forall\{V_k(t)\}, V(t) \in \overline{B}$

$$\lim_{k \longrightarrow \infty} \|V_k(t) - V(t)\| = 0 \Longrightarrow \lim_{k \longrightarrow \infty} \left\|(\overline{T}V_k)(t) - \overline{T}(V)(t)\right\| = 0.$$

For every $V(t) \in \overline{B}$, and every sequence $\{V_k(t)\} \in \overline{B}, k = 1, 2, \ldots$, suppose that

$$\lim_{k \longrightarrow +\infty} \|V_k(t) - V(t)\| = 0, \tag{2.19}$$

☞ We have

- $0 \leq \tau \leq t$ (from (H_3)),

- $V(t) \in \overline{B}$ (i. e., continuous),
- $V_k(t) \in \overline{B}$ (i. e., continuous),

which implies

$$\begin{cases} V(0) \leq V(\tau) \leq V(t), \\ V_k(0) \leq V_k(\tau) \leq V_k(t), \end{cases} \Longrightarrow \lim_{k \longrightarrow +\infty} \|V_k(\tau) - V(\tau)\| = 0,$$

☞ From (H_1), we get

$$\lim_{k \longrightarrow +\infty} \|g(t, V_k(\tau)) - g(t, V(\tau))\| = 0, \tag{2.20}$$

(2.19)(2.20)(using definition of the limit) means that: for every $\varepsilon > 0$ (which can be arbitrary small), there exists a positive integer number A_1, such that $\forall k > A_1$,

$$\|V_k(\tau) - V(\tau)\| < \frac{\alpha}{2\mathbf{c}\alpha_1\omega^\alpha}\varepsilon, \tag{2.21}$$

and

$$\|g(t, V_k(\tau)) - g(t, V(\tau))\| < \frac{\alpha}{2\omega^\alpha}\varepsilon. \tag{2.22}$$

From (2.18), we have

$$\begin{aligned} \left\|(\overline{T}V_k)(t) - (\overline{T}V)(t)\right\| \leq \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} \|V_k(s) - V(s)\| \, ds \\ + \int_0^t (t-s)^{\alpha-1} \|g(s, V_k(\tau)) - g(s, V(\tau))\| \, ds, \end{aligned}$$

using(2.21)and(2.22), we obtain

$$\begin{aligned} \left\|(\overline{T}V_k)(t) - (\overline{T}V)(t)\right\| &\leq \alpha_1 \mathbf{c} \frac{\alpha}{2\mathbf{c}\alpha_1\omega^\alpha}\varepsilon \int_0^t (t-s)^{\alpha-1} ds + \frac{\alpha}{2\omega^\alpha}\varepsilon \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{\alpha}{2\omega^\alpha}\varepsilon\frac{\omega^\alpha}{\alpha} + \frac{\alpha}{2\omega^\alpha}\varepsilon\frac{\omega^\alpha}{\alpha} \\ &= \varepsilon, \end{aligned}$$

So

$$\lim_{k \longrightarrow +\infty} \left\|(\overline{T}V_k)(t) - (\overline{T}V)(t)\right\| = 0,$$

which means that $\overline{T}$ is continuous.

☞ $\overline{T}\,\overline{B}$ is relatively compact,

1. $\overline{T}V : V \in \overline{B}$ is uniformly bounded:
   From(2.17), for all $V \in \overline{B}$, we obtain

   $$\left\|\overline{T}V\right\| \leq \|V(0)\| + \ell,$$

   which force that $\overline{T}V : V \in \overline{B}$ is uniformly bounded.

2. $\overline{T}V : V \in \overline{B}$ is equicontinuous on $[0,\omega]$: Going back again to (2.17), and using similar proof of (2.14), for every $t_1, t_2 \in [0,\omega]$, and $t_1 < t_2$, we get

