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An efficient algorithm for solving the conformable time-space fractional telegraph equations

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ABSTRACT. In this paper, an efficient algorithm is proposed for solving one dimensional time-space-fractional telegraph equations. The fractional derivatives are described in the conformable sense. This algorithm is based on shifted Chebyshev polynomials of the fourth kind. The time-space fractional telegraph equations is reduced to a linear system of second order differential equations and the Newmark's method is applied to solve this system. Finally, some numerical examples are presented to confirm the reliability and effectiveness of this algorithm.

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1. Introduction

Telegraph equation is introduced by Oliver Heaviside and is a linear second-order hyperbolic partial differential equations, see [14]. This equation describe the current and voltage on an electrical transmission line with distance and demonstrates that the electromagnetic waves can

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be reflected on the wire, and that appear wave patterns along the transmission line [22]. The nonhomogeneous linear telegraph equation can be written as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \left(\frac{R}{L} + \frac{G}{C}\right)\frac{\partial u(x,t)}{\partial t} + \frac{RG}{LC} u(x,t) = \frac{1}{LC}\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \tag{1.1}$$

where $u(x,t)$ can be voltage or current through the wire at position x and time t, R and G are, respectively, the resistance and the conductance of resistor, C is the capacitance of capacitor, and L is the inductance of coil. This telegraph equation is applicable in several fields such as wave propagation [43], random walk theory [2], signal analysis [17].

Various numerical scheme such as splines radial basis function [13], Chebyshev Tau method [34], Legendre multiwavelet Galerkin method [45], homotopy perturbation method [24], Chebyshev spectral collocation method [16], differential quadrature method [30], B-spline collocation method [38], Haar wavelet method [3], Bessel functions [10] and dual reciprocity boundary integral equation method [12] have been applied to solve telegraph equation.

Fractional calculus, as an extension of the classical derivatives and integrals to non-integer orders, has been frequently used to model many fundamental problems in various branches of sciences and engineering [31, 36, 8]. More recently, it has been found that fractional operators are more suitable for modeling phenomena in sciences and engineering.

In 2014, a new definition of fractional derivative, named "conformable fractional derivative", is introduced by Khalil et al. [18]. This novel fractional derivative is compatible with the classical derivative and it is excellent for studying non regular solutions. The subject of the conformable fractional derivative has attracted the attention of many authors in domains such as mechanics, electronic, and anomalous diffusion. We are interested in studying in this paper the telegraph model (1.1) in framework of the conformable time-space-fractional derivative. Precisely, we will propose the following transformations:

$$\begin{cases} \frac{\partial}{\partial t} \to \mathcal{D}_t^{(\alpha)},\ \frac{\partial^2}{\partial t^2} \to \mathcal{D}_t^{(1+\alpha)} \text{ and } \frac{\partial^2}{\partial x^2} \to \mathcal{D}_x^{(\beta)}, \\ 2a = R/L + G/C,\ \ b^2 = RG/LC,\ \ \omega = 1/LC,\ \ 0 < \alpha \leq 1,\ \ 1 < \beta \leq 2, \end{cases} \tag{1.2}$$

where $\mathcal{D}_t^{(\alpha)}$ and $\mathcal{D}_x^{(\beta)}$ are the conformable time and space fractional derivative operators [18]. Then, we get the conformable time-space-fractional telegraph model associated with the transformation (1.2) as follows:

$$\mathcal{D}_t^{(1+\alpha)} u(x,t) + 2a\mathcal{D}_t^{(\alpha)} u(x,t) + b^2 u(x,t) = \omega \mathcal{D}_x^{(\beta)} u(x,t) + f(x,t), \tag{1.3}$$

subject to the initials conditions

$$u(x,0) = \varphi(x),\ \ \mathcal{D}_t^{(\alpha)} u(x,0) = \psi(x),\ \ 0 \leq x \leq 1, \tag{1.4}$$

and the Dirichlet boundary condition

$$u(0,t) = g(t) \text{ and } u(1,t) = h(t),\ \ 0 \leq t \leq T, \tag{1.5}$$

where $\varphi, \psi \in \mathcal{C}^2(0,1)$, $g, h \in \mathcal{C}^2(0,T)$ and $f \in \mathcal{C}([0,1] \times [0,T])$ are given functions.

Numerical solution of fractional telegraph equation have been investigated by many authors. Mollahasani et. al considered the time-fractional telegraph equations and used hybrid Legendre functions to approximate their solutions [25]. In order to solve two-dimensional fractional telegraph equation a spectral meshless radial point interpolation method was proposed in [39]. Bhrawy et al. proposed a Chebyshev Tau method for numerical solution of the two-sided fractional-order telegraph equation [4]. A computational Tau method based on the Legendre polynomials has been proposed to solve time-fractional telegraph equations by Saadatmandi and Mohabbati [35]. Suleman et. al [41] have been used a new projected differential transform method for space and time fractional telegraph equations. In [9] the method of separation of variables has been applied for deriving the analytical solutions of time-fractional telegraph equations with different kind of boundary conditions. Sweilam et. al [42] considered the Sinc-Legendre collocation method for solving time-fractional telegraph equations. Fourier and Laplace transforms have been also applied to derive the analytical solutions of time-fractional telegraph equations [19, 20]. Heydari et. al used an efficient Legendre wavelets method for numerical solution of time-fractional telegraph equations [15]. Moreover, semi-analytical methods have been employed by the researchers in [26, 44, 32, 37] for solving time-fractional telegraph equations.

