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List of publications

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Introduction

Binary relations are among the oldest acquaintances of modern mathematics. Since their introduction, they have become omnipresent, in both pure and applied mathematics, as well as in computer science and engineering. Ternary and, more generally, n -ary relations, have received far less attention. One prominent counterexample is that of geometry, where the ternary relation of betweenness has fascinated researchers for a long time [56, 74].

However, in recent years, the interest in ternary relations is on the rise. For instance, in mathematics, ternary relations appear in algebra (e.g. in median algebras [2, 22]), order theory (e.g. cyclic orders [21, 61, 64]), group theory (e.g. in hypergroups associated with ternary relations [25, 26]) and logic (e.g. in the Routley–Meyer semantics of relevant logics [6, 38, 67]). Also in models for decision making, the use of ternary relations is becoming quite common [77], such as in the aggregation of rankings where monometrics w.r.t. a betweenness relation play a central role [68, 69]. Other applications can be found in computational biology, where ternary relations are used for modelling phylogenies [81], qualitative spatial reasoning [24, 47] and string matching [49].

Ternary relations have become essential tools for information modelling. One prominent use case is the Resource Description Framework (RDF) [76], a family of World Wide Web Consortium specifications for the conceptual description or modelling of information that is implemented in web resources, and which is based on the idea of making statements in the form of subject-predicate-object expressions, known as triples, similar to classical conceptual modeling approaches (such as entity-relationship or class diagrams) (see e.g. [1] for a discussion of related data structures). Another example is the Ternary Relations Model [75] designed to cover a number of differing and isolated information management paradigms, such as hypertext navigation and workflow modelling.

Inspired by the importance of the ternary relations and the increasing interest in them, our thinking has led to the addition of a contribution to this framework. More specifically, we aim to investigate and study the compatibility of a ternary relation with a binary fuzzy relation, similarly to the binary case [48].

The notion of compatibility of a binary fuzzy relation with a fuzzy equality relation is due to Bělohlávek [7]. It turns out to be equivalent to the older notion of extensionality introduced by Höhle and Blanchard [41]. Daňková [27] has further generalized this notion to the extensionality of an n -ary fuzzy relation w.r.t. n binary fuzzy relations. The compatibility property has been extensively studied by Kheniche et al. [48] for arbitrary binary fuzzy relations. In particular, they have shown that Fodor’s notion [39] of traces of a binary fuzzy relation facilitates the characterization of the compatibility of fuzzy relations in terms of fuzzy relational inclusions. In general, the notion of compatibility is witnessing increasing attention and appears in various studies on binary fuzzy relations. For instance, in the study of fuzzy ordered structures [7, 9, 35, 57, 82, 91], fuzzy functions [19, 34, 73]

and L -algebras [10, 34, 36]. Given the importance of this notion, De Baets et al. [33] have characterized the fuzzy tolerance and fuzzy equivalence relations that a given strict order relation is compatible with. Later on, and in [32], De Baets et al. have extended this characterization by considering an arbitrary binary relation instead of a strict order relation. In a recent paper, and in order to highlight other points related to this notion of compatibility, Zedam et al. [90] have extended this study by considering the left- and right-compatibility of a(n) (strict) order relation with a fuzzy tolerance relation and vice versa. They characterized the fuzzy tolerance relations that are compatible with a given (strict) order relation. Conversely, they provided a representation of the fuzzy tolerance relations that a given strict order relation is left- or right-compatible with. Specific attention is paid to the case of fuzzy equivalence relations.

To any strict order relation $<$ on a set X , we can associate a ternary relation $B_<$ on X , called strict order betweenness relation, as follows [55]:

$$(a, b, c) \in B_< \text{ if and only if } a < b < c \text{ or } c < b < a.$$

As a consequence, the compatibility of the strict order relation $<$ with a binary fuzzy relation S implies the compatibility of $B_<$ with S . More specifically, if

$$\tau(x < y) * S(x, a) * S(y, b) \leq \tau(a < b),$$

for any $x, y, a, b \in X$, then it holds that

$$\tau(B_<(x, y, z)) * S(x, a) * S(y, b) * S(z, c) \leq \tau(B_<(a, b, c)),$$

for any $x, y, z, a, b, c \in X$. Here, $*$ denotes a t-norm and the symbol τ is used to refer to the characteristic mapping of a relation. In particular, $\tau(B_<(x, y, z)) = 1$ if $(x, y, z) \in B_<$, while $\tau(B_<(x, y, z)) = 0$ if $(x, y, z) \notin B_<$.

Inspired by the above example, in this work we study the compatibility of an arbitrary ternary relation with a binary fuzzy relation.

The study of the compatibility requires effective tools for this purpose. In the present thesis, we focus on two of them. We aim to extend the interesting notions of traces and clone relations of a binary relation to the setting of ternary relations. The first study, allow us to characterize the compatibility in terms of traces. The second one, allow us to represent all the binary fuzzy tolerance relations that a crisp ternary relation is compatible with.

The notions of traces have been started by Doignon et al. [37] who have introduced them for the purpose of characterizing biorder relations. In this initial approach, the traces associated with both crisp and fuzzy binary relations were defined as crisp binary relations. However, Fodor [39] showed that these traces are not appropriate for characterizing fuzzy biorder relations. For this reason, he redefined the traces of a binary fuzzy relation as fuzzy relations and demonstrated their usefulness in characterizing basic properties of binary fuzzy relations. Various authors have built upon this work, e.g. in the context of property indicators of fuzzy relations [84, 83] and the theory of conjoint measurement [18]. More specifically, we generalize the notions of left and right traces of a binary relation in a natural way to the notions of left, middle and right traces of a ternary relation, and we investigate their basic properties as well as their usefulness to study and characterize the properties of a ternary relation. Furthermore, in the various works on traces of a binary

relation, a major role is played by the relational composition, as it is the operation that allows to characterize the properties of a binary relation through its traces [39, 84, 83]. In our attempt to obtain similar results for ternary relations, we have stumbled upon four types of composition of a ternary relation with a binary relation and vice versa.

In 2016, De Baets et al. [33] introduced the clone relations of a given strict order relation and provided a representation of all fuzzy tolerance/equivalence relations a given order relation or a strict order relation is compatible with. Later on, Bouremel et al. [17] have generalized the notion of clone relation to any type of binary relation, and they extended this characterization by considering an arbitrary binary relation instead of a strict order relation. In this thesis, we follow the same direction and introduce the notion of clone relations of a ternary relation. More precisely, we generalize the notions of left and right clone relations of a binary relation in a natural way to the notions of left, middle and right clone relations of a ternary relation, and we investigate their basic properties as well as their usefulness to study and characterize the properties of a ternary relation.

This dissertation is structured as follows.

- In Chapter 1, we provide generalities on binary relations, ordered sets, residuated lattices, Binary L -relations and ternary relation, that we need throughout this thesis.
- In Chapter 2, we focus on one hand, on the main proprieties of traces of a ternary relation. On the other hand, on the role of traces in the characterization of a ternary relation in terms of its traces. We extend the notion of traces of a binary relation to the setting of ternary relations, where we introduce the left, middle and right traces of a ternary relation. However we introduce various relational compositions and we show that the traces represent the greatest solutions of certain relational inequalities associated with these compositions.
- In Chapter 3, we introduce the left, middle and right clone relations of a ternary relation with their essentially properties and we study the main properties of a ternary relation in terms of its clone relations. We investigate the interaction of the left, middle and right clone relations of ternary relations with the basic set operations. Also, we characterize the clone relations of union of two ternary relations defined on disjoint sets.
- In Chapter 4, we focus on the compatibility of a ternary relation with a binary L -relation. Interestingly, we will prove that this compatibility can be expressed in terms of inclusions of the binary L -relation in the traces of the ternary relation. We introduce and study the notions of left-, middle- and right-compatibility of an arbitrary ternary relation with a binary L -relation. We provide a characterization of the left-, middle- and right-compatibility of a ternary relation with a binary L -relation in terms of inclusions of the binary L -relation in the left, middle and right traces of the ternary relation. Subsequently, several basic properties of these compatibility relations are discussed. Note that this study of compatibility represents one of the important applications of the notion of traces. Finally, we study the representation of all the binary fuzzy tolerance a crisp ternary relation is compatible with.
- Finally, general conclusions and future research lines are given.

Most parts of results presented in this thesis have already been published or submitted for publication in peer-reviewed international journals. Results included in Chapter 2 have been described in [89] and those included in Chapter 4 have been described in [3].

1

Generalities on relations, posets and residuated lattices

The purpose of this first chapter is to provide a basic introduction to the binary relations, ordered sets, residuated lattices, Binary L -relations, as well as to the ternary relations. Next, we recall the notion of fuzzy relations and some of their basic properties. Note that many of the properties of these concepts will be used in the next chapters.

1.1/ Binary relations

This section contains the basic definitions and properties of binary relations.

A binary relation R on a set X is a subset of X^2 , i.e., it is a set of couples $(x, y) \in X^2$. For a given binary relation R on a set X , we denote:

- (i) the transpose (converse) of R by R^t , i.e., for any $x, y \in X$, $(x, y) \in R^t$ means that $(y, x) \in R$;
- (ii) the complement of R by R^c , i.e., for any $x, y \in X$, $(x, y) \in R^c$ means that $(x, y) \notin R$;
- (iii) the **dual** of R by R^d , i.e., for any $x, y \in X$, $(x, y) \in R^d$ means that $(y, x) \notin R$.

A binary relation R on a set X is called:

- (i) reflexive, if, for any $x \in X$, it holds that $(x, x) \in R$;
- (ii) irreflexive, if, for any $x \in X$, it holds that $(x, x) \in R^c$;
- (iii) symmetric, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies that $(y, x) \in R$;
- (iv) asymmetric, if, for any $x, y \in X$, it holds that $(x, y) \in R$ implies that $(y, x) \in R^c$;

- (v) antisymmetric, if, for any $x, y \in X$, it holds that $(x, y) \in R \wedge (y, x) \in R$ implies that $x = y$;
- (vi) transitive, if, for any $x, y, z \in X$, it holds that $(x, y) \in R \wedge (y, z) \in R$ implies that $(x, z) \in R$;
- (vii) negatively transitive, if, for any $x, y, z \in X$, it holds that $(x, z) \in R$ implies that $(x, y) \in R \vee (y, z) \in R$;
- (viii) complete, if, for any $x, y \in X$, it holds that $(x, y) \in R \vee (y, x) \in R$.

A binary relation R_1 on a set X is said to be included in a binary relation R_2 on the same set X , denoted by $R_1 \subseteq R_2$, if, for any $x, y \in X$, $(x, y) \in R_1$ implies that $(x, y) \in R_2$. Two elements x and y of a set X equipped with a relation R are called comparable elements, if it holds that $(x, y) \in R$ or $(y, x) \in R$. Otherwise, they are called incomparable elements. The union of two binary relations R_1 and R_2 on X is the binary relation $R_1 \cup R_2$ on X defined as $R_1 \cup R_2 = \{(x, y) \in X^2 \mid (x, y) \in R_1 \vee (x, y) \in R_2\}$. Similarly, the intersection of two binary relations R_1 and R_2 on X is the binary relation $R_1 \cap R_2$ on X defined as $R_1 \cap R_2 = \{(x, y) \in X^2 \mid (x, y) \in R_1 \wedge (x, y) \in R_2\}$. If $R_1 \cap R_2 = \emptyset$, then R_1 and R_2 are called disjoint relations. The composition of two relations R_1 and R_2 on X is the relation $R_1 \circ R_2$ on X defined as:

$$R_1 \circ R_2 = \{(x, z) \in X^2 \mid (\exists y \in X)((x, y) \in R_1 \wedge (y, z) \in R_2)\}.$$

A binary relation R on a set X is called:

- (i) a pseudo-order relation if it is reflexive and antisymmetric;
- (ii) a strict order if it is irreflexive and transitive;
- (iii) an order relation if it is reflexive, antisymmetric and transitive;
- (iv) a total order relation if it is reflexive, antisymmetric, transitive and complete;
- (v) a tolerance relation if it is reflexive and symmetric;
- (vi) an equivalence relation if it is reflexive, symmetric and transitive.

For any tolerance/equivalence relation R on a set X , the tolerance/equivalence class of an element $x \in X$ is given by $[x]_R = \{y \in X \mid (x, y) \in R\}$.

For more details on binary relations, we refer to [14, 28, 78].

1.2/ Posets and Lattices

1.2.1/ Posets

A partial order (order, for short) is a binary relation $\leq$ over a set X which is reflexive ($a \leq a$, for any $a \in X$), antisymmetric ($a \leq b$ and $b \leq a$ implies $a = b$, for any $a, b \in X$) and transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$, for any $a, b, c \in X$). A set with an order relation is called an ordered set (also called a poset).

A strict order relation $<$ on a set X is a relation that is irreflexive (i.e., $a < a$ does not hold for any $a \in X$) and transitive, implying that it is asymmetric (i.e., $a < b$ implies $\neg(b < a)$, for any $a, b \in X$). To any order relation $\leq$ corresponds a strict order relation $<$ (its strict part or irreflexive kernel): $a < b$ if $a \leq b$ and $a \neq b$. Conversely, to any strict order relation $<$ corresponds an order relation $\leq$ (its reflexive closure): $a \leq b$ if $a < b$ or $a = b$.

Definition 1: [28]

Let $(X, \leq_1)$ and $(Y, \leq_2)$ two posets, then the mapping $f : X \rightarrow Y$ is monotone (or order-preserving) if $x \leq_1 y \Rightarrow f(x) \leq_2 f(y)$, for any $x, y \in X$.

For more details about order, strict order we refer to [78].

1.2.2/ Lattices

It is well known that there are many important properties of an order set $(L, \leq)$. These are expressed in term of the existence of certain upper bounds or lower bounds of subsets of X . We will be particularly interested in two of the most important classes of ordered sets: lattice and complete lattice. We often write $x \vee y$ instead of $\sup\{x, y\}$ when it exists and $x \wedge y$ instead of $\inf\{x, y\}$ when it exists. Similarly we write $\bigvee S$ (the join of S) and $\bigwedge S$ (the meet of S) instead of $\sup S$ and $\inf S$ when these exist.

Definition 2: [28]

Let $(X, \leq)$ be an ordered set.

- (i) If $x \vee y$ exists for all $x, y \in X$, then $(X, \leq)$ is called a $\vee$ -semi-lattice.
- (ii) If $x \wedge y$ exists for all $x, y \in X$, then $(X, \leq)$ is called a $\wedge$ -semi-lattice.
- (iii) $(X, \leq)$ is called a lattice if it is both a $\wedge$ -semi-lattice and a $\vee$ -semi-lattice.
- (iv) If $\bigvee S, \bigwedge S$ exist for all $S \subseteq X$, then $(X, \leq)$ is called a complete lattice.

A non-empty subset L' of a lattice $(L, \wedge, \vee)$ is called sub-lattice of L if, for any $a, b \in L'$, it holds that $a \wedge b \in L'$ and $a \vee b \in L'$. A sub-lattice L' of a lattice L is called a convex sub-lattice of L , if $a \leq c \leq b$ and $a, b \in L'$ implies that $c \in L'$, for any $a, b, c \in L$.

A bounded lattice is a lattice that additionally has a greatest element 1 and a smallest element 0, which satisfy $0 \leq x \leq 1$ for any x in X .

A lattice $(L, \leq, \wedge, \vee)$ is distributive if the following additional condition holds

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \text{ for any } x, y, z \in L.$$

This means that the meet operation preserves non-empty finite joins. It is known that the above condition is equivalent to its dual

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \text{ for any } x, y, z \in L.$$

A complemented lattice is a bounded lattice $(L, \wedge, \vee, 0, 1)$, in which any element x has a complement, i.e., there exists an element $y \in L$ such that

$$x \vee y = 1 \text{ and } x \wedge y = 0.$$

An element may have more than one complement in general. However, if $(L, \wedge, \vee, 0, 1)$ is distributive then every element will have at most one complement.

For more information on lattices, complete lattices can be found in [14, 15, 28, 78].

1.3/ T-norms

The history of triangular-norms (*t-norms* for short) started with Menger [58]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements. Schweizer and Sklar [79] provided the axioms of *t-norms*, as they are used today.

Definition 3: [60]

A *t-norm* T on $[0, 1]$ is a function $T : [0, 1]^2 \rightarrow [0, 1]$ satisfying the following four axioms:

- (T1) Commutativity: $(\forall x, y \in [0, 1])(T(x, y) = T(y, x))$;
- (T2) Associativity: $(\forall x, y, z \in [0, 1])(T(x, T(y, z)) = T(T(x, y), z))$;
- (T3) Monotonicity: $(\forall x, y, z \in [0, 1])(x \leq y \Rightarrow T(x, z) \leq T(y, z))$;
- (T4) Boundary condition: $(\forall x \in [0, 1])(T(x, 1) = x)$.

Conditions (T4) and (T3) imply that for any *t-norm* T it holds that $T(x, y) \leq x$, $T(x, y) \leq y$, $T(x, y) \leq \min(x, y)$ and $T(x, 0) = 0$.

The following definition of a *t-norm* on a bounded partially ordered set $(L, \leq)$ is analogous to the definition of a *t-norm* on the real unit interval $[0, 1]$.

Definition 4: [30]

A *t-norm* T on a bounded poset $(L, \leq)$ is a function $T : L^2 \rightarrow L$ satisfies the following four axioms:

- (i) Commutativity: $(\forall x, y \in L)(T(x, y) = T(y, x))$;
- (ii) Associativity: $(\forall x, y, z \in L)(T(x, T(y, z)) = T(T(x, y), z))$;
- (iii) Monotonicity: $(\forall x, y, z \in L)(x \leq y \Rightarrow T(x, z) \leq T(y, z))$;
- (iv) Boundary condition $(\forall x \in L)(T(x, 1) = x)$.

Many *t-norms* can be found in the literature. The following four operations are the most common ones:

- (T5) Minimum: $T_M(x, y) = \min\{x, y\}$
- (T6) Product: $T_P(x, y) = x \cdot y$
- (T7) Lukasiewicz: $T_L(x, y) = \max\{x + y - 1, 0\}$
- (T8) Drastic product:

$$T_D(x, y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{if } x, y < 1. \end{cases}$$

Let T be a t -norm on $[0, 1]$.

An element $a \in]0, 1[$ is called a zero divisor of T if there exists some $b > 0$ such that $T(a, b) = 0$.

An element $a \in [0, 1]$ is called an idempotent element of T if $T(a, a) = a$.

T is called Archimedean if $T(x, x) < x$ for every $x \in [0, 1]$.

Each $a \in [a, b]$ is an idempotent element of the Minimum t -norm T_M (Actually T_M is the only t -norm whose set of idempotent is equal $[0, 1]$), T_M has no zero divisor. Each $a \in]0, 1[$ is a zero divisor of the Lukasiewicz t -norm T_L as well of the Drastic product t -norm T_D . For two t -norms T_1 and T_2 on $[0, 1]$, we define:

$$T_1 \leq T_2 \Leftrightarrow (\forall x, y \in [0, 1])(T_1(x, y) \leq T_2(x, y)).$$

Let be T_1 and T_2 two t -norms. If $T_1 \leq T_2$, then T_1 is called weaker than T_2 (or, equivalently, T_2 is called stronger than T_1). Note that T_D is the weakest t -norm, and T_M is the strongest t -norm, i.e. for any t -norm it holds: (T9) $T_D \leq T \leq T_M$. Since $T_L \leq T_P$, it obviously holds: (T10) $T_D \leq T_L \leq T_P \leq T_M$.

Definition 5: [16]

Let T_1 and T_2 be two t -norms. T_1 is said to dominate T_2 if and only if, for any $x, y, z, t \in [0, 1]$, it holds that:

$$T_1(T_2(x, y), T_2(z, t)) \geq T_2(T_1(x, z), T_1(y, t)).$$

Lemma 1: [29, 50]

The following statements hold:

- (i) Any t -norm T dominates itself.
- (ii) The minimum t -norm T_M dominates any other t -norm.
- (iii) If a t -norm T_1 dominates another t -norm T_2 , then T_1 is stronger than T_2 .

Lemma 1 particularly implies that dominance is a reflexive, antisymmetric and not transitive relation on the set of t -norms.

1.4/ Residuated lattices

1.4.1/ Definitions

Residuated lattices, introduced by Dilworth and Ward [85] and some related algebraic systems [71] play an important role because they provide an algebraic frameworks to fuzzy logic and fuzzy reasoning. In this chapter, we recall the notion of residuated lattice which is related to our work in Chapter three with some of its important properties in order to clarify this concept more.

Definition 6:

A residuated lattice is an algebra $(L, \wedge, \vee, *, \rightarrow, 0, 1)$, or simply, $(L, *, \rightarrow)$ where [8]:

- (i) $(L, \wedge, \vee, 0, 1)$ is a lattice (the corresponding order will be denoted by $\leq$) with the least element 0 and the greatest element 1;
- (ii) $(*, \rightarrow)$ forms an adjoint couple on L , i.e. for any $a, b, c \in L$:
 - (R1) If $a \leq b$ and $c \leq d$ then $a * c \leq b * d$,
 - (R2) If $b \leq c$ then $a \rightarrow b \leq a \rightarrow c$,
 - (R3) If $a \leq b$ then $b \rightarrow c \leq a \rightarrow c$,
 - (R4) $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$ (adjointness condition);
- (iii) $(L, *, 1)$ forms a commutative monoid, i.e. for any $a, b, c \in L$:
 - (R5) $(a * b) * c = a * (b * c)$,
 - (R6) $a * b = b * a$,
 - (R7) $1 * a = a$.

Residuated lattice L is called complete if $(L, \wedge, \vee, 0, 1)$ is a complete lattice.

$*$ and $\rightarrow$ called multiplication and residuum, respectively. Multiplication is isotone, residuum is isotone in the first and antitone in the second argument (w.r.t. lattice order $\leq$).

Example 1:

Consider $L = (0, a, b, c, d, e, 1)$ with $0 < a < b < e < 1$ and $0 < c < d < e < 1$, but elements $\{a, c\}$ and $\{b, d\}$ are pairwise incomparable, the Hasse diagram of L is shown in Figure 1.1.

Then ([46], page 23) $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ becomes a residuated lattice relative to the following operations:

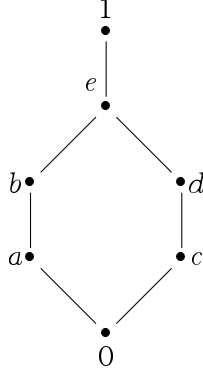
$*$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
e	0	a	a	c	c	e	e
1	0	a	b	c	d	e	1

$\rightarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	e	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	e	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

1.4.2/ Main properties

In the following, we use $\mathcal{L}$ to denote the class of all residuated lattices, and we always suppose that L is a bounded lattice with the smallest element 0 and the greatest element 1, $*$ and $\rightarrow$ are two binary operations on L . In addition, we often use the following derived operations: $a^0 = 1$, $a^n = a^{n-1} * a$, where $n \in \mathbb{N}$ and $a \in L$.

The following Proposition lists the fundamental properties of residuated lattices.

Figure 1.1: Hasse diagram of L .

Proposition 1: [70, 71]

If $(L, *, \rightarrow) \in \mathcal{L}$, then the following statements hold:

R8 $a \leq b \rightarrow a * b$,

R9 $(a \rightarrow b) * a \leq b$,

R10 $f_a : L \rightarrow L, x \mapsto x * a$ preserves all joins existing in L , i.e.

$$\left(\bigvee_{i \in I} a_i \right) * a = \bigvee_{i \in I} (a_i * a),$$

R11 $g_a : L \rightarrow L, x \mapsto a \rightarrow x$ preserves all meets existing in L , i.e.

$$a \rightarrow \left(\bigwedge_{i \in I} a_i \right) = \bigwedge_{i \in I} (a \rightarrow a_i),$$

R12 $h_a : L \rightarrow L, x \mapsto a * x$ preserves all joins existing in L , i.e.

$$a * \left(\bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} (a * a_i),$$

R13 $k_a : L \rightarrow L, x \mapsto x \rightarrow a$ changes all joins existing in L to meets, i.e.

$$\left(\bigvee_{i \in I} a_i \right) \rightarrow b = \bigwedge_{i \in I} (a_i \rightarrow b),$$

R14 $b \rightarrow c \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$ and $b \rightarrow a \leq (a \rightarrow c) \rightarrow (b \rightarrow c)$,

R15 $a = 1 \rightarrow a$,

R16 $a \leq b \Leftrightarrow a \rightarrow b = 1$,

R17 $a \leq b \rightarrow c \Leftrightarrow b \leq a \rightarrow c$,

R18 $a \rightarrow b \leq a * c \rightarrow b * c$ and $(a \rightarrow b) * (b \rightarrow c) \leq (a \rightarrow c)$,

R19 $a * b \rightarrow c = a \rightarrow (b \rightarrow c)$,

R20 $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$,

R21 $a * b \leq a \wedge b$,

R22 $a^n \leq a^m, n, m \in \mathbb{N}, m \leq n$.

Proposition 2: [53]

Let $(L, *, \rightarrow)$ be a residuated lattice. The following two statements hold:

- (i) $x * (y \wedge z) \leq (x * y) \wedge (x * z)$;
- (ii) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

The following proposition shows that under suitable conditions, the operations $*$ and $\rightarrow$ are not independent.