$$\begin{aligned}
\left\|(\overline{T}V)(t_1)-(\overline{T}V)(t_2)\right\| &\leq \alpha_1 \mathbf{c}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|\|V(s)\|\,ds\right. \\
&\left.+\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|\|V(s)\|\,ds\right) \\
&+\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|\|g(s,V(\tau))\|\,ds \\
&+\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|\|g(s,V(\tau))\|\,ds \\
&\leq (\ell+\|V(0)\|)\alpha_1\mathbf{c}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda} \\
&\times\left(\int_0^{t_1}1^{\frac{1}{\lambda}}ds\right)^{\lambda} \\
&+(\ell+\|V(0)\|)\alpha_1\mathbf{c}\left(\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda} \\
&\times\left(\int_{t_1}^{t_2}1^{\frac{1}{\lambda}}ds\right)^{\lambda} \\
&+(\ell+\|V(0)\|)^{k_1}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda} \\
&\times\left(\int_0^{t_1}m_1^{\frac{1}{\lambda}}(s)ds\right)^{\lambda} \\
&+(\ell+\|V(0)\|)^{k_1}\left(\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda}\left(\int_{t_1}^{t_2}m_1^{\frac{1}{\lambda}}(s)ds\right)^{\lambda} \\
&\leq (\ell+\|V(0)\|)\alpha_1\mathbf{c}\,|\Omega|^{\lambda}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda} \\
&+(\ell+\|V(0)\|)\alpha_1\mathbf{c}\,|\Omega|^{\lambda}\left(\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda} \\
&+(\ell+\|V(0)\|)^{k_1}c_2\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda} \\
&+(\ell+\|V(0)\|)^{k_1}c_2\left(\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda}
\end{aligned}$$

$$
\begin{aligned}
&= (\ell + \|V(0)\|)\alpha_1 \mathbf{c} \, |\Omega|^\lambda \left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda} \left[-(t_2-t_1)^{\frac{\alpha-\lambda}{1-\lambda}} + \left(t_2^{\frac{\alpha-\lambda}{1-\lambda}} - t_1^{\frac{\alpha-\lambda}{1-\lambda}} \right) \right]^{1-\lambda} \\
&+ (\ell + \|V(0)\|)\alpha_1 \mathbf{c} \, |\Omega|^\lambda \left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda} \left[(t_2-t_1)^{\frac{\alpha-\lambda}{1-\lambda}} \right]^{1-\lambda} \\
&+ (\ell + \|V(0)\|)^{k_1} c_2 \left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda} \left[-(t_2-t_1)^{\frac{\alpha-\lambda}{1-\lambda}} + \left(t_2^{\frac{\alpha-\lambda}{1-\lambda}} - t_1^{\frac{\alpha-\lambda}{1-\lambda}} \right) \right]^{1-\lambda} \\
&+ (\ell + \|V(0)\|)^{k_1} c_2 \left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda} \left[(t_2-t_1)^{\frac{\alpha-\lambda}{1-\lambda}} \right]^{1-\lambda}
\end{aligned} \tag{2.23}
$$

Observe that $0 < \lambda < \alpha < 1$, which means that $\frac{\alpha-\lambda}{1-\lambda} > 0$. So

$$\lim_{t_1 \longrightarrow t_2} (t_2 - t_1)^{\frac{\alpha-\lambda}{1-\lambda}} = 0, \tag{2.24}$$

$$\lim_{t_1 \longrightarrow t_2} \left(\left(t_2^{\frac{\alpha-\lambda}{1-\lambda}} - t_1^{\frac{\alpha-\lambda}{1-\lambda}} \right)^{1-\lambda} - (t_2-t_1)^{\frac{\alpha-\lambda}{1-\lambda}} \right) = 0. \tag{2.25}$$

Combining with(2.23), one find

$$\lim_{t_1 \longrightarrow t_2} \left\| (\overline{T}V)(t_1) - (\overline{T}V)(t_2) \right\| = 0.$$

Thus $\overline{T}V : V \in \overline{B}$ is equicontinuous on $[0, \omega]$. Therefore $\overline{T}\ \overline{B}$ is relatively compact. Summarizing by Leary-Schauder fixed point theorem, $\overline{T}$ has at least a fixed point $\overline{V} \in \overline{B}$, that is,

$$\overline{V}(t) = \overline{V}(0) - \alpha_1 \mathbf{c} \int_0^t (t-s)^{\alpha-1} \overline{V}(s) ds + \int_0^t (t-s)^{\alpha-1} g\left(s, \overline{V}(\tau)\right) ds, \ t \in [0, \omega].$$

It is easy to show that $\overline{V}$ is a solution of problem (P1) on $[0, \omega]$. This ends the proof.

