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وزارة التعليم العــالي والبحث العلمـــي

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Université Mohamed Boudiaf - M'sila

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قــــسم الـهنـــدسة الميكانيكية

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مستخلص اللجنة العلمية لقسم الهندسة الميكانيكية
بخصوص تقييم مطبوعة جامعية

بناءا على التقارير الإيجابية الواردة من السادة أعضاء لجنة دراسة وتقييم مطبوعة جامعية , والآتية أسماؤهم :

- الأستاذ : عمرون صالح — أستاذ — جامعة محمد بوضياف - المسيلة
- الأستاذ : فارسي شوقي — أستاذ محاضر أ — جامعة محمد بوضياف - المسيلة
- الأستاذ : بقار عبد الحكيم — أستاذ محاضر أ — جامعة بسكرة

صادق أعضاء اللجنة العلمية على قبول المطبوعة المنجزة باللغة الإنجليزية من طرف الأستاذ :
سايب شريف أستاذ محاضر- أ - قسم الهندسة الميكانيكية , كلية التكنولوجيا , جامعة محمد بوضياف بالمسيلة. تحت عنوان :

- Continuum Mechanics

وفق البرنامج المقترح لطلبة السنة الأولى ماستر صناعة ميكانيكية وإنتاجية.
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رئيس اللجنة العلمية للقسم

People's Democratic Republic of Algeria

Ministry of High Education and Scientific Research

MOHAMED Boudiaf University of M'sila

Faculty of Technology

Mechanical Engineering Department

Course handout:

CONTINUUM MECHANICS

First year Master in Mechanical Manufacturing and Production

(1ème Année Master Fabrication Mécanique et Production)

Realized by:

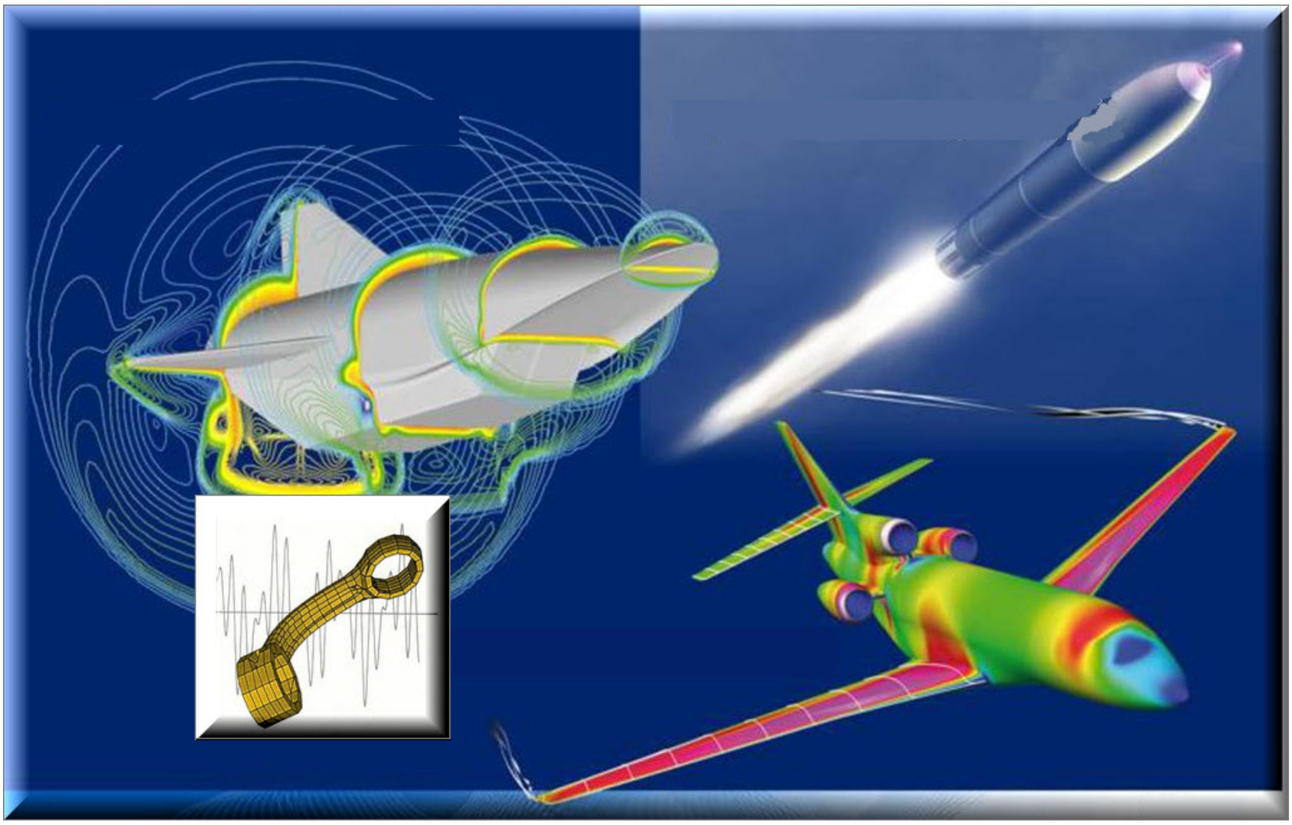
H.D.R. Dr. Cherif SAIB

Academic year: 2024/2025

COURSE HANDOUT

Continuum Mechanics

First Year Master in Mechanical and Production Manufacturing



Directed by: Dr. Saib Cherif

Academic year: 2024/2025.

My hope !! Algeria in future days

DEDICATIONS

I dedicate this work as a call for recognition and reflection to:

♣ *My very dear wife and my dear children for their moral support which allowed me to complete this work.*

I also dedicate it to:

♣ *My dear parents for their encouragement,*

♣ *My dear brothers and sisters*

♣ *My uncles and aunts*

♣ *All my family and loved ones*

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Preface

Continuous media mechanics (Continuum Mechanics-CM) allows us to analyze physical phenomena using a rigorous mathematical description.

This work is mainly inspired by the courses in continuous media mechanics given by the author at the mechanical engineering department of the University of M'sila. This work intended for students of the mechanical engineering specialty who are pursuing Master's studies particularly in mechanical and production manufacturing, actually represents the effort of several years of teaching.

The content and progression of this manuscript were designed with three objectives:

1. To provide the student of the discipline with the necessary elements to address problems of solid mechanics.

2. To support and contribute to the efforts of teaching this discipline (CM) by providing the student with a rigorous and practical document.

3. To present to the students, this discipline as well as its capacities and its limits so that they enrich their culture in the field.

This manuscript is divided into seven chapters:

The first was devoted to a reminder of mathematics including the basic tools necessary for the development of the different concepts of CM. The second chapter we presented the concept of continuum mechanics and Lagranfian and Eulerian descriptions. The stress analysis is devoted to the third chapter. We also presented in the fourth chapter the theory of deformations. The basic equations of the kinematics of continuous media are discussed in the fifth chapter. To conclude, the sixth chapter presents the relationship between stresses and deformations and the fundamental concepts of the behavior laws for an elastic continuous medium. Some applications in solid mechanics are solved in the seventh chapter.

In the last chapter we discuss the failure criteria applied in continuum mechanics.

Chapter I:

Tensor calculation reminders

Chapter 1 : Tensor calculation reminders

- Introduction
- Vector algebra
- Einstein's notation
- Delta kronecker: Properties
- Levi-cevita symbol: Properties
- Transformation law
- Matrices
- Derivative of tensors
- Gauss and stokes integral theorems

I. Introduction :

The continuum mechanics: is the part of physics which studies the mechanics of deformable solid bodies, it is a physical and mathematical framework allowing a concrete problem to be modeled. Once the mathematical model has been established, it can be solved by an analytical or numerical method.…

II. Vector algebra :

When we study physical phenomena we encounter three kinds of magnitudes, the scalars, vectors and tensors.

The physical quantity can be represented mathematically by:

A **scalar** is defined by a ***single number*** independently of the example reference axes: (mass, temperature, volume, pressure....etc.)[1] or a quantity defined by only a magnitude such as: length, time, density, energy, temperature, potential..,[2].

A **vector** is inseparable from the concept of direction; it is defined by its direction, its length, its point of application and by its supporting axis [1] or quantities defined by the couple, a magnitude and a direction such as: displacement, speed, acceleration, volume forces... [2].

A **Tensor:** is a generic word of mathematical entities that designates a physical quantity. Tensors are the generalization of scalars, vectors and matrices.

The description of a tensor depends on the reference, we interested in Cartesian tensors, and the reference system is an orthonormal reference which can be rotated by transformation matrix and change into another reference [3].

As a **tensor of order 2:** stresses, deformations, heat flow, electric field..., a tensor of order greater than 2: for example: tensor of elasticity which is a tensor of order 4 [2].

Order	Entity	Example [1]	Dimension	Nbr components	Notation
0	Scalar	Temperature	2D 3D	$2^0=1$ $3^0=1$	α
1	Vector	Displacement	2D 3D	$2^1=2$ $3^1=3$	v_i
2	Matrix	Stress	2D 3D	$2^2=4$ $3^2=9$	a_{ij}

We can say that as another definition a tensor A is a linear operator (application) on R whose components are defined relative to a basis.

$\forall X, Y \in R^3 \quad A(X+Y) = A(X) + A(Y)$,

$\forall \lambda \in R; X \in R^3 \quad A(\lambda X) = \lambda A(X)$,

The relation Y=A(X) can be written

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

(A) is the matrix associated with the tensor A

Every tensor can be decomposed, and in a unique way, into the sum of a symmetric tensor and an antisymmetric tensor.

$$A = \frac{A+A^t}{2} + \frac{A-A^t}{2} = A^s + A^a = \begin{cases} A^s = \frac{A+A^t}{2} \\ A^a = \frac{A-A^t}{2} \end{cases} [4].$$

II.1 Coordinate system, base vectors:

A vector may be defined with respect to a particular coordinate system by specifying the components of the vector in that system. The choice of coordinate system is arbitrary, but in certain situations a particular choice may be advantageous. The reference system of coordinate axes provides units for measuring vector magnitudes and assigns directions in space by which the orientation of vectors may determinate.

The well-known rectangular Cartesian coordinate system is often represented by the mutually perpendicular axes,(O, X_1, X_2, X_3) shown in Figure .I.1.

The most frequent choice of base vectors for rectangular Cartesian system is the set of unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ along the coordinate axes as shown in Figure .I.1.

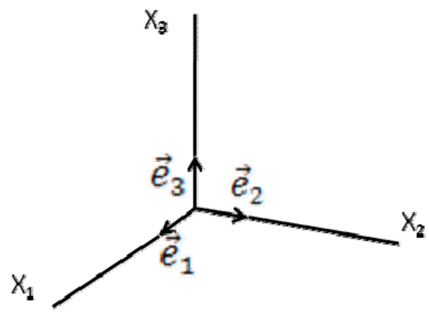


Figure .I.1. The coordinate axes reference.

The orthonormal vector basis$(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$, must satisfy the following conditions:

- $\|\vec{e}_1\| = \|\vec{e}_2\| = \|\vec{e}_3\| = 1$,
- $\vec{e}_1 . \vec{e}_2 = \vec{e}_2 . \vec{e}_3 = \vec{e}_3 . \vec{e}_1 = 0$,
- $\vec{e}_1 . \vec{e}_1 = \vec{e}_2 . \vec{e}_2 = \vec{e}_3 . \vec{e}_3 = 1$ and the following :
- $\vec{e}_1 \times \vec{e}_1 = \vec{e}_2 \times \vec{e}_2 = \vec{e}_3 \times \vec{e}_3 = \vec{0}$,
- $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$, $\vec{e}_2 \times \vec{e}_1 = -\vec{e}_3$,
- $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$, $\vec{e}_3 \times \vec{e}_2 = -\vec{e}_1$,
- $\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$, $\vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$,

II.2 Position vector:

In a reference space we define an orthonormal vector basis $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$. A point P in space can be defined as a **position vector** $\overrightarrow{OP}$, which gives the position of the point P in space with respect to the defined reference.

The position vector can be expressed by: $\overrightarrow{OP} = X_1\vec{e}_1 + X_2\vec{e}_2 + X_3\vec{e}_3$;

where: X_1, X_2, X_3 are the components of the vector $\overrightarrow{OP}$ in the orthonormal basis $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$.

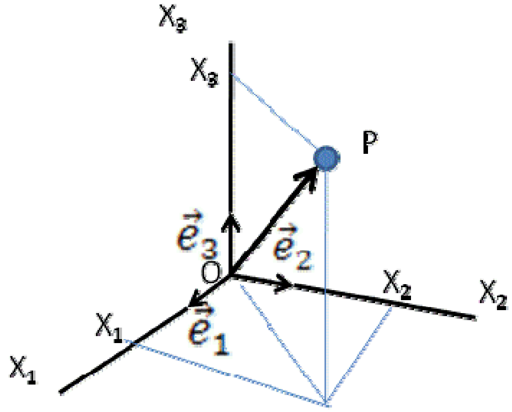


Figure .I.2. Projection of a point P defined in an orthonormal vector basis $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$.

The magnitude or the module of the position vector is given by: $\|\overrightarrow{OP}\| = \sqrt{X_1^2 + X_2^2 + X_3^2}$

Or: $\overrightarrow{OP}.\overrightarrow{OP} = X_1^2 + X_2^2 + X_3^2$

_**Unit vector:** is a vector of length “magnitude, modulus” equal to one (unit).

$$\|\vec{e}\| = 1.$$

_ We can extract a **unit vector** from a non-zero vector, as follows:

$$\vec{e}_u = \frac{\vec{u}}{\|\vec{u}\|}, \quad \text{or: } \vec{u} = \|\vec{u}\|.\vec{e}_u.$$

_ **Null vector**: is a zero-length vector. $\vec{0}$, or: $\|\vec{0}\| = 0$.

II.3 Vector operations :

Let be the non null vectors: $\vec{u}, \vec{v}$ and $\vec{w}$,

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutative),
- $\vec{u} + \overrightarrow{(v + \vec{w})} = (\vec{u} + \vec{v}) + \vec{w}$ (associativity),
- $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ (null element),
- $\vec{u} + (-\vec{u}) = \vec{0}$ (opposing element).

- <u>**Multiplying vectors by a scalar:**</u>

  Let be the scalars α, β and the vectors $\vec{u}, \vec{v}$;

  ** $\alpha(\beta\,\vec{u}) = (\alpha\beta)\vec{u}$ (associativity),
  ** $(\alpha+\beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$ (distributive scalar addition),
  ** $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$ (addition distribution vector),
  ** $1.\vec{u} = \vec{u}.1 = \vec{u}$,
  ** $0.\vec{u} = \vec{0}$,

II.4 The dot or inner product or the scalar product of the vectors:

The result is a scalar (number).

Let be the vectors $\vec{u}$, $\vec{v}$ and their successive components (u_1, u_2, u_3) and (v_1, v_2, v_3)

➔ $\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$, $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$; then :

The scalar product of two vectors $\vec{u}$, $\vec{v}$ is then written in matrix form:

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = [u_1 \quad u_2 \quad u_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$ [5].

Which can be written as:

** $\vec{u}.\vec{v} = \|\vec{u}\|.\|\vec{v}\|.\cos\theta = u_1v_1 + u_2v_2 + u_3v_3$

** The angle θ between the two vectors $\vec{u}$, $\vec{v}$ can be determined by:

$$\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|.\|\vec{v}\|} ;$$

The scalar product is the algebraic measure of the projection of $\vec{u}$ onto the line carried and oriented by $\vec{v}$ as represented by Figure .I.3.

The vectors $\vec{u}$ and $\vec{v}$ being permutable in this definition.

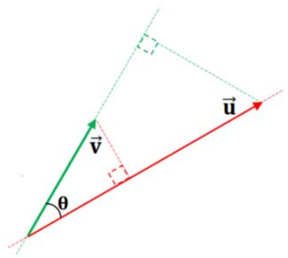


Figure.I.3. Projection of the vectors $\vec{u}$, $\vec{v}$.

Then we can write: $\vec{u}.\vec{v} = projection\frac{\vec{u}}{\vec{v}} = projection\frac{\vec{v}}{\vec{u}}$ [6].

Concept of projection is introduced.

Properties :

** $\vec{u}.\vec{v} = \vec{v}.\vec{u}$ (commutative),

** soit le vecteur $\vec{w}$: $\vec{u}.(\vec{v} + \vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$ (distribution)

** soit le scalaire α : $\alpha(\vec{u}.\vec{v}) = \alpha\vec{u}.\vec{v} = \vec{u}.\alpha\vec{v}$

** $\vec{0}.\vec{v} = 0$,

** $\vec{v}.\vec{v} = \|\vec{v}\|^2$,

II.5 The cross or vector product: The result of the cross product of two vectors is a vector perpendicular to the two vectors.

Let be the vectors $\vec{u}$, $\vec{v}$ and their successive components (u_1, u_2, u_3) and (v_1, v_2, v_3)

➔ $\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$,and

$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$; then :

** $\vec{u} \times \vec{v} = \|\vec{u}\|.\|\vec{v}\|.\sin\theta\,.\vec{n}$

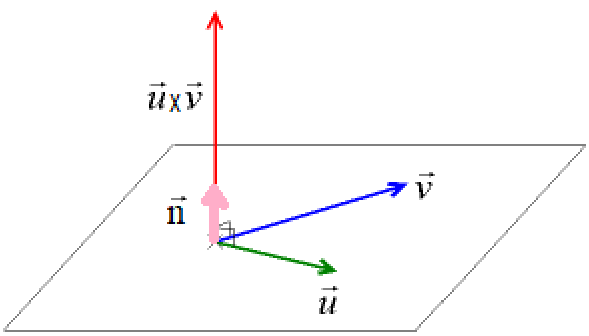


Figure .I.3. The vector product of two vectors.

Noted as:

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{Bmatrix} \overrightarrow{e_1} \\ -\overrightarrow{e_2} \\ \overrightarrow{e_3} \end{Bmatrix}$$

Else, the result is similar to a calculation of the determinant obtained from the relationship associated with the cofactors and presented in the following form:

$$\vec{u} \times \vec{v} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} \vec{e}_1 - \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} \vec{e}_2 + \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \vec{e}_3 \,;$$

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\, \vec{e}_1 + (u_3 v_1 - u_1 v_3)\, \vec{e}_2 + (u_1 v_2 - u_2 v_1)\, \vec{e}_3$$

The magnitude of the result vector corresponds to the area of a parallelogram having $\vec{u}$ and $\vec{v}$ as adjacent sides,

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|.\|\vec{v}\|.\sin\theta = Area\,(\vec{u}, \vec{v})$$

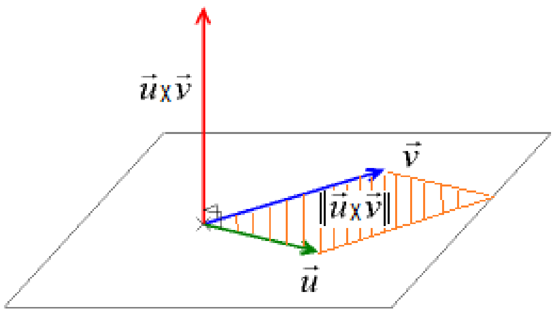


Figure.I.4. Area between $\vec{u}$ and $\vec{v}$.

Properties :

** $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$,

** $\vec{u} \times \vec{v} = \vec{0}$ ➔ the two vectors $\vec{u}$ and $\vec{v}$ are linearly dependent,

** let be the scalar α : $(\alpha\vec{u}) \times \vec{v} = \vec{u} \times (\alpha\vec{v}) = \alpha(\vec{u} \times \vec{v})$,

** let be the vector $\vec{w}$: $\vec{u}.(\vec{v} \times \vec{w}) = \vec{v}.(\vec{w} \times \vec{u}) = \vec{w}.(\vec{u} \times \vec{v})$,

** $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$,

** $(\vec{u} \times \vec{v}) \times \vec{w} = (\vec{u}.\vec{w})\vec{v} - (\vec{v}.\vec{w})\vec{u}$,

II.6 Tensor product of vectors:

If $\vec{u}$ and $\vec{v}$ are vectors of length M and N respectively, their tensor product $\vec{u} \otimes \vec{v}$ is a vector $\vec{w}$ defined as the M×N matrix defined by $\vec{u} \otimes \vec{v} = \vec{w} = \{u_i v_j\}$.

Example: $\vec{u} \otimes \vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \left\{ u_1 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, u_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\} = \begin{Bmatrix} u_1 v_1 \\ u_1 v_2 \\ u_1 v_3 \\ u_2 v_1 \\ u_2 v_2 \\ u_2 v_3 \end{Bmatrix}$ [4].

Properties:

- $\vec{u}\otimes\vec{v} = \begin{pmatrix}1\\2\end{pmatrix}\otimes\begin{pmatrix}3\\4\\5\end{pmatrix} = \left\{1\begin{pmatrix}3\\4\\5\end{pmatrix}, 2\begin{pmatrix}3\\4\\5\end{pmatrix}\right\} = \begin{Bmatrix}3\\4\\5\\6\\8\\10\end{Bmatrix} \neq \vec{v}\otimes\vec{u} = \begin{pmatrix}3\\4\\5\end{pmatrix}\otimes\begin{pmatrix}1\\2\end{pmatrix} =$

$$\left\{3\begin{pmatrix}1\\2\end{pmatrix}, 4\begin{pmatrix}1\\2\end{pmatrix}, 5\begin{pmatrix}1\\2\end{pmatrix}\right\} = \begin{Bmatrix}3\\6\\4\\8\\5\\10\end{Bmatrix} \text{(not commutative)}$$

- The tensor product is linear on the right and on the left.
- $(\lambda\vec{u})\otimes\vec{v} = \lambda(\vec{u}\otimes\vec{v}) = \vec{u}\otimes(\lambda\vec{v})$,
- $(\vec{u}_1 + \vec{u}_2)\otimes\vec{v} = \vec{u}_1\otimes\vec{v} + \vec{u}_2\otimes\vec{v}$,
- $\vec{u}\otimes(\vec{v}_1 + \vec{v}_2) = \vec{u}\otimes\vec{v}_1 + \vec{u}\otimes\vec{v}_2$,
- $(\vec{u}_1 + \vec{u}_2)\otimes(\vec{v}_1 + \vec{v}_2) = \vec{u}_1\otimes\vec{v}_1 + \vec{u}_1\otimes\vec{v}_2 + \vec{u}_2\otimes\vec{v}_1 + \vec{u}_2\otimes\vec{v}_2$

III. Einstein's "notation" summation convention:

In ordinary physical space a basis is composed of three axes, noncoplanar vectors, and so any vector in this space is completely specified by its three components.

$$\overrightarrow{OP} = X_1\vec{e}_1 + X_2\vec{e}_2 + X_3\vec{e}_3$$

Can be written in the form: $\overrightarrow{OP} = \sum_{i=1}^{3} X_i\vec{e}_i$;

Where $\overrightarrow{OP}$ is the position vector and X_i are the components X_1, X_2, X_3.

By convention we can write:

$$\overrightarrow{OP} = X_i\vec{e}_i \quad ,$$

with i as index and can take the values 1, 2, 3.

Accordingly the symbol X_i is understood to represent the three components X_1, X_2, X_3 and $\vec{e}_i$ is understood to represent the three vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$[2].

So, an index which occurs twice in an expression is summed over. Such an index is called **dummy**. The rule is called the Einstein summation convention. So, now we write simply [6]:

For a range of three on both indices, the symbol A_{ii} with i as dummy indice represents the sum of $A_{11} + A_{22} + A_{33}$[2].

The i-component of the vector $\overrightarrow{OP}$ we write X_i. This index is a **free** index. It is not summed over. It is very important to distinguish between free indices and dummy indices.

- Look at the expression: $a_i b_j c_i$

Here, j occurs once only, so it is a **free** index. But "i" occurs twice. "i" must be a **dummy** index and is summed over. So this expression has one free index.

The order of the factors can be changed arbitrarily: writing out the expression explicitly, we have: $a_i b_j c_i = a_i c_i b_j = (a_i c_i) b_j = (a_1 c_1 + a_2 c_2 + a_3 c_3) b_j$ [7].

Example expressions:

- $F_i = A_i B_j C_j$, in this expression: i is a free indice however j is dummy one.
- $G_k = H_k(2 - 3A_i B_i) + P_j Q_j F_k$: k is a free indice and i,j are dummy indices.

IV. Kronecker-delta symbol:

Let us take the scalar product of the unit vectors of a basis $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$, we conclude :

- $\vec{e}_1 . \vec{e}_2 = \vec{e}_2 . \vec{e}_3 = \vec{e}_3 . \vec{e}_1 = 0$,
- $\vec{e}_1 . \vec{e}_1 = \vec{e}_2 . \vec{e}_2 = \vec{e}_3 . \vec{e}_3 = 1.$

This allowed Kronecker, for simplification; to use the symbol that bears its name: Kronecker delta symbol, which is defined:

$\delta_{ij} = \begin{cases} 1, & if \quad i = j \\ 0, & if \quad i \neq j \end{cases}$. and we can also write: $\delta_{ij} = \vec{e}_i . \vec{e}_j$,with *i*,*j* = 1, 2, 3.

The Kronecker delta is sometimes called the substitution operator, since, for example;

$$\delta_{ij} b_j = \delta_{i1} b_1 + \delta_{i2} b_2 + \delta_{i3} b_3 = b_i$$

Properties:

- $u_i \delta_{ij} = u_j$, • $u_i v_j \delta_{ij} = u_i\, v_i = u_j\, v_j$,• $\delta_{ij} \delta_{ik} = \delta_{jk}$

The dot or the scalar product of two vectors $\vec{u}, \vec{v}$ can be written:

$\vec{u} . \vec{v} = (u_i \vec{e}_i) . \left(v_j \vec{e}_j\right) = u_i\, v_j \delta_{ij} = u_i\, v_i = u_j\, v_j$

V. Levi-Civita permutation symbol:

If we mix the vector products of the unit vectors with the scalar products, such that:

$(\vec{e}_1 \times \vec{e}_1) . \vec{e}_1$, $(\vec{e}_2 \times \vec{e}_2) . \vec{e}_2$ and so on

$(\vec{e}_1 \times \vec{e}_2) . \vec{e}_3$, $(\vec{e}_2 \times \vec{e}_3) . \vec{e}_1$ and so on

$(\vec{e}_2 \times \vec{e}_1) . \vec{e}_3$, $(\vec{e}_3 \times \vec{e}_2) . \vec{e}_1$and so on, or write: $\left(\vec{e}_i \times \vec{e}_j\right) . \vec{e}_k$ with : *i* ,*j* , *k* =1, 2, 3.