Proposition 3: [70, 71]

Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice.

- (i) If $*$ is a binary operation on L satisfying conditions (R1) and (R10), then there exists a binary operation $\rightarrow$ satisfying conditions (R2), (R3) and (R4). Such operation is unique and determined by the following formula:

$$a \rightarrow b = \bigvee \{x \in L \mid x * a \leq b\}, \quad a, b \in L;$$

- (ii) If $\rightarrow$ is a binary operation on L satisfying conditions (R2), (R3) and (R11), then there exists a binary operation $*$ satisfying conditions (R1) and (R4), and such operation is unique and determined by the following formula:

$$a * b = \bigwedge \{x \in L \mid a \leq b \rightarrow x\}, \quad a, b \in L.$$

Now we characterize the residuated lattices by the following propositions.

Proposition 4: [71]

Let $(L, \wedge, \vee, 0, 1)$ be a lattice. $(L, *, \rightarrow) \in \mathcal{L}$ if and only if the following conditions hold, for all $a, b, c \in L$:

- (i) (R4) $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$;
- (ii) (R7) $1 * a = a$;
- (iii) (R20) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$.

Proposition 5: [71]

Let $(L, \wedge, \vee, 0, 1)$ be a lattice. $(L, *, \rightarrow) \in \mathcal{L}$ if and only if the following conditions hold, for all $a, b, c \in L$:

- (i) (R8) $(a \rightarrow b) * a \leq b$;
- (ii) (R9) $a \leq b \rightarrow a * b$;
- (iii) (R7) $1 * a = a$.
- (iv) (R20) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$;
- (v) (R23) $(a \vee b) * c = (a * c) \vee (b * c)$;
- (vi) (R24) $a \rightarrow b \wedge c = (a \rightarrow b) \wedge (a \rightarrow c)$.

Proposition 6:

Let $(L, \wedge, \vee, 0, 1)$ be a lattice. $(L, *, \rightarrow) \in \mathcal{L}$ if and only if the following conditions hold, for all $a, b, c \in L$:

- (i) (R4) $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$;
- (ii) (R'7) $1 \rightarrow a = a$;
- (iii) (R20) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$.

Proof. It is easy to see that (i), (ii) and (iii) hold in any residuated lattice. Conversely, it suffices to show that (i), (ii) and (iii) imply that $(L, *, 1)$ is a commutative monoid. We have $x * 1 \leq t$ iff $x \leq (1 \rightarrow t)$ iff (by (R'7)) $x \leq t$, which implies $x * 1 = x$. Furthermore, $x * y \leq t$ iff $x \leq (y \rightarrow t)$ iff $1 \rightarrow x \leq y \rightarrow t$ iff (by (R2)) $x \rightarrow (1 \rightarrow x) \leq x \rightarrow (y \rightarrow t)$ iff (by (R20)) $(1 \rightarrow (x \rightarrow x)) \leq y \rightarrow (x \rightarrow t)$ iff $1 \leq y \rightarrow (x \rightarrow t)$ iff (by (R4)) $1 * y \leq x \rightarrow t$ iff $y \leq 1 \rightarrow (x \rightarrow t)$ iff $y \leq x \rightarrow t$ iff $y * x \leq t$, i.e. $x * y = y * x$. Finally, $(x * y) * z \leq t$ iff ...iff $1 \rightarrow x \leq y \rightarrow (z \rightarrow t)$ iff $1 \leq x \rightarrow (y \rightarrow (z \rightarrow t))$ iff (by (R20)) $1 \leq z \rightarrow (y \rightarrow (x \rightarrow t))$ iff ...iff $x * (y * z) \leq t$, i.e. $(x * y) * z = x * (y * z)$. Therefore $(L, *, 1)$ is a commutative monoid. $\square$

A residuated lattice satisfies the prelinearity axiom [8] if and only if $(x \rightarrow y) \vee (y \rightarrow x) = 1$ holds. A residuated lattice is *divisible* [8] if and only if $x \wedge y = x * (x \rightarrow y)$. It can be shown [8] that divisibility is equivalent to the following condition: for each $x \leq y$ there is z such that $x = y * z$. A residuated lattice satisfies the law of double negation (and is called integral, commutative Girard-monoid [8]) if and only if $x = (x \rightarrow 0) \rightarrow 0$ holds.

Several important algebras are special residuated lattices: Boolean algebras (algebraic counterpart of classical logic). Heyting algebra is a residuated lattice where $x * y = x \wedge y$. A BL-algebra [8] is a residuated lattice which is divisible and satisfies the prelinearity axiom. An MV-algebra [8, 23] is a residuated lattice in which $x \vee y = (x \rightarrow y) \rightarrow y$ holds. Equivalently [42], an MV-algebra is a residuated lattice which is divisible and satisfies the law of double negation. Thus, each BL-algebra satisfying the law of double negation is an MV-algebra (which is the way MV-algebras are defined in [44]). A Π -algebra (product algebra) [44] is a BL-algebra satisfying $(z \rightarrow 0) \rightarrow 0 \leq ((x * z) \rightarrow (y * z)) \rightarrow (x \rightarrow y)$ and $x \wedge (x \rightarrow 0) = 0$. A G-algebra (Gödel algebra) is a BL-algebra which satisfies $x * x = x$ (i.e. a Heyting algebras satisfying the prelinearity axiom). A Boolean algebras is a residuated lattice which is both a Heyting algebras and an MV-algebra (relation to the usual axiomatization is $x \rightarrow y = x' \vee y$).

More information about residuated and complete residuated lattices can be found in [15, 7, 8, 28, 44, 43, 70, 78].

1.5/ Fuzzy sets and fuzzy relations

Analogously to the bivalent case, one can start developing a naive set theory with truth values in an (appropriately chosen) complete residuated lattice L (the classical bivalent case being a special case for $L = 2$). In this section we recall the basic notions of fuzzy logic. In the following, L will be a (complete) residuated lattice.

1.5.1/ Fuzzy sets

An L -fuzzy subset (L -set, for short) in a universe set X is a mapping $A : X \rightarrow L$ assigning to every element $x \in X$ an element $A(x) \in L$ interpreted as the truth degree to which x belongs to A . Ordinary crisp subset of X are considered as fuzzy subset of X (or an L -subset) of X , taking membership values in the set $\{0, 1\} \subseteq L$. For later, L^X denotes the set of all L -subsets of X , i.e. the set of all mappings from X to L . Let A and B be two L -subsets of X .

The equality of A and B is defined as the usual equality of mappings, i.e. $A = B$ if and only if $A(x) = B(x)$, for every $x \in X$.

The inclusion $A \leq B$ is also defined pointwise: $A \leq B$ if and only if $A(x) \leq B(x)$, for every $x \in X$.

Endowed with this partial order the set $\mathcal{L}(X)$ of all L -subsets of X forms a complete residuated lattice, in which the meet (intersection) $\bigwedge_{i \in I} A_i$ and the join (union) $\bigvee_{i \in I} A_i$ of an arbitrary family $\{A_i\}_{i \in I}$ of L -subsets of X , are mappings from X into L defined by

$$\left(\bigwedge_{i \in I} A_i\right)(x) = \bigwedge_{i \in I} A_i(x), \quad \left(\bigvee_{i \in I} A_i\right)(x) = \bigvee_{i \in I} A_i(x).$$

The product $A \otimes B$ is an L -subset defined by $A \otimes B(x) = A(x) \otimes B(x)$, for every $x \in X$. The crisp part of an L -subset A of X is a crisp subset $\hat{A} = \{x \in X \mid A(x) = 1\}$ of X . We will also consider $\hat{A}$ as a mapping $\hat{A} : X \rightarrow L$ defined by $\hat{A}(x) = 1$, if $A(x) = 1$, and $\hat{A}(x) = 0$, if $A(x) \neq 1$.

More information on L -set can be found in [7, 44, 86].

1.5.2/ Binary fuzzy relations

Binary fuzzy relations were first introduced by Lotfi Zadeh [86] as a natural generalization of the usual crisp relations. They have applications in several fields such as economics, sociology, psychology, medicine, etc. Throughout the last chapter of this thesis, we will use the binary fuzzy relations where the ternary relations are compatible with.

The notion of L -fuzzy relation (an L -relation, for short) on a set X generalizes the notion of a relation by expressing degrees of relationship in some bounded lattice $(L, \leq, \vee, \wedge, 0, 1)$. A binary L -relation S on a set X is a mapping $S : X \times X \rightarrow L$. Obviously, if $L = \{0, 1\}$, then binary relations are retrieved, often referred to as crisp relations. As is commonly done, we will further restrict ourselves to the use of a complete residuated lattice $(L, \leq, \vee, \wedge, *, \rightarrow, 0, 1)$ [15]. The corresponding infimum and supremum operations are denoted as $\inf$ and $\sup$, respectively. The operations $*$ and $\rightarrow$ are known as a t-norm and its residual implication, and form an adjoint pair, i.e. $x * z \leq y$ if and only if $z \leq x \rightarrow y$, for any $x, y, z \in L$.

A binary L -relation S_1 is said to be included in a binary L -relation S_2 , denoted by $S_1 \subseteq S_2$, if $S_1(x, y) \leq S_2(x, y)$, for any $x, y \in X$. The intersection of two binary L -relations S_1 and S_2 on X is the binary L -relation $S_1 \cap S_2$ on X defined by $S_1 \cap S_2(x, y) = S_1(x, y) \wedge S_2(x, y)$. Similarly, the union of two binary L -relations S_1 and S_2 on X is the binary L -relation $S_1 \cup S_2$ on X defined by $S_1 \cup S_2(x, y) = S_1(x, y) \vee S_2(x, y)$, for any $x, y \in X$. The composition of two binary L -relations S_1 and S_2 on X is the binary L -relation $S_1 \circ S_2$ on X defined by

$S_1 \circ S_2(x, z) = \sup_{y \in X} S_1(x, y) * S_2(y, z)$. The transpose S^t of a binary L -relation S is the binary L -relation defined by $S^t(x, y) = S(y, x)$. A binary L -relation S on a set X is called:

- (i) reflexive, if, for any $x \in X$, it holds that $S(x, x) = 1$;
- (ii) symmetric, if $S^t = S$;
- (iii) $*$ -transitive, if, for any $x, y, z \in X$, it holds that $S(x, y) * S(y, z) \leq S(x, z)$;
- (iv) $*$ -Asymmetric, if, for any $x, y \in X$, $S(x, y) * S(y, x) = 0$;
- (v) $*$ -Antisymmetric, if, for any $x, y \in X$, $x \neq y$ implies $S(x, y) * S(y, x) = 0$;
- (vi) separability, if, for any $x, y \in X$, $S(x, y) = 1$ implies that $x = y$.

For more details on binary L -relations and their properties, we refer to [20, 7, 16, 87].

1.6/ Ternary relations

In this section, we recall the definition of a ternary relation, the basic operations on ternary relations and their properties.

1.6.1/ Definitions and examples

A ternary (or triadic) relation T on a set X is a subset of X^3 . Three special ternary relations on a set X are the empty relation $\emptyset$, the ternary identity relation $I_{X^3} = \{(x, x, x) \mid x \in X\}$ and the universal ternary relation X^3 .

For a given ternary relation T on a set X , we denote:

- (i) the **transpose** of T by T^t , i.e., for any $x, y, z \in X$, $(x, y, z) \in T^t$ means that $(z, y, x) \in T$;
- (ii) the **complement** of T by T^c , i.e., for any $x, y, z \in X$, $(x, y, z) \in T^c$ means that $(x, y, z) \notin T$;
- (iii) the **dual** of T by T^d , i.e., for any $x, y, z \in X$, $(x, y, z) \in T^d$ means that $(z, y, x) \notin T$.

A ternary relation T_1 on a set X is said to be included in a ternary relation T_2 on the same set X , denoted by $T_1 \subseteq T_2$, if, for any $x, y, z \in X$, $(x, y, z) \in T_1$ implies that $(x, y, z) \in T_2$. The union of two ternary relations T_1 and T_2 on X is the ternary relation $T_1 \cup T_2$ on X defined as $T_1 \cup T_2 = \{(x, y, z) \in X^3 \mid (x, y, z) \in T_1 \vee (x, y, z) \in T_2\}$. Similarly, the intersection of two ternary relations T_1 and T_2 on X is the ternary relation $T_1 \cap T_2$ on X defined as $T_1 \cap T_2 = \{(x, y, z) \in X^3 \mid (x, y, z) \in T_1 \wedge (x, y, z) \in T_2\}$. If $T_1 \cap T_2 = \emptyset$, then T_1 and T_2 are called disjoint ternary relations.

Example 2:

Let us consider $(X, \leq)$ a partially ordered set. We can define a ternary relation as:

$$T = \{(x, y, z) \in X^3 \mid y \leq x \text{ or } y \leq z\}.$$

Example 3:

Let T_1 and T_2 be two ternary relations on $X = \{a, b, c, d, e\}$, given by $T_1 = \{(a, b, c), (a, b, d), (a, b, e), (c, b, e), (c, d, e)\}$ and $T_2 = \{(a, b, c), (b, c, a), (c, a, b), (a, b, d), (a, b, e), (b, d, a), (c, b, e), (e, c, d), (c, d, e)\}$. We easily verify that $T_1 \subseteq T_2$. Therefore, $T_1 \cup T_2 = T_2$ and $T_1 \cap T_2 = T_1$.

Further information on ternary relations can be found in [21, 25, 26, 65, 66].

The following proposition contains trivial properties, but will be useful in the sequel.

Proposition 7:

For any ternary relation T and any family of ternary relations $(T_i)_{i \in I}$ on a set X , the following equalities hold:

- (i) $(T^t)^t = T$ and $(T^c)^c = T$;
- (ii) $(\bigcup_{i \in I} T_i)^t = \bigcup_{i \in I} T_i^t$ and $(\bigcup_{i \in I} T_i)^c = \bigcup_{i \in I} T_i^c$;
- (iii) $(\bigcap_{i \in I} T_i)^t = \bigcap_{i \in I} T_i^t$ and $(\bigcap_{i \in I} T_i)^c = \bigcap_{i \in I} T_i^c$.

1.6.2/ Cubic matrix - representation of a ternary relation

We first recall the notion of cubic matrix. For more details, we refer to [13].

An $(n \times n \times n)$ -matrix (cubic matrix, for short) over a field F is an ordered object $A = (a_{ijk})$ of elements of F referred to through three indices i, j, k such that $1 \leq i, j, k \leq n$. The indices i, j and k indicate the position of the element a_{ijk} , i.e., the row, the column and the depth of the element a_{ijk} . Any cubic matrix A can be written as a list of $(n \times n)$ -matrices, i.e.

$$A = (A_1, \dots, A_n),$$

where $A_i = (a_{ijk})$ are usual $(n \times n)$ -matrices of the elements $a_{ijk} \in F$, for any $1 \leq i, j, k \leq n$.

A ternary relation T on a finite set X of cardinality n can be represented using a cubic matrix. For any $x_i, x_j, x_k \in X = \{x_1, x_2, \dots, x_n\}$ with $1 \leq i, j, k \leq n$, it holds that

$$T_{ijk} = \begin{cases} 1 & , \text{ if } (x_i, x_j, x_k) \in T, \\ 0 & , \text{ otherwise.} \end{cases}$$

Example 4:

Let T be the ternary relation on $X = \{x_1, x_2, x_3, x_4\}$ given by:

$$T = \{(x_1, x_1, x_3), (x_1, x_2, x_4), (x_2, x_2, x_2), (x_2, x_4, x_4), (x_3, x_1, x_1), \\ (x_3, x_1, x_2), (x_3, x_4, x_1), (x_4, x_4, x_2), (x_4, x_4, x_4)\}.$$

The matrix representation of T can be written as follows:

$$M_T = \left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right).$$

1.6.3/ Ternary relations obtained by permutation

A permutation σ of a set X is a bijection from X to itself. For a set $X = \{x, y, z\}$, we write $\sigma(x, y, z)$ instead of $(\sigma(x), \sigma(y), \sigma(z))$. The possible permutations of $\{x, y, z\}$ are listed as follows:

$$\sigma_0(x, y, z) = (x, y, z), \quad \sigma_1(x, y, z) = (x, z, y), \quad \sigma_2(x, y, z) = (y, x, z),$$

$$\sigma_3(x, y, z) = (z, x, y), \quad \sigma_4(x, y, z) = (y, z, x), \quad \sigma_5(x, y, z) = (z, y, x).$$

For a ternary relation T on X , we define the ternary relation T^σ as

$$T^\sigma = \{\sigma(x, y, z) \in X^3 \mid (x, y, z) \in T\}.$$

It is clear that $T^{\sigma_0} = T$ and $T^{\sigma_5} = T^t$.

Definition 7:

Let T be a ternary relation on a set X . We define:

- (i) the right-converse of T , as $T^\perp := T^{\sigma_1}$;
- (ii) the left-converse of T , as $T^{\perp} := T^{\sigma_2}$;
- (iii) the right-rotation of T , as $T^+ := T^{\sigma_3}$;
- (iv) the left-rotation of T , as $T^- := T^{\sigma_4}$.

The following proposition is immediate.

Proposition 8:

Let T be a ternary relation on a set X . The composition table for the considered permutations reads as follows: for given σ_i and σ_j , it lists $T^{\sigma_i \circ \sigma_j}$.

$\sigma_i \backslash \sigma_j$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
σ_0	T	$T^\neg$	$T^\vdash$	T^+	T^-	T^t
σ_1	$T^\neg$	T	T^+	$T^\vdash$	T^t	T^-
σ_2	$T^\vdash$	T^-	T	T^t	$T^\neg$	T^+
σ_3	T^-	$T^\vdash$	T^t	T	T^+	$T^\neg$
σ_4	T^+	T^t	$T^\neg$	T^-	T	$T^\vdash$
σ_5	T^t	T^+	T^-	$T^\neg$	$T^\vdash$	T

2

Traces of a ternary relation

In this chapter, we generalize the notion of traces of a binary relation to the setting of ternary relations. With a given ternary relation, we associate three binary relations: its left, middle and right trace. As in the binary case, these traces facilitate the study and characterization of properties of a ternary relation. Interestingly, the traces themselves turn out to be the greatest solutions of relational inequalities associated with newly introduced compositions of a ternary relation with a binary relation (and vice versa). Note that, the traces will play a key role to provide us with the proper tools for characterizing the binary fuzzy relations that a crisp ternary relation is left-, middle- or right-compatible with, as we shall see in the last chapter of this work.

2.1/ Traces of a binary fuzzy relation

In this section, we reformulate the notions of left and right traces of a binary L -relation introduced in [39] and recall some of their properties.

Definition 8: [39]

Let S be a binary L -relation on a set X .

- (i) The left trace of S is the L -relation S^ℓ on X defined by

$$S^\ell(x, y) = \inf_{z \in X} S(x, z) \rightarrow S(y, z);$$

- (ii) The right trace of S is the L -relation S^r on X defined by

$$S^r(x, y) = \inf_{z \in X} S(z, x) \rightarrow S(z, y).$$

Note that the traces of a binary relation R on X , which have been introduced in [37], can be obtained as a special case of Definition 8, as follows:

Definition 9: [37]

Let R be a binary relation on a set X .

- (i) The left trace of R is the binary relation R^ℓ on X defined as

$$R^\ell = \{(x, y) \in X^2 \mid (\forall z \in X)((x, z) \in R \Rightarrow (y, z) \in R)\};$$

- (ii) The right trace of R is the binary relation R^r on X defined as

$$R^r = \{(x, y) \in X^2 \mid (\forall z \in X)((z, x) \in R \Rightarrow (z, y) \in R)\}.$$

What we call left trace here corresponds to the transpose of the right trace by Doignon et al. [37] and Fodor [39]. Also, what we call right trace was called left trace by Doignon et al. and Fodor. The reason for this unfortunate change of terminology will become clear when introducing traces of ternary relations further on. The following results have been adopted from [39] taking into account the change of terminology just discussed. Further details can be found in [37, 39].

Proposition 9: [39]

Let S be a binary L -relation on a set X . The following statements hold:

- (i) S^ℓ and S^r are reflexive;
- (ii) S^ℓ and S^r are transitive;
- (iii) $(S^t)^\ell = S^r$ and $(S^t)^r = S^\ell$.

Theorem 1: [39]

Let S be a binary L -relation on a set X . The following statements are equivalent:

- (i) S is reflexive;
- (ii) $(S^\ell)^t \subseteq S$;
- (iii) $S^r \subseteq S$.

Theorem 2: [39]

Let S be a binary L -relation on a set X . The following statements are equivalent:

- (i) S is transitive;
- (ii) $S \subseteq (S^\ell)^t$;
- (iii) $S \subseteq S^r$.

Corollary 1:

[39] Let S be a binary L -relation on a set X . The following statements are equivalent:

- (i) S is reflexive and transitive;
- (ii) $S = (S^\ell)^t = S^r$.

Theorem 3: [39]

Let S be a binary L -relation on a set X . Then it holds that:

- (i) S^ℓ is the greatest binary fuzzy relation U on X such that $U^t \circ S \subseteq S$;
- (ii) S^r is the greatest binary fuzzy relation V on X such that $S \circ V \subseteq S$.

Theorem 4: [39]

Let S be a binary L -relation on a set X , it holds that

$$(S^\ell)^t \circ S = S \circ S^r = S.$$

2.2/ Useful properties of ternary relations

The following properties of a ternary relation T are of interest in this work. Some of them are discussed in [21, 25, 26, 65, 66].

A ternary relation T on a set X is called:

- (i) reflexive, if, for any $x \in X$, it holds that $(x, x, x) \in T$;
- (ii) strongly reflexive, if, for any $x, y, z \in X$ with $\text{card}\{x, y, z\} \leq 2$, it holds that $(x, y, z) \in T$;
- (iii) irreflexive, if, for any $x \in X$, it holds that $(x, x, x) \notin T$;
- (iv) strongly irreflexive, if, for any $x, y, z \in X$ with $\text{card}\{x, y, z\} \leq 2$, it holds that $(x, y, z) \notin T$;
- (v) symmetric, if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ implies that $(z, y, x) \in T$, i.e., $T = T^t$;
- (vi) strongly symmetric, if $T = T^{\sigma_i}$, for any $i \in \{0, \dots, 5\}$;
- (vii) asymmetric, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T$ implies that $(z, y, x) \notin T$;
- (viii) strongly asymmetric, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T$ implies that $\sigma_i(x, y, z) \notin T$, for any $i \in \{1, \dots, 5\}$;
- (ix) cyclic, if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ implies that $(y, z, x) \in T$;
- (x) complete, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $(x, y, z) \in T \vee (z, y, x) \in T$;

- (xi) strongly complete, if, for any $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$, it holds that $\sigma_i(x, y, z) \in T$, for some $i \in \{0, \dots, 5\}$.

Where the above definitions of (ir)reflexivity, (a)symmetry and completeness are quite standard, there is much less agreement in literature on the definition of transitivity of a ternary relation. Indeed, various alternative definitions, each with its own motivation, have been proposed [21, 61, 64, 74]. The ones we have selected below are those that are closest in spirit to the definition of transitivity of a binary relation:

- (xii) c_1 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, t) \in T$ and $(s, t, z) \in T$ implies that $(x, y, z) \in T$;
- (xiii) c_2 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T$ and $(s, t, z) \in T$ implies that $(x, t, z) \in T$;
- (xiii) c_3 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T$ and $(s, t, z) \in T$ implies that $(x, y, z) \in T$;
- (xivi) c_4 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T$ and $(s, t, z) \in T$ implies that $(x, y, t) \in T$;
- (xvi) c_5 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, y, s) \in T$ and $(s, t, z) \in T$ implies that $(y, t, z) \in T$;
- (xvii) c_6 -transitive, if, for any $x, y, z, s, t \in X$, it holds that $(x, s, t) \in T$ and $(s, y, z) \in T$ implies that $(x, y, z) \in T$;
- (xviii) transitive, if it is c_i -transitive, for all $i \in \{1, \dots, 6\}$;
- (xviii) c_i -negatively transitive if its complement T^c is c_i -transitive, for any $i \in \{1, \dots, 6\}$;
- (xixi) negatively transitive, if it is c_i -negatively transitive, for all $i \in \{1, \dots, 6\}$, i.e., its complement T^c is transitive.

A ternary relation T on X is called:

- (i) a preorder relation if it is reflexive and transitive;
- (ii) an order relation if it is reflexive, antisymmetric and transitive;
- (iii) a total order relation if it is reflexive, antisymmetric, transitive and complete;
- (iv) a tolerance relation if it is reflexive and symmetric;
- (v) an equivalence relation if it is reflexive, symmetric and transitive;
- (vi) a cyclic order relation if it is asymmetric, transitive and cyclic;
- (vii) betweenness if it satisfies the following statements:
 - (a) for any $x, y, z \in X$, $(x, y, z) \in T$ if and only if $(z, y, x) \in T$,
 - (b) for any $x, y, z \in X$, $(x, y, z) \in T \wedge (x, z, y) \in T$ if and only if $y = z$;
- (viii) strict order-betweenness if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ if and only if $x < y < z$ or $z < y < x$;

- (ix) order-betweenness if, for any $x, y, z \in X$, it holds that $(x, y, z) \in T$ if and only if $x \leq y \leq z$ or $z \leq y \leq x$.

Now, we introduce some interesting properties of ternary relations that we need through this thesis.