□

We have an analogue results for the problem (P2)

2.2.2 The existence of solution of problem (P2)

Our purpose now is to prove the existence of solution of the problem (P2). Before that, we shall show the following lemma.

Lemma 2.3. *For $(H_1) - (H_4)$ holds, if there exists a$\lambda \in (0, \alpha)$. For any constant $\ell_1 > 0$, we suppose that*

$$\omega_1 < min \left\{ a, T_0, \left[\frac{\ell_1}{(\ell_1 + \|U(0)\|)^{k_1} c_2} \left(\frac{\alpha-\lambda}{1-\lambda} \right)^{1-\lambda} \right]^{\frac{1}{\alpha-\lambda}} \right\}, k_1 \geq 0.$$

Then for all $t \in (0, \omega_1)$, IVP (P_2) has a solution $\Longleftrightarrow$ the equation bellow has a solution,

$$U(t) = U(0) + \int_0^t (t-s)^{\alpha-1} g(s, U(\tau)) ds, \ t \in [0, \omega_1] \tag{2.26}$$

Proof. **Necessary**($\Longleftarrow$):Suppose that $U(t)$ is solution of (2.9). Let $\nu(x,t) = U(t)$. Combining (2.1)(2.3)(2.9), it is easy to show that $\nu(x,t)$is a solution of(2.2). It is obvious that $\nu(x,t)$ is a solution of(P_0)(from lemma2.1).
Sufficiency:($\Longrightarrow$)Suppose that(P_1) has a solution $\nu(x,t)$. According to lemma2.1, and multiplying by $\varphi(x)$ then integrating on Ω both sides of (2.2), it comes

$$\begin{aligned}\int_\Omega \nu(x,t)dx &= \int_\Omega \psi(x)dx + \int_\Omega\int_0^t (t-s)^{\alpha-1}[\mathbf{c}\Delta\nu(x,s) + g(s,\nu(x,\tau))]dsdx\\ &= \int_\Omega \psi(x)dx + \int_0^t (t-s)^{\alpha-1}\left[\mathbf{c}\int_\Omega \Delta\nu(x,s) + \int_\Omega g(s,\nu(x,\tau))dx\right]ds\end{aligned}$$

$$= \int_\Omega \psi(x)dx + \int_0^t (t-s)^{\alpha-1}\left[\mathbf{c}\int_\Omega \Delta\nu(x,s)dx + \int_\Omega g(s,\nu(x,\tau))dx\right]ds \tag{2.27}$$

By using Green's formula, we obtain

$$\int_\Omega \Delta\nu dx = 0, \quad t \in [0,T_0]. \tag{2.28}$$

combining(2.27)(2.28) and dividing both sides by $\int_\Omega dx$, we get

$$\begin{aligned}U(t) &= \frac{\int_\Omega \nu(x,t)dx}{\int_\Omega dx}\\ &= \frac{\int_\Omega \psi(x)dx}{\int_\Omega dx} + \int_0^t (t-s)^{\alpha-1}\frac{\int_\Omega g(s,\nu(x,\tau))dx}{\int_\Omega dx}ds\end{aligned}$$

using Jensen's inequality[21], we find

$$U(t) \le U(0) + \int_0^t (t-s)^{\alpha-1} g\left(s, \underbrace{\frac{\int_\Omega \nu(x,\tau)dx)}{\int_\Omega dx}}_{U(\tau)}\right)ds$$

which follows that $U(t)$ is a solution of

$$U(t) \le U(0) + \int_0^t (t-s)^{\alpha-1} g\left(s, U(\tau)\right)ds. \tag{2.29}$$

Let

$$\widetilde{B} = \left\{U : U \in \mathcal{C}([0,T_0],\mathbb{R}), \|U(t) - U(0)\| \le \ell_1\right\}. \tag{2.30}$$

we can easy show that $\widetilde{B}$ is nonempty(take for instance $U(t) = U(0) \in \widetilde{B}$), Closed and Convex set.