In the last decade, considerable attention was paid to the application of orthogonal polynomials in the solution of fractional differential equations, integral equations and fractional partial differential equations. Numerical method based on Legendre [11, 35], Chebyshev [5, 34, 16], second kind of Chebyshev [33], fourth kind Chebyshev [6] and Jacobi [40, 27] polynomials were proposed. The rest of this paper is structured as follows: Section 2 deals with some description of conformable fractional derivative and its properties. In Section 3, deals with some properties of shifted Chebyshev polynomials of the fourth kind. Section 4 is devoted to evaluation of the conformable fractional derivative using shifted Chebyshev polynomials of the fourth kind. In Section 5, a Chebyshev collocation method based on the shifted Chebyshev polynomials of the fourth kind has been proposed to solve the problem (1.3)-(1.5). In Section 6, the Newmark's method has been proposed to solve this system. Various illustrative examples are considered to confirm accuracy of the Chebyshev collocation method in Section 7.

2. Description of conformable fractional derivative and its properties

Definition 2.1 ([18]). *For a function $f :]0, +\infty[\to \mathbb{R}$, the conformable fractional derivative of f of order $0 < \alpha < 1$ in the variable t is defined as:*

$$\mathcal{D}_t^{(\alpha)} f(t) := \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}.$$

Some important properties of the conformable fractional derivative are as follows:

Theorem 2.1 ([18, 1]). *(1) The conformable fractional derivative satisfies linearity properties similar to integer order differentiation*

$$\mathcal{D}_x^{(\alpha)} (\lambda f + \mu g)(x) = \lambda \mathcal{D}_x^{(\alpha)} f(x) + \mu \mathcal{D}_x^{(\alpha)} g(x), \quad \forall \lambda, \mu \in \mathbb{R}. \tag{2.1}$$

(2) If f is n times differentiable, then we have

$$\mathcal{D}_t^{(\alpha)} f(t) = t^{n+1-\alpha} f^{(n+1)}(t), \quad \text{for all } n < \alpha \leq n+1. \tag{2.2}$$

(3) For conformable fractional derivative, we have

$$\mathcal{D}^{(\alpha)}K = 0,\ K \text{ is a constant}, \tag{2.3}$$

$$\mathcal{D}_t^{(\alpha)} t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-n)} t^{p-\alpha} & \text{if } p \in \mathbb{N} \text{ and } p > \alpha, \\ 0 & \text{if } p \in \mathbb{N} \text{ and } p < \alpha, \end{cases} \tag{2.4}$$

where Γ denotes Gamma function and $n < \alpha \leq n+1$.

3. Some properties of shifted Chebyshev polynomials of the fourth kind

Shifted Chebyshev polynomials of the fourth kind $W_n^*(x)$ of degree n in x are defined on $[0,1]$ and can be determined the following recurrence formula, see [21].

$$\begin{cases} W_0^*(x) = 1, \\ W_1^*(x) = 4x - 1, \\ W_n^*(x) = 2(2x-1)W_{n-1}^*(x) - W_{n-2}^*(x),\ n = 2, 3, \ldots \end{cases}$$

The analytical form of the shifted Chebyshev polynomials of the fourth kind $W_n^*(x)$ of degree n in x is given by

$$W_n^*(x) = \sum_{k=0}^{n} (-1)^k 2^{2n-2k} \frac{\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+1)} x^{n-k},\ n \in \mathbb{N}^*. \tag{3.1}$$

These polynomials are orthogonal on the support interval $[0,1]$ as the following inner product:

$$\langle W_n^*, W_m^* \rangle := \int_0^1 \sqrt{\frac{1-x}{x}} W_n^*(x) W_m^* dx = \begin{cases} 0 & \text{if } n \neq m, \\ \pi/2 & \text{if } n = m, \end{cases}$$

where $x \mapsto \sqrt{\frac{1-x}{x}}$ is weight function corresponding to W_n^*.

Any function $y(x)$ belongs to the space of square integrable functions in $[0,1]$, may be expressed in terms of shifted Chebyshev polynomials of the fourth kind as

$$y(x) = \sum_{i=0}^{+\infty} c_i W_i^*(x),$$

where the coefficients c_i, $i \in \mathbb{N}$ are given by

$$c_i = \frac{2}{\pi} \int_0^1 y(x) \sqrt{\frac{1-x}{x}} W_i^*(x)\, dx. \tag{3.2}$$

For practical purpose we take only first $(m+1)$-terms of $W_n^*(x)$ in approximation which is given:

$$y_m(x) = \sum_{i=0}^{m} c_i W_i^*(x). \tag{3.3}$$

4. Evaluation of the conformable fractional derivative using shifted Chebyshev polynomials of the fourth kind

The main approximate formula for the function $y_m(x)$ given in (3.3) is presented in the following theorem.

Theorem 4.1. *Let $y_m(x)$ be approximated function in terms of shifted Chebyshev polynomials of the fourth kind as given in (3.3), suppose $\gamma > 0$, then we have*

$$\mathcal{D}^{(\gamma)} y_m(x) = \sum_{i=n+1}^{m} \sum_{k=0}^{i-n-1} c_i \mathcal{N}_{i,k}^{n} x^{i-k-\gamma}, \; n < \gamma \leq n+1, \tag{4.1}$$

where $\mathcal{N}_{i,k}^{n}$ is given by

$$\mathcal{N}_{i,k}^{n} = (-1)^k \, 2^{2i-2k} \frac{\Gamma(2i-k+1)\,\Gamma(i-k+1)}{\Gamma(k+1)\,\Gamma(2i-2k+1)\,\Gamma(i-k-n)}. \tag{4.2}$$

Proof. By using the linearity of the conformable fractional derivative given in (2.1) and by using definition of approximation function $y_m(x)$ as in (3.3), we obtain

$$\mathcal{D}_x^{(\gamma)} y_m(x) = \sum_{i=0}^{m} c_i \mathcal{D}_x^{(\gamma)} W_i^*(x), \quad \text{for all } \gamma > 0. \tag{4.3}$$

Moreover, from (2.1), (2.2), (2.4) and (3.1), we get

$$\mathcal{D}_x^{(\gamma)} W_i^*(x) = 0, \;\; i = 0, 1, \ldots, n, \; n \in \mathbb{N}, \; n < \gamma \leq n+1. \tag{4.4}$$