Due to definitions of c_i -transitive, for all $i \in \{1, \dots, 6\}$, the following result is immediate.

Proposition 10:

Let T be a ternary relation on a set X . The following equivalences hold:

- (i) T is c_1 -transitive if and only if T^t is c_6 -transitive;
- (ii) T is c_3 -transitive if and only if T^t is c_2 -transitive;
- (iii) T is c_5 -transitive if and only if T^t is c_4 -transitive.

Proof. We prove the first equivalence. Assume that T is c_1 -transitive. Let $(x, s, t) \in T^t$ and $(s, y, z) \in T^t$. Then it holds that $(t, s, x) \in T$ and $(z, y, s) \in T$. Since T is c_1 -transitive, it follows that $(z, y, x) \in T$. Hence, $(x, y, z) \in T^t$. Thus, T^t is c_6 -transitive. The converse is similar.

The proofs of equivalences (ii) and (iii) are analogous. □

The following proposition expresses the relationships between the properties of a ternary relation and those of its dual.

Proposition 11:

Let T be a ternary relation on a set X . The following equivalences hold:

- (i) T is reflexive if and only if T^d is irreflexive;
- (ii) T is strongly reflexive if and only if T^d is strongly irreflexive;
- (iii) T is symmetric if and only if T^d is symmetric;
- (iv) T is strongly symmetric if and only if T^d is strongly symmetric;
- (v) T is asymmetric if and only if T^d is complete;
- (vi) T is strongly asymmetric if and only if T^d is strongly complete;
- (vii) T is cyclic if and only if T^d is cyclic;
- (viii)
 - (a) T is c_1 -transitive if and only if T^d is c_6 -negatively transitive,
 - (b) T is c_3 -transitive if and only if T^d is c_2 -negatively transitive,
 - (c) T is c_5 -transitive if and only if T^d is c_4 -negatively transitive.

Proof. We easily verify the equivalences (i)-(ix) as:

- (i) Suppose that T is reflexive and let $x \in X$. It means that $(x, x, x) \in T$. This is equivalent to $(x, x, x) \in T^t$. Hence, $(x, x, x) \notin (T^t)^c = T^d$. Therefore, T^d is irreflexive.
- (ii) The proof is similar to that of (i).
- (iii) Let T be symmetric and suppose that $(x, y, z) \in T^d$ such that $x, y, z \in X$. Since $(x, y, z) \in T^d$, it follows that $(z, y, x) \in T^c$. This implies that $(z, y, x) \notin T$. The fact that T is symmetric, implies that $(z, y, x) \notin T^t$, and hence $(z, y, x) \in (T^t)^c = T^d$. Thus, T^d is symmetric.
- (iv) The proof is similar to that of (iii).
- (v) Suppose that T is asymmetric and let $x, y, z \in X$ such that $\text{card}\{x, y, z\} \geq 2$. It means that $(x, y, z) \in T$ implies $(z, y, x) \notin T$. This is equivalent to $(z, y, x) \notin T^d$, this implies $(x, y, z) \in T^d$. Hence, T^d is complete.
- (vi) The proof is similar to that of (v).
- (vii) Let T is cyclic and suppose that $(x, z, y) \in T$ such that $x, y, z \in X$. Since T is cyclic, it follows that $(z, y, x) \in T$. In other words, $(z, y, x) \notin T$ implies $(x, z, y) \notin T$. It means that $(x, y, z) \in T^d$ implies $(y, z, x) \in T^d$. Hence, T^d is cyclic.
- (viii) We only prove (a), the other equivalences can be proved analogously. Suppose that T^d is c_6 -negatively transitive. This is equivalent to $(T^d)^c = T^t$ is c_6 -transitive, which is equivalent from proposition 10 to T is c_1 -transitive.

□

Proposition 12:

Let T be a ternary relation on a set X . Then it holds that T is strongly reflexive and c_i -transitive for any $i \in \{1, \dots, 6\}$ if and only if $T = X^3$.

Proof. We will show only that if T is strongly reflexive and c_1 -transitive, then $T = X^3$. The other cases can be proved analogously. Let $(x, y, z) \in X^3$. Since T is strongly reflexive, it follows that $(x, y, y) \in T$ and $(y, y, z) \in T$. From the c_1 -transitivity of T , it follows that $(x, y, z) \in T$. Hence, $X^3 \subseteq T$ and thus $T = X^3$. Conversely, it trivially holds that X^3 is strongly reflexive and c_i -transitive for any $i \in \{1, \dots, 6\}$. □

Corollary 2:

Let T be a ternary relation on a set X . Then it holds that T is strongly irreflexive and c_i -negatively transitive for any $i \in \{1, \dots, 6\}$ if and only if $T = \emptyset$.

2.3/ Traces of ternary relations

In this section, we review the definitions of traces of a ternary relation introduced in [89], and discuss their properties.

2.3.1/ Definitions and particular cases

Definition 10: [89]

Let T be a ternary relation on a set X .

- (i) The left trace of T is the binary relation T^ℓ on X defined as

$$T^\ell = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((x, a, b) \in T \Rightarrow (y, a, b) \in T)\};$$

- (ii) The middle trace of T is the binary relation T^m on X defined as

$$T^m = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((a, x, b) \in T \Rightarrow (a, y, b) \in T)\};$$

- (iii) The right trace of T is the binary relation T^r on X defined as

$$T^r = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((a, b, x) \in T \Rightarrow (a, b, y) \in T)\}.$$

It is important to note that the traces of a ternary relation are defined as binary relations.

The following remark and proposition discuss the traces of some particular ternary relations.

Remark 1:

Let T be a ternary relation on a set X . The following statements hold:

- (i) If $T = X^3$ or $T = \emptyset$, then $T^\ell = T^m = T^r = X^2$.
(ii) If $T = I_{X^3}$, then $T^\ell = T^m = T^r = I_{X^2} = \{(x, x) \mid x \in X\}$.

Proposition 13:

Let T be a ternary relation on a set X . If $T^\ell = T^m = T^r = X^2$, then $T = X^3$ or $T = \emptyset$.

Proof. Suppose that $T^\ell = T^m = T^r = X^2$. If we suppose that $T \neq \emptyset$, then we need to show that $T = X^3$. Since $T \neq \emptyset$, there exists $(a, b, c) \in T$. Let $x, y, z \in X$, then the fact that $T^\ell = T^m = T^r = X^2$ leads to the following statements:

- (i) $(a, x) \in T^\ell$ and $(a, b, c) \in T$ imply that $(x, b, c) \in T$,
(ii) $(b, y) \in T^m$ and $(x, b, c) \in T$ imply that $(x, y, c) \in T$ and
(iii) $(c, z) \in T^r$ and $(x, y, c) \in T$ imply that $(x, y, z) \in T$.

Hence, $X^3 \subseteq T$. Thus, $T = X^3$. □

2.3.2/ Properties of the traces of a ternary relation

Proposition 14:

Let T be a ternary relation on a set X , then the traces T^ℓ , T^m and T^r are reflexive and transitive.

Proof. We only prove that T^ℓ is reflexive and transitive. The other cases can be proved analogously.

It trivially holds that T^ℓ is reflexive. Next, we will show that T^ℓ is transitive. Indeed, let $x, y, z \in X$ such that $(x, y), (y, z) \in T^\ell$. Let $a, b \in X$ such that $(x, a, b) \in T$. The fact that $(x, y), (y, z) \in T^\ell$ implies that $(y, a, b) \in T$, and hence $(z, a, b) \in T$. Thus, $(x, z) \in T^\ell$. Therefore, T^ℓ is transitive. $\square$

Combining Proposition 14 and Corollary 1 leads to the following corollary.

Corollary 3:

Let T be a ternary relation on a set X . The following statements hold:

- (i) $T^\ell = ((T^\ell)^\ell)^t = (T^\ell)^r$;
- (ii) $T^m = ((T^m)^\ell)^t = (T^m)^r$;
- (iii) $T^r = ((T^r)^\ell)^t = (T^r)^r$.

The following proposition expresses the traces of the transpose, the complement and the dual of a ternary relation in terms of the traces of that relation.

Proposition 15:

Let T be a ternary relation on a set X . The left, middle and right traces of its transpose T^t , its complement T^c and its dual T^d are listed in the following table:

	$(.)^\ell$	$(.)^m$	$(.)^r$
T^t	T^r	T^m	T^ℓ
T^c	$(T^\ell)^t$	$(T^m)^t$	$(T^r)^t$
T^d	$(T^r)^t$	$(T^m)^t$	$(T^\ell)^t$

Proof. We only prove the equalities in the first column. The results in the second and the third column can be proved similarly.

- (i) By definition,

$$(T^t)^\ell = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((x, a, b) \in T^t \Rightarrow (y, a, b) \in T^t)\}.$$

Since $((x, a, b) \in T^t \Leftrightarrow (b, a, x) \in T)$ and $((y, a, b) \in T^t \Leftrightarrow (b, a, y) \in T)$, it follows that

$$(T^t)^\ell = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((b, a, x) \in T \Rightarrow (b, a, y) \in T)\} = T^r.$$

(ii) By definition,

$$(T^c)^\ell = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((x, a, b) \in T^c \Rightarrow (y, a, b) \in T^c)\}.$$

Since $((x, a, b) \in T^c \Rightarrow (y, a, b) \in T^c)$ is equivalent to $((y, a, b) \in T \Rightarrow (x, a, b) \in T)$, it follows that

$$(T^c)^\ell = \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((y, a, b) \in T \Rightarrow (x, a, b) \in T)\} = (T^\ell)^t.$$

(iii) Follows from (i) and (ii). □

The following proposition discusses the traces of the ternary relations obtained by permutation of a ternary relation.

Proposition 16:

Let T be a ternary relation on a set X . The left, middle and right traces of the corresponding ternary relations obtained by permutation are listed in the following table:

	$(.)^\ell$	$(.)^m$	$(.)^r$
$T^\perp$	T^ℓ	T^r	T^m
$T^\vdash$	T^m	T^ℓ	T^r
T^+	T^r	T^ℓ	T^m
T^-	T^m	T^r	T^ℓ

Proof. We only prove that $(T^\perp)^\ell = T^\ell$. The other results can be proved analogously using Proposition 8. Since $((x, a, b) \in T^\perp \Rightarrow (y, a, b) \in T^\perp)$ is equivalent to $((x, b, a) \in T \Rightarrow (y, b, a) \in T)$, it follows that

$$\begin{aligned} (T^\perp)^\ell &= \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((x, a, b) \in T^\perp \Rightarrow (y, a, b) \in T^\perp)\} \\ &= \{(x, y) \in X^2 \mid (\forall (a, b) \in X^2)((x, b, a) \in T \Rightarrow (y, b, a) \in T)\} \\ &= T^\ell. \end{aligned}$$
□

Proposition 17:

For any family of ternary relations $(T_i)_{i \in I}$ on a set X , the following statements hold:

- (i) $\bigcap_{i \in I} T_i^\ell \subseteq (\bigcap_{i \in I} T_i)^\ell$, $\bigcap_{i \in I} T_i^m \subseteq (\bigcap_{i \in I} T_i)^m$ and $\bigcap_{i \in I} T_i^r \subseteq (\bigcap_{i \in I} T_i)^r$;
- (ii) $\bigcap_{i \in I} T_i^\ell \subseteq (\bigcup_{i \in I} T_i)^\ell$, $\bigcap_{i \in I} T_i^m \subseteq (\bigcup_{i \in I} T_i)^m$ and $\bigcap_{i \in I} T_i^r \subseteq (\bigcup_{i \in I} T_i)^r$.

Proof. We only prove that $\bigcap_{i \in I} T_i^\ell \subseteq (\bigcap_{i \in I} T_i)^\ell$. The other inclusions can be proved analogously.

Let $(x, y) \in \bigcap_{i \in I} T_i^\ell$ and $(x, a, b) \in \bigcap_{i \in I} T_i$. It holds that $(x, a, b) \in T_i$, for any $i \in I$. Since $(x, y) \in T_i^\ell$, it holds that $(y, a, b) \in T_i$ for any $i \in I$. Hence, $(x, y) \in (\bigcap_{i \in I} T_i)^\ell$. Thus, $\bigcap_{i \in I} T_i^\ell \subseteq (\bigcap_{i \in I} T_i)^\ell$. $\square$

2.4/ Interaction of the traces with projections and cylindrical extensions

In this section, we investigate the interaction of the traces with the projections and the cylindrical extensions.

2.4.1/ Interaction with binary projections

The binary projections of a ternary relation are introduced in the following definition.

Definition 11:

Let T be a ternary relation on a set X .

- (i) The left projection of T is the binary relation $P_\ell^\vee(T)$ on X defined as

$$P_\ell^\vee(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T)\};$$

- (ii) The middle projection of T is the binary relation $P_m^\vee(T)$ on X defined as

$$P_m^\vee(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, z, y) \in T)\};$$

- (iii) The right projection of T is the binary relation $P_r^\vee(T)$ on X defined as

$$P_r^\vee(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, y, z) \in T)\}.$$

In the following definition, we consider a more restrictive version of the projections, called strong projections.

Definition 12:

Let T be a ternary relation on a set X .

- (i) The left strong projection of T is the binary relation $P_\ell^\wedge(T)$ on X defined as

$$P_\ell^\wedge(T) = \{(x, y) \in X^2 \mid (\forall z \in X)((z, x, y) \in T)\};$$

- (ii) The middle strong projection of T is the binary relation $P_m^\wedge(T)$ on X defined as

$$P_m^\wedge(T) = \{(x, y) \in X^2 \mid (\forall z \in X)((x, z, y) \in T)\};$$

- (iii) The right strong projection of T is the binary relation $P_r^\wedge(T)$ on X defined as

$$P_r^\wedge(T) = \{(x, y) \in X^2 \mid (\forall z \in X)((x, y, z) \in T)\}.$$

The following results are immediate.

Proposition 18:

Let T be a ternary relation on a set X . The left, middle and right (strong) projections of its transpose T^t , its complement T^c and its dual T^d are listed in the following tables:

(i)		$P_\ell^\vee(\cdot)$	$P_m^\vee(\cdot)$	$P_r^\vee(\cdot)$
	T^t	$(P_r^\vee(T))^t$	$(P_m^\vee(T))^t$	$(P_\ell^\vee(T))^t$
	T^c	$(P_\ell^\wedge(T))^c$	$(P_m^\wedge(T))^c$	$(P_r^\wedge(T))^c$
	T^d	$(P_r^\wedge(T))^d$	$(P_m^\wedge(T))^d$	$(P_\ell^\wedge(T))^d$

(ii)		$P_\ell^\wedge(\cdot)$	$P_m^\wedge(\cdot)$	$P_r^\wedge(\cdot)$
	T^t	$(P_r^\wedge(T))^t$	$(P_m^\wedge(T))^t$	$(P_\ell^\wedge(T))^t$
	T^c	$(P_\ell^\vee(T))^c$	$(P_m^\vee(T))^c$	$(P_r^\vee(T))^c$
	T^d	$(P_r^\vee(T))^d$	$(P_m^\vee(T))^d$	$(P_\ell^\vee(T))^d$

Proposition 19:

Let T be a ternary relation on a set X , then it holds that

- (i) $T^\ell \subseteq (P_m^\vee(T))^\ell$;
- (ii) $T^m \subseteq (P_\ell^\vee(T))^\ell \cap (P_r^\vee(T))^r$;
- (iii) $T^r \subseteq (P_m^\vee(T))^r$.

Proof.

- (i) Let $(x, y) \in T^\ell$ and $(x, t) \in P_m^\vee(T)$. It holds that $(\exists s \in X)((x, s, t) \in T)$. Since $(x, y) \in T^\ell$, it follows that $(y, s, t) \in T$. This implies that $(y, t) \in P_m^\vee(T)$. Hence, $(x, y) \in (P_m^\vee(T))^\ell$. Thus, $T^\ell \subseteq (P_m^\vee(T))^\ell$.
- (ii) Let $(x, y) \in T^m$ and $(x, t) \in P_\ell^\vee(T)$. It holds that $(\exists s \in X)((s, x, t) \in T)$. Since $(x, y) \in T^m$, it holds that $(s, y, t) \in T$. This implies that $(y, t) \in P_\ell^\vee(T)$. Hence, $(x, y) \in (P_\ell^\vee(T))^\ell$. Thus, $T^m \subseteq (P_\ell^\vee(T))^\ell$. In a similar way, we obtain that $T^m \subseteq (P_r^\vee(T))^r$. Thus, $T^m \subseteq (P_\ell^\vee(T))^\ell \cap (P_r^\vee(T))^r$.

- (iii) Follows from Proposition 15, (i) and Proposition 18.

□

Proposition 20:

Let T be a ternary relation on a set X , then it holds that

- (i) $T^\ell \subseteq (P_m^\wedge(T))^\ell$;
- (ii) $T^m \subseteq (P_\ell^\wedge(T))^\ell \cap (P_r^\wedge(T))^r$;
- (iii) $T^r \subseteq (P_m^\wedge(T))^r$.

Proof. The proof is similar to that of Proposition 19.

□

2.4.2/ Interaction with cylindrical extensions

Definition 13:

Let R be a binary relation on a set X .

- (i) The left cylindrical extension of R is the ternary relation $C_\ell(R)$ on X defined as

$$C_\ell(R) = \{(x, y, z) \in X^3 \mid (y, z) \in R\};$$

- (ii) The middle cylindrical extension of R is the ternary relation $C_m(R)$ on X defined as

$$C_m(R) = \{(x, y, z) \in X^3 \mid (x, z) \in R\};$$

- (iii) The right cylindrical extension of R is the ternary relation $C_r(R)$ on X defined as

$$C_r(R) = \{(x, y, z) \in X^3 \mid (x, y) \in R\}.$$

The following proposition follows immediately from the above definition.

Proposition 21:

Let R be a binary relation on a set X . The left, middle and right cylindrical extensions of its transpose R^t , its complement R^c and its dual R^d are listed in the following table:

	$C_\ell(\cdot)$	$C_m(\cdot)$	$C_r(\cdot)$
R^t	$(C_r(R))^t$	$(C_m(R))^t$	$(C_\ell(R))^t$
R^c	$(C_\ell(R))^c$	$(C_m(R))^c$	$(C_r(R))^c$
R^d	$(C_r(R))^d$	$(C_m(R))^d$	$(C_\ell(R))^d$

The following proposition shows that any binary relation R coincides with the strong projections of its cylindrical extensions. Also, any ternary relation T coincides with the cylindrical extensions of its strong projections.

Proposition 22:

Let R be a binary relation on a set X . The following equalities hold:

- (i) $R = P_\ell^\wedge(C_\ell(R)) = P_m^\wedge(C_m(R)) = P_r^\wedge(C_r(R));$
(ii) $T = C_\ell(P_\ell^\wedge(T)) = C_m(P_m^\wedge(T)) = C_r(P_r^\wedge(T)).$

Proof.

- (i) We only prove that $R = P_\ell^\wedge(C_\ell(R))$, as the other equalities can be proved analogously.

$$\begin{aligned} P_\ell^\wedge(C_\ell(R)) &= \{(x, y) \in X^2 \mid (\forall z \in X)((z, x, y) \in C_\ell(R))\} \\ &= \{(x, y) \in X^2 \mid (x, y) \in R\} = R. \end{aligned}$$

- (ii) We only prove that $T = C_\ell(P_\ell^\wedge(T))$, as the other equalities can be proved analogously.

$$\begin{aligned} C_\ell(P_\ell^\wedge(T)) &= \{(x, y, z) \in X^3 \mid (y, z) \in P_\ell^\wedge(T)\} \\ &= \{(x, y, z) \in X^3 \mid (x, y, z) \in T\} \\ &= T. \end{aligned}$$

□

Proposition 23:

Let R be a binary relation on a set X , then it holds that

- (i) $R^\ell = (C_r(R))^\ell = (C_m(R))^\ell = (C_\ell(R))^m$;
- (ii) $R^r = (C_\ell(R))^r = (C_m(R))^r = (C_r(R))^m$.

Proof.

- (i) We only prove the equality $R^\ell = (C_r(R))^\ell$, as the other equalities can be proved analogously. Let $(x, y) \in R^\ell$, then we need to show that $(x, y) \in (C_r(R))^\ell$. Let $(x, a, b) \in C_r(R)$, then it holds that $(x, a) \in R$. Since $(x, y) \in R^\ell$, it follows that $(y, a) \in R$. This implies that $(y, a, b) \in C_r(R)$. Hence, $(x, y) \in (C_r(R))^\ell$. Thus, $R^\ell \subseteq (C_r(R))^\ell$.
Conversely, let $(x, y) \in (C_r(R))^\ell$ and $(x, a) \in R$, then it follows that $(x, a, b) \in C_r(R)$, for any $b \in X$. Since $(x, y) \in (C_r(R))^\ell$, it follows that $(y, a, b) \in C_r(R)$, for any $b \in X$. Hence, $(y, a) \in R$. Thus, $(x, y) \in R^\ell$. Therefore, $R^\ell = (C_r(R))^\ell$.
- (ii) Follows from Proposition 9, (i), Propositions 21 and 15.

□

2.5/ Compositions of ternary and binary relations and their interaction with the traces

In this section, we recall the definition of four types of composition of a ternary relation with a binary relation and vice versa. These compositions will turn out to be intimately linked to the traces introduced before.

Definition 14: [89]

Let T be a ternary relation and R be a binary relation on a set X .

- (i) The $\ltimes$ -composition of T and R is the ternary relation $T \ltimes R$ on X defined as

$$T \ltimes R = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T \wedge (t, z) \in R)\};$$
- (ii) The $\ltimes$ -composition of R and T is the ternary relation $R \ltimes T$ on X defined as

$$R \ltimes T = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t) \in R \wedge (t, y, z) \in T)\};$$
- (iii) The $\blacktriangleleft$ -composition of T and R is the ternary relation $T \blacktriangleleft R$ on X defined as

$$T \blacktriangleleft R = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t, z) \in T \wedge (t, y) \in R)\};$$
- (iv) The $\blacktriangleleft$ -composition of R and T is the ternary relation $T \blacktriangleleft R$ on X defined as

$$R \blacktriangleleft T = \{(x, y, z) \in X^3 \mid (\exists t \in X)((y, t) \in R \wedge (x, t, z) \in T)\}.$$

The following proposition lists some properties of the compositions of a ternary relation with a binary relation.

Proposition 24:

Let T, T_1 and T_2 be three ternary relations on a set X and R, R_1 and R_2 be three binary relations on X . The following statements hold:

- (i) if $R_1 \subseteq R_2$ and $T_1 \subseteq T_2$, then
 - (a) $T_1 \ltimes R_1 \subseteq T_2 \ltimes R_2$ and $R_1 \rtimes T_1 \subseteq R_2 \rtimes T_2$;
 - (b) $T_1 \ltimes R_1 \subseteq T_2 \ltimes R_2$ and $R_1 \rtimes T_1 \subseteq R_2 \rtimes T_2$;
- (ii) $(T \ltimes R)^t = R^t \rtimes T^t$;
- (iii) $(R \rtimes T)^t = T^t \ltimes R^t$;
- (iv) $T \ltimes R = R^t \rtimes T$;
- (v) $(T_1 \cup T_2) \ltimes R = (T_1 \ltimes R) \cup (T_2 \ltimes R)$ and $(T_1 \cap T_2) \ltimes R = (T_1 \ltimes R) \cap (T_2 \ltimes R)$;
- (vi) $(R_1 \cup R_2) \rtimes T = (R_1 \rtimes T) \cup (R_2 \rtimes T)$ and $(R_1 \cap R_2) \rtimes T = (R_1 \rtimes T) \cap (R_2 \rtimes T)$;
- (vii) $(R_1 \cup R_2) \rtimes T = (R_1 \rtimes T) \cup (R_2 \rtimes T)$ and $(R_1 \cap R_2) \rtimes T = (R_1 \rtimes T) \cap (R_2 \rtimes T)$;
- (viii) $(T_1 \cup T_2) \ltimes R = (T_1 \ltimes R) \cup (T_2 \ltimes R)$ and $(T_1 \cap T_2) \ltimes R = (T_1 \ltimes R) \cap (T_2 \ltimes R)$;
- (ix) $(T \ltimes R_1) \ltimes R_2 = T \ltimes (R_1 \circ R_2)$;
- (x) $R_1 \rtimes (R_2 \rtimes T) = (R_1 \circ R_2) \rtimes T$;
- (xi) $(T \ltimes R_1) \ltimes R_2 = T \ltimes (R_1 \circ R_2)$.

Proof.

- (i) Suppose that $R_1 \subseteq R_2$ and $T_1 \subseteq T_2$. We only prove that $T_1 \ltimes R_1 \subseteq T_2 \ltimes R_2$, as the other cases can be proved analogously. Let $(x, y, z) \in T_1 \ltimes R_1$, then $(\exists t \in X)((x, y, t) \in T_1 \wedge (t, z) \in R_1)$. Since $R_1 \subseteq R_2$ and $T_1 \subseteq T_2$, it follows that $(\exists t \in X)((x, y, t) \in T_2 \wedge (t, z) \in R_2)$. Hence, $(x, y, z) \in T_2 \ltimes R_2$. Thus, $T_1 \ltimes R_1 \subseteq T_2 \ltimes R_2$.

- (ii) We easily verify that

$$\begin{aligned}
 (T \ltimes R)^t &= \{(x, y, z) \in X^3 \mid (z, y, x) \in T \ltimes R\} \\
 &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((z, y, t) \in T \wedge (t, x) \in R)\} \\
 &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((t, y, z) \in T^t \wedge (x, t) \in R^t)\} \\
 &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, t) \in R^t \wedge (t, y, z) \in T^t)\} \\
 &= R^t \rtimes T^t.
 \end{aligned}$$

- (iii) Analogous to (ii).

(iv) Applying a similar reasoning, we find

$$\begin{aligned}
(T \ltimes R)^t &= \{(x, y, z) \in X^3 \mid (z, y, x) \in T \ltimes R\} \\
&= \{(x, y, z) \in X^3 \mid (\exists t \in X)((z, t, x) \in T \wedge (t, y) \in R)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists t \in X)((z, t, x) \in T \wedge (y, t) \in R^t)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists t \in X)((y, t) \in R^t \wedge (z, t, x) \in T)\} \\
&= R^t \rtimes T.
\end{aligned}$$

(v) Similarly,

$$\begin{aligned}
(T_1 \cup T_2) \ltimes R &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T_1 \cup T_2 \wedge (t, z) \in R)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists t \in X) \\
&\quad ((x, y, t) \in T_1 \wedge (t, z) \in R) \vee ((x, y, t) \in T_2 \wedge (t, z) \in R)\} \\
&= (T_1 \ltimes R) \cup (T_2 \ltimes R).
\end{aligned}$$

$$\begin{aligned}
(T_1 \cap T_2) \ltimes R &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T_1 \cap T_2 \wedge (t, z) \in R)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists t \in X) \\
&\quad ((x, y, t) \in T_1 \wedge (t, z) \in R) \wedge ((x, y, t) \in T_2 \wedge (t, z) \in R)\} \\
&= (T_1 \ltimes R) \cap (T_2 \ltimes R).
\end{aligned}$$

(vi) - (viii) analogous to (v).

(ix) Finally,

$$\begin{aligned}
(T \ltimes R_1) \ltimes R_2 &= \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in T \ltimes R_1 \wedge (t, z) \in R_2)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists s, t \in X)((x, y, s) \in T \wedge (s, t) \in R_1 \wedge (t, z) \in R_2)\} \\
&= \{(x, y, z) \in X^3 \mid (\exists s, t \in X)((x, y, s) \in T \wedge ((s, t) \in R_1 \wedge (t, z) \in R_2))\} \\
&= \{(x, y, z) \in X^3 \mid (\exists s \in X)((x, y, s) \in T \wedge (s, z) \in R_1 \circ R_2)\} \\
&= T \ltimes (R_1 \circ R_2).
\end{aligned}$$

The proofs of (x) and (xi) are analogous to that of (ix). $\square$

The following result shows that the traces of a ternary relation are the greatest binary relations that satisfy the following relational inequalities corresponding to the relational compositions introduced above.

Theorem 5:

Let T be a ternary relation on a set X , then it holds that

- (i) T^ℓ is the greatest binary relation R that satisfies $R^t \rtimes T \subseteq T$;
- (ii) T^m is the greatest binary relation R that satisfies $T \ltimes R = R^t \rtimes T \subseteq T$;
- (iii) T^r is the greatest binary relation R that satisfies $T \ltimes R \subseteq T$.

Proof.

- (i) Let $(x, y, z) \in (T^\ell)^t \bowtie T$, then $(\exists t \in X)((x, t) \in (T^\ell)^t \wedge (t, y, z) \in T)$. Since $(t, x) \in T^\ell$ and $(t, y, z) \in T$, it follows that $(x, y, z) \in T$. Hence, $(T^\ell)^t \bowtie T \subseteq T$.
Next, we show that any binary relation R that satisfies $R^t \bowtie T \subseteq T$ is contained in T^ℓ . Let R be a binary relation that satisfies $R^t \bowtie T \subseteq T$ and consider $(x, y) \in R$. Let $a, b \in X$ such that $(x, a, b) \in T$. Since $(y, x) \in R^t$, it follows that $(y, a, b) \in R^t \bowtie T$. The fact that $R^t \bowtie T \subseteq T$ implies that $(y, a, b) \in T$. Hence, $(x, y) \in T^\ell$. Thus, $R \subseteq T^\ell$.
- (ii) Note that Proposition 24 (iv) implies that $T \blacktriangleleft T^m = (T^m)^t \bowtie T$. Hence, we need to show that $T \blacktriangleleft T^m \subseteq T$. Let $(x, y, z) \in T \blacktriangleleft T^m$, then $(\exists t \in X)((x, t, z) \in T \wedge (t, y) \in T^m)$. This implies that $(x, y, z) \in T$. Hence, $T \blacktriangleleft T^m \subseteq T$. Now, let R be a binary relation that satisfies $T \blacktriangleleft R \subseteq T$ and consider $(x, y) \in R$. Let $a, b \in X$ such that $(a, x, b) \in T$. Since $(x, y) \in R$, it follows that $(a, y, b) \in T \blacktriangleleft R \subseteq T$. Hence, $(x, y) \in T^m$. Thus, $R \subseteq T^m$.
- (iii) Follows from Propositions 15 and 24, and (i).

□

The following theorem expresses that the greatest solutions of the relational inequalities in Theorem 5 are the (greatest) solutions of the corresponding relational equations.

Theorem 6:

Let T be a ternary relation on a set X , then it holds that

- (i) $(T^\ell)^t \bowtie T = T \bowtie T^r = T$;
- (ii) $T \blacktriangleleft T^m = (T^m)^t \bowtie T = T$.

Proof.

- (i) From Theorem 5, it follows that $(T^\ell)^t \bowtie T \subseteq T$ and $T \bowtie T^r \subseteq T$. Next, we show that $T \subseteq (T^\ell)^t \bowtie T$ and $T \subseteq T \bowtie T^r$. Let $(x, y, z) \in T$. Since T^ℓ and T^r are reflexive, it holds that $(x, x) \in T^\ell$ and $(z, z) \in T^r$, for any $x, z \in X$. The fact that $(x, y, z) \in T$ and $(x, x) \in (T^\ell)^t$ implies that $(x, y, z) \in (T^\ell)^t \bowtie T$. Hence, $T \subseteq (T^\ell)^t \bowtie T$. Thus, $(T^\ell)^t \bowtie T = T$. In the same way, the fact that $(x, y, z) \in T$ and $(z, z) \in T^r$ implies that $(x, y, z) \in T \bowtie T^r$. Hence, $T \subseteq T \bowtie T^r$. Thus, $T \bowtie T^r = T$.
- (ii) Theorem 5 guarantees that $T \blacktriangleleft T^m = T^m \bowtie T \subseteq T$. Next, we show that $T \subseteq T \blacktriangleleft T^m$. Let $(x, y, z) \in T$. Since T^m is reflexive, it holds that $(y, y) \in T^m$, for any $y \in X$. Consequently, $(x, y, z) \in T \blacktriangleleft T^m$. Hence, $T \subseteq T \blacktriangleleft T^m$. Thus, $T = T \blacktriangleleft T^m$.

□

Proposition 25:

Let T be a ternary relation on a set X and R be a binary relation on X . The following statements hold:

- (i) $T^\ell \subseteq (T \ltimes R)^\ell$, $T^m \subseteq (T \ltimes R)^m \cap (R \ltimes T)^m$ and $T^r \subseteq (R \ltimes T)^r$;
- (ii) $T^\ell \subseteq (T \ltimes R)^\ell \cap (R \ltimes T)^\ell$, $T^m \subseteq (T \ltimes R)^m \cap (R \ltimes T)^m$ and $T^r \subseteq (R \ltimes T)^r \cap (R \ltimes T)^r$;
- (iii) $R^\ell \subseteq (R \ltimes T)^\ell$ and $R^r \subseteq (T \ltimes R)^r$;
- (iv) $R^\ell \subseteq (R \ltimes T)^m$ and $R^r \subseteq (T \ltimes R)^m$.

Proof. We only prove that $T^\ell \subseteq (T \ltimes R)^\ell$. The other inclusions can be proved analogously or by using Propositions 15, 16 and 24, and (i). Let $(x, y) \in T^\ell$ and let $(x, a, b) \in T \ltimes R$. Then $(\exists t \in X)((x, a, t) \in T \wedge (t, b) \in R)$. Since $(x, y) \in T^\ell$, it follows that $(\exists t \in X)((y, a, t) \in T \wedge (t, b) \in R)$. This implies that $(y, a, b) \in T \ltimes R$. Hence, $(x, y) \in (T \ltimes R)^\ell$. Thus, $T^\ell \subseteq (T \ltimes R)^\ell$. $\square$

2.6/ Properties of a ternary relation in terms of its traces

In this section, we show how the traces of a given ternary relation can be used to study the properties of this relation. To that end, we associate the following binary relations with a given ternary relation T :

- (i) $W_\ell(T) = \{(x, y) \in X^2 \mid (x, y, y) \in T\}$;
- (ii) $W_m(T) = \{(x, y) \in X^2 \mid (x, y, x) \in T\}$;
- (iii) $W_r(T) = \{(x, y) \in X^2 \mid (x, x, y) \in T\}$.

Remark 2:

Note that the binary relation $W_m(T)$ can also be defined as

$$W_m(T) = \{(x, y) \in X^2 \mid (y, x, y) \in T\}.$$

The following proposition can easily be verified.

Proposition 26:

Let T be a ternary relation on a set X .

- (i) The following table expresses $W_\ell(\cdot)$, $W_m(\cdot)$ and $W_r(\cdot)$ of its transpose T^t , its complement T^c and its dual T^d :

	$W_\ell(\cdot)$	$W_m(\cdot)$	$W_r(\cdot)$
T^t	$(W_r(T))^t$	$W_m(T)$	$(W_\ell(T))^t$
T^c	$(W_\ell(T))^c$	$(W_m(T))^c$	$(W_r(T))^c$
T^d	$(W_r(T))^d$	$(W_m(T))^c$	$(W_\ell(T))^d$

- (ii) The following table expresses $W_\ell(\cdot)$, $W_m(\cdot)$ and $W_r(\cdot)$ of the permutations of T :

	$W_\ell(\cdot)$	$W_m(\cdot)$	$W_r(\cdot)$
T^{-1}	$W_\ell(T)$	$W_r(T)$	$W_m(T)$
$T^\perp$	$(W_m(T))^t$	$(W_\ell(T))^t$	$W_r(T)$
T^+	$(W_r(T))^t$	$(W_\ell(T))^t$	$W_m(T)$
T^-	$(W_m(T))^t$	$W_r(T)$	$(W_\ell(T))^t$

Proposition 27:

Let T be a ternary relation on a set X . The following statements are equivalent:

- (i) T is reflexive;
- (ii) $W_\ell(T)$ is reflexive;
- (iii) $W_m(T)$ is reflexive;
- (iv) $W_r(T)$ is reflexive.

Proof. We only prove (i) $\Leftrightarrow$ (ii), as the other equivalences can be proved similarly. Since T is reflexive, it holds that $(x, x, x) \in T$, for any $x \in X$. Hence, $(x, x) \in W_\ell(T)$, for any $x \in X$. Thus, $W_\ell(T)$ is reflexive. Conversely, since $(x, x) \in W_\ell(T)$, for any $x \in X$, it holds that $(x, x, x) \in T$, for any $x \in X$. Thus, T is reflexive. $\square$

Proposition 28:

Let T be a ternary relation on a set X , then it holds that

- (i) $T^\ell \subseteq (W_\ell(T))^\ell$;
- (ii) $T^m \subseteq ((W_m(T))^t)^\ell = (W_m(T))^r$;
- (iii) $T^r \subseteq (W_r(T))^r$.

Proof.

- (i) Let $(x, y) \in T^\ell$. Suppose that $(x, a) \in W_\ell(T)$, then $(x, a, a) \in T$. Since $(x, y) \in T^\ell$, it holds that $(y, a, a) \in T$. Hence, $(y, a) \in W_\ell(T)$. Thus, $(x, y) \in (W_\ell(T))^\ell$. Therefore, $T^\ell \subseteq (W_\ell(T))^\ell$.

Statements (ii) and (iii) follow from Proposition 16, (i) and Proposition 26. $\square$

The following result characterizes the reflexivity of a ternary relation in terms of its traces.

Theorem 7:

Let T be a ternary relation on a set X . The following statements are equivalent:

- (i) T is reflexive;
- (ii) $(T^\ell)^t \subseteq W_\ell(T)$;
- (iii) $T^m \subseteq W_m(T)$;
- (iv) $T^r \subseteq W_r(T)$.

Proof. (i) $\Rightarrow$ (ii): From Proposition 28 it follows that $T^\ell \subseteq (W_\ell(T))^\ell$. Since T is reflexive, it follows from Proposition 27 that $W_\ell(T)$ is reflexive. Theorem 1 guarantees that $((W_\ell(T))^\ell)^t \subseteq W_\ell(T)$. Hence, $(T^\ell)^t \subseteq W_\ell(T)$.

(ii) $\Rightarrow$ (iii): From Proposition 16, it follows that $T^m = (T^\perp)^\ell$. Since $T^\perp$ is reflexive, it follows from (ii) that $((T^\perp)^\ell)^t \subseteq W_\ell(T^\perp)$. Hence, $T^m \subseteq (W_\ell(T^\perp))^t$. Proposition 26 guarantees that $T^m \subseteq W_m(T)$.

(iii) $\Rightarrow$ (iv): Follows from Propositions 16 and 26, and (iii).

(iv) $\Rightarrow$ (i): From Proposition 14 we know that T^r is reflexive. Since $T^r \subseteq W_r(T)$, it holds that $W_r(T)$ is reflexive. Proposition 27 then guarantees that T is reflexive. $\square$

The following result expresses the transitivity of a ternary relation in terms of its traces.

Theorem 8:

Let T be a ternary relation on a set X . It holds that

- (i) If T is c_2 -transitive, then $W_\ell(T) \subseteq (T^\ell)^t$;
- (ii) If T is c_1 - and c_6 -transitive, then $W_m(T) \subseteq T^m$;
- (iii) If T is c_1 -transitive, then $W_r(T) \subseteq T^r$.

Proof.

- (i) Let $(x, y) \in W_\ell(T)$, i.e., $(x, y, y) \in T$. Let $a, b \in X$ such that $(y, a, b) \in T$. The c_2 -transitivity of T implies that $(x, a, b) \in T$, for any $a, b \in X$. Hence, $(y, x) \in T^\ell$. Thus, $W_\ell(T) \subseteq (T^\ell)^t$.

The proofs of statements (ii) and (iii) are analogous to that of (i). $\square$

In general, the converse of above result does not hold, as can be seen in the following example.

Example 5:

Let T be a ternary relation on $X = \{x, y, z, t\}$, given by:

$$T = \{(y, y, y), (y, t, t), (t, y, y), (z, x, y), (z, t, x), (t, t, t)\}.$$

- (i) We easily verify that $T^\ell = \{(x, x), (y, y), (z, z), (t, t), (y, t), (t, y)\}$ and $W_\ell(T) = \{(y, y), (y, t), (t, y), (t, t)\}$. Therefore, $W_\ell(T) \subseteq (T^\ell)^t$. Since the assertion $[(z, x, y) \in T \wedge (y, t, t) \in T] \wedge (z, t, t) \notin T$ is true, it follows that T is not c_2 -transitive. Hence, the converse of (i) is false.
- (ii) Applying a similar reasoning, we find $T^m = \{(x, x), (y, y), (z, z), (t, t), (z, x), (z, y), (z, t)\}$ and $W_m(T) = \{(y, y), (t, t)\}$. Therefore, $W_m(T) \subseteq T^m$. Since the assertion $[(y, t, t) \in T \wedge (z, t, x) \in T] \wedge (y, t, x) \notin T$ is true, it follows that T is not c_1 -transitive. Hence, the converse of (ii) is false. Similarly, it follows that T is not c_6 -transitive.
- (iii) Similarly, we have $T^r = T^m$ and $W_r(T) = \{(y, y), (t, t)\}$. Therefore, $W_r(T) \subseteq T^r$. Since the assertion $[(y, t, t) \in T \wedge (z, t, x) \in T] \wedge (y, t, x) \notin T$ is true, it follows that T is not c_1 -transitive. Hence, the converse of (iii) is false.

Combining Proposition 10, Theorem 8, Propositions 26 and 15 leads to the following result.

Corollary 4:

Let T be a ternary relation on a set X . It holds that

- (i) If T is c_3 -transitive, then $W_r(T) \subseteq T^r$;
- (ii) If T is c_6 -transitive, then $W_\ell(T) \subseteq (T^\ell)^t$.

Combining Theorems 7 and 8, and Corollary 4 leads to the following result.

Corollary 5:

Let T be a ternary relation on a set X . It holds that

- (i) If T is reflexive and c_2 - or c_6 -transitive, then $(T^\ell)^t = W_\ell(T)$;
- (ii) If T is reflexive and c_1 - and c_6 -transitive, then $T^m = W_m(T)$;
- (iii) If T is reflexive and c_1 - or c_3 -transitive, then $T^r = W_r(T)$.

The following result characterizes the strong reflexivity of a ternary relation in terms of its traces.

Theorem 9:

Let T be a ternary relation on a set X . The following statements are equivalent:

- (i) T is strongly reflexive;
- (ii) $C_\ell(T^r) \cup C_m((T^\ell)^t \cup T^r) \cup C_r((T^\ell)^t) \subseteq T$.

Proof. We only prove that $C_\ell(T^r) \subseteq T$, as the other inclusions $C_m((T^\ell)^t \cup T^r) \subseteq T$ and

$G_r((T^\ell)^t) \subseteq T$ can be proved similarly. Let $(x, y, z) \in G_\ell(T^r)$, then $(y, z) \in T^r$. Since T is strongly reflexive, it holds that $(x, y, y) \in T$. The fact that $(y, z) \in T^r$ then implies that $(x, y, z) \in T$. Thus, $G_\ell(T^r) \subseteq T$.

Conversely, we know from Proposition 14 that T^ℓ and T^r are reflexive, i.e., $(x, x) \in T^\ell \cap T^r$, for any $x \in X$. Then it follows that $(z, x, x) \in G_\ell(T^r)$, $(x, z, x) \in G_m((T^\ell)^t \cup T^r)$ and $(x, x, z) \in G_r((T^\ell)^t)$, for any $x, z \in X$. Since $G_\ell(T^r) \cup G_m((T^\ell)^t \cup T^r) \cup G_r((T^\ell)^t) \subseteq T$, it holds that $(x, y, z) \in T$ for any $x, y, z \in X$ with $\text{card}\{x, y, z\} \leq 2$. Thus, T is strongly reflexive. $\square$

The following result expresses the symmetry and cyclicity of a ternary relation in terms of its traces.

Proposition 29:

Let T be a ternary relation on a set X . It holds that

- (i) If T is symmetric, then $T^\ell = T^r$.
- (ii) If T is cyclic, then $T^\ell = T^m = T^r$.

Proof.

- (i) Follows from Proposition 15.
- (ii) Follows from Proposition 16.

$\square$

The following result expresses the asymmetry of a ternary relation in terms of its traces.

Theorem 10:

Let T be a ternary relation on a set X . The following statements are equivalent:

- (i) T is asymmetric;
- (ii) $T \ltimes (T^\ell)^t \subseteq T^d$;
- (iii) $T \ltimes (T^m)^t = T^m \rtimes T \subseteq T^d$;
- (iv) $T^r \rtimes T \subseteq T^d$.

Proof. We only prove (i) $\Leftrightarrow$ (ii), as the other equivalences can be proved similarly. Let $(x, y, z) \in T \ltimes (T^\ell)^t$. Then it holds that $(\exists t \in X)((x, y, t) \in T \wedge (t, z) \in (T^\ell)^t)$. Since T is asymmetric, it follows that $(t, y, x) \in T^c$. From Proposition 15 it follows that $(T^\ell)^t = (T^c)^\ell$. Since $(t, y, x) \in T^c$ and $(t, z) \in (T^\ell)^t = (T^c)^\ell$, it follows that $(z, y, x) \in T^c$. Hence, $(x, y, z) \in T^d$. Thus, $T \ltimes (T^\ell)^t \subseteq T^d$. Conversely, let $(x, y, z) \in T$ such that $\text{card}\{x, y, z\} \geq 2$. Since T^ℓ is reflexive, it follows that $(z, z) \in (T^\ell)^t$. Hence, $(x, y, z) \in T \ltimes (T^\ell)^t$. Using the fact that $T \ltimes (T^\ell)^t \subseteq T^d$, it follows that $(x, y, z) \in T^d$. This implies that $(z, y, x) \notin T$. Thus, T is asymmetric. $\square$

Note that in view of Proposition 11, similar results can be formulated for the properties of irreflexivity, strong irreflexivity, c_i -negative transitivity and completeness as can be seen in the following corollaries.

Combining Propositions 11, Propositions 15, 26 and Theorem 7 leads to characterize the irreflexivity of a ternary relation in terms of its traces.

Corollary 6:

Let T be a ternary relation on a set X . The following statements are equivalent:

- (i) T is irreflexive;
- (ii) $(T^\ell)^t \subseteq (W_\ell(T))^d$;
- (iii) $(T^m)^t \subseteq (W_m(T))^c$;
- (iv) $T^r \subseteq (W_r(T))^d$.

Combining Propositions 11, 15 and 21, and Theorem 9 leads to characterize the strong irreflexivity of a ternary relation in terms of its traces.

Corollary 7:

Let T be a ternary relation on a set X . The following statements are equivalent:

- (i) T is strong irreflexive;
- (ii) $G_r(T^\ell) \cup G_m(T^\ell \cup T^r) \cup G_\ell(T^r) \subseteq (T^d)^t$.

Combining Propositions 11, 10, and 15, and Theorem 8 with its Corollary 4 leads to characterize the c_i -negative transitivity of a ternary relation in terms of its traces.

Corollary 8:

Let T be a ternary relation on a set X . The following implications hold:

- (i) If T is c_1 - or c_3 -negatively transitive, then $W_\ell(T^d) \subseteq T^r$;
- (ii) If T is c_2 - or c_6 -negatively transitive, then $W_r(T^d) \subseteq (T^\ell)^t$;
- (iii) If T is c_1 - and c_6 -negatively transitive, then $W_\ell(T^d) \subseteq (T^m)^t$.

Combining Propositions 11, 15 and Theorem 10 gives a characterization of the completeness of a ternary relation in terms of its traces.

Corollary 9:

Let T be a ternary relation on a set X . The following statements are equivalent:

- (i) T is complete;
- (ii) $T^d \ltimes T^r \subseteq T$;
- (iii) $T^d \ltimes T^m = (T^m)^t \ltimes T^d \subseteq T$;
- (iv) $(T^\ell)^t \ltimes T^d \subseteq T$.

3

The clone relations of a ternary relation

In this chapter, we generalize the notions of left and right clone relations [17] of a binary relation to the notions of left, middle and right clone relations of a ternary relation. Subsequently, we investigate the most important of their properties. Importantly, these clone relations played an important role in the study and characterization of the properties of a ternary relation. The interaction of clone relations of a ternary relation with the basic set operations, based on their properties, is studied. The characterization of the clone relations of disjoint union is detailed.

3.1/ The clone relations of a binary relation

In this section, we recall the notion of the left and right clone relations of a binary relation introduced by Bouremel et al. [17].

Definition 15: [17]

Let R be a relation on a set X .

- (i) The left clone relation of R is the binary relation $\approx_R^l$ defined as

$$\approx_R^l = \{(x, y) \in X^2 \mid (\forall z \in X \setminus \{x, y\})((x, z) \in R \Leftrightarrow (y, z) \in R)\};$$

- (ii) The right clone relation of R is the binary relation $\approx_R^r$ defined as

$$\approx_R^r = \{(x, y) \in X^2 \mid (\forall z \in X \setminus \{x, y\})((z, x) \in R \Leftrightarrow (z, y) \in R)\}.$$

The clone relation $\approx_R$ of R is the binary relation on X defined by $\approx_R = \approx_R^l \cap \approx_R^r$.

Proposition 30: [17]

Let R be a relation on a set X , then it holds that $\approx_R^l = \approx_{R^l}^r$.

Proposition 31: [17]

Let R be a relation on a set X . If R is symmetric, then it holds that $\approx_R$ is an equivalence relation.

Proposition 32: [17]

Let R be a relation on a set X . If R is an equivalence relation, then it holds that $R = \approx_R$ if and only if there do not exist $x, y \in X$ such that $x \neq y$, $[x]_R = \{x\}$ and $[y]_R = \{y\}$.

3.2/ The clone relations of a ternary relation

In this section, we introduce the notions of left, middle and right clone relation of a ternary relation and discuss their properties.

3.2.1/ Definition and examples

Definition 16:

Let T be a ternary relation on a set X .

- (i) The left clone relation of T is the binary relation $\approx_T^l$ defined as

$$\approx_T^l = \{(x, y) \in X^2 \mid (\forall (a, b) \in (X \setminus \{x, y\})^2)((x, a, b) \in T \Leftrightarrow (y, a, b) \in T)\};$$

- (ii) The middle clone relation of T is the binary relation $\approx_T^m$ defined as

$$\approx_T^m = \{(x, y) \in X^2 \mid (\forall (a, b) \in (X \setminus \{x, y\})^2)((a, x, b) \in T \Leftrightarrow (a, y, b) \in T)\};$$

- (iii) The right clone relation of T is the binary relation $\approx_T^r$ defined as

$$\approx_T^r = \{(x, y) \in X^2 \mid (\forall (a, b) \in (X \setminus \{x, y\})^2)((a, b, x) \in T \Leftrightarrow (a, b, y) \in T)\}.$$

The clone relation $\approx_T$ of T is the binary relation on X defined by:

$$\approx_T = \approx_T^l \cap \approx_T^m \cap \approx_T^r.$$

If $(x, y) \in \approx_T$, then we say that x and y are clones w.r.t. the relation T .

Example 6:

Let T be a ternary relation on $X = \{x, y, z\}$, given by $T = \{(x, z, z)\}$. The clone relations of T are defined as:

$\approx_T^l = \{(x, x), (y, y), (z, z), (x, z), (z, x), (y, z), (z, y)\}$ and $\approx_T^m = \approx_T^r = X^2$. Hence, $\approx_T = \approx_T^l$.

Example 7:

Let $B_<$ be the strict order-betweenness relation on $X = \{1, 2, 3, 4\}$. It is easy to verify that, $\approx_{B_<}^\ell = \approx_{B_<}^r = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (3, 4)\}$ and $\approx_{B_<}^m = \approx_{B_<}^\ell \cup \{(1, 4)\}$.

The following proposition is straightforward.

Proposition 33:

Let T be a ternary relation on a set X , then for any $\lambda \in \{\ell, m, r\}$, the following implications hold:

- (i) If $T = \emptyset$ or $T = X^3$ or $T = I_{X^3}$, then $\approx_T^\lambda = X^2$;
- (ii) If $\text{card}(X) \leq 2$, then $\approx_T^\lambda = X^2$.

Remark 3:

The converse of the above proposition does not necessarily hold. Indeed, let T be a ternary relation on $X = \{x, y, z\}$, given by $T = \{(x, x, y), (y, y, x), (y, y, z), (x, x, z)\}$. We have, $\approx_T^\lambda = X^2$, for any $\lambda \in \{\ell, m, r\}$, but $T \neq X^3$, $T \neq \emptyset$ and $T \neq I_{X^3}$. Also, $\text{card}(X) > 2$.

3.2.2/ Basic properties of the clone relations of a ternary relation

In this subsection, we study the most relevant properties of the clone relations of a ternary relation T .

It is trivial to prove that the left, middle and right clone relations of any ternary relation T are reflexive and symmetric. Therefore, the following result is straightforward.

Proposition 34:

Let T be a ternary relation on a set X . The left, middle and right clone relations of T are tolerance relations.

In general, the clone relations of T do not need to be equivalence relations as can be seen in the following example.

Example 8:

Let $\mathbb{N}^* = \{1, 2, 3, \dots\}$ be the set of natural numbers with the usual strict order relation $<$ and $B_<$ be the strict order-betweenness relation on X [54]. It is clear that, for any $n \in \mathbb{N}^*$, it holds that n and $n+1$ are left clones, i.e., $(n, n+1) \in \approx_{B_<}^\ell$. However, it holds that $\approx_{B_<}^\ell$ is not transitive. For instance, it holds that $(1, 2) \in \approx_{B_<}^\ell$ and $(2, 3) \in \approx_{B_<}^\ell$, while $(1, 3) \notin \approx_{B_<}^\ell$. In a similar way, we easily verify that the relations $\approx_{B_<}^m$ and $\approx_{B_<}^r$ are not transitive.

Remark 4:

Although the strict order-betweenness $B_<$ is a symmetric relation, but Example 8 asserted that the symmetry of a ternary relation is not a sufficient condition to be its clone relations are equivalence relations, as is known in the binary case (see, Proposition 31).

The following corollary is immediate from the definition of the clone relation $\approx_T$ of T and Proposition 34.

Corollary 10:

Let T be a ternary relation on a set X . The clone relation $\approx_T$ of T is a tolerance relation.

Next, the following proposition shows when the clone relation $\approx_T$ is transitive. In other words, when it is an equivalence relation. To that end, the following definition of projections of T is needed.

Definition 17: [89]

Let T be a ternary relation on a set X .

- (i) The left projection of T is the binary relation $P_\ell(T)$ on X defined as

$$P_\ell(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((z, x, y) \in T)\};$$

- (ii) The middle projection of T is the binary relation $P_m(T)$ on X defined as

$$P_m(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, z, y) \in T)\};$$

- (iii) The right projection of T is the binary relation $P_r(T)$ on X defined as

$$P_r(T) = \{(x, y) \in X^2 \mid (\exists z \in X)((x, y, z) \in T)\}.$$

Proposition 35:

Let T be a ternary relation on a set X . If there do not exist $x, y \in X$ such that $(x, y) \in \approx_T$ and $(x, y) \in \bigcup_{i \in \{\ell, m, r\}} (P_i(T) \cap (P_i(T^c))^t)$, then it holds that $\approx_T$ is an equivalence relation.

Proof. Corollary 10 guarantees that $\approx_T$ is a tolerance relation. Next, it suffices to prove that $\approx_T$ is transitive. Let $x, y, z \in X$ such that $(x, y) \in \approx_T$ and $(y, z) \in \approx_T$. Suppose that $(x, z) \notin \approx_T$, i.e., $(x, z) \notin \approx_T^\ell$ or $(x, z) \notin \approx_T^m$ or $(x, z) \notin \approx_T^r$. We only show the case $(x, z) \notin \approx_T^\ell$ as the other cases can be done similarly.

Let $x, y, z \in X$ such that $(x, y) \in \approx_T^\ell$, $(y, z) \in \approx_T^\ell$ and $(x, z) \notin \approx_T^\ell$. It follows that there exists $a, b \in X \setminus \{x, z\}$ such that $((x, a, b) \in T$ and $(z, a, b) \in T^c)$ or $((x, a, b) \in T^c$ and $(z, a, b) \in T)$.

- (i) Suppose that $(x, a, b) \in T$ and $(z, a, b) \in T^c$. It follows that there are four cases to consider. $(a \neq y \text{ and } b \neq y)$, $(a = y \text{ and } b \neq y)$, $(a \neq y \text{ and } b = y)$ and $(a = b = y)$.
- (a) Let us suppose that $a \neq y$ and $b \neq y$. Since $(x, y) \in \approx_T^\ell$ and $(y, z) \in \approx_T^\ell$, it follows that $(y, a, b) \in T$ and $(y, a, b) \in T^c$, a contradiction.
- (b) Suppose that $a = y$ and $b \neq y$. Then we have by hypothesis that $(x, y, b) \in T$ and $(z, y, b) \in T^c$. It follows that $(y, z) \in (P_r(T^c))^t$. Since $(x, y) \in \approx_T$, $(y, z) \in \approx_T$ and $(x, z) \notin \approx_T$, it follows that $x \neq y \neq z \neq x$. Since $(x, y, b) \in T$ and $(y, z) \in \approx_T^m$, it holds that $(x, z, b) \in T$. Moreover, as $(x, z, b) \in T$ and $(x, y) \in \approx_T^\ell$, it follows that $(y, z, b) \in T$ and, hence, $(y, z) \in P_r(T)$, a contradiction with the fact that $(y, z) \in P_r(T)$ and $(z, y) \in P_r(T^c)$, i.e., $(y, z) \in P_r(T) \cap (P_r(T^c))^t$.
- (c) Suppose that $a \neq y$ and $b = y$. Then we have by hypothesis that $(x, a, y) \in T$ and $(z, a, y) \in T^c$. It follows that $(y, z) \in (P_m(T^c))^t$. Since $(x, a, y) \in T$ and $(y, z) \in \approx_T^r$, it follows that $(x, a, z) \in T$. Moreover, as $(x, a, z) \in T$ and $(x, y) \in \approx_T^\ell$, it holds that $(y, a, z) \in T$. Hence, $(y, z) \in P_m(T) \cap (P_m(T^c))^t$, a contradiction with the hypothesis.
- (d) Suppose that $a = b = y$. Then we have by hypothesis that $(x, y, y) \in T$ and $(z, y, y) \in T^c$. It follows that $(y, y) \in P_\ell(T)$ and $(y, y) \in P_\ell(T^c)$. Hence, $(y, y) \in P_\ell(T) \cap (P_\ell(T^c))^t$, a contradiction with the hypothesis.
- (ii) The proof is analogous to (i). □

The following proposition shows the impact of a condition on a three elements of a given set, in order to get two clones of them.

Proposition 36:

Let T be a ternary relation on a set X and let us consider two sets defined by $H = \{x \in X \mid (\forall y, z \in (X \setminus \{x\}))((x, y, z) \in T^c \wedge (x, y, z) \in (T^t)^c)\}$ and $H' = \{x \in X \mid (\forall y, z \in (X \setminus \{x\}))((y, x, z) \in T^c)\}$, then the following statements hold:

- (i) $(a, b) \in \approx_T^\lambda$, for any $a, b \in H$ and $\lambda \in \{\ell, r\}$;
- (ii) $(a, b) \in \approx_T^m$, for any $a, b \in H'$.

Proof. Follows from the definition of the clone relations of T . □

It is easy to express the relationship between the left, middle and right clone relations of the transpose, complement and permutations of T and that of its clone relations.

Proposition 37:

Let T be a relation on a set X .

- (i) The following table expresses the clone relations of its transpose T^t and its complement T^c :

	$\approx^\ell$	$\approx^m$	$\approx^r$
T^t	$\approx_T^r$	$\approx_T^m$	$\approx_T^\ell$
T^c	$\approx_T^\ell$	$\approx_T^m$	$\approx_T^r$

- (ii) The following table expresses the clone relations of its permutations:

	$\approx^\ell$	$\approx^m$	$\approx^r$
T^{-1}	$\approx_T^\ell$	$\approx_T^r$	$\approx_T^m$
$T^\perp$	$\approx_T^m$	$\approx_T^\ell$	$\approx_T^r$
T^+	$\approx_T^m$	$\approx_T^r$	$\approx_T^\ell$
T^-	$\approx_T^r$	$\approx_T^\ell$	$\approx_T^m$

The following proposition shows a particular case of two ternary relations have the same clone relations.

Proposition 38:

Let T_1 and T_2 be two ternary relations on a set X . If for any $x, y, z \in X$ such that $x \neq y$ and $x \neq z$ it holds that $(x, y, z) \in T_1 \Leftrightarrow (x, y, z) \in T_2$, then $\approx_{T_1}^\lambda = \approx_{T_2}^\lambda$, for any $\lambda \in \{\ell, m, r\}$.

Proof. We only prove the case $\lambda = \ell$, the other cases being analogous.

Let $(x, y) \in \approx_{T_1}^\ell$ and $(x, a, b) \in T_2$ such that $a, b \in X \setminus \{x, y\}$. It holds that $x \neq a$ and $x \neq b$. This implies that $(x, a, b) \in T_1$. Since $(x, y) \in \approx_{T_1}^\ell$, it follows that $(y, a, b) \in T_1$. Hence, $(y, a, b) \in T_2$. Thus, $(x, y) \in \approx_{T_2}^\ell$. Therefore, $\approx_{T_1}^\ell \subseteq \approx_{T_2}^\ell$. The proof that $\approx_{T_2}^\ell \subseteq \approx_{T_1}^\ell$ is analogous. $\square$

Next, we consider the set $A = \{(x, y, z) \in X^3 \mid (x = y) \vee (x = z)\}$. The following proposition shows that A has no impact on the clone relations of a given ternary relation on X .

Proposition 39:

Let T be a ternary relation on a set X . For any $\lambda \in \{\ell, m, r\}$, it holds that $\approx_T^\lambda = \approx_{T \setminus A}^\lambda = \approx_{T \cup A}^\lambda$.

Proof. Let $(x, y, z) \in T$ such that $x \neq y$ and $x \neq z$. It holds that $(x, y, z) \in T \Leftrightarrow (x, y, z) \in T \setminus A$. From Proposition 38, it follows that $\approx_T^\lambda = \approx_{T \setminus A}^\lambda$, for any $\lambda \in \{\ell, m, r\}$. The proof of $\approx_T^\lambda = \approx_{T \cup A}^\lambda$ is follows from Proposition 37 as:
 $\approx_T^\lambda = \approx_{T^c}^\lambda = \approx_{T^c \setminus A}^\lambda = \approx_{T^c \cap A^c}^\lambda = \approx_{(T \cup A)^c}^\lambda = \approx_{T \cup A}^\lambda$. $\square$

In particular, we conclude that the clone relations of a ternary relation do not take (ir)reflexivity into account.

Corollary 11:

Let T be a ternary relation on a set X . For any $\lambda \in \{\ell, m, r\}$, it holds that $\approx_T^\lambda = \approx_{T \setminus I_{X^3}}^\lambda = \approx_{T \cup I_{X^3}}^\lambda$.

Example 9:

Let $(X, \leq)$ be a poset and $B_<$ be the strict order betweenness relation associated with the order betweenness relation $B_\leq$ on X . From Corollary 11, it holds that $\approx_{B_<}^\lambda = \approx_{B_<}^\lambda$, for any $\lambda \in \{\ell, m, r\}$.

3.3/ Properties of a ternary relation in terms of its clone relations

In this section, we characterize some properties of a ternary relation in terms of its clone relations. In words, we explain how to use the clone relations of a given ternary relation to study some properties of this relation. To that end, the following definition is needed.

Definition 18: [89]

For a ternary relation T on a set X , we associate the following binary relations:

- (i) $W_\ell(T) = \{(x, y) \in X^2 \mid (x, y, y) \in T\}$;
- (ii) $W_m(T) = \{(x, y) \in X^2 \mid (x, y, x) \in T\}$;
- (iii) $W_r(T) = \{(x, y) \in X^2 \mid (x, x, y) \in T\}$.

The following proposition expresses the symmetry and cyclicity of a ternary relation in terms of its clone relations.

Proposition 40:

Let T be a ternary relation on a set X , then the following statements hold:

- (i) If T is symmetric, then $\approx_T^\ell = \approx_T^r$;
- (ii) If T is cyclic, then $\approx_T^\ell = \approx_T^m = \approx_T^r = \approx_T$.

Next, the following result shows that the clone relations of a ternary relation T are parts of the clone relations of $W_\ell(T)$, $W_m(T)$ and $W_r(T)$.

Proposition 41:

Let T be a ternary relation on a set X , then the following statements hold:

- (i) $\approx_T^\ell \subseteq \approx_{W_\ell(T)}^\ell$;
- (ii) $\approx_T^m \subseteq \approx_{(W_m(T))^t}^\ell = \approx_{W_m(T)}^r$;
- (iii) $\approx_T^r \subseteq \approx_{W_r(T)}^r$.

Proof.

- (i) Let $(x, y) \in \approx_T^\ell$ and $(x, a) \in W_\ell(T)$ such that $a \in X \setminus \{x, y\}$, then $(x, a, a) \in T$. Since $(x, y) \in \approx_T^\ell$, it holds that $(y, a, a) \in T$. Hence, $(y, a) \in W_\ell(T)$. Thus, $(x, y) \in \approx_{W_\ell(T)}^\ell$.

Therefore, $\approx_T^\ell \subseteq \approx_{W_\ell(T)}^\ell$.

Statements (ii) and (iii) follow from Proposition 37 (ii) and (i). $\square$

The following result expresses the symmetry and transitivity of a ternary relation in terms of its clone relations.

Proposition 42:

Let T be a ternary relation on a set X . If T is both symmetric and c_i -transitive, for any $i \in \{1, \dots, 6\}$, then the following statements hold:

- (i) $W_\ell(T) \subseteq \approx_T^\ell$;
- (ii) $W_m(T) \cap (W_m(T))^t \subseteq \approx_T^m$;
- (iii) $W_r(T) \subseteq \approx_T^r$.

Proof. We only prove (i), as (ii) and (iii) can be proved similarly. Let $(x, y) \in W_\ell(T)$, then it holds that $(x, y, y) \in T$. Since T is symmetric, it follows that $(y, y, x) \in T$. On the one hand, suppose that $(x, a, b) \in T$, for any $a, b \in X \setminus \{x, y\}$. Since $(y, y, x) \in T$ and $(x, a, b) \in T$, it follows from the fact T is c_2 -transitive that $(y, a, b) \in T$. On the other hand, suppose that $(y, a, b) \in T$, for any $a, b \in X \setminus \{x, y\}$. Since $(x, y, y) \in T$ and T is c_2 -transitive, it follows that $(x, a, b) \in T$. Hence, $(x, y) \in \approx_T^\ell$. Thus, $W_\ell(T) \subseteq \approx_T^\ell$. $\square$

In general, the converse of statements of Proposition 42 does not hold as can be seen in the following example.

Example 10:

Let T be a ternary relation on $X = \{x, y, z, t\}$, given by:

$$T = \{(y, y, y), (y, t, t), (t, y, y), (z, x, y), (z, t, x), (t, t, t)\}.$$

- (i) We can easily verify that $\approx_T^\ell = \{(x, x), (y, y), (z, z), (t, t), (y, t), (t, y), (x, t), (t, x), (x, z), (z, x), (x, y), (y, x)\}$ and $W_\ell(T) = \{(y, y), (y, t), (t, t), (t, y)\}$. Therefore, $W_\ell(T) \subseteq \approx_T^\ell$. Since T is not symmetric, it holds that the converse of (i) is false. Also, we get the same result as follows. Since the assertion $[(y, t, t) \in T \wedge (z, t, x) \in T] \wedge (y, t, x) \notin T$ is true, it follows that T is not c_1 -transitive. Hence, the converse of (i) is false.
- (ii) Applying a similar reasoning, we find $\approx_T^m = \{(x, x), (y, y), (z, z), (t, t), (z, x), (x, z), (z, y), (y, z), (z, t), (t, z), (x, y), (y, x)\}$ and $W_m(T) = \{(y, y), (t, t)\}$. Therefore, $W_m(T) \cap (W_m(T))^t \subseteq \approx_T^m$. Since T is not symmetric, it holds that the converse of (ii) is false.
- (iii) Similarly, we have $\approx_T^r = \{(x, x), (y, y), (z, z), (t, t), (z, y), (y, z), (z, t), (t, z), (t, y), (y, t), (t, x), (x, t), (z, x), (x, z)\}$ and $W_r(T) = \{(y, y), (t, t)\}$. Therefore, $W_r(T) \subseteq \approx_T^r$. Since T is not symmetric, it holds that the converse of (iii) is false.

Combining Propositions 42 and 41 easily leads to the following corollary.

Corollary 12:

Let T be a ternary relation on a set X . If T is both symmetric and c_i -transitive, for any $i \in \{1, \dots, 6\}$, then the following statements hold:

- (i) $W_\ell(T) \subseteq \approx_{W_\ell(T)}^\ell$;
- (ii) $W_m(T) \cap (W_m(T))^t \subseteq \approx_{(W_m(T))^t}^\ell = \approx_{W_m(T)}^r$;
- (iii) $W_r(T) \subseteq \approx_{W_r(T)}^r$.

The following result is provided.

Proposition 43:

Let T be a ternary relation on a set X . If T is an equivalence relation, then for any $\lambda \in \{\ell, m, r\}$, the following equalities are equivalent:

- (i) $\approx_T^\lambda = W_\lambda(T)$;
- (ii) $W_\lambda(T) = \approx_{W_\lambda(T)}^\lambda$.

Proof. We only prove the case $\lambda = \ell$, as the other cases can be proved similarly.

($\Rightarrow$) Let T be an equivalence relation and suppose that $\approx_T^\ell = W_\ell(T)$. Due to Corollary 12, it holds that $W_\ell(T) \subseteq \approx_{W_\ell(T)}^\ell$. It remains to prove that $\approx_{W_\ell(T)}^\ell \subseteq W_\ell(T)$. Let $(x, y) \in \approx_{W_\ell(T)}^\ell$. It holds that $(\forall t \in X \setminus \{x, y\}) \mid ((x, t) \in W_\ell(T) \Leftrightarrow (y, t) \in W_\ell(T))$. This implies that $(\forall t \in X \setminus \{x, y\}) \mid ((x, t, t) \in T \Leftrightarrow (y, t, t) \in T)$. Hence, $(x, y) \in \approx_T^\ell = W_\ell(T)$.

($\Leftarrow$) Let T be an equivalence relation and suppose that $W_\ell(T) = \approx_{W_\ell(T)}^\ell$. Due to Proposition 42, it holds that $W_\ell(T) \subseteq \approx_T^\ell$. Proposition 41 guarantees that $\approx_T^\ell \subseteq \approx_{W_\ell(T)}^\ell = W_\ell(T)$. $\square$

Combining Propositions 43 and 32 leads to the following corollary.

Corollary 13:

Let T be a ternary relation on a set X . If T is an equivalence relation, then for any $t \in X \setminus \{x, y\}$, the following equivalences hold:

- (i) $\approx_T^\ell = W_\ell(T)$ if and only if $\nexists x, y \in X \mid (x \neq y) \wedge ((x, t, t) \in T^c) \wedge ((y, t, t) \in T^c)$;
- (ii) $\approx_T^m = W_m(T)$ if and only if $\nexists x, y \in X \mid (x \neq y) \wedge ((t, x, t) \in T^c) \wedge ((t, y, t) \in T^c)$;
- (iii) $\approx_T^r = W_r(T)$ if and only if $\nexists x, y \in X \mid (x \neq y) \wedge ((t, t, x) \in T^c) \wedge ((t, t, y) \in T^c)$.

3.4/ Interaction of the clone relations with the basic set operations

This section is devoted to discussing the interaction of the clone relations of ternary relations with the most common set operations.

Proposition 44:

Let T_1 and T_2 be two ternary relations on a set X . If $T_1 \subseteq T_2$, then for any $\lambda \in \{\ell, m, r\}$ the following statements hold:

- (i) $\approx_{T_1}^\lambda \cap \approx_{T_2 \setminus T_1}^\lambda \subseteq \approx_{T_2}^\lambda$;
- (ii) $\approx_{T_2}^\lambda \subseteq (\approx_{T_1}^\lambda \cap \approx_{T_2 \setminus T_1}^\lambda) \cup ((\approx_{T_1}^\lambda)^c \cap (\approx_{T_2 \setminus T_1}^\lambda)^c)$.

Proof.

- (i) We only prove that $\approx_{T_1}^\ell \cap \approx_{T_2 \setminus T_1}^\ell \subseteq \approx_{T_2}^\ell$. The other inclusions can be proved analogously. Suppose that $T_1 \subseteq T_2$ and let $x, y \in X$ be such that $(x, y) \in (\approx_{T_1}^\ell \cap \approx_{T_2 \setminus T_1}^\ell)$. It follows that $(x, y) \in \approx_{T_1}^\ell$ and $(x, y) \in \approx_{T_2 \setminus T_1}^\ell$. Therefore, for any $(s, t) \in (X \setminus \{x, y\})^2$, it holds that

$$\begin{aligned} (x, s, t) \in T_2 &\Leftrightarrow (x, s, t) \in T_1 \vee (x, s, t) \in T_2 \setminus T_1 \\ &\Leftrightarrow (y, s, t) \in T_1 \vee (y, s, t) \in T_2 \setminus T_1 \\ &\Leftrightarrow (y, s, t) \in T_2. \end{aligned}$$

Hence, $(x, y) \in \approx_{T_2}^\ell$.

- (ii) We only prove that $\approx_{T_2}^\ell \subseteq (\approx_{T_1}^\ell \cap \approx_{T_2 \setminus T_1}^\ell) \cup ((\approx_{T_1}^\ell)^c \cap (\approx_{T_2 \setminus T_1}^\ell)^c)$. The other inclusions can be proved analogously.

Suppose that $T_1 \subseteq T_2$ and let $x, y \in X$ be such that $(x, y) \in \approx_{T_2}^\ell$. Since it trivially holds that $X^2 = (\approx_{T_1}^\ell \cup (\approx_{T_1}^\ell)^c) \cap (\approx_{T_2 \setminus T_1}^\ell \cup (\approx_{T_2 \setminus T_1}^\ell)^c)$, it follows that one of the following statements holds:

$$(x, y) \in (\approx_{T_1}^\ell \cap \approx_{T_2 \setminus T_1}^\ell) \text{ or } (x, y) \in (\approx_{T_1}^\ell \cap (\approx_{T_2 \setminus T_1}^\ell)^c) \text{ or } (x, y) \in ((\approx_{T_1}^\ell)^c \cap \approx_{T_2 \setminus T_1}^\ell) \text{ or } (x, y) \in ((\approx_{T_1}^\ell)^c \cap (\approx_{T_2 \setminus T_1}^\ell)^c).$$

We will prove that $(x, y) \in [((\approx_{T_1}^\ell)^c \cap \approx_{T_2 \setminus T_1}^\ell)^c]$ and $(x, y) \in [(\approx_{T_1}^\ell \cap (\approx_{T_2 \setminus T_1}^\ell)^c)^c]$.

- (a) Suppose that $((x, y) \in (\approx_{T_1}^\ell)^c \text{ and } (x, y) \in \approx_{T_2 \setminus T_1}^\ell)$. Since $(x, y) \in (\approx_{T_1}^\ell)^c$, it follows that there exists $(s, t) \in (X \setminus \{x, y\})^2$ such that one of the following statements holds: $((x, s, t) \in T_1 \text{ and } (y, s, t) \in T_2^c)$ or $((y, s, t) \in T_2 \text{ and } (x, s, t) \in T_1^c)$. Any of these cases contradicts the fact that $((x, y) \in \approx_{T_2}^\ell \text{ and } (x, y) \in \approx_{T_2 \setminus T_1}^\ell)$. For instance, if $((x, s, t) \in T_1 \text{ and } (y, s, t) \in T_2^c)$, then, since $T_1 \subseteq T_2$, it follows that $(x, s, t) \in T_2$. Since $(x, y) \in \approx_{T_2}^\ell$ and $(s, t) \in (X \setminus \{x, y\})^2$, it follows that $(y, s, t) \in T_2$. On the one hand, since $(y, s, t) \in T_2$ and $(y, s, t) \in T_1^c$, it follows that $(y, s, t) \in T_2 \setminus T_1$. On the other hand, since $(x, s, t) \in T_1$, it follows that $(x, s, t) \in (T_2 \setminus T_1)^c$, a contradiction with the fact that $(x, y) \in \approx_{T_2 \setminus T_1}^\ell$. The other case where $((y, s, t) \in T_2 \text{ and } (x, s, t) \in T_1^c)$ is analogously proved.
- (b) We apply a similar reasoning. Suppose that $((x, y) \in \approx_{T_1}^\ell \text{ and } (x, y) \in (\approx_{T_2 \setminus T_1}^\ell)^c)$. Since $(x, y) \in (\approx_{T_2 \setminus T_1}^\ell)^c$, it follows that there exists $(s, t) \in (X \setminus \{x, y\})^2$ such that one of the following statements holds: $((x, s, t) \in T_2 \setminus T_1 \text{ and } (y, s, t) \in (T_2 \setminus T_1)^c)$ or $((y, s, t) \in T_2 \setminus T_1 \text{ and } (x, s, t) \in (T_2 \setminus T_1)^c)$. Any of these cases contradicts the fact that $((x, y) \in \approx_{T_2}^\ell \text{ and } (x, y) \in \approx_{T_1}^\ell)$. For instance, if $((x, s, t) \in T_2 \setminus T_1 \text{ and } (y, s, t) \in (T_2 \setminus T_1)^c)$, then it follows that

$((x, s, t) \in T_2 \text{ and } (x, s, t) \in T_1^c) \text{ and } ((y, s, t) \in T_2^c \text{ or } (y, s, t) \in T_1)$. Therefore, it holds that $((x, s, t) \in T_2 \text{ and } (y, s, t) \in T_2^c) \text{ or } ((x, s, t) \in T_1^c \text{ and } (y, s, t) \in T_1)$, a contradiction with the fact that $((x, y) \in \approx_{T_2}^\ell \text{ and } (x, y) \in \approx_{T_1}^\ell)$. The other case where $((y, s, t) \in T_2 \setminus T_1 \text{ and } (x, s, t) \in (T_2 \setminus T_1)^c)$ is analogously proved.

Hence, it holds that $((x, y) \in \approx_{T_1}^\ell \text{ and } (x, y) \in \approx_{T_2 \setminus T_1}^\ell) \text{ or } ((x, y) \in (\approx_{T_1}^\ell)^c \text{ and } (x, y) \in (\approx_{T_2 \setminus T_1}^\ell)^c)$. Therefore, it holds that $\approx_{T_2}^\ell \subseteq (\approx_{T_1}^\ell \cap \approx_{T_2 \setminus T_1}^\ell) \cup ((\approx_{T_1}^\ell)^c \cap (\approx_{T_2 \setminus T_1}^\ell)^c)$.

□

The following corollaries follow immediately from the above proposition.

Corollary 14:

Let T_1 and T_2 be two ternary relations on a set X . If $T_1 \subseteq T_2$, then the following statements hold:

- (i) $\approx_{T_1} \cap \approx_{T_2 \setminus T_1} \subseteq \approx_{T_2}$;
- (ii) $\approx_{T_2} \subseteq (\approx_{T_1} \cap \approx_{T_2 \setminus T_1}) \cup ((\approx_{T_1})^c \cap (\approx_{T_2 \setminus T_1})^c)$.

Corollary 15:

Let T_1 and T_2 be two ternary relations on a set X . Then it holds that $\approx_{T_2 \cup T_1} \subseteq (\approx_{T_1} \cap \approx_{T_2 \setminus T_1}) \cup ((\approx_{T_1})^c \cap (\approx_{T_2 \setminus T_1})^c)$.

Remark 5:

If T_1 and T_2 are two ternary relations on a set X such that $T_1 \subseteq T_2$, then it does not necessarily hold that $\approx_{T_1}^\lambda \subseteq \approx_{T_2}^\lambda$, for any $\lambda \in \{\ell, m, r\}$, as can be seen in the following example.

Example 11:

Let us consider two ternary relations T_1 and T_2 on $X = \{x, y, z, t\}$ defined by $T_1 = \{(y, y, y), (t, t, t)\}$ and $T_2 = \{(y, y, y), (y, t, t), (t, y, y), (z, x, y), (z, t, x), (t, t, t), (y, x, z)\}$. Then it holds that $T_1 \subseteq T_2$ and,

- (i) $(y, t) \in \approx_{T_1}^\ell$, while $(y, t) \in (\approx_{T_2}^\ell)^c$. Hence, $\approx_{T_1}^\ell \not\subseteq \approx_{T_2}^\ell$.
- (ii) $(x, t) \in \approx_{T_1}^m$, while $(x, t) \in (\approx_{T_2}^m)^c$. Hence, $\approx_{T_1}^m \not\subseteq \approx_{T_2}^m$.
- (iii) $(x, y) \in \approx_{T_1}^r$, while $(x, t) \in (\approx_{T_2}^r)^c$. Hence, $\approx_{T_1}^r \not\subseteq \approx_{T_2}^r$.

Using Proposition 37 (i) leads immediately to the following results.

Corollary 16:

Let $(T_i)_{i \in I}$ be a finite family of ternary relations on a set X . Then for any $\lambda \in \{\ell, m, r\}$, the following equalities hold:

$$\approx_{\bigcup_{i \in I} T_i}^\lambda = \approx_{\bigcap_{i \in I} T_i^c}^\lambda \quad \text{and} \quad \approx_{\bigcap_{i \in I} T_i}^\lambda = \approx_{\bigcup_{i \in I} T_i^c}^\lambda.$$

Corollary 17:

Let $(T_i)_{i \in I}$ be a finite family of ternary relations on a set X , then the following equalities hold:

$$\approx_{\bigcup_{i \in I} T_i} = \approx_{\bigcap_{i \in I} T_i^c} \quad \text{and} \quad \approx_{\bigcap_{i \in I} T_i} = \approx_{\bigcup_{i \in I} T_i^c}.$$

Next, we show more the interaction of the clone relations of ternary relations with the intersection and the union.

Proposition 45:

Let T_1 and T_2 be two ternary relations on a set X . Then for any $\lambda \in \{\ell, m, r\}$, the following statements hold:

- (i) $\approx_{T_1}^\lambda \cap \approx_{T_2}^\lambda = \approx_{T_1 \cap T_2}^\lambda \cap \approx_{T_1 \setminus T_2}^\lambda \cap \approx_{T_2 \setminus T_1}^\lambda$;
- (ii) $\approx_{T_1}^\lambda \cap \approx_{T_2}^\lambda = \approx_{T_1 \cup T_2}^\lambda \cap \approx_{T_1 \setminus T_2}^\lambda \cap \approx_{T_2 \setminus T_1}^\lambda$.

Proof. We only prove that $\approx_{T_1}^\ell \cap \approx_{T_2}^\ell = \approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus T_2}^\ell \cap \approx_{T_2 \setminus T_1}^\ell$. The other equalities can be proved analogously.

- (i) We need to prove that $\approx_{T_1}^\ell \cap \approx_{T_2}^\ell \subseteq \approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus T_2}^\ell \cap \approx_{T_2 \setminus T_1}^\ell$ and that $\approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus T_2}^\ell \cap \approx_{T_2 \setminus T_1}^\ell \subseteq \approx_{T_1}^\ell \cap \approx_{T_2}^\ell$.
- (a) First, we prove that $\approx_{T_1}^\ell \cap \approx_{T_2}^\ell \subseteq \approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus T_2}^\ell \cap \approx_{T_2 \setminus T_1}^\ell$. Let $x, y \in X$ be such that $(x, y) \in (\approx_{T_1}^\ell \cap \approx_{T_2}^\ell)$. It follows that $(x, y) \in \approx_{T_1}^\ell$ and $(x, y) \in \approx_{T_2}^\ell$. Therefore, for any $(s, t) \in (X \setminus \{x, y\})^2$, it holds that

$$\begin{aligned} (x, s, t) \in T_1 \cap T_2 &\Leftrightarrow (x, s, t) \in T_1 \wedge (x, s, t) \in T_2 \\ &\Leftrightarrow (y, s, t) \in T_1 \wedge (y, s, t) \in T_2 \\ &\Leftrightarrow (y, s, t) \in T_1 \cap T_2. \end{aligned}$$

Hence, $(x, y) \in (\approx_{T_1 \cap T_2}^\ell)$. Thus, $\approx_{T_1}^\ell \cap \approx_{T_2}^\ell \subseteq \approx_{T_1 \cap T_2}^\ell$. Moreover, for any $(s, t) \in (X \setminus \{x, y\})^2$, it holds that

$$\begin{aligned} (x, s, t) \in T_1 \setminus T_2 &\Leftrightarrow (x, s, t) \in T_1 \wedge (x, s, t) \in T_2^c \\ &\Leftrightarrow (y, s, t) \in T_1 \wedge (y, s, t) \in T_2^c \\ &\Leftrightarrow (y, s, t) \in T_1 \setminus T_2. \end{aligned}$$

Hence, $(x, y) \in \approx_{T_1 \setminus T_2}^\ell$. Thus, $\approx_{T_1}^\ell \cap \approx_{T_2}^\ell \subseteq \approx_{T_1 \setminus T_2}^\ell$. The fact that $\approx_{T_1}^\ell \cap \approx_{T_2}^\ell \subseteq \approx_{T_2 \setminus T_1}^\ell$ is proved in an analogous way.

- (b) Second, we prove that $\approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus T_2}^\ell \cap \approx_{T_2 \setminus T_1}^\ell \subseteq \approx_{T_1}^\ell \cap \approx_{T_2}^\ell$. We have that

$$\begin{aligned} \approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus T_2}^\ell \cap \approx_{T_2 \setminus T_1}^\ell &= \approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus (T_1 \cap T_2)}^\ell \cap \approx_{T_2 \setminus (T_1 \cap T_2)}^\ell \\ &= (\approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus (T_1 \cap T_2)}^\ell) \cap (\approx_{T_1 \cap T_2}^\ell \cap \approx_{T_2 \setminus (T_1 \cap T_2)}^\ell). \end{aligned}$$

From Proposition 44, it follows that $(\approx_{T_1 \cap T_2}^\ell \cap \approx_{T_1 \setminus (T_1 \cap T_2)}^\ell) \subseteq \approx_{T_1}^\ell$ and $(\approx_{T_1 \cap T_2}^\ell \cap \approx_{T_2 \setminus (T_1 \cap T_2)}^\ell) \subseteq \approx_{T_2}^\ell$.

- (ii) From (i), it follows that $\approx_{T_1^c}^\ell \cap \approx_{T_2^c}^\ell = \approx_{T_1^c \cap T_2^c}^\ell \cap \approx_{T_1^c \setminus T_2^c}^\ell \cap \approx_{T_2^c \setminus T_1^c}^\ell$.
 Since $\approx_{T_1^c}^\ell = \approx_{T_1}^\ell$, $\approx_{T_2^c}^\ell = \approx_{T_2}^\ell$, $\approx_{T_1^c \cap T_2^c}^\ell = \approx_{(T_1 \cup T_2)^c}^\ell = \approx_{T_1 \cup T_2}^\ell$, $T_1^c \setminus T_2^c = T_2 \setminus T_1$ and $T_2^c \setminus T_1^c = T_1 \setminus T_2$, it follows that $\approx_{T_1}^\ell \cap \approx_{T_2}^\ell = \approx_{T_1 \cup T_2}^\ell \cap \approx_{T_2 \setminus T_1}^\ell \cap \approx_{T_1 \setminus T_2}^\ell$.

□

Using Proposition 45 leads immediately to the following results.

Corollary 18:

Let $(T_i)_{i \in I}$ be a finite family of ternary relations on a set X . Then for any $\lambda \in \{\ell, m, r\}$, the following statements hold:

$$\bigcap_{i \in I} \approx_{T_i}^\lambda \subseteq \approx_{\bigcap_{i \in I} T_i}^\lambda \quad \text{and} \quad \bigcap_{i \in I} \approx_{T_i}^\lambda \subseteq \approx_{\bigcup_{i \in I} T_i}^\lambda.$$

Corollary 19:

Let $(T_i)_{i \in I}$ be a finite family of ternary relations on a set X . The following statements hold:

$$\bigcap_{i \in I} \approx_{T_i} \subseteq \approx_{\bigcap_{i \in I} T_i} \quad \text{and} \quad \bigcap_{i \in I} \approx_{T_i} \subseteq \approx_{\bigcup_{i \in I} T_i}.$$

Corollary 20:

Let T_1 and T_2 be two ternary relations on a set X . The following statements hold:

- (i) $\approx_{T_1} \cap \approx_{T_2} = \approx_{T_1 \cap T_2} \cap \approx_{T_1 \setminus T_2} \cap \approx_{T_2 \setminus T_1}$;
- (ii) $\approx_{T_1} \cap \approx_{T_2} = \approx_{T_1 \cup T_2} \cap \approx_{T_1 \setminus T_2} \cap \approx_{T_2 \setminus T_1}$.

3.5/ Characterization of the clone relations of disjoint union

This section provides principally the characterization of the clone relations of union of two ternary relations defined on disjoint sets.

For a given ternary relation T_E defined on a set E , we write $\mathbb{E} = (E, T_E)$ and we call $\mathbb{E}$ a ternary structure.

Definition 19:

A ternary structure $\mathbb{E} = (E, T_E)$ is called a reduction of another ternary structure $\mathbb{F} = (F, T_F)$ if the following two statements hold:

- (i) $E \subseteq F$.
- (ii) For any $x, y, z \in E$, it holds that $(x, y, z) \in T_E$ if and only if $(x, y, z) \in T_F$.

The following proposition shows that if a ternary structure is a reduction of another ternary structure, then the clone relation of the second one is included in that of the first.

Proposition 46:

Let $\mathbb{E} = (E, T_E)$ be a reduction of $\mathbb{F} = (F, T_F)$. For any $x, y \in E$, it holds that $(x, y) \in \approx_{T_F}^\lambda$ implies that $(x, y) \in \approx_{T_E}^\lambda$, for any $\lambda \in \{\ell, m, r\}$.

Proof. We only prove the case $\lambda = \ell$, the other cases being analogous. Let $x, y \in E$ be such that $(x, y) \in \approx_{T_F}^\ell$. It holds that $(x, a, b) \in T_F \Leftrightarrow (y, a, b) \in T_F$, for any $a, b \in F \setminus \{x, y\}$. Since $\mathbb{E}$ is a reduction of $\mathbb{F}$, it follows from Definition 19 that, for any $a, b \in E \setminus \{x, y\}$, it holds that $(x, a, b) \in T_E \Leftrightarrow (y, a, b) \in T_E$ and hence, $(x, y) \in \approx_{T_E}^\ell$. $\square$

Note that throughout this section, we mean by $\approx_{T_E}^\lambda$, for any $\lambda \in \{\ell, m, r\}$, as the left (resp. middle and right) clone relation of T_E in E and not in $E \cup F$. The same applies to $\approx_{T_F}^\lambda$.

Remarkably, the converse of the statement in Proposition 46 does not hold, as can be seen in the following example.

Example 12:

Let $E = \mathbb{N}$ and $F = \mathbb{R}$ be the sets of natural and real numbers respectively, with the usual strict order relation $<$ and, $B_<$ be the strict order-betweenness relation. It is clearly that, $\mathbb{E} = (\mathbb{N}, B_{<\mathbb{N}})$ is a reduction of $\mathbb{F} = (\mathbb{R}, B_{<\mathbb{R}})$. However, it holds that $(0, 1) \in \approx_{B_{<\mathbb{N}}}^\ell$, while $(0, 1) \notin \approx_{B_{<\mathbb{R}}}^\ell$. Hence, if $(x, y) \in \approx_{T_E}^\ell$ for some $x, y \in E$, then it does not necessarily hold that $(x, y) \in \approx_{T_F}^\ell$. In a similar way, we easily verify the same result for the middle and right clone relations.

Let us consider two arbitrary ternary structures $\mathbb{E} = (E, T_E)$ and $\mathbb{F} = (F, T_F)$. Then we say that $\mathbb{E}$ and $\mathbb{F}$ are disjoint if E and F are disjoint. The union of two disjoint ternary structures is called a disjoint union. Note that in the disjoint union, the relations between elements in the same original set are conserved, but we cannot consider that if the elements in different original sets.

Definition 20:

Let $\mathbb{E} = (E, T_E)$ and $\mathbb{F} = (F, T_F)$ be two disjoint ternary structures. The disjoint union $\mathbb{E} \cup \mathbb{F}$ of $\mathbb{E}$ and $\mathbb{F}$ is the ternary structure $\mathbb{E} \cup \mathbb{F} = (E \cup F, T_E \cup T_F)$. In words, the relation $T = T_E \cup T_F$ is defined on $E \cup F$ by

$$T = \{(x, y, z) \in E \cup F \mid (x, y, z) \in T_E \vee (x, y, z) \in T_F\}.$$

The following proposition is straightforward.

Proposition 47:

Let $\mathbb{E} = (E, T_E)$ and $\mathbb{F} = (F, T_F)$ be two disjoint ternary structures such that $\mathbb{E} \cup \mathbb{F}$ is the disjoint union and, suppose that $x, y, z \in E \cup F$. If each two elements a, b of the set $\{x, y, z\}$ belong to E or F and in the same time the third element c of it belongs to other one, then it holds that $(a, b, c) \notin T = T_E \cup T_F$.

Next, we show in the following proposition how to characterize the clone relations of the disjoint union.

Proposition 48:

Let $\mathbb{E} = (E, T_E)$ and $\mathbb{F} = (F, T_F)$ be two disjoint ternary structures. The clone relation $\approx_T^\lambda$ of the disjoint union $T = T_E \cup T_F$ is given by

$$\approx_T^\lambda = \approx_{T_E}^\lambda \cup \approx_{T_F}^\lambda \cup (E' \times F') \cup (F' \times E'),$$

for any $\lambda \in \{\ell, m, r\}$, where $E' = \{x \in E \mid (\forall y, z \in (E \setminus \{x\}))((x, y, z) \in T_E^c)\}$ and $F' = \{x \in F \mid (\forall y, z \in (F \setminus \{x\}))((x, y, z) \in T_F^c)\}$.

Proof. We only prove the case $\lambda = \ell$, the other cases being analogous.

- (i) First, we prove that $\approx_T^\ell \subseteq \approx_{T_E}^\ell \cup \approx_{T_F}^\ell \cup (E' \times F') \cup (F' \times E')$.
 Let $x, y \in E \cup F$ be such that $(x, y) \in \approx_T^\ell$. Here, we distinguish four cases: $(x \in E$ and $y \in E)$ or $(x \in F$ and $y \in F)$ or $(x \in E$ and $y \in F)$ or $(x \in F$ and $y \in E)$.
 - (a) If $x, y \in E$, then, since $\mathbb{E}$ is a reduction of $\mathbb{E} \cup \mathbb{F}$ and $(x, y) \in \approx_T^\ell$, it follows from Proposition 46 that $(x, y) \in \approx_{T_E}^\ell$.
 - (b) If $x, y \in F$, then, again from Proposition 46, it follows that $(x, y) \in \approx_{T_F}^\ell$.
 - (c) If $x \in E$ and $y \in F$, then one of the following cases holds: $(x \in E \setminus E'$ and $y \in F)$ or $(x \in E$ and $y \in F \setminus F')$ or $(x \in E'$ and $y \in F')$. We will show that the two first cases lead to a contradiction.
 - (1) Suppose that $x \in E \setminus E'$ and $y \in F$, then there exists $z, z' \in E \setminus \{x\}$ such that $(x, z, z') \in T_E$. This implies that $(x, z, z') \in T$. Since $y \in F$, it follows from Proposition 47 that $(y, z, z') \in T^c$ such that $z, z' \in E \setminus \{y\}$. From $(x, z, z') \in T$ and $(y, z, z') \in T^c$, it follows that $(x, y) \notin \approx_T^\ell$, a contradiction.
 - (2) Suppose that $x \in E$ and $y \in F \setminus F'$, then as in (1), it follows that $(x, y) \notin \approx_T^\ell$, a contradiction.
 - (d) If $x \in F$ and $y \in E$, it follows analogously to (c) that $(x, y) \in F' \times E'$.
- (ii) Second, we prove that $\approx_{T_E}^\ell \cup \approx_{T_F}^\ell \cup (E' \times F') \cup (F' \times E') \subseteq \approx_T^\ell$.
 - (a) Let $x, y \in E$ be such that $(x, y) \in \approx_{T_E}^\ell$. From Proposition 47, it follows that, for any $z_F, z'_F \in F$, $(x, z_F, z'_F) \in T^c$ and $(y, z_F, z'_F) \in T^c$. Therefore, it holds that $(x, z_F, z'_F) \in T \Leftrightarrow (y, z_F, z'_F) \in T$. Since E and F are disjoint sets and $(x, y) \in \approx_{T_E}^\ell$, it follows that $x, y \notin F$ and, for any $z_E, z'_E \in E \setminus \{x, y\}$, $(x, z_E, z'_E) \in T_E \Leftrightarrow (y, z_E, z'_E) \in T_E$. As $\mathbb{E}$ is a reduction of $\mathbb{E} \cup \mathbb{F}$, it follows that, for any $z, z' \in (E \cup F) \setminus \{x, y\}$, $(x, z, z') \in T \Leftrightarrow (y, z, z') \in T$. Hence, it holds that $(x, y) \in \approx_T^\ell$, and, thus, $\approx_{T_E}^\ell \subseteq \approx_T^\ell$.
 - (b) Let $x, y \in F$ be such that $(x, y) \in \approx_{T_F}^\ell$. Applying a similar reasoning as in (a), we can prove that $\approx_{T_F}^\ell \subseteq \approx_T^\ell$.
 - (c) Let $x \in E$ and $y \in F$ be such that $(x, y) \in (E' \times F')$. On the one hand, by definition of E' and F' , it holds that for any $z, z' \in (E \cup F) \setminus \{x, y\}$, $(x, z, z') \in T_E^c$ and $(y, z, z') \in T_F^c$. On the other hand, by definition of the disjoint union, it holds that $(y, z, z') \in T_E^c$ and $(x, z, z') \in T_F^c$. Therefore, it follows that $(x, z, z') \in (T_E \cup T_F)^c = T^c$ and $(y, z, z') \in (T_E \cup T_F)^c = T^c$, for any $z, z' \in (E \cup F) \setminus \{x, y\}$, which implies that $(x, y) \in \approx_T^\ell$, and, hence, $E' \times F' \subseteq \approx_T^\ell$.
 - (d) Let $x \in F$ and $y \in E$ be such that $(x, y) \in (F' \times E')$. Applying a similar reasoning as in (c), we can prove that $F' \times E' \subseteq \approx_T^\ell$.

□

Example 13:

Let T_E and T_F be two ternary relations respectively on $E = \{x, y, z, t\}$ and $F = \{a, b, c\}$ defined by: $T_E = \{(y, x, x), (y, z, z), (x, y, y), (z, x, y), (z, x, x), (z, y, y)\}$ and $T_F = \{(a, b, b), (c, b, b)\}$. We have $T_E \cup T_F = T = \{(y, x, x), (y, z, z), (x, y, y), (z, x, y), (z, x, x), (z, y, y), (a, b, b), (c, b, b)\}$ defined on $E \cup F = \{x, y, z, t, a, b, c\}$. We easily verify that $\approx_T^\ell = \{(x, x), (y, y), (z, z), (t, t), (a, a), (b, b), (c, c), (x, z), (y, z), (a, c)\}$, $\approx_{T_E}^\ell = \{(x, x), (y, y), (z, z), (t, t), (x, z), (y, z)\}$, $\approx_{T_F}^\ell = \{(a, c), (a, a), (b, b), (c, c)\}$, $E' = \{t\}$ and $F' = \emptyset$. Finally, we conclude that

$$\approx_{T_E \cup T_F}^\ell = \approx_{T_E}^\ell \cup \approx_{T_F}^\ell \cup (E' \times F') \cup (F' \times E').$$

In particular, the necessary and sufficient condition that the clone relations of the disjoint union of T_E and T_F relation needs to satisfy in order to coincide with the union of the clone relations of T_E and T_F is provided in the following theorem.

Theorem 11:

Let $\mathbb{E} = (E, T_E)$ and $\mathbb{F} = (F, T_F)$ be two disjoint ternary structures. Then it holds that $\approx_{T_E \cup T_F}^\lambda = \approx_{T_E}^\lambda \cup \approx_{T_F}^\lambda$, for any $\lambda \in \{\ell, m, r\}$ if and only if $(\forall x \in E)(\exists y, z \in E \setminus \{x\})((x, y, z) \in T_E) \vee (\forall x \in F)(\exists y, z \in F \setminus \{x\})((x, y, z) \in T_F)$.

Proof. It obviously follows that, by definition of E' and F' , $(E' = \emptyset) \vee (F' = \emptyset)$ is equivalent to $(\forall x \in E)(\exists y, z \in E \setminus \{x\})((x, y, z) \in T_E) \vee (\forall x \in F)(\exists y, z \in F \setminus \{x\})((x, y, z) \in T_F)$. This is equivalent from Proposition 48 to $\approx_{T_E \cup T_F}^\lambda = \approx_{T_E}^\lambda \cup \approx_{T_F}^\lambda$, for any $\lambda \in \{\ell, m, r\}$. □

Corollary 21:

Let $\mathbb{E} = (E, T_E)$ and $\mathbb{F} = (F, T_F)$ be two disjoint ternary structures. Then it holds that if either T_E or T_F is complete, then $\approx_{T_E \cup T_F}^\lambda = \approx_{T_E}^\lambda \cup \approx_{T_F}^\lambda$, for any $\lambda \in \{\ell, m, r\}$.

4

Compatibility of a ternary relation with a binary fuzzy relation

The main aim of this chapter is to study the compatibility of an arbitrary ternary relation with a binary L -relation. We prove that this compatibility can be expressed in terms of inclusions of the binary L -relation in the traces of the ternary relation. For this purpose, we introduce and study the notions of left-, middle- and right-compatibility of an arbitrary ternary relation with a binary L -relation. we provide a characterization of the left-, middle- and right-compatibility of a ternary relation with a binary L -relation in terms of inclusions of the binary L -relation in the left, middle and right traces of the ternary relation. Subsequently, several basic properties of these compatibility relations are discussed. Finally, we investigate the representation of all the binary fuzzy tolerance a crisp ternary relation is compatible with.

4.1/ Definitions and example

In this section, we study the compatibility of binary L -relations by replacing one of the binary L -relations by a (crisp) ternary relation. To that end, we first recall the original notion of compatibility of binary L -relations as introduced and studied by Kheniche et al. [48]. For more details, we refer to [9, 33].

Definition 21: [48]

Let S_1 and S_2 be two binary L -relations on a set X .

- (i) S_1 is called left-compatible with S_2 if it holds that

$$S_1(x, y) * S_2(x, a) \leq S_1(a, y),$$

for any $x, y, a \in X$;

- (ii) S_1 is called right-compatible with S_2 if it holds that

$$S_1(x, y) * S_2(y, b) \leq S_1(x, b),$$

for any $x, y, b \in X$;

- (iii) S_1 is called compatible with S_2 if it holds that

$$S_1(x, y) * S_2(x, a) * S_2(y, b) \leq S_1(a, b),$$

for any $x, y, a, b \in X$.

Note that this definition applies in particular when the binary L -relation S_1 is crisp. We now generalize this special case by replacing this (crisp) binary relation by a (crisp) ternary relation.

Definition 22:

Let T be a ternary relation and S be a binary L -relation on a set X .

- (i) T is called left-compatible with S if it holds that

$$\tau(T(x, y, z)) * S(x, a) \leq \tau(T(a, y, z)), \quad (4.1)$$

for any $x, y, z, a \in X$;

- (ii) T is called middle-compatible with S if it holds that

$$\tau(T(x, y, z)) * S(y, b) \leq \tau(T(x, b, z)), \quad (4.2)$$

for any $x, y, z, b \in X$;

- (iii) T is called right-compatible with S if it holds that

$$\tau(T(x, y, z)) * S(z, c) \leq \tau(T(x, y, c)), \quad (4.3)$$

for any $x, y, z, c \in X$;

- (iv) T is called compatible with S if it holds that

$$\tau(T(x, y, z)) * S(x, a) * S(y, b) * S(z, c) \leq \tau(T(a, b, c)), \quad (4.4)$$

for any $x, y, z, a, b, c \in X$.

Example 14:

Let us consider $X = \{x, y\}$, $L = [0, 1]$ and $* = \wedge$ (the minimum operation). Let T be the ternary relation and S be the binary L -relation on X defined as:
 $T = \{(x, y, x), (y, x, y)\}$ and

S	x	y
x	α	0
y	0	β

with $\alpha \neq 0$ and $\beta \neq 0$. It is clear that T is left-/middle-/right-compatible with S . Moreover, T is compatible with S .

The following proposition identifies two important implications that will be helpful in the study of the compatibility property.

Proposition 49:

Let T be a ternary relation and S be a binary L -relation on a set X . The following implications hold:

- (i) If T is left-, middle- and right-compatible with S , then T is compatible with S ;
- (ii) If T is compatible with S and S is reflexive, then T is left-, middle- and right-compatible with S .

Proof.

- (i) Suppose that T is left-, middle- and right-compatible with S .
 Let $x, y, z, a, b, c \in X$. Since T is left-compatible with S , it holds that

$$\tau(T(x, y, z)) * S(x, a) \leq \tau(T(a, y, z)).$$

Furthermore, since $*$ is increasing and T is middle-compatible with S , it follows that

$$\tau(T(x, y, z)) * S(x, a) * S(y, b) \leq \tau(T(a, y, z)) * S(y, b) \leq \tau(T(a, b, z)).$$

Similarly, since $*$ is increasing and T is right-compatible with S , we obtain

$$\tau(T(x, y, z)) * S(x, a) * S(y, b) * S(z, c) \leq \tau(T(a, b, z)) * S(z, c) \leq \tau(T(a, b, c)).$$

Hence, T is compatible with S .

- (ii) Setting $b = y$ and $c = z$ in (4.4) and taking into account the reflexivity of S , we obtain (4.1). Similarly, we can obtain (4.2) and (4.3).

□

The following proposition is immediate.

Proposition 50:

Let T be a ternary relation and S be a binary L -relation on a set X . The following equivalences hold:

- (i) T is left-compatible with S if and only if T^t is right-compatible with S ;
- (ii) T is middle-compatible with S if and only if T^t is middle-compatible with S ;
- (iii) T is right-compatible with S if and only if T^t is left-compatible with S .

The following proposition relates the middle-compatibility of a ternary relation with the left- and right-compatibility of its cyclic permutations.

Proposition 51:

Let T be a ternary relation and S be a binary L -relation on a set X . The following equivalences hold:

- (i) T is middle-compatible with S if and only if T^- is left-compatible with S ;
- (ii) T is middle-compatible with S if and only if T^+ is right-compatible with S .

Proof. We prove (i) only, as both cases are analogous. Let $x, y, z, a \in X$. Suppose that T is middle-compatible with S , then it holds that

$$\tau(T(z, x, y)) * S(x, a) \leq \tau(T(z, a, y)).$$

From the definition of T^- , it follows that

$$\tau(T^-(x, y, z)) * S(x, a) \leq \tau(T(z, a, y)) = \tau(T^-(a, y, z)).$$

Hence, T^- is left-compatible with S . In a similar way, we can prove the converse implication. $\square$

4.2/ Properties of the compatibility of a ternary relation and a binary fuzzy relation

In this section, we characterize the left-, middle- and right-compatibility of a ternary relation with a binary fuzzy relation in terms of inclusions of the binary fuzzy relation in the left, middle and right traces of the ternary relation. It is important to note that, just as for binary relations, the traces of a ternary relation are defined as binary relations.

The following proposition states that a ternary relation is compatible with its traces, just as is the case for a binary relation or a binary L -relation.

Proposition 52:

Let T be a ternary relation on a set X . The following statements hold:

- (i) T is left-compatible with T^ℓ ;
- (ii) T is middle-compatible with T^m ;
- (iii) T is right-compatible with T^r .

Proof. As the proofs of the different statements are similar, we only prove the first one. Let $x, y, z, a \in X$, then we need to show that

$$\tau(T(x, y, z)) * \tau(T^\ell(x, a)) \leq \tau(T(a, y, z)). \quad (4.5)$$

If $(x, y, z) \notin T$ or $(x, a) \notin T^\ell$, then inequality (4.5) trivially holds. If $(x, y, z) \in T$ and $(x, a) \in T^\ell$, then it holds that $(a, y, z) \in T$, and inequality (4.5) holds again. $\square$

The above proposition allows to prove an important theorem stating that left-, middle- and right-compatibility can be expressed in terms of inclusions involving traces. This result nicely generalizes corresponding results on the compatibility of two L -relations recalled further in Theorem 13.

Theorem 12:

Let T be a ternary relation and S be an L -relation on a set X . The following equivalences hold:

- (i) T is left-compatible with S if and only if $S \subseteq T^\ell$;
- (ii) T is middle-compatible with S if and only if $S \subseteq T^m$;
- (iii) T is right-compatible with S if and only if $S \subseteq T^r$.

Proof.

- (i) Suppose that T is left-compatible with S . Let $x, a \in X$, then we need to show that

$$S(x, a) \leq \tau(T^\ell(x, a)). \quad (4.6)$$

If $S(x, a) = 0$, then inequality (4.6) trivially holds; if $S(x, a) > 0$, then we need to show that $(x, a) \in T^\ell$. Let $y, z \in X$ such that $(x, y, z) \in T$, then the left-compatibility of T with S implies that

$$\tau(T(x, y, z)) * S(x, a) = S(x, a) \leq \tau(T(a, y, z)).$$

Hence, $\tau(T(a, y, z)) = 1$, i.e., $(a, y, z) \in T$. Hence, $(x, a) \in T^\ell$.

For the converse implication, it suffices to realise that if T is left-compatible with a binary L -relation S_2 , then it is also left-compatible with any binary L -relation $S_1 \subseteq S_2$. Since Proposition 52 (i) states that T is left-compatible with T^ℓ and $S \subseteq T^\ell$, it follows that T is left-compatible with S .

(ii) Follows from (i), Propositions 51, 15, 52.

(iii) Follows from (i), Propositions 50, 15, 52.

□

Combining Proposition 49 and Theorem 12 leads to the following corollary.

Corollary 22:

Let T be a ternary relation and S be a binary L -relation on a set X . The following implications hold:

- (i) If $S \subseteq T^l \cap T^m \cap T^r$, then T is compatible with S ;
- (ii) If T is compatible with S and S is reflexive, then $S \subseteq T^l \cap T^m \cap T^r$.

Combining Theorem 12 and Proposition 15 leads to the following corollaries.

Corollary 23:

Let T be a ternary relation and S be a binary L -relation on a set X . The following equivalences hold:

- (i) If T is left-compatible with S if and only if T^c is right-compatible with S ;
- (ii) If T is middle-compatible with S if and only if T^c is middle-compatible with S ;
- (iii) If T is right-compatible with S if and only if T^c is left-compatible with S ;
- (iv) If S is reflexive, then T is compatible with S if and only if T^c is compatible with S .

Corollary 24:

Let T be a ternary relation and S be a binary L -relation on a set X . The following equivalences hold:

- (i) If T is left-compatible with S if and only if T^d is left-compatible with S ;
- (ii) If T is middle-compatible with S if and only if T^d is middle-compatible with S ;
- (iii) If T is right-compatible with S if and only if T^d is right-compatible with S ;
- (iv) If S is reflexive, then T is compatible with S if and only if T^d is compatible with S .

Next, we recall some trivial properties of the compatibility of binary L -relation for further use.

Proposition 53: [48]

For any two binary L -relations S_1 and S_2 on a set X , the following equivalences hold:

- (i) S_1 is left-compatible with S_2 if and only if S_1^t is right-compatible with S_2 ;
- (ii) S_1 is right-compatible with S_2 if and only if S_1^t is left-compatible with S_2 .

Note that the following theorem has been adopted from Kheniche et al. [48] taking into account the change of terminology of traces of a binary L -relation introduced in [89] to facilitate a more elegant formulation of this theorem and the proper introduction of the traces of a ternary relation.

Theorem 13: [48]

For any two binary L -relations S_1 and S_2 on a set X , the following equivalences hold:

- (i) S_1 is left-compatible with S_2 if and only if $S_2 \subseteq S_1^l$;
- (ii) S_1 is right-compatible with S_2 if and only if $S_2 \subseteq S_1^r$.

The following proposition shows that the compatibility of a ternary relation with a binary L -relation S_1 that is itself compatible with a binary L -relation S_2 implies the compatibility of that ternary relation with S_2 .

Proposition 54:

Let T be a ternary relation and S_1 and S_2 be two binary L -relations on a set X such that S_1 is reflexive. The following implications hold:

- (i) (a) If T is left-compatible with S_1 , S_1 is left-compatible with S_2 and S_2 is symmetric, then T is left-compatible with S_2 ;
- (b) If T is left-compatible with S_1 and S_1 is right-compatible with S_2 , then T is left-compatible with S_2 ;
- (ii) (a) If T is middle-compatible with S_1 , S_1 is left-compatible with S_2 and S_2 is symmetric, then T is middle-compatible with S_2 ;
- (b) If T is middle-compatible with S_1 and S_1 is right-compatible with S_2 , then T is middle-compatible with S_2 ;
- (iii) (a) If T is right-compatible with S_1 , S_1 is left-compatible with S_2 and S_2 is symmetric, then T is right-compatible with S_2 ;
- (b) If T is right-compatible with S_1 and S_1 is right-compatible with S_2 , then T is right-compatible with S_2 .

Proof. We only prove (i)-(a), the other implications can be proved analogously. Suppose that T is left-compatible with S_1 , S_1 is left-compatible with S_2 and S_1 is reflexive and S_2 is symmetric. Since T is left-compatible with S_1 , it follows from Theorem 12 that $S_1 \subseteq T^l$. Since S_1 is left-compatible with S_2 , it follows from Theorem 13 that $S_2 \subseteq S_1^l$. The symmetry of S_2 guarantees that $S_2 \subseteq (S_1^l)^t$. Since S_1 is reflexive, it follows from Theorem 1 that $(S_1^l)^t \subseteq S_1$. Hence, $S_2 \subseteq S_1 \subseteq T^l$. Thus, T is left-compatible with S_2 . $\square$

4.3/ Compatibility of a ternary relation with a binary fuzzy relation in terms of traces

In this section, we discuss the relationship between the compatibility of a ternary relation and that of its traces. The following theorem expresses that the compatibility of a ternary relation with a given binary L -relation is equivalent to the compatibility of its traces with that binary L -relation.

Theorem 14:

Let T be a ternary relation and S be a binary L -relation on a set X . The following equivalences hold:

- (i) T is left-compatible with S if and only if $(T^l)^t$ is left-compatible with S ;
- (ii) T is middle-compatible with S if and only if T^m is right-compatible with S ;
- (iii) T is middle-compatible with S if and only if $(T^m)^t$ is left-compatible with S ;
- (iv) T is right-compatible with S if and only if T^r is right-compatible with S .

Proof.

- (i) Suppose that T is left-compatible with S . Due to Theorem 12 and Corollary 3, it holds that $S \subseteq T^l = (T^l)^r$, i.e. T^l is right-compatible with S . From Proposition 53 (ii), it follows that $(T^l)^t$ is left-compatible with S . The converse goes in the same way.
- (ii) Suppose that T is middle-compatible with S . Due to Theorem 12 and Corollary 3, it holds that $S \subseteq T^m = (T^m)^r$, i.e. T^m is right-compatible with S . The converse goes in the same way.
- (iii) Follows from (ii) and Proposition 53 (ii).
- (iv) Suppose that T is right-compatible with S . Due to Theorem 12 and Corollary 3, it holds that $S \subseteq T^r = (T^r)^r$, i.e. T^r is right-compatible with S . The converse goes in the same way.

□

4.4/ Relationship between the compatibility of a ternary relation and that of its projections

The following result expresses that the compatibility of a ternary relation with a given binary L -relation implies the compatibility of its projections with that binary L -relation.

Combining Theorem 13 and Proposition 19 leads to the following proposition.

Proposition 55:

Let T be a ternary relation and S be a binary L -relation on a set X . The following implications hold:

- (i) If T is left-compatible (resp. right-compatible) with S , then $P_m(T)$ is left-compatible (resp. right-compatible) with S ;
- (ii) If T is middle-compatible with S , then $P_l(T)$ is left-compatible with S and $P_r(T)$ is right-compatible with S .

Note that in chapter 2, we showed how the traces of a given ternary relation can be used to study the properties of this relation. For this purpose, we associated the binary relations $W_l(T)$, $W_m(T)$ and $W_r(T)$ with a given ternary relation T . Here, we can define a relationship between the compatibility of T and that of its particular associated relations.

Combining Theorem 12, Proposition 28 and Theorem 13 leads to the following result.

Proposition 56:

Let T be a ternary relation and S be a binary fuzzy relation on a set X . The following statements hold:

- (i) If T is left-compatible with S , then $W_l(T)$ is left-compatible with S ;
- (ii) If T is middle-compatible with S , then $(W_m(T))^t$ is left-compatible with S and $W_m(T)$ is right-compatible with S ;
- (iii) If T is right-compatible with S , then $W_r(T)$ is right-compatible with S .

4.5/ Relationship between the compatibility of a binary relation and that of its cylindrical extensions

In this section, we discuss the relationship between the compatibility of a binary relation and that of its cylindrical extensions.

The following theorem expresses that the compatibility of a binary relation with a given binary L -relation is equivalent to the compatibility of its cylindrical extensions with that binary L -relation.

Combining Theorem 13 and Proposition 23 leads to the following result.

Theorem 15:

Let R be a binary relation and S be a binary L -relation on a set X .

- (i) The following statements are equivalent:
 - (a) R is left-compatible with S ;
 - (b) $C_\ell(R)$ is middle-compatible with S ;
 - (c) $C_m(R)$ is left-compatible with S ;
 - (d) $C_r(R)$ is left-compatible with S ;
- (ii) The following statements are equivalent:
 - (e) R is right-compatible with S ;
 - (f) $C_\ell(R)$ is right-compatible with S ;
 - (g) $C_m(R)$ is right-compatible with S ;
 - (h) $C_r(R)$ is middle-compatible with S .

Proof. We only prove (a) $\Leftrightarrow$ (b), the other equivalences can be proved similarly. Suppose that R is left-compatible with S . Due to Theorem 13 and Proposition 23 (i), this is equivalent to $S \subseteq R^\ell = (C_\ell(R))^m$, i.e. $C_\ell(R)$ is middle-compatible with S . $\square$

4.6/ Interaction of compatibility with inclusion and basic set operations

In this section, we study the interaction of the compatibility relations with inclusion and set-theoretical operations. The characterizations in terms of inclusions expressed by Theorem 12 turn out to play an important role in this study.

First, we show the interaction with inclusion as follows.

Proposition 57:

Let T be a ternary relation and S_1 and S_2 be two binary L -relations on a set X such that $S_1 \subseteq S_2$. The following implications hold:

- (i) If T is left-compatible with S_2 , then T is left-compatible with S_1 ;
- (ii) If T is middle-compatible with S_2 , then T is middle-compatible with S_1 ;
- (iii) If T is right-compatible with S_2 , then T is right-compatible with S_1 ;
- (iv) If T is compatible with S_2 and S_2 is reflexive, then T is compatible with S_1 .

Proof. We only prove (i), the other implications can be proved analogously. Suppose that T is left-compatible with S_2 . From Theorem 12 (i), it follows that $S_2 \subseteq T^\ell$. Since $S_1 \subseteq S_2 \subseteq T^\ell$, it then follows that T is left-compatible with S_1 . $\square$

Second, we show the interaction with intersection and union in the following proposition.

Proposition 58:

For any ternary relation T and family $(T_i)_{i \in I}$ of ternary relations and any binary L -relations S and family $(S_i)_{i \in I}$ of binary L -relations on a set X , the following implications hold:

- (i) If T_i is left-/middle-/right-compatible with S , for any $i \in I$, then $\bigcap_{i \in I} T_i$ and $\bigcup_{i \in I} T_i$ are left-/middle-/right-compatible with S ;
- (ii) If T is left-/middle-/right-compatible with S_i , for any $i \in I$, then T is left-/middle-/right-compatible with $\bigcap_{i \in I} S_i$ and $\bigcup_{i \in I} S_i$.

Proof. We only prove (i) for left-compatibility and intersection, the other cases being analogous. Suppose that T_i is left-compatible with S , for any $i \in I$. From Theorem 12 (i), it follows that $S \subseteq T_i^\ell$, for any $i \in I$, and, hence, $S \subseteq \bigcap_{i \in I} T_i^\ell$. From Proposition 17 (i), it then follows that $S \subseteq (\bigcap_{i \in I} T_i)^\ell$, i.e. $\bigcap_{i \in I} T_i$ is left-compatible with S . $\square$

Proposition 59:

For any ternary relation T and family $(T_i)_{i \in I}$ of ternary relations and any binary L -relations S and family $(S_i)_{i \in I}$ of binary L -relations on a set X , the following implications hold:

- (i) If T_i is compatible with S , for any $i \in I$, and S is reflexive, then $\bigcap_{i \in I} T_i$ and $\bigcup_{i \in I} T_i$ are compatible with S ;
- (ii) If T is compatible with S_i and S_i is reflexive, for any $i \in I$, then T is compatible with $\bigcap_{i \in I} S_i$ and $\bigcup_{i \in I} S_i$.

Proof. We only prove (i) for intersection, the other implications can be proved analogously. Suppose that S is reflexive and that T_i is compatible with S , for any $i \in I$. From Proposition 49 (ii), it follows that T_i is left-, middle- and right-compatible with S , for any $i \in I$. Since T_i is left-compatible with S , for any $i \in I$, it follows from Proposition 58 (i) that $\bigcap_{i \in I} T_i$ is left-compatible with S . In a similar way, we prove that $\bigcap_{i \in I} T_i$ is middle- and right-compatible with S , for any $i \in I$. Therefore, $\bigcap_{i \in I} T_i$ is compatible with S . $\square$

4.7/ Interaction of compatibility with relational compositions

In this section, we show that compatibility is preserved under appropriate relational compositions.

Proposition 60:

Let T , R and S be a ternary, binary and binary L -relation on a set X , respectively. The following implications hold:

- (i) If T is left-compatible with S , then $T \ltimes R$, $T \blacktriangleleft R$ and $R \blacktriangleright T$ are left-compatible with S ;
- (ii) If T is middle-compatible with S , then $T \ltimes R$, $R \ltimes T$, $T \blacktriangleleft R$ and $R \blacktriangleright T$ are middle-compatible with S ;
- (iii) If T is right-compatible with S , then $R \ltimes T$, $T \blacktriangleleft R$ and $R \blacktriangleright T$ are right-compatible with S ;
- (iv) If R is left-compatible with S , then $R \ltimes T$ is left-compatible with S and $R \blacktriangleright T$ is middle-compatible with S ;
- (v) If R is right-compatible with S , then $T \ltimes R$ is right-compatible with S and $T \blacktriangleleft R$ is middle-compatible with S .

Proof. We only prove (i), the other cases being analogous. Suppose that T is left-compatible with S , which implies that $S \subseteq T^\ell$. From Proposition 25 (i), it follows that $S \subseteq (T \ltimes R)^\ell$. Hence, $T \ltimes R$ is left-compatible with S . $\square$

From Propositions 49 and 60, the following result follows immediately.

Corollary 25:

Let T , R and S be a ternary, binary and binary L -relation on a set X , respectively. If T and R are compatible with S and S is reflexive, then $T \ltimes R$, $R \ltimes T$, $T \blacktriangleleft R$ and $R \blacktriangleright T$ are compatible with S .

4.8/ Interaction of compatibility with a particular type of composition

It is well known that there is much less agreement in literature on the definition of composition of ternary relations. Indeed, various alternative definitions, each with its own motivation, have been proposed for instance [47].

In this section, we defined the $\circ$ -composition of two ternary relations T_1 and T_2 on a set X by:

$$T_1 \circ T_2 = \{(\alpha, \gamma, z) \in X^3 \mid (\exists(t, s) \in X^2)((\alpha, \gamma, t) \in T_1 \text{ and } (s, t, z) \in T_2)\},$$

as an example only and not limited to this type of compositions in order to study its compatibility.

We first introduce some results related of this composition that will be needed throughout this study.

Proposition 61:

For any family of ternary relations $(T_i)_{i \in I}$ on a set X , then it holds that $\bigcap_{i \in I} T_i^\lambda \subseteq (\bigcirc_{i \in I} T_i)^\lambda$, for any $\lambda \in \{\ell, m, r\}$.

Proof. We only prove that $\bigcap_{i \in I} T_i^\ell \subseteq (\bigcirc_{i \in I} T_i)^\ell$, the other cases of λ can be proved analogously. Let $(x, y) \in \bigcap_{i \in I} T_i^\ell$, then $(x, y) \in T_i^\ell$, for any $i \in I$. It holds that, $(x, a, b) \in T_i$ implies that $(y, a, b) \in T_i$, for any $i \in I$. Hence, $(x, a, b) \in \bigcirc_{i \in I} T_i$ implies that $(y, a, b) \in \bigcirc_{i \in I} T_i$. Thus, $(x, y) \in (\bigcirc_{i \in I} T_i)^\ell$. Therefore, $\bigcap_{i \in I} T_i^\ell \subseteq (\bigcirc_{i \in I} T_i)^\ell$. $\square$

Proposition 62:

Let T_1 and T_2 be two ternary relations on a set X , then

$$T_1^\ell \subseteq (T_1 \circ T_2)^\ell, T_1^m \subseteq (T_1 \circ T_2)^m \text{ and } T_1^r \subseteq (T_2 \circ T_1)^r.$$

Proof. We only prove that $T_1^\ell \subseteq (T_1 \circ T_2)^\ell$, as the other inclusions can be proved similarly. Suppose that $(x, y) \in T_1^\ell$ and let $(x, a, b) \in T_1 \circ T_2$ be such that $a, b \in X$. Then $(\exists(t, s) \in X^2)((x, a, t) \in T_1 \text{ and } (s, t, b) \in T_2)$. Since $(x, y) \in T_1^\ell$ and $(x, a, t) \in T_1$, it follows that $(y, a, t) \in T_1$, which implies that $(y, a, b) \in T_1 \circ T_2$. Hence, $(x, y) \in (T_1 \circ T_2)^\ell$. Therefore, $T_1^\ell \subseteq (T_1 \circ T_2)^\ell$. $\square$

Next, we show that the left-, middle- and right-compatibility are preserved by the $\circ$ -composition of ternary relations.

Proposition 63:

Let T_1, T_2 be two ternary relations and S be a binary L -relation on a set X . The following implications hold:

- (i) If T_1 is left-compatible with S , then $T_1 \circ T_2$ is left-compatible with S ;
- (ii) If T_1 is middle-compatible with S , then $T_1 \circ T_2$ is middle-compatible with S ;
- (iii) If T_1 is right-compatible with S , then $T_2 \circ T_1$ is right-compatible with S .

Proof.

- (i) Suppose that T_1 is left-compatible with S , which implies that $S \subseteq T_1^\ell$. From Proposition 62, it follows that $S \subseteq T_1^\ell \subseteq (T_1 \circ T_2)^\ell$. Hence, $T_1 \circ T_2$ is left-compatible with S . The statements (ii) and (iii) are proved analogously to (i).

$\square$

The following corollary is immediate.

Corollary 26:

Let T_1, T_2 be two ternary relations on a set X and S be a binary L -relation on X . The following statements hold:

- (i) If T_1 and T_2 are left-compatible with S , then $T_1 \circ T_2$ and $T_2 \circ T_1$ are left-compatible with S ;
- (ii) If T_1 and T_2 are middle-compatible with S , then $T_1 \circ T_2$ and $T_2 \circ T_1$ are middle-compatible with S ;
- (iii) If T_1 and T_2 are right-compatible with S , then $T_1 \circ T_2$ and $T_2 \circ T_1$ are right-compatible with S .

Consecutively, Proposition 61 allows to the following general result.

Corollary 27:

For any family $(T_i)_{i \in I}$ of ternary relations and S a binary L -relation on a set X , the following implications hold:

- (i) If T_i is left-compatible with S , for any $i \in I$, then $\bigcirc_{i \in I} T_i$ is left-compatible with S ;
- (ii) If T_i is middle-compatible with S , for any $i \in I$, then $\bigcirc_{i \in I} T_i$ is middle-compatible with S ;
- (iii) If T_i is right-compatible with S , for any $i \in I$, then $\bigcirc_{i \in I} T_i$ is right-compatible with S .

Next, we need to recall the following result concerning the n -th power of a binary L -relation.

Proposition 64: [31]

Let S be a binary L -relation on a set X . The following statements hold:

- (i) If S is reflexive, then it holds that $(\forall n \in \mathbb{N}^*)(S^n \subseteq S^{n+1})$.
- (ii) If S is transitive, then it holds that $(\forall n \in \mathbb{N}*)(S^{n+1} \subseteq S^n)$.
- (iii) If S is reflexive and transitive, then it hold that $(\forall n \in \mathbb{N}*)(S^n = S)$.

Note that the n -th power S^n of S is recursively defined as follows.

Definition 23: [31]

Let S be a binary L -relation on a set X . For any $n \in \mathbb{N}^*$,

$$S^1 = S \text{ and } S^{n+1} = S^n \circ S,$$

such that $\mathbb{N}^*$ is the set of non-zero natural numbers.

Proposition 65:

For any family $(S_i)_{i \in I}$ of binary L -relations and T a ternary relation on a set X , the following implications hold:

- (i) If T is left-compatible with S_i , for any $i \in I$, then T is left-compatible with $\bigcirc_{i \in I} S_i$;
- (ii) If T is middle-compatible with S_i , for any $i \in I$, then T is middle-compatible with $\bigcirc_{i \in I} S_i$;
- (iii) If T is right-compatible with S_i , for any $i \in I$, then T is right-compatible with $\bigcirc_{i \in I} S_i$.

Proof.

- (i) Suppose that T is left-compatible with S_i , for any $i \in I$. From Theorem 12 (i), it holds that $S_i \subseteq T^\ell$, for any $i \in I$. This implies that $\bigcirc_{i \in I} S_i \subseteq \bigcirc_{i \in I} T^\ell$. Since T^ℓ is reflexive and transitive, it follows from Proposition 64 that $\bigcirc_{i \in I} S_i \subseteq T^\ell$. Thus, T is left-compatible with $\bigcirc_{i \in I} S_i$.

Statements (ii) and (iii) are proved analogously to (i). □

Combining Proposition 49, Corollary 27 and Proposition 65 leads to the following corollary.

Corollary 28:

Let $T, (T_i)_{i \in I}$ be ternary relations and $S, (S_i)_{i \in I}$ be binary L -relations on a set X . The following implications hold:

- (i) If T is compatible with S_i such that S_i is reflexive, for any $i \in I$, then T is compatible with $\bigcirc_{i \in I} S_i$;
- (ii) If T_i is compatible with S , for any $i \in I$ and S is reflexive, then $\bigcirc_{i \in I} T_i$ is compatible with S .

4.9/ Compatibility of particular kinds of ternary relations with a binary fuzzy relation

In this section, we study the compatibility of particular types of ternary relations with a binary L -relation. To that end, we need the definition of transitivity of ternary relations. Note that there exist various definitions of transitivity (see, e.g., [26, 63]), but here we restrict to those definitions from our work on traces of ternary relations [89], which we mentioned in chapter 2.

Combining Theorem 12 and Corollary 5 easily leads to the following result.

Proposition 66:

Let T be a ternary relation and S be a binary L -relation on a set X . The following implications hold:

- (i) If T is reflexive and c_2 - or c_6 -transitive, then T is left-compatible with S if and only if $S \subseteq (W_l(T))^t$;
- (ii) If T is reflexive and c_1 - and c_6 -transitive, then T is middle-compatible with S if and only if $S \subseteq W_m(T)$;
- (iii) If T is reflexive and c_1 - or c_3 -transitive, then T is right-compatible with S if and only if $S \subseteq W_r(T)$.

Next, we study the case of reflexive and antisymmetric ternary relations. The following proposition generalizes a remarkable result of De Baets et al. [33].

Proposition 67:

Let T be a reflexive and antisymmetric ternary relation and S be a reflexive binary L -relation on a set X . It holds that T is compatible with S if and only if S is the crisp equality on X .

Proof. Suppose that T is compatible with S . Let $x, z \in X$, then it holds that

$$\tau(T(z, z, z)) * S(z, x) * S(z, z) * S(z, z) \leq \tau(T(x, z, z)).$$

Since T and S are reflexive, it then follows that $S(z, x) \leq \tau(T(x, z, z))$.

Since $\min(\tau(T(x, z, z)), \tau(T(z, z, x))) = 0$ for $x \neq z$, it follows that $S(z, x) = 0$. Since S is reflexive, it follows that S is the crisp equality on X . Conversely, if S is the crisp equality on X , then it is obvious that T is compatible with S . $\square$

Combining Proposition 29 and Theorem 12 easily leads to the following result. It expresses that for particular types of ternary relations, two or all notions of compatibility coincide.

Proposition 68:

Let T be a ternary relation S be a binary L -relation on a set X . The following implications hold:

- (i) If T is symmetric, then T is left-compatible with S if and only if T is right-compatible with S ;
- (ii) If T is cyclic, then the following statements are equivalent:
 - (a) T is left-compatible with S ;
 - (b) T is middle-compatible with S ;
 - (c) T is right-compatible with S .

4.10/ Representation of all binary fuzzy tolerance relations a crisp ternary relation is compatible with

In this section, we provide a representation of all the binary fuzzy tolerance relations that a crisp ternary relation is compatible with. This representation is largely draws upon the notions of the clones relations of a crisp ternary relation introduced in Chapter 3.

Definition 24:

Let T be a ternary relation on a set X , for any $\lambda \in \{\ell, m, r\}$ and $I = \{0, \dots, 5\}$, we associate the following binary relations on X with T defined by:

- (i) $A_T^\lambda = \{(x, y) \in X^2 \mid (x, y) \in \approx_T^\lambda \wedge (\forall z \in X)((x, y, z) \in \bigcap_{i \in I} T^{\sigma^i})\}$;
- (ii) $B_T^\lambda = \{(x, y) \in X^2 \mid (x, y) \in \approx_T^\lambda \wedge (\forall z \in X)((x, y, z) \in \bigcap_{i \in I} (T^c)^{\sigma^i})\}$;
- (iii) $A_T = \{(x, y) \in X^2 \mid (x, y) \in \approx_T \wedge (\forall z \in X)((x, y, z) \in \bigcap_{i \in I} T^{\sigma^i})\}$;
- (iv) $B_T = \{(x, y) \in X^2 \mid (x, y) \in \approx_T \wedge (\forall z \in X)((x, y, z) \in \bigcap_{i \in I} (T^c)^{\sigma^i})\}$.

Note that $A_T = \bigcap_{\lambda \in \{\ell, m, r\}} A_T^\lambda$ and $B_T = \bigcap_{\lambda \in \{\ell, m, r\}} B_T^\lambda$.

Remark 6:

For a given ternary relation T on a set X , one can easily verify that $A_T^\lambda \cup \delta$, $B_T^\lambda \cup \delta$, $A_T^\lambda \cup B_T^\lambda \cup \delta$ are binary tolerance relations on X , for any $\lambda \in \{\ell, m, r\}$. Furthermore, $A_T \cup \delta$, $B_T \cup \delta$, $A_T \cup B_T \cup \delta$ are also binary tolerance relations on X . Here, we mention that $\delta = \{(x, y) \in X^2 \mid x = y\}$.

The following proposition shows the compatibility of a given ternary relation T with A_T^λ , where $\lambda \in \{\ell, m, r\}$.

Proposition 69:

Let T be a ternary relation on a set X . The following statements hold:

- (i) T is left-compatible with A_T^ℓ ;
- (ii) T is middle-compatible with A_T^m ;
- (iii) T is right-compatible with A_T^r ;
- (iv) T is compatible with A_T .

Proof. As the proofs of the different statements are similar, we only prove the first one. Let $x, y, z, a \in X$, then we need to show that

$$\tau(T(x, y, z)) * \tau(A_T^\ell(x, a)) \leq \tau(T(a, y, z)). \quad (4.7)$$

If $(x, y, z) \notin T$ or $(x, a) \notin A_T^\ell$, then inequality (4.7) trivially holds. If $(x, y, z) \in T$ and $(x, a) \in A_T^\ell$, then we distinguish five cases.

- (a) If $y, z \in X \setminus \{x, a\}$, then the fact that $(x, a) \in \approx_T^\ell$ and $(x, y, z) \in T$ imply $(a, y, z) \in T$. Hence, the inequality (4.7) holds.
- (b) If $y = x$, then the fact that $(x, a) \in A_T^\ell$ implies that $\tau(A_T^\ell(x, a)) \leq \tau(\bigcap_{i \in I} T^{\sigma i}(x, a, z)) \leq \tau(T(a, x, z))$. Hence, the inequality (4.7) holds.
- (c) If $y = a$, then the fact that $(x, a) \in A_T^\ell$ implies that $\tau(A_T^\ell(x, a)) \leq \tau(\bigcap_{i \in I} T^{\sigma i}(a, a, z)) \leq \tau(T(a, a, z))$. Hence, the inequality (4.7) holds.

In a similar way, we prove that the inequality (4.7) holds for the two last cases (d) $z = x$ and (e) $z = a$. $\square$

In similar way, the following proposition shows the compatibility of a given ternary relation T with B_T^λ , where $\lambda \in \{\ell, m, r\}$.

Proposition 70:

Let T be a ternary relation on a set X . The following statements hold:

- (i) T is left-compatible with B_T^ℓ ;
- (ii) T is middle-compatible with B_T^m ;
- (iii) T is right-compatible with B_T^r ;
- (iv) T is compatible with B_T .

Corollary 29:

Let T be a ternary relation on a set X . The following statements hold:

- (i) T is left-compatible with $A_T^\ell \cup B_T^\ell$;
- (ii) T is middle-compatible with $A_T^m \cup B_T^m$;
- (iii) T is right-compatible with $A_T^r \cup B_T^r$;
- (iv) T is compatible with $A_T \cup B_T$.

Theorem 16:

Let T be a ternary relation and E be an L -tolerance relation on a set X . Then it holds that T is compatible with E if and only if $E \subseteq A_T \cup B_T \cup \delta$.

Proof. Suppose that T is compatible with E , i.e.,

$$\tau(T(x, y, z)) * E(x, r) * E(y, s) * E(z, t) \leq \tau(T(r, s, t)), \quad (4.8)$$

for any $x, y, z, r, s, t \in X$. Let $a, b \in X$, if $a = b$ or $E(a, b) = 0$, then it trivially holds that $E(a, b) \leq \tau((A_T \cup B_T \cup \delta)(a, b))$. If $a \neq b$ and $E(a, b) > 0$, then we need to prove that $\tau((A_T \cup B_T)(a, b)) = 1$, i.e., it holds that $(\forall z \in X) ((a, b, z) \in T^{\sigma i}$, for any $i \in \{0, \dots, 5\}$) or $((a, b, z) \in (T^c)^{\sigma i}$, for any $i \in \{0, \dots, 5\})$, as well as $(a, b) \in \approx_T$.

- (i) First, we prove that $(\forall z \in X) ((a, b, z) \in T^{\sigma^i}$, for any $i \in \{0, \dots, 5\}$) or $((a, b, z) \in (T^c)^{\sigma^i}$, for any $i \in \{0, \dots, 5\})$. On the one hand, since E is reflexive and symmetric and T is compatible with E , it follows that

$$\tau(T(a, b, z)) * E(a, b) * E(b, a) * E(z, z) \leq \tau(T(b, a, z)).$$

>0 >0 =1

This implies that $\tau(T(a, b, z)) \leq \tau(T(b, a, z))$. On the other hand, it follows that

$$\tau(T(b, a, z)) * E(a, b) * E(b, a) * E(z, z) \leq \tau(T(a, b, z)),$$

>0 >0 =1

which implies also that $\tau(T(b, a, z)) \leq \tau(T(a, b, z))$. Hence,

$$\tau(T(a, b, z)) = \tau(T(b, a, z)) = \tau(T^{\sigma^1}(b, a, z)).$$

Using the inequality (4.8), we obtain in an analogous way that

$$\tau(T(a, b, z)) = \tau(T(b, a, z)) = \tau(T^{\sigma^i}(b, a, z)),$$

for any $i \in \{0, \dots, 5\}$. Therefore, $(\forall z \in X) ((a, b, z) \in T^{\sigma^i}$, for any $i \in \{0, \dots, 5\})$ or $((a, b, z) \in (T^c)^{\sigma^i}$, for any $i \in \{0, \dots, 5\})$.

- (ii) Second, it remains to prove that $(a, b) \approx_T$. Let $s, t \in X \setminus \{a, b\}$. We distinguish four subcases.

- (a) If $(a, s, t) \in T$, then since E is reflexive and T is compatible with E , it follows that

$$0 < E(a, b) = \tau(T(a, s, t)) * E(a, b) * E(s, s) * E(t, t) \leq \tau(T(b, s, t)).$$

=1 >0 =1 =1

This implies that $\tau(T(b, s, t)) > 0$, whence $(b, s, t) \in T$.

- (b) If $(b, s, t) \in T$, then since E is reflexive and T is compatible with E , it follows that

$$0 < E(a, b) = \tau(T(b, s, t)) * E(b, a) * E(s, s) * E(t, t) \leq \tau(T(a, s, t)).$$

=1 >0 =1 =1

This implies that $\tau(T(a, s, t)) > 0$, whence $(a, s, t) \in T$.

Hence, $(a, b) \approx_T^l$. In similar way, we get that $(a, b) \approx_T^m$ and $(a, b) \approx_T^r$. Thus, $(a, b) \approx_T$.

Finally, we conclude that $E \subseteq A_T \cup B_T \cup \delta$.

Conversely, if $E \subseteq A_T \cup B_T \cup \delta$, then due to the monotonicity of $*$, it holds that

$$\begin{aligned} \tau(T(x, y, z)) * E(x, r) * E(y, s) * E(z, t) &\leq \tau(T(x, y, z)) * \tau((x, r) \in (A_T \cup B_T \cup \delta)) \\ &\quad * \tau((y, s) \in (A_T \cup B_T \cup \delta)) * \tau((z, t) \in (A_T \cup B_T \cup \delta)), \end{aligned}$$

for any $x, y, z, r, s, t \in X$. From Corollary 29, it follows that

$$\tau(T(x, y, z)) * E(x, r) * E(y, s) * E(z, t) \leq \tau(T(r, s, t)),$$

for any $x, y, z, r, s, t \in X$. Thus, T is compatible with E . □

Theorem 16 leads to the following corollary.

Corollary 30:

Let T be a ternary relation on a set X . Then it holds that $A_T \cup B_T \cup \delta$ is the greatest L -tolerance relation on X that T is compatible with.

Theorem 17:

Let T be a ternary relation and E be an L -tolerance relation on a set X . Then it holds that T is compatible with E if and only if there exist two L -tolerance relations α and β on X with $\alpha \subseteq A_T \cup \delta$ and $\beta \subseteq B_T \cup \delta$ such that $E = \alpha \cup \beta$.

Proof. Suppose that T is compatible with E . Consider the L -relations α and β on X defined by

$$\alpha(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in A_T \cup \delta, \\ 0 & , \text{ otherwise,} \end{cases}$$

$$\beta(x, y) = \begin{cases} E(x, y) & , \text{ if } (x, y) \in B_T \cup \delta, \\ 0 & , \text{ otherwise.} \end{cases}$$

Note that if $(x, y) \in A_T \cup \delta$, then it also holds that $(y, x) \in A_T \cup \delta$. Similarly, if $(x, y) \in B_T \cup \delta$, then also $(y, x) \in B_T \cup \delta$. Hence, α and β are L -tolerance relations on X . Since T is compatible with E , it follows from Theorem 16 that $E \subseteq A_T \cup B_T \cup \delta$, whence $E(x, y) = 0$ if $(x, y) \in (A_T \cup B_T \cup \delta)^c$. Hence, it follows that $E = \alpha \cup \beta$.

Conversely, let $\alpha \subseteq A_T \cup \delta$ and $\beta \subseteq B_T \cup \delta$ be two L -tolerance relations on X . One can easily verify that $E = \alpha \cup \beta$ is an L -tolerance relation on X . Since $E \subseteq A_T \cup B_T \cup \delta$, it follows from Theorem 16 that T is compatible with E . $\square$

General conclusions and future research

In this thesis, we have extended the notions of left and right traces of a binary relation to the left, middle and right traces of a ternary relation, and we have investigated their properties in detail. We have also studied the properties of a ternary relation in terms of its traces. Moreover, we have introduced new compositions for combining a ternary relation with a binary relation and vice versa. The different traces of a ternary relation turn out to be the greatest solutions of appropriate relational inequalities associated with these compositions.

Similarly to that of the traces of a ternary relation, we have extended the notions of left and right clone relations of a binary relation to the left, middle and right clone relations of a ternary relation. Throughout this thesis, the important properties of these clone relations have been analysed. We have also studied the properties of a ternary relation in terms of its clone relations. Moreover, we have investigated the interaction of the clone relations of a ternary relation with the basic set operations based on their properties. The characterization of the clone relations of disjoint union have been detailed.

In this work, we have generalized the notion of compatibility of a binary relation with a binary L -relation to the notion of compatibility of a ternary relation with a binary L -relation. More precisely, we have introduced the notions of left-, middle- and right-compatibility and have shown that they can be expressed in terms of inclusions of the binary L -relation in the left, middle and right traces of the ternary relation. Moreover, the relationship between the compatibility of a ternary relation and that of its traces and its projections, as well as the relationship between the compatibility of a binary relation and that of its cylindrical extensions have been investigated. Also, we have studied the interaction of the compatibility notions with basic set operations and relational compositions, and we have paid attention to the compatibility of several classes of ternary relations. Finally, we have studied the representation of binary fuzzy relations a crisp ternary relation is compatible with.

Proceeding from the idea of our determination to continue the approach that we have initiated in supporting research into ternary relations, future work is anticipated in multiple directions. First, the future work will be directed towards how extend the notion of clonal sets from binary relations introduced in [88] to ternary relations. In other words, we propose to introduce the notion of clonal set of a ternary relation. Second, we intend to extend our work and all that we have proposed as future works to ternary fuzzy relations.

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تلاؤم العلاقات الثلاثية مع العلاقات الثنائية الضبابية

ملخص: في السابق، لم تحظ العلاقات الثلاثية بالاهتمام الكافي من الدراسات مثل العلاقات الثنائية، غير أنه في الأونة الأخيرة شوهد إقبال ملحوظ على ذلك. انطلاقاً من أهمية العلاقات الثلاثية والاهتمام المتزايد عليها، اتجه تفكيرنا نحو المساهمة بإضافة في هذا الإطار. هذا يعني أن الهدف الرئيسي لهذا العمل تمثل في إضافة دراسة جديدة في مجال العلاقات الثلاثية. بشكل أكثر تحديداً، تم تركيز الهدف نحو تحقيق و دراسة مدى تلاؤم علاقة ثلاثية مع علاقة ثنائية ضبابية، على غرار الحالة الثنائية.

خلال البحث وجدنا أن دراسة التلاؤم المذكور أعلاه، يتطلب دراسة تفصيلية لأدوات فعالة لتحقيق ذلك الغرض. في هذه الأطروحة، ركزنا على اثنين منهم يهدفان إلى تمديد المفاهيم الهامة من علاقات الآثار و الاستنساخ لعلاقة ثنائية نحو ما يقابلها في العلاقات الثلاثية. الدراسة الأولى، سمحت لنا بتوصيف علاقة التلاؤم بدلالة علاقات الآثار، حيث أثبتنا أن هذا التلاؤم يمكن التعبير عنه من حيث تضمين العلاقة الثنائية الضبابية في علاقات الآثار لعلاقة ثلاثية. أما الدراسة الثانية، فقد سمحت لنا بتمثيل كل العلاقات الضبابية التوافقية التي تتلاءم مع علاقة كلاسيكية ثلاثية.

الكلمات المفتاحية : العلاقات الثنائية؛ العلاقات الثلاثية؛ الآثار؛ تراكيب العلاقات؛ علاقات الاستنساخ؛ التلاؤم.

Compatibility of ternary relations with binary fuzzy relations

Abstract: Previously, ternary relations did not get enough attention from studies like binary relations, but recently there has been a noticeable turnout. Inspired by the importance of them, our thinking has led to the addition of a contribution to this framework. It means that the main aim of this work is to contribute to the addition of a new study in the field of ternary relations. More specifically, we aim to study the compatibility of a ternary relation with a binary fuzzy relation, similarly to the binary case.

The study of above compatibility requires us to study effective tools to achieve this. In the present thesis, we focused on two of them. We aim to extend the interesting notions of traces and clone relations of a binary relation to the setting of ternary relations. The first study, allow us to characterize the compatibility in terms of traces, where we proved that this compatibility can be expressed in terms of inclusions of the binary fuzzy relation in the traces of the ternary relation . The second one, allow us to represent all the binary fuzzy tolerance relations that a crisp ternary relation is compatible with.

Keywords: Binary relation; Ternary relation; Traces; Relational compositions; Clone relations; Compatibility.

Compatibilité des relations ternaires avec des relations binaires floues

Résumé: Auparavant, les relations ternaires n'ont pas reçu une attention suffisante d'études, telles que les relations binaires, mais récemment, il y a eu un taux de participation notable. Inspirée par leur importance, notre réflexion a conduit à l'ajout d'une contribution à ce cadre. Cela signifie que l'objectif principal de ce travail est de contribuer à l'ajout d'une nouvelle étude dans le domaine des relations ternaires. Plus spécifiquement, nous cherchons à étudier la compatibilité d'une relation ternaire avec une relation floue binaire, de la même manière que dans le cas binaire.

L'étude de la compatibilité ci-dessus nécessite que nous étudions des outils efficaces pour y parvenir. Dans la présente thèse, Nous nous sommes concentrés sur deux d'entre eux. Nous visons à étendre les notions intéressantes de traces et de relations de clone d'une relation binaire à l'établissement de relations ternaires. La première étude, nous permet de caractériser la compatibilité en termes de traces, où nous avons prouvé que cette compatibilité peut être exprimée en termes d'inclusions de la relation floue binaire dans les traces de la relation ternaire. La seconde, permet de représenter toutes les relations binaires tolérance floues avec lesquelles une relation ternaire classique est compatible avec.

Mots-clés : Relation binaire; Relation ternaire; Traces; Compositions relationnelles; Relations clone; Compatibilité.