Now,we define an operator T as follows

$$(\widetilde{T}U)(t) = \begin{cases} U(0), & t = 0,\\ U(0) + \int_0^t (t-s)^{\alpha-1} g\left(s, U(\tau)\right)ds, & 0 \le t \le min\left\{a, T_0\right\}. \end{cases} \tag{2.31}$$

First, we prove that $\widetilde{T}$ a maps $\widetilde{B}$ into itself (i. e., $\forall U \in \widetilde{B}, \widetilde{T}U \in \widetilde{B}$). For all $U \in \widetilde{B}$, and by following the same ideas as in the proof of lemma2.2, we obtain

$$\left\|(\widetilde{T}U)(t) - U(0)\right\| = \left\|(\widetilde{T}U)(t) - (\widetilde{T}U)(0)\right\|$$
$$= \left\|\int_0^t (t-s)^{\alpha-1} g\left(s, U(\tau)\right) ds\right\|$$

$$\leq \left(\ell_1 + \|U(0)\|\right)^{k_1} c_2 \left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda} \omega_1^{\alpha-\lambda} \leq \ell_1, \quad t \leq \omega_1, \tag{2.32}$$

So $\widetilde{T}U \in \widetilde{B}$, which means that $\widetilde{T}(\widetilde{B}) \subset \widetilde{B}$.

Now, we construct a sequence $\{U_m(t)\}$ in $\widetilde{B}$ as follows

$$\begin{cases} U_0 \equiv U(0), & t \in [0, min\{a, T_0\}] \\ U_{m+1}(t) = (\widetilde{T}U_m)(t), & t \in [0, min\{a, T_0\}], m \in \{1, 2, \dots\} \end{cases}$$

Since $\widetilde{B}$ is closed and bounded, then we may extract a subsequence $\left\{U_{m_j}(t)\right\}$ from $\{U_m(t)\}$ such that

$$\lim_{m_j \to \infty} U_{m_j}(t) = \mathcal{U}(t) \in \widetilde{B}, \quad 0 \leq t \leq min\{a, T_0\}.$$

From theorem1.3 it follows that $\mathcal{U}(t)$ satisfies

$$\mathcal{U}(t) = \begin{cases} \mathcal{U}(0), & t = 0 \\ \mathcal{U}(0) + \int_0^t (t-s)^{\alpha-1} g\left(s, \mathcal{U}(\tau)\right) ds, & 0 \leq t \leq \omega_1, \end{cases}$$

which explains that $U(t)$ is a solution of (2.9). This ends the proof. □

The existence of the solution for the problem (P2) is ensured in the theorem below.

Theorem 2.2. *Assume that $(H_1) - (H_4)$ holds. Suppose that ℓ_1, ω_1 are defined in lemma2.3. Then, the problem (P_1) has at least a solution on $\Omega \times [0, \omega_1]$.*

Proof. According to lemma2.1 and lemma2.3, we just need to prove that (2.26) has at least a solution.
For all $t \in (0, \omega_1)$,(it implies that $t \in [0, a]$,$t \in [0, T_0]$). Let

$$\widehat{B} = \{U : U \in \mathcal{C}([0, \omega_1], \mathbb{R}), \|U(t) - U(0)\| \le \ell_1\}. \tag{2.33}$$

We have $\widehat{B}$ is nonempty(only show that $U(t) = U(0) \in \widehat{B}$), closed, and convex set(for every $U_1, U_2 \in \widehat{B}$ and every$\lambda_1, \lambda_2 \ge 0$ such that$\lambda_1 + \lambda_2 = 1$, we have

$$\|(\lambda_1 U_1 + \lambda_2 U_2) - 1U_0\| = \|(\lambda_1 U_1 + \lambda_2 U_2) - (\lambda_1 + \lambda_2)U_0\|$$

Using triangular inequality,we get

$$\|(\lambda_1 U_1 + \lambda_2 U_2) - U_0\| \le \lambda_1 \|U_1 - U_0\| + \lambda_2 \|U_2 - U_0\|$$

so $\|(\lambda_1 U_1 + \lambda_2 U_2) - U_0\| \le \lambda_1 \ell_1 + \lambda_2 \ell_1 = \ell_1$
Thus,$\lambda_1 U_1 + \lambda_2 U_2 \in \widehat{B}$.)