For $i - k > n$ and from (2.4), we have

$$\mathcal{D}_x^{(\gamma)} x^{i-k} = \frac{\Gamma(i-k+1)}{\Gamma(i-k-n)} x^{i-k-\gamma}, \; n \in \mathbb{N}, \; n < \gamma \leq n+1. \tag{4.5}$$

Substituting (4.5) into (3.1), we obtain

$$\mathcal{D}_x^{(\gamma)} W_i^*(x) = \sum_{k=0}^{i-n-1} (-1)^k \, 2^{2i-2k} \frac{\Gamma(2i-k+1)\,\Gamma(i-k+1)}{\Gamma(k+1)\,\Gamma(2i-2k+1)\,\Gamma(i-k-n)} x^{i-k-\gamma}. \tag{4.6}$$

Substituting (4.6) into (4.3), we obtain for $n < \gamma \leq n+1$:

$$\mathcal{D}_x^{(\gamma)} y_m(x) = \sum_{i=n+1}^{m} \sum_{k=0}^{i-n-1} c_i \, (-1)^k \, 2^{2i-2k} \frac{\Gamma(2i-k+1)\,\Gamma(i-k+1)}{\Gamma(k+1)\,\Gamma(2i-2k+1)\,\Gamma(i-k-n)} x^{i-k-\gamma},$$

where can be rewritten as the form

$$\mathcal{D}_x^{(\gamma)} y_m(x) = \sum_{i=n+1}^{m} \sum_{k=0}^{i-n-1} c_i \mathcal{N}_{i,k}^{n} x^{i-k-\gamma},$$

where $\mathcal{N}_{i,k}^{n}$ is defined by (4.2).

Example 4.1. *Consider $g(x) = x^2$ with $m = 2$, $\gamma = 1.8$ and $n = 1$. Using (2.3) and (2.4), we obtain*

$$\mathcal{D}_x^{(1.8)} x^2 = \mathcal{D}_x^{(0.8)} (2x) = 2\mathcal{D}_x^{(0.8)} (x) = 2x^{0.2}.$$

Now, using Theorem 4.1 we obtain:

$$\mathcal{D}_x^{(1.8)} x^2 = c_2 \mathcal{N}_{2,0}^1 x^{0.2}.$$

From (3.2) and (4.2), we obtain:

$$c_2 = \frac{1}{16} \text{ and } \mathcal{N}_{2,0}^1 = 32.$$

Then, $\mathcal{D}_x^{(1.8)} x^2 = 2x^{0.2}$.

4.1. Error Analysis. The main goal of this section is study the truncating error and its convergence analysis.

Theorem 4.2. *(Uniformly converges theorem, [6]). Let $y \in L^2(0,1)$ be a function twice differentiable on $[0,1]$ and the second derivative is bounded on $[0,1]$ i.e.*

$$\exists M > 0,\ \forall x \in [0,1] : \left|y''(x)\right| \leq M.$$

If $y(x) = \sum_{i=0}^{+\infty} c_i W_i^*(x)$ is the sum of the series of shifted Chebyshev polynomials of the fourth kind, then the sequence of partial sums (y_m) with $y_m(x) = \sum_{i=0}^{m} c_i W_i^*(x)$ uniformly converges to $y(x)$ on $[0,1]$.

Proof. Using the variable change $2x - 1 = \cos(\theta)$ in (3.2), we obtain:

$$c_i = \frac{2}{\pi} \int_0^\pi y\left(\frac{1+\cos(\theta)}{2}\right) \sin\left[\left(i + \frac{1}{2}\right)\theta\right] \sin(\theta/2)\, d\theta.$$

Now integration by parts two times, we get:

$$c_i = \frac{1}{4\pi} \int_0^\pi y''\left(\frac{1+\cos(\theta)}{2}\right) \delta_i(\theta)\, d\theta,$$

where

$$\delta_i(\theta) = \sin(\theta)\left[\frac{1}{i}\left(\frac{\sin(i-1)\theta}{i-1} - \frac{\sin(i+1)\theta}{i+1}\right) + \frac{1}{i+1}\left(\frac{\sin i\theta}{i} - \frac{\sin(i+2)\theta}{i+2}\right)\right].$$

$$|c_i| = \left|\frac{1}{4\pi}\int_0^\pi y''\left(\frac{1+\cos(\theta)}{2}\right)\delta_i(\theta)\, d\theta\right| \leq \frac{M}{4\pi}\int_0^\pi |\delta_i(\theta)|\, d\theta = \frac{M\left(i^2 + 2i - 1\right)}{i\left(i^2-1\right)(i+2)} \leq \frac{M}{i^2}.$$

Other hand, we have:

$$|y(x) - y_m(x)| \leq \sum_{i=m+1}^{+\infty} |c_i|\,|W_i^*(x)| \leq \sum_{i=m+1}^{+\infty} |c_i| \leq \sum_{i=m+1}^{+\infty} \frac{M}{i^2}.$$

Then, $\sum_{i=1}^{+\infty} \frac{1}{i^2}$ is the Riemann series converges, then the rest of this series converges to zero, so the sequence $(y_m(x))$ is uniformly converges to $y(x)$ on $[0,1]$.