We will find the results as:

$$(\vec{e}_1 \times \vec{e}_1) . \vec{e}_1 = (\vec{e}_2 \times \vec{e}_2) . \vec{e}_2 = (\vec{e}_3 \times \vec{e}_3) . \vec{e}_3 = 0$$
$$(\vec{e}_1 \times \vec{e}_1) . \vec{e}_2 = (\vec{e}_2 \times \vec{e}_2) . \vec{e}_3 = (\vec{e}_3 \times \vec{e}_3) . \vec{e}_1 = 0$$
$$(\vec{e}_1 \times \vec{e}_1) . \vec{e}_3 = (\vec{e}_2 \times \vec{e}_2) . \vec{e}_1 = (\vec{e}_3 \times \vec{e}_3) . \vec{e}_2 = 0$$
$$(\vec{e}_1 \times \vec{e}_2) . \vec{e}_3 = (\vec{e}_2 \times \vec{e}_3) . \vec{e}_1 = (\vec{e}_3 \times \vec{e}_1) . \vec{e}_2 = 1$$
$$(\vec{e}_2 \times \vec{e}_1) . \vec{e}_3 = (\vec{e}_3 \times \vec{e}_2) . \vec{e}_1 = (\vec{e}_1 \times \vec{e}_3) . \vec{e}_2 = -1$$

This allowed Levi-Civita to define its "epsilon" permutation symbol as follows:

$$\varepsilon_{ijk}=\begin{cases}1 & \begin{cases}\text{if } i,j,k \text{ are an even permutation of } 1,2,3\\ (i.e.\ \text{if they appear in sequence as in the arrangement } 12312).\end{cases}\\ -1 & \begin{cases}\text{if } i,j,k \text{ are an odd permutation of } 1,2,3\\ (i.e.\ \text{if they appear in sequence as in the arrangement } 32132).\end{cases}\\ 0 & \begin{cases}\text{if } i,j,k \text{ are not a permutation of } 1,2,3\\ (i.e. \text{ if two or more of the indices have the same value}).\end{cases}\end{cases}$$

Or write: $\varepsilon_{ijk}=\left(\vec{e}_i\times\vec{e}_j\right).\vec{e}_k$ with: i, j, k =1, 2, 3.
It represents the orientation of the coordinate system as shown in Figure.I.5.
The concept of the orientation or the direction of the coordinate reference (space) was introduced.

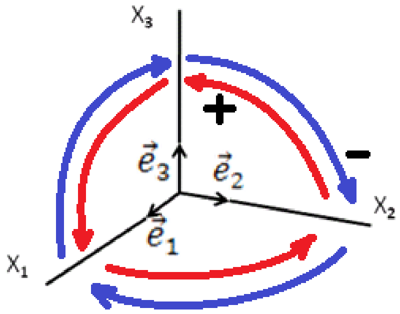


Figure .I.5. The orientation of the coordinate system representation.

Properties:

- $\varepsilon_{ijk}=\varepsilon_{kij}=\varepsilon_{jki}$;
- $\varepsilon_{ijk}=-\varepsilon_{jik}=\varepsilon_{jki}=-\varepsilon_{kji}$,

The cross or the vector product of two vectors $\vec{u}$, $\vec{v}$ can be written:
$\vec{u}\times\vec{v}=(u_i\vec{e}_i)\times\left(v_j\vec{e}_j\right)=u_i\, v_j\varepsilon_{ijk}\,\vec{e}_k$

Relationship between permutation symbol and Konecker-delta:

$$\varepsilon_{ijk}=\det\begin{bmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{bmatrix}, \text{ et : } \varepsilon_{ijk}\varepsilon_{pqr}=\det\begin{bmatrix}\delta_{ip} & \delta_{iq} & \delta_{ir}\\ \delta_{jp} & \delta_{jq} & \delta_{jr}\\ \delta_{kp} & \delta_{kq} & \delta_{kr}\end{bmatrix}$$

VI. Basis Change - Transformation law:

Let the two orthonormal (rectangular) Cartesian coordinate systems$(O,\vec{e}_1,\vec{e}_2,\vec{e}_3)$ and $(O',\vec{e}'_1,\vec{e}'_2,\vec{e}'_3)$ with common origin at an arbitrary point O as shown in Figure .I.6.

The primed system may be imagined to be obtained from the unprimed by a rotation of the axes about the origin, or by a reflection of axes in one of the coordinate planes, or by a combination of these. If the symbol a_{ij} denotes the cosine of the angle between the i^{th} primed and j^{th} unprimed coordinate axes,

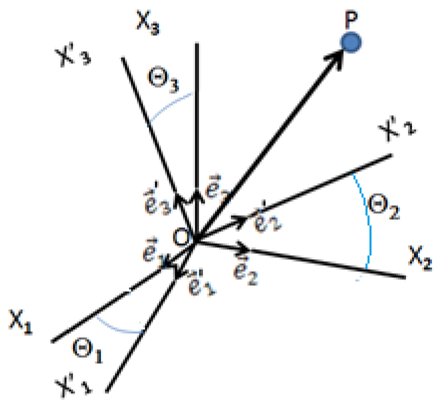


Figure .I.6. Rotation of two orthonormal Cartesian coordinate systems.

i.e. $a_{ij} = \vec{e}_i.\overrightarrow{e'}_j = \cos(\vec{e}_i, \vec{e}'_j) = \cos\theta_{ij}$, Figure.I.7., the relative orientation of the individual axes of each system with respect to the other is conveniently given by the table :

	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$
$\overrightarrow{e'}_1$	a_{11}	a_{12}	a_{13}
$\overrightarrow{e'}_2$	a_{21}	a_{22}	a_{23}
$\overrightarrow{e'}_3$	a_{31}	a_{32}	a_{33}

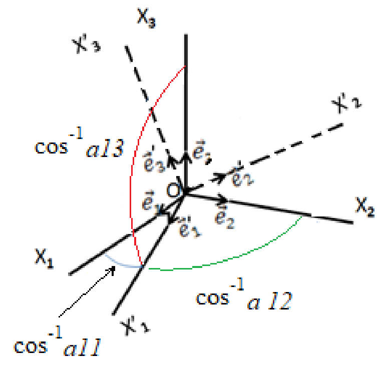


Figure.I.7. The transformation tensor representation.

Or alternatively by the transformation tensor:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$$

From this definition the unit vectors $\overrightarrow{e'}_1$ along the X'_1axis is given by:

$$\overrightarrow{e'}_1 = a_{11}\vec{e}_1 + a_{12}\vec{e}_2 + a_{13}\vec{e}_3$$

An obvious generalization of this equation gives the arbitrary unit base vector $\overrightarrow{e'}_i$ as:

$$\overrightarrow{e'}_i = a_{ij}\vec{e}_j$$

In component form, the arbitrary vector $\vec{v}$ may be expressed in the unprimed system by the equation: $\vec{v} = v_j\vec{e}_j$ and in the primed system by: $\vec{v} = v'_i\overrightarrow{e'}_i = v'_i a_{ij}\vec{e}_j$ by comparison we can conclude that the vector components in the primed and unprimed systems are related by the equations: $v_j = a_{ij}v'_i$

This expression is the transformation law for first- order Cartesian tensors, by interchanging the roles of the primed and unprimed base vectors in the above development, the inverse is found to be: $v'_i = a_{ij}v_j$

It is important to note that the free index on a_{ij} appears as the second index in the second equation, the free index appear as the first index.

By an appropriate choice of dummy indices may be combined to produce: $v_j = a_{ij}a_{ik}v_k$

Since vector $\vec{v}$ is arbitrary, which allow to write: $a_{ij}a_{ik} = \delta_{ik}$, this last equation is known as the orthogonality or orthonormality conditions on the direction cosines a_{ij}.

According to the transformation law, the unprimed vector $\overrightarrow{u'v'}$ which has components in the primed coordinate system given by:

$$u'_i v'_j = (a_{ip}u_p)(a_{jq}v_q) = a_{ip}a_{jq}u_p v_q$$

In an obvious generalization, any second-order Cartesian tensor T_{ij} obeys to the transformation law: $T'_{ij} = a_{ip}a_{jq}T_{pq}$;

With the help of the orthogonality conditions it is a simple calculation to invert:

$$T_{ij} = a_{pi}a_{qj}T'_{pq}.$$

Working with tensors requires knowledge of matrix operations.

VII. Matrices, Matrix representation of Cartesian tensors:

A rectangular array of elements, enclosed by square brackets and subject to certain laws of combination, is called matrix.

An M×N matrix is one having M (horizontal) rows and N (vertical) columns of elements. In the symbol A_{ij} used to represent the typical element of a matrix, the first subscript denotes the row, the second subscript the column occupied by the element, said of two dimensions (M, N) noted for example:

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$, with: i=1, 2, 3 and j=1, 2, 3.

M=3 (three rows), N=3 (three columns), number of array elements: M×N=3X3=9 scalar elements.

• Generally, in Continuum mechanics, M=N, in this case the matrices are said to be "square".

a- Some special square matrices

Unit matrix: Sometimes denoted In, n is the dimension of the matrix

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Diagonal matrix : where the elements are only on the diagonal of the matrix

Example : $E_3 = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix}$

Symmetric matrix: A square matrix A is said to be symmetric if Aji = Aij for all (i) different from (j).[3]

Matrix transpose: The transpose A^T of a matrix A is the matrix obtained by exchanging its rows for its columns **(** Aji ≠ Aij).

The transpose of a column vector is a row vector. $\vec{X}(x_1, x_2, x_3)$ ➔ $\vec{X}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Matrix determinant: is a scalar given by:

$$det(A) = \varepsilon_{ijk}\, A_{1i}A_{2j}A_{3k}$$

$$det(A) = \frac{1}{6}\varepsilon_{ijk}\, \varepsilon_{mnp}\, A_{im}A_{jn}A_{kp} \text{ [7]}$$

Matrix operations:

Let the square matrices of the same dimensions be: A, B, C,

- $A + B = B + A$; commutative addition,
- $A + (B + C) = (A + B) + C$; associative addition,
- $A + 0 = 0 + A = A$, the neutral matrix 0 is a matrix where the elements are null (known as zero elements).
- $A + (-A) = 0$, for each matrix A, there is a unique matrix –A.
- Let be the scalar α then: $\alpha(A + B) = \alpha A + \alpha B$,

- $(A+B)C = AC + BC$: distribution on the addition,
- Transposed from a matrix noted A^T : $A = [a_{ij}] \Rightarrow A^T = [a_{ji}]$, with :

$$(A^T)^T = A.$$

Properties:

- $(A+B)^T = A^T + B^T$ and • $(AB)^T = A^T B^T$.

Matrix Inverse: if a square matrix is non-singular, it possesses a unique inverse matrix. A square matrix whose determinant is zero is called a singular matrix.

The inverse matrix is a matrix denoted by A^{-1}, such that:

$$AA^{-1} = A^{-1}A = I$$

Where I is the "identity" unit matrix, having ones on the principal diagonal elements and zeros elsewhere.

Properties:

- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = A^{-1}B^{-1}$,

• The matrix inverse can be calculated using the cofactor method:

$$A^{-1} = \frac{1}{\det(A)} . com(A)^T$$

$$com(A) = (-1)^{i+j} \det\ (A_{i,j})$$

If A is a matrix which size is N then $(A_{i,j})$is the square sub-matrix of size N–1 deduced from A by deleting the i-th row and from the j-th column (its determinant is therefore one of the minors of (A).

Principal values and principal directions of matrix or second-order tensors:

Let be the matrix $A = [a_{ij}]$, there is associated with each direction(specified by the unit normal n_i) an arbitrary vector $\vec{n}\begin{Bmatrix} n_1 \\ n_2 \\ \vdots \\ n_n \end{Bmatrix}$, a vector given by inner product $\vec{v}$ where its components will be $v_i = a_{ij}\, n_j$.

Here a_{ij} may be envisioned as a linear vector operator which produces the vector v_i conjugate to the direction n_i. If the direction is one for which v_i is parallel to n_i , the inner product may be expressed as a scalar multiple of n_i. For this case,

$$A\vec{n} = \lambda\, \vec{n}$$

And the direction n_i is called a principal direction, or principal axis of a_{ij}. With the help of the identity $n_i = \delta_{ij}\, n_i$ can be put in the form:

$(A - \lambda\delta)\vec{n} = \vec{O}$ or $(a_{ij} - \lambda\delta_{ij})n_j = 0$.

Which represents a system of three equations for four unknown n_i and λ, associated with each principal direction.

Note first that for every λ, the trivial solution $n_i = 0$ satisfies the equations. The purpose here, however, is to obtain non-trivial solutions. Also, for the homogeneity of the system it follows that no loss of generally is incurred by restricting attention to solutions for which:

$$n_i n_i = 1,$$

and this condition is imposed from now on.
To have a non trivial solution, the determinant of coefficients must be zero, that is;

$$|a_{ij} - \lambda\delta_{ij}| n_j = 0$$

Expansion of this determinant leads to a cubic polynomial in λ, namely;

$$\lambda^3 - I\,\lambda^2 + II\,\lambda\text{-III=0}$$

Which is known as **the characteristic equation** of a_{ij}, and for which the scalar coefficients,

$$I = a_{ii} = tr\ a_{ij} \text{ (trace of } a_{ij}\text{)}$$

$$II = {}^{1}\!/_{2}\,(a_{ii}a_{jj} - a_{ij}a_{ij})$$

$$III = |a_{ii}| = det\ (a_{ij})$$

Are called the first, second and third invariants, respectively of a_{ii}. The three roots of the cubic characteristic equation, labeled λ_I, λ_{II}, λ_{III}, are called the principal values of a_{ii}.
For a symmetric tensor with real components, the principal values are real; and if these values are distinct, the three principal directions are mutually orthogonal.
When referred to principal axes, both the tensor appears diagonal form as:

$$A^* = [a^*{}_{ij}] = \begin{bmatrix} \lambda_I & 0 & 0 \\ 0 & \lambda_{II} & 0 \\ 0 & 0 & \lambda_{III} \end{bmatrix}$$

The principal values are ordered as in $\lambda_I > \lambda_{II} > \lambda_{III}$ [6].

An eigenvector $\overrightarrow{n^*}$ of components $\begin{Bmatrix} n_1 \\ n_2 \\ \vdots \\ n_n \end{Bmatrix}$ associated with the eigenvalue λ must satisfy the relation:

$$(A - \lambda\mathrm{I})\overrightarrow{n^*} = \vec{O} \Leftrightarrow (A - \lambda\mathrm{I})\begin{Bmatrix} n_1 \\ n_2 \\ \vdots \\ n_n \end{Bmatrix} = \vec{O}$$

If the matrix A admits λ_n eigenvalues, distinct two by two, the $\overrightarrow{n^*}_n$ associated eigenvectors are linearly independent.
For principal axes labeled (O, X_1^*, X_2^*, X_3^*) the transformation from (O, X_1, X_2, X_3) axes is given by the elements of the table:

	X_1	X_2	X_3
X_1^*	$a_{11} = n_1^{(1)}$	$a_{11} = n_2^{(1)}$	$a_{11} = n_3^{(1)}$
X_2^*	$a_{11} = n_1^{(2)}$	$a_{11} = n_2^{(2)}$	$a_{11} = n_3^{(2)}$
X_3^*	$a_{11} = n_1^{(3)}$	$a_{11} = n_2^{(3)}$	$a_{11} = n_3^{(3)}$

In which $n_i^{(j)}$ are the direction cosines of the jth principal direction.

VIII. Derivative of tensors:

1/ The temporal differentiation of a vector is given by:

$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial v_1}{\partial t}\,\vec{e}_1 + \frac{\partial v_2}{\partial t}\,\vec{e}_2 + \frac{\partial v_3}{\partial t}\,\vec{e}_3 = v_{i,t}\vec{e}_i = \dot{v}_i\vec{e}_i$$

Properties:

- $\frac{\partial(\lambda\vec{A})}{\partial t} = \frac{\partial\lambda}{\partial t}\vec{A} + \lambda\frac{\partial\vec{A}}{\partial t} = \dot{\lambda}\vec{A} + \lambda\dot{\vec{A}}$
- $\frac{\partial(\vec{A}.\vec{B})}{\partial t} = \frac{\partial\vec{A}}{\partial t}.\vec{B} + \vec{A}.\frac{\partial\vec{B}}{\partial t} = \dot{\vec{A}}.\vec{B} + \vec{A}.\dot{\vec{B}}$
- $\frac{\partial(\vec{A}\times\vec{B})}{\partial t} = \frac{\partial\vec{A}}{\partial t}\times\vec{B} + \vec{A}\times\frac{\partial\vec{B}}{\partial t} = \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}}$

2/ the spatial differentiation of a vector:

- $\frac{\partial\vec{v}}{\partial x_i} = \frac{\partial v_i}{\partial x_i}\,\vec{e}_i = v_{i,i}\vec{e}_i$
- $\frac{\partial\vec{v}}{\partial x_j} = \frac{\partial v_i}{\partial x_j}\,\vec{e}_i = v_{i,j}\vec{e}_i$
- $\frac{\partial^2 v_i}{\partial x_j\partial x_k} = v_{i,jk}$

3/ the spatial differentiation of a tensor:

- $\frac{\partial a_{ij}}{\partial x_k} = a_{ij,k}$
- $\frac{\partial^2 a_{ij}}{\partial x_k\partial x_m} = a_{ij,km}$

Several important differential operators appear in continuum mechanics and are given here for reference.

The divergence: the divergence of a vector is a scalar given by the sum of the partial derivatives of its components in the three directions of space:

- $div\,\vec{v} = \nabla\vec{v}$ or : $\partial_i v_i = v_{i,i}$

The divergence of a tensor will produce a vector of coordinates:

- $(\overrightarrow{div}\,\bar{\bar{a}})_i = \frac{\partial a_{i1}}{\partial x_1} + \frac{\partial a_{i2}}{\partial x_2} + \frac{\partial a_{i3}}{\partial x_3}$ or : $\partial_p a_{ip} = \frac{\partial a_{ip}}{\partial x_p} = a_{ip,p}$

The gradient: it includes the concept of "slope" which exists in two dimensions. It describes the variation of a quantity according to the three directions of space.

- $\overrightarrow{grad}f = \frac{\partial f}{\partial x_1}\,\vec{e}_1 + \frac{\partial f}{\partial x_2}\vec{e}_2 + \frac{\partial f}{\partial x_3}\vec{e}_3 = \frac{\partial f}{\partial x_i}\,\vec{e}_i = f_{,i}\vec{e}_i$

We define the gradient operator applied to a vector. It is about a tensor of order 2, whose components are:

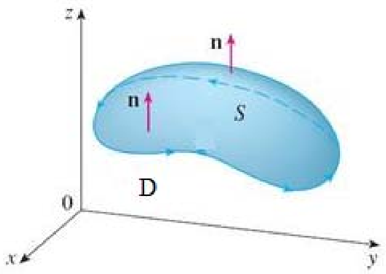
- $\overrightarrow{grad}\vec{U}_{ij} = \frac{\partial U_i}{\partial x_j} = U_{i,j}$

The Laplacian: noted Δ, is the sum of the second derivatives in the three directions of space.it preserves the tensorial order of which it is applied, because it is the composition of a divergence and a gradient, applied to a scalar, we therefore have:

- $\Delta f = div(\overrightarrow{grad}f) = \frac{\partial}{\partial x_p}\left(\frac{\partial f}{\partial x_p}\right) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$
- $\Delta\vec{U} = \overrightarrow{div}(\overrightarrow{grad}\vec{U}) = \vec{U}_{,ii} = \frac{\partial}{\partial x_p}\left(\frac{\partial U_i}{\partial x_p}\right)$

IX. Gauss and stokes integral theorems:

Integral transformation formulas are mathematical tools used to match an integral over D (volume integral) to an integral over S (surface integral).



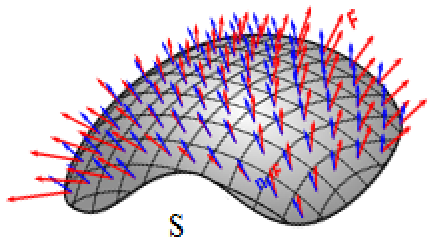


Figure .I.8. The volume integral over D to surface integral over S

They are given here without proof, applied to a scalar, vector or to the tensor:

- $\int_D \overrightarrow{grad} f \, dv = \int_S f\vec{n}dS$
- $\int_D div\,\vec{U} \, dv = \int_S \vec{U}.\vec{n}dS$
- $\int_D \overrightarrow{div} T \, dv = \int_S T\vec{n}dS$

These formulas have a computational interest, but have a strong meaning when they present physical quantities. They make it possible to write the behavior of a quantity on a domain from its behavior on the surface [3].

Bibliographic references:

[1] DEGHBOUDJ Samir, 2016, POLYCOPIE DE COURS « MECANIQUE DES MILIEUX CONTINUS COURS ET APPLICATIONS ».
[2] Nicholas Moés, 2011, Mécanique des milieux continus, Ecole centrale de Nantes.
[3] Slim KAMMOUN, 2020-2021, Plan de la MMC, Ecole nationale d'ingénieurs de Sfax, Tunis
[4] HECINI Mabrouk, 2015, Ouvrage pédagogique, « Mécanique des Milieux Continus Exercices corrigés avec résumé du cours ».
[5] Lars H. Söderholm, 2008, BASIC CONTINUUM MECHANICS, KTH Royal Institute of Technology, Sweden
[6] ADMAN Redouane, Théorie de l'élasticité : Cours et applications, FGC/USTHB/2020
[7] François Sidoroff, 2010, Mécanique des milieux continus, Ecole centrale de Lyon.
[8] http://uel.unisciel.fr/physique/outils_nancy/outils_nancy_ch11/co/apprendre_ch11_20.html

Chapitre II:

Introduction to continuum mechanics

Chapter II: Introduction to continuum mechanics

- From the mechanics of the material point to the mechanics of the continuum…
- Notion of the continuous medium…
- Referentials – Systems
- Configuration
- Lagrangian description
- Eulerian description
- Reversibility condition

I. From the mechanics of the material point to the mechanics of the continuum …

The mechanics of the material point makes it possible to predict the movement of a point subjected to a set of forces. One distinguishes from this theory of description of the kinematics: position, speed, acceleration of the point…. On the other hand, the “dynamics” makes the relation between forces and movement.

With the mechanics of the material point, we cannot describe the rotations of a body on itself…problem! We have recourse to the mechanics of undeformable solid bodies which integrates rotations, inertias and moments….

It makes it possible to reduce major problems such as those resulting from robotics…

But for the calculation of structures, manufacturing processes, biomechanics, fluid mechanics, civil engineering, the design of new materials… we need continuum mechanics (C.M.).

Continuum mechanics: this is the part of physics that studies the mechanics of deformable solid bodies; it is a physical and mathematical framework for modeling a concrete problem. Once the mathematical model has been established, it can be solved by an analytical or numerical method. …

II. Notion of the continuous medium …

We say that a domain contains a continuous material medium, if at each moment and at each point of this domain we can define physical quantities relating to this material medium.

III. Referentials – Coordinate systems:

The study of a material domain requires that we proceed to its description and identification throughout its evolution over time. The concept of reference must be developed to specify the evolutions.

The R referential is linked to the observer. It represents the set of animated points of the motion of the observer's rigid body.

To carry out the spatial locations of the material points in a frame of reference R, a vector base associated with an origin point O is used. A frame of reference R is thus obtained. This frame, animated by the movement of the rigid body of the frame of reference R, makes it possible to materialize this frame of reference.

For the purposes of the study, it may be necessary to make changes of systems. The same transformation of the spatial coordinates will thus be carried out on the components of the mathematical beings (vector, tensor, etc.) used to describe the domain. It should be noted that several systems can be associated with the same referential.

At the same time, one can imagine a change of observer, which will result in a change of frame of reference. We can imagine such a transformation by associating a system with each of the reference frames. The two systems being chosen so that they coincide at a given instant, their evolution over time is examined.

Finally, to describe the configuration of a material domain, it is possible to choose from two types of variables [1].

III.1 Configuration:

A body can take on many different shapes or **configurations** depending on the loading applied to it. We choose one of these configurations to be the reference configuration of the body and label it B_0.

Any possible configuration of the body can be taken as its reference. Typically the choice is dictated by convenience to the analysis. Often, it corresponds to the state where no external loading is applied to the body.

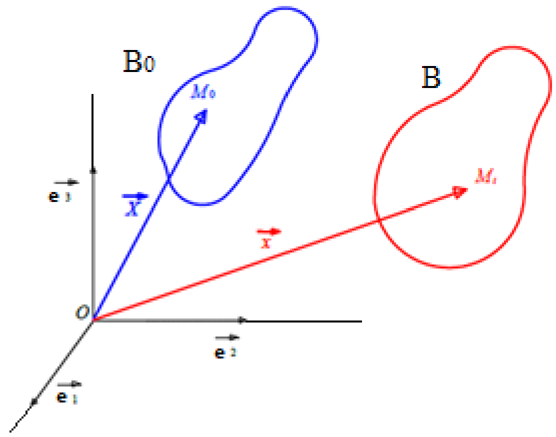


Figure .1. Lagrange and Euler configurations

We denote the position of a particle M_0 in the referen (see section II.2. Lagrangian description)

The deformed configuration occupied by the body is described in terms of the union of all positions x –of all the points occupied by M_t - (see Figure .1. and section II.3. Eulerian description).

We adopted the convention of denoting all things associated with the reference configuration with uppercase letters (as in X) or with a subscript 0 (as in B_0) and all things associated with the deformed configuration in lower-case (as in x) or without a subscript (as in B)[2].

III.2 Lagrangian description:

Consider an orthonormal system R associated with a referential R. The classical kinematics of a continuous medium is built from the notions:

* of time, which can be represented by a real variable t determined by two extreme values.

* of physical space, which can be represented by an affine space of dimension 3. The points of this space are called "material points".

In the system R, at a time t=0, the point M_0 has coordinates X_1, X_2, X_3 which define the position of the material point M. This coordinate system is also called the “material” coordinate system in the reference configuration C_0. We can thus write: $\vec{OM}_0 = X_1 \vec{e}_1 + X_2 \vec{e}_2 + X_3 \vec{e}_3 = X_i \vec{e}_i = \vec{X}$.

To describe the movement of the domain, it is therefore necessary to give the law of evolution over time of the positions of all the material particles constituting the domain. We therefore obtain the current configuration C_t. Thus it is necessary to define the coordinates x_1, x_2, x_3 of the point M_t which at time t represents the position of the material point M.

$$\vec{OM}_t = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3 = x_i \vec{e}_i = \vec{x}$$

Which amounts to saying that we must give ourselves the following scalar functions : $x_i = \Phi_i(X_J, t)$;

In this description, the independent variables X_1, X_2, X_3 and t are called "Lagrange variables or coordinates".

The functions Φi represent the Lagrangian description of the motion of our domain with respect to the referential R.

Knowing the position at each instant of the material point M, it is then possible to define its speed and its acceleration with respect to the referential R.

Velocity vector: $\boxed{\overline{V(M,t)} = \frac{d\vec{OM}_t}{dt}}$

In an orthonormal cartesian basis, its components are $v_i = \frac{d\,\Phi_i}{dt}(X_J, t) = \frac{\partial \Phi_i}{\partial t}(X_J, t)$

.

In this last formula, the symbol $\frac{\partial}{\partial t}$ represents the partial derivation with respect to time, ie the derivation considering the time-independent position variables X_J .

Acceleration vector: $\boxed{\overline{\gamma(M,t)} = \frac{d\,\overline{V(M,t)}}{dt} = \frac{d^2\vec{OM}_t}{dt^2}}$

In an orthonormal cartesian basis, its components are:.

$$\gamma_i = \frac{d^2\,\Phi_i}{dt}(X_J, t) = \frac{\partial^2 \Phi_i}{\partial t^2}(X_J, t)$$

Displacement vector:

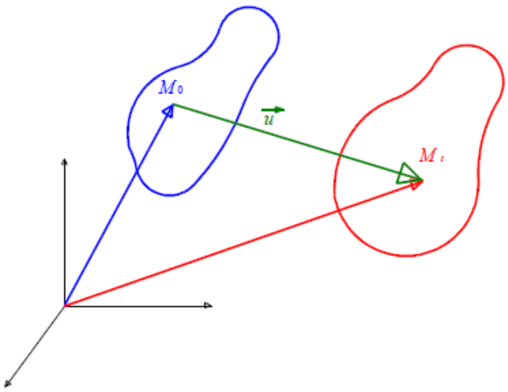


Figure .2. Displacement vector

Often we prefer to use the displacement vector instead of the position vector:

$$\boxed{\overrightarrow{u(X_J,t)}=\overrightarrow{OM_t}-\overrightarrow{OM_0}=\vec{x}-\vec{X}}$$

We can then observe the equality:

$$\overrightarrow{V(M,t)}=\frac{\partial\vec{u}}{\partial t}(X_J,t)=\frac{d\vec{u}}{dt}(X_J,t)$$

$$\overrightarrow{\gamma(M,t)}=\frac{\partial^2\vec{u}}{\partial t^2}(X_J,t)=\frac{d^2\vec{u}}{dt^2}(X_J,t)$$

III.3 Eulerian description:

The continuity hypotheses (medium and transformation) impose that the functions Φi are bijections from the reference configuration C_0 onto the current configuration C_t. this bijectivity imposes the existence of an inverse relationship between the reference position variables and the current position variables. So we have: $X_I=\psi_I(x_j,t)$;

It can therefore be seen that it is possible to change spatial variables. The so-called Eulerian description consists in considering the variables x_1,x_2,x_3 and t as independent and using them in the form of "variables or Euler coordinates".

In the Eulerian description, we are not concerned with knowing what happens to each particle. In fact, we study what is happening, at every moment, at every point in space.

We can express velocity and acceleration as a function of Euler variables:

Velocity vector: $\overrightarrow{V(M,t)}=\dfrac{d\overrightarrow{OM_t}}{dt}$ with $v_i=\dfrac{\partial\Phi_i}{\partial t}(X_J,t)=\dfrac{\partial\Phi_i}{\partial t}(\psi_J(x_k,t),t)$,

Acceleration vector: $\overrightarrow{\gamma(M,t)}=\dfrac{d^2\overrightarrow{OM_t}}{dt^2}$ with $\gamma_i=\dfrac{\partial^2\Phi_i}{\partial t^2}(X_J,t)=\dfrac{\partial^2\Phi_i}{\partial t^2}(\psi_J(x_k,t),t)$.

Practically, we can say that in Lagrangian description, we follow the domain in its movement, whereas in Eulerian description, we observe the evolution of the system in a fixed geometric point for the observer [3].

III.4 Reversibility condition:

The two descriptions are unique inverses of each other. A necessary and sufficient condition for the inverse functions to exist is that the Jacobian $J = \left|\frac{\partial x_i}{\partial X_j}\right|$ is not zero [4].

As a simple example, the Lagrangian description given by the following equations:

$$\begin{aligned} x_1 &= X_1 + X_2(e^t - 1) \\ x_2 &= X_1(e^{-t} - 1) + X_2 \\ x_3 &= X_3 \end{aligned} \quad ;$$

Has the inverse Eulerian formulation;

$$\begin{aligned} X_1 &= \frac{-x_1 + x_2(e^t - 1)}{1 - e^t - e^{-t}} \\ X_2 &= \frac{x_1(e^{-t} - 1) - x_2}{1 - e^t - e^{-t}} \\ X_3 &= x_3 \end{aligned} \quad .$$

Bibliographic references:

[1] Nichoals Moés, 2011. Mécanique des milieux continus. Ecole centrale de Nantes.
[2] E.B. TADMOR, R.E.MILLER and R.S. ELLIOT, 2012, Continuum Mechanics and Thermodynamics.
[3] Michel Maya, 2014. Cours de Mécanique des Milieux continus. Arts et Métiers Paris Tech, Centre d'Enseignement et de recherche de CLUNY.
[4] George E. Mase, 1970. Theory and problems of continuum mechanics. McGraw-Hill Book Company.

Chapitre III:

Stress Analysis

Chapter III: Stress Analysis

- Introduction
- Assumptions on the continuous medium
- Forces in a continuous medium
- Concept of the stress vector
- Stresses in 1 dimension
- Stresses in 2 dimensions
- The use of plane stress in engineering problems
- Plane stress law transformation
- Principal stresses; maximum shearing stress
- The determination of the principal stresses and maximum shear stress methods
- Stresses in 3D
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- 3D Stresses transformation law
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- Mohr's circles for 3D stress state
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- Deviator and Spherical Stress State
- Compatibility equations for linear strain

I. Introduction :

The main objective of the study of the mechanics of materials is to give the future engineer the means to analyze and design various machines and structures. Both the analysis and the design of a given structure involve the determination of stresses and strains. This chapter is devoted to the concept of stress [1].

II. Assumptions on the continuous medium:

The **continuity** concept: Adopting the point of view as a mathematical description of the continuous medium of a behavior of a material means that the quantities of the field as stress and displacement are expressed as continuous functions on pieces for spatial coordinates and of time.

Homogeneity:
a material is homogeneous it is a material which has identical properties in all its points.

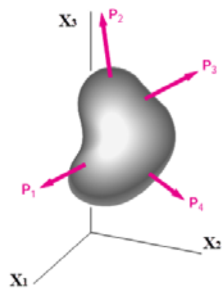


Figure.III.1. Body in 3D.

Isotropic: a material is said to be isotropic if one of its properties is **the same in all directions** in a point of this material.
Anisotropic: a material if one of its properties is directional in a point.

The density in an internal point P of elementary volume ΔV is given

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV},$$

The density ρ is a scalar quantity [2].

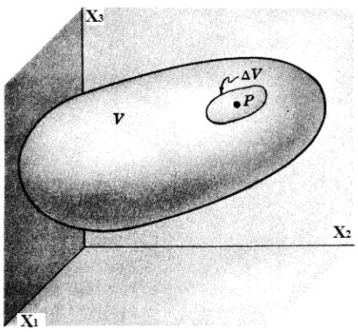


Figure.III.2. Point P from the body.

Forces in a continuous medium:

There are three types of forces namely:

-**The inner forces:** These are the forces that are exerted between the particles that form the medium, they are of the same modulus and in opposite directions, so they can be neglected.

-**External mass (volume) forces**: Appelés forces de pesanteur, elles sont proportionnelles à la masse du milieu et s'exercent à travers son volume.

$$d\vec{F}_v = \rho.\vec{g}.dV$$

ρ : density, $\vec{g}$: acceleration of gravity, dV: volume element

-**Surface forces**:

Also called contact forces, they are exerted according to Cauchy's principle which states (the effect of the material outside (S) on the material inside (S) is equipollent to the existence of a surface force $d\vec{F}$ which acts on each element of (S).

$$d\vec{F}_s = \vec{\tau}.dS$$

$\vec{\tau}$: stress vector,

dS: surface element.

The stress vector $\vec{\tau}$ can be defined by passing to the limit of the ratio of $d\vec{F}$ to dS when this tends towards the zero value:

$$\vec{\tau} = \lim_{ds\to 0} \frac{d\vec{F}}{dS}.$$

III. Concept of the stress vector:

A solid is in a state of stress if it is subjected to the action of external forces.

Consider a domain (D) delimiting a solid (Ω) which is in equilibrium under the action of several external forces Fi . Under the action of its forces, the solid is in equilibrium. We see the birth of stresses inside (Ω). Let (π) be a virtual plane which is defined by its perpendicular vector n , this plane divides (Ω) into two parts (Figure .III.3).

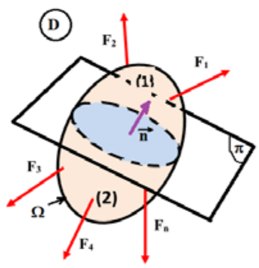


Figure.III.3. Cut on one facet.

Virtually isolating part (2) and revealing the internal stresses (Figure .III.4)

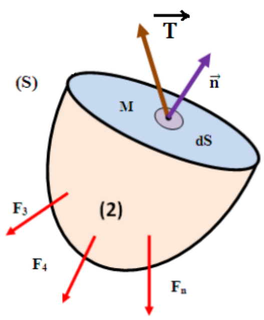


Figure.III.4. State of stress on a facet.

Let us call $d\vec{F}$ the resultant of the forces exerted by part (1) on part (2) of (Ω), through the surface element (dS), let be the ratio $\frac{d\vec{F}}{dS}$, passing to the limit when $dS \to 0$, we obtain the stress vector on the section at point (M) as shown in Fig.4, noted by:

$$\vec{T}(M,\vec{n}) = \lim_{dS\to 0}\frac{d\vec{F}}{dS} ;$$

$\vec{T}(M,\vec{n})$ represents the stress at point M on facet dS, whose orientation is defined by the unit normal $\vec{n}$ outside the facet (dS) [3].

The stress vector $\vec{T}$ can be decomposed into two components along two directions. The first is normal to the surface (dS), whose unit normal is ($\vec{n}$). The second is tangential to (dS) and has unit normal ($\vec{t}$) (Figure.III. 5). as we will see in the following paragraphs.

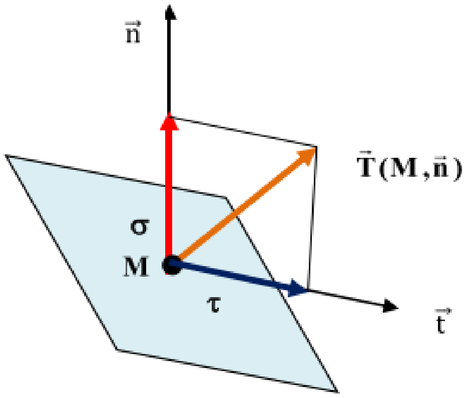


Figure.III.5. Components of the stress vector.

Let us project the vector $\vec{T}$ onto the normal ($\vec{n}$) and onto the plane perpendicular to this normal ($\vec{t}$), i.e. then: $\vec{T} = \vec{\sigma}_n + \vec{\tau}_t$ where: $\vec{\sigma}_n$: Normal stress and $\vec{\tau}_t$: Tangential (shear) stress [4].

IV. Stresses in 1 dimension:

V.1 The normal stress in a tensile bar:

Let we consider a bar under an axial load under the effect of two forces P and P' (which are equal and in opposite directions). The normal stress in this bar is obtained by dividing the magnitude P of the axial load by the cross-sectional area A of the bar (Figure.III.6.).

For which the normal $\vec{n}$ is parallel to the load axis.

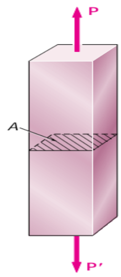


Figure.III.6. Bar under an axial load.

We wrote:

$$\sigma_n = \frac{P}{A}$$

If P is in [N] and A [m²] then the normal stress σ_n is in Pascal [Pa], in resistance of materials often we use as unit the GPa [Giga Pascal: 1 GPa= 1000 N/mm²] or the MPa [Miga Pascal: 1MPa=1N/mm²].

It will be said that "σ_n" is positive in tension, the negative stress is said to be of compression.

The stress vector in this case unidirectional: $\vec{T}_{\vec{n}} = \sigma_n \vec{e}_1$

V.2 The shear stress:

In the previous section the applied forces were perpendicular to the surface. Another different type of stress is obtained when the loads P and P' are applied transversely passing through the section surface A for which the normal $\vec{n}$ is perpendicular to the load axis➔ the normal stress is null, and the tangential $\vec{t}$ is parallel to the load axis.

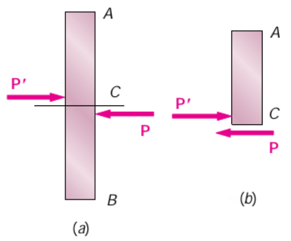


Figure.III.7. Bar under a shear load.

Internal loads must exist in the plane of the section and their resultant must be equal to *P*. These elementary internal loads are called shear loads and the magnitude *P* of their resultant which is the shear (tangential stress) in the section, noted:

$\tau = \frac{P}{A}$; The tangential stress τ is in [Pa].

The stress vector in this case unidirectional: $\vec{\boldsymbol{T}}_{\vec{\boldsymbol{n}}} = \tau \vec{e}_2$

V. Stresses in 2 dimensions:

VI.1 Stress vector on any facet:

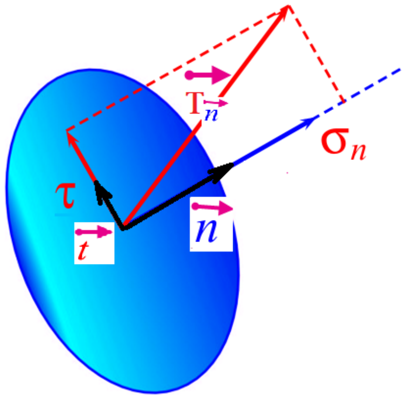


Figure.III.8. Decomposition of the stress-vector $\vec{T}_{\vec{n}}$ into normal and tangential stresses on a facet of normal $\vec{n}$.

In general cases, we now decompose the vector–stress $\vec{\boldsymbol{T}}_{\vec{\boldsymbol{n}}}$ applying to a facet of normal $\vec{n}$ into a normal component $\sigma_{\mathbf{n}}$ and a tangential component τ contained in the plane of the facet of norm τ called shear stress (Figurte.III.8.). **Often, in Continuum Mechanics we give more importance to know the vector $\vec{\boldsymbol{n}}$ normal to the surface, the other tangential vector (parallel to the surfaces) is determined automatically. The stress vector $\vec{\boldsymbol{T}}_{\vec{\boldsymbol{n}}}$ depends linearly on $\vec{\boldsymbol{n}}$. There thus exists a linear application, the tensor of the stress, to pass from $\vec{\boldsymbol{T}}_{\vec{\boldsymbol{n}}}$ to $\vec{\boldsymbol{n}}$.**

$\vec{T}_{\vec{n}} = \sigma_n.\vec{n} + \tau.\vec{t}$ noted by : : $\vec{\boldsymbol{T}}_{\vec{\boldsymbol{n}}} = \begin{Bmatrix} \sigma_n \\ \tau \end{Bmatrix}$

With : $\sigma_n = \vec{n} \,.\, \sigma.\vec{n} = n_i\, \sigma_{ij} n_j$

and : $\tau^2 = \vec{T}_{\vec{n}}.\vec{T}_{\vec{n}} - \sigma_n^2$

When the facet is normal to a principal direction of the stress $\vec{n} = \vec{n}_i$, the component of the shear stress τ is zero and σ_n is equal to the principal stress σ_i [2].

VI.2 The normal and the shear stresses on an oblique plane under axial loading

Let we consider the element subjected to two axial loads *P* and *P′* (Figure.III.9.). If we pass a section forming an angle θ with a normal plane (Figure.III.9a.), and drawing the free part of the element located to the left of this section (Figure.III.9b.), we find according to the conditions of equilibrium of the free body that the distributed loads acting on the section must be equivalent to the load *P*.

By substitution P in its components *F* and *V*, respectively normal and tangential to the section (Figure.III.9c.), we will have:

$$F = P\cos\theta \text{ , and: } V = P\sin\theta.$$

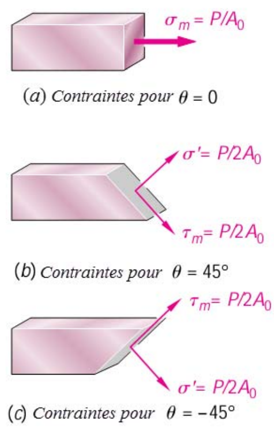


Figure.III.9. Element with an oblique plane under axial loading.

The load F presents the resultant of the normal loads distributed over the section and the load V resulting from the shear loads (Figure.III.10d.).

The average values of the corresponding normal and shearing stresses are obtained by dividing respectively F and V by the surface A_θ of the section:

$$\sigma = \frac{F}{A_\theta},\ \tau = \frac{V}{A_\theta};$$

By substituting the values of F and V previously calculated, we will have:

$$\boxed{\sigma = \frac{P}{A_0}\cos^2\theta} \quad \text{and} \quad \boxed{\tau = \frac{P}{A_0}\sin\theta\cos\theta}$$

With : $A_\theta = \frac{A_0}{\cos\theta}$

And A_0: is the section perpendicular to the axis of the element.

We can conclude that the stress vector $\vec{\boldsymbol{T}}_{\vec{\boldsymbol{n}}}$ is composed by a normal σ and a tangential τ stresses, noted as: $\vec{\boldsymbol{T}}_{\vec{\boldsymbol{n}}} = \begin{Bmatrix} \sigma \\ \tau \end{Bmatrix}$.

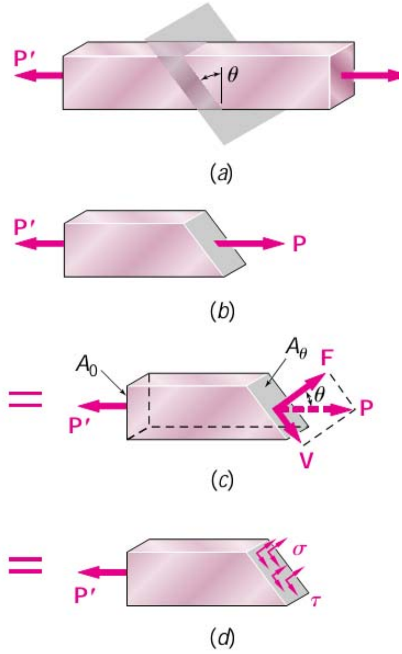


Figure.III.10. Element with an oblique plane under axial.

We note from this equation that the normal stress σ is maximal when θ = 0, i.e.; when the plane of the section is perpendicular to the axis of the element, and that it approaches zero when θ approaches to 90°. We check that the value of σ when is θ=0.

$$\sigma_m = \frac{P}{A_0};$$

The shear stress τ is zero for θ=0° and θ=90°, and for θ=45° it reaches its maximum value:

$\tau_m = \frac{P}{A_0}\sin 45°\cos 45° = \frac{P}{2A_0}$, when the angle is equal to 45° then:

$$\sigma'_m = \frac{P}{A_0}\cos^2 45° = \frac{P}{2A_0}.$$

VI.2 2D Cauchy Stress Tensor and vector stress - plane stress:

A condition of plane stress occurs when all of the stresses acting on the component are in the same plane.

Normal and shear stresses at a free surface are zero, or the stresses in one of the directions are zero.

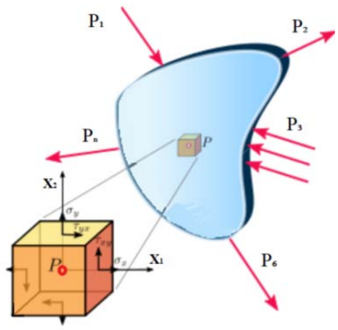


Figure .III.11.2D Chauchy body element.

Let we consider a body subjected to several loads P_1, P_2, etc. (Figure.III.11).

Two facets are considered here the facet denoted ① and ②.

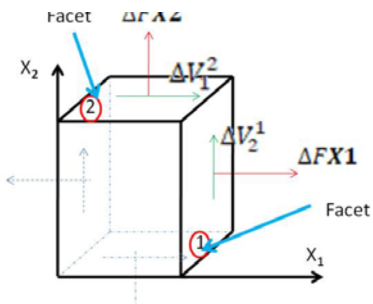


Figure .III.12. 2D Chauchy stress tensor.

The part of the body at the left of the section is subjected to some of the original loads, the normal and shear loads distributed over the section. We denote by ΔF_{X2} and ΔV_{X2} respectively the normal and shear loads acting on a small surface ΔA surrounding the point P (Figure.III.12). Note that the subscript X_2 is used to indicate that the forces ΔF_{X2} and ΔV_{X2} act on a surface perpendicular to the X_2 axis.

Although the normal load ΔF_{X2} has a well-defined direction, the shear load ΔV_{X2} can have any direction in the plane of the section. We therefore solve ΔV_{X2} in two load components, ΔV_{X1}^{X2} and ΔV_{X3}^{X2} in directions parallel to X_2X_1, (Figure.III.12). Now dividing the magnitude of each load by the area ΔA, we define two stress components as:

$\sigma_{X2} = \frac{\Delta F\mathbf{X2}}{\Delta A}$ and : $\tau_{X2X1} = \frac{\Delta V_{X1}^{X2}}{\Delta A}$;

These stresses are determined in the case of a plane perpendicular to the X_2 axis ($\vec{e}_2$).

The stress vector $\vec{T}_{\vec{n}2}$ will be composed with a normal stress σ_{X2} and a shear stress τ_{X2X1} noted as:

$\vec{T}_{\vec{n}2} = \begin{Bmatrix} \sigma_{X2} \\ \tau_{X2X1} \end{Bmatrix}$;

In the same way, we choose a plane passing through the point P perpendicular to X_1 illustrated in Figure.III.13.

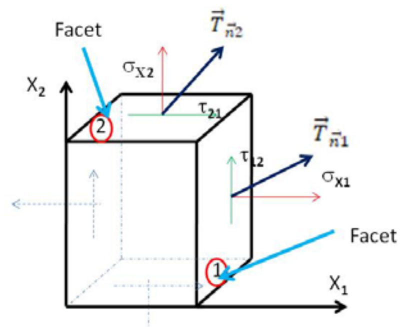


Figure .III.13. 2D Couchy vector stress.

$\sigma_{X1} = \frac{\Delta F\mathbf{X1}}{\Delta A}$ and : $\tau_{X1X2} = \frac{\Delta V_{X2}^{X1}}{\Delta A}$

These stresses are determined in the case of a plane perpendicular to the X_1 axis ($\vec{e}_1$).

The stress vector $\vec{T}_{\vec{n}1}$ will be composed with a normal stress σ_{X1} and a shear stress τ_{X1X2} , noted as :

$\vec{T}_{\vec{n}1} = \begin{Bmatrix} \sigma_{X1} \\ \tau_{X1X2} \end{Bmatrix}$;

The three components required to specify stress state at a point P are:

$\sigma_{X1}, \sigma_{X2}, \tau_{X1X2} = \tau_{X2X1}$ (by symmetry).

Finally, we can say that to define the state of stress σ_{ij} in the point P, we must consider an elementary cube of unit as edge subjected to the following stress tensor:

$\sigma_{ij} = \begin{bmatrix} \sigma_{X1} & \tau_{X1X2} \\ \tau_{X2X1} & \sigma_{X2} \end{bmatrix}$;

What's the significance of σ_{ij} ? it means that the value of stress of the facet number i in the direction of axis j,

for example: σ_{11} that means the stress value on the facet number in the direction of axe 1(X1) $=\sigma_{X1}$.

σ_{12} that means the stress value on the facet 1 in the direction of axe 2(X2)= τ_{X1X2} and so on.

VI.3 The use of plane stress in engineering problems:

The plane stress is a stress state where the stress components related to one direction, usually the X_3 direction in the Cartesian coordinate system, the two-dimensional object is assumed to have no thickness, which transforms the three-dimensional stress state into a less complex two-dimensional problem.

Let us consider the case where $\tau_{X2X3} = \tau_{X3X2} = \sigma_{X3X3} = 0$.

Often the plane stress is used to turn a 3D problem into 2D problem, in simple terms, plane stress is mainly used in materials and structures where one dimension is significantly smaller than the other two.

Such practical applications of the plane stress concept remind us that well-established fundamental engineering concepts play a crucial role in everyday life, ensuring the safety and efficiency of commonly used items.

Mechanical engineers frequently use the plane stress concept. Components such as thin-walled pressure vessels, shafts, thin rotating disks, and springs are often analyzed under plane stress conditions. These prototypes allow engineers to design more efficient and safer machines.

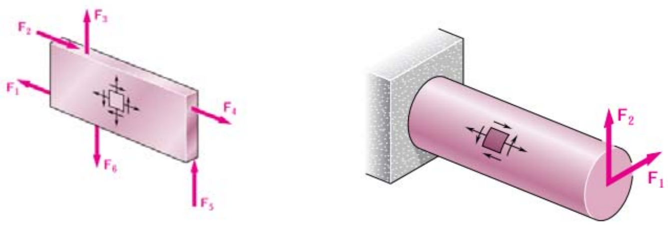
For example, the torsion of circular bars, often analyzed in mechanical engineering, involves the plane stress condition.

Thin plates and shells used in aerospace structures are analyzed under plane stress conditions. The fuselage skin, wings or tail structures of aircraft can be considered thin relative to their other dimensions. For example, the stress state in the midplane of the wing skin due to aerodynamic pressure can be accurately modeled using the plane stress assumption.

Real-world complexities often require engineering problems to be simplified to make them easier to solve. In cases involving slender bodies where the dimensions in one direction are significantly small compared to other directions, plane stress is the basic assumption often considered. The formula that makes the concept of plane stress meaningful is to simplify the stress state into a two-dimensional scenario.

Plane stress analysis is a key principle in engineering fields because it provides valuable insights into how stresses vary with orientation and helps in understanding complex stress states in practical engineering problems [5].

Such a situation occurs in a thin plate subjected to forces acting in the midplane of the plate (Figure.III.14a). It also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface of that element or component that is not subjected to an external force (Figure.III.14b.).



a) *b)*

Figure.III.14. Torsion and circular bars.

Examples of tanks for liquids and cylindrical or spherical pressure vessels.

Figure .III.15. Thin-walled pressure vessels.

VI.4 Plane stress transformation law:

Let us assume that a state of plane stress exists at point Q and that it is defined by the stress components σ_{X1}, σ_{X2} and τ_{X1X2}associated with the element shown in Figure.III.16a. We propose to determine the stress components σ_{X1}, σ_{X2} and τ'_{X1X2} associated with the element after it has been rotated through an angle θ about the z axis (Figure.III.16b), and to express these components in terms of σ_{X1}, σ_{X2} ,τ_{X1X2} and θ.

Which means that to find the new coordinates of stresses in the plane basis (O,X'$_1$, X'$_2$) (Figure.III.16b) from the initial plane basis (O,X$_1$, X$_2$) (Figure.III.16a).
The transformation matrix is given by:

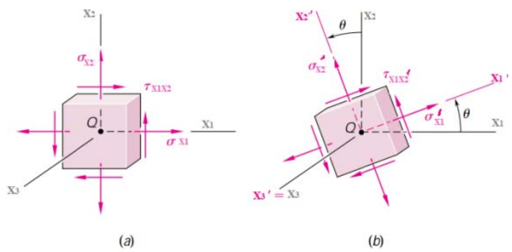


Figure .III.16. Plane transformation law.

	$\vec{e}_1$	$\vec{e}_2$
$\vec{e}'_1$	$\cos\theta$	$\sin\theta$
$\vec{e}'_2$	$-\sin\theta$	$\cos\theta$

With the law transformation of a stress tensor, we know: $\sigma'_{ij} = A^t \sigma_{ij} A$:

$$\sigma'_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{X1} & \tau_{X1X2} \\ \tau_{X2X1} & \sigma_{X2} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow \begin{cases} \sigma'_{11} = \sigma_{X1}\cos\theta^2 - 2\tau_{X1X2}\cos\theta\sin\theta + \sigma_{X2}\sin\theta^2 \;\ldots..(\times) \\ \sigma'_{22} = \sigma_{X1}\sin\theta^2 + 2\tau_{X1X2}\cos\theta\sin\theta + \sigma_{X2}\cos\theta^2 \\ \tau'_{X1X2} = \sigma_{X1}\cos\theta\sin\theta + \tau_{X1X2}\cos\theta^2 - \tau_{X1X2}\sin\theta^2 - \sigma_{X2}\sin\theta\cos\theta \;\ldots.(\times\times) \end{cases}$$

With using the following trigonometric formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin\theta^2 = \frac{1-\cos 2\theta}{2};$$
$$\cos\theta^2 = \frac{1+\cos 2\theta}{2};$$
$$\cos\theta^2 - \sin\theta^2 = \cos 2\theta.$$

$$\begin{cases} \sigma'_{X11} = \dfrac{\sigma_{X11}+\sigma_{X22}}{2} + \dfrac{\sigma_{X11}-\sigma_{X22}}{2}\cos 2\theta + \tau_{X1X2}\sin 2\theta \\ \sigma'_{X22} = \dfrac{\sigma_{X11}+\sigma_{X22}}{2} - \dfrac{\sigma_{X11}-\sigma_{X22}}{2}\cos 2\theta - \tau_{X1X2}\sin 2\theta \\ \tau'_{X1X2} = -\dfrac{\sigma_{X11}-\sigma_{X22}}{2}\sin 2\theta + \tau_{X1X2}\cos 2\theta \end{cases}$$

For the critic case when ($\tau_{X1X2} = \sigma_{X2} = 0$).
We find the same results as before when we take one facet:

$$\begin{cases} \sigma'_{11} = \sigma_{X1}\cos\theta^2 \\ \tau'_{X1X2} = \sigma_{X1}\cos\theta\sin\theta \end{cases} \quad \text{equations (x,xx)}$$

The formula to calculate the maximum shear stress is:

$$\tau'_{max} = \sqrt{(\sigma'_{11} - \sigma'_{22})^2 + (\tau'_{X1X2})^2}$$

Rotating a structural element through 180°:

If the 2D element is rotated through an angle θ, the normal and shear stress components will change, as will the coordinates.
We can calculate the normal and shear stress components on a structural element through any angle of rotation using Cauchy's equations.

Example:

Let consider a stress state at point P given by:

$$\sigma_{ij} = \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix} MPa\,;$$

An element is rotated through 180° at 10° intervals; the initial and calculated values of σ_{X1}, σ_{X2}, and τ_{X1X2} are in units of MPa (megapascals). Plotting the stress values for each σ_{X1}', σ_{X2}', and τ_{X1X2}' against the rotation angle produces sinusoidal curves as represent on Figure.III.19 and Figure.III.20.

On the facet ❶ we find:

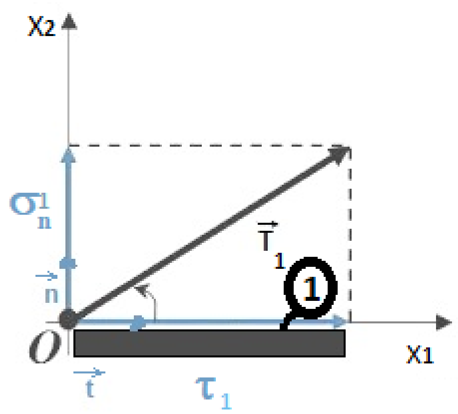


Figure.III.17. The vector stress as a resultant of σn and τ on facet 1.

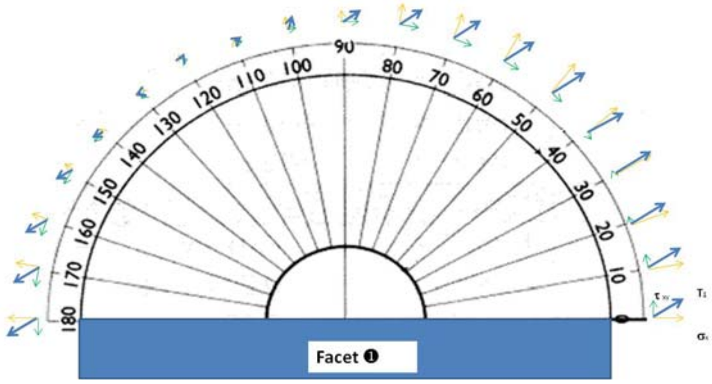


Figure.III.18. The rotation of the normal and shear stress with the stress vector.

The maximum normal stress σ_{X1}' is 13.7 MPa at the angle 30°, its minimum value is 0.3 MPa at 120°
The maximum shear stress τ_{X1X2}' is 6.7 MPa located at the angle 80°.

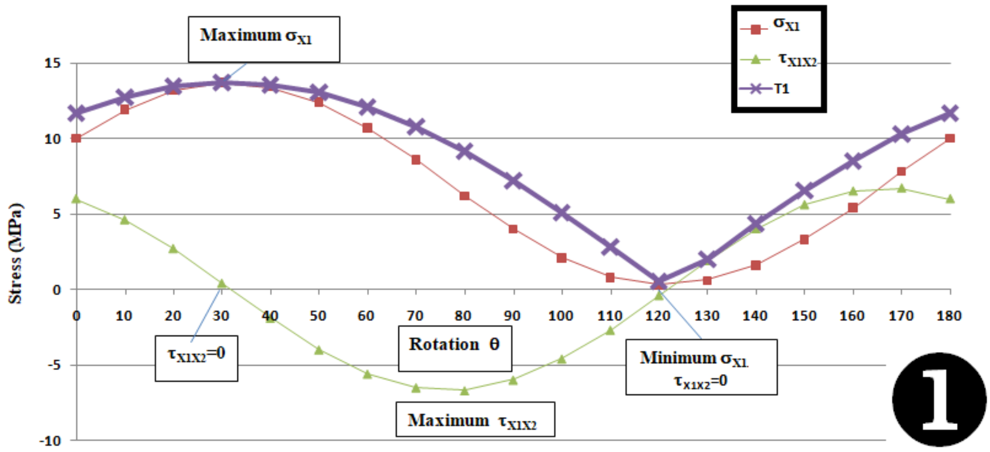


Figure .III.19. Sinusoidal curves of normal and shear stress with the magnitude of the stress vector on facet (1).

On the facet ❷ we have:

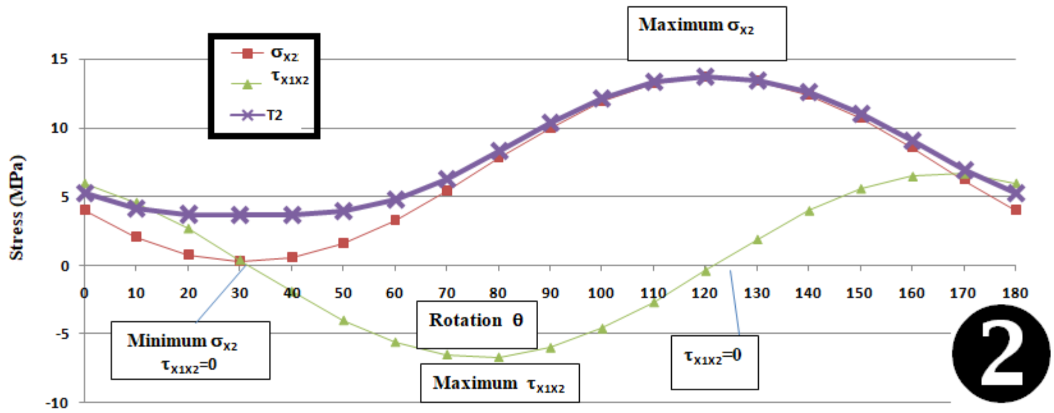


Figure .III.20. Sinusoidal curves of normal and shear stress with the magnitude of the stress vector on facet (2).

The maximum normal stress σ_{X2}' is 13.7 MPa at the angle 120°, its minimum value is 0.3 MPa at 30°
The maximum shear stress τ_{X1X2}' is 6.7 MPa located at the angle 80°.

We can visit the web site to verify:
http://strengthandstiffness.com/tools/stress_trans_calc/stress_transform_calculator.htm

To visualize the rotating structural element through 360°, we can go to the following site:
https://drbuc2jl8158i.cloudfront.net/shared/Engeneering/mecmovies/ch12/m12_01_s040.html

VI.4. Principal Stresses; maximum shearing stress

The previous equations (×,××) can be written by manipulating the trigonometric relations as:

$$\sigma'_{11} = \frac{\sigma_{X1}+\sigma_{X2}}{2} + \frac{\sigma_{X1}-\sigma_{X2}}{2}\cos 2\theta + \tau_{X1X2}\sin 2\theta$$

$$\tau'_{X1X2} = -\frac{\sigma_{X1}-\sigma_{X2}}{2}\sin 2\theta + \tau_{X1X2}\cos 2\theta$$

This means that, if we choose a set of rectangular axes and plot a point M of abscissa and ordinate for any given value of the parameter all the points thus obtained will lie on a circle. To establish this property we eliminate θ from Equations; and finally adding member to member the two equations obtained in this fashion. We have

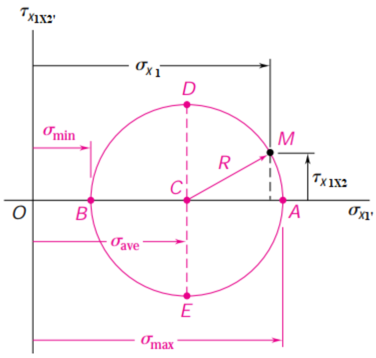


Figure .III.21. Plot a point M of abscissa and ordinate for any given value of the parameter all the points thus obtained will lie on a circle.

$$\left(\sigma'_{11} - \frac{\sigma_{X1}+\sigma_{X2}}{2}\right)^2 + \tau'^{\,2}_{X1X2} = \left(\frac{\sigma_{X1}-\sigma_{X2}}{2}\right)^2 + \tau_{X1X2}^{\,2};$$

Setting $\sigma_{av} = \frac{\sigma_{X1}+\sigma_{X2}}{2}$

and $R = \sqrt{\left(\frac{\sigma_{X1}-\sigma_{X2}}{2}\right)^2 + \tau_{X1X2}^{\,2}}$

We write $(\sigma'_{11} - \sigma_{av})^2 + \tau'^{\,2}_{X1X2} = (R)^2$

Which is the equation of a circle of radius R centered at the point C of abscissa and ordinate 0 (Fig.21.).

The two points A and B where the circle of Figure. III.21.intersects the horizontal axis are of special interest: Point A corresponds to the maximum value of the normal stress while point s B corresponds to its minimum value. Besides, both points correspond to a zero value of the shearing stress τ'_{X1X2}. We write

$$tg\ 2\theta_p = \frac{2\tau_{X1X2}}{\sigma_{X1}-\sigma_{X2}} \dots\dots (\times\times\times)$$

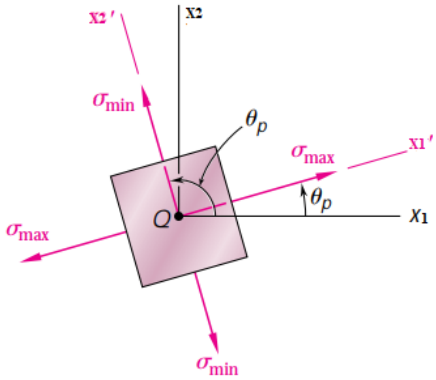


Figure .III.22.*The principal planes of stress* at point Q.

This equation defines two values that are apart, either of these values can be used to determine the orientation of the corresponding element (Figure.III.22.).
The planes containing the faces of the element obtained in this way are called *the principal planes of stress* at point Q, and the corresponding values of the normal stress exerted on these planes are called the principal stresses at Q.
We observe: $\sigma_{max} = \sigma_{av} + R$ and $\sigma_{min} = \sigma_{av} - R$

Referring again to the circle, we note that the points D and E located on the vertical diameter of the circle correspond to the largest numerical value of the shearing stress τ'_{X1X2}.

Maximum Shearing Stress:

Formula yields $\tau_{max} = \sqrt{\left(\frac{\sigma_{X1}-\sigma_{X2}}{2}\right)^2 + \tau^2_{X1X2}}$

Since σ_{max} and σ_{min} have opposite signs, the value obtained for τ_{max} actually represents the maximum value of the shearing stress at the point considered.

VI.5 The determination of the principal stresses and maximum shear stress methods:

Two methods are available the easiest one is the graphical “approximate” method (Mohr’s circle) and the other is the analytical “exact” method, let’s take a look on them:

VI.5.1.The analytical method for principal stresses of plane stresses:

The stress tensor σ_{ij} is symmetric; we can make its rotation to diagonalize it. We find two orthogonal principal directions associated with three eigen values σ_I, σ_{II} , called the principal stresses.
$\vec{t}_{\vec{n}} = \sigma_{ij}.\vec{n}_j = \sigma_i.\vec{n}_i$ and since: $\vec{n}_i = \delta_{ij}.\vec{n}_j$ then:
$(\sigma_{ij} - \delta_{ij}.\sigma_i)\,\vec{n}_j = \vec{0}$; this called the specific equation for the rotation of σ_{ij}

For $\vec{n}_j = \vec{0}$ the solution is trivial, and since $(\sigma_{ij} - \delta_{ij}.\sigma_i)$ is a matrix where the unknowns are the principal stresses σ_i; this allows us to say that its determinant must be zero.

$$\begin{vmatrix} \sigma_{11} - \sigma & \tau_{X1X2} \\ \tau_{X2X1} & \sigma_{22} - \sigma \end{vmatrix} = 0$$

We will obtain a polynomial equation of order two called the characteristic equation:

$$\sigma^2 - I\,\sigma - II = 0$$

A second order ordinary equation which can be solved by using the determinant.
Once the principal stresses σ_I, σ_{II} ,were found we can replace them case by case in the specific equation to find the principal directions $\vec{n}_I$ and $\vec{n}_{II}$.

Example 1:

A single horizontal force P of magnitude 150 N is applied to end D of lever ABD. Knowing that portion AB of the lever has a diameter of 1.2 mm, determine (a) the normal and shearing stresses on an element located at point H and having sides parallel to the *x* and *y* axes, (b) the principal planes and the principal stresses at point H, (c) the maximum shear stress?.

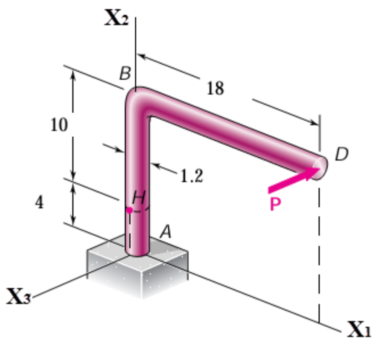


Figure .III.23.A horizontal force P is applied to end D of lever ABD.

Solution:

Force-Couple System: We replace the force P by an equivalent force-couple system at the center *C* of the transverse section containing point *H*:

P=150 N.
Torque: *T*= (150N) (18 mm)=2.7 KN.mm
Bending moment: *Mx*= (150N) (10mm) = 1.5 KN.mm

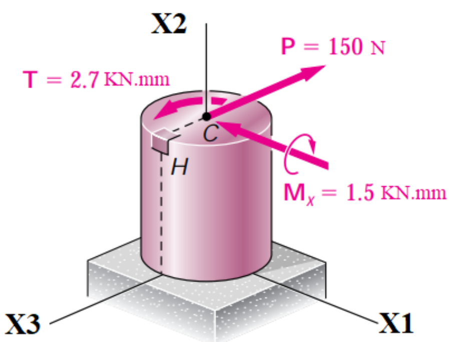


Figure .III.24. Forecse and couples applied at Point H.

The stresses σ_{X1}, $\sigma_{X2}$and τ_{X1X2} at Point H :
Using the sign convention shown in Figure.III.20.we determine the sense and the sign of each stress component by carefully examining the sketch of the force-couple system at point C:

$$\sigma_{X1} = 0,\ \sigma_{X2} = -\frac{M_x C}{I} = -\frac{1.5\ KN.mm * 0.6mm}{\frac{1}{4} * \pi * (0.6mm)^4} = -8.84 \frac{KN}{mm^2} = -8840\ MPa$$

$$\tau_{X1X2} = +\frac{TC}{J} = \frac{2.7\ KN.mm * 0.6mm}{\frac{1}{2} * \pi * (0.6mm)^4} = 7.96 \frac{KN}{mm^2} = 7960\ MPa$$

We note that the shearing force P does not cause any shearing stress at point H.

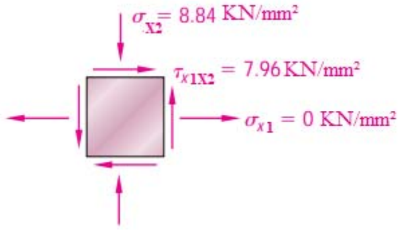


Figure .III.25. The stress state defined in point H.

b. Principal stresses and principal planes

Substituting into equation:

$$\sigma_{max,min} = \frac{\sigma_{X1}+\sigma_{X2}}{2} \pm \sqrt{\left(\frac{\sigma_{X1}-\sigma_{X2}}{2}\right)^2 + \tau_{X1X2}^2},$$

We determine the magnitudes of the principal stresses:

$$\sigma_{max,min} = \frac{0+8.84}{2} \pm \sqrt{\left(\frac{0+8.84}{2}\right)^2 + 7.96^2} = +4.42 \pm 9.10$$

$$\sigma_{max} = 13.52\ KN/mm^2$$
$$\sigma_{min} = -4.68\ KN/mm^2$$

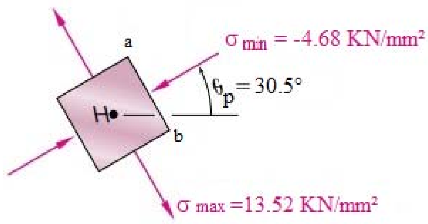


Figure .III.26. The principal stresses and planes.

Substituting the values of the stress components into the equation:

$$tg\ 2\theta = \frac{2.\tau_{X1X2}}{\sigma_{X1}-\sigma_{X2}} \ldots (\times\times\times),$$

We determine the orientation of the principal planes:

$$tg\ 2\theta_p = \frac{2 * 7.96}{0+8.84} = +1.8$$

$2\theta_p = +61.0°$ and $180° + 61.0 = +241°$
$\theta_p = +30.5°$ and $120.5°$

(c) The maximum shear stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_{X1}-\sigma_{X2}}{2}\right)^2 + \tau^2{}_{X1X2}} = \sqrt{\left(\frac{8.84}{2}\right)^2 + (7.96)^2} = 9.1\ GPa$$

Since σ_{max} and σ_{min} have opposite signs, the value obtained for τ_{max} actually represents the maximum value of the shearing stress at the point considered.

Which we can verify by using ***the analytical method***:

(a) The principal stresses:

At the beginning we have: the stress state at point H defined by: $\sigma_{ij} = \begin{bmatrix} 0 & 7.96 \\ 7.96 & 8.84 \end{bmatrix} GPa$;

Writing the specific equation that rotates the stress state at point H by: $(\sigma_{ij} - \delta_{ij}.\sigma_i)\,\vec{n}_j = \vec{0}$;

$\begin{bmatrix} 0\text{-}\sigma & 7.96 \\ 7.96 & 8.84\text{-}\sigma \end{bmatrix}.\vec{n} = \vec{0}$ in maths this is possible when : $\det\left(\begin{bmatrix} \text{-}\sigma & 7.96 \\ 7.96 & 8.84\text{-}\sigma \end{bmatrix}\right)=0$ that means : $\begin{vmatrix} \text{-}\sigma & 7.96 \\ 7.96 & 8.84\text{-}\sigma \end{vmatrix} = 0$

- ⇨ $(-\sigma)(8.84 - \sigma) - 7.96^2 = 0$ which gives a second order ordinary equation:
- ⇨ $\sigma^2 - 8.84\sigma - 63.36 = 0$;
- ⇨ We get the principal stresses : $\sigma_I = 13.52\ GPa$ and $\sigma_{II} = -4.68\ GPa$ ordred $\sigma_I > \sigma_{II}$

(b) The principal directions:

To find the principal planes (directions) we should substitute the principal stresses one by one in the equation: $(\sigma_{ij} - \delta_{ij}.\sigma_i)\,\vec{n}_j = \vec{0}$;

Case I: $\sigma_I = 13.52\ GPa$

⇨ $\begin{bmatrix} \text{-}13.52 & 7.96 \\ 7.96 & \text{-}9.71 \end{bmatrix}.\vec{n_I} = \vec{0}$ where $\vec{n_I}$ is the direction unit vector when $\sigma_I = 13.52\ GPa$;

We consider $\vec{n_I}\begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix}$ ➔ $\begin{bmatrix} \text{-}13.52 & 7.96 \\ 7.96 & \text{-}4.68 \end{bmatrix}.\begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ we obtain the system:

$\begin{cases} -13.52 \cdot n_1 + 7.96 \cdot n_2 = 0 \\ 7.96 \cdot n_1 - 4.68 \cdot n_2 = 0 \end{cases}$ which gives the same equation, $n_1 = \frac{4.68}{7.96} n_2$, to get their values we shall use LAMÉ's formula, it states in plane : $\vec{n_I} \cdot \vec{n_I} = 1$ or : $n_1^2 + n_1^2 = 1$.

$\left(\frac{4.68}{7.96}\right)^2 \cdot n_2^2 + n_2^2 = 1$ ➔ $1.35 \cdot n_2^2 = 1$ ➔ $n_2 = 0.86$ then $n_1 = 0.5$ ➔ $\vec{n_I}\begin{Bmatrix} 0.5 \\ 0.86 \end{Bmatrix}$

The principal plan angle is 30°.

Case II: $\sigma_{II} = -4.68\ GPa$

⇨ $\begin{bmatrix} 4.68 & 7.96 \\ 7.96 & 13.52 \end{bmatrix}.\vec{n_{II}} = \vec{0}$ where $\vec{n_{II}}$ is the direction unit vector when $\sigma_{II} = -4.68\ GPa$;

We consider $\vec{n_{II}}\begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix}$ ➔ $\begin{bmatrix} 4.68 & 7.96 \\ 7.96 & 13.52 \end{bmatrix}.\begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ we obtain the system:

$\begin{cases} 4.68 \cdot n_1 + 7.96 \cdot n_2 = 0 \\ 7.96 \cdot n_1 + 13.52 \cdot n_2 = 0 \end{cases}$ which gives the same equation, $n_1 = -\frac{7.96}{4.68} n_2$, to get their values we shall use LAMÉ's formula, it states in plane : $\vec{n_{II}} \cdot \vec{n_{II}} = 1$ or : $n_1^2 + n_1^2 = 1$.

$\left(\frac{7.96}{4.68}\right)^2 \cdot n_2^2 + n_2^2 = 1$ ➔ $3.9 \cdot n_2^2 = 1$ ➔ $n_2 = 0.5$ then $n_1 = -0.85$ ➔ $\vec{n_I}\begin{Bmatrix} -0.85 \\ 0.5 \end{Bmatrix}$.

The principal plan angle is 60°.

(c) The maximum shear stress:

$$\tau_{max} = \frac{\sigma_I - \sigma_{II}}{2} = \frac{13.52+4.68}{2} = 9.1\ GPa.$$

Example 2:

The axle of an automobile is acted upon by the forces and couple shown. Knowing that the diameter of the solid axle is 15mm, determine (a) the principal planes and principal stresses at point H located on top of the axle, (b) the maximum shearing stress at the same point.

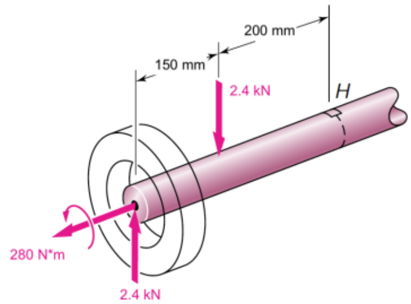


Figure .III.27. The axle of an automobile.

For this part, what we need to do is evaluate it the same way we did in the torsion problem, noting that H is located on the outer portion of the rod. So that the shear due to torsion becomes

$$\tau_{X1X2} = +\frac{TC}{J} = \frac{2T}{\pi * (\emptyset)^3} = \frac{2 * 280 * 1000}{\pi * (15)^3} = 52.8\,MPa$$

$\sigma_{X1} = -\frac{M_x C}{I} = -\frac{2400\,N.150mm.}{\frac{1}{4}*\pi*(15mm)^3} = -188.4\,MPa$; $\sigma_{X2} = 0$

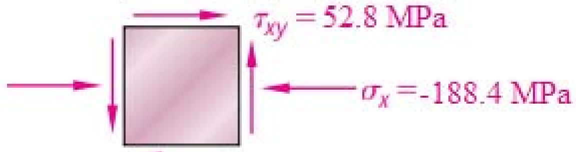


Figure .III.28. Stress state at point H.

So now if we do a drawing of what is going on in term of the stress state at H we see the following:

$$\sigma_{ij} = \begin{bmatrix} -188.4 & 52.8 \\ 52.8 & 0 \end{bmatrix}$$

So we than obtain the average stress as $\sigma_{av} = \frac{(\sigma_{X1}+\sigma_{X2})}{2} = -94.2\,MPa$

The radius of the circle

$$R = \sqrt{\left(\frac{\sigma_{X1} - \sigma_{X2}}{2}\right)^2 + \tau_{X1X2}^2} = 107.98\,MPa$$

So to calculate the first principal stress we use

$$\sigma_I = \sigma_{av} + R = -94.2 + 107.98 = 13.78\,MPa$$

and to calculate the second principal stress we use

$$\sigma_{II} = \sigma_{av} - R = -94.2 - 107.98 = -202.18\,MPa$$

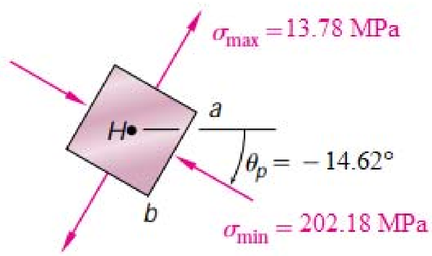


Figure .III.29. Principal stresses.

With : $tg\,2\theta_p = \frac{2*52.8}{-188.4} = -0.56$

$2\theta_p = -29.24$ and $180° - 29.24 = +150.76°$

Then: $\theta_p = -14.62\,°$ and $75.38°$

b) The maximum shearing stress:

$$\tau_{max} = R = 107.98\,MPa$$

VI.5.2. Graphical method (Mohr's circle) for plane stress:

Before starting to explain the graphical method we shall begin with the stresses sign convention:

Stresses sign convention:

The Mohr's circle construction for plane stresses is greatly simplified if we consider separately each face of the element used to define the components of the stress. We can see that, when the shear stress exerted on a given face tends to rotate the element clockwise sense, the point of the Mohr circle corresponding to this face is located above the axis σ_n.

When the shear stress on a given face tends to rotate the element counterclockwise sense, the point corresponding to this face is located below the axis σ_n. (Figure .III.30a).

When it comes to normal stresses, the usual convention holds, i.e. a tensile stress is considered positive and is plotted on the right, while a compressive stress is considered negative and is drawn on the left.

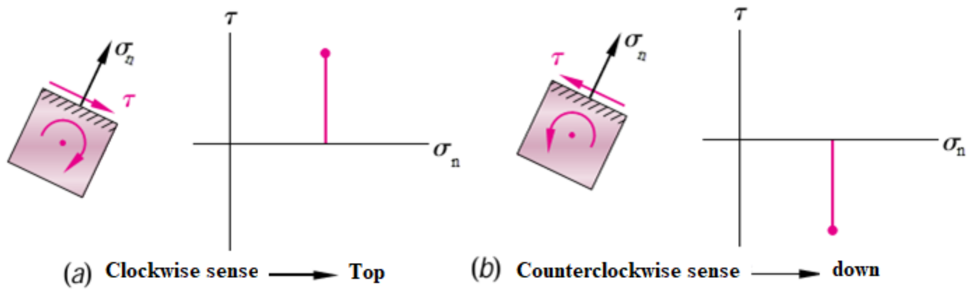


Figure .III.30. Stress sign convention.

The circle used in the preceding section to derive some of the basic formulas relating to the transformation of plane stress was first introduced by the German engineer Otto Mohr (1835–1918) and is known as Mohr's circle for plane stress. As you will see presently, this circle can be used to obtain an alternative method for the solution of the various problems. This method is based on simple geometric considerations and does not require the use of specialized formulas. While originally designed for graphical solutions, it lends itself well to the use of a calculator.

Consider a square element of a material subjected to plane stress and let σ_X, σ_y and τ_{xy} be the components of the stress exerted on the element. We plot a point *X* of coordinates $\sigma_X$and -τ_{xy} and a point Y of coordinates σ_y and +τ_{xy} (Figure.III.31b). If is positive, as assumed in Figure.III.31a, point X is located below the axis and point Y above, as shown in Figure.III.31b.

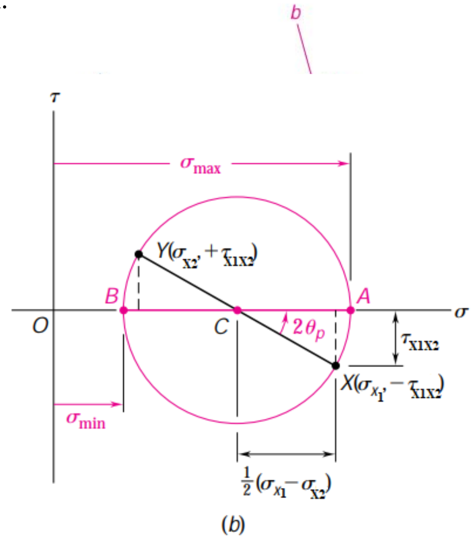


Figure .III.31. Steps to design Mohr's plane circle.

Joining *X* and *Y* by a straight line, we define the point *C* of intersection of line *XY* with the axis and draw the circle of center *C* and diameter *XY*. Noting that the abscissa of *C* and the radius of the circle are respectively equal to the quantities and R defined previously, we conclude that the circle obtained is Mohr's circle for plane stress.

Thus the abscissas of points A and B where the circle intersects the axis represent respectively the principal stresses σ_{max} and σ_{min} at the point considered.

The angle of rotation θ can be determinate by using directly the right side which its center is σ_{II} or dividing by 2 the angle at the left side which its center is C as shown by Figure.III.32.

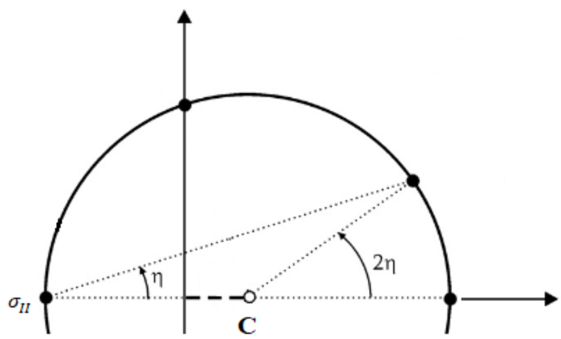


Figure.III.32. Choice of the rotation angle.

Example Using Mohr's circle:

The stresses state at the point H is given by : $\sigma_{ij} = \begin{bmatrix} -188.4 & 52.8 \\ 52.8 & 0 \end{bmatrix}$

We can visit the web site : https://mechanicalc.com/calculators/mohrs-circle/

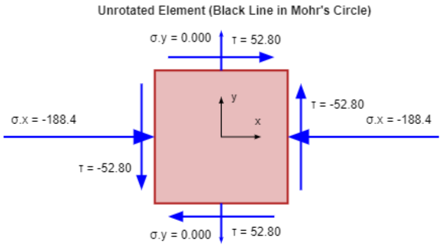


Figure .III.33. The given Stress state.

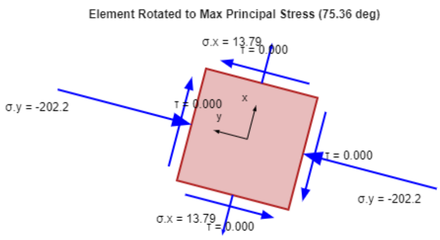


Figure .III.34. The element rotated to the principal stresses.

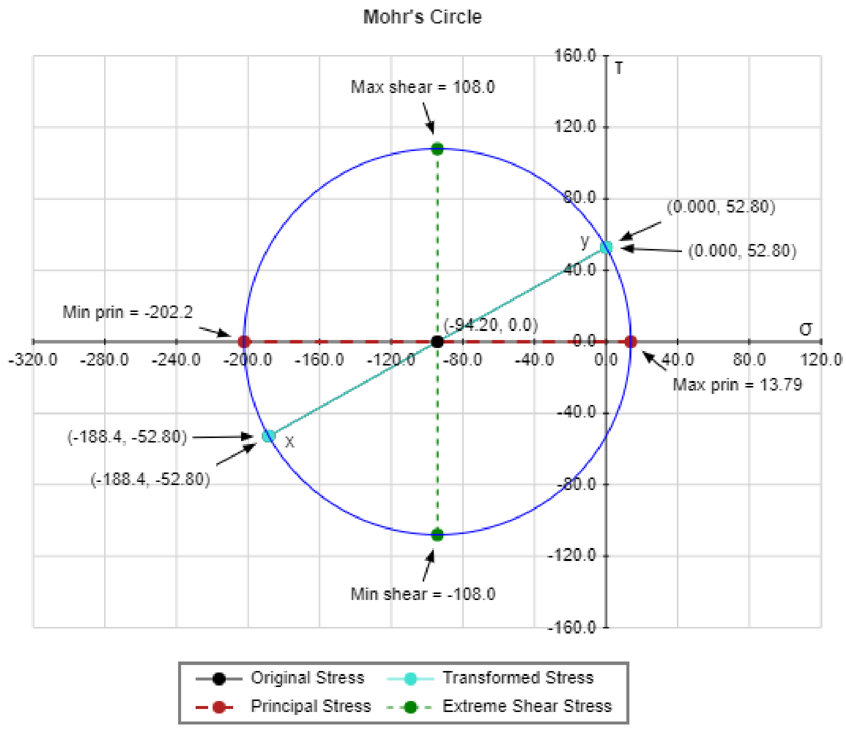


Figure .III.35. The principal stresses and max,min.shear stresses.

IV. Stresses in 3 dimensions:

IV.1 Stresses under general loading conditions; stress components:

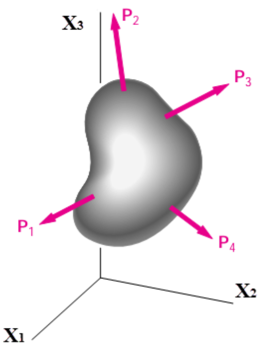


Figure .III.36. Body under forces.

Let we consider a body subjected to several loads P_1, P_2, etc. (Figure.III.36).
To understand the stress condition created by these loads at a point Q of the body, we will first cut through a fictitious section through Q, using a plane parallel to the X_1X_2 plane.
The part of the body at the left of the section is subjected to some of the original loads, the normal and shear loads distributed over the section. We denote by ΔF_{X2} and ΔV_{X2} respectively the normal and shear loads acting on a small surface ΔA surrounding the point Q (Figure.III.37a). Note that the subscript X_2 is used to indicate that the forces ΔF_{X2} and ΔV_{X2} act on a surface perpendicular to the X_2 axis.

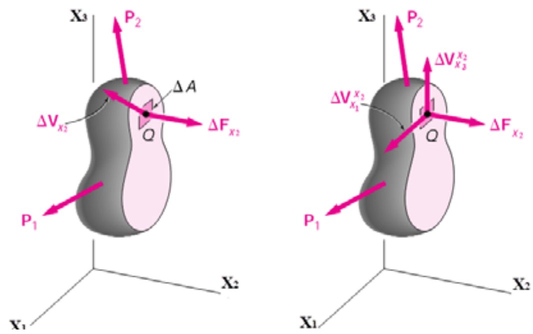


Figure .III.37. Section through Q, using a plane parallel to the X_1X_2 lane.

Although the normal load ΔF_{X2} has a well-defined direction, the shear load ΔV_{X2} can have any direction in the plane of the section. We therefore solve ΔV_{X2} in two load components, ΔV_{X1}^{X2} and ΔV_{X3}^{X2} in directions parallel to X_2X_1 and X_2X_3, respectively (Figure.III.37b). Now dividing the magnitude of each load by the area ΔA, we define the three stress components as shown in Figure.III.38:

In Figure.III.38., we notice that there is three principal facets denoted ①,② and ③ ;

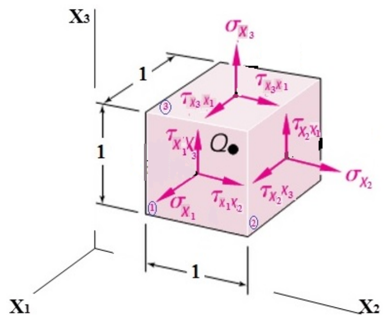


Figure .III.38. Three principal facets.

$\sigma_{X2} = \frac{\Delta F\mathbf{X2}}{\Delta A}$ and : $\tau_{X2X1} = \frac{\Delta V_{X1}^{X2}}{\Delta A}$; $\tau_{X2X3} = \frac{\Delta V_{X3}^{X2}}{\Delta A}$;

These stresses are determined in the case of a plane perpendicular to the X_2 axis ($\vec{e}_2$).

The stress vector $\vec{T}_{\vec{e}2}$ will be composed with a normal stress σ_{X2} and two shear stresses τ_{X2X1} , τ_{X2X3}, noted as :

$\vec{T}_{\vec{e}2} = \begin{Bmatrix} \sigma_{X2} \\ \tau_{X2X1} \\ \tau_{X2X3} \end{Bmatrix};$

In the same way, we choose a plane passing through the point Q perpendicular to X_1 illustrated in Fig.39.

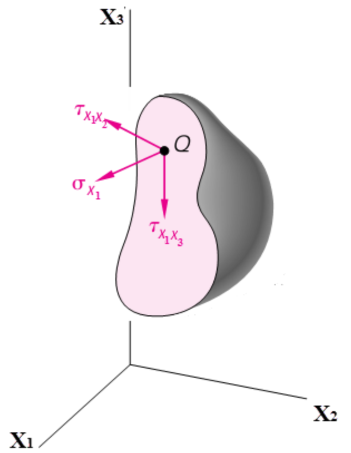


Figure .III.39. Choosing a plane passing through the point Q perpendicular to X_1.

$\sigma_{X1} = \frac{\Delta F\mathbf{X1}}{\Delta A}$ and : $\tau_{X1X2} = \frac{\Delta V_{X2}^{X1}}{\Delta A}$; $\tau_{X1X3} = \frac{\Delta V_{X3}^{X1}}{\Delta A}$;

These stresses are determined in the case of a plane perpendicular to the X_1 axis ($\vec{e}_1$).

The stress vector $\vec{T}_{\vec{e}1}$ will be composed with a normal stress σ_{X1} and two shear stresses τ_{X1X2} , τ_{X1X3}, noted as :

$\vec{T}_{\vec{e}1} = \begin{Bmatrix} \sigma_{X1} \\ \tau_{X1X2} \\ \tau_{X1X3} \end{Bmatrix};$

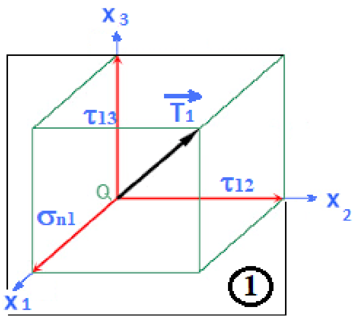


Figure .III.40. Presentation of the stress vector on facet 1.

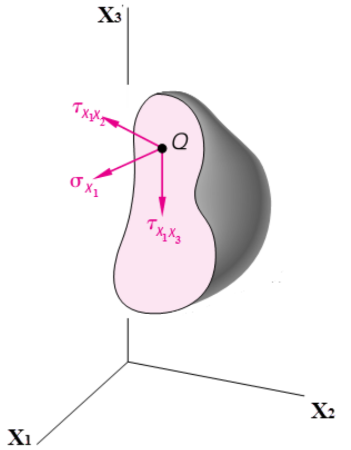


Figure .III.41. Normal stress σ_{X1} and two shear stresses τ_{X1X2} , τ_{X1X3}.

and so for the last axis X_3.

$\sigma_{X3} = \frac{\Delta F\mathbf{X3}}{\Delta A}$ and : $\tau_{X3X1} = \frac{\Delta V_{X1}^{X3}}{\Delta A}$; $\tau_{X3X2} = \frac{\Delta V_{X2}^{X3}}{\Delta A}$;

These stresses are determined in the case of a plane perpendicular to the X_3 axis ($\vec{e}_3$).

The stress vector $\vec{T}_{\vec{e}3}$ will be composed with a normal stress σ_{X3} and two shear stresses τ_{X3X2} , τ_{X3X1}, noted as :

$$\vec{T}_{\vec{e}3} = \begin{Bmatrix} \sigma_{X3} \\ \tau_{X3X2} \\ \tau_{X3X1} \end{Bmatrix};$$

Finally, we can say that to define the state of stress σ_{ij} in the point Q, we must consider an elementary cube of unit as edge subjected to the following stress tensor:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{X1} & \tau_{X1X2} & \tau_{X1X3} \\ \tau_{X2X1} & \sigma_{X2} & \tau_{X2X3} \\ \tau_{X3X1} & \tau_{X3X2} & \sigma_{X3} \end{bmatrix}$$

With the diagonal stresses $\sigma_{X1}, \sigma_{X2}, \sigma_{X3}$ they are called normal stresses, and the other stresses $\tau_{X1X2}, \tau_{X1X3}, \tau_{X1X2}, \tau_{X2X1}, \tau_{X2X3}, \tau_{X3X1}; \tau_{X3X2}$ they are called as “shear” stresses.

The stress tensor σ_{ij} is called as “3D Cauchy stress tensor” at point Q, the tensor is symmetric, i.e.;

$$\tau_{X1X2} = \tau_{X2X1}, \tau_{X1X3} = \tau_{X3X1}, \tau_{X2X3} = \tau_{X3X2}.$$

The component σ_{ij} of the Cauchy stress tensor, it represents the component along the direction $\vec{e}_i$ of the vector stress on the facet of normal $\vec{e}_j$ [1].

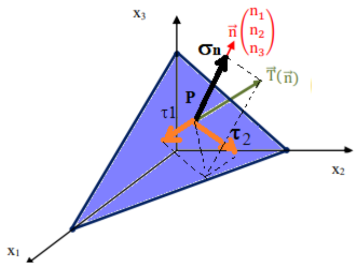


Figure .III.42. The stress vector $\vec{T}_{\vec{n}}$ on a plane (facet) of arbitrary orientation.

IV.2 The stress vector in 3D:

The relationship between the stress tensor σ_{ij} at a point P and the stress vector $\vec{T}_{\vec{n}}$ on a plane of arbitrary orientation (facet) at that point by:

$$\vec{T}_{\vec{n}} = \begin{Bmatrix} \sigma_n \\ \tau_1 \\ \tau_2 \end{Bmatrix} \text{ where : } \vec{T}_{\vec{n}} = \sigma.\vec{n}$$

But in general case the stresses state is defined by an elementary cube with three facets, for each facet we have a stress vector and an elementary direction (Figure.III.43.);

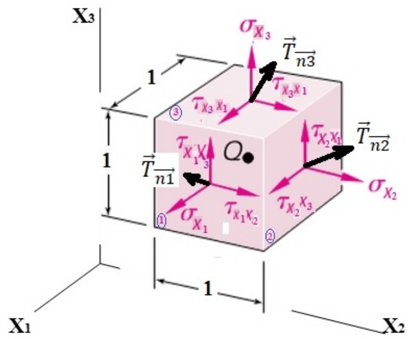


Figure .III.43. The stresses state is defined by an elementary cube.

The vector $\vec{T}_{\vec{n}}$ depends linearly with $\vec{n}$, therefore exists a linear application, moving from $\vec{T}_{\vec{n}}$ to $\vec{n}$ with :

$\vec{T}_{\vec{n}} = \sigma.\vec{n}$ ➔

$$\begin{Bmatrix} T_{\vec{n}1} \\ T_{\vec{n}2} \\ T_{\vec{n}3} \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11}n_1 + \tau_{12}n_2 + \tau_{13}n_3 \\ \tau_{21}n_1 + \sigma_{22}n_2 + \tau_{23}n_3 \\ \tau_{31}n_1 + \tau_{32}n_2 + \sigma_{33}n_3 \end{Bmatrix}$$

The component σ_{ij} of the Cauchy stress tensor, it represents the component along the direction $\vec{e}_i$ of the vector stress on the facet of normal $\vec{e}_j$ [1].

IV. 3D Stress transformation law:

At the point P let the rectangular Cartesian coordinate systems $(O, \vec{X}_1, \vec{X}_2, \vec{X}_3)$ et $(O, \vec{x}'_1, \vec{x}'_2, \vec{x}'_3)$ of Figure.III.44. be related to one another by the table of direction cosines:

	$\vec{x}'_1$	$\vec{x}'_2$	$\vec{x}'_3$
$\vec{X}_1$	a_{11}	a_{12}	a_{13}
$\vec{X}_2$	a_{21}	a_{22}	a_{23}
$\vec{X}_3$	a_{31}	a_{32}	a_{33}

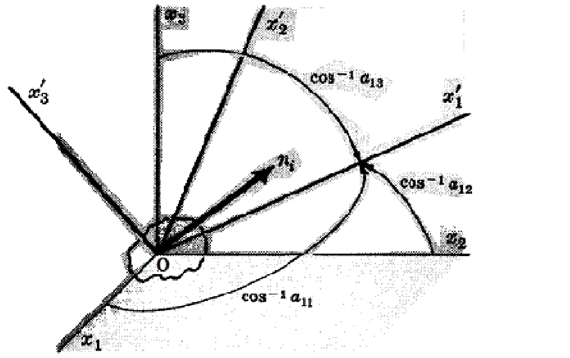


Figure .III.44. Axis transformation direct cosines.

The stress vector in the two bases can be written: $\vec{T}'_{\vec{n}} = a_{ij}\vec{T}_{\vec{n}}$.

$$\begin{Bmatrix} T'_{n1} \\ T'_{n2} \\ T'_{n3} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} T_{n1} \\ T_{n2} \\ T_{n3} \end{Bmatrix}$$

⇨ The stress tensor components in the two systems are related by : $\sigma'_{ij} = a_{ip}\,\sigma_{pq} a_{qj}$ ➔ $\sigma' = A\,\sigma\,A^T$

$$\begin{bmatrix} \sigma'_{11} & \tau'_{12} & \tau'_{13} \\ \tau'_{21} & \sigma'_{22} & \tau'_{23} \\ \tau'_{31} & \tau'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{31} & a_{23} & a_{33} \end{bmatrix} [2].$$

The complexity of 3D Stress Transformation equations comes from the presence of multiple stress components in all three dimensions. Whereas in 2D stress transformation we focus on a single plane (often X_1-X_2 plane), in 3D stress transformation we have to consider not only the X_1X_2 plane but also X_1X_3 and X_2X_3 planes. This general stress state comprises normal stresses (σ_{X1}, σ_{X2}, and σ_{X3}) and shear stresses in each plane.

The transformation equations for 3D stresses are derived following a similar method to 2D transformation but are more complicated due to the involvement of more stress components and mathematical operations. The intent remains the same: we transform the known state of stress into another coordinate system, rotated by certain angles with respect to the original coordinate system.

The orientation of plane ABC may be defined in terms of the angles between a unit normal $\vec{n}$ to the plane and the X_1, X_2, and X_3 directions. The direction cosines associated with these angles are:

$$\cos(\phi) = \cos(\vec{n}, \acute{X}_1) = l$$
$$\cos(\beta) = \cos(\vec{n}, \acute{X}_2) = m$$
$$\cos(\theta) = \cos(\vec{n}, \acute{X}_3) = n$$

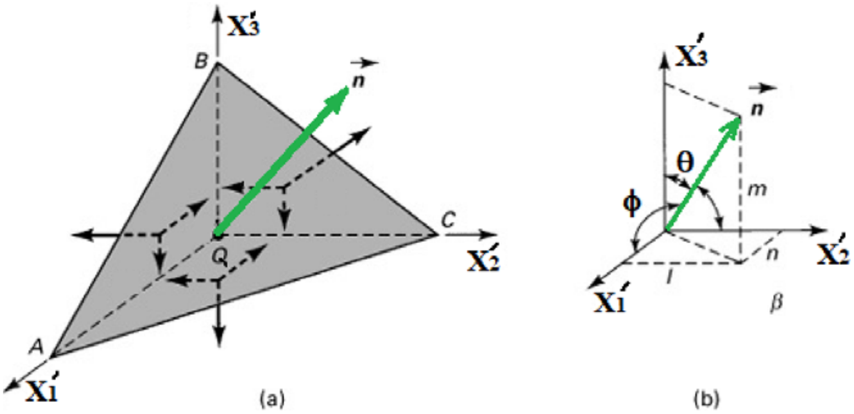


Figure.III.45. Stress components on a tetrahedron.

The three direction cosines for the $\vec{n}$ direction are related by: $l^2 + m^2 + n^2 = 1$ Known as the Lamé Formula in 3dimensions.

Notation for Direction Cosines:

	X_1	X_2	X_3
X'_1	l_1	m_1	n_1
X'_2	l_2	m_2	n_2
X'_3	l_3	m_3	n_3

The six transformation equations for the normal and shear stresses in a new coordinate system are as follows:

$$\begin{cases} \acute{\sigma}_{X1'} = \sigma_{X1}l_1^2 + \sigma_{X2}m_1^2 + \sigma_{X3}n_1^2 + 2(\tau_{X1X2}l_1m_1 + \tau_{X2X3}m_1n_1 + \tau_{X1X3}l_1n_1) \\ \acute{\sigma}_{X2'} = \sigma_{X1}l_2^2 + \sigma_{X2}m_2^2 + \sigma_{X3}n_2^2 + 2(\tau_{X1X2}l_2m_2 + \tau_{X2X3}m_2n_2 + \tau_{X1X3}l_2n_2) \\ \acute{\sigma}_{X3'} = \sigma_{X1}l_3^2 + \sigma_{X2}m_3^2 + \sigma_{X3}n_3^2 + 2(\tau_{X1X2}l_3m_3 + \tau_{X2X3}m_3n_3 + \tau_{X1X3}l_3n_3) \\ \acute{\tau}_{X1'X2'} = \sigma_{X1}l_1l_2 + \sigma_{X2}m_1m_2 + \sigma_{X3}n_1n_2 + \tau_{X1X2}(l_1m_2 + m_1l_2) + \tau_{X2X3}(m_1n_2 + n_1m_2) + \tau_{X1X3}(n_1l_2 + l_1n_2) \\ \acute{\tau}_{X1'X3'} = \sigma_{X1}l_1l_3 + \sigma_{X2}m_1m_3 + \sigma_{X3}n_1n_3 + \tau_{X1X2}(l_1m_3 + m_1l_3) + \tau_{X2X3}(m_1n_3 + n_1m_3) + \tau_{X1X3}(n_1l_3 + l_1n_3) \\ \acute{\tau}_{X2'X3'} = \sigma_{X1}l_2l_3 + \sigma_{X2}m_2m_3 + \sigma_{X3}n_2n_3 + \tau_{X1X2}(m_2l_3 + l_2m_3) + \tau_{X2X3}(n_2m_3 + m_2n_3) + \tau_{X1X3}(l_2n_3 + n_2l_3) \end{cases}$$ [6].

Example:

A stress state at a point Q is given by : $\sigma_{ij} = \begin{bmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{bmatrix} MPa;$

The rotation (cosine) matrix is given by: $a_{ij} = \begin{bmatrix} 0.57 & 0.57 & 0.57 \\ 0.7 & -0.7 & 0 \\ 0.41 & 0.41 & 0.82 \end{bmatrix}$

What are the stresses on the new orthonormal axes?

Solution:

Using the stress transformation law:

$$\begin{bmatrix} \sigma'_{11} & \tau'_{12} & \tau'_{13} \\ \tau'_{21} & \sigma'_{22} & \tau'_{23} \\ \tau'_{31} & \tau'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$$

We find: $\sigma'_{ij} = \begin{bmatrix} 93.3 & 32.7 & -18.9 \\ 32.7 & 30 & 40.4 \\ -18.9 & 40.4 & 76.7 \end{bmatrix} MPa$[7].

Uses and applications of 3D Stress transformation equations in Engineering:

3D Stress Transformation equations play a pivotal role in solving complex engineering problems. As real-world engineering objects are composed of three-dimensional elements, the application of 3D equations is ubiquitous in engineering. They help in designing materials that can withstand complex loading conditions in material science; in mechanical engineering, these equations facilitate to solve problems related to the design of machine elements, structures, and mechanical systems as well as in civil engineering and aerospace engineering. Their understanding can significantly enhance your problem-solving ability and creative thinking in the field of engineering [8].

IV.3 Principal Stresses, stress invariants, stress ellipsoid:

By using the «Analytical method "; the symmetry of the stress tensor means that it has three eigen values $\sigma_I, \sigma_{II}, \sigma_{III}$, real, called principal stresses. By taking these three vectors (principal directions) as a coordinate system (principal coordinate system) (Figure .III.46.), the stress tensor becomes diagonal.

- The stress tensor at any coordinate system :

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}$$

- The stress tensor at a principal coordinate system:

$$\sigma^*_{ij} = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

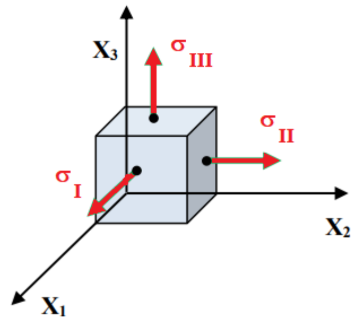


Figure .III.46. Principal coordinate system.

The knowledge of principal stresses is very important and essentially concerns the calculation of structures in RDM. An important example is the application of the criterion for the appearance of the rupture.

And we can find a particle containing the point (P) and oriented in such a way that these facets will only be subject to the normal stresses (principal stresses) in exclusion of all shear stresses.

As conclusion: we find a particle oriented as the stress vector of the facets is the same as the normal-principal- stresses.

$\overrightarrow{T^*}_{\vec{n}} = \sigma_{ij}.\vec{n}_j = \sigma_i.\vec{n}_i$ with $\vec{n}_i = \delta_{ij}.\vec{n}_j$

Then: $(\sigma_{ij} - \delta_{ij}.\sigma_i)\,\vec{n}_j = \vec{0}$

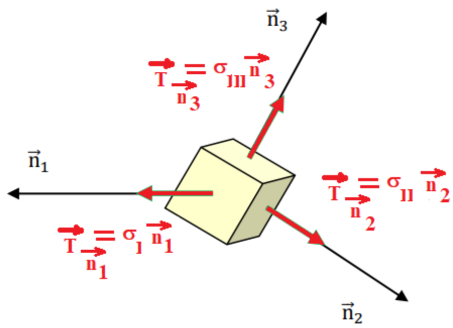


Figure .III.47.A particle oriented as the stress vector of the facets is the same as the normal principal stresses.

For $\vec{n}_j = \vec{0}$ the solution is trivial, and since $(\sigma_{ij} - \delta_{ij}.\sigma_i)$ is a matrix where the unknowns are the principal stresses σ_i; this allows us to say that its determinant must be zero.

$$\rightarrow \begin{vmatrix} \sigma_{11}-\sigma & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22}-\sigma & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0,$$

We will obtain a polynomial equation of order three called the characteristic equation:

$$\sigma^3 - I\,\sigma^2 + II\,\sigma - III = 0$$

In convention, the values of the principal constraints are ordered as follows: $\sigma_I > \sigma_{II} > \sigma_{III}$.

Where I, II and III are called respectively the first, second and third stress invariant, with:

$$I = trace(\sigma_{ij}) = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$
$$II = \frac{1}{2}(\sigma_{ii}\,\sigma_{jj} - \sigma_{ij}\sigma_{ij}) = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - \tau_{12}^2 - \tau_{23}^2 - \tau_{13}^2$$
$$III = |\sigma_{ij}| = \det(\sigma_{ij}) = \sigma_{11}\sigma_{22}\sigma_{33} + 2\tau_{12}\tau_{23}\tau_{13} - \tau_{12}^2\sigma_{33} - \tau_{23}^2\sigma_{11} - \tau_{13}^2\sigma_{22}$$

The three roots of the characteristic equation are the principal stress values; each value of the principal stresses is associated with a principal direction for which the direction cosines are the solutions of the equations:

$$(\sigma_{ij} - \delta_{ij}.\sigma_i)\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \vec{0}$$

In the principal stress space where the axes are the principal directions, the stress vector will have the following three components:

$$\overrightarrow{T^*}_{\vec{n}} = \begin{Bmatrix} T^*{}_{n1} = \sigma_I.n_1 \\ T^*{}_{n2} = \sigma_{II}.n_2 \\ T^*{}_{n3} = \sigma_{III}.n_3 \end{Bmatrix}$$

We take into consideration the orthogonality condition in 3 dimensions: $\vec{n}\cdot\vec{n} = 1$ i.e. $n_i.n_i = 1$ or:

$$(n_1)^2 + (n_2)^2 + (n_3)^2 = 1,$$

The stress vector must satisfy:

$\frac{(T_{n1})^2}{(\sigma_I)^2} + \frac{(T_{n2})^2}{(\sigma_{II})^2} + \frac{(T_{n3})^2}{(\sigma_{III})^2} = 1$, this equation is known as the Lamé's stress ellipsoid **[4]**.

Maximum shear stress:

The equation: $\tau_{max} = \frac{(\sigma_I - \sigma_{III})}{2}$ gives the maximum shear value, which is equal to half the difference of the largest and smallest principal stresses.

IV.3 MOHR's circles for 3D stress state:

Where the state of stress at a point P is given by the following stress tensor:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}$$

Each line represents the state of stresses on each facet of elementary cube as showed which will be described in each circle.

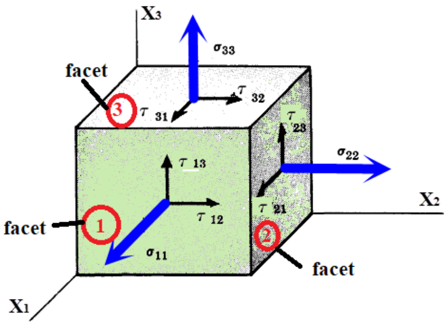


Figure .III.48. The state of stress at a point P.

The stress vector has normal and shear components whose magnitudes satisfy the equations:

$$\sigma_N = \sigma_I n_1^2 + \sigma_{II} n_2^2 + \sigma_I n_3^2$$

And

$$\sigma_N^2 + \tau^2 = \sigma_I{}^2 n_1^2 + \sigma_{II}{}^2 n_2^2 + \sigma_{III}{}^2 n_3^2;$$

Combining these two expressions with the orthogonality condition: $n_i n_j = 1$ and solving for the direction cosine n_i, results in the equations:

$$(n_1)^2 = \frac{(\sigma_N - \sigma_{II})(\sigma_N - \sigma_{III}) + (\tau)^2}{(\sigma_I - \sigma_{II})(\sigma_I - \sigma_{III})};$$

$$(n_2)^2 = \frac{(\sigma_N - \sigma_{III})(\sigma_N - \sigma_I) + (\tau)^2}{(\sigma_{II} - \sigma_{III})(\sigma_{II} - \sigma_I)};$$

$$(n_3)^2 = \frac{(\sigma_N - \sigma_I)(\sigma_N - \sigma_{II}) + (\tau)^2}{(\sigma_{III} - \sigma_I)(\sigma_{III} - \sigma_{II})};$$

These equations serve as the basis for Mohr's stress circles, as discussed in the "stress plane "section, for which the σ_n axis is the abscissa, and the τ axis is the ordinate.

To develop this graphical interpretation of the three-dimensional stress state in terms of σ_n and τ, we note that the denominators of the above equations are positive also that $n_i^2 > 0$, all of which tells us that

$$(\sigma_N - \sigma_{II})(\sigma_N - \sigma_{III}) + (\tau)^2 \geq 0$$

For the case where the equality sign holds, this equation may be rewritten, after some simple algebraic manipulations, to read

$$\left[\sigma_N - \frac{1}{2}(\sigma_{II} + \sigma_{III})\right]^2 + \tau^2 = \left[\frac{1}{2}(\sigma_{II} - \sigma_{III})\right]^2$$

Which is the equation of a circle in the σ_n, τ plane, with its center at the point on the σ_n axis, and having a radius $\frac{1}{2}(\sigma_{II} - \sigma_{III})$. We label this circle C1 and display it in Figure.III.49.

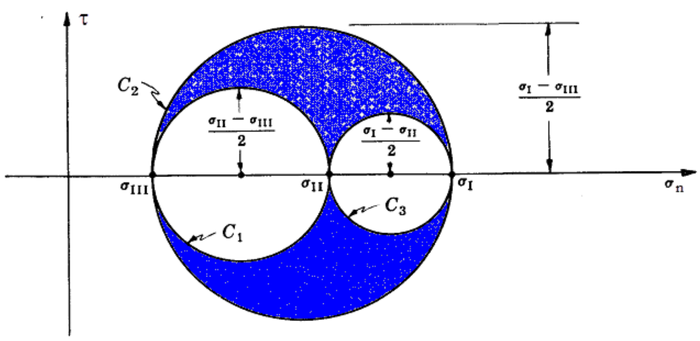


Figure .III.49. Mohr's stress tricicles.

We observe that conjugate pairs of values of σ_n and τ which satisfy this relationship result in stress points having coordinates exterior to circle C1.
Same work for the second and the third equations is done.
The three circles defined above, and shown in Figure.III.38, are called *Mohr's circles for stress*.

In order to relate a typical stress point having coordinates σ_n and τ in the stress plane of Figure.III.49. to the orientation of the area element ΔS upon which the stress components σ_n and τ act, we consider a small spherical portion of the continuum body centered at P. As the unit normal n_i assumes all possible directions at P, the point of intersection of its line of action with the sphere will move over the surface of the sphere.

Accordingly, we may restrict our attention to the first octant of the spherical body, as shown in Figure.III.50. (Octant of small spherical portion of body together with plane at P with normal n_i referred to principal axes (O, x_1^*, x_2^*, x_3^*).

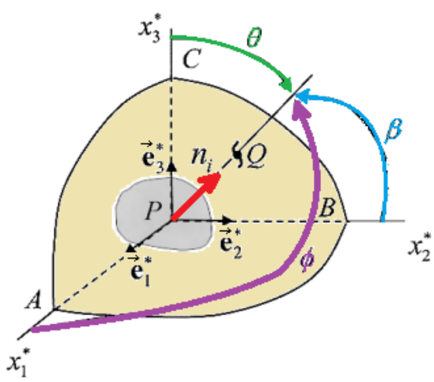


Figure .III.50. The first octant of the spherical body.

Let Q be the point of intersection of the line of action of

n_i with the spherical surface ABC in Figure.III.50, and note that

$$\vec{n} = \cos\emptyset\,.\,\vec{e}_1^* + \cos\beta\,.\,\vec{e}_2^* + \cos\theta\,.\,\vec{e}_3^*$$

The Mohr's stress semicircles for octant of small spherical portion are presented in Figure.III.51.

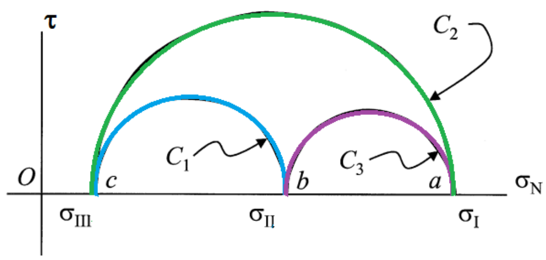


Figure .III.51. The Mohr's stress semicircles for octant of small spherical portion.

IV.4. The stress vector for a given direction from the Mohr's stress tricircles:

The Figure.III.52 shows the reference angles φ and β for intersection point Q on surface of body octant.

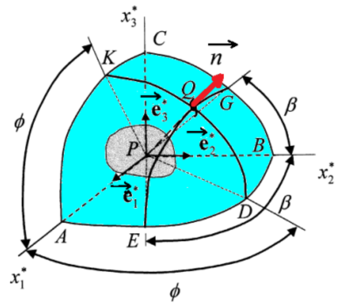


Figure .III.52. The reference angles φ and β for intersection point Q on surface of body octant.

The Mohr's semicircle for the precedent stress state displayed in Figure.III.53.

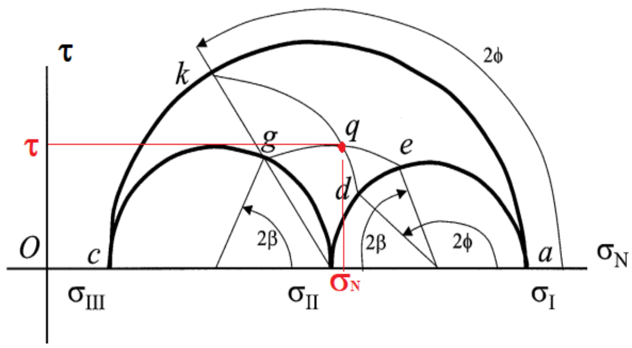


Figure .III.53. The Mohr's stress semicircles for octant of small spherical portion.

In summary, for a specific at point P in the body, point Q, where the line of action of intersects the spherical octant of the body (Figure.III. 52.), is located at the common point of circle arcs KD and EG, and at the same time, the corresponding stress point q (having coordinates σ_N and τ) is located at the intersection of circle arcs kd and eg, in the stress plane of Figure.III.54. The following example provides details of the procedure.

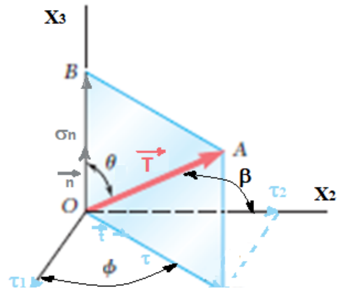


Figure .III.54. Construction of the stress vector from a given vector on the Mohr's stresses tricircles.

Example:

The state of stress at point P is given in MPa with respect to axes $Px_1x_2x_3$ by the matrix:

$$\sigma_{ij} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & -60 \\ 0 & -60 & 5 \end{bmatrix}$$

1/ determine the stress vector on the plane whose normal is $\vec{n} = \frac{1}{3}(2.\vec{e}_1 + \vec{e}_2 + 2.\vec{e}_3)$?

2/ determine the normal stress component σ_N and shear component τ on the same plane?

3/ Verify the results of part (2) by the Mohr's circle construction?

Solution:

1/ using the equation of the stress vector we find: $\overrightarrow{T^*}_{\vec{n}} = \sigma_{ij}.\vec{n}_j$

$$\begin{Bmatrix} T_{1\vec{n}} \\ T_{2\vec{n}} \\ T_{3\vec{n}} \end{Bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & -60 \\ 0 & -60 & 5 \end{bmatrix} \begin{Bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 50 \\ -150 \\ -50 \end{Bmatrix}$$

2/ the normal stress σ_N is determinate by:

$$\sigma_N = \begin{Bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{Bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & -30 & -60 \\ 0 & -60 & 5 \end{bmatrix} \begin{Bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{Bmatrix} = -\frac{150}{9} = -16.67\text{MPa} ;$$

The shear stress we use:

$$\tau^2 = \vec{T}_{\vec{n}}.\vec{T}_{\vec{n}} - \sigma_N^2 = \frac{1}{9}\left((50 * 50) + (150 * 150) + (50 * 50)\right) - (16.67)^2 = 2777;$$

Finally; $\tau = 52.7$ MPa

3/ we can verify that for the stress tensor σ_{ij} given here the principal stress values are $\sigma_I = 50, \sigma_{II} = 25$ and $\sigma_{III} = -75$.

Therefore, $\phi = \cos^{-1}\left(\frac{1}{3}\right) = 70.53\,^{o}$; $\theta = \beta = \cos^{-1}\left(\frac{2}{3}\right) = 48.19°$, so that — following the procedure outlined for construction of Fig.44. — we obtain Fig.45., from which we may measure the coordinates of the stress point q and confirm the values $\sigma_N = -16.7$ and $\tau = 52.7$, both in MPa.

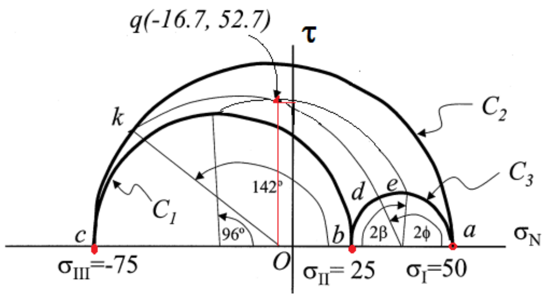


Figure .III.55. Measure the coordinates of the stress at point q.

We can check the results by visiting the site: https://valdivia.staff.jade-hs.de/mohr3d_en.html

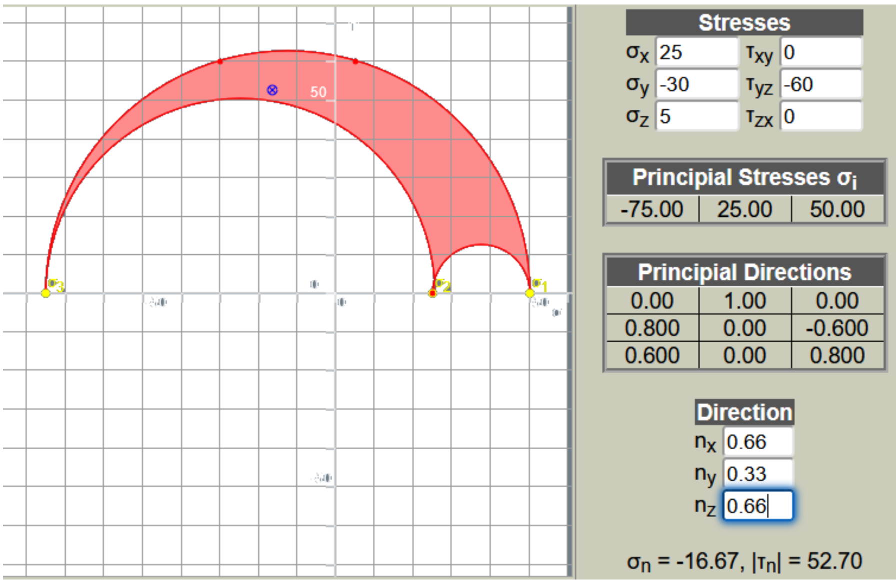


Figure.III.56. Checking the results.

Deviator and Spherical Stress State:

The arithmetic mean of the normal stresses,

$$\sigma_M = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii}$$

The state of stress having all three principal stresses equal (and therefore equal to σ_M) is called a *spherical state of stress*, represented by the diagonal matrix

$$\sigma_{ij} = \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix}$$

for which all directions are principal directions.

Every state of stress σ_{ij} may be decomposed into a spherical portion S_{ij} and a portion D_{ij} known as the *deviator stress* in accordance with the equation

$$\sigma_{ij} = D_{ij} + \delta_{ij}\sigma_M = D_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}$$

We notice immediately that the first invariant of the deviator stress is (the deviator diagonal is null)➔

$$D_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ii}\sigma_{kk} = 0$$

Example:

Decompose the following stress tensor:

$$\sigma_{ij} = \begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix}$$

into its deviator and spherical portions and determine the principal stress values of the deviator portion?

The arithmetic of mean normal stress is

$$\sigma_M = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}(57 + 50 + 43) = 50$$

The spherical state of stress

$$S_{ij} = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

The *deviator stress*

$\sigma_{ij} = D_{ij} + S_{ij}$➔ $D_{ij} = \sigma_{ij} + \delta_{ij}\sigma_M$

$$D_{ij} = \begin{bmatrix} 57 & 0 & 24 \\ 0 & 50 & 0 \\ 24 & 0 & 43 \end{bmatrix} - \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 24 \\ 0 & 0 & 0 \\ 24 & 0 & -7 \end{bmatrix}$$

A convenient two-dimensional graphical representation of three-dimensional state of stress at a point is provided by the well-known *Mohr's stress circles*[2].

Bibliographic references:

[1] Nichoals Moés, 2011. Mécanique des milieux continus. Ecole centrale de Nantes.

[2] George E. Mase, 1970. Theory and problems of continuum mechanics. McGraw-Hill Book Company.

[3] Samir DEGHBOUDJ,2016, POLYCOPIE DE COURS « MECANIQUE DES MILIEUX CONTINUS COURS ET APPLICATIONS ».

[4] Michel Maya, 2014. Cours de Mécanique des Milieux continus. Arts et Métiers Paris Tech, Centre d'Enseignement et de recherche de CLUNY.

[5] https://www.studysmarter.fr/resumes/ingenierie/mecanique-des-solides/contrainte-plane/

[6]https://www.informit.com/articles/article.aspx?p=1729271&seqNum=12

[7] https://cdnx.uobabylon.edu.iq/lectures/ehzS21OxB0mXpp7V5q3K4w.pdf

[8] https://www.studysmarter.co.uk/explanations/engineering/solid-mechanics/stress-transformation-equations/

[9] file:///F:/MMC%20X/Continuum%20Mechanics%20for%20Engineers%20(1999).pdf

Chapitre IV:

Strain Analysis

Chapter IV: Strain Analysis

I. Introduction :

Deformation is the change in the shape or size of an object. It is quantified as the residual displacement of particles in a non-rigid body, from an initial configuration to a final configuration.
A deformation can occur because of external loads, intrinsic activity (e.g. muscle contraction), body forces (such as gravity or electromagnetic forces), or changes in temperature, moisture content, or chemical reactions, etc.
In a continuous body, a deformation field results from a stress field due to applied forces or because of some changes in the conditions of the body [1].

Elastic and plastic deformation:
Deformations which cease to exist after the stress field is removed are termed as **elastic** deformation. In this case, the continuum completely recovers its original configuration. On the other hand, irreversible deformations remain. They exist even after stresses have been removed. One type of irreversible deformation is **plastic** deformation, which occurs in material bodies after stresses have attained a certain threshold value known as the elastic limit or yield stress.
Limiting the range of deformations in certain structures and machines proves to be very essential to protect them from failure (disaster) so their deformations must be respected something which pushes us to study and know how to manipulate the deformations. In this chapter we will discuss the dimensional part and introduce notions such as the Eulerian and Lagrangian descriptions of the movements of a continuous medium, the notion of deformation of a continuous medium, the 'deformation rate' tensor, etc. …the temporal part will be addressed in the following chapter [2].

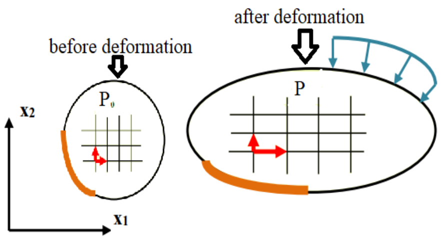


Figure.IV.1. Body before and after deformation.

II. Definitions:

a. Point and particle:

A point is a place in space, but a particle is a small part of a continuous medium or as a material point which have a mass and neglected dimensions.

b. Configuration:

At any instant of time t, a continuous medium having volume V and surface S will occupy a certain region R of physical space. Identifying the particles of the continuous medium with the points in the space it occupies at time t by reference to an appropriate set of coordinate axis is said to specify the configuration of the continuous medium at that time.
The term **deformation (strain)** refers to a change in shape of the continuous medium between an initial (undeformed) configuration and a later (deformed) configuration.

Flow: state of continuity in the movement of a continuous medium.

c. Movement of a continuous medium:

To completely describe the movement of a continuous medium, in a given reference system (a Cartesian coordinate system) it is enough to give the position of the different material particles at each instant. A simple way to distinguish the different material particles that make up a portion of matter is to locate them by means of their position X at an arbitrary reference time t=0. We will therefore agree to use capital letters such as X to designate the material particles,

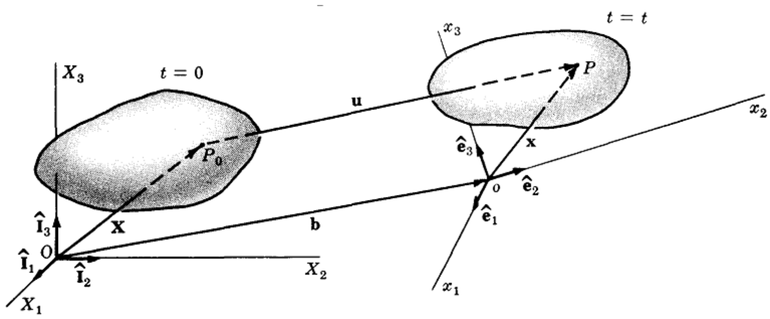


Figure .IV.2. Description of the motion of continuum medium.

or their position at an initial instant t=0.

Given these conventions, the movement of a continuous medium is described entirely by the data of the positions x at any instant t of all the material particles X, that is to say by the data of the relations: x =x(X , t) [1].

d. Lagrangian and Eulerian description :

When a continuous medium deforms, these particles move following variable trajectories in space. The movement can be expressed by equations of the form: $x_i = x_i(X_1, X_2, X_3, t)$, which represents the position x_i of the particle occupied by the point (X_1,X_2,X_3) at time t=0. They can be interpreted as a mapping from the initial configuration to the current configuration. Known as the Lagrangian "description" formulation.

If, on the contrary, the movement or deformation is given by equations of the form: $X_i = X_i(x_1, x_2, x_3, t)$.

In which the independent variables are the ordinates x_i and t, the description is called an Eulerian "description" formulation. This description can be thought of as a description to trace the original position of the particle which now occupies location (x_1, x_2, x_3). If the Eulerian description is a continuous one-to-one mapping with continuous partial derivatives, as was also assumed for the Lagrangian description, the two descriptions – mappings – are unique inverses of each other. A necessary and sufficient condition for inverse functions to exist is that the Jacobian $J = \left|\frac{\partial x_i}{\partial X_j}\right|$, should not disappear (equal to zero).

As a simple example, the Lagrangian description is given by the equations:

$$x_1 = X_1 + X_2(e^t - 1)$$

$$x_2 = X_1(e^{-t} - 1) + X_2$$

$$x_3 = X_3$$

Has an inverse, the Eulerian description is given by the equations:

$$X_1 = \frac{-x_1+x_2(e^t-1)}{1-e^t-e^{-t}} \ ;$$

$$X_2 = \frac{x_1(e^{-t}-1)-x_2}{1-e^t-e^{-t}} \ ;$$

$$X_3 = x_3.$$

e. Position Vector:

In the initial configuration, a particle representative of the continuous medium occupies a point in space and has the *position vector*: $\vec{X} = X_k\vec{I}_k$; compared to the rectangular Cartesian axes $OX_1X_2X_3$. The coordinates ($X_1X_2X_3$) , called *the material coordinates*.
In the deformed configuration, the particle originally located at P_0 is located at point P and has the **position vector**: $\vec{x} = x_i\vec{e}_i$.
Referring to the Cartesian coordinates $ox_1x_2x_3$ often called *the spatial coordinates*.

The relative orientation of the material and spatial axes is specified by:
$\vec{e}_k.\vec{I}_K = \vec{I}_k.\vec{e}_K = \alpha_{kK} = \alpha_{Kk}$ where : α_{kK} are the directing cosines between the material axes and the spatial axes.
The orthogonality condition between the two system of coordinates is expressed by:

$$\alpha_{Kk}\alpha_{Kp} = \alpha_{kK}\alpha_{pK} = \delta_{kp} \ ;$$
$$\alpha_{Kp}\alpha_{Mp} = \alpha_{pK}\alpha_{pM} = \delta_{KM}$$

f. Displacement vector:

The vector linking the points P_0 and P (the initial and final position, respectively of the particle) is known as the displacement vector which can be noted: $\vec{u} = u_k\vec{e}_k$ or $\vec{U} = U_K\vec{I}_K$.

The components U_K and u_k are interconnected by the direction cosines by: $\vec{e}_k = \alpha_{kK}\vec{I}_K$, by substitution we will have:

$\vec{u} = u_k\vec{e}_k = u_k(\alpha_{kK}\vec{I}_K) = U_k\vec{I}_K = \vec{U}$ ➔ $U_k = \alpha_{kK}u_k$.
As the direction cosines are constants, the components of the displacement vector are observed to obey the degree of transformation of the first order Cartesian tensors.
The vector $\vec{b}$ is used to position the origin o relative to O, hence: $\vec{u} = \vec{b} + \vec{x} - \vec{X}$.

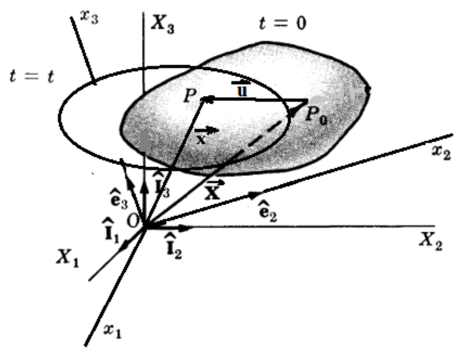


Figure.IV.3. The displacement vector.

Very often in continuum mechanics it is possible to consider the coordinate systems $OX_1X_2X_3$ and $ox_1x_2x_3$ superimposed, with $\vec{b}=\vec{0}$, so that: $\vec{u} = \vec{x} - \vec{X}$.
The Cartesian components will be: $u_k = x_k - \alpha_{kK} X_K$,
However, for the superimposed axes, the triads of the basis vectors of the two systems are identical, which gives: $u_k = x_k - X_K$ [1].

g. The concept of the strain vector:

In order to describe the deformation of a body by changes in length of line segments and the changes in the angles between them, we will develop the concept of strain vector.

The strain vector is defined by: $\vec{\boldsymbol{E}}_{\vec{n}}$ with its components the normal strain denoted (ε_n) and the shearing strains (γ).
This can be written as: $\vec{\boldsymbol{E}}_{\vec{n}} = \vec{\varepsilon}_n + \vec{\gamma} = \varepsilon_n.\vec{n} + \gamma.\vec{t}$

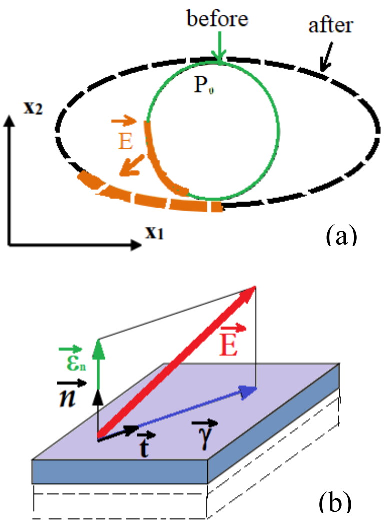


Figure.IV.4. The components of the strain vector.

Where $\vec{\varepsilon}_n$ is the *normal strain* and $\vec{\gamma}$ the *tangential* or *the shear strain*, the key difference between them is the direction of deformation relative to the applied force.

- $\vec{\varepsilon}_n$: the normal strain occurs parallel to the force and it causes a change in the length of the material without a change in its shape,
- $\vec{\gamma}$: the shear strain occurs perpendicular to the force and it causes a change in the shape of the material without a change in its volume [3].

III. The strain vector in 1 dimension:

1. The normal strain in a tensile bar:

The simplest case is that of a bar suspended vertically and subjected to a force (F) on its two ends (the force acts on one facet (which its normal unit vector is $\vec{n}$, which is parallel to the load axis).
If we note by (L_0) its length before loading and by (L) that after deformation, then the quantity ($\frac{L-L_0}{L}$) is called longitudinal deformation or extension or the normal strain denoted (ε_n).
For which the normal $\vec{n}$ is parallel to the load axis.
We wrote:

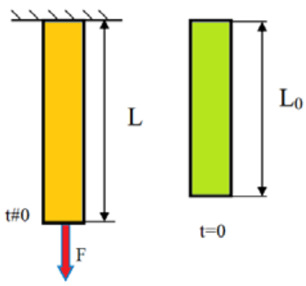


Figure.IV.5. Tensile bar.

$$\varepsilon_n = \frac{L - L_0}{L} = \frac{\Delta L}{L}$$

The strain vector in this case is unidirectional: $\vec{\boldsymbol{E}}_{\vec{n}} = \varepsilon_n \vec{e}_1^{\,*}$ where $\vec{e}_1^{\,*}$ is the unit vector perpendicular to the facet 1.
Note that the normal strain is a dimensionless quantity, since it is a ratio of two lengths. Although this is the case, it is sometimes stated in terms of a ratio of length units. If the SI system is used, then the basic unit for length is the meter (m). Ordinarily for most engineering applications ε_n will be very small, so measurements of strain are in micrometers or (μm/μm). Sometimes for experimental work, strain is expressed as a percent (0.001 m/m=0.1%).

Let we consider, for example; the line AB, which is contained within the undeformed body shown in Figure.IV.6a. This line lies along the $\vec{n}$ axis and has an original length of Δs. After deformation, points A and B are displaced to A' and B', and the line becomes a curve having a length of $\Delta s'$, Figure.IV.6b. The change in length of the line is therefore $\Delta s'$- Δs . If we define the average normal strain ε then: $\varepsilon = \frac{\Delta s' - \Delta s}{\Delta s'}$
Hence when ε is positive the initial line will elongate, whereas if ε is negative the line contracts.

(a) Undeformed body (b) deformed body

Figure.IV.6. The change in length of line before and after deformation [4].

2. The tangential or shear strain:

Deformations not only cause line segments to elongate or contract, but they also cause them to change direction. If we select two line segments that are originally perpendicular to one another, then the change in angle that occurs between these two line segments is referred to as **shear strain**. This angle is denoted by γ and is always measured in radians (rad), which are dimensionless. For example, consider the line segments AB and AC originating from the same point A in a body, and directed along the perpendicular n and t axes, Figure.IV.6a.
After deformation, the ends of both lines are displaced, and the lines themselves become curves, such that the angle between them at A is θ', Figure.IV.6b. Hence the shear strain at point A associated with the n and t axes becomes in this case:

$$\gamma = \frac{\pi}{2} - \theta';$$

(a) Undeformed body (b) deformed body

Figure.IV.7. The change in angle between two lines before and after deformation [4].

The strain vector in this case unidirectional: $\boldsymbol{E}_{\vec{n}} = \gamma \vec{e}_2^{\,*}$ where $\vec{e}_2^{\,*}$ is the tangential vector to the facet 1.
We can by keeping in mind, that the shear stress τ acting on the facet 1 may be related to the shear strain in the elastic domain of the material by the relation: $\gamma = \frac{\tau}{G}$ where G: is the coulomb elasticity transversal module, which is linked to the Young's longitudinal modulus of elasticity E by the formula G=E/2×(1+υ), involving Poisson's ratio υ, it is also called the Lamé's second modulus μ.

Or by considering the as shown in Figure.IV.8a we can put:

- the normal strains: $\varepsilon_{11} = \frac{\partial U_{X1}}{\partial X_1}$; and $\varepsilon_{22} = \frac{\partial U_{X2}}{\partial X_2}$,
- for the shear strains $tg\gamma_1 = \frac{\partial U_{x1}}{\partial X_2}$ and $tg\gamma_2 = \frac{\partial U_{x2}}{\partial X_1}$ for the small displacements we can put: $\gamma_1 = \frac{\partial U_1}{\partial X_2}$ and $\gamma_2 = \frac{\partial U_1}{\partial X_2}$, finally: $\gamma_{12} = \gamma_1 + \gamma_2 = \frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1}$.

And by superposition, for each facet we have the strain vector: $\vec{\boldsymbol{E}}_{\vec{n}} = \vec{\varepsilon}_n + \vec{\gamma} = \varepsilon_n.\vec{n} + \gamma_{12}.\vec{t}$.

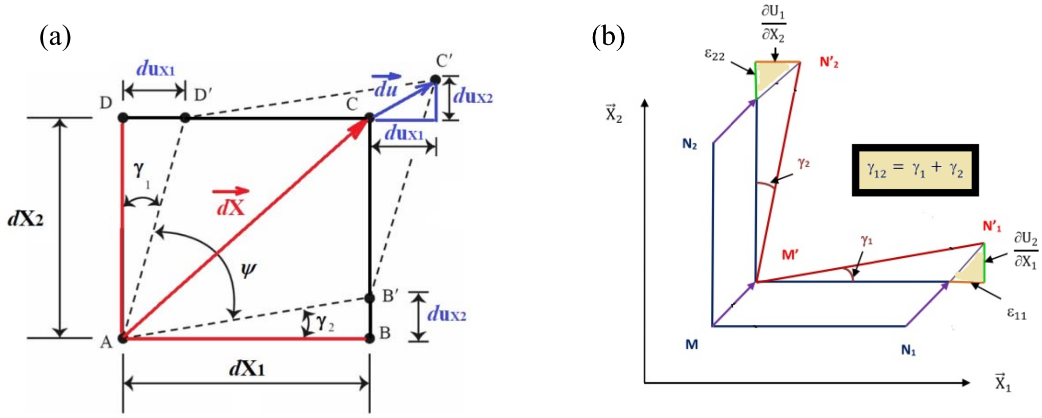


Figure.IV.8. Normal and shear strains in plane presentation [3].

Example:

Due to a loading, the plate is deformed into the dashed shape shown in Figure.IV.7. Determine:
- The normal strain along the side AB and AC?
- The shear strain in the plate at A relative to the X_1 and X_2 axes?

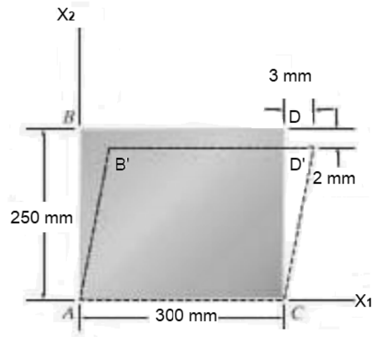


Figure.IV.9. Deformed plate.

Line AB, coincident with the X_2 axis, becomes line AB' after deformation, the length of AB' is:

$AB' = 250 - 2 = 248\ mm$

The normal strain for AC is :

$$\varepsilon_{11} = \frac{BD' - BD}{BD} = \frac{303 - 300}{300} = 1 \times 10^{-2} mm/mm$$

The positive sign indicates the strain causes a elongation of BD.

The normal strain for AB is therefore:

$$\varepsilon_{22} = \frac{AB' - AB}{AB} = \frac{248 - 250}{250} = -8 \times 10^{-3} mm/mm$$

The negative sign indicates the strain causes a contraction of AB.

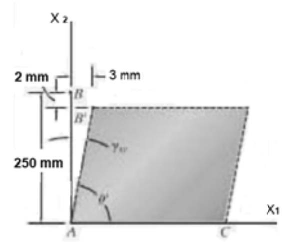


Figure.IV.10. The strains.

The once 90° angle BAC between the sides of the plate at A changes to θ' due to the displacement of B to B'. Since $\gamma_{xy} = \frac{\pi}{2} - \theta'$, then γ_{xy} is the angle in the Figure.IV.10.

Thus,

$$\gamma_{xy} = \tan^{-1}\left(\frac{3}{250-2}\right) = 0.0121\ rd\ [4].$$

IV. The strain vector in 2 dimensions (plane strain):

1. Strain vector on any facet:

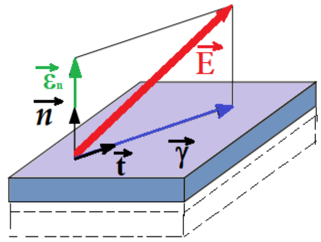


Figure.IV.11. The strain vector.

In some cases we have the strain applied on one area which has two directions as presented in the Figure.IV.11.

This means that the strain vector has two components ε_n as normal strain and γ as shear strain.

$$\vec{E}_{\vec{n}} = \varepsilon_n.\vec{n} + \gamma.\vec{t} \quad \text{noted by :} \quad \vec{E}_{\vec{n}} = \begin{Bmatrix} \varepsilon_n \\ \gamma \end{Bmatrix}$$

Example:

The rubber bearing support under the concrete bridge girder is subjected to both normal and shear strain. The normal strain is caused by the weight and bridge loads on the girder, and the shear strain is caused by the horizontal movement of the girder due to temperature changes.

Figure.IV.12. The strains in The rubber bearing support under the concrete bridge girder [4].

2. 2D (plane) strains:

Let us assume that a state of plane strain exists at point Q and the strain vectors are acting on two perpendicular facets, and that it defined by the strain components ε_{11} , $\varepsilon_{22}$and γ_{12}, associated with the X_1 and X_2 axes.

Important assumption:

Some authors arbitrarily assume that the actual deformation of the element is accompanied by a rigid-body rotation such that the horizontal faces of the element do not rotate.

The strain γ_{12} is then represented by the angle through which the other two faces have rotated (Figure.IV.13.).

Others assume a rigid-body rotation such that the horizontal faces rotate through $\frac{1}{2}\gamma_{12}$ counterclockwise and the vertical faces through $\frac{1}{2}\gamma_{12}$ clockwise (Figure.IV.13.), but often we prefer to use the second assumption in graphic method by Mohr's circle.

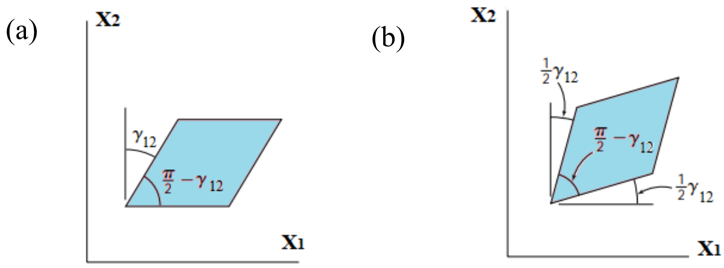


Figure.IV.13. The shear strain assumptions.

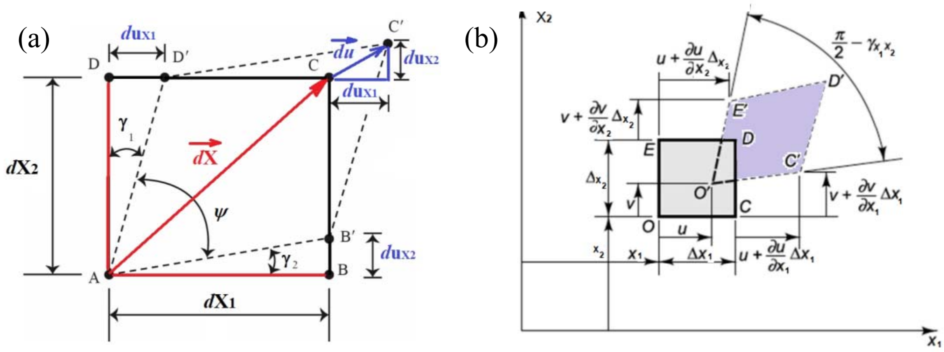


Figure.IV.14. Relation between strain and displacement vector in plane strain.

The linear strain tensor is given by :

$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \frac{\gamma_{12}}{2} \\ \frac{\gamma_{21}}{2} & \varepsilon_{22} \end{bmatrix}$ where ε_{11} and $\boldsymbol{\varepsilon_{22}}$ are called the normal strains they are perpendicular to the factes 1 and 2, $\boldsymbol{\gamma_{12}}$ and $\boldsymbol{\gamma_{21}}$ are called the shear strains on the facets 1 and 2 as represented on the Figure.IV.15.

In this case we have two important facets than we have two vector strains

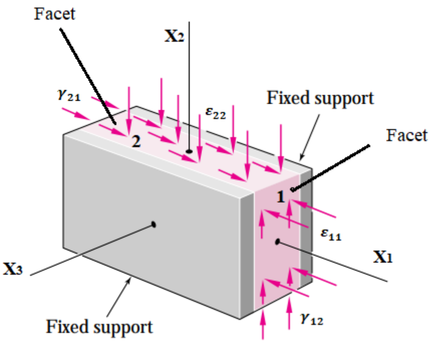


Figure.IV.15. The normal and shear strains in fixed support.

On the facet 1 we have: $\vec{\boldsymbol{E}}_{\overrightarrow{\boldsymbol{n1}}} = \begin{Bmatrix} \varepsilon_{11} \\ \gamma_{12} \end{Bmatrix}$ and on the facet 2 we have : $\vec{\boldsymbol{E}}_{\overrightarrow{\boldsymbol{n2}}} = \begin{Bmatrix} \varepsilon_{22} \\ \gamma_{21} \end{Bmatrix}$

3. Strain components associated with arbitrary sets of axes:

In this section we examine the problem of geometric compatibility at a point and determine relationships which must exist between strain components associated with the axes which are not perpendicular to each other.

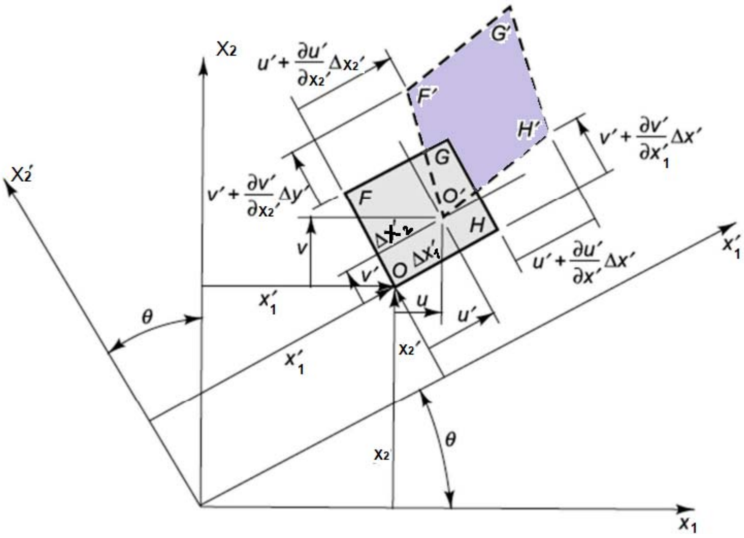


Figure.IV.16 . Plane strain with arbitrary sets axes [5].

Which means that to find the new coordinates of strains in the plane basis (O,X'$_1$, X'$_2$) (Figue.IV.16.) from the initial plane basis (O,X$_1$, X$_2$).

The transformation matrix is given by:

	$\vec{e}_1$	$\vec{e}_2$
$\vec{e}'_1$	$\cos\theta$	$\sin\theta$
$\vec{e}'_2$	$-\sin\theta$	$\cos\theta$

With the law transformation of a stress tensor, we know: $\varepsilon'_{ij} = A^t \varepsilon_{ij} A$:

$$\begin{bmatrix} \varepsilon'_{X1} & \frac{\gamma'_{X1X2}}{2} \\ \frac{\gamma'_{X2X1}}{2} & \varepsilon'_{X2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{X1} & \frac{\gamma_{X1X2}}{2} \\ \frac{\gamma_{X2X1}}{2} & \varepsilon_{X2} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\Rightarrow \begin{cases} \varepsilon'_{X1} = \frac{\varepsilon_{X1}}{2}(1+\cos 2\theta) + \frac{\varepsilon_{X2}}{2}(1-\cos 2\theta) - \frac{\gamma_{X1X2}}{2}\sin 2\theta \\ \varepsilon'_{X2} = \frac{\varepsilon_{X1}}{2}(1-\cos 2\theta) + \frac{\varepsilon_{X2}}{2}(1+\cos 2\theta) + \frac{\gamma_{X1X2}}{2}\sin 2\theta \\ \frac{\gamma'_{X2X1}}{2} = \frac{\varepsilon_{X1}}{2}\sin 2\theta - \frac{\varepsilon_{X2}}{2}\sin 2\theta + \frac{\gamma_{X1X2}}{2}\cos 2\theta \end{cases}$$

Using the following trigonometric formulas:

$$\cos\theta^2 = \frac{1}{2}(1+\cos 2\theta);$$
$$\sin\theta^2 = \frac{1}{2}(1-\cos 2\theta);$$
$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta;$$
$$\cos\theta^2 - \sin\theta^2 = \cos 2\theta;$$

We obtain:

$$\Rightarrow \begin{cases} \varepsilon'_{X1} = \left(\frac{\varepsilon_{X1}+\varepsilon_{X2}}{2}\right) + \left(\frac{\varepsilon_{X1}-\varepsilon_{X2}}{2}\right)\cos 2\theta + \frac{\gamma_{X1X2}}{2}\sin 2\theta \\ \varepsilon'_{X2} = \left(\frac{\varepsilon_{X1}+\varepsilon_{X2}}{2}\right) - \left(\frac{\varepsilon_{X1}-\varepsilon_{X2}}{2}\right)\cos 2\theta - \frac{\gamma_{X1X2}}{2}\sin 2\theta; \\ \frac{\gamma'_{X2X1}}{2} = -\left(\frac{\varepsilon_{X1}-\varepsilon_{X2}}{2}\right)\sin 2\theta + \frac{\gamma_{X1X2}}{2}\cos 2\theta \end{cases}$$

Maximum shear strain: $\frac{\gamma'_{max}}{2} = \sqrt{\left(\frac{\varepsilon_{X1}-\varepsilon_{X2}}{2}\right)^2 + \left(\frac{\gamma_{X1X2}}{2}\right)^2}$ [6].

Example:

Let consider a strain state at point Q given by:

$\varepsilon_{ij} = \begin{bmatrix} 340 & 180 \\ 180 & 110 \end{bmatrix} \times 10^{-6} \mu m/\mu m$; as represented in the following figure :

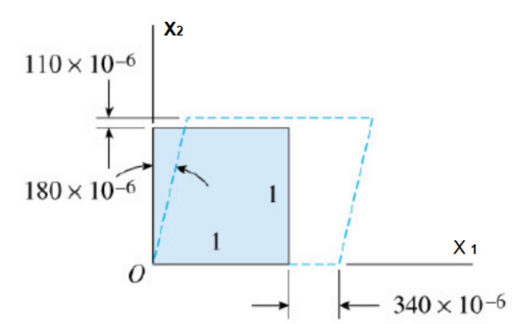


Figure.IV.17. Plane strain.

$\frac{\varepsilon_{X1}+\varepsilon_{X2}}{2} = \frac{(340+110)\times 10^{-6}}{2} = 225 \times 10^{-6}$ µm/µm;

$\frac{\varepsilon_{X1}-\varepsilon_{X2}}{2} = \frac{(340-110)\times 10^{-6}}{2} = 115 \times 10^{-6}$ µm/µm;

$\frac{\gamma_{X1X2}}{2} = 90 \times 10^{-6}$;

$$\Rightarrow \begin{cases} \varepsilon'_{X1} = (225 + 115\cos 2\theta + 90\sin 2\theta) \times 10^{-6} \\ \varepsilon'_{X2} = (225 - 115\cos 2\theta - 90\sin 2\theta) \times 10^{-6} \\ \frac{\gamma'_{X2X1}}{2} = (-115\sin 2\theta + 90\cos 2\theta) \times 10^{-6} \end{cases} ;[7].$$

4. Rotating a structural element through 180°:

An element is rotated through 180° at 10° intervals; the initial and calculated values of ε_{X1}, ε_{X2}, and $\gamma_{X1X2}/2$ are in units (µm/µm). Plotting the strain values for each ε_{X1}', ε_{X2}', and $\gamma'_{X1X2}/2$ against the rotation angle produces sinusoidal curves as represent on Figure.IV.18. and Figure.IV.19.

On the facet ❶ we find:

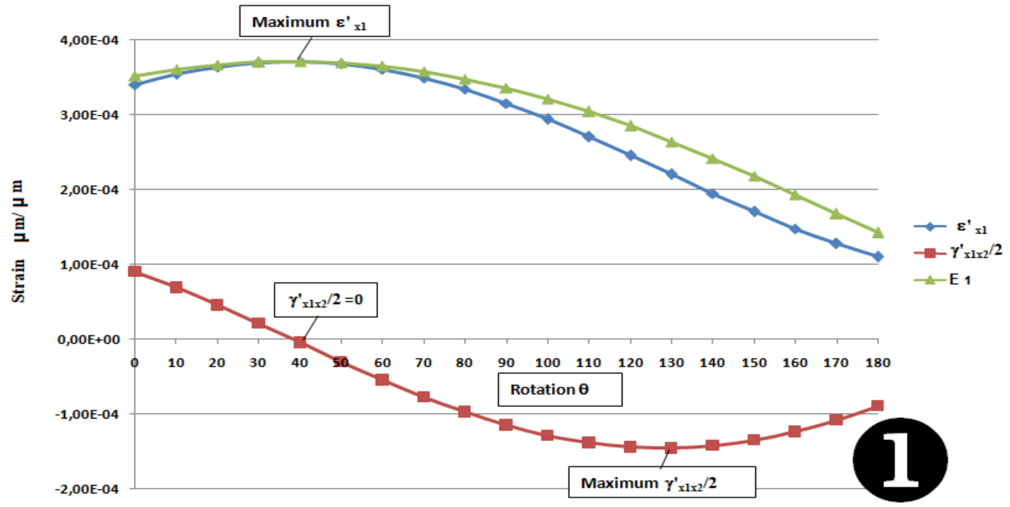


Figure.IV.18. Sinusoidal curves of normal and shear strains with the magnitude of the strain vector on facet (1).

The maximum normal strain ε_{X1}' is 3.71×10^{-4} µm/µm at the angle 40°, the maximum shear strain $\gamma'_{X1X2}/2$ is 1.46×10^{-4} µm/µm located at the angle 130°.

On the facet ❷ we have:

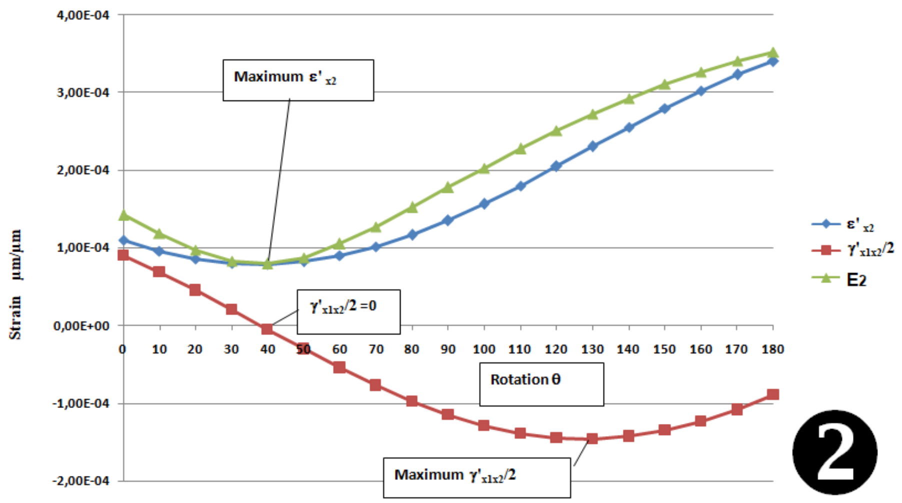


Figure.IV.19. Sinusoidal curves of normal and shear strains with the magnitude of the strain vector on facet (2).

The maximum normal strain ε_{X2}' is 7.91×10^{-5} µm/µm at the angle 40°,
The maximum shear strain $\gamma'_{X1X2}/2$ is 1.46×10^{-4} µm/µm located at the angle 130°, the same as on the facet 1.
We can verify The **Maximum shear strain by using the formula:**

$$\frac{\gamma'_{max}}{2}=\sqrt{\left(\frac{\varepsilon_{X1}-\varepsilon_{X2}}{2}\right)^2+\left(\frac{\gamma_{X1X2}}{2}\right)^2}=\sqrt{(115\times10^{-6})^2+(90\times10^{-6})^2}=\mathbf{1.46\times10^{-6}}\mu\frac{m}{\mu m}[6].$$

To verify the different results you can visit the web site:
https://www.efunda.com/formulae/solid_mechanics/mat_mechanics/calc_strain_transform.cfm#calc

5. Principal strains in plane (2D) :

To determine the principal strains in 2D, we have to use the analytical method as discussed for plane stresses state, since, we know the strain state in a point and we find the principal strains:

Let consider a plane strain state: $\varepsilon_{ij} = \begin{bmatrix} -240 & -330 \\ -330 & 320 \end{bmatrix} \times 10^6 \mu m/\mu m$ ➔ $(\varepsilon_{ij} - \delta_{ij}\varepsilon)\vec{n} = \vec{0}$

$\begin{pmatrix} -240-\varepsilon & -330 \\ -330 & 320-\varepsilon \end{pmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ ➔ we got the characteristic equation: $\varepsilon^2 - 80\varepsilon - 185700 = 0$

The principal strains are:

$$\begin{cases} \varepsilon_I = 472.8 \times 10^6 \mu m/\mu m \\ \varepsilon_{II} = -392.8 \times 10^6 \mu m/\mu m \end{cases}$$

By substitution of the two values of the principal strains we find the rotation angle is: $\theta = 65.8°$.

Maximum Shear Strain $\frac{\gamma}{2}_{max} = 433 \times 10^6 \mu m/\mu m.$

Visit the web site to verify:

https://www.efunda.com/formulae/solid_mechanics/mat_mechanics/calc_principal_strain.cfm

6. Mohr's circle for plane strain:

From the plane state of deformation as define at a point P the strain tensor by:

$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} \\ \frac{1}{2}\gamma_{12} & \varepsilon_{22} \end{bmatrix}$; we plot a point $X(\varepsilon_{11}, -\frac{\gamma_{12}}{2})$ of abscissa equal to the normal stain ε_x and of ordinate equal to minus the shearing strain γ_{12} , and point $Y(\varepsilon_{22}, +\frac{\gamma_{12}}{2})$ (See Figure.IV.20.).

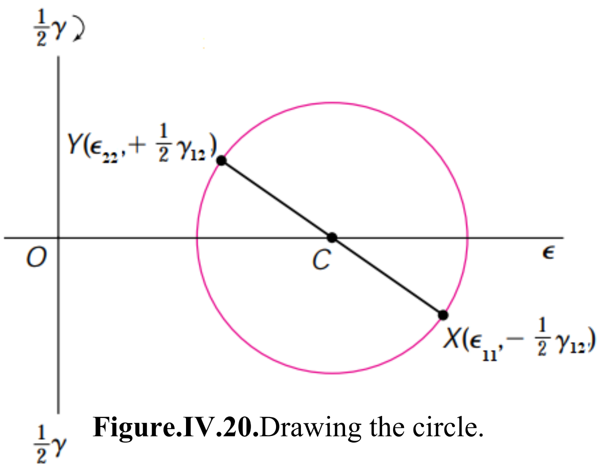


Figure.IV.20. Drawing the circle.

Drawing the diameter XY, we define the center C of Mohr's circle for plane strain. The abscissa of C and the radius R of the circle are respectively equal to:

$\varepsilon_{ave} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2}$; and $R = \sqrt{\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\right)^2 + \left(\frac{\gamma_{12}}{2}\right)^2}$.

Points A and B where Mohr's circle intersects the horizontal axis corresponds to the principal strains ε_{max} and ε_{min} (Figure.IV.21.), we find:

$$\varepsilon_{max} = \varepsilon_{ave} + R \quad \text{and} \quad \varepsilon_{min} = \varepsilon_{ave} - R$$

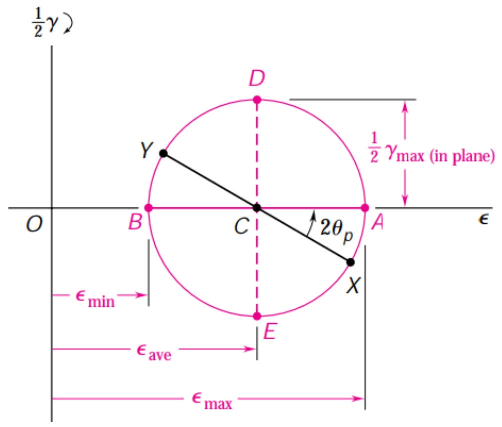


Figure.IV.21. The max., min. strains and the the max. shear strain.

The corresponding value θ_p of the angle θ is obtained by observing that the shearing strain is zero for A and B. Setting $\gamma_{1'2'} = 0$, we have :

$$tg\ 2\theta_p = \frac{\gamma_{12}}{\varepsilon_{11} - \varepsilon_{22}}$$

The corresponding axes a and b in Figure.IV.16. are the *principal axes* of strain. The angle θ_p which defines the direction of the principal axis *Oa* in Figure.IV.20. corresponding to point *A* in Figure.IV.20., is equal to half of the angle *XCA* measured on Mohr's circle, and the rotation that brings *Ox* into *Oa* has the same sense as the rotation that brings the diameter *XY* of Mohr's circle into the diameter AB.

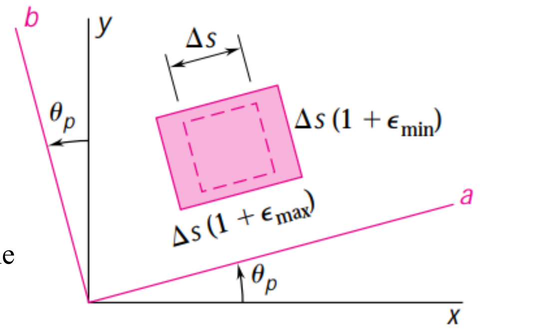


Figure.IV.22. The principal plane (orientation).

We recall that the principal axes of strain coincide with the principal axes of stress.
The maximum in-plane shearing strain is defined by points *D* and *E* in Figure21. It is equal to the diameter of Mohr's circle.

$$\gamma_{\max\,(in\,plane)} = 2R = \sqrt{(\varepsilon_{11}-\varepsilon_{22})^2+\gamma_{12}{}^2}$$

Finally, we note that the points *X'* and *Y'* that define the components of strain corresponding to a rotation of the coordinate axes through an angle θ (Figure.IV.23.) are obtained by rotating the diameter *XY* of Mohr's circle in the same sense through an angle 2θ (Figure.IV.23.).

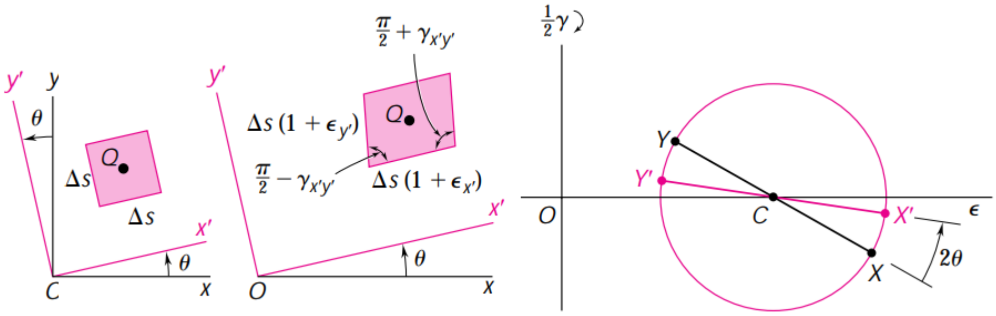


Figure.IV.23. The components of strain corresponding to a rotationθ.

Example:

A material in a state of plane strain, it is known that the horizontal side of a 10×10 mm square elongates by 4µm, while its vertical side remains unchanged, and that the angle at the lower left corner increases by 0.4×10^{-3} rad (Figure.IV.24). Determine (a) the principal axes and principal strains, (b) the maximum shearing strain and the corresponding normal strain?

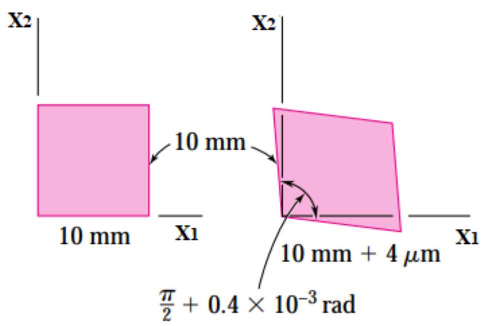


Figure.IV.24.A material in a state of plane strains.

(a) Principal axes and principal strains:

We first determine the coordinates of points X and Y on Mohr's circle for strain. We have:

$$\varepsilon_{X1} = +4 \times \frac{10^{-6}}{10} \times 10^3 = +400; \varepsilon_{X2} = 0$$

Since γ_{max} is equal to the diameter of the circle, for that we have the formula: $\left|\frac{\gamma_{X1X2}}{2}\right| = \varepsilon_{X1} - \varepsilon_{X2}$

$$\left|\frac{\gamma_{X1X2}}{2}\right| = 200$$

Since the side of the square associated with ε_{X1} rotates clockwise, point X of the coordinates ε_{X1} and $\left|\frac{\gamma_{X1X2}}{2}\right|$ is plotted above the horizontal axis. since $\varepsilon_{X2} = 0$ and the corresponding side rotates counterclockwise, point Y is plotted directly below, as showed in Figure.IV.25.

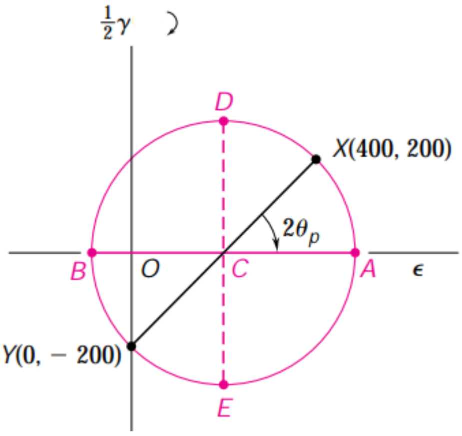


Figure.IV.25. The principal strains values are the points A, B.

Drawing the diameter XY, we determine the center C of Mohr's circle and its radius R. We have:

$OC = \frac{\varepsilon_{X1}+\varepsilon_{X2}}{2} = 200;\ OY = 200$

$$R = \sqrt{(OC)^2 + (OY)^2} = 283$$

The principal strains are defined by the abscissas of points A and B. We write:

$$\varepsilon_I = OA = OC + R = 200 + 283 = 483$$
$$\varepsilon_{II} = OB = OC - R = 200 - 283 = -83$$

b) Maximum shearing strain:

Points D and E define the maximum in-plane shearing strain which is:

$$\frac{\gamma_{max}}{2} = R = 283 \rightarrow \gamma_{max} = 566$$

The corresponding normal strains are both equal to: $\varepsilon' = OC = 200$.

We can verify by visiting:

https://www.efunda.com/formulae/solid_mechanics/mat_mechanics/calc_principal_strain.cfm

V. Relation between strain and displacement

1. Deformation gradients, displacement gradients:

The partial derivative of the Lagrangian description with respect to X_j gives us the tensor: $\frac{\partial x_i}{\partial X_j}$ which is called the gradient of material deformations, represented by: $F = x\nabla X = \frac{\partial x}{\partial X_1}\vec{e}_1 + \frac{\partial x}{\partial X_2}\vec{e}_2 + \frac{\partial x}{\partial X_3}\vec{e}_3$, represented by which can develop:

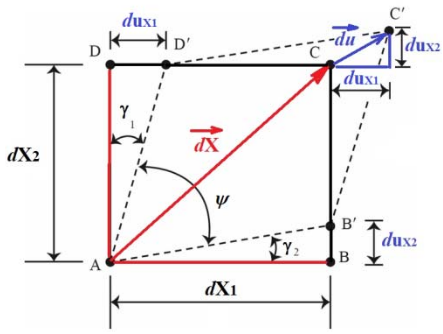


Figure.IV.26. The position and the displacement vectors.

$$F=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\begin{bmatrix}\dfrac{\partial}{\partial X_1} & \dfrac{\partial}{\partial X_2} & \dfrac{\partial}{\partial X_3}\end{bmatrix}=\begin{bmatrix}\partial x_1/\partial X_1 & \partial x_1/\partial X_2 & \partial x_1/\partial X_3\\ \partial x_2/\partial X_1 & \partial x_2/\partial X_2 & \partial x_2/\partial X_3\\ \partial x_3/\partial X_1 & \partial x_3/\partial X_2 & \partial x_3/\partial X_3\end{bmatrix}=\left[\partial x_i/\partial X_j\right]=x_{i,j}$$

The partial derivative of the Eulerian description with respect to x_j gives us the tensor: $\frac{\partial X_i}{\partial x_j}$ which is called the spatial deformation gradient, represented by: $H=X\nabla x=\frac{\partial X}{\partial x_1}\vec{e}_1+\frac{\partial X}{\partial x_2}\vec{e}_2+\frac{\partial X}{\partial x_3}\vec{e}_3$ which can develop:

$$H=\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\begin{bmatrix}\dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3}\end{bmatrix}=\begin{bmatrix}\partial X_1/\partial x_1 & \partial X_1/\partial x_2 & \partial X_1/\partial x_3\\ \partial X_2/\partial x_1 & \partial X_2/\partial x_2 & \partial X_2/\partial x_3\\ \partial X_3/\partial x_1 & \partial X_3/\partial x_2 & \partial X_3/\partial x_3\end{bmatrix}=\left[\partial X_i/\partial x_j\right]=X_{i,j}$$

Partial differentiation of the displacement vector u_i with respect to the coordinates produces either the material displacement gradient $\partial u_i/\partial X_j$ or the spatial displacement gradient $\partial u_i/\partial x_j$. These tensors are given in terms of the deformation gradients as the material gradient : $\frac{\partial u_i}{\partial X_j}=\frac{\partial x_i}{\partial X_j}-\delta_{ij}$ or : $J=F-I$

And the spatial gradient

$$\frac{\partial u_i}{\partial x_j}=\delta_{ij}-\frac{\partial X_3}{\partial x_1}\quad\text{or}\quad K=I-H$$

In the usual manner, the matrix forms of J and K are respectively:

$$J=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\begin{bmatrix}\dfrac{\partial}{\partial X_1} & \dfrac{\partial}{\partial X_2} & \dfrac{\partial}{\partial X_3}\end{bmatrix}=\begin{bmatrix}\partial u_1/\partial X_1 & \partial u_1/\partial X_2 & \partial u_1/\partial X_3\\ \partial u_2/\partial X_1 & \partial u_2/\partial X_2 & \partial u_2/\partial X_3\\ \partial u_3/\partial X_1 & \partial u_3/\partial X_2 & \partial u_3/\partial X_3\end{bmatrix}$$

And

$$K=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\begin{bmatrix}\dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3}\end{bmatrix}=\begin{bmatrix}\partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3\\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3\\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3\end{bmatrix}$$

<u>* Large deformations</u> (distance between two points of the continuous medium):

2. Deformation Tensor, finite strain tensor :

In the initial configuration and the deformed configuration of a continuous medium are assumed to be superimposed. The points P_0 and Q_0 of the continuous medium before deformation, move to the points P and Q respectively in the deformed configuration.

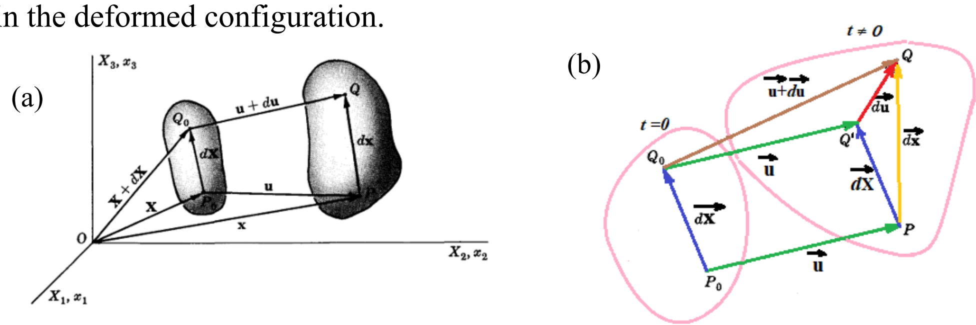


Figure.IV.27. The initial and deformed configurations.

The square of the distance of the differential element between the points P_0 and Q_0 is:

$$(dX)^2 = d\vec{X}.d\vec{X} = dX_i dX_i = \delta_{ij} dX_i dX_j$$

Since : $\vec{X} = \vec{X}(x,t)$:➔ $dX_i = \frac{\partial X_i}{\partial x_j} dx_j$ or : $d\vec{X} = H.d\vec{x}$

So : $(dX)^2 = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} dx_i dx_j = C_{ij}\, dx_i dx_j$ or : $\left(d\vec{X}\right)^2 = d\vec{x}.C.d\vec{x}$

with C : is a second order tensor: $C_{ij} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}$, or : $C = H^t H$ which is known as: Cauchy's deformation tensor.

In the deformed configuration, we will have:

$$(dx)^2 = d\vec{x}.d\vec{x} = dx_i dx_i = \delta_{ij} dx_i dx_j$$

And since : $\vec{x} = \vec{x}(X,t)$:➔ $dx_i = \frac{\partial x_i}{\partial X_j} dX_j$ or : $d\vec{x} = F.d\vec{X}$, in the same way :

$(dx)^2 = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} dX_i dX_j = G_{ij}\, dX_i dX_j$ or : $(d\vec{x})^2 = d\vec{X}.G.d\vec{X}$

with G : is a second order tensor : $G_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}$, or : $G = F^t F$ which is known as: Green's deformation tensor.

3. Finite strain tensor :

The difference magnitude $(dx)^2 - (dX)^2$ for the neighboring particles of the continuous medium is used to measure the deformation that occurs by the neighboring particles between the initial and final configuration. If this difference equals zero, a rigid displacement occurs (Solid body).

$(dx)^2 - (dX)^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) dX_i dX_j = 2L_{ij} dX_i dX_j$;

or : $(dx)^2 - (dX)^2 = d\vec{X}.(F^t F - I).d\vec{X} = d\vec{X}.2L_G.d\vec{X}$ where L_G : is a second order tensor :

$L_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)$ ou : $L_G = \frac{1}{2}(F^t F - I)$, is called the Lagrangian (or Green's) finite strain tensor.

By the same method: $(dx)^2 - (dX)^2 = \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i dx_j = 2E_{ij} dx_i dx_j$

or : $(dx)^2 - (dX)^2 = d\vec{x}.(I - H^t H).d\vec{x} = d\vec{x}.2E_A.d\vec{x}$ where E_A : is a second order tensor :

$E_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right)$ or : $E_A = \frac{1}{2}(I - H^t H)$, Called the Eulerian (or Almansi's) finite strain tensor.

A second technique is used, this time using the displacement gradients, which gives:

$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_i} + \frac{\partial u_j}{\partial X_j} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$ or : $L_G = \frac{1}{2}(J + J^t + J^t J)$

And for: $E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$ or : $E_A = \frac{1}{2}(K + K^t - K^t K)$

*** Infinitesimal strain tensors**(Small deformation theory-**SDT**) :

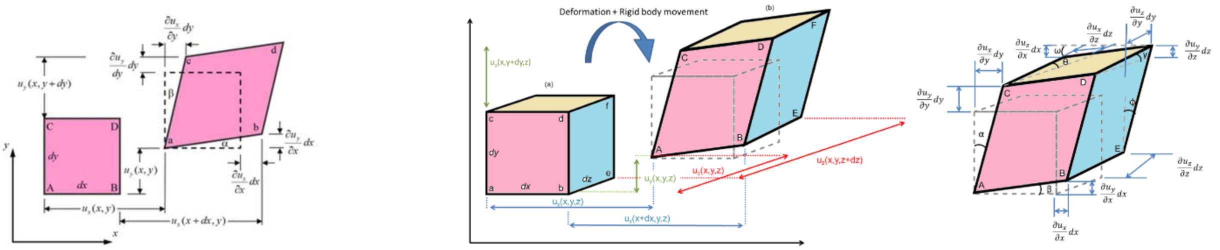
The theory of small deformations is based on the condition that the displacement gradients are very small compared to unity. In this case the following approximations are allowed:

For $\frac{\partial u_i}{\partial X_j} \ll 1 \rightarrow\ l_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_i} + \frac{\partial u_j}{\partial X_j}\right)$ or : $L = \frac{1}{2}(J + J^t)$ called the Lagrangian infinitesimal strain tensor.

With the same way :

If $\frac{\partial u_i}{\partial x_j} \ll 1 \rightarrow\ \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ or : $E = \frac{1}{2}(K + K^t)$ named the Eulerian infinitesimal strain tensor.

In the case where **the displacement gradients and the displacements are very small** in the 2D or 3D strains as shown in Figure.IV.28., the existence of a small difference between the material and spatial coordinates of a particle of the continuous medium, this allows us to admit: $l_{ij} = \epsilon_{ij}$ or : $L = E$.



(a) Plane (2d) strains (b) 3D strains

Figure.IV.28. The plane and 3D strains.

h.Relative displacements, linear rotation tensor, rotation vector :

The movement of two neighboring particles is represented by the vectors $u_i^{(P_0)}$ et $u_i^{(Q_0)}$, the vector :

$du_i = u_i^{(Q_0)} - u_i^{(P_0)}$ or : $d\vec{u} = \vec{u}^{(Q_0)} - \vec{u}^{(P_0)}$ is called **the relative displacement vector** of a particle initially at Q_0 with respect to the particle initially at P_0.

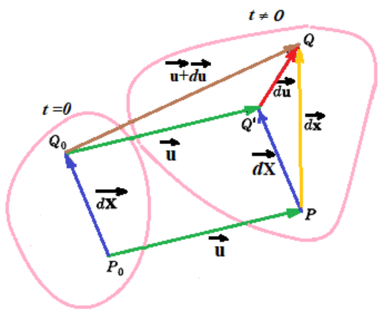


Figure.IV.29. The relative displacement vector.

Often we define the unit vector of relative displacement du_i/dX or dX is the magnitude the differential vector distance dX_i, if v_i is the unit vector in the direction of dX_i, so that: $dX_i = v_i\, dX$ then:

$$\frac{du_i}{dX} = \frac{\partial u_i}{\partial X_j}\frac{\partial X_j}{\partial X} = \frac{\partial u_i}{\partial X_j} v_i \text{ or : } \frac{d\vec{u}}{dX} = J.\vec{v}$$

Since the material displacement gradient ${\partial u_i}/{\partial X_j}$ can be decomposed into two parts: a symmetric part and an anti-symmetric part, the relative displacement vector can be written:

$$du_i = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)\right] dX_j$$

The first term is known as the **linear Lagrange deformation tensor** l_{ij}, the second term is known as **the linear Lagrange rotation tensor** and is expressed by:

$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)$$

A side from the Eulerian description, we find: **the relative displacement vector** is given by:

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j \text{ or: } d\vec{u} = K.d\vec{x}$$

And **the unit vector of relative displacement** is:

$$du_i = \frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial x} = \frac{\partial u_i}{\partial x_j}\mu_i \text{ or: } \frac{d\vec{u}}{dX} = K.\vec{\mu}$$

By decomposition of the gradient of Euler displacements (spatial):

$$\frac{du_i}{dx} = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] dx_j$$

The first term is known by the name: **linear Euler deformation tensor** e_{ij}, the second term is **the linear Euler rotation tensor**, is denoted by:

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

With the theory of small deformations, the finite Lagrange deformation tensor L_{ij} can be replaced by the linear Lagrange deformation tensor l_{ij} and the following expressions become:

$$(dx)^2 - (dX)^2 = (dx - dX)(dx + dX) = 2l_{ij}dX_i dX_j = d\vec{X}.2L.d\vec{X}$$

Since : $dx \sim dX$ for small deformations, then : $\frac{dx-dX}{dX} = l_{ij}v_i v_j$ ou : $\frac{dx-dX}{dX} = \vec{v}.L.\vec{v}$

This is known as **the change in length** by the initial unit length and also known as **the normal strain** for a line element initially having the direction cosine dX_i/dX .

i. Stretch ratio :

Another important measure of extensional strain of a differential line element is the ratio: dx/dX, known as **the stretch ratio**, the squared stretch at point P_0 for a line element along the unit vector $\vec{m} = d\vec{X}/dX$ is given by:

$$\left(\frac{dx}{dX}\right)^2_{P_0} = \Delta^2_{(\vec{m})} = G_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} \text{ ou : } \Delta^2_{(\vec{m})} = \vec{m}.G.\vec{m}\ .$$

By similarity, the squared stretch for a line element at the point P following the unit vector: $\vec{n} = d\vec{x}/dx$ is given by:

$$\left(\frac{dX}{dx}\right)^2_{P} = \frac{1}{\lambda^2_{(\vec{n})}} = C_{ij}\frac{dx_i}{dx}\frac{dx_j}{dx} \text{ ou : } \frac{1}{\lambda^2_{(\vec{n})}} = \vec{n}.C.\vec{n}\ .$$

j. Change of volume and area:

The relationships between the areas and volumes in the reference and current configurations are important in formulating the equations that define the effect of the inertia and elastic forces of the continuum.

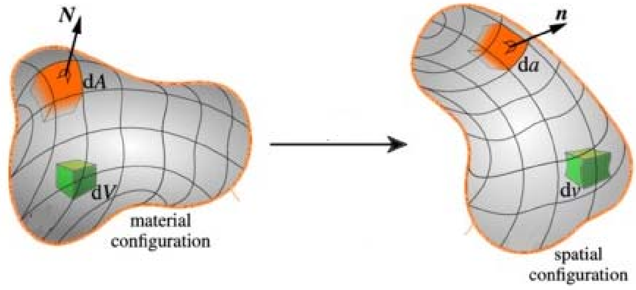


Figure.IV.30. The change of area and volume from intial to deformed configurations.

Volume The volume of an element can be obtained using the scalar triple product. Consider an element whose base is defined by the two vectors b and c, while another side is defined by the vector a. In fact, the volume can be written using any cyclic permutation of the vectors a, b, and c, that is,

$$\boxed{V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}$$

Now consider a volume element that has sides of length dX_1, dX_2, and dX_3 in the reference configuration dV.

The transformation of volume element from the reference configuration to the present configuration:

$$\boxed{dv = JdV}$$

J: represents the Jacobian transformation. $J = \det(F)$

Let's call θ the relative change in volume V *then*: $\boxed{\theta = \frac{\Delta V}{V_0} = J - 1}$.

Area The relationship between the volumes in the reference and current configurations can be used to obtain the relationship between the areas. To this end, consider an area element defined in the reference and current configurations, respectively. The oriented surface element in the reference configuration is defined by:

$$d\vec{A} = d\overrightarrow{X_1} \times d\overrightarrow{X_2} \, ;$$

after deformation becomes the oriented surface element by $d\vec{a} = d\overrightarrow{x_1} \times d\overrightarrow{x_2}$.

The transformation of a surface element from the reference configuration to the present configuration is:

$$d\vec{a} = JF^{-T}.d\vec{A}$$

Where: $F^{-T} \equiv (F^T)^{-1} = (F^{-1})^T$[8].

1. The strain vector in 3 dimensions (3D strain):

In a point of material we have a cubic element which has three principal facets symmetric to three facets and we have on the facets three strain vectors as represented in Figure.IV.31.

The strain tensor (symbol ε_{ij}) is defined in Mechanics, as a "tensor quantity representing the deformation of matter caused by stress. Strain tensor is symmetric and has three linear strain ε_{11}, ε_{22} and ε_{33} with three shear strain γ_{12}, γ_{13} and γ_{23} its components are written as:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\ \gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{bmatrix}$$

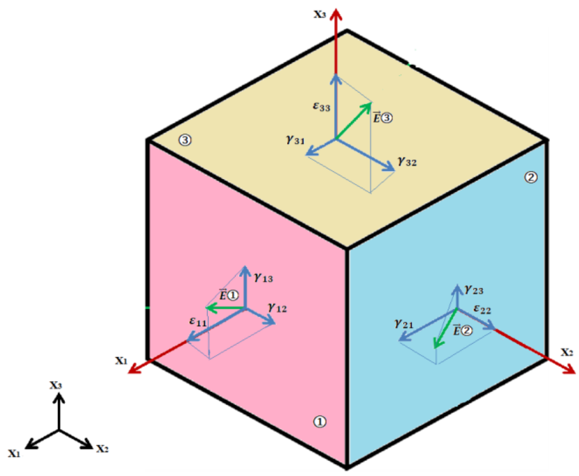


Figure.IV.31. The strain vectors and its components on the facets.

A normal strain is perpendicular to the face of an element, and a shear strain is parallel to it. These definitions are consistent with those of normal stress and shear stress.

2. Principal strains in 3D :

To determine the principal strains in 3D, we have to use the analytical method as discussed for 3D stresses state, since, we know the strain state in a point and we find the principal strains:

Let consider a 3D strain state at a point: $\varepsilon_{ij} = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix} \times 10^6 \mu m/\mu m$ ➔ $(\varepsilon_{ij} - \delta_{ij}\varepsilon)\vec{n} = \vec{0}$

$\begin{pmatrix} 5-\varepsilon & -1 & -1 \\ -1 & 4-\varepsilon & 0 \\ -1 & 0 & 4-\varepsilon \end{pmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ ➔ we got the characteristic equation: $(4-\varepsilon)(\varepsilon^2 - 9\varepsilon + 18) = 0$

The principal strains are:

$$\begin{cases} \varepsilon_I = 6 \times 10^6 \mu m/\mu m \\ \varepsilon_{II} = 46 \times 10^6 \mu m/\mu m \\ \varepsilon_{III} = 36 \times 10^6 \mu m/\mu m \end{cases}$$

By substitution; we find the directions vectors:

$$\vec{n}_I \begin{cases} 0.57 \\ 0.57 \\ 0.57 \end{cases} ; \quad \vec{n}_{II} \begin{cases} 0 \\ -0.71 \\ 0.71 \end{cases} ; \quad \vec{n}_{III} \begin{cases} 0.82 \\ -0.41 \\ -0.41 \end{cases}$$

Vist the web site to confirm: https://valdivia.staff.jade-hs.de/mohr3d_en.html

3. Mohr's circle for 3D strain

We can, therefore, draw Mohr's circle through the points *A* and *B* corresponding to the principal axes *a* and *b* (Figure.IV.32.). similarly, circles of diameters *BC* and *CA* can be used to analyze the transformation of strain as the cube is rotated about the *a* and *b* axes, respectively.

The three-dimensional analysis of strain by means of Mohr's circle is limited here to rotations about principal axes (as was the case for the analysis of stress) and is used to determine the maximum shearing strain at point Q. Since is equal to the diameter of the largest of the three circles shown in Figure.IV.32., we have

$$\gamma_{\max} = |\varepsilon_{max} - \varepsilon_{min}|$$

Where ε_{max} and ε_{min} represent the algebraic values of the maximum and minimum strains at point Q.

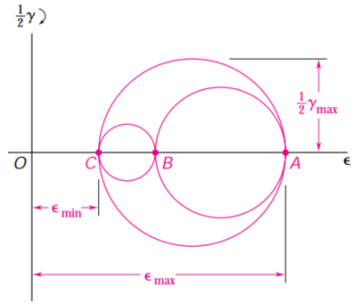


Figure.IV.32. Mohr's tricircles of the