We define an operator $\widehat{T}$ as follow

$$(\widehat{T}U)(t) = U(0) + \int_0^t (t-s)^{\alpha-1} g\left(s, U(\tau)\right) ds. \tag{2.34}$$

First, similarly to (2.32) we can show that $\widehat{T}$ a maps $\widehat{B}$ into itself (i. e., for all $U \in \widehat{B}, \widehat{T}U \in \widehat{B}$)

Next, we show that $\widehat{T}$ is completely continuous, i. e.,
(a)$\widehat{T}$ is continuous: For every$U(t) \in \widehat{B}$, and every sequence $\{U_k(t)\} \in \widehat{B}, k = 1, 2, \ldots$, suppose that

$$\lim_{k \longrightarrow +\infty} \|U_k(t) - U(t)\| = 0,$$

• Since
$U(t) \in \widehat{B}$ (i.e., continuous),
$U_k(t) \in \widehat{B}$ (i.e., continuous),
Then, according to (H_3), we obtain

1. $U(0) \le U(\tau) \le U(t)$,

2. $U_k(0) \le U_k(\tau) \le U_k(t)$, which implies that

$$\lim_{k \longrightarrow +\infty} \|U_k(\tau) - U(\tau)\| = 0.$$

• From (H_1), we get

$$\lim_{k \longrightarrow +\infty} \|g(t, U_k(\tau)) - g(t, U(\tau))\| = 0,$$

Which means that: there exists $\varepsilon_1 > 0$ (can be arbitrary small) and $A_2 \in \mathbb{N}^*$ such that $\forall k > A_2$,

$$\|g(t, V_k(\tau)) - g(t, V(\tau))\| < \frac{\alpha}{\omega_1^\alpha} \varepsilon_1. \tag{2.35}$$

From (2.34)(2.35), we obtain

$$\begin{aligned}
\left\|(\widehat{T}U_k)(t) - (\widehat{T}U)(t)\right\| &\leq \int_0^t (t-s)^{\alpha-1} \left\|g\left(s, U_k(\tau)\right) - g\left(s, U(\tau)\right)\right\| ds \\
&\leq \frac{\alpha}{\omega_1^\alpha} \varepsilon_1 \int_0^t (t-s)^{\alpha-1} ds \\
&\leq \frac{\alpha \varepsilon_1}{\omega_1^\alpha} \frac{\omega_1^\alpha}{\alpha} \\
&= \varepsilon_1
\end{aligned}$$

Thus

$$\lim_{k \longrightarrow +\infty} \left\|(\widehat{T}U_k)(t) - (\widehat{T}U)(t)\right\| = 0,$$

Therefore $\widehat{T}$ is continuous.
(b)$\widehat{T}$ $\widehat{B}$ is relatively compact:

- $\widehat{T}U : U \in \widehat{B}$ is uniformly bounded: According to (2.34), for all $U \in \widehat{B}$, we obtain $\left\|\widehat{T}U\right\| \leq \|U(0)\| + \ell$, which means that$\widehat{T}U : U \in \widehat{B}$ is uniformly bounded.

- $\widehat{T}\, U : U \in \widehat{B}$ is equicontinuous on $[0, \omega_1]$: From (2.33), and using similar proof of (2.32), for every $t_1, t_2 \in [0, \omega_1]$,and $t_1 < t_2$, we get

$$
\begin{aligned}
\left\|(\widehat{T}U)(t_1)-(\widehat{T}U)(t_2)\right\| &= \left\|\int_0^{t_1}(t_1-s)^{\alpha-1}g\left(s,U(\tau)\right)ds-\int_0^{t_2}(t_2-s)^{\alpha-1}g\left(s,U(\tau)\right)ds\right\| \\
&\leq \int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|\left\|g\left(s,U(\tau)\right)\right\|ds \\
&+ \int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|\left\|g\left(s,U(\tau)\right)\right\|ds \\
&\leq (\ell_1+\|U(0)\|)^{k_1}\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda}\left(\int_0^{t_1}m_1^{\frac{1}{\lambda}}(s)ds\right)^{\lambda} \\
&+ (\ell_1+\|U(0)\|)^{k_1}\left(\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda}\left(\int_{t_1}^{t_2}m_1^{\frac{1}{\lambda}}(s)ds\right)^{\lambda} \\
&\leq (\ell_1+\|U(0)\|)^{k_1}c_2\left(\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda} \\
&+ (\ell_1+\|U(0)\|)^{k_1}c_2\left(\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}\right|^{\frac{1}{1-\lambda}}ds\right)^{1-\lambda} \\
&= (\ell_1+\|U(0)\|)^{k_1}c_2\left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda}\left[-(t_2-t_1)^{\frac{\alpha-\lambda}{1-\lambda}}+\left(t_2^{\frac{\alpha-\lambda}{1-\lambda}}-t_1^{\frac{\alpha-\lambda}{1-\lambda}}\right)\right]^{1-\lambda} \\
&+ (\ell_1+\|U(0)\|)^{k_1}c_2\left(\frac{1-\lambda}{\alpha-\lambda}\right)^{1-\lambda}\left[(t_2-t_1)^{\frac{\alpha-\lambda}{1-\lambda}}\right]^{1-\lambda}
\end{aligned}
$$