5. Chebyshev Collocation Method

In this section, we apply Chebyshev collocation method to problems (1.3)-(1.5) based on the shifted Chebyshev polynomials of the fourth kind. Let us denote $u_m(x,t)$ as the approximation to $u(x,t)$ in the following form

$$u_m(x,t) = \sum_{i=0}^{m} c_i(t) W_i^*(x). \tag{5.1}$$

We apply Theorem 4.1, (1.3) and (5.1), we obtain

$$\begin{aligned}\sum_{i=0}^{m} t^{1-\alpha} c_i''(t) W_i^*(x) + 2a \sum_{i=0}^{m} t^{1-\alpha} c_i'(t) W_i^*(x) + b^2 \sum_{i=0}^{m} c_i(t) W_i^*(x) \\ = \omega \sum_{i=2}^{m} \sum_{k=0}^{i-2} c_i(t) \mathcal{N}_{i,k}^{1} x^{i-k-\beta} + f(x,t)\end{aligned} \tag{5.2}$$

In order to find the unknown coefficients, Chebyshev collocation method with the following collocation points

$$x_p = \frac{1}{2}\left(1 + \cos(p\pi/m)\right),\ p = 1, 2, \ldots, m-1,$$

are applied. Then, we have

$$\sum_{i=0}^{m} \left[t^{1-\alpha} c_i''(t) W_i^*(x_p) + 2a t^{1-\alpha} c_i'(t) W_i^*(x_p) + c_i(t) R_i(x_p) \right] = f(x_p, t), \tag{5.3}$$

where

$$\begin{cases} S_0(x_p) = S_1(x_p) = 0, \\ S_i(x_p) = \sum_{k=0}^{i-2} \mathcal{N}_{i,k}^{1} x_p^{i-k-\beta},\ i\ = 2, 3, ..., m, \\ R_i(x_p) = b^2 W_i^*(x_p) - \omega S_i(x_p),\ i = 1, 2, \ldots, m. \end{cases}$$

Also put (5.1) in (1.5), we get

$$\sum_{i=0}^{m} (-1)^i c_i(t) = g(t), \quad \sum_{i=0}^{m} (2i+1) c_i(t) = h(t). \tag{5.4}$$

We introduce the vectors $V(t)$ and $F(t)$ defined by

$$\begin{aligned} V(t) &= (c_0(t), c_1(t), \ldots, c_m(t))^T, \\ F(t) &= (f(x_1,t), f(x_2,t), \ldots, f(x_{m-1},t), g(t), h(t))^T. \end{aligned}$$

Substituting (3.2) and (5.1) into the initial conditions (1.4), we can compute

$$V(0) = (c_0(0), c_1(0), \ldots, c_m(0))^T \text{ and } \frac{d}{dt}V(0) = \left(c_0'(0), c_1'(0), \ldots, c_m'(0)\right)^T.$$

Let the matrices $M(t)$, $C(t)$ and K given by

$$M(t) = t^{1-\alpha}\begin{pmatrix} W_0^*(x_1) & W_1^*(x_1) & \ldots & W_m^*(x_1) \\ W_0^*(x_2) & W_1^*(x_2) & \ldots & W_m^*(x_2) \\ \vdots & \ddots & \ldots & \vdots \\ W_0^*(x_{m-1}) & W_1^*(x_{m-1}) & \ldots & W_m^*(x_{m-1}) \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$C(t) = 2aM,$$

$$K = \begin{pmatrix} R_0(x_1) & R_1(x_1) & \ldots & R_m(x_1) \\ R_0(x_2) & R_1(x_2) & \ldots & R_m(x_2) \\ \vdots & \ldots & \ddots & \vdots \\ R_0(x_{m-1}) & R_1(x_{m-1}) & \ldots & R_m(x_{m-1}) \\ 1 & -1 & \ldots & (-1)^m \\ 1 & 3 & \ldots & 2m+1 \end{pmatrix}$$

By combining the equations (5.3) and (5.4), we find the following matrix form

$$\begin{cases} M(t)\ddot{V}(t) + C(t)\dot{V}(t) + KV(t) = F(t), \\ V(0) = (c_0(0), c_1(0), \ldots, c_m(0))^T, \\ \dot{V}(0) = (c_0'(0), c_1'(0), \ldots, c_m'(0))^T. \end{cases} \tag{5.5}$$

To solve the system of second order differential equations (5.5), we use Newmark's method.

6. Newmark's method

For positive integer N, $\Delta t = T/N$, denotes the step size of the variable t. So we define $t_j = j\Delta t$ in which $j = 0, 1, \ldots, N$, and we introduce the following notations: $c_i(t_j) = c_i^j$, $g(t_j) = g^j$, $h(t_j) = h^j$, $V^j = \left(c_0^j, c_1^j, \ldots, c_m^j\right)^T$ and $F^j = \left(f(x_1, t_j), f(x_2, t_j), \ldots, f(x_{m-1}, t_j), g^j, h^j\right)^T$.

In 1959 Newmark proposed a method, see [29] which links the accelerations, the velocities and the displacements of the nodes at the instants t_{j+1} and t_j as follows:

$$V^{j+1} = V^j + \Delta t\dot{V}^j + \frac{(\Delta t)^2}{2}\left[(1-2\theta)\ddot{V}^j + 2\theta\ddot{V}^{j+1}\right], \tag{6.1}$$

$$\dot{V}^{j+1} = \dot{V}^j + \Delta t\left[(1-\gamma)\ddot{V}^j + \gamma\ddot{V}^{j+1}\right]. \tag{6.2}$$

where $0 \leq \gamma \leq 1$ and $0 \leq \theta \leq 1/2$. The most commonly used method is the average acceleration method ($\theta = 1/4$ and $\gamma = 1/2$), see [7].

