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Sur les espaces de Banach non Archimédiens

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Table of contents

1 Non-Archimedean Valued Fields **2**
1.1 Valuation 2
1.1.1 Definitions and Properties 2
1.1.2 The Topology Induced by a Valuation on $\mathbb{K}$ 4
1.1.3 Non-Archimedean Valuations 6
1.2 Examples of Archimedean Valuation 6
1.2.1 Examples of Non-Archimedean Valued Fields 6

2 Non-Archimedean Banach Spaces **8**
2.1 Non-Archimedean Norms 8
2.2 Non-Archimedean Banach Spaces 10
2.3 Link between $\|\mathbb{E}\|$ and $|\mathbb{K}|$ 14
2.4 Free Banach Spaces 15

3 Bounded Linear Operators in Non-Archimedean Banach Spaces **18**
3.1 Bounded Linear Operators 18
3.1.1 Definitions and Examples 18
3.1.2 Properties of bounded linear operators 22
3.2 Non bounded linear operators 24
3.3 Closed linear operators 24
3.4 Bounded linear operators in free Banach spaces 26

Bibliography **27**

Introduction

Early 1940s, non-Archimedean field analysis has been attempted from different perspectives. In 1943, Monna [17] outlined the non-Archimedean normed vector spaces. One of the successful applications of p-adic functional analysis was the use by Dwork of an ad hoc linear operator in his study of the rationality of the zeta function of a hypersurface over finite fields (a part of Weil conjectures). Immediately, Serre has given a general setting of this operator by constructing the Fredholm determinant of completely continuous operators which applies very well to Dwork's operator (see [21]). Several authors such as Gruson [14, 15], van der Put [22], Schikhof [20], Perez-Garcia and co-authors [19] referred to the early works of Monna [23] and van Rooij [30], to determine the properties of some classes of operators on non-Archimedean Banach spaces. Also, in the book [13] the author clarified the authors' recent work on linear operators on non-Archimedean Banach spaces as well as their spectral theory. Aymen et all in [1] find a non-Archimedean counterpart of the generalized convergence of closable unbounded linear operators as defined by Kato.

This memory is organized as follows.

In the first chapter, we based on the sources Artin [2], Cassels [6], Miranda [16], Attimu [3], Attimu and Diagana [4] and Endler [5], by givng some definitions and properties of non-Archimedean valued fields, the topology tnduced by a Valuation on $\mathbb{K}$, examples of non-Archimedean valued fields. In the second chapter we use the references [7, 8], Diagana et al. [11], Diarra [12, 13], Perez-Garcia [19], Perez-Garcia and Vega [18] for give, some basic notions about non-archimedean norms, non-archimedean Banach spaces, free Banach spaces, also some examples. Finaly, in the last chapter we take the souces Diagana [7, 8] nd Diagana and Ramaroson [9], we devote to basic properties of bounded linear and non bounded operators on non-archimedean Banach spaces.

Chapter 1

Non-Archimedean Valued Fields

1.1 Valuation

1.1.1 Definitions and Properties

Definition 1.1.1 *Let $\mathbb{K}$ be a field. A valuation on $\mathbb{K}$ is a map $|.| : \mathbb{K} \longrightarrow \mathbb{R}$such that for some real number $C \geq 1$, the following hold:*

1) $|x| \geq 0$ for all $x \in \mathbb{K}$ with equality only for $x = 0$.

2) $|xy| = |x|.|y|$ for any x, y in $\mathbb{K}$.

3) For x in $\mathbb{K}$ if $|x| \leq 1$, then $|x + 1| \leq C$.

The valuation $|.|$ such that $|x| = 1$ for every non-zero x and $|0| = 0$ is called the trivial valuation.

Proposition 1.1.1 *The following hold:*

1) $|1| = 1$.

2) For x in $\mathbb{K}$, if $|x^n| = 1$then $|x| = 1$.

3) $|-1| = 1$.

4) $|-x| = |x|$.

Proposition 1.1.2 *Let $|.| : \mathbb{K} \longrightarrow \mathbb{R}$ be a valuation on $\mathbb{K}$ and λ a positive real number then $|.|_\lambda$defined by:*

$$|x|_\lambda = |x|^\lambda$$

for any x in $\mathbb{K}$ is a valuation on $\mathbb{K}$.

Proof. Properties (1) and (2) of Definition 1.1.1 are clear. For (3) of Definition 1.1.1, if $|x|_\lambda \leq 1$then $|x|^\lambda \leq 1$, hence $|x| \leq 1$and since $|.|$ is a valuation, $|x+1| \leq C$ and $|x+1|_\lambda = |x+1|^\lambda \leq C^\lambda$ hence (3) of Definition 1.1.1 holds with the constant C^λ. ■

Definition 1.1.2 *Two valuations $|.|_1$ and $|.|_2$ on the field $\mathbb{K}$ are equivalent if there exists a positive real numbers λ such that $|.|_2 = |.|_1^\lambda$.*

Proposition 1.1.3 *This is an equivalence relation on the set of valuations on the field $\mathbb{K}$.*

Definition 1.1.3 *A valuation $|.|$on the field $\mathbb{K}$ satisfies the triangle inequality if for any x , y in $\mathbb{K}$,*

$$|x+y| \leq |x| + |y|.$$

Proposition 1.1.4 *For any x , y in $\mathbb{K}$,*

$$||x| - |y||_\infty \leq |x-y|$$

where $|.|_\infty$ is the absolute value on $\mathbb{R}$

Proof. *Let*

$$\begin{aligned} |x| &= |(x-y)+y| \\ &\leq |x-y| + |y|, \end{aligned}$$

which implies that

$$|x| - |y| \leq |x-y|.$$

We have also

$$\begin{aligned} |x| &= |(y-x)+x| \\ &\leq |y-x| + |x|, \end{aligned}$$

then

$$|y| - |x| \leq |y-x|$$

and therefore,

$$||x| - |y||_\infty \leq |x - y|.$$

■

Definition 1.1.4 *A valuation* $|.|$ *on* $\mathbb{K}$ *satisfies the ultrametric inequality if for any* x , y *in* $\mathbb{K}$

$$|x + y| \leq \max\{|x|, |y|\}.$$

Proposition 1.1.5 *A valuation* $|.|$ *on* $\mathbb{K}$ *satisfies the ultrametric inequality if and only if one can take* $C = 1$ *in (3) of Definition 1.1.1.*

Definition 1.1.5 *A valuation on* $\mathbb{K}$ *is called non-archimedean if it satisfies the ultrametric inequality.*

Proposition 1.1.6 *Every valuation on* $\mathbb{K}$ *that is equivalent to a non-archimedean valuation is itself non-archimedean.*

Proposition 1.1.7 *Let* $|.|$*be a non-archimedean valuation on* $\mathbb{K}$*. Let* x , y *be in* $\mathbb{K}$ *such that* $|x| < |y|$*, then*

$$|x + y| = |y|.$$

Proof. First $|x + y| \leq |y|$, next, $|y| = |(x + y) - x| \leq \max(|x + y|,|x|)$. If $|x + y| < |x|$ which is against our assumption, therefore $|x + y| \geq |x|$ and hence $|y| \leq |x + y|$.We can conclude that $|x + y| = y$. ■

1.1.2 The Topology Induced by a Valuation on $\mathbb{K}$

Proposition 1.1.8 *Let* d*:* $\mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{R}_+$ *be defined by*

$$d(x, y) = |x - y|$$

then, d *is a distance function on* $\mathbb{K}$ *and* $(\mathbb{K}, d)$ *is a metric space.*

Proof. Suppose $d(x,y)=0$, then $|x-y|=0 \Longrightarrow x=y$ and

$$\begin{aligned} d(x,y) &= |x-y| \\ &= |-(x-y)| \\ &= d(y,x) \end{aligned}$$

for all x, y in $\mathbb{K}$.

For all x, y, z in $\mathbb{K}$,

$$\begin{aligned} d(x,z) &= |x-z| \\ &= |(x-y)+(y-z)| \\ &\leq |x-y|+|y-z| \\ &= d(x,y)+d(y,z) \end{aligned}$$

and hence $(\mathbb{K}, d)$ is a metric space. ■

Corollary 1.1.1 *For any x, y, z in $\mathbb{K}$,*

$$|d(x,z)-d(y,z)|_\infty \leq d(x,y).$$

Since $\mathbb{K}$ is a metric space, the fundamental system of neighborhoods of every element a in $\mathbb{K}$ consists of the open balls of the form

$$B(a,r)=\{x \in \mathbb{K} : |x-a|<r\},$$

where r is a positive real number.

Remark 1.1.1 *For any open ball $B(a,R)$ is such that any element in it is its center, in other words, for any $b \in B(a,r)$, $B(b,r)=B(a,r)$.*

Proposition 1.1.9 *Let $|.|_1$ and $|.|_2$ be two non-trivial valuations which induce the same topology on $\mathbb{K}$, then they are equivalent.*

Definition 1.1.6 *Let $|.|$ be a valuation on $\mathbb{K}$. A completion of $\mathbb{K}$ is a field $\mathbb{F}$ containing $\mathbb{K}$ together with a valuation $||.||$ on it, such that:*

a) $\mathbb{F}$is a complete metric space with respect to the distance induced by $||.||$.

b) The valuation $||.||$extends $|.|$, meaning that for any x in $\mathbb{K}$, $||x||=|x|$.

c) $\mathbb{F}$is the closure of $\mathbb{K}$ with respect to the topology induced by $||.||$.

1.1.3 Non-Archimedean Valuations

Definition 1.1.7 *Let $\mathbb{K}$ be a field with a non-archimedean valuation $|.|$, then*

$$A = \{x \in \mathbb{K} \ |x| \leq 1\}.$$

is called the valuation ring (or the ring of integers) of $\mathbb{K}$.

Proposition 1.1.10 *[10] The following hold:*

a) A is a local ring.

b) $U = \{x \in A : |x| = 1\}$ is the group of units in A.

c) $M = \{x \in A : |x| \leq 1\}$ is the unique maximal ideal of A.

Definition 1.1.8 *[10] The value group of $\mathbb{K}$ is the image of$\mathbb{K}^*$ under the valuation map $|.|$ It is denoted $|\mathbb{K}^*|$.*

The value group $|\mathbb{K}^*|$ is a multiplicative group of positive real numbers, hence it is either:

a) everywhere dense.

b) infinite cyclic.

In the case where the value group is infinite cyclic, the valuation is called a discrete valuation and in the case where the value group is everywhere dense, the valuation is called a dense valuation.

1.2 Examples of Archimedean Valuation

The ordinary absolute value on $\mathbb{C}$ on $\mathbb{R}$ and on any subfield, is the typical example of archimedean valuations. In fact one can prove the following theorem

Theorem 1.2.1 *[10] Let $\mathbb{K}$be complete with respect to an archimedean valuation $|.|$, then $\mathbb{K}$is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ and $|.|$ is equivalent to the ordinary absolute value.*

1.2.1 Examples of Non-Archimedean Valued Fields

Example 1.2.1 *[10] (The field $\mathbb{Q}$of rational numbers). This is a classic example and we will work out the details. Let p be a prime number, then, because of the unique factorization* *in $\mathbb{Z}$, every non-zero rational number x can be written as*

$$x = \frac{a}{b}p^n$$

where n, a, b are integers, and $gcd(p, ab) = 1$.

Put

$$|x|_p = p^{-n} if\ x \neq 0\ and\ |0|_p\ = 0.$$

Then we have the following

Proposition 1.2.1 *[10] $|.|_p$is a valuation on $\mathbb{Q}$,called the p-adic valuation.*

Proof. *From the definition $|x|_p = 0$ if and only if $x = 0$. If $x = p^n\frac{a}{b}$ and $y = p^m\frac{c}{d}$ then $xy = p^{n+m}\frac{ac}{bd} gcd(p, abcd) = 1$, therefore*

$$|xy|_p = p^{-(n+m)} = |x|_p|y|_p.$$

If $n \leq m$ then $x + y = p^n(\frac{a+p^{m-n}c}{bd})$and hence

$$|x + y| \leq p^{-n} = \max\{|x|_p, |y|_p\}.$$

The case $m \leq n$ is handled similarly.

It is useful to also use the additive valuation, or order function in this case. The order function is denoted ord_p. The relationship between the two approaches is: for all $x \in \mathbb{Q}$,

$$|x|_p = p^{-ord_p(x)}.$$

■

Proposition 1.2.2 *[10] The following hold*

1) $ord_p(x) = \infty$ ifand only if $x = 0$.

2) $ord_p(xy) = ord_p(x) + ord_p(y)$.

3) $ord_p(x + y) \geq \min\{ord_p(x), ord_p(y)\}$.

Proposition 1.2.3 *[10] The p-adic valuation is a discrete valuation.*

Chapter 2

Non-Archimedean Banach Spaces

2.1 Non-Archimedean Norms

Definition 2.1.1 *Let* $\mathbb{E}$ *be a vector space over* $\mathbb{K}$*. A non-archimedean norm on* $\mathbb{E}$ *is a map* $||.|| : \mathbb{E} \longrightarrow \mathbb{R}_+^*$ *satisfying*

1) $||x|| = 0$ *and only if* $x = 0$*.*

2) $||\lambda x|| = |\lambda| ||x||$ *for any* $x \in \mathbb{E}$*and any* $\lambda \in \mathbb{K}$*.*

3) $||x + y|| \leq \max\{||x||, ||y||\}$*for any* $x, y \in \mathbb{E}$*.*

A non-archimedean normed space is a pair .$(\mathbb{E}, ||.||)$*where* $\mathbb{E}$ *is a vector space over* $\mathbb{K}$ *and* $||.||$ *is a non-archimedean norm on* $\mathbb{E}$*.*

Remark 2.1.1 *Property (3) of Definition 2.1.1 is referred to as the ultrametric or strong triangle inequality.*

Proposition 2.1.1 *[10] Let* $(\mathbb{E}, ||.||)$ *be a non-archimedean normed space. For* $x, y \in \mathbb{E}$*,*

$$||x + y|| = \max\{||x||, ||y||\} \text{ if } ||x|| \neq ||y||.$$

Proof. Suppose that $||x|| < ||y||$ so that $\max\{||x||, ||y||\} = ||y||$, then by Definition 1.1.1 $||x + y|| \leq ||y||$. Now

$$||y|| = ||x + y - x|| \leq \max\{||x + y||, ||x||\}.$$

But since $||x|| < ||y||$, we must have

$$\max\{||x+y||, ||x||\} = ||x+y||,$$

and therefore

$$||y|| \leq ||x+y||$$

and the conclusion follows. ■

Definition 2.1.2 *Let* $(\mathbb{E}, ||.||)$ *be non-archimedean normed space and* S *be a nonempty subset of* $\mathbb{E}$. *The set* S *is said to be bounded if the set of real numbers* $\{||x|| : x \in S\}$ *is bounded.*

Definition 2.1.3 *A sequence* $(x_i)_{i\in\mathbb{N}}$*in the normed space* $(\mathbb{E}, ||.||)$*converges to* $x \in \mathbb{E}$ *and we write:*

$$\lim x_i = x$$

if the sequence of real numbers $(||x_i - x||)_{i\in\mathbb{N}}$*converges to* 0.

Definition 2.1.4 *A series* $\sum_{i=0}^{\infty} x_i$ *in* $(\mathbb{E}, ||.||)$ *converges to* $x \in \mathbb{E}$*and we write:*

$$\sum_{i=0}^{\infty} x_i = x$$

if the sequence of partial sums

$$(S_n)_{n\in\mathbb{N}} \, S_n = \sum_{i=0}^{n} x_i, n \in \mathbb{N}$$

converges to x.

Proposition 2.1.2 *[10] Let* $(\mathbb{E},||.||)$*be a non-archimedean normed space over* $\mathbb{K}$. *If the sequence* $(x_i)_{i\in\mathbb{N}}$ *converges in* $\mathbb{E}$, *then it is bounded.*

Proof. *Suppose* $(x_i)_{i\in\mathbb{N}}$ *converges to* x*,then the sequence of real numbers*$(||x_i - x||)_{i\in\mathbb{N}}$*converges in* $\mathbb{R}$, *therefore is bounded. It follows that the set* $\{x_i : i \in \mathbb{N}\}$*is bounded as a subset of* $\mathbb{E}$. ■

2.2 Non-Archimedean Banach Spaces

Definition 2.2.1 *Let* $(\mathbb{E}, ||.||)$*be a non-archimedean normed space, then a metric d can be defined on* $\mathbb{E}$ *to give it the topology of a metric space. This metric is defined by*

$$x, y \in \mathbb{E}, d(x, y) = ||x - y||.$$

Proposition 2.2.1 *The strong triangle inequality translates as follows,*

$$\text{for } x, y, z \in \mathbb{E},\ d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

Definition 2.2.2 *A normed space*$(\mathbb{E},||.||)$*is called a Banach space if it is complete with respect to the natural metric induced by the norm*

$$d(x, y) = ||x - y||.\ x, y \in \mathbb{E}.$$

Example 2.2.1 *1) The spaces* $\mathbb{K}, \mathbb{K}^n, \sum_{i=0}^{\infty} \mathbb{K}_i, L^{\infty}(\mathbb{K}), B(\mathbb{E}, \mathbb{K})$ *with their respective norms are Banach spaces.*
2) Let $C_0(\mathbb{K})$ *the set of all sequences* $(x_i)_{i\in\mathbb{N}}$ *such that*

$$\lim_{i\to\infty} |x_i| = 0.$$

Then, $C_0(\mathbb{K})$*is a vector space over* $\mathbb{K}$ *and*

$$\|x_i\| = \sup_{i\in\mathbb{N}} |x_i| = 0$$

is a non-archimedean norm for which $(C_0(\mathbb{K}), \|\cdot\|)$ *is a Banach space.*
3) Let $t = (t_i)_{i\in\mathbb{N}}$ *is a sequence of non zero elements in* $\mathbb{K}$ *we define the set* $\mathbb{E}_t$ *by*

$$\mathbb{E}_t = \left\{x = (x_i)_{i\in\mathbb{N}} \text{ for all } i,\ x_i \in \mathbb{K} \text{ and} \lim_{i\to\infty} \left(|t_i|^{\frac{1}{2}} |x_i|\right) = 0\right\}.$$

We define the norm on $\mathbb{E}_t$ *by*

$$\|x_i\| = \sup_{i\in\mathbb{N}} \left(|t_i|^{\frac{1}{2}} |x_i|\right) = 0,\ \text{for all } (x_i)_{i\in\mathbb{N}} \text{ in } \mathbb{E}_t.$$

Then, $(\mathbb{E}_t, \|\cdot\|)$ *is a non-archimedean Banach space*

Proposition 2.2.2 *1) A close subspace of a Banach space is a Banach space.*

2) The direct sum of two Banach spaces is a Banach space.

Proof. 1) It is clear.

2) The norm on the direct sum is defined by $||(x, y)|| = \max\{||x||, ||y||\}$.From there, the proof is also clear. ■

Definition 2.2.3 *Let $\mathbb{E}$ be a Banach space and $\mathbb{V}$ a closed subspace of $\mathbb{E}$. Let $P : \mathbb{E} \longrightarrow \mathbb{E}/\mathbb{V}$ be the quotient map. Define:*

$$||Px|| = d(x, \mathbb{V}), x \in \mathbb{E}. \tag{2.2.1}$$

where

$$d(x, \mathbb{V}) = \inf\{d(x, z) : z \in \mathbb{V}\} = \inf\ \{||x - z|| : z \in \mathbb{V}\}$$

is the distance from x to $\mathbb{V}$.

Remark 2.2.1 *The norm defined in the Equation 2.2.1 is well defined because $Px = Py$ if and only if $x - y \in \mathbb{V}$, moreover $\|Px\| \leq \|x\|$, $x \in \mathbb{E}$.*

Proposition 2.2.3 *The norm defined in the Equation 2.2.1 is a non-archimedean norm on $\mathbb{E}/\mathbb{V}$.*

Proof. 1) First $||0|| = ||P(0)|| = 0$ since $0 \in \mathbb{V}$. Next, if $||Px|| = 0$then $d(x, \mathbb{V}) = 0$hence $x \in \mathbb{V}$, and $Px = 0$.

2) For $\lambda \in \mathbb{K}^*$

$$\begin{aligned} ||\lambda Px|| &= ||P(\lambda x)|| \\ &= \inf\ \{||\lambda x - z|| : z \in \mathbb{V}\} \\ &= \inf\ \{||x - z/\lambda|| : z \in \mathbb{V}\} \\ &= \inf\ \{||x - y|| : y \in \mathbb{V}\} \\ &= |\lambda|||Px||. \end{aligned}$$

3) For $x, y \in \mathbb{E}$, since $\mathbb{V}$is closed, there exist $z_1, z_2, z_3 \in \mathbb{V}$ such that

$$||Px|| = ||x - z_1||, ||Py|| = ||x - z_2||, ||P(x + y)|| = ||x + y - z_3||,$$

then

$$\begin{aligned}
||P(x)+P(y)|| &= ||P(x+y)|| \\
&= ||x+y-z_3|| \\
&\leq ||(x+y)-(z_1+z_2)|| \text{ (because } z_1+z_2 \in \mathbb{V}) \\
&= ||(x-z_1)+(y-z_2)|| \\
&\leq \max\{||x-z_1||, ||y-z_2||\} \\
&= \max\{||Px||, ||Py||\}.
\end{aligned}$$

■

Definition 2.2.4 *Let $\mathbb{E}$ be a vector space over $\mathbb{K}$ and $||.||_1$ and $||.||_2$ two non archimedean norms on $\mathbb{E}$ for each of which $\mathbb{E}$ is a Banach space. The two norms are said to be equivalent if there exist positive constants c_1 and c_2 such that for any $x \in \mathbb{E}$,*

$$c_1||x||_1 \leq ||x||_2 \leq c_2||x||_1.$$

Proposition 2.2.4 *On a finite dimensional Banach space over $\mathbb{K}$, all non archimedean norms are equivalent.*

Proof. We use induction on the dimension n. If $n = 1$, let $||x||_0 = |x|$ be the norm determined by the absolute value. Now let $||.||$ be any norm on $\mathbb{K}$, then for any $x \in \mathbb{K}$,

$$||x|| = |x|||1|| = c||x||_0, \text{with } c = ||1||$$

which implies that $||.||$ is equivalent to $||.||_0$.

Suppose that the proposition is true for a space of dimension $(n-1)$.

Let $\mathbb{E}$ be of dimension n and let $\{e_1, ..., e_n\}$ be a basis for $\mathbb{E}$. First we have the natural norm on $\mathbb{E}$ which is

$$x \in \mathbb{E}, x = \sum_{i=1}^{n} x_i e_i, ||x||_0 = \max\{|x_i| : 1 \leq i \leq n\}.$$

Let $||.||$ be any norm on $\mathbb{E}$. We want to show that $||.||$ is equivalent to $||.||_0$.

For any $x = \sum_{i=1}^{n} x_i e_i$ we have

$$||x|| = ||\sum_{i=1}^{n} x_i e_i|| \leq \max\{|x_i|||e_i|| : 1 \leq i \leq n\} \leq c||x||_0$$

where $c = \max\{||e_i|| : 1 \le i \le n\}$ and we find

$$||x|| \le c||x||_0.$$

To obtain the other inequality which will complete the equivalence, we let$\mathbb{V}$ be the subspace of $\mathbb{E}$ generated by$\{e_1, .., e_{n-1}\}$,then

$$x = y + x_n e_n$$

where $y = \sum_{i=1}^{n-1} x_i e_i \in \mathbb{V}$. We note that $\mathbb{V}$ is a closed subspace of $\mathbb{E}$, being the set of all vectors in $\mathbb{E}$ whose n-th component is zero. Therefore, it follows that

$$a = \inf\{||z + e_n|| : z \in \mathbb{V}\} > 0$$

then

$$||x_n^{-1} y + e_n|| \ge a > 0.$$

Put

$$b = a||e_n||^{-1} sothatb \le 1.$$

Suppose first that $x_n \neq 0$, then

$$||e_n||^{-1}||x_n^{-1} y + e_n|| \ge b.$$

Now

$$||x|| = |x|||e_n||(||e_n||^{-1}||x_n^{-1} y + e_n||) \ge b||x_n e_n||$$

and we find

$$||x|| \ge b||x_n e_n||.$$

■

For complete the proof of above proposition we have need the following lemma.

Lemma 2.2.1 $||x|| \ge b||y||$.

Proof. Suppose that $||x|| < b||y||$ hence $||y + x_n e_n|| < b||y||$ and since $b \le 1$we find that

$$||y + x_n e_n|| < ||y||.$$

which implies that

$$||x_n e_n|| = ||(y + x_n e_n) - y|| = ||y||$$

and since $||y + x_n e_n|| \geq b||x_n e_n||$we get a contradiction.
Now we have

$$||x|| \geq b|x_n|||e_n||and||x|| \geq b||y||.$$

By induction, there exist constants b' and b'' such that

$$||x|| \geq bb'|x|and||x|| \geq bb'' \max\{|x_i| : 1 \leq i \leq (n-1)\}.$$

Let $c = \min\{bb', bb''\}$.Then,

$$||x|| \geq c \max\{|x_i| : 1 \leq i \leq n\} = c||x||_0.$$

Suppose next that $x_n = 0$.In this case, we still have

$$||x|| \geq b||y||.$$

and the same argument carries on, hence, $||.||$ is equivalent to $||.||_0$. ∎

2.3 Link between $\|\mathbb{E}\|$ and $|\mathbb{K}|$

Definition 2.3.1 *Let* $(\mathbb{E}, \|.\|)$ *be a non-archimedean Banach space, we define*

$$\|\mathbb{E}\| = \{\|x\|, x \in \mathbb{E}\}$$

$$|\mathbb{K}| = \{|\lambda|, \lambda \in \mathbb{K}\}.$$

Remark 2.3.1 *i) As the valuation of* $\mathbb{K}$ *to values in* $\mathbb{R}_+$ *and the norm of* $\mathbb{E}$ *also to values in* $\mathbb{R}_+$*, then* $|\mathbb{K}| \subseteq \mathbb{R}_+$ *and* $\|\mathbb{E}\| \subseteq \mathbb{R}_+$*. This result is not always true in theory of non-archimedean, for example if we take* $\mathbb{K} = \mathbb{Q}_p$ *then*

$$|\mathbb{K}| = \left\{1, \frac{1}{2}, ..., \frac{1}{p}\right\} \subsetneqq \mathbb{R}_+.$$

ii) $\|\mathbb{E}\| \subseteq |\mathbb{K}|$ *i.e., for every* $x \in \mathbb{E}$ *there exists a constant* $c \in \mathbb{K}$*, such that* $\|x\| = |c|$.

2.4 Free Banach Spaces

Definition 2.4.1 *A family $(V_i)_{i\in I}$ of vectors in $\mathbb{E}$indexed by a set Iconverges to 0 and we write:*$\|x\| = |c|$

$$\lim_{i\in I} V_i = 0.$$

if

$$\forall \varepsilon > 0, \{i \in I : ||V_i|| \geq \varepsilon\} is finite :$$

Definition 2.4.2 *Let $V \in \mathbb{E}$and let $(V_i)_{i\in I}$ be a family of elements of $\mathbb{E}$ indexed by the set I. We say that v is the sum of the family $(V_i)_{i\in I}$ and we write:*

$$\sum_{i\in I} V_i = v$$

if $\forall \varepsilon > 0$,there exists a finite subset $J_0 \subset I$ such that for any finite $J \subset I, J \supseteq J_0$

$$|| \sum_{i\in J} V_i - v|| \leq \varepsilon.$$

In this situation, we also say that the family $(V_i)_{i\in I}$ is summable and its sum is v.

Proposition 2.4.1 *Let the family $(V_i)_{i\in I}$ be summable in $\mathbb{E}$with sum $v \in \mathbb{E}$, then*

$$\lim_{i\in I} V_i = 0.$$

Proof. *Given $\varepsilon > 0$,let $H = \{i \in I : ||V_i|| \geq \varepsilon\}$.Since the family $(V_i)_{i\in I}$is summable with sum v, there exists a finite subsetJ_0 of Isuch that for any finite subset $J$of Icontaining J_0,*

$$|| \sum_{i\in J} V_i - v|| \leq \varepsilon.$$

Let $j \in I\backslash J_0$and consider $J = J_0 \cup \{j\}$, then

$$|| \sum_{i\in J} V_i - v|| \leq \varepsilon.$$

Since

$$|| \sum_{i\in J_0} V_i - v|| \leq \varepsilon$$

it follows that

$$\max\{||\sum_{i\in J} V_i - v||, ||\sum_{i\in J_0} V_i - v||\} \leq \varepsilon,$$

which implies that

$$||V_i|| \leq \varepsilon$$

Since this holds for any $j \notin J_0$, we conclude that $H \in J_0$, hence H is finite and therefore $\lim_{i\in I} V_i = 0$. ■

Definition 2.4.3 *For each $i \in \mathbb{N}$ we define the Mahler function M_i to be the function: $M_i : \mathbb{Z}_p \longrightarrow \mathbb{K}, M_0(x) = 1$, and for $i > 0$ $M_i(x) = \frac{x(x-1)...(x-i+1)}{i!}, x \in \mathbb{Z}_p$. The function M_i satisfies the following:*

1) $M_i(j) = 0$ for j is an integer with $i < n$.

2) $M_i(i) = 1$.

3) $M_i(x)$ is a polynomial function of degree i.

Let $C(\mathbb{Z}_p, \mathbb{K})$ be the $\mathbb{K}$-vector space of continuous functions from the compact set $\mathbb{Z}_p$ to $\mathbb{K}$,equipped with the sup-norm

$$||f||_\infty = \sup_{z\in\mathbb{Z}_p} |f(z)|.$$

Theorem 2.4.1 *The following hold:*

1) For each $i \in \mathbb{N}$,$M_i \in C(\mathbb{Z}_p, \mathbb{K})$ and $||M_i|| = 1$.

2)For each $f \in C(\mathbb{Z}_p, \mathbb{K})$there exists a unique sequence $(a_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ such that

$$f(x) = \sum_{i=0}^{\infty} a_i M_i(x), x \in \mathbb{Z}_p.$$

The series converges uniformly and

$$||f||_\infty = \max\{|a_i| : i \in \mathbb{N}\}.$$

3) If $(a_i)_i \in \mathbb{N} \in c_0(\mathbb{K})$then, the function

$$f(x) = \sum_{i=0}^{\infty} a_i M_i(x), x \in \mathbb{Z}_p.$$

defines an element of$C(\mathbb{Z}_p, \mathbb{K})$.

Definition 2.4.4 *We say that $x, y \in \mathbb{E}$ are orthogonal to each other if:*

$$||ax + by|| = \max\{||ax||, ||by||\}, \textit{for any} \;\; a, b \in \mathbb{K}.$$

This definition is clearly symmetric and generalizes as follows.

Definition 2.4.5 *Let $(V_i)_{i\in I}$ be a family of vectors in $\mathbb{E}$. We say that the family is orthogonal if for any $J \subset I$ and for any family $(V_i)_{i\in J}$ of elements of $\mathbb{K}$ such that $\lim_{i\in J} a_i V_i = 0$,*

$$||\sum_{i\in J} a_i x_i|| = \max\{||a_i x_i|| : i \in J\}.$$

Remark 2.4.1 *An orthogonal basis for the Banach space $\mathbb{E}$ is a base which is an orthogonal family. This means, then, that a family $\{e_i : i \in I\}$ is an orthogonal basis if and only if:*
1)For every $x \in E$,there exists a unique family $(x_i)_{i\in I} \subset \mathbb{K}$ such that

$$x = \sum_{i\in I} x_i e_i.$$

2) $||x|| = \max\{||x_i e_i|| : i \in I\}$.
The orthogonal basis $\{e_i : i \in I\}$ is called an orthonormal basis if:

$$||e_i|| = 1, forall i \in I.$$

Remark 2.4.2 *The sequence of Mahler functions $\{M_i : i = 0, 1, ...\}$ forms an orthonormal basis of $C(\mathbb{Z}_p, \mathbb{K})$.*

Chapter 3

Bounded Linear Operators in Non-Archimedean Banach Spaces

3.1 Bounded Linear Operators

3.1.1 Definitions and Examples

Definition 3.1.1 *A mapping* $f:\mathbb{X}\to\mathbb{Y}$ *that associates every* $x\in\mathcal{D}(f)\subset\mathbb{X}$ *a unique* $y\in\mathbb{Y}$ *is called operator (transformation). The set* $\mathcal{D}(f)$ *designates the domain of* f.

Definition 3.1.2 *A mapping* f *is said to be a linear operator if,*
i) $\mathcal{D}(f)$ *vector subspace*
ii) For all $\alpha,\beta\in\mathbb{K}$ *and* $x,y\in\mathbb{X}$,

$$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y).$$

Definition 3.1.3 *A mapping* $f:\mathbb{X}\to\mathbb{Y}$ *is said to be bounded if* $\mathbb{X}=\mathcal{D}(f)$ *and there exists* $c\geq 0$ *such that*

$$||fx||\leq c||x||, \text{for all } x\in\mathbb{X}.$$

Remark 3.1.1 *We denote by* $B(\mathbb{X},\mathbb{Y})$ *the collection of all bounded linear operators from* $\mathbb{X}$ *into* $\mathbb{Y}$. *In the case* $\mathbb{X}=\mathbb{Y}$, *then* $B(\mathbb{X},\mathbb{Y})$ *denote by* $B(\mathbb{X})$. *It is clear that if* $f\in B(\mathbb{X})$, *then the quantity, called the norm-operator of* f,

$$||f||=\sup_{x\in\mathbb{X}\setminus\{0\}}\frac{||fx||}{||x||}<\infty.$$

Remark 3.1.2 *If $f \in B(\mathbb{X})$, then the following identity holds,*

$$||fx|| \leq ||f||.||x||, \text{for all } x \in \mathbb{X}. \tag{3.1.1}$$

Remark 3.1.3 *Let $\mathbb{X}, \mathbb{Y}$ be a non-archimedean Banach spaces. If $f \in B(\mathbb{X}, \mathbb{Y})$, we define the two norms of f by*

$$||f|| = \sup_{x \in \mathbb{X}\setminus\{0\}} \frac{||fx||_{\mathbb{Y}}}{||x||_{\mathbb{X}}} \text{ and } ||f||_0 = \sup_{\substack{x \in \mathbb{X} \\ ||x|| \leq 1}} ||fx||_{\mathbb{Y}}.$$

This two norms are not always equivalent or equal. For example, let $\mathbb{X} = \mathbb{Y} = \mathbb{Q}_3$ such that $\mathbb{X} = (\mathbb{Q}, ||x|| = 2\,|x|_3)$ and $\mathbb{Y} = (\mathbb{Q}, ||x|| = |x|_3)$ with

$$|x|_3 = \begin{cases} 3^{-n} \text{ if } x \neq 0, n \in \mathbb{N}^* \\ 0 \text{ if } x = 0. \end{cases}$$

For the identity operator $I : \mathbb{X} \to \mathbb{Y}$ we have

$$\begin{aligned} ||I|| &= \sup_{x \in \mathbb{X}\setminus\{0\}} \frac{||Ix||_{\mathbb{Y}}}{||x||_{\mathbb{X}}} = \frac{|x|_3}{2\,|x|_3} = \frac{1}{2} \\ &\quad \text{and} \\ ||I||_0 &= \sup_{\substack{x \in \mathbb{X} \\ ||x|| \leq 1}} ||Ix||_{\mathbb{Y}} = \sup_{\substack{x \in \mathbb{X} \\ ||x|| \leq \frac{1}{2}}} |x|_3 = \frac{1}{3}. \end{aligned}$$

Hence, $||I|| \neq ||I||_0$.

Proposition 3.1.1 *Let $\mathbb{X}, \mathbb{Y}$ be a non-archimedean Banach spaces. If $f \in B(\mathbb{X}, \mathbb{Y})$, the two norms are not always equivalent or equal if $\|\mathbb{X}\| \subseteq |\mathbb{K}|$ or the valuation of $\mathbb{K}$ is dense.*

Remark 3.1.4 *i) We suppose that $\|\mathbb{X}\| \neq \mathbb{R}_+$, then for $\alpha \in \mathbb{R}_+ \setminus \|\mathbb{X}\|$ we consider on $\mathbb{X}$ the following norm*

$$\|x\|_\alpha = \alpha^{-1} \|x\|, \forall x \in \mathbb{X}.$$

As $\alpha \in \mathbb{R}_+ \setminus \|\mathbb{X}\|$, then $\alpha \neq \|x\|$ this implies that $\alpha^{-1}\|x\| \neq 1$. Therefore

$$\{x \in \mathbb{X}, \|x\|_r = 1\} = \varnothing.$$

So, if $\|\mathbb{X}\| \subseteq |\mathbb{K}|$, then for all $x \in \mathbb{X}$ there exists $c \in \mathbb{K}\setminus\{0\}$ such that $\|x\| = |c|$ and $|c|^{-1}\|x\| = 1$. Hence the unit sphere non empty.

ii) We have $0 \in B(\mathbb{X}, \mathbb{Y})$ *with* $\|0\| = 0$. *Moreover if* $f, g \in B(\mathbb{X}, \mathbb{Y})$ *and* $\lambda \in \mathbb{K}$ *then* $f + g, \lambda g, fg \in B(\mathbb{X}, \mathbb{Y})$ *and we have*

a) $\|f + g\| \leq \|f\| + \|g\|$

b) $\|\lambda f\| = |\lambda| \, \|f\|$

c) $\|fg\| \leq \|f\| \, \|g\|$.

Therefore, $(B(\mathbb{X}, \mathbb{Y}), \|\cdot\|)$ *is a normed vector space.*

Definition 3.1.4 *An operator* f *is called continuous at* $x_0 \in \mathbb{X}$ *if* (x_n) *converge to* x_0*, then* $f(x_n)$ *converge to* $f(x_0)$.

Remark 3.1.5 *Every bounded linear operator* f *on* $\mathbb{X}$ *is continuous. Indeed, if* $(x_n)_{n\in\mathbb{N}} \subset \mathbb{X}$ *is a sequence which converges strongly to some* $x \in \mathbb{X}$*, i.e.,* $\|x_n - x\| \underset{n\to\infty}{\to} 0$. *Then by using 3.1.1 it follows that*

$$\|f(x_n - x)\| \leq \|f\| \, \|x_n - x\| .$$

Which yields,

$$\|f(x_n - x)\| \underset{n\to\infty}{\to} 0.$$

Theorem 3.1.1 *Every continuous linear operator* $f : \mathbb{X} \to \mathbb{X}$ *is bounded.*

Proof. Suppose f is continuous. Consequently, f is continuous at $x = 0$. Hence, there exists $\eta > 0$ such that $||fx|| \leq 1$whenever $||x|| \leq \eta$. Suppose the valuation of the non-archimedean field $\mathbb{K}$ is dense. Consequently, there exists $z_n \in \mathbb{K} \setminus \{0\}$such that $|z_n| = \eta$. If $0 \neq x \in \mathbb{X}$, then let $z_x \in \mathbb{K} \setminus \{0\}$such that $|z_x| = ||x||$ we have

$$||\frac{z_n x}{z_x}|| = \eta.$$

Now

$$\begin{aligned} ||f(\frac{z_n x}{z_x})|| &= \frac{|z_n|||fx||}{z_x} \\ &= \frac{\eta||fx||}{||x||} \\ &\leq 1, \end{aligned}$$

and hence

$$||fx|| \leq \eta^{-1}||x||$$

which yields f is bounded. ∎

Remark 3.1.6 *One should point out that the proof is similar in the case when the valuation of* $\mathbb{K}$ *is discrete and hence is omitted.*

Example 3.1.1 *Let* $\mathbb{X} = \mathbb{K}^n = \{(x_1, x_2, ..., x_n) : z_k \in \mathbb{K}, k = 1, 2, .., n\}$ *be equipped with its natural non-archimedean norm given by*

$$||x|| = \max_{i=1,...,n} |x_i|$$

for all $x = (x_1, x_2, ..., x_n) \in \mathbb{K}^n$*. Let* $(e_1, e_2, ..., e_n)$ *be the canonical basis of* $\mathbb{K}^n$ *defined by,* $e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, 0, ..., 1)$*. For all* $x = (x_1, x_2, ..., x_n) \in \mathbb{K}^n$*, we have*

$$x = \sum_{j=1}^{n} x_j e_j \text{ for some } x_j \in \mathbb{K}, j = 1, 2, ..., n.$$

Let $f : \mathbb{K}^n \longmapsto \mathbb{K}^n$ *be a linear mapping. Clearly,* $fe_i \in \mathbb{K}^n$ *and hence there exists* $a_{ij} \in \mathbb{K}$ *for* $i, j = 1, ..., n$ *such that*

$$fe_j = \sum_{i=1}^{n} a_{ij} e_i.$$

In what follows, we show that the arbitrary linear operator $\mathbb{A}$ *given above is necessarily bounded. Indeed, for all*
$x = \sum_{j=1}^{n} x_j e_j$ *and* $y = \sum_{j=1}^{n} y_j e_j$*, we have*

$$\begin{aligned} ||fx - fy|| &= || \textstyle\sum_{j=1}^{n} (x_j - y_j) A e_j || \\ &\leq C \max(|x_1 - y_1|, ..., |x_n - y_n|) \\ &= C||x - y||, \end{aligned}$$

where

$$C = \max_{j=1,...,n} ||fe_j|| = \max_{j=1,...,n} (\max_{i=1,...,n} |a_{ij}|) < \infty.$$

Consequently, $f : \mathbb{K}^n \longmapsto \mathbb{K}^n$ *is a bounded linear operator.*

3.1.2 Properties of bounded linear operators

Theorem 3.1.2 *The space* $(B(\mathbb{X}), ||.||)$*of bounded linear operator on* $\mathbb{X}$*is a Banach space.*

Proof. Indeed, Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(\mathbb{X})$. Equivalently, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$||f_n - f_m|| < \epsilon \text{ for all } m, n > N.$$

Another hand, we prove that there exists a bounded linear operator $f : \mathbb{X} \to \mathbb{X}$ such that $||f_n - f|| \underset{n\mapsto\infty}{\to} 0$.
For all $x \in \mathbb{X} \diagdown \{0\}$, we have

$$||\left(f_n - f_m\right)(x)|| < \epsilon \|x\| \text{ for all } m, n > N. \tag{3.1.2}$$

Consequently, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{X}$. Since $\mathbb{X}$ is a non-archimedean Banach space, there exists $\zeta \in \mathbb{X}$ such that $||(f_n)(x) - \zeta|| \underset{n\mapsto\infty}{\to} 0$.
Setting

$$fx = \zeta = \lim_{n\mapsto\infty} f_n x,$$

one defines a linear operator $f : \mathbb{X} \to \mathbb{X}$.
Letting $m \to \infty$ in 3.1.2, one obtains,

$$||\left(f_n - f\right)(x)|| < \epsilon \|x\| \text{ for all } n > N. \tag{3.1.3}$$

Consequently, for $n > N$

$$\begin{aligned} ||f(x)|| &= ||f(x) - f_n(x) + f_n(x)|| \\ &\leq \max(||f(x) - f_n(x)||, ||f_n(x)||) \\ &\leq \max(\epsilon \|x\|, ||f_n(x)||) \\ &\leq \max(\epsilon \|x\|, ||f_n|| \|x\|) \\ &= \max(\epsilon, ||f_n||) \|x\|. \end{aligned}$$

This yields f is a bounded operator. Further,

$$||f - f_n|| \leq \epsilon for n > N.$$

Equivalently, $||f_n - f|| \underset{n\mapsto\infty}{\to} 0$. ■

Definition 3.1.5 *Let f a linear operator, we say*
i) The kernel of f and we denote by $\mathcal{N}(f)$, the set defined by

$$\mathcal{N}(f) = \{x \in \mathcal{D}(f) : fx = 0.\}$$

ii) The range (image) of f and it denoted by $\mathcal{R}(f)$, the set defined by

$$\mathcal{R}(f) = \{fx : x \in \mathcal{D}(f).\}$$

Definition 3.1.6 *If $f \in B(\mathbb{X}, \mathbb{Y})$, then*
i) f is injective if $\mathcal{N}(f) = \{0\}$.
ii) f is surjective if $\mathcal{R}(f) = \mathbb{Y}$.
ii) f is invertible if it is injective and surjective in the same time.

Definition 3.1.7 *If f is invertible, then there exists an unique bounded linear operator noted by $f^{-1} : \mathbb{Y} \to \mathbb{X}$ called the inverse of f such that*

$$f^{-1}f = I_{\mathbb{X}} \text{ and } ff^{-1} = I_{\mathbb{Y}},$$

where $I_{\mathbb{X}}$ and I are respectively the identity operators on $\mathbb{X}$ and $\mathbb{Y}$. If $\mathbb{X} = \mathbb{Y}$ the identity operator noted by I.

Theorem 3.1.3 *Let $f \in B(\mathbb{X})$, and suppose that $\|I - f\| \leq 1$, then Then the followin hold*
i) For all $x \in \mathbb{X}$, the series $\sum_{n=0}^{\infty} (I - f)^n x$ converges in $\mathbb{X}$ and $\sum_{n=0}^{\infty} (I - f)^n \in B(\mathbb{X})$
ii) f is invertible and

$$f^{-1} = \sum_{n=0}^{\infty} (I - f)^n.$$

Proof. i) let $g = I - f$, so that $\|g\| \leq 1$. Let $x \in \mathbb{X}$, to show the convergence of the series $\sum_{n=0}^{\infty} g^n$ we need to show that $\lim_{n\to\infty} g^n x = 0$, and we have for all $n \in \mathbb{N}$

$$\|g^n x\| \leq \|g\|^n \|x\|$$

But since $\|g\| \leq 1$, then $\|g\|^n \|x\| \underset{n\mapsto\infty}{\to} 0$. Hence the series converges. Moreover,

$$\begin{aligned} \left\| \sum_{n=0}^{\infty} g^n \right\| &= \left\| \lim_{n\to\infty} \sum_{k=0}^{n} g^k \right\| \\ &\leq \lim_{n\to\infty} \left\| \sum_{k=0}^{n} g^k \right\| \\ &\leq \lim_{n\to\infty} \max\left(1, \|g\|, ..., \|g^n\|\right) \\ &\leq \lim_{n\to\infty} \max\left(1, \|g\|, ..., \|g\|^n\right) \\ &= 1. \end{aligned}$$

and $\sum_{n=0}^{\infty} g^n \in B(\mathbb{X})$.

ii) We have need to see that

$$\begin{aligned} f\left(\lim_{n\to\infty} \sum_{k=0}^{n} g^k\right) &= (I-g)\lim_{n\to\infty} \sum_{k=0}^{n} g^k \\ &= I, \end{aligned}$$

and, hence

$$f^{-1=} \sum_{n=0}^{\infty} g^n.$$

■

3.2 Non bounded linear operators

Definition 3.2.1 *A non bounded linear operator f on non-archimedean Banach space is the pair $(\mathcal{D}(f), f)$ consisting of a subspace $\mathcal{D}(f) \subset \mathbb{X}$ and a linear transformation $f : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{Y}$ (may be non bonded).*

We denote by $\mathcal{U}(\mathbb{X}, \mathbb{Y})$ the set of all non bounded linear operators. If $\mathbb{X} = \mathbb{Y}$, then $\mathcal{U}(\mathbb{X}, \mathbb{Y}) = \mathcal{U}(\mathbb{X})$.

3.3 Closed linear operators

Definition 3.3.1 *We recall that $X \times Y$ equipped with the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$, for all $(x, y) \in X \times Y$ is non-archimedean Banach space.*

Definition 3.3.2 *An operator $f \in \mathcal{U}(\mathbb{X}, \mathbb{Y})$ is called closed if its graph*

$$G(f) = \{(x, f(x)) \in X \times Y : x \in \mathcal{D}(f)\}$$

is a closed subspace of $X \times Y$.

We denote by $\mathcal{C}(\mathbb{X}, \mathbb{Y})$ the set of all non closed linear operators. If $\mathbb{X} = \mathbb{Y}$, then $\mathcal{C}(\mathbb{X}, \mathbb{Y}) = \mathcal{C}(\mathbb{X})$.

Proposition 3.3.1 *Let $f : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{Y}$is closed linear operator. If $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(f)$ such that $\|x_n - x\| \underset{n\to\infty}{\to} 0$ and $\|f(x_n) - y\| \underset{n\to\infty}{\to} 0$, for some $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, then $x \in \mathcal{D}(f)$ and $y = f(x)$.*

Definition 3.3.3 *We call an operator $f : \mathcal{D}(f) \subset \mathbb{X} \to \mathbb{Y}$ is closable if admits a closed extension. i.e., there exists a closed operator $g : \mathcal{D}(g) \subset \mathbb{X} \to \mathbb{Y}$ such that*

$$\begin{cases} \mathcal{D}(f) \subset \mathcal{D}(g) \\ g(x) = f(x), \text{ for all } x \in \mathcal{D}(f). \end{cases}$$

Remark 3.3.1 *If f is closable, there exists a closed operator $\overline{f}$ such that $G(\overline{f}) = \overline{G(f)}$. It follows immediately that $\overline{f}$ is the smallest closed extension.*

Proposition 3.3.2 *Let $\mathbb{X}, \mathbb{Y}$ be non-archimedean Banach spaces. We suppose that $g \in B(\mathbb{X}, \mathbb{Y})$ and $f \in \mathcal{U}(\mathbb{X}, \mathbb{Y})$.*
i) For that $g + f$ be closed, it is necessary and sufficient f be closed.
ii) For that $g + f$ be closable, it is necessary and sufficient f be closable and in this case we have $\overline{g + f} = g + \overline{f}$.

Proof. i) Let $(x_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{D}(f + g)$ such that

$$\begin{cases} x_n \to x \\ (f + g)\, x_n \to y \end{cases}$$

Since g is bounded, then we have $\mathcal{D}(g) = \mathbb{X}$ and $\mathcal{D}(f + g) = \mathcal{D}(f)$. And as

$$\begin{cases} x_n \to x \\ g \in B(\mathbb{X}, \mathbb{Y}) \end{cases} \text{alors } gx_n \to gx.$$

We have $fx_n = (f+g)\,x_n - gx_n$, or f is closed hence $x \in \mathcal{D}(f)$ and $fx = y - gx$.
ii) If f is closable operator, then $f+g$ has a closed extension $g+\overline{f}$. Then $g+f$ is closable and

$$\overline{g+f} \subset g+\overline{f}.$$

By replacing f by $f+g$ and g by $-g$, we have if $f+g$ is closed or closable it is the same for f et consequently, we have

$$\overline{f} \subset \overline{g+f} - g,$$

then

$$g+\overline{f} \subset g+\overline{g+f} - g = \overline{g+f}.$$

■

3.4 Bounded linear operators in free Banach spaces

Definition 3.4.1 *Let* $\mathbb{X}$*be a free Banach space over the non-archimedean field* $(\mathbb{K}, |.|)$*with canonical orthogonal basis* $(e_j)_{j\in I}$ *Define* $e_i^{'} \in \mathbb{X}^*$*by setting*

$$x = \sum_{i\in I} x_i e_i, \;\; e_i^{'}(x) = x_i.$$

It turns out that $||e_i^{'}||_* = ||e_i||^{-1}$*. Furthermore, every* $x^{'} \in \mathbb{X}^*$*can be expressed as*

$$x^{'} = \sum_{i\in I} < x^{'}, e_i > e_i^{'}$$

with

$$||x^{'}||_* = \sup_{i\in I} \frac{| < x^{'}, e_i > |}{||e_i||}.$$

For each $f \in \mathbb{X}^*$*,define the linear operator* $v^{'} \otimes u : \mathbb{X} \to \mathbb{X}$*by*

$$(v^{'} \otimes u)(w) := < v^{'}, w > u.$$

Clearly, the operator$(v\otimes u)$*is bounded as* $||v^{'}\otimes u|| = ||v^{'}||_{*}||u||$*.Among other things,if*$(e_i^{'})_{i\in I}$*is the dual canonical orthogonal basis for* $\mathbb{X}^{*}$*,then*$(e_i^{'}\otimes e_j)_{(i,j)\in I\times I} \in \mathbb{B}(\mathbb{X})$*and its operator-norm is given by*

$$||e_i^{'}\otimes e_j|| = \frac{||e_j||}{||e_i||}.$$

Proposition 3.4.1 *let* $A \in \mathbb{B}(\mathbb{X})$*,then it can be written in a unique fashion as a pointwise convergent series*

$$A = \sum_{(i,j)\in I\times I} a_{ij}e_i^{'}\otimes e_j, i \in I, \lim_i |a_{ij}|||e_i|| = 0.$$

Moreover,

$$||A|| = \sup_{i\in I}\sup_{j\in I}\frac{|a_{ij}|||e_i||}{||e_i||}.$$

Proof. *For all*$j \in I, Ae_j = \sum_{i\in I} a_{ij}e_i$*where*$a_{ij} \in \mathbb{K}, |a_{ij}|||e_j|| = 0$*.Now for any* $x = \sum_{i\in I} x_je_j \in \mathbb{X}$.

$$Ax = \sum_{j\in I}\sum_{i\in I} a_{ij}x_je_i = \sum_{j\in I}\sum_{i\in I} a_{ij}(e_j^{'}\otimes e_i)x.$$

It remains to show that

$$||A|| = \sup_{j\in I}\frac{||Ae_j||}{||e_j||} = \sup_{j\in I}\sup_{i\in I}\frac{|a_{ij}|||e_i||}{||e_j||}.$$

Indeed,

$$\frac{||Ae_j||}{||e_j||} \leq ||A||.$$

Next, for any $x = \sum_{j\in I} x_je_j$;

$$\begin{aligned} ||Ax|| &= ||\textstyle\sum_{j\in I} x_jAe_j|| \\ &\leq \sup_{j\in I}(|x_j|.||Ae_j||) \\ &= \sup_{j\in I}(|x_j|.||e_j||.\tfrac{||Ae_j||}{||e_j||}) \\ &\leq ||x||\sup_{j\in I}\tfrac{||Ae_j||}{||e_j||}. \end{aligned}$$

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Abstract:

In this memory we will present and study the non-Archimedean Banach spaces by giving some examples. we devote to the basic properties of bounded and unbounded linear operators on non-Archimedean Banach spaces.

Keywords: Banach spaces, non-Archimedean Banach spaces, bounded and unbounded linear operators.

Résumé:

Dans ce mémoire on va présenter et étudier les espaces de Banach non-archimédiens en donnant quelques exemples. nous consacrons aux propriétés de base des opérateurs linéaires bornés et non bornés sur les espaces de Banach non-archimédiens.

Mots clés: Espaces de Banach, espaces de Banach non-archimédiens, opérateurs linéaires bornés et non bornés.

ملخص:

في هذه الأطروحة سوف نقدم وندرس فضاءات باناخ غير الأرخميدية و إعطاء بعض الأمثلة عنها. كذلك ندرس بعض الخصائص الأساسية للتطبيقات الخطية المحدودة وغير المحدودة في فضاءات باناخ غير الأرخميدية .

الكلمات المفتاحية:

فضاءات باناخ , فضاءات باناخ غير الأرخميدية, فضاء Arens-Eells, التطبيقات الخطية المحدودة وغير المحدودة.