By using (2.24) and (2.25), we obtain

$$\lim_{t_1\longrightarrow t_2}\left\|(\widehat{T}U)(t_1)-(\widehat{T}U)(t_2)\right\|=0.$$

Thus $\widehat{T}U : U\in\widehat{B}$ is equicontinuous on $[0,\omega_1]$. Therefore $\widehat{T}\ \widehat{B}$ is relatively compact.Summarizing by Leary-Schauder fixed point theorem, $\widehat{T}$ has at least a fixed point $\widetilde{U}\in\widehat{B}$, that is,

$$\widetilde{U}(t)=\widetilde{U}(0)+\int_0^t(t-s)^{\alpha-1}g\left(s,\widetilde{U}(\tau)\right)ds,\ t\in[0,\omega_1].$$

It is easy to show that $\widetilde{U}$ is a solution of problem (P1) on $[0,\omega_1]$, from which we have the conclusion of the proof. □

We have seen in the previous section that a solution of problems (P1) and (P2) may not be the unique one. However, we can show uniqueness via the Banach contraction theorem, precisely: lemma 1.3.

2.2.3 Uniqueness of solution

We want to discus here the uniqueness of the solution of the problem (P1).

Theorem 2.3. *We suppose that $(H_1)-(H_4)$ hold. Then, There exists a unique solution of problem (P1).*

Proof. Let V_1 and V_2 in $\overline{B}$. Set $W_1=V_1-V_2$, then W_1 satisfies

$$
\begin{cases}
\mathcal{D}_t^{\alpha}W_1(x,t)=\mathbf{c}\Delta W_1(x,t)+g(t,V_1(x,\tau))-g(t,V_2(x,\tau)), & t\in[0,T_0],\\
W_1(x,0)=0 \qquad for\ x\in\Omega, \\
W_1(x,t)=0 \qquad for\ x\in\partial\Omega, t\in[0,T_0]
\end{cases}
$$

We have

$$\begin{aligned}\left|(\overline{T}V_1)(t)-(\overline{T}V_2)(t)\right| &\le \left|W_1(0)-\alpha_1\mathbf{c}\int_0^t (t-s)^{\alpha-1}(V_1(s)-V_2(s))ds\right| \\ &+\left|\int_0^t (t-s)^{\alpha-1}[g\left(s,V_1(\tau)\right)-g\left(s,V_2(\tau)\right)]ds\right| \\ &\le \alpha_1\mathbf{c}\int_0^t (t-s)^{\alpha-1}\left|(V_1(s)-V_2(s))\right|ds \\ &+\int_0^t (t-s)^{\alpha-1}\left|g\left(s,V_1(\tau)\right)-g\left(s,V_2(\tau)\right)\right|ds \\ &\le \alpha_1\mathbf{c}\left\|(V_1(s)-V_2(s))\right\|\int_0^t (t-s)^{\alpha-1}ds \\ &+\left\|g\left(s,V_1(\tau)\right)-g\left(s,V_2(\tau)\right)\right\|\int_0^t (t-s)^{\alpha-1}ds\end{aligned}$$

After using (H_4) on $\overline{T}$, we obtain

$$\left|(\overline{T}V_1)(t)-(\overline{T}V_2)(t)\right|\le k\left\|V_1-V_2\right\|,$$

where $k=(\alpha_1\mathbf{c}+L)(\frac{1-\beta}{\alpha-\beta})^{1-\beta}w^{\alpha-\beta}<1$. □

For the moment, we state in the following theorem the uniqueness of the solution to the problem (P2).

Theorem 2.4. *Assume that $(H_1)-(H_4)$ hold. Then, there exists a unique solution of problem (P2).*

Proof. Let U_1 and U_2 in $\widehat{B}$. Set $W_2=U_1-U_2$, then W_2 satisfies

$$\begin{cases}\mathcal{D}_t^{\alpha}W_2(x,t)=\mathbf{c}\Delta W_2(x,t)+g(t,U_1(x,\tau))-g(t,U_2(x,\tau)), & t\in[0,T_0],\\ W_2(x,0)=0 \quad for\ x\in\Omega, \\ W_2(x,t)=0 \quad for\ x\in\partial\Omega, t\in[0,T_0]\end{cases}$$

We have

$$\begin{aligned}\left|(\widehat{T}U_1)(t)-(\widehat{T}U_2)(t)\right| &\le \left|W_1(0)-\alpha_1\mathbf{c}\int_0^t (t-s)^{\alpha-1}(U_1(s)-U_2(s))ds\right| \\ &+\left|\int_0^t (t-s)^{\alpha-1}[g\left(s,U_1(\tau)\right)-g\left(s,U_2(\tau)\right)]ds\right| \\ &\le \alpha_1\mathbf{c}\int_0^t (t-s)^{\alpha-1}\left|(U_1(s)-U_2(s))\right|ds \\ &+\int_0^t (t-s)^{\alpha-1}\left|g\left(s,U_1(\tau)\right)-g\left(s,U_2(\tau)\right)\right|ds \\ &\le \alpha_1\mathbf{c}\left\|(U_1(s)-U_2(s))\right\|\int_0^t (t-s)^{\alpha-1}ds \\ &+\left\|g\left(s,U_1(\tau)\right)-g\left(s,U_2(\tau)\right)\right\|\int_0^t (t-s)^{\alpha-1}ds\end{aligned}$$

After using (H_4) on $\widehat{T}$, we obtain

$$\left|(\widehat{T}U_1)(t) - (\widehat{T}U_2)(t)\right| \leq k \left\|U_1 - U_2\right\|,$$

where

$$k' = (\alpha_1 \mathbf{c} + L)(\frac{1-\beta}{\alpha-\beta})^{1-\beta} w_1^{\alpha-\beta} < 1.$$

Our theorem is proved. □

APPENDIX A

BASIC CONCEPTS IN NORMS AND BANACH SPACES

For the proofs of the theorems and proposition stated in this appendix, the interested readers can consult, e.g.[26] and references therein.

A.1 Norms and spaces of Banach

Let Y be a linear space over the scalar field $\mathbb{C}$ or $\mathbb{R}$. A norm in Y defined as a real function

$$\|.\| : Y \longrightarrow \mathbb{R} \tag{A.1}$$

such that, for all scalar λ_1 and every $y_1, y_2 \in Y$, the following properties holds:

1. $\|y_1\| \geq 0; \|y_1\| = 0 \quad$ *if and only if* $\quad y_1 = 0 \quad$ *(positivity).*
2. $\|\lambda_1 y_1\| = |\lambda_1| \, \|y_1\| \quad$ *(homogeneity).*
3. $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\| \quad$ *(triangular inequality).*

A norm is introduced to measure the size (or "length") of each vector $y_1 \in Y$, for this, properties 1, 2, 3 should appear as natural requirements.

A normed space is a linear space Y endowed with a norm $\|.\|$. A norm induces a distance between two vectors given by

$$d(y_1, y_2) = \|y_1 - y_2\| \tag{A.2}$$

which make Y as a metric space.

More precisely, a metric space is a set M, endowed with a distance $d : M \times M \longrightarrow \mathbb{R}$, satisfying the three properties below; for every y_1, y_2, y_3 in M:

- $d(y_1, y_2) = 0, \quad$ *if and only if* $\quad y_1 = 0 \quad$ *(positivity).*
- $d(y_1, y_2) = d(y_2, y_1) \quad$ *(symmetry).*
- $d(y_1, y_3) \leq d(y_1, y_2) + d(y_2, y_3) \quad$ *(triangular inequality).*

Observe that a metric space does not need to be a linear space. We introduce the ball of radius r', and centered at y_1 as follow

$$B_{r'}(y_1) = \{y_2 \in M : d(y_1, y_2) < r'\}$$

*We say that a sequence $y_n \in M$ is **convergent** to an element y_1 in M, and we denote it by $y_n \longrightarrow y_1$ in M, if*

$$d(y_n, y_1) \longrightarrow 0 \qquad as\ n \longrightarrow \infty.$$

If $d(y_n, y_1) \longrightarrow 0$ and $d(y_n, y_2) \longrightarrow 0$ as $n \longrightarrow \infty$, then, by using the triangular inequality, we get

$$d(y_1, y_2) \leq d(y_n, y_1) + d(y_n, y_2) \quad as\ n \longrightarrow \infty$$

which implies that $y_1 = y_2$. Thus, the limit is unique. (see more in Sect6.2 pg 368 [26])

***Complete space**. A metric space in which each Cauchy sequence() converges is called **complete**.*

Definition A.1. [26]A complete, normed linear space is called a Banach space.

Proposition A.1. *[26]Let Y be a linear space. Every norm in Y is continuous.*

A.2 Some principal theorems and inequalities

Let Ω be an open, bounded set of $\mathbb{R}^n$ with a smooth boundary Γ_1.

Theorem A.1. *[20](Holder's inequality) Let $1 \leq q \leq +\infty$, and p its conjugate exponent. Suppose that $h \in L^q(\Omega)$, and $g \in L^p(\Omega)$, then $hg \in L^1(\Omega)$ and*

$$\int_\Omega |h(x)g(x)|\, dx \leq \|h\|_q \|g\|_p .$$

***Newton inequality.** For all reel numbers $a, b \geq 0$ and every $n \in \mathbb{N}$:*

$$\frac{1}{2^{n-1}} (a+b)^n \leq a^n + b^n.$$

General Conclusion

In this memory, two systems of a nonlinear time-conformable fractional reaction-diffusion equations with delay with the same initial condition and two different boundary conditions are considered. For both systems, and under suitable hypotheses, we showed the existence and uniqueness of solution by using Leary-Schauder fixed point and Contraction mapping theorems.

We have studied the existence and uniqueness of the solutions of the fractional reaction-diffusion equations with a delay using the concept of the fractional conformable derivative with respect to time. This study enables us to search for analytical or numerical methods to find solutions to this equation with delay, especially since it has great importance in reality in several scientific fields.

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ملخص:

في هذه المذكرة، قمنا بدراسة وجود ووحدانية الحلول لجملتي معادلات مشكلة من معادلة تفاعل- انتشار الكسرية غير الخطية بمفهوم المشتق الكسري المطابق. بفرض بعض الشروط على المعطيات، نبرهن أولا وجود الحل وذلك باستخدام نظرية النقطة الصامدة Leary-Schauder. بعد ذلك، نبرهن وحدانية الحل لكلتا جملتي المعادلات السابقة وذلك باستخدام نظرية contraction mapping theorem.

كلمات مفتاحية: معادلة تفاعل-انتشار غير خطية، حساب كسري متطابق، حد التأخر، نظرية النقطة الثابتة ليري-شودر، contraction mapping theorem.

In this work, two coupled systems for nonlinear time-conformable fractional reaction-diffusion equation with delay have been considered. Under some hypotheses in given data we have proved first the existence of the solution for both systems by using Leary-Schauder fixed point theorem. After, we have showed the uniqueness of the solution for the same two systems by using the contraction mapping theorem.

Keywords: *Nonlinear reaction-diffusion equation, fractional conformable calculus, delay, Leary-Schauder fixed point theorem, contraction mapping theorem.*

Dans ce travail, deux systèmes gouvernés par l'équation non linéaire de réaction-diffusion fractionnaire avec retard en temps et des conditions aux bord sont considérées. On décrites les dérivée fractionnaires au sens conformable. Sous certaines hypothèses sur les données, nous avons montré l'existence de la solution de deux problèmes en utilisant le théorème de point fixe de Leary-Schauder. Ensuite, nous avons prouvé l''unicité de la solution du même problèmes précédent en utilisant le théorème des opérateurs contractantes.

Mots-Clés: *Equation de réaction-diffusion non linéaire, calcule fractionnel conformable, terme de retard, théorème de point fixe de Leary-Schauder, théorème des opérateurs contractantes.*