We then try to solve the system (5.5) at the instant t_{j+1}:

$$M^{j+1}\ddot{V}^{j+1} + C^{j+1}\dot{V}^{j+1} + KV^{j+1} = F^{j+1}, \ j = 0, 1, \ldots, N-1. \tag{6.3}$$

Rearranging (6.1) to provide an expression for $\ddot{V}^{j+1}$ gives:

$$\ddot{V}^{j+1} = \frac{V^{j+1} - V^{j}}{\theta \Delta t^{2}} - \frac{\dot{V}^{j}}{\theta \Delta t} - \left(\frac{1}{2\theta} - 1\right)\ddot{V}^{j}. \tag{6.4}$$

Inserting the expression for $\ddot{V}^{j+1}$ from (6.4) into (6.2) gives:

$$\dot{V}^{j+1} = \frac{\gamma}{\theta \Delta t}\left(V^{j+1} - V^{j}\right) + \left(1 - \frac{\gamma}{\theta}\right)\dot{V}^{j} + \Delta t\left(1 - \frac{\gamma}{2\theta}\right)\ddot{V}^{j}. \tag{6.5}$$

We can now insert the explicit expressions for $\ddot{V}^{j+1}$ and $\dot{V}^{j+1}$ from (6.4) and (6.5) into (6.3) to obtain the following linear system:

$$A^{j+1}V^{j+1} = B^{j+1}, \tag{6.6}$$

where

$$\begin{aligned}
A^{j+1} &= \frac{M^{j+1}}{\theta\left(\Delta t\right)^{2}} + \frac{\gamma C^{j+1}}{\theta \Delta t} + K, \\
B^{j+1} &= F^{j+1} + M^{j+1}\left[\frac{V^{j}}{\theta\left(\Delta t\right)^{2}} + \frac{\dot{V}^{j}}{\theta \Delta t} + \left(\frac{1}{2\theta} - 1\right)\ddot{V}^{j}\right] \\
&\quad + C^{j+1}\left[\frac{\gamma V^{j}}{\theta \Delta t} - \left(1 - \frac{\gamma}{\theta}\right)\dot{V}^{j} - \Delta t\left(1 - \frac{\gamma}{2\theta}\right)\ddot{V}^{j}\right].
\end{aligned}$$

The complete algorithm using the Newmark's method is given in the following:

Algorithm 1 (Newmark's method)

1: **Initial calculations:**

(1) Form stiffness matrix K, mass matrix M^{j+1}, and damping matrix C^{j+1}.

(2) Initialize V^0, $\dot{V}^0$, and $\ddot{V}^0$.

(3) Select time step $\Delta t = T/N$, parameter θ, γ and calculate integration constants:

$$\gamma \geq 1/2; \quad \theta \geq \frac{(\gamma + 1/2)^2}{4};$$
$$a_0 = \frac{1}{\theta (\Delta t)^2}; \; a_1 = \frac{\gamma}{\theta \Delta t}; \; a_2 = \frac{1}{\theta \Delta t}; \; a_3 = \frac{1}{2\theta} - 1;$$
$$a_4 = \frac{\gamma}{\theta} - 1; \; a_5 = \Delta t \left(\frac{\gamma}{2\theta} - 1 \right); \; a_6 = \Delta t (1 - \gamma); \; a_7 = \gamma \Delta t.$$

(4) Form effective stiffness matrix $A^{j+1} = a_0 M^{j+1} + a_1 C^{j+1} + K$.

2: **For each time step:**

(1) Calculate effective loads at time $t_{j+1} = t_j + \Delta t$:

$$B^{j+1} = F^{j+1} + M^{j+1} \left(a_0 V^j + a_2 \dot{V}^j + a_3 \ddot{V}^j \right) + C^{j+1} \left(a_1 V^j + a_4 \dot{V}^j + a_5 \ddot{V}^j \right).$$

(2) Solve for displacements at time $t_{j+1} = t_j + \Delta t$:

$$A^{j+1} V^{j+1} = B^{j+1}.$$

(3) Calculate accelerations and velocities at time $t_{j+1} = t_j + \Delta t$:

$$\ddot{V}^{j+1} = a_0 \left(V^{j+1} - V^j \right) - a_2 \dot{V}^j - a_3 \ddot{V}^j,$$
$$\dot{V}^{j+1} = \dot{V}^j + a_6 \ddot{V}^j + a_7 \ddot{V}^{j+1}.$$

7. Applications

In this section, we present some numerical results of the conformable time-space fractional telegraph equations (1.3) with initial (1.4) and boundary conditions (1.5). To test the efficiency of Algorithm 1 several numerical examples for different values of a, b, ω, α and β are given in this section with L_2, L_∞ and root mean square (RMS) errors are calculated by

$$L_2 = \|u_e - u_m\|_2 = \sqrt{h \sum_{i=0}^{N} |u_e(x_i, t) - u_m(x_i, t)|^2},$$
$$L_\infty = \|u_e - u_m\|_\infty = \max_{0 \leq i \leq N} |u_e(x_i, t) - u_m(x_i, t)|,$$
$$RMS = \sqrt{\frac{\sum_{i=0}^{N} |u_e(x_i, t) - u_m(x_i, t)|^2}{N+1}},$$

where u_e is the exact solution and u_m is the numerical solution.

We compare the numerical solutions obtained by Algorithm 1 with known exact solutions and those numerical methods available in the literature. All numerical computations were carried out by MATLAB R2014b in OS Windows 10 (64 bits) with Intel(R) Core(TM) i7-2670QM, 2.20 GHz CPU machine and 8 GB of memory.

Example 7.1. *In this example, we consider the telegraph Eq.(1.3) with $a = 1/2$, $b = 1$, $\omega = 1$ and $\beta = 2$ in the domain $0 \le x \le 1$, with following initial and boundary conditions,*

$$\begin{cases} u(x,0) = \mathcal{D}_t^{(\alpha)} u(x,0) = 0, & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & 0 < t \le 1. \end{cases}$$

and $f(x,t) = \left(\alpha + \alpha^2 - (1+\alpha)t + t^{1+\alpha}\right)\left(x - x^2\right)e^{-t} + 2t^{1+\alpha}e^{-t}$. The exact solution of this example is $u_e(x,t) = \left(x - x^2\right)t^{1+\alpha}e^{-t}$. If $\alpha = 1$, this example was studied in [23, 28] by different numerical methods. We apply Algorithm 1 for $m = 5$ and approximate to the solution $u(x,t)$ as follows:

$$u_5(x,t) = \sum_{i=0}^{5} c_i(t)\, W_i^*(x).$$

Using the linear system (6.6) for $m = 5$ and $N = 1000$ with the initial data $V^0 = \dot{V}^0 = \ddot{V}^0 = (0,0,0,0,0,0)^T$. The L_2 and L_∞ errors and CPU time in seconds is shown in Table 1. Numerical results are compared with the obtained results in [23] and [28]. It can be concluded that the numerical solutions obtained by our algorithm are good. In Table 2, we present L_2 and L_∞ errors at different t for $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$, with $\Delta t = 0.0001$ and $h = 0.01$. The graph of exact and numerical solutions for $\alpha = 1$ at $t = 1,2,3,4,5$ and for $t = 2$ at $\alpha = 0.2, 0.4, 0.6, 0.8, 1$ is shown in Figure 1 and the time-space graphs of numerical solution and absolute error up to $t = 5$ are presented in Figure 2.

TABLE 1. L_2 and L_∞ errors and CPU time of Example 7.1 with $\Delta t = 0.001, h = 0.01$ and $\alpha = 1$

	Proposed Algorithm			CuTBSM, [28]			CuBSM, [23]		
T	L_2	L_∞	CPU(s)	L_2	L_∞	CPU(s)	L_2	L_∞	CPU(s)
1.0	8.15E-07	1.46E-06	0.24	6.31E-05	8.76E-05	0.34	4.55E-05	5.91E-05	0.43
2.0	1.30E-05	1.83E-05	0.45	2.34E-05	3.29E-05	0.57	1.43E-05	1.78E-05	0.77
3.0	4.75E-06	7.12E-06	0.65	4.62E-06	5.90E-06	1.05	6.42E-06	1.43E-05	1.15
4.0	9.09E-06	1.26E-05	0.86	2.19E-05	3.04E-05	1.11	8.92E-06	1.35E-05	1.29
5.0	4.43E-06	6.03E-06	1.07	5.18E-06	6.92E-06	1.26	3.01E-06	5.20E-06	1.46

TABLE 2. L_2 and L_∞ errors of Example 7.1 at different values of T and α with $\Delta t = 0.0001$ and $h = 0.01$.

	$T=1$		$T=2$		$T=3$		$T=4$	
α	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.2	2.51E-03	3.53E-03	2.08E-03	2.93E-03	7.99E-04	1.12E-03	6.14E-04	8.58E-04
0.4	5.27E-04	7.44E-04	4.10E-04	5.77E-04	1.06E-04	1.51E-04	1.63E-04	2.27E-04
0.6	7.65E-05	1.09E-04	6.24E-05	8.91E-05	1.51E-05	2.21E-05	2.82E-05	3.91E-05
0.8	8.11E-06	1.14E-05	6.73E-06	9.72E-06	3.05E-06	4.50E-06	1.69E-06	2.26E-06
1.0	6.95E-08	1.26E-07	1.25E-06	1.75E-06	5.64E-07	8.38E-07	1.03E-06	1.43E-06

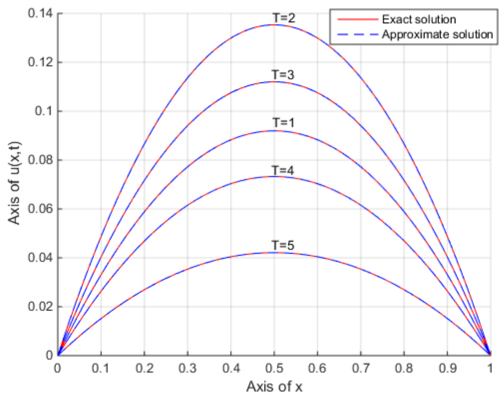

FIGURE (A). Exact and numerical solutions (Left) for $\alpha = 1$ at $t = 1, 2, 3, 4, 5$.

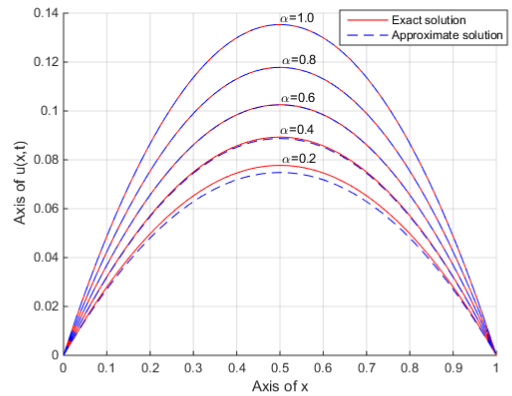

FIGURE (B). Exact and numerical solutions (Right) for $t = 2$ at $\alpha = 0.2, 0.4, 0.6, 0.8, 1$.

FIGURE 1. Comparison of numerical and exact solutions of Example 7.1 at different time and α levels with $h = 0.01$, $\Delta t = 0.001$.

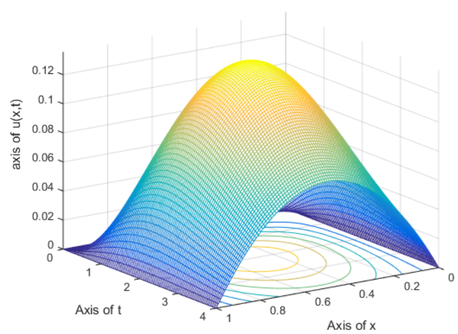

FIGURE (A). Exact solution.

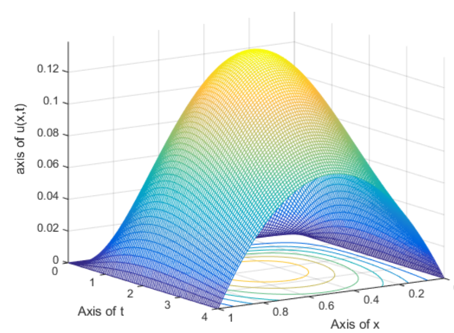

FIGURE (B). Numerical solution.

FIGURE 2. Time-space graphs of Exact and numerical solutions for Example 7.1 at $t = 4$ and $\alpha = 1$.

Example 7.2. *We consider Eq.(1.3) in the domain* $[0, 1]$ *with* $a = 6$, $b = 2$, $\omega = 1$ *and* $\alpha = 1$. *The initial and boundary conditions are given by,*

$$u(x, 0) = \sin((\beta - 1)x), \; u_t(x, 0) = 0, \; 0 \leq x \leq 1,$$
$$u(0, t) = 0, \; u(1, t) = \cos(t)\sin(\beta - 1), \; t \geq 0$$

and $f(x, t) = \left[3\cos(t) - 12\sin(t) + (\beta - 1)^2 x^{2-\beta}\cos(t)\right]\sin(\beta - 1)x$. *The exact solution of this example is* $u_e = \cos(t)\sin((\beta - 1)x)$. *If* $\beta = 2$, *this example was studied in* [23, 28] *by different* *numerical methods. We apply Algorithm 1 for* $m = 5$ *and approximate to the solution* $u(x,t)$ *as follows:*

$$u_5(x,t) = \sum_{i=0}^{5} c_i(t) W_i^*(x).$$

Using the linear system (6.6) *for* $m = 5$ *and* $N = 1000$ *with the initial data*

$$V^0 = (0.2373, 0.2273, -0.0124, -0.0023, 0.0001, 0)^T, \ \dot{V}^0 = \ddot{V}^0 = (0,0,0,0,0,0)^T,$$

we compute the approximate solution $u_5(x,t)$.

The efficiency of Algorithm 1 is measured using L_2, L_∞ *and root mean square errors with* $\Delta t = 0.0001$, $h = 0.01$ *which are shown in Table 3.The numerical results are compared with the obtained results in* [28, 23]. *It can be concluded that the numerical solutions obtained by our algorithm are good. In Figure 3, we present the comparison of numerical and exact solutions for different time levels with* $\Delta t = 0.001$ *and* $h = 0.01$. *Time-space graph of numerical solution up to* $t = 1$ *and* $t = 2$ *are shown in Figure 4.*

TABLE 3. L_2, L_∞ and RMS errors of Example 7.2 with $\Delta t = 0.0001$ and $h = 0.01$ and $\beta = 2$.

	Proposed Algorithm			CuTBSM, [28]			CuBSM, [23]		
T	L_2	L_∞	RMS	L_2	L_∞	RMS	L_2	L_∞	RMS
0.2	5.07E-07	1.03E-06	5.09E-07	2.96E-06	4.63E-06	2.94E-06	2.69E-06	5.24E-06	2.67E-06
0.4	3.48E-07	6.12E-07	3.50E-07	6.77E-06	1.01E-05	6.73E-06	5.61E-06	8.61E-06	5.59E-06
0.6	9.23E-07	1.70E-06	9.28E-07	9.81E-06	1.42E-05	9.76E-06	9.75E-06	1.25E-05	9.70E-06
0.8	1.69E-06	2.69E-06	1.70E-06	1.20E-05	1.71E-05	1.19E-05	1.38E-05	2.03E-05	1.37E-05
1.0	2.01E-06	2.91E-06	2.02E-06	1.34E-05	1.90E-05	1.33E-05	1.73E-05	2.75E-05	1.72E-06

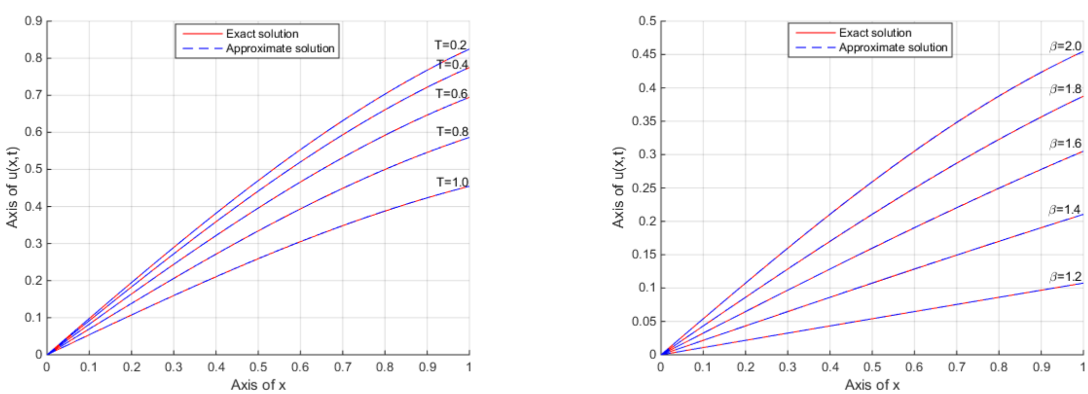

FIGURE (A). Exact and numerical solutions (Left) for $\beta = 2$ at $T = 0.2$ to $T = 1.0$.

FIGURE (B). Exact and numerical solutions (Right) for $t = 1$ at $\beta = 1.2$ to $\beta = 2$.

FIGURE 3. Comparison of numerical and exact solutions of Example 7.2 at different time and β levels with $h = 0.01$, $\Delta t = 0.001$.

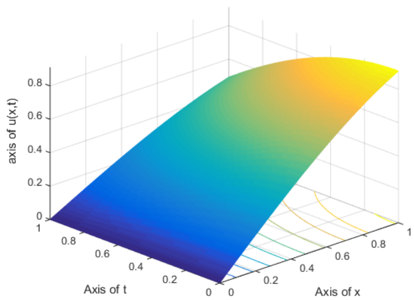

FIGURE (A). Numerical solution at $t = 1$.

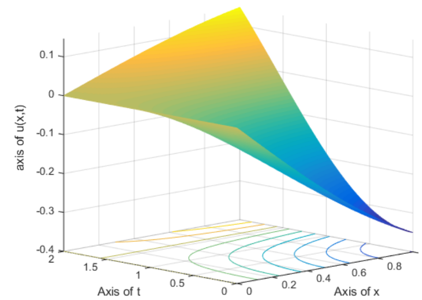

FIGURE (B). Numerical solution at $t = 2$.

FIGURE 4. Time-space numerical solution for Example 7.2 at $t = 1, 2$ with $\beta = 2$.

Example 7.3. *In this example, we consider the telegraph Eq.(1.3) with* $a = 10$, $b = 5$, $\omega = 1$ *and* $\beta = 1 + \alpha$ *in the domain* $0 \leq x \leq 1$, *with following initial and boundary conditions,*

$$u(x,0) = \mathcal{D}_t^{(\alpha)} u(x,0) = 0,\ 0 < x < 1,$$
$$u(0,t) = 0,\ \ u(1,t) = t^{1+\alpha} \sin(\pi(1+\alpha)),\ 0 < t \leq T.$$

and $f(x,t) = \left[\alpha(\alpha+1) + 20(\alpha+1)t + 25t^{1+\alpha} + \pi^2(1+\alpha)^2 x^{1-\alpha} t^{1+\alpha}\right] \sin(\pi(1+\alpha)x)$. *The exact solution to this example is* $u_e = t^{1+\alpha} \sin(\pi(1+\alpha)x)$.

L_2 and L_∞ errors at different time step sizes t for $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$, with $\Delta t = 0.001$ and $h = 0.01$ are reported in Table 4. From Figure 5, it is clear that numerical solution coincides with the exact solution for different time and α levels with $\Delta t = 0.001$ and $h = 0.01$. The time-space graphs of numerical solution and absolute error are presented in Figure 6.

TABLE 4. L_2 and L_∞ errors of Example 7.3 at different values of T and α with $\Delta t = 0.0001$ and $h = 0.01$.

	$T = 0.3$		$T = 0.5$		$T = 0.7$		$T = 1.0$	
α	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.2	5.79E-04	1.22E-03	9.57E-04	1.74E-03	1.43E-03	2.53E-03	2.20E-03	3.84E-03
0.4	8.39E-04	1.59E-03	1.72E-03	3.07E-03	2.77E-03	4.87E-03	4.59E-03	8.02E-03
0.6	1.01E-03	1.76E-03	2.36E-03	4.17E-03	4.10E-03	7.33E-03	7.37E-03	1.33E-02
0.8	9.04E-04	1.93E-03	2.41E-03	5.20E-03	4.61E-03	9.96E-03	9.11E-03	1.97E-02
1.0	7.65E-04	1.42E-03	2.44E-03	4.48E-03	5.29E-03	9.60E-03	1.20E-02	2.15E-02

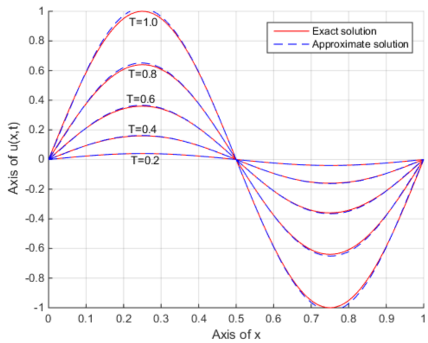

FIGURE (A). Exact and numerical solutions (Left) for $\alpha = 1$ at $T = 0.2$ to $T = 1.0$.

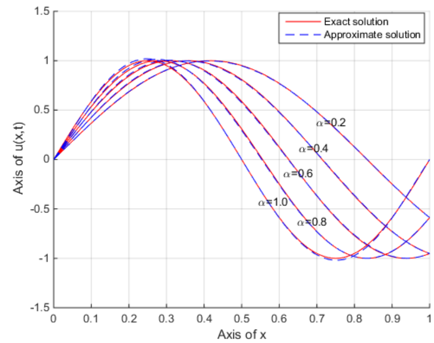

FIGURE (B). Exact and numerical solutions (Right) for $T = 1$ at $\alpha = 0.2$ to $\alpha = 1$.

FIGURE 5. Comparison of numerical and exact solutions of Example 7.3 at different time and α levels with $h = 0.01$, $\Delta t = 0.0001$.

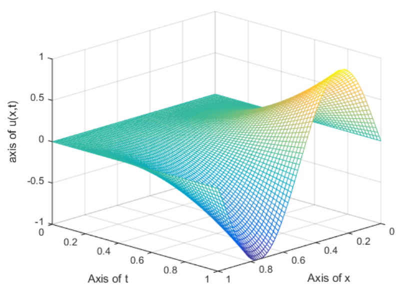

FIGURE (A). Numerical solution.

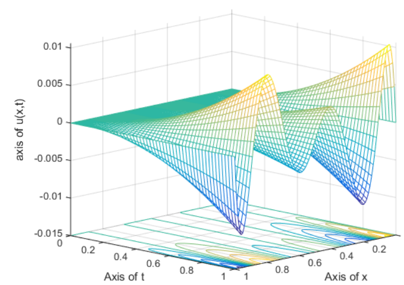

FIGURE (A). Absolute error.

FIGURE 6. Time-space of numerical solution and absolute error for Example 7.3 at $t = 1$ and $\alpha = 1$.

8. Conclusion

An efficient algorithm has been presented to solve the conformable time-space fractional telegraph equations. The properties of the Chebyshev polynomials of the fourth kind and conformable fractional calculus are used to reduce fractional telegraph equations into a linear system of second order differential equations and the Newmark's method is applied to solve this system. Three examples are presented to confirm the reliability and effectiveness of this algorithm.

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