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Theme

Representations and constructions of fuzzy weak orders

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إهداء

إلى أحبائي إلى من وضعني على طريق الحياة ، رعنتني ، علمتني ، حاربت من أجلي وجعلتني أشعر بقيمتي ، والأهم أحببتني دون قيد أو شرط. أمي، والدتي .

إلى جدي الذي لا ينسى والذي وافته المنية قبل عام ، رحمه الله .

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Introduction

A classically set can be described by means of the characteristic function ($\chi_A : A \rightarrow \{0, 1\}$) i.e., that takes two values: $\chi_A(x) = 1$, for x belonging to a set A , and $\chi_A(x) = 0$, for x not belonging to a set A .

Zadeh in [39] give a definition of a new concept of **fuzzy set** of a non-empty set A , by substitution the $\{0, 1\}$ in characteristic function χ_A by a closed unit interval $[0, 1]$ called degrees of member-ship. It is denoted by a symbol μ_A . i.e., $\mu : A \rightarrow [0, 1]$, for $x \in A$ we have $\mu_A(x) \in [0, 1]$.

Baets and Walle had introduced the notion of T-fuzzy interval orders and studied two particular types of T-fuzzy interval orders: weak and strong fuzzy interval orders [15]. The study of fuzzy weak orders with respect to a left continuous t-norm T and their representability by the residual implication operator associated with T (called T -representable fuzzy weak orders) has been done by several authors (see e.g. Refs. [11, 10]).

The representability of a fuzzy total preorder additive fuzzy preference structure without incomparability and a compatible fuzzy semiorder, in terms of the α -cuts of their corresponding fuzzy weak preference relation have been respectively studied by Agud et al. and Indurain et al. [1, 23].

In this memory we studied representability of the fuzzy weak orders, it is observed that representable and constriction of fuzzy weak orders with respect to T (t-norm). Important recent results concerning fuzzy orders and their representations are obtained

by De Baets and Bodenhofer, we mention that we use a weak fuzzy order relation (the reflexivity is weak) which differs from the notion of weak order used in [4].

1

Preliminaries

In this chapter, we recall some useful notions on fuzzy set and triangular norms and conorms, weak order, fuzzy weak orders, residual implication of T , the bi-implication of T , T -preorders, T -equivalences, weak T -orders and T -E-order.

1.1 Fuzzy Sets

Definition 1.1.1 [39] *Let X be an arbitrary non empty set. A fuzzy subset A of X is uniquely represented by its membership function,*

$$\mu_A : X \rightarrow [0, 1]$$

Where the value $\mu_A(x)$ is interpreted as the degree to which the value x is contained in A . The set of all fuzzy subsets of X is called fuzzy power set of X and denoted by the symbol $F(X)$.

Definition 1.1.2 [39] Let A, B be two fuzzy subsets of X .

A and B are called equal, denoted $A = B$ if and only if their membership functions coincide.

A is called subset of B short $A \subseteq B$ if and only if for all $x \in X$:

$$\mu_A(x) \leq \mu_B(x)$$

Consequently B is called superset of A .

1.2 Triangular norms and conorms

At the beginning of the 1980s triangular norms were discovered as a unifying framework for fuzzy conjunctions [19, 38], originally they were used to formulate triangle inequalities in probabilistic metric spaces [33, 35] .

Triangular norms (t-norms), **triangular conorms** (t-conorms), and (strong) **negations** are classes of connectives for fuzzy logic analogous to the connectives in classical two-valued logic. The t-norms are models for “and,” the t-conorms are models for “or,” and negations are models for “not.”

Definition 1.2.1 A binary operations $T : [0, 1] \rightarrow [0, 1]$ or $S : [0, 1] \rightarrow [0, 1]$ is a **t-norm** and **t-conorm**, respectively, if it satisfies the following:

t-norm

1. (Boundary condition) $\forall x \in [0, 1] : T(1, x) = x$
2. (Commutativity) $\forall x, y \in [0, 1] : T(x, y) = T(y, x)$
3. (Associativity) $\forall x, y, z \in [0, 1] : T(x, T(y, z)) = T(T(x, y), z)$
4. (Monotonicity) $\forall x, y, z, w \in [0, 1] : \text{If } w \leq x \text{ and } y \leq z \text{ then } T(w, y) \leq T(x, z)$

t-conorm

1. (Boundary condition) $\forall x \in [0, 1] : S(0, x) = x$
2. (Commutativity) $\forall x, y \in [0, 1] : S(x, y) = S(y, x)$
3. (Associativity) $\forall x, y, z \in [0, 1] : S(x, S(y, z)) = S(S(x, y), z)$
4. (Monotonicity) $\forall x, y, z, w \in [0, 1] : \text{If } w \leq x \text{ and } y \leq z \text{ then } S(w, y) \leq S(x, z)$

Definition 1.2.2 A t -norm or t -conorm is **Archimedean** if for each $x, y \in (0, 1)$, there is a positive integer n for which

$$T(\overbrace{x, x, \dots, x}^{n \text{ times}}) < y$$

A t -norm is **strict** if it is continuous and satisfies the cancellation law: $T(x, y) = 0$ implies $x = 0$ or $y = 0$.

A t -conorm is **strict** if it is continuous and satisfies the law: $S(x, y) = 1$ implies $x = 1$ or $y = 1$. Strict t -norms and t -conorms are Archimedean. A continuous Archimedean t -norm or t -conorm that is not strict is **nilpotent**.

Theorem 1.2.3 A binary operation T on $[0, 1]$ is a strict t -norm if and only if it is isomorphic to the product t -norm; that is, there is an increasing function $f : [0, 1] \rightarrow [0, 1]$ that is one-to-one and onto, such that

$$T(x, y) = f^{-1}(f(x), f(y))$$

If h is another such function and $T(x, y) = h^{-1}(h(x)h(y))$ then $h(x) = f(x)^r$ for some $r > 0$.

Theorem 1.2.4 A binary operation T on $[0, 1]$ is a nilpotent t -norm if and only if it is isomorphic to the Lukasiewicz t -norm; that is, there is an increasing function $h : [0, 1] \rightarrow [0, 1]$ that is one-to-one and onto, such that

$$T(x, y) = h^{-1}(\max\{h(x) + h(y) - 1, 0\})$$

The function h is unique.

The functions f in the representation theorem for strict t -norms are called (multiplicative) **generators**. The functions h in the representation theorem for nilpotent t -norms are called **L-generators**.

- *Additive* generators F for strict t -norms are related to multiplicative generators f by $F(x) = -\ln f(x)$ and $f(x) = e^{-\ln F(x)}$.
- *Additive* generators F for nilpotent t -norms are related to L-generators f by $F(x) = 1 - f(x)$ and $f(x) = 1 - F(x)$.
- *Multiplicative* generators g for nilpotent t -norms are related to L-generators h by $g(x) = e^{h(x)-1}$ and $h(x) = 1 + \ln g(x)$.

Definition 1.2.5 A function $\eta : [0, 1] \rightarrow [0, 1]$ is a (strong) **negation** if it satisfies the following:

1. $\eta(0) = 1$ and $\eta(1) = 0$.
2. $x \leq y$ implies $\eta(x) \geq \eta(y)$.
3. $\eta(\eta(x)) = x$.

If $T(x, y)$ is a t-norm and η is a strong negation, then $S(x, y) = T(\eta(\eta(x)), \eta(y))$ is a **t-conorm**. This t-conorm is called **dual to T with respect to η** .

A binary operation T is a strict Archimedean t-conorm if and only if it is isomorphic to the dual of the product t-norm; that is, there is a decreasing function $g : [0, 1] \rightarrow [0, 1]$ that is one-to-one and onto, such that

$$S(x, y) = g^{-1}(g(x) + g(y) - g(x)g(y))$$

If k is another such function and $xTy = k^{-1}(k(x) + k(y) - k(x)k(y))$ then $k(x) = g(x)^r$ for some $r > 0$.

Theorem 1.2.6 A binary operation T is a nilpotent Archimedean t-conorm if and only if it is isomorphic to the Lukasiewicz t-norm; that is, there is a decreasing function $k : [0, 1] \rightarrow [0, 1]$ that is one-to-one and onto, such that

$$S(x, y) = k^{-1}(\min\{k(x) + k(y), 1\})$$

The function k is unique.

For the strict and nilpotent t-conorms S , the functions g and k are **multiplicative generators** and **L-cogenerators**, respectively.

1.2.1 Prominent examples

1.2.1.1 Minimum and Maximum

t-norm: $T(x, y) = x \wedge y$ Multiplicative generator: none

t-conorm: $S(x, y) = x \vee y$ Multiplicative cogenerator: none

duality relative to negation: $\alpha(x) = 1 - x$

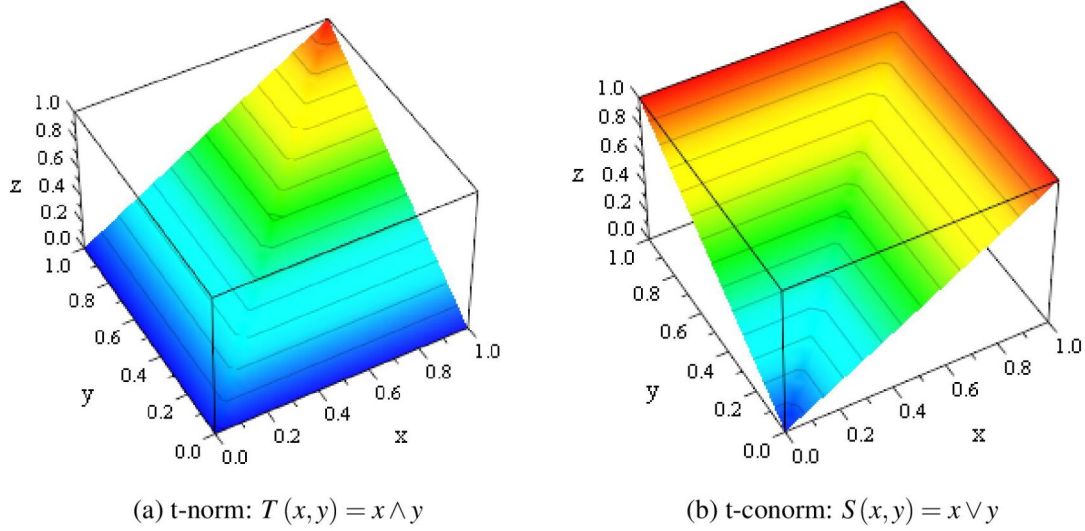


Figure 1.1: Minimum and Maximum.

1.2.1.2 Algebraic Product and Algebraic Sum

The algebraic product and algebraic sum are the prototypes for all strict t-norms and strict t-conorms, respectively.

t-norm: xy generator: $f(x) = x$
t-conorm: $x + y - xy$ cogenerator: $g(x) = 1 - x$
duality relative to negation: $1 - x$

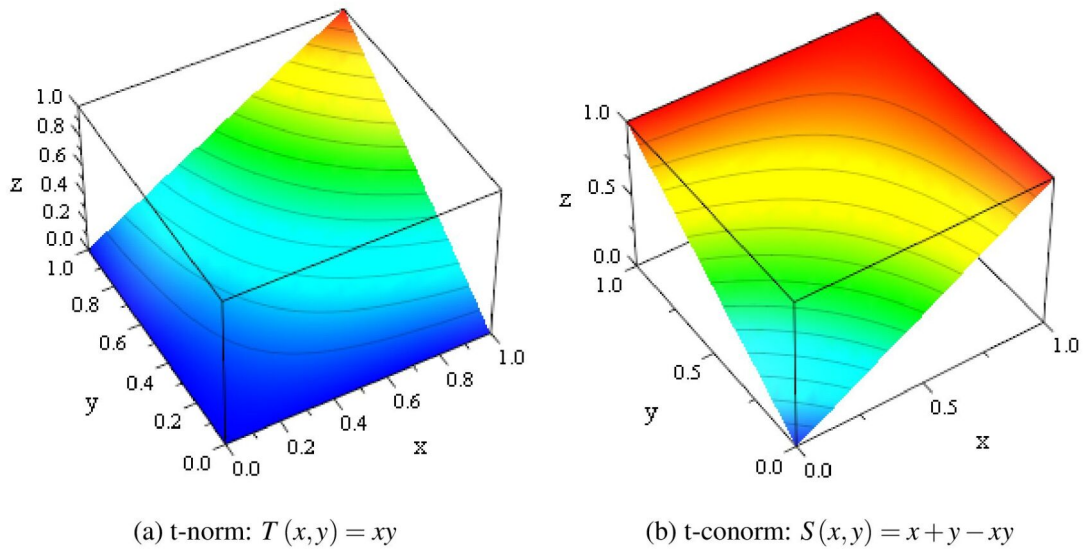


Figure 1.2: Algebraic Product and Algebraic Sum

1.2.1.3 Łukaciewicz t-norm and t-conorm

The bounded product and bounded sum are the prototypes for all nilpotent t-norms and nilpotent t-conorms, respectively.

$$\text{t-norm: } (x + y - 1) \vee 0 \quad \text{L-generator: } f(x) = x$$

$$\text{t-conorm: } (x + y) \wedge 1 \quad \text{L-cogenerator: } g(x) = 1 - x$$

$$\text{duality relative to negation: } 1 - x$$

$$\text{natural negation for t-norm and t-conorm: } 1 - x$$

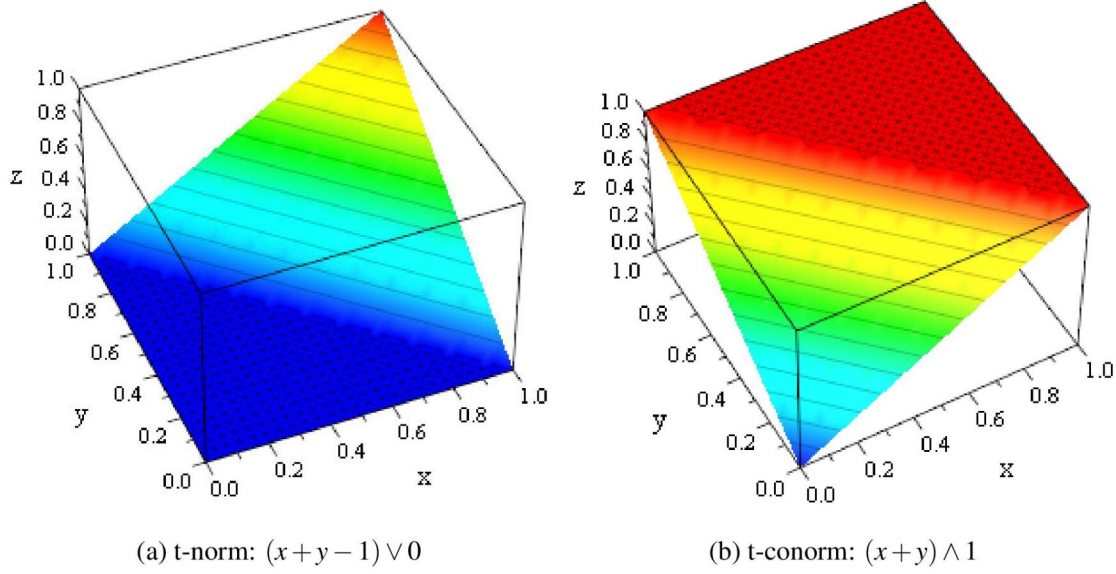


Figure 1.3: Łukaciewicz t-norm and t-conorm

1.2.1.4 Drastic Product and Drastic Sum

$$\text{t-norm: } T_D(x, y) = \begin{cases} x \wedge y & \text{if } x \vee y = 1 \\ 0 & \text{if } x \vee y < 1 \end{cases}$$

$$\text{t-conorm: } S_D(x, y) = \begin{cases} x \vee y & \text{if } x \wedge y = 0 \\ 1 & \text{if } x \wedge y > 0 \end{cases}$$

Multiplicative generator: none

Multiplicative cogenerator: none

$$\text{duality relative to negation: } \alpha(x) = 1 - x$$

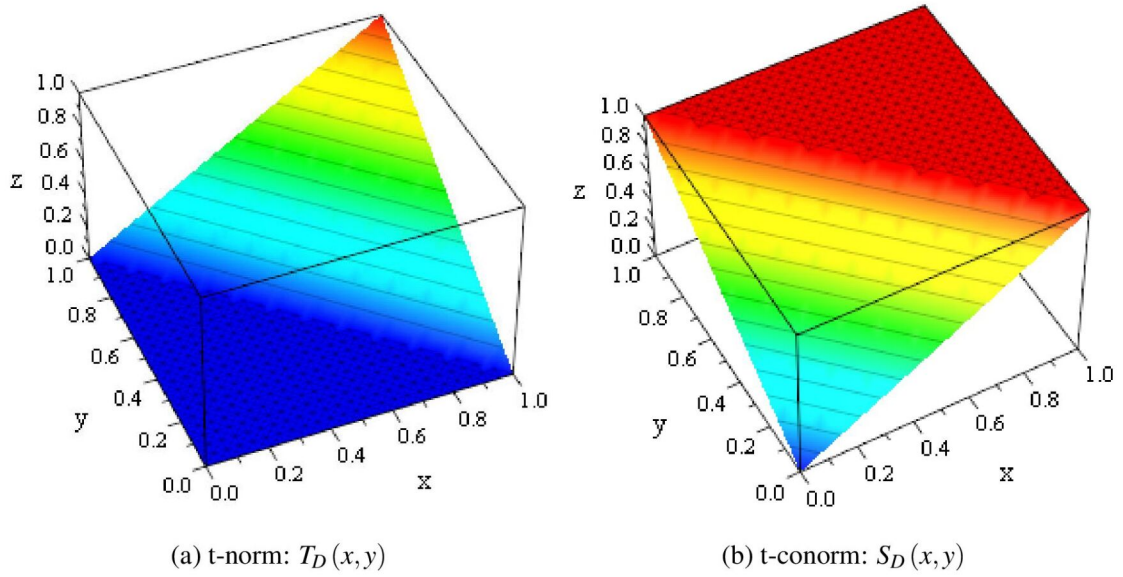


Figure 1.4: Drastic Product and Drastic Sum

Lemma 1.2.7 *Lemma For a t-norm T a sequence $(x_i)_{i \in I}$ and a value y in the unit interval the following inequalities hold*

$$\begin{aligned} T\left(\inf_{i \in I} x_i, y\right) &\leq \inf_{i \in I} T(x_i, y) \\ T\left(\sup_{i \in I} x_i, y\right) &\geq \sup_{i \in I} T(x_i, y) \end{aligned}$$

Proof. Follows directly from the monotonicity of t norms. ■

Definition 1.2.8 *Of course we can restrict the well known partial ordering of real valued functions such that a partial ordering of t-norms is obtained*

$$T_1 \leq T_2 \Leftrightarrow \forall x, y \in [0, 1] : T_1(x, y) \leq T_2(x, y)$$

If $T_1 \leq T_2$ we will say that T_1 is weaker than T_2 or equivalently that T_2 is stronger than T_1

1.3 Weak order

In mathematics, especially order theory, a weak ordering is a mathematical formalization of the intuitive notion of a ranking of a set, some of whose members may be tied with each other. Weak orders are a generalization of totally ordered sets (rankings without ties) and are in turn generalized by partially ordered sets and preorders [2].

Weak orders are among the most fundamental concepts in preference modeling.

Definition 1.3.1 A binary relation $\lesssim$ on a given non-empty domain X is called a weak order if it has the following two properties for all $x, y, z \in X$:

1. If $x \lesssim y$ and $y \lesssim z$ then $x \lesssim z$ (It's called **Transitivity**).
2. $x \lesssim y$ or $y \lesssim x$ (It's called **Completeness**).

Remark 1.3.2 Note that completeness trivially implies reflexivity, i.e., $x \lesssim x \quad \forall x \in X$.

Indeed that, if it was $\lesssim$ weak order then, the propriety of completeness verified, i.e.,

$\forall x \in X \quad x \lesssim x$ or $x \lesssim x$ hold, therefore $x \lesssim x$.

Remark 1.3.3 The difference between weak orders and linear orders is that weak orders need not be antisymmetric, i.e., a weak order $\lesssim$ is a linear order **if and only if** $x \lesssim y$ and $y \lesssim x$ then $x = y$ holds for all $x, y \in X$ (It's called **Antisymmetry**).

1.4 Fuzzy weak orders

Definition 1.4.1 A binary fuzzy relation $R : X^2 \rightarrow [0, 1]$ is a fuzzy weak order if it has the following two properties for all $x, y, z \in X$, where T denotes a t -norm:

- $T(R(x, y), R(y, z)) \leq R(x, z)$ (It's called T -transitivity),
- $R(x, y) = 1$ or $R(y, x) = 1$ (It's called strong completeness).

Remark 1.4.2 Analogously to the crisp case above (Remark 1.3.2), fuzzy weak orders are necessarily reflexive, i.e., $R(x, x) = 1$ holds for all $x \in X$.

Remark 1.4.3 *Further note that the Definition 1.4.1 of completeness is unique in the crisp case, whereas, in the fuzzy case, several concepts of completeness are possible.*

In this memory, we consider a straightforward generalization of the crisp definition using the maximum t-conorm. As this is supposed to be the strongest possible definition, the term "strong completeness" has emerged in literature [7].

1.5 Residual implication of T

Definition 1.5.1 *Let T is a left-continuous t-norm. The residual implication of a T (Note that $\overrightarrow{T}$) is a defined for all $x, y, \in [0, 1]$ by:*

$$\overrightarrow{T}(x, y) = \sup\{\alpha \in [0, 1], T(x, \alpha) \leq y\}$$

Example 1.5.2

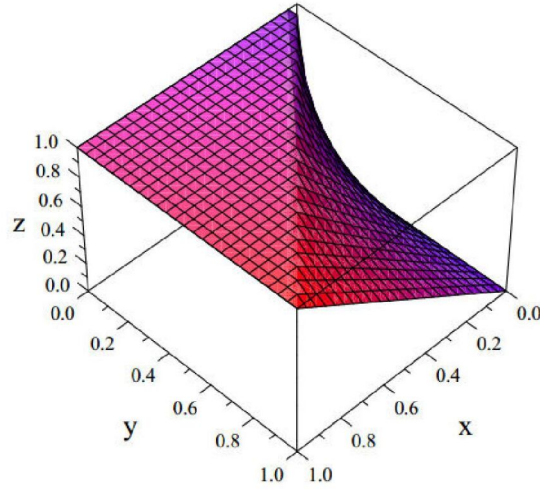
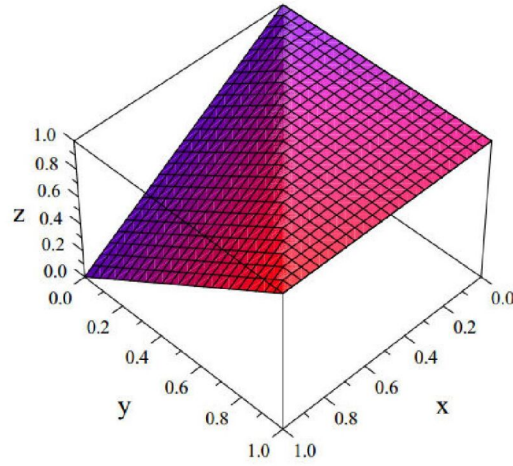
1. *If T is a continuous Archimedean t-normes with additive genertor t . Then,*

$$\overrightarrow{T}(x, y) = T^{-1}(t(y) - t(x)) \text{ for all } x, y \in [0, 1]$$

As special cases,

(a) *If T is the Łukasiewicz t-norm, then*

$$\overrightarrow{T}(x, y) = \min(1 - x + y, 1) \text{ for all } x, y \in [0, 1]$$

Figure 1.6: Residual implication If T is the product t-normFigure 1.5: Residual implication If T is the Łukasiewicz t-norm

(b) If T is the product t-norm, then

$$\vec{T}(x,y) = \min\left(\frac{x}{y}, 1\right) \text{ for all } x,y \in [0,1]$$

(c) If T is the minimum t-norm, then

$$\vec{T}(x,y) = \begin{cases} y & \text{if } y \leq x \\ 1 & \text{otherwise} \end{cases} \text{ for all } x,y \in [0,1]$$

Regarding the basic case, we include the following basic characteristics:

Proposition 1.5.3 (Gottwald, [20], Hajek, P. [22], Klement, E.P., Mesiar, R. [25]) for all $x, y, z \in]0, 1]$.

$$(I.1) \quad x \leq y \text{ if and only if } \overrightarrow{T}(x, y) = 1.$$

$$(I.2) \quad T(x, y) \leq z \text{ if and only if } x \leq \overrightarrow{T}(y, z).$$

$$(I.3) \quad T(\overrightarrow{T}(x, y), \overrightarrow{T}(y, z)) \leq \overrightarrow{T}(x, z).$$

$$(I.4) \quad \overrightarrow{T}(1, y) = y.$$

$$(I.5) \quad T(x, \overrightarrow{T}(x, y)) \leq y.$$

$$(I.6) \quad y \leq \overrightarrow{T}(x, T(x, y)).$$

Proof.

$$(I.1) \quad " \Rightarrow "$$

Let $x \leq y$. We have $\mu \leq 1$ because $\mu \in [0, 1]$. Then,

$T(x, \mu) \leq T(y, 1) = y$, for all $\mu \in [0, 1]$ implies, $\bigvee \{\mu \in [0, 1], T(x, \mu) \leq y\} = 1$. Then,

$$\overrightarrow{T}(x, y) = 1.$$

$$" \Leftarrow "$$

We have $\overrightarrow{T}(x, y) = 1 \Rightarrow \bigvee \{\mu \in [0, 1], T(x, \mu) \leq y\} = 1 \Rightarrow T(x, 1) \leq y \Rightarrow x \leq y$.

$$(I.2) \quad " \Rightarrow " \quad T(x, y) \leq z \Leftrightarrow x \leq \overrightarrow{T}(y, z). \quad \text{Let } x, y, z \in [0, 1] \text{ and let } T(x, y) \leq z. \text{ We have } \overrightarrow{T}(y, z) =$$

$\bigvee \{\mu \in [0, 1], T(y, \mu) \leq z\}$ and we have $T(x, y) \leq z$. Then, $x \in \{\mu \in [0, 1], T(y, \mu) \leq z\}$. So,

$x \leq \bigvee \{\mu \in [0, 1], T(y, \mu) \leq z\}$. So, $x \leq \overrightarrow{T}(y, z)$.

$" \Leftarrow "$. We have $x \leq \overrightarrow{T}(y, z) = \bigvee \{\mu \in [0, 1], T(y, \mu) \leq z\} = \mu_0$.

Do, $x \leq \mu_0$. Then, $T(x, y) \leq T(y, \mu_0) \leq z$ implies, $T(x, y) \leq z$.

(I.3) We put:

$$\begin{cases} u_0 = \overrightarrow{T}(x, y) = \bigvee \{u \in [0, 1] : T(x, u) \leq y\}; \\ v_0 = \overrightarrow{T}(y, z) = \bigvee \{v \in [0, 1] : T(y, v) \leq z\}; \\ w_0 = \overrightarrow{T}(x, z) = \bigvee \{w \in [0, 1] : T(x, w) \leq z\}. \end{cases}$$

We know that, $T(x, u_0) \leq y$.

So, $T(T(x, u_0), v_0) \leq T(y, v_0) \leq z$, from where $T(x, T(u_0, v_0)) \leq z$.

Then, $T(u_0, v_0) \in \{w \in [0, 1], T(x, w) \leq z\}$, then $T(u_0, v_0) \leq w_0$.

So, $T(\vec{T}(x, y), \vec{T}(y, z)) \leq \vec{T}(x, z)$.

(I.4) For all $y \in [0, 1]$ we have $\vec{T}(1, y) = \vee \{u \in [0, 1], T(1, u) \leq y\} = y$.

(I.5) $T(x, \vec{T}(x, y)) \leq y$, we have $T(\vec{T}(1, x), \vec{T}(x, y)) \leq \vec{T}(1, y) \leq y$, (According to (I.3)).

(I.6) $y \leq \vec{T}(x, T(x, y))$ we have $\vec{T}(x, T(x, y)) = \vee \{u \in [0, 1] : T(x, u) \leq T(x, y)\} \geq y$.

■

Remark 1.5.4 In addition, $\vec{T}$ is not increasing, and its first component is continuous on the left and, non-decreasing, and its second component is continuous on the right.

Proposition 1.5.5 Let T is t -norm left continue, then

$$\forall x, y, z \in [0, 1], z \geq x \implies \vec{T}(x, y) = \vec{T}(\vec{T}(z, x), \vec{T}(z, y)) \quad (1.1)$$

Proof. Let $x, y, z \in [0, 1]$, then $\vec{T}(\vec{T}(z, x), \vec{T}(z, y)) = \vee \{u \in [0, 1] / T(\vec{T}(z, x), u) \leq \vec{T}(z, y)\}$.

(According to (I.3)) we have, $T(\vec{T}(z, x), \vec{T}(x, y)) \leq \vec{T}(z, y)$.

So, $\vec{T}(x, y) \in \{u \in [0, 1] / T(\vec{T}(z, x), u) \leq \vec{T}(z, y)\}$,

then, $\vec{T}(x, y) \leq \vec{T}(\vec{T}(z, x), \vec{T}(z, y))$. ■

1.6 The bi-implication of T

Definition 1.6.1 (Gottwald, [20], Hajek, P. [22], Klement, E.P., Mesiar, R. [25]) The bi-implication of T is defined as:

$$\overleftrightarrow{T}(x, y) = T(\vec{T}(x, y), \vec{T}(y, x)), \forall x, y, z \in [0, 1].$$

Proposition 1.6.2 The following assertions hold for all $x, y, z \in [0, 1]$, see [20, 22]:

(B.1) $\overleftrightarrow{T}(x, y) = 1$ if and only if $x = y$.

(B.2) $\overleftrightarrow{T}(x, y) = \overleftrightarrow{T}(y, x)$.

$$(B.3) \quad \overleftarrow{T}(x, y) = \min(\overrightarrow{T}(x, y), \overrightarrow{T}(y, x)).$$

$$(B.4) \quad T(\overleftarrow{T}(x, y), \overleftarrow{T}(y, z)) \leq \overleftarrow{T}(x, z).$$

$$(B.5) \quad \overleftarrow{T}(x, y) = \overleftarrow{T}(\max(x, y), \min(x, y)).$$

Proof.

$$\begin{aligned}
 (B.1) \quad \text{We have } \overleftarrow{T}(x, y) = 1 &\Leftrightarrow T(\overrightarrow{T}(x, y), \overrightarrow{T}(y, x)) = 1, \\
 &\Leftrightarrow \overrightarrow{T}(x, y) = 1 \text{ and } \overrightarrow{T}(y, x) = 1, \\
 &\Leftrightarrow x \leq y \text{ and } y \leq x, \\
 &\Leftrightarrow x = y.
 \end{aligned}$$

(B.2) Trivial by definition.

(B.3) We have $\overleftarrow{T}(x, y) = T(\overrightarrow{T}(x, y), \overrightarrow{T}(y, x))$, It is correct for $T = \min$.

(B.4) Let $x, y, z \in [0, 1]$:

$$\begin{aligned}
 T(\overleftarrow{T}(x, y), \overleftarrow{T}(y, z)) &= T(T(\overrightarrow{T}(x, y), \overrightarrow{T}(y, x)), T(\overrightarrow{T}(y, z), \overrightarrow{T}(z, y))) \\
 &= T(T(\overrightarrow{T}(x, y), \overrightarrow{T}(y, z)), T(\overrightarrow{T}(z, y), \overrightarrow{T}(y, x))) \\
 &\leq T(\overrightarrow{T}(x, z), \overrightarrow{T}(z, x)) \\
 &\leq \overleftarrow{T}(x, z).
 \end{aligned}$$

(B.5) Let $x, y \in [0, 1]$.

$$\text{If } x \leq y \text{ we have: } \overleftarrow{T}(\max(x, y), \min(x, y)) = \overleftarrow{T}(y, x) = \overleftarrow{T}(x, y).$$

$$\text{If } y \leq x \text{ we have: } \overleftarrow{T}(\max(x, y), \min(x, y)) = \overleftarrow{T}(x, y).$$

■

1.7 T-preorders, T-equivalences, weak T-orders and T-E-order

The fuzzy power set of X will be denoted with $F(X) = \{A | A : X \rightarrow [0, 1]\}$.

Definition 1.7.1

A binary fuzzy relation $R : X^2 \rightarrow [0, 1]$ is called

1. *Reflexive: if $R(x, x) = 1$ for all $x \in X$,*
2. *Symmetric: if $R(x, y) = R(y, x)$ for all $x, y \in X$,*
3. *T-transitive: if $T(R(x, y), R(y, z)) \leq R(x, z)$ for all $x, y, z \in X$,*
4. *Strongly complete: if $\max(R(x, y), R(y, x)) = 1$ for all $x, y \in X$.*

Definition 1.7.2

1. *Fuzzy relations that are reflexive and T-transitive are called fuzzy preorders with respect to T, short T-preorders.*
2. *Symmetric T-preorders are called fuzzy equivalence relations with respect to T, short T-equivalences. As mentioned in*
3. *Strongly complete T-preorders are called fuzzy weak orders with respect to T, short weak T-orders.*

Definition 1.7.3

Given a T-equivalence $E : X^2 \rightarrow [0, 1]$, a binary fuzzy relation $L : X^2 \rightarrow [0, 1]$ is called a fuzzy order with respect to T and E, short T-E-order, if it is T-transitive and additionally has the following two properties:

- *E-reflexivity: $E(x, y) \leq L(x, y)$ for all $x, y \in X$.*
- *T-E-antisymmetry: $T(L(x, y), L(y, x)) \leq E(x, y)$ for all $x, y \in X$.*

In analogy to the crisp case, we defined the afterset and foreset of $x \in X$ as follows.

Definition 1.7.4 [3]

Let a binary fuzzy relation $R : X^2 \rightarrow [0, 1]$ and $x \in X$.

- *The afterset of x is defined as the fuzzy set $D(x) \in F(X)$ that expresses the degree to which a given value $y \in X$ is greater than or equivalent to x , i.e., $D(x)(y) = R(x, y)$. We will also use the nonchalant notation $R(x, \cdot)$.*

- In the same way defines the following set (forset) of x as the fuzzy set $C(x) \in F(X)$ expresses the degree to which a value gives $y \in X$ is smaller than or equivalent to x , that is to say, $C(x)(y) = R(y, x)$. We will also use the notation $R(., x)$.

Theorem 1.7.5 [6, Theorem 1.1.] A binary relation $\lesssim$ on a non-empty domain X is a **weak order** if and only if there exists a linearly ordered non-empty set $(Y, \preceq)$ and a mapping $f : X \rightarrow Y$ such that $\lesssim$ can be represented in the following way for all $x, y \in X$:

$$x \lesssim y \text{ if and only if } f(x) \preceq f(y) \quad (1.2)$$

Proof. " $\Leftarrow$ " $\lesssim$ is a relation weak order

We can define $C(x) = \{y \in X; y \lesssim x\} = \downarrow x$, we take $Y = \{C(x) / x \in X\}$, and we define on Y the relation $\preceq$ as follows: $\forall x, y \in X : C(x) \subset C(y) \Leftrightarrow x \lesssim y$.

Let's prove that $\subset$ is an order relation on Y .

Let $x \in X$, we have $x \lesssim x$, so $C(x) \subset C(x)$ then $\subset$ is reflexive.

Let $x, y \in X$ such that

$$x \lesssim y \text{ and } y \lesssim x \Leftrightarrow C(x) \subset C(y) \text{ et } C(y) \subset C(x) \Leftrightarrow C(x) = C(y).$$

So, $\subset$ is antisymmetric.

Let $C(x), C(y), C(z) \in Y$ such that :

$$\begin{cases} C(x) \subset C(y) \\ \text{and} \\ C(y) \subset C(z) \end{cases} \Leftrightarrow \begin{cases} x \lesssim y \\ \text{and} \\ y \lesssim z \end{cases} \Leftrightarrow x \lesssim z \Leftrightarrow C(x) \subset C(z).$$

So, $\subset$ is transitive.

Let's prove that $\subset$ is linear.

Let $C(x), C(y) \in Y$ we have

$$x \lesssim y \text{ or } y \lesssim x \Rightarrow C(x) \subset C(y) \text{ or } C(y) \subset C(x)$$

Then $\subset$ is linear.

And we define the application $f : X \rightarrow Y$ such that: $f(x) = C(x)$ Which verifies (1.2). ■

Another way to prove of Theorem 1.7.5

The proof that a relation defined as in Equation (1.2) is a weak order is straightforward. To prove the existence of a linearly ordered set $(Y, \preceq)$ and a mapping f such that representation (1.2) holds for a given weak order $\preceq$, one has to follow the following steps:

- (1) Define an equivalence relation $\sim$ as the symmetric kernel,
- (2) Define Y as the factor set $X / \sim$,
- (3) Define f as the projection $f(x) = x \sim$,
- (4) Prove that the projection of $\preceq$ onto $X / \sim$ is a linear order on $X / \sim$,
- (5) Prove the representation (1.2) holds.

From this perspective, we can view weak orders as linear orders of equivalence classes. In the context of Theorem 1.7.5, the equivalence classes contain exactly those elements that share the same property, i.e., those elements for which f yields the same value.

Remark 1.7.6 *In the case where X is at most countable, Theorem 1.7.5 can be fortified in the following sense: It is always possible to choose $Y = [0, 1]$, that is to say, for each weak order, we can find an $f : X \rightarrow [0, 1]$; such as the representation (1.2) is holds.*

In other words, the weak orders on the countable domains can always be injected in the natural linear order over the interval of the unit. It's a classic result that goes back to Chantre [9, 26, 31].

Weak orders are not only simple and fundamental concepts (as the examples above illustrate), they are also the basis for the representation of other fundamental concepts and order theory: it is known that preorders, ie, reflexive and transitive binary relationships, are uniquely characterized as intersections of weak orders. In the analogy of this classic case, fuzzy weak orders are fundamental concepts in fuzzy preference which models [12, 18, 17, 16].

2

Representations and constructions by score function-based

Before turning to our actual objects of study, fuzzy weak orders, let us briefly mention a result that will be useful later (compare with [17, 37]).

Lemma 2.0.1 [6] *For all T -preorder $R : X^2 \longrightarrow [0, 1]$, satisfies the following equality, for all $x, y \in X$*

$$R(x, y) = \inf_{z \in X} \overrightarrow{T}(R(z, x), R(z, y)) \quad (2.1)$$

Proof. We have, $T(\alpha, \beta) \leq \gamma \Leftrightarrow \alpha \leq \overrightarrow{T}(\beta, \gamma)$ and according to (I.2) , so

$$\begin{aligned}
T(R(z, x), R(x, y)) \leq R(z, y) &\Rightarrow T(R(x, y), R(z, x)) \leq R(z, y), \\
&\Rightarrow R(x, y) \leq \overrightarrow{T}(R(z, x), R(z, y)), \text{ for all } x, y, z \in [0, 1], \\
&\Rightarrow R(x, y) \leq \inf_{z \in X} \overrightarrow{T}(R(z, x), R(z, y)).
\end{aligned}$$

On the other hand if we take $z = x$ then, $\inf_{z \in X} \overrightarrow{T}(R(z, x), R(z, y)) \leq \overrightarrow{T}(R(x, x), R(x, y)) = R(x, y)$
 So, $R(x, y) = \inf_{z \in X} \overrightarrow{T}(R(z, x), R(z, y))$. ■

2.1 Representations and constructions by score function-based

The question asked is whether there is a generalization of this Theorem 1.7.5 to the case of fuzzy weak orders.

Theorem 2.1.1 [6] *A binary fuzzy relation $R : X^2 \rightarrow [0, 1]$ is a weak T -order if and only if there exist a non-empty domain Y , a T -equivalence $E : Y^2 \rightarrow [0, 1]$, a strongly complete $T - E$ -order $L : Y^2 \rightarrow [0, 1]$, and a mapping $f : X \rightarrow Y$ such that the following equality holds for all $x, y \in X$:*

$$R(x, y) = L(f(x), f(y)) \quad (2.2)$$

Proof. Let us first assume that we have a non-empty domain Y equipped with a T -equivalence E and a $T - E$ -order L and a mapping $f : X \rightarrow Y$ such that representation 2.2 holds. As every strongly complete $T - E$ -order is a weak T -order, R is trivially a weak T -order.

Now assume that R is a weak T -order. Let us define $Y = X$ and $f(x) = x$ (i.e., $f \equiv id_X$). Then $L(x, y) = R(x, y)$ is obviously a reflexive, T -transitive, and strongly complete fuzzy relation. What remains to be shown is how to define an appropriate T -equivalence E such that L is a $T - E$ -order. If we define,

$$E(x, y) = T(L(x, y), L(y, x)) = T(R(x, y), R(y, x))$$

Then, L is a $T - E$ -order [4, 5], and the proof is completed. ■

Remark 2.1.2 *Theorem 2.1.1[6] can be viewed from two different angles.*

- *It is a nice straightforward generalization of Theorem 1.7.5 and demonstrates the smooth interplay between fuzzy weak orders and strongly complete fuzzy orders (analogously to the crisp case).*

- *The trivial proof reveals that fuzzy weak orders and strongly complete fuzzy orders are basically the same concepts.*

Theorem 2.1.1 does not provide us with a new construction method or any new insight. More insight would potentially be obtained if we could restrict the choice of Y or E to certain standard cases that could be utilized for constructions in an easier way.

An interesting question is whether Y , L , and f can be chosen such that Theorem 2.1.1 holds for E being the crisp equality, (i.e., with L being a so-called T -order [4, 17, 21], which, in the case that $T = T_M$, is nothing else but a fuzzy partial order in the sense of Zadeh [29, 16, 40]).

The answer is quick and negative: as demonstrated in [16, Subsection 2.3], strongly complete fuzzy orders with respect to some t-norm T and the crisp equality can only be crisp orders. Thus, it is never possible to embed a non-crisp weak order into a strongly complete T -order, so it is impossible to strengthen Theorem 2.1.1 by fixing E as the crisp equality.

The question remains whether there is any standard choice Y , E , L , f into which we can embed all, or at least a sub-class of, weak T -orders. As shown by Ovchinnikov, it is possible to embed every weak T -order into one particular continuous weak T -order on the real numbers $\mathbb{R}$, but it is necessary to restrict to strict t-norms and finite domains X [30]. Since this is outside the scope of this chapter, we turn our attention to a different investigation. The standard crisp case consists of the unit interval $[0, 1]$ equipped with its natural linear order.

Given a left-continuous t-norm T , the canonical fuzzification of the natural linear order on $[0, 1]$ is the residual implication $\overrightarrow{T}$ [4, 20, 22].

The following proposition, therefore, provides us with a construction that can be considered as a straightforward counterpart of 1.2.

Proposition 2.1.3 [6] *Given a function $f : X \rightarrow [0, 1]$, the binary fuzzy relation $R : X^2 \rightarrow [0, 1]$ defined by*

$$R(x, y) = \overrightarrow{T}(f(x), f(y)) \quad (2.3)$$

is a weak T -order.

Proof.

(I.1) implies that defines a strongly complete, thus reflexive, fuzzy relation.

(I.3) implies that R is T -transitive. ■

The function f in Proposition 2.1.3 can also be understood as a fuzzy set on X .

This section, we rather leave this aspect aside and adopt the classical interpretation as a score function.

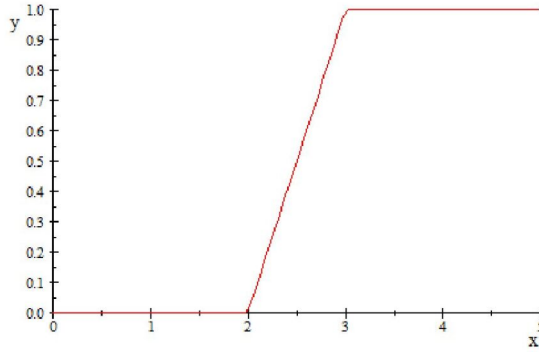
Proposition 2.1.3 could also be inferred directly from Theorem 2.1.1 and the fact that $\overrightarrow{T}$ is fuzzy order on $[0, 1]$ with respect to T and $\overleftarrow{T}$ [4, Subsection 2.1].

Definition 2.1.4 Consider a weak T -order $R : X_2 \rightarrow [0, 1]$. R is called representable if there exists a function $f : X \rightarrow [0, 1]$, called generating score function, such that Eq. 2.3 holds. A representable weak T_M -order is called Gödel-representable [12].

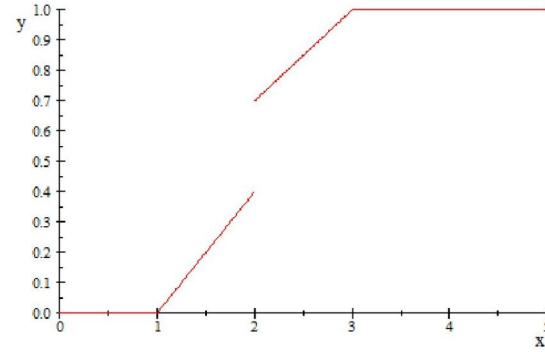
Example 2.1.5 [6] Let us consider $X = [0, 5]$ and the following two score functions $f_1, f_2 : X \rightarrow [0, 1]$:

$$f_1(x) = \min(1, \max(0, x - 2)),$$

$$f_2(x) = \begin{cases} 0 & \text{if } x \in [0, 1[\\ 0.4 \times (x - 1) & \text{if } x \in [1, 2[\\ 0.7 + 0.3 \times (x - 2) & \text{if } x \in [2, 3[\\ 1 & \text{if } x \in [3, 5] \end{cases}$$



(a) Graphic of function f_1



(b) Graphic of function f_2

Figure 2.1: Graphic of score functions f_1 and f_2

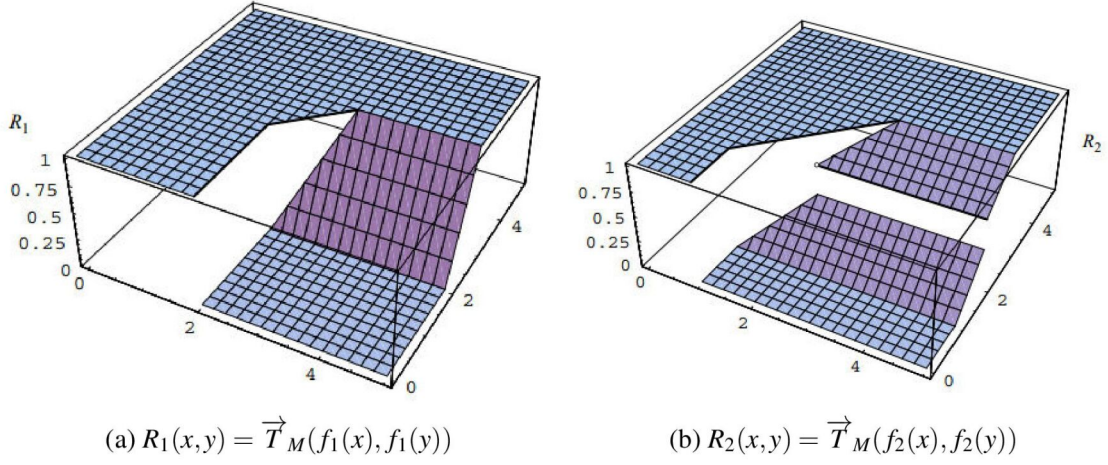


Figure 2.2: Fuzzy weak orders constructed from the score functions f_1 and f_2 by means of Proposition 2.1.3 using the T_M t-norms.

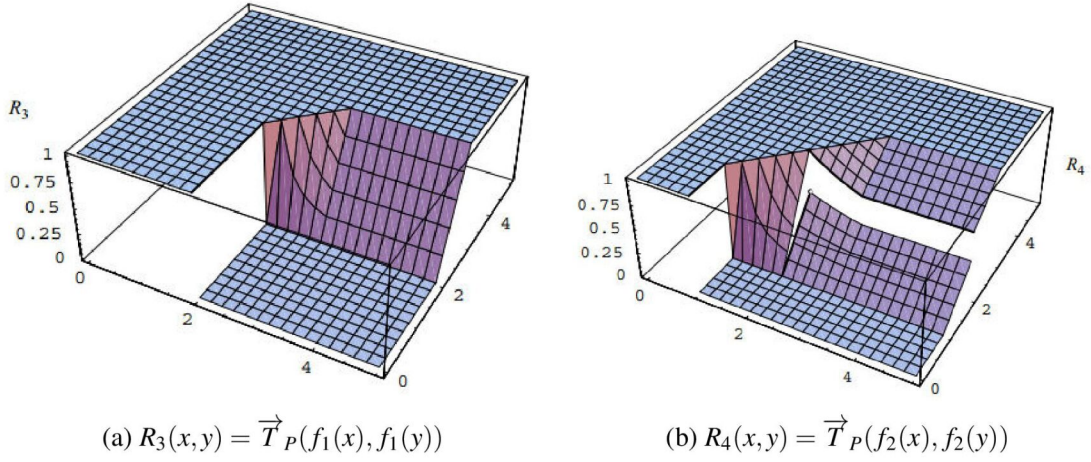


Figure 2.3: Fuzzy weak orders constructed from the score functions f_1 and f_2 by means of Proposition 2.1.3 using the T_P t-norms.

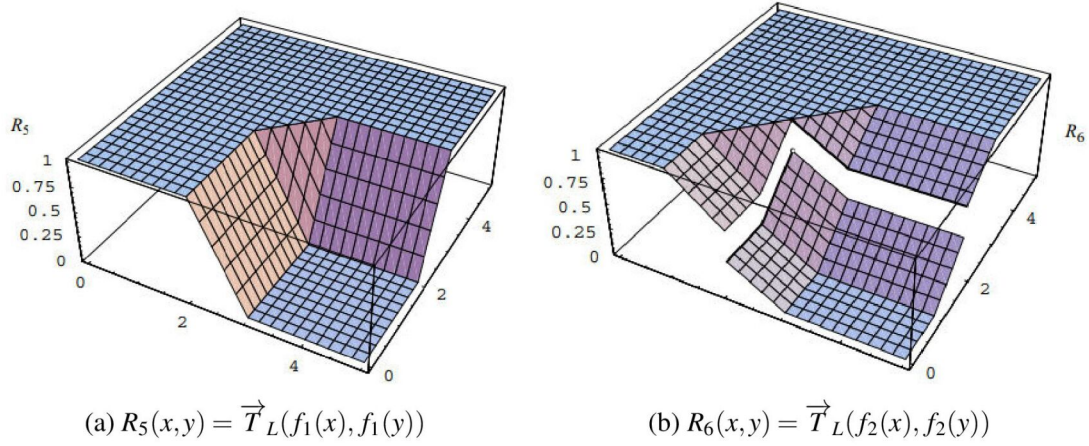


Figure 2.4: Fuzzy weak orders constructed from the score functions f_1 and f_2 by means of Proposition 2.1.3 using the T_L t-norms.

The fuzzy relations plotted in Figures. 2.2, 2.3 and 2.4 have one common feature: the lower right edge always corresponds to the generating score function. More specifically, all fuzzy weak orders in the left column fulfill $R(5, y) = f_1(y)$, while $R(5, y) = f_2(y)$ holds for the fuzzy weak orders in the right column. Note that this is true independent of the t -norm chosen (at least for the three basic t -norms). The question arises whether this is a coincidence or whether there is a principle behind. The following theorem tells us that the latter is the case, but even more than that, we obtain a unique characterization of representable fuzzy weak orders (at least for continuous t -norms).

Theorem 2.1.6 [6] *Assume that T is continuous. Then a weak T -order R is representable if and only if the following function is a generating score function of R :*

$$\bar{f}(x) = \inf_{z \in X} R(z, x)$$

Proof. First assume that $\bar{f}$ is a generating score function of R . Then R is trivially representable and nothing needs to be proved.

Therefore assume that we are given an arbitrary representable weak T -order R with generating score function f .

It is sufficient to prove that $\bar{f}$ is a generating score function, too. This means that we have to prove the

following equality for all $x, y \in X$:

$$R(x, y) = \overrightarrow{T} \left(\inf_{z \in X} R(z, x), \inf_{z \in X} R(z, y) \right)$$

Using the fact that R is representable, we obtain

$$\overrightarrow{T} \left(\inf_{z \in X} R(z, x), \inf_{z \in X} R(z, y) \right) = \overrightarrow{T} \left(\inf_{z \in X} \overrightarrow{T}(f(z), f(x)), \inf_{z \in X} \overrightarrow{T}(f(z), f(y)) \right) = (*)$$

Taking into account that T is non-increasing and left-continuous in the first argument, we get the following:

$$(*) = \overrightarrow{T} \left(\overrightarrow{T} \left(\sup_{z \in X} f(z), f(x) \right), \overrightarrow{T} \left(\sup_{z \in X} f(z), f(y) \right) \right) = (*')$$

As $\sup_{z \in X} f(z)$ is trivially greater than or equal to $f(x)$ and since T is supposed to be continuous, we can apply (I7),

$$(*') = \overrightarrow{T}(f(x), f(y)) = R(x, y).$$

And the proof is completed. ■

Theorem 2.1.6 provides us with an easy-to-use tool for checking whether a fuzzy weak order is representable we only have to check whether one specific function is a generating score function. Note, however, that the generating score function need not be unique, i.e., it may happen that a fuzzy weak order R is generated by some score function f that does not coincide with f defined as in 2.1.1 Let us shortly consider this issue and ask ourselves under which condition f coincides with some generating score function f .

Therefore assume that R is representable as

$$R(x, y) = \overrightarrow{T}(f(x), f(y))$$

Then we obtain the following:

$$\overline{f}(x) = \inf_{z \in X} R(z, x) = \inf_{z \in X} \overrightarrow{T}(f(z), f(x)) = \overrightarrow{T} \left(\sup_{z \in X} f(z), f(x) \right).$$

Then, $\sup_{z \in X} f(z) = 1$ is a sufficient criterion for f and $\bar{f}$ to coincide:

$$\bar{f}(x) = \overrightarrow{T}(\sup_{z \in X} f(z), f(x)) = \overrightarrow{T}(1, f(x)) \stackrel{(I4)}{=} f(x)$$

It should be clear now that by far not all fuzzy weak orders are representable by single score functions for all left continuous t-norms, there exist non-representable fuzzy weak orders. What has not been answered so far is the question whether fuzzy weak orders can be represented by more than one score function. The following well-known theorem provides us with a starting point of this investigation.

Theorem 2.1.7 (Valverde [37]). *Consider a binary fuzzy relation $R : X^2 \rightarrow [0, 1]$. Then the following two statements are equivalent:*

- (i) *R is a T -preorder.*
- (ii) *There exists a non-empty family of $X \rightarrow [0, 1]$ score functions $(f_i)_{i \in I}$ such that the following representation holds:*

$$R(x, y) = \inf_{i \in I} \left(\overrightarrow{T}(f_i(x), f_i(y)) \right) \quad (2.4)$$

Proof. Although this is a well-known result [37], we shortly sketch the proof in order to describe the basic idea that will be useful later.

The proof that a fuzzy relation defined as in Eq 2.4 is a T -preorder is a routine matter that mainly consists of applying basic properties of residual implications. To prove the converse implication, one can define $I = X$ and $f_z(x) = R(z, x)$, i.e., for each $z \in X$ to its afterset. Then representation 2.4 follows from Lemma 2.0.1. ■

Theorem 2.1.7 is essential for two main reasons:

1. It shows that every T -preorder is an intersection of representable weak T -orders,
2. As weak T -orders are a special kind of T -preorders, we know for sure that, for each weak T -order R , there exists a family of score functions such that R can be represented as in Eq 2.4. Be aware, however, that this is only a representation of theoretical nature.

Remark 2.1.8 *We do not know yet how to choose a family of score functions $(f_i)_{i \in I}$ such that fuzzy relation defined as in 2.4 is guaranteed to fulfill strong completeness.*

Lemma 2.1.9 [6] *Let R be a weak T -order. Then its kernel relation, defined as*

$$x \lesssim_R y \text{ if and only if } R(x, y) = 1$$

Is a crisp weak order.

Moreover, every afterset of R is non-decreasing with respect to $\lesssim$,

i.e., for all $x, y, z \in X$,

$$x \lesssim_R y \text{ implies } R(z, x) \leq R(z, y)$$

Proof. That $\lesssim_R$ is a crisp weak order follows trivially from elementary properties of fuzzy weak orders and t -norms.

Now suppose that $x R y$ holds for some pair $x, y \in X$, i.e., $R(x, y) = 1$. Then T -transitivity implies

$$R(z, x) = T(R(z, x), R(x, y) = 1) R(z, y),$$

which completes the proof. ■

The following theorem provides us with a unique characterization of weak T -orders.

Theorem 2.1.10 [6] *Consider a binary fuzzy relation $R : X^2 \rightarrow [0, 1]$. Then the following two statements are equivalent:*

(i) *R is a weak T -order.*

(ii) *There exists a crisp weak order $\lesssim$ and a non-empty family of $X \rightarrow [0, 1]$ score functions $(f_i)_{i \in I}$ that are nondecreasing with respect to $\lesssim$ such that representation 2.4 holds.*

Proof.

1. (i) $\Rightarrow$ (ii): Lemma 2.1.9 guarantees that the kernel relation $\lesssim_R$ is a crisp weak order. Analogously to the proof of the Theorem 2.1.7, let us define $I = X$ and $f_z(x) = R(z, x)$. Lemma 2.1.9 then guarantees that these score functions are non-decreasing with respect to $\lesssim_R$. Finally, Lemma 2.0.1 ensures that representation 2.4 holds.

2. (ii) $\Rightarrow$ (i): Theorem 2.1.7 states that R defined as in Eq 2.4 is a T -preorder. Therefore, it only remains to prove that R is strongly complete.

Consider arbitrary $x, y \in X$. Since $\lesssim$ is complete, at least one of the two inequalities $x \lesssim y$ and

$y \lesssim x$ holds. If we assume that $x \lesssim y$ is fulfilled, the non-decreasingness of all f_i guarantees that $f_i(x) \leq f_i(y)$ holds for all $i \in I$. From (I1), we can conclude that $T(f_i(x), f_i(y)) = 1$ for all $i \in I$. Therefore,

$$R(x, y) = \inf_{i \in I} (f_i(x), f_i(y)) = 1$$

Conversely, if we assume that $y \lesssim x$, we obtain analogously that $R(y, x) = 1$.

Hence, in any case, the maximum of $R(x, y) = 1$ and $R(y, x) = 1$.

■

Lemma 2.1.11 [6] *For every fuzzy weak order R , there exists a crisp linear order that is a sub-relation of the kernel relation $\lesssim_R$.*

Moreover, every afterset of R is non-decreasing with respect to $\preceq$,

i.e., for all $x, y, z \in X$, $x \preceq y$ implies $R(z, x) \leq R(z, y)$.

Proof. We already know from Lemma 2.1.9 that the kernel relation $\lesssim_R$ is always a weak order. Now we define a relation $\sim$ as the symmetric kernel of $\lesssim_R$ in the following way (for all $x, y \in X$): $x \sim y$ if and only if $(x \lesssim_R y \text{ and } y \lesssim_R x)$. Of course, $x \sim y$ holds if and only if $R(x, y) = 1$ and $R(y, x) = 1$. So $\sim$ is trivially an equivalence relation.

Now consider the factor set $X / \sim$. It is easy to see, as a consequence of the fact that R is complete, that the projection of R onto the factor set $X / \sim$ is a linear order. Then we can choose arbitrary linear orders of all equivalence classes $x \in X / \sim$ [32, 36] and perform lexicographic composition to obtain a linear order on X that is a sub-relation of $\lesssim_R$. For technical details, see [4, Theorem 4.1]. Then the monotonicity of the aftersets follows trivially from Lemma 2.1.9 and the fact that $\preceq$ is a sub-relation of $\lesssim_R$. ■

The following theorem finally characterizes weak T -orders as intersections of representable weak T -orders that are generated by score functions that are monotonic at the same time with respect to the same crisp linear order.

Theorem 2.1.12 [6] *Consider a binary fuzzy relation $R : X^2 \rightarrow [0, 1]$. Then the following two statements are equivalent:*

- (i) *R is a weak T -order.*

- (ii) *There exists a crisp linear order $\preceq$ and a non-empty family of $X \rightarrow [0, 1]$ score functions $(f_i)_{i \in I}$ that are non-decreasing with respect to $\preceq$ such that representation 2.4 holds.*

Proof.

1. (i) $\Rightarrow$ (ii): Given a weak T -order R , Lemma 2.1.11 guarantees the existence of a crisp linear order $\preceq$ such that the aftersets of R are non-decreasing with respect to $\preceq$. The remaining part of the proof can be carried out analogously to the proof of Theorem 2.1.10.
2. (ii) $\Rightarrow$ (i): Every crisp linear order is a crisp weak order, hence this implication follows trivially from Theorem 2.1.10.

■

Example 2.1.13 [6] *We consider $X = [0, 5]$ again and a family of five functions that are defined as follows:*

- $g_1(x) = \min(1, x), \quad g_2(x) = \min(1, \max(0, x - 1)),$
- $g_3(x) = \min(1, \max(0, x - 2)), \quad g_4(x) = \min(1, \max(0, x - 3)),$
- $g_5(x) = \min(1, \max(0, x - 4)).$

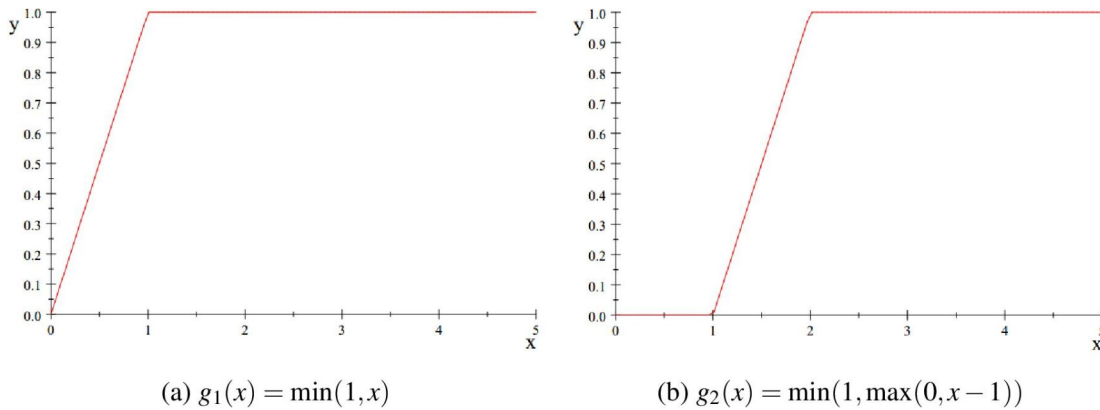
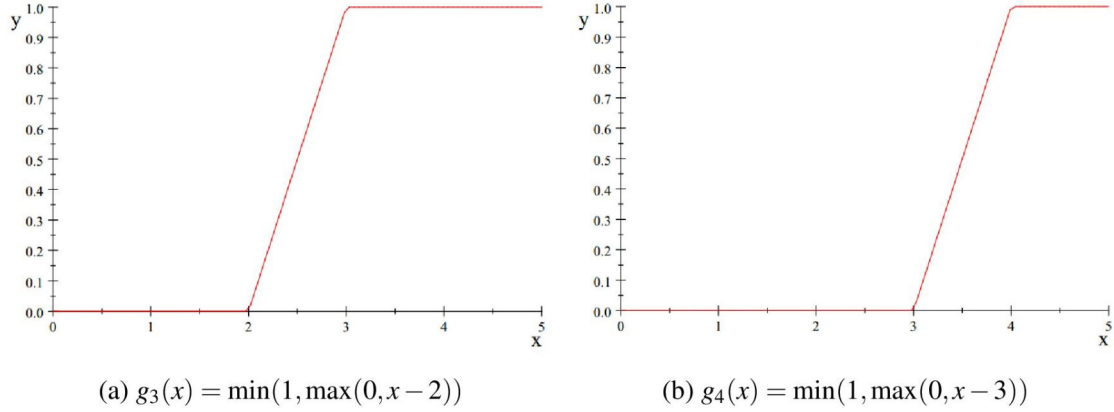
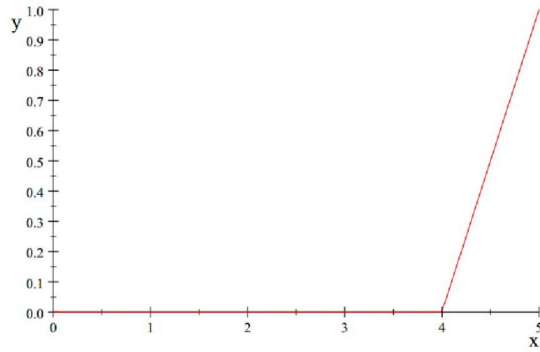


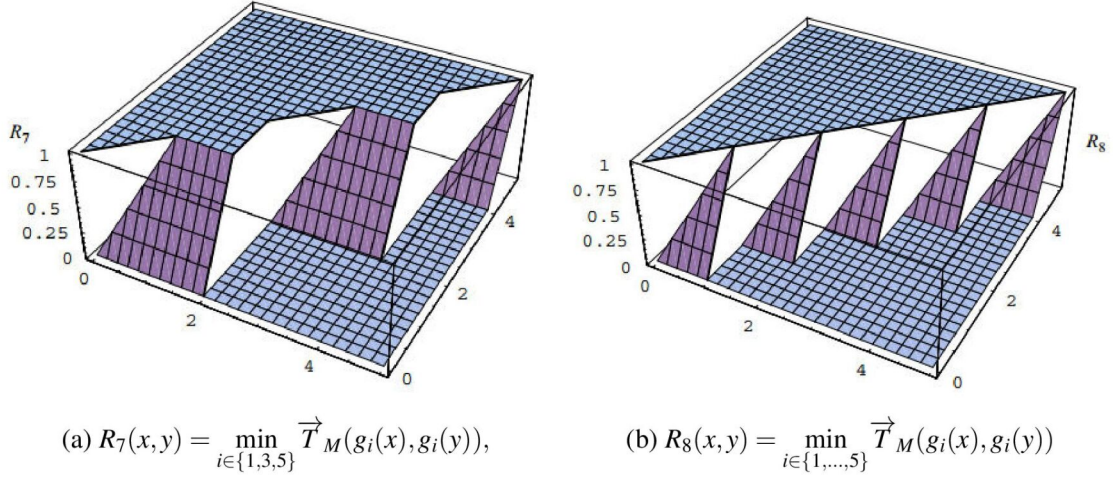
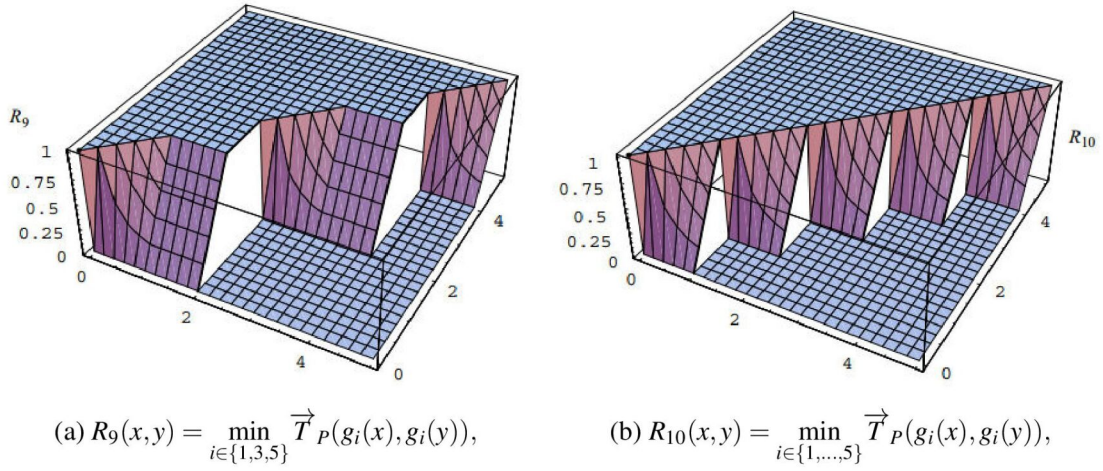
Figure 2.5: Graphic of score functions g_1 and g_2

Figure 2.6: Graphic of score functions g_3 and g_4 Figure 2.7: Graphic of score functions $g_5(x) = \min(1, \max(0, x-4))$

It is immediate that all five functions are non-decreasing with respect to the natural order of real numbers.

Figures. 2.8, 2.9 and 2.10 depicts six fuzzy weak orders defined in accordance with :

- $R_7(x, y) = \min_{i \in \{1, 3, 5\}} \vec{T}_M(g_i(x), g_i(y)), \quad R_8(x, y) = \min_{i \in \{1, \dots, 5\}} \vec{T}_M(g_i(x), g_i(y)),$
- $R_9(x, y) = \min_{i \in \{1, 3, 5\}} \vec{T}_P(g_i(x), g_i(y)), \quad R_{10}(x, y) = \min_{i \in \{1, \dots, 5\}} \vec{T}_P(g_i(x), g_i(y)),$
- $R_{11}(x, y) = \min_{i \in \{1, 3, 5\}} \vec{T}_L(g_i(x), g_i(y)), \quad R_{12}(x, y) = \min_{i \in \{1, \dots, 5\}} \vec{T}_L(g_i(x), g_i(y)).$

Figure 2.8: Fuzzy weak orders constructed by means of Theorem 2.1.12 using the T_M t-normsFigure 2.9: Fuzzy weak orders constructed by means of Theorem 2.1.12 using the T_P t-norms

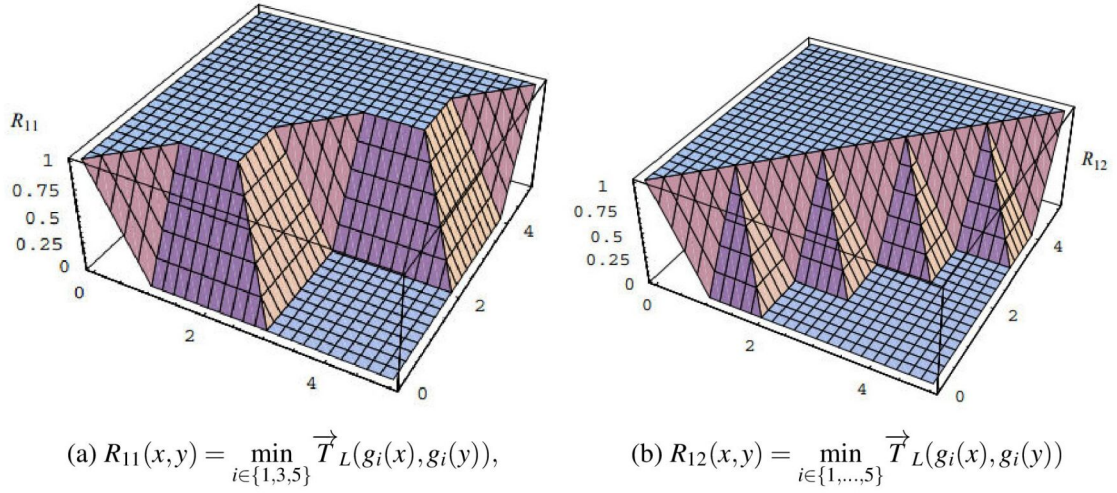


Figure 2.10: Fuzzy weak orders constructed by means of Theorem 2.1.12 using the T_L t-norms

Example 2.1.13 uses the natural linear order of real numbers and rather simple monotonic score functions. The next example constructs some more complicated weak T_L -orders on the basis of a non-trivial linear order on the real numbers.

Example 2.1.14 [6] Let us consider the following transformation function:

$$\varphi(x) = \begin{cases} 4-x & \text{if } x \in [1, 3], \\ x & \text{otherwise.} \end{cases}$$

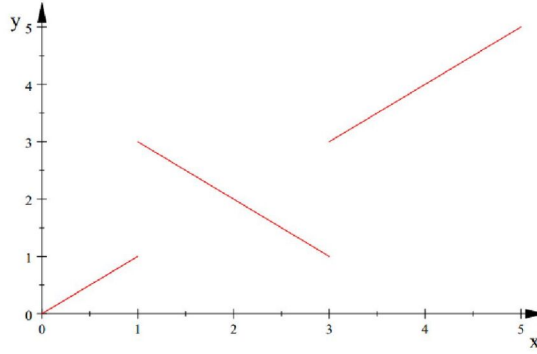


Figure 2.11: The transformation function

It is immediate that is a self-inverse bijective $\mathbb{R} \rightarrow \mathbb{R}$ mapping that equals the identity in $] -\infty, 1[\cup]3, +\infty[$ and flips the values in $[1, 3]$. It is clear, therefore, that the binary relation

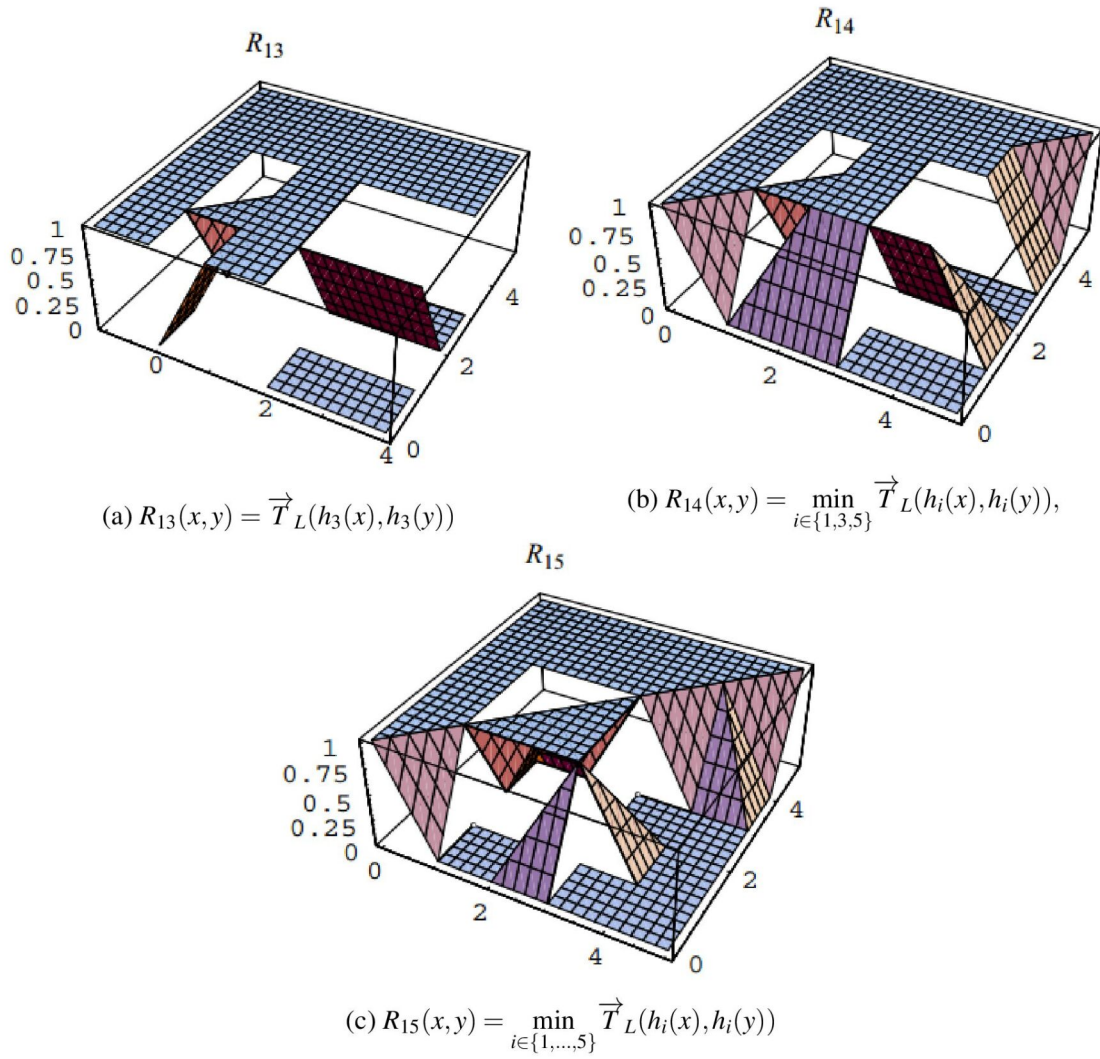
$$x \preccurlyeq y \text{ if and only if } \varphi(x) \leq \varphi(y)$$

is a linear order on the real numbers. Taking the score functions $g_1, \dots, g_5$ from Example 2.1.13, we can define another family of score functions $h_1, \dots, h_5$ as $h_i(x) = g_i(\varphi(x))$ (for all $x \in [0, 5]$). It is easy to see that all functions h_i are non-decreasing with respect to $\preccurlyeq$. Thus, we can use them to define fuzzy weak orders.

Figures 2.12 shows three weak, T_L -orders defined as follows:

- $R_{13}(x, y) = \overrightarrow{T}_L(h_3(x), h_3(y)),$
- $R_{14}(x, y) = \min_{i \in \{1, 3, 5\}} \overrightarrow{T}_L(h_i(x), h_i(y)),$
- $R_{15}(x, y) = \min_{i \in \{1, \dots, 5\}} \overrightarrow{T}_L(h_i(x), h_i(y))$

Remark 2.1.15 As a final remark concerning score function-based representations, let us note that the construction methods provided in this section are only of practical relevance if the number of score functions is finite.

Figure 2.12: Weak T_L -orders with a non-trivial underlying crisp linear order

3

Representation and construction by inclusion-based

3.1 Inclusion-based representations

As mentioned in Section 1, Theorem 1.7.5 can also be proved by embedding the given crisp weak order into the partially ordered set $(P(X), \subseteq)$. Technically, this is done by mapping the elements $x \in X$ to their foresets $C(x)$.

The question arises whether an analogous technique works for fuzzy weak orders as well.

This section is devoted to that topic.

Definition 3.1.1 [39] *Consider the fuzzy power set $\mathcal{F}(X)$. Then the well-known crisp inclusion of fuzzy sets.*

$A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$ is a crisp partial order on $\mathcal{F}(X)$.

Definition 3.1.2 [2, 4, 20] Given a left-continuous t -norm T , we can define the following two binary fuzzy relations on $\mathcal{F}(X)$:

$$\begin{aligned} INCL_T(A, B) &= \inf_{x \in X} \overrightarrow{T}(A(x), B(x)), \\ SIM_T(A, B) &= \inf_{x \in X} \overleftrightarrow{T}(A(x), B(x)). \end{aligned}$$

Proposition 3.1.3 [4] Let $A, B \in \mathcal{F}(X)$. Then,

- (i) SIM_T is a T -equivalence on $\mathcal{F}(X)$.
- (ii) $INCL_T$ is a $T - SIM_T$ -order on $\mathcal{F}(X)$.
- (iii) $INCL_T(A, B) = 1$ if and only if $A \subseteq B$.
- (vi) $SIM_T(A, B) = 1$ if and only if $A = B$.

Proof. It is easy to see from elementary properties of residual (bi)implications. ■

The following theorem gives us a unique description of fuzzy weak orders, which are based on the inclusion of the specified fuzzy weak order in the set of fuzzy powers.

Theorem 3.1.4 [6] Consider a binary fuzzy relation $R : X^2 \rightarrow [0, 1]$. Then the following two statements are equivalent:

- (i) R is a weak T -order.
- (ii) There exists a non-empty family of fuzzy sets $S \subseteq \mathcal{F}(X)$ that are linearly ordered with respect to the inclusion relation $\subseteq$ and a mapping $\varphi : X \rightarrow S$ such that the following representation holds for all $x, y \in X$:

$$R(x, y) = INCL_T(\varphi(x), \varphi(y)). \quad (3.1)$$

Proof. (i) $\Rightarrow$ (ii): Let us define S as the set of foresets, i.e., $S = \{C \in \mathcal{F}(X) \mid \exists x \in X : C(y) = R(y, x), \forall y \in X\}$. To prove that S is linearly ordered with respect to inclusion, consider an arbitrary pair $x, y \in X$. As

R is strongly complete, either $R(x, y) = 1$ or $R(y, x) = 1$ holds. Assume that $R(x, y) = 1$ holds, then we obtain the following for all $z \in X$:

$$R(z, x) = T \left(R(z, x), \underbrace{R(x, y)}_{=1} \right) = 1 \leq R(z, y)$$

This means that the foreset of x is fully contained in the foreset of y . If $R(y, x) = 1$, we can see analogously that the foreset of y is a subset of the foreset of x . Thus, we have proved that S is linearly ordered with respect to the inclusion relation $\subseteq$. Now define φ as the mapping that maps each element x to its foreset $R(., x)$, i.e., $\varphi(x)(z) = R(z, x)$. Then Lemma 2.0.1 gives us the following:

$$R(x, y) = \inf_{z \in X} \vec{T}(R(z, x), R(z, y)) = INCL_T(\varphi(x), \varphi(y)).$$

(ii) $\Rightarrow$ (i): It has been mentioned above that SIM_T is a T -equivalence on $F(X)$ and that $INCL_T$ is a $T - SIM_T$ -order on $\mathcal{F}(X)$. If we restrict to a subset $S \subseteq F(X)$ that is linearly ordered with respect to inclusion, $INCL_T$ is a strongly complete $T - SIM_T$ -order on S . Then it follows from Theorem 3.1 that $R(x, y) = INCL_T(\varphi(x), \varphi(y))$ is a weak T -order. ■

With only cosetic modifications of the proof, we can formulate an equivalent result that appears a bit more appealing than 3.1.4

Corollary 3.1.5 [5, 6] *Consider a binary fuzzy relation $R : X^2 \rightarrow [0, 1]$. Then the following two statements are equivalent:*

- (i) R is a weak T -order.
- (ii) *There exists a mapping $\varphi : X \rightarrow \mathcal{F}(X)$ fulfilling $\varphi(x) \subseteq \varphi(y)$ or $\varphi(y) \subseteq \varphi(x)$ for all $x, y \in X$ such that representation 3.1 holds.*

If we omit the linearity conditions in Theorem 3.1.4 and Corollary 3.1.5, a unique representation of T -preorders is obtained: a binary fuzzy relation R is a T -preorder if and only if there exists a mapping $\varphi : X \rightarrow \mathcal{F}(X)$ such that Eq. 3.1 holds. In this sense, the T -preorder $INCL_T$ on $\mathcal{F}(X)$ “contains” all T -preorders that can be defined on X . Weak T -orders are then the sub-class that is obtained by restricting to linearly ordered subsets of $\mathcal{F}(X)$.

Although the use of foresets is essential for proving Theorem 3.1.4 (and Corollary 3.1.5), there is no restriction to use only foresets in the construction Eq 3.1, as long as the range of the embedding mapping $\varphi(X)$ is linearly ordered. Thus, Theorem 3.1.4 and Corollary 3.1.5 give rise to potentially interesting constructions. For infinite domains, however, $INCL_T(A, B)$ is mostly difficult to compute, as an infimum over an infinite set has to be determined. Only under very restrictive assumptions, for instance, that all membership functions of the fuzzy sets $\varphi(x)$ are piecewise linear or differentiable, practically feasible constructions are imaginable. One can overall conclude that Theorem 3.1.4 and Corollary 3.1.5 provide us with nice theoretical insight, but they do not have much practical value. That is why we do not provide an example in this section.

4

Representations by decomposition into crisp linear orders and fuzzy equivalence relations

4.1 Decomposition into crisp linear orders and T-equivalences

The standard proof of Theorem 1.7.5 is based on factorization with respect to the symmetric kernel of a given weak order (cf. Section 1).

This section follows this idea and presents corresponding results for fuzzy weak orders. Before coming to the main result, let us shortly introduce an important prerequisite.

Definition 4.1.1 [6] *Let $\preceq$ be a crisp order on X and let $E : X^2 \rightarrow [0, 1]$ be a fuzzy equivalence relation (regardless of the underlying t -norm T). E is called compatible with $\preceq$ if the following inequality holds*

for all ascending three-element chains $x \preccurlyeq y \preccurlyeq z$ in X :

$$E(x, z) \leq \min(E(x, y), E(y, z))$$

Remark 4.1.2 *Compatibility of a crisp order and a fuzzy equivalence relation can be understood as follows: the two outer elements of an ascending three-element chain are at most as similar as any two elements of this chain.*

Theorem 4.1.3 [6] *Consider a binary fuzzy relation $R : X^2 \rightarrow [0, 1]$. Then the following two statements are equivalent:*

(i) *R is a weak T -order.*

(ii) *There exists a crisp linear order $\preccurlyeq$ and a T -equivalence E that is compatible with $\preccurlyeq$ such that R can be represented as follows:*

$$R(x, y) = \begin{cases} 1 & \text{if } x \preccurlyeq y, \\ E(x, y) & \text{otherwise.} \end{cases} \quad (4.1)$$

Proof. (i) $\Rightarrow$ (ii): On the one hand, we know from Lemma 2.1.11 that R contains a crisp linear order $\preccurlyeq$. On the other hand, we can define a fuzzy relation E as the symmetric kernel of R , i.e.,

$$E(x, y) = T(R(x, y), R(y, x)).$$

It is easy to show that E is a T -equivalence[37]. Now we have to show that representation 4.1 really holds. First of all, assume that $x \preccurlyeq y$. Since $\preccurlyeq$ is a sub-relation of the kernel relation $\lesssim_R$, it directly follows that $R(x, y) = 1$, and the first case of 4.1 follows. Conversely, assume that $x \not\preccurlyeq y$. Since $\preccurlyeq$ is linear, it follows that $y \preccurlyeq x$, hence $R(y, x) = 1$, and we obtain

$$R(x, y) = T(R(x, y), \underbrace{R(y, x)}_{=1}) = E(x, y),$$

which shows the second case. It remains to prove that E is really compatible with $\preccurlyeq$. For this purpose, assume we have a chain $x \preccurlyeq y \preccurlyeq z$, which implies $R(x, y) = 1$, $R(y, z) = 1$, and $R(x, z) = 1$. Then we

obtain from T -transitivity

$$R(z, x) = T(R(z, x), \overbrace{R(x, y)}^{=1}) \leq R(z, y),$$

$$R(z, x) = T(\underbrace{R(y, z)}_{=1}, R(z, x)) \leq R(y, x),$$

which finally implies compatibility:

$$E(x, z) = T(\overbrace{R(x, z)}^{=1}, R(z, x)) = R(z, x) \leq R(z, y) = T(\overbrace{R(y, z)}^{=1}, R(z, y)) = E(y, z),$$

$$E(x, z) = T(\underbrace{R(x, z)}_{=1}, R(z, x)) = R(z, x) \leq R(y, x) = T(\underbrace{R(x, y)}_{=1}, R(y, x)) = E(x, y).$$

(ii) $\Rightarrow$ (i): It is sufficient to show that 4.1 defines a weak T -order. Reflexivity is a trivial consequence of the reflexivity of $\preccurlyeq$. In order to prove T -transitivity, let us first assume $x \preccurlyeq z$. Then $R(x, z) = 1$ holds and T -transitivity can never be violated, no matter which y we choose. So assume that $z \prec x$ holds, which implies $R(x, z) = E(x, z)$. If $y = x$ or $y = z$, T -transitivity follows trivially. So it remains to consider the following three cases:

1. $z \prec y \prec x$: Here, $R(x, y) = E(x, y)$, $R(y, z) = E(y, z)$, and $R(x, z) = E(x, z)$, hence T -transitivity follows directly from the fact that E is T -transitive.
2. $z \prec x \prec y$: In this case, $R(x, y) = 1$, $R(y, z) = E(y, z)$, and $R(x, z) = E(x, z)$. The compatibility of R and E implies that $E(y, z) \leq E(x, z)$. Thus, we obtain $T(R(x, y), R(y, z)) = T(1, E(y, z)) = E(y, z) \leq E(x, z) = R(x, z)$.
3. $y \prec z \prec x$: Analogous to the previous case.

Strong completeness of R follows directly from the linearity of $\preccurlyeq$.

■

Representation 4.1 simply says the following: weak T -orders are characterized as unions of crisp linear orders and compatible T -equivalences. In other words, we can say that weak T -orders are a fuzzification of crisp linear orders, and the fuzzy component can solely be attributed to a T -equivalence. To utilize Theorem 4.1.3 for constructing weak T -orders, we have to know more about how to construct

T -equivalences that are compatible with a given crisp linear order. Let us start with a well-known result on T -equivalences. We cite this result without its proof which is analogous to the proof of Theorem 2.1.7

Theorem 4.1.4 (Valverde [37]). *Consider a binary fuzzy relation $E : X^2 \rightarrow [0, 1]$. Then the following two statements are equivalent:*

- (i) E is a T -equivalence.
- (ii) *There exists a non-empty family of $X \rightarrow [0, 1]$ functions $(f_i)_{i \in I}$ such that the following representation holds:*

$$E(x, y) = \inf_{i \in I} \overleftrightarrow{T}(f_i(x), f_i(y)). \quad (4.2)$$

The following theorem finally provides a unique characterization of T -equivalences that are compatible with a given crisp linear order.

Theorem 4.1.5 [6] *Consider a crisp linear order $\preceq$ on X and a binary fuzzy relation $E : X^2 \rightarrow [0, 1]$. Then the following two statements are equivalent:*

- (i) E is a T -equivalence that is compatible with $\preceq$.
- (ii) *There exists a non-empty family of $X \rightarrow [0, 1]$ functions $(f_i)_{i \in I}$ that are non-decreasing with respect to $\preceq$ such that representation 4.2 holds.*

Proof. (i) $\Rightarrow$ (ii): Given a T -equivalence E that is compatible with $\preceq$, we can apply Theorem 4.1.3 and define a weak T -order R given as in Then Lemma 2.0.1 guarantees that R can be represented as $z \in X$

$$R(x, y) = \inf_{z \in X} \overrightarrow{T}(R(z, x), R(z, y)).$$

As R is a weak T -order, $R(x, y) = 1$ or $R(y, x) = 1$ must hold for all pairs $x, y \in X$, and we can infer $E(x, y) = \min(R(x, y), R(y, x))$. Therefore, we obtain the following:

$$\begin{aligned} E(x, y) &= \min(R(x, y), R(y, x)) \\ &= \min\left(\inf_{z \in X} \overrightarrow{T}(R(z, x), R(z, y)), \inf_{z \in X} \overrightarrow{T}(R(z, y), R(z, x))\right) \\ &= \inf\left(\min(\overrightarrow{T}(R(z, x), R(z, y)), \overrightarrow{T}(R(z, y), R(z, x)))\right) \\ &\stackrel{(B3)}{=} \inf_{z \in X} \overleftrightarrow{T}(R(z, x), R(z, y)). \end{aligned}$$

As $\preccurlyeq$ is a sub-relation of the kernel relation of R , we can infer from 2.1.9 that all aftersets $R(z, \cdot)$ are non-decreasing with respect to $\preccurlyeq$. Hence, with the setting $I = X$, $f_z(x) = R(z, x)$ (for all $x, z \in X$), we have found a family of non-decreasing functions such that representation Eq 4.2 holds.

(ii) $\Rightarrow$ (i): Theorem 4.1.4 guarantees that Eq 4.2 defines a T -equivalence, and it only remains to prove that E is compatible with $\preccurlyeq$. Suppose we have an ascending chain $x \preccurlyeq y \preccurlyeq z$. If all f_i are non-decreasing, we have $f_i(x) \leq f_i(y)$,

$f_i(y) \leq f_i(z)$, and $f_i(x) \leq f_i(z)$ for all $i \in I$. Therefore, by (B5), we have

$$E(x, y) = \inf_{i \in I} \overleftrightarrow{T}(f_i(x), f_i(y)) = \inf_{i \in I} \overrightarrow{T}(f_i(y), f_i(x)),$$

$$E(y, z) = \inf_{i \in I} \overleftrightarrow{T}(f_i(y), f_i(z)) = \inf_{i \in I} \overrightarrow{T}(f_i(z), f_i(y)),$$

$$E(x, z) = \inf_{i \in I} \overleftrightarrow{T}(f_i(x), f_i(z)) = \inf_{i \in I} \overrightarrow{T}(f_i(z), f_i(x)).$$

Then the inequalities $E(x, z) \leq E(x, y)$ and $E(x, z) \leq E(y, z)$ follow from the fact that residual implications are non-increasing in the first and non-decreasing in the second argument. ■

Remark 4.1.6 *Note that Theorem 4.1.5 remains valid if we replace (non-decreasing) in (ii) by (non-increasing). One should be aware of the fact that the insight provided by combining Theorem 4.1.3 and Theorem 4.1.5 is only of theoretical interest.*

5

Constructions the fuzzy weak order by pseudo-metric-based

This chapter we try to take advantage of correspondence between pseudo-metrics and fuzzy equivalence relations that holds if we restrict to continuous Archimedean t -norms. Unlike most of the existing literature, we have not assumed that the underlying domain is finite.

Moreover, we noted above that Theorem 4.1.5 is only of little help for deploying new constructions. Therefore, we would like to change our viewpoint now and turn to the well-known correspondence between pseudo-metrics and T -equivalences, which we can combine with Theorem 4.1.3 in order to achieve really new and practically feasible constructions of weak T -orders. This correspondence holds only for continuous Archimedean t -norms, thus, we restrict to this class of t -norms in this section. The following fundamental representation result is known for continuous Archimedean t -norms [25, 27, 34]:

a binary operation on the unit interval $T : [0, 1]^2 \rightarrow [0, 1]$ is a continuous Archimedean t -norm if and only if there exists a continuous, strictly decreasing function $t : [0, 1] \rightarrow [0, \infty]$ with $t(1) = 0$ called additive generator such that, for all $x, y \in [0, 1]$, the following holds:

$$T(x, y) = t^{-1}(\min(t(x) + t(y), t(0))).$$

The generator t is uniquely determined up to a positive multiplicative constant.

5.1 Constructions the fuzzy weak order by pseudo-metric-based

Definition 5.1.1 A mapping $d : X^2 \rightarrow [0, \infty]$ is called a pseudo-metric on X if it fulfills the following properties for all $x, y, z \in X$:

1. $d(x, x) = 0$ (homogeneity),
2. $d(x, y) = d(y, x)$ (symmetry),
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Remark 5.1.2 For all $x \in X$ and $\lambda \geq 0$, an open-ball function of radius λ and around x is defined as

$$B_d(x, \lambda) = \{x \in X : d(x, K) < \lambda\}$$

The following theorem provides a simple, yet fundamental, correspondence between pseudo-metrics and fuzzy equivalence relations with respect to continuous Archimedean t -norms that will be essential for this paper.

Theorem 5.1.3 (De Baets and Mesiar [13][14], Moser [28]). Consider a continuous Archimedean t -norm T with additive generator t .

(1) For every pseudo-metric $d : X^2 \rightarrow [0, \infty]$, the mapping

$$E_d(x, y) = t^{-1}(\min(d(x, y), t(0))) \tag{5.1}$$

defines a T -equivalence on X .

(2) Provided that $E : X^2 \rightarrow [0, 1]$ is a T -equivalence, we can define a pseudo-metric d_E as

$$d_E(x, y) = t(E(x, y)). \quad (5.2)$$

Now let us introduce a dual notion of compatibility between pseudo-metrics and orders.

Definition 5.1.4 Let $\preceq$ be a crisp order on X and let $d : X^2 \rightarrow [0, \infty]$ be a pseudo-metric. d is called compatible with $\preceq$ if the following inequality holds for all ascending three-element chains $x \preceq y \preceq z$ in X

$$d(x, z) \geq \max(d(x, y), d(y, z)).$$

Remark 5.1.5 [6] Compatibility of a crisp order and a pseudo-metric can be understood in perfect analogy to the compatibility of fuzzy equivalence relations and orders: the two outer elements of an ascending three-element chain are at least as far away from each other as any two elements of this chain. Now we establish the link between Theorem 5.1.3 and the notion of compatibility.

Lemma 5.1.6 [6] . Let T be a continuous Archimedean t -norm with additive generator t and let $\preceq$ be a crisp order on X .

- (1) Consider a pseudo-metric $d : X^2 \rightarrow [0, \infty]$. If d is compatible with $\preceq$, then its induced fuzzy equivalence relation E_d , defined as in Eq 5.1, is compatible with $\preceq$.
- (2) Consider a fuzzy equivalence relation $E : X^2 \rightarrow [0, 1]$. If E is compatible with $\preceq$, then its induced pseudo-metric d_E , defined as in Eq. 5.2, is also compatible with $\preceq$.

Proof. Follows directly from the fact that the additive generator t and its inverse are non-increasing functions. ■

Example 5.1.7 [6] It is easy to see that $d(x, y) = |x - y|$ is a metric on the real numbers $X = \mathbb{R}$ that is compatible with the natural order $\leq$ [8]. Further note that $t_P(x) = -\ln x$ is an additive generator of the product t -norm T_P with inverse $t_P^{-1}(x) = e^{-x}$ and that $t_L(x) = 1 - x$ is a self-inverse additive generator of

the Łukasiewicz t -norm T_L . Hence, Theorem 4.1.3 and Lemma 5.1.6 entail that

$$R_{16}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \exp(-\min(|x-y|, \infty)) & \text{otherwise} \end{cases}$$

$$= \min(1, \exp(y-x))$$

is a weak T_P -order and

$$R_{17}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - \min(|x-y|, 1) & \text{otherwise } x \leq y \end{cases}$$

$$= \min(1, \max(1-x+y, 0))$$

is a weak T_L -order.

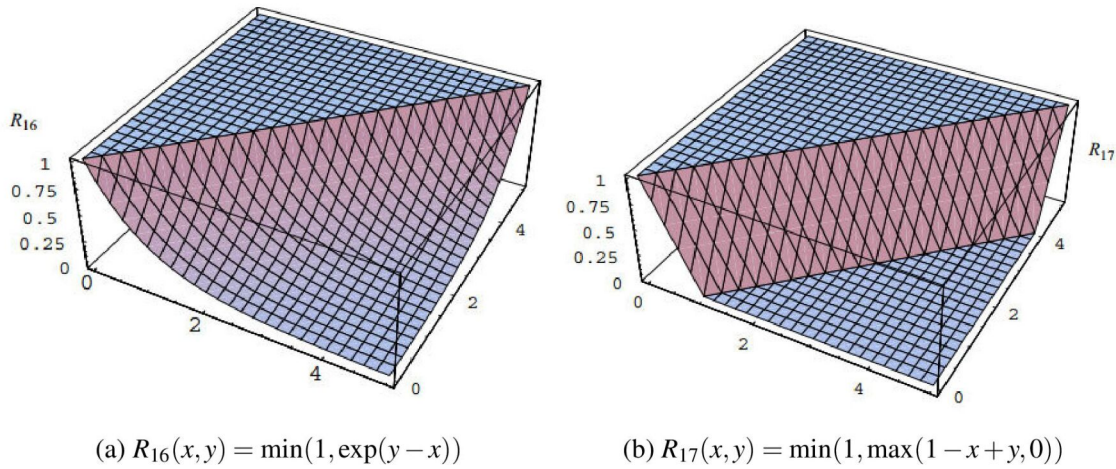


Figure 5.1: Fuzzy weak orders constructed from the absolute distance of real numbers.

Figures 5.1 shows these two fuzzy weak orders, where the plots are restricted to $[0, 5]^2$. Of course, multiplying $d(x,y)$ with $1/C$ (where C is a positive constant) does not change the compatibility with $\leq$. At least for $T = T_L$, the constant C then corresponds to a “radius of interest”. This simple construction is highly useful for designing fuzzy relations for flexible query answering systems [8].

The following lemma demonstrates that compatibility of a pseudo-metric with a crisp linear order is preserved even by monotonic transformations of the underlying domain. Note that we do not make any assumption concerning bijectivity or continuity.

Lemma 5.1.8 [6] Consider a pseudo-metric $d : X^2 \rightarrow [0, \infty]$ that is compatible with a crisp linear order $\preceq$ on X and a monotonic (non-decreasing or non-increasing) mapping $\psi : X \rightarrow X$. Then the mapping

$$d'(x,y) = d(\psi(x), \psi(y))$$

defines a pseudo-metric on X that is compatible with $\preceq$.

Proof. Take an ascending three-element chain $x \preceq y \preceq z$. Since ψ is non-decreasing, we obtain $\psi(x) \leq \psi(y) \leq \psi(z)$

and compatibility of d with entails

$$d(\psi(x), \psi(z)) \geq \max(d(\psi(x), \psi(y)), d(\psi(y), \psi(z))).$$

If ψ is non-increasing, the same follows analogously taking into account the symmetry of d . ■

Putting all the above results together, we obtain the following construction theorem.

Theorem 5.1.9 [6] Consider a crisp linear order $\preceq$ on X , a pseudo-metric d that is compatible with $\preceq$, a continuous

Archimedean t -norm T with additive generator t , and a monotonic function $\psi : X \rightarrow X$. Then

$$R(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ t^{-1}(\min(d(\psi(x), \psi(y)), t(0))) & \text{otherwise} \end{cases}$$

is a weak T -order.

Proof. Is an immediate consequence of Theorems 4.1.3 and 5.1.3, Lemma 5.1.6 and Lemma 5.1.8. ■

Remark 5.1.10 [6] We can also revert this construction: starting from an arbitrary fuzzy weak order R with respect to a continuous Archimedean t -norm, by means of Theorem 4.1.3, we obtain a linear order $\preceq$ and a T -equivalence E that is compatible with $\preceq$. Then, by Theorem 5.1.3 and Lemma 5.1.6, we can construct a pseudo-metric that is compatible with $\preceq$.

As constructions of pseudo-metrics are not within the scope of this memory, we do not follow these ideas any further. One issue still remains open. While monotonic functions are simple and well-known objects (in particular, if the real numbers are considered), we have not yet addressed the question how to design an application-tailored monotonic transformation. To tackle this problem, we adopt the idea of scaling functions that has been introduced for constructing fuzzy equivalence relations [24, 34]. Suppose that our domain of interest X is some measurable subset of the real numbers and let $c : X \rightarrow \mathbb{R}_0^+$ be a non-negative integrable weighting function. Then, for an arbitrarily chosen $x_0 \in X$, the following function is trivially

non-decreasing with respect to the natural linear order $\leq$:

$$\psi(x) = \int_{x_0}^x c(y) dy \quad (5.3)$$

Hence, we have a simple method to construct a monotonic transformation that is easily interpreted as follows: the higher the values of the function c around some point x , the more the elements around x should be distinguished. If, for instance, $c(x) = 0$ in some interval, then we do not distinguish between those values at all. It is obvious that ψ is a kind of antiderivative of the weighting function c , whereas c can be considered correspondingly as a kind of (generalized) derivative of ψ .

Example 5.1.11 [6] *Let us consider the following two weighting functions:*

$$c_1 = \begin{cases} \frac{1}{2} & \text{if } x < 2 \\ 0 & \text{if } x \in [2, 3] \\ 2 & \text{otherwise,} \end{cases}$$

$$c_2 = \frac{1}{\sqrt{\frac{\pi}{8}}} \times e^{(-8 \cdot (x-2.5)^2)}$$

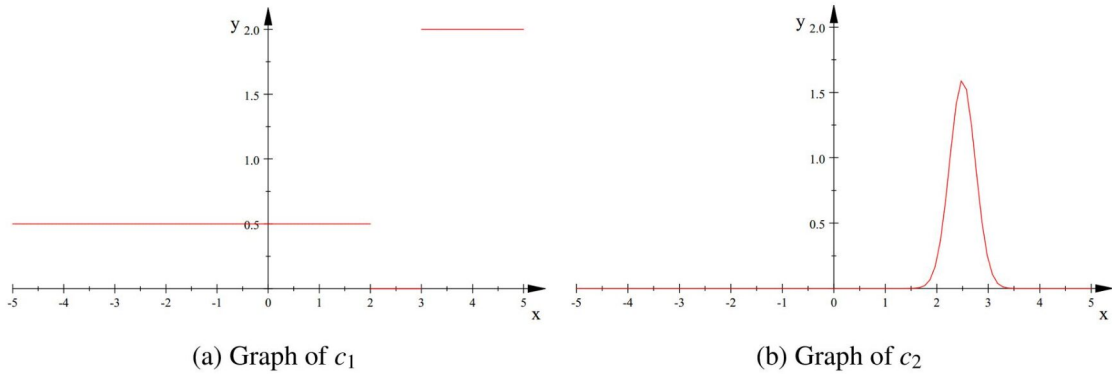


Figure 5.2

Obviously, c_1 is designed not to make any difference between two values from the interval $[2, 3]$ and to weigh differences of values above 3 four times as strongly as differences of values below 2. The function c_2 is obviously nothing else but the density function of the ordinary normal distribution with mean value 2.5 and standard deviation 0.25.

Using 5.3, we can define two scaling functions 1 and 2 from c_1 and c_2 , respectively, which allows us to define the following four fuzzy weak orders (compare also with Example 5.1.7):

- $R_{18}(x, y) = \min(1, \exp(\psi_1(y) - \psi_1(x)))$,
- $R_{19}(x, y) = \min(1, \exp(\psi_2(y) - \psi_2(x)))$,
- $R_{20}(x, y) = \min(1, \max(1 - \psi_1(x) + \psi_1(y), 0))$,
- $R_{21}(x, y) = \min(1, \max(1 - \psi_2(x) + \psi_2(y), 0))$.

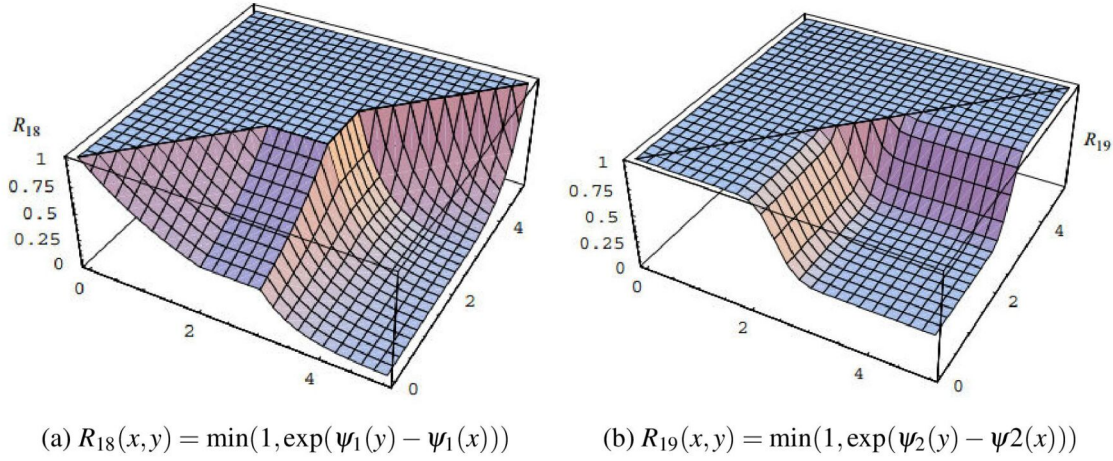


Figure 5.3: Fuzzy weak orders on the real numbers constructed from the absolute difference of real numbers and two distortion functions that were defined from weighting functions.

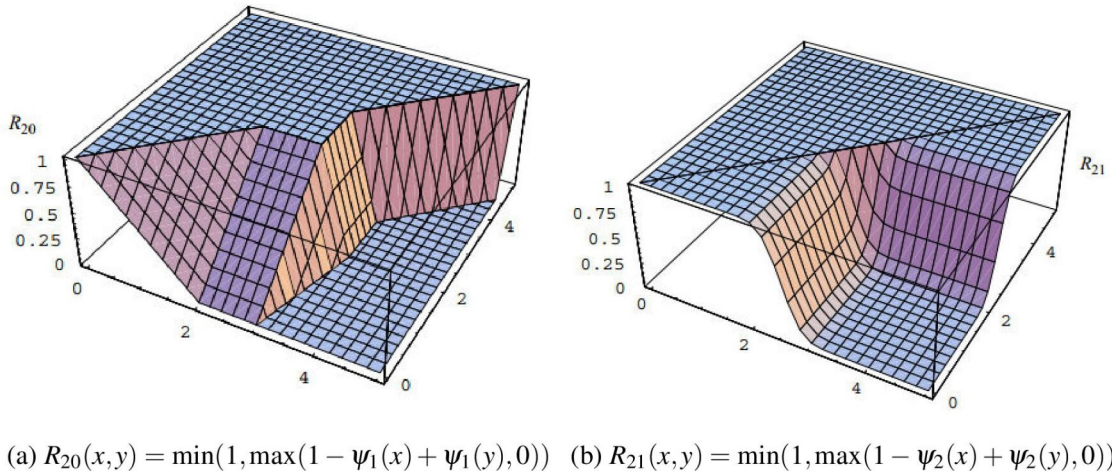


Figure 5.4: Fuzzy weak orders on the real numbers constructed from the absolute difference of real numbers and two distortion functions that were defined from weighting functions.

It is clear from the results above that R_{18} and R_{19} are weak T_p -orders, while R_{20} and R_{21} are weak T_L -orders. Figures 5.4 and Figures 5.3 shows these four fuzzy weak orders (plots are again restricted to $[0, 5]^2$).

The applicability of construction Eq 5.3 is rather limited, as we may only use the natural order $\leq$ for the construction.

If we have a non-trivial order $\preccurlyeq$, the function defined in Eq 5.3 is not necessarily monotonic and, therefore, useless for the purpose of constructing fuzzy weak orders. But there is rather simple solution to this problem. Let $\preccurlyeq$ be some linear order on $X \subseteq \mathbb{R}$ such that all “intervals” $[a, b]_{\preccurlyeq} = \{x \in X | a \preccurlyeq x \preccurlyeq b\}$ are measurable sets. Then, for an arbitrarily chosen $x_0 \in X$, the following function is non-decreasing with respect to $\preccurlyeq$:

$$\psi(x) = \begin{cases} \int_{[x_0, x]_{\preccurlyeq}} c(y) dy & \text{if } x_0 \preccurlyeq x \\ -\int_{[x_0, x]_{\preccurlyeq}} c(y) dy & \text{otherwise} \end{cases} \quad (5.4)$$

Thus, we have a simple principle to construct interpretable application-specific monotonic transformations with respect to an arbitrary crisp linear order $\preccurlyeq$. Then this construction can be used directly to construct fuzzy weak orders with respect to continuous Archimedean t-norms using Theorem 4.1.3 The restriction that all intervals need to be measurable is not a serious restriction-in fact, it is practically impossible to violate it.

General conclusions and future research

This memory presents three fundamental representation results, each of which also provides a construction method:

1. Representations and construction of fuzzy weak orders by score function-based.
2. Representations and construction of fuzzy weak orders by inclusion-based.
3. Representations by decomposition into crisp linear orders and fuzzy equivalence relations,

which also facilitates a pseudo-metric-based construction. i.e., we have exploited the correspondence between pseudo-metrics and fuzzy equivalence relations that holds if we restrict to continuous Archimedean t -norms. Unlike most of the existing literature, we have not assumed that the underlying domain is finite. The examples in this memory demonstrate that meaningful constructions are possible even for non-countable domains.

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خلاصة

في هذه المذكرة ، أبرزنا تمثيلات مختلفة للأوامر الضعيفة الغامضة. سجل التمثيل القائم على الوظيفة وتحلل الأوامر الضعيفة الغامضة إلى أوامر خطية واضحة وعلاقات تكافؤ غامض أيضًا زودتنا بأساليب البناء الممكنة عمليا. بالنسبة للآخر ، فقد استغلنا المراسلات بين المقاييس الزائفة وعلاقات التكافؤ الغامضة التي تتمسك بها إذا اقتصرنا على معايير أرخميدس المستمرة. على عكس معظم الأدبيات الموجودة ، لم نفترض أن المجال الأساسي محدود. الأمثلة في هذا تالمذكرة وتوضح أن الإنشاءات ذات المغزى ممكنة حتى بالنسبة للمجالات غير القابلة للعد.

الكلمات المفتاحية: ترتيب ضعيف غامض ؛ علاقة غامضة .

Abstract EN:

In this memory, we have highlighted various representations of fuzzy weak orders. Score function-based representations and the decomposition of fuzzy weak orders into crisp linear orders and fuzzy equivalence relations also provided us with practically feasible construction methods. For the latter, we have exploited the correspondence between pseudo-metrics and fuzzy equivalence relations that holds if we restrict to continuous Archimedes t-norms. Unlike most of the existing literature, we have not assumed that the underlying domain is finite. The examples in this memory demonstrate that meaningful constructions are possible even for non-countable domains.

Key EN: Fuzzy weak order; Fuzzy relation.

Abstract FR:

Dans ce mémoire, nous avons mis en évidence diverses représentations d'ordres faibles flous. Évaluez également les représentations basées sur les fonctions et la décomposition des ordres flous faibles en ordres linéaires nets et relations d'équivalence floue. nous a fourni des méthodes de construction pratiquement réalisables. Pour ces derniers, nous avons exploité la correspondance entre pseudo-métriques et relations d'équivalence floue qui tient si nous nous limitons aux normes t d'Archimède continues. Contrairement à la plupart de la littérature existante, nous n'avons pas supposé que le domaine sous-jacent est fini. Les exemples de ce mémoire démontre que des constructions significatives sont possibles même pour les domaines non dénombrables.

Key FR: Ordre faible flou; Relation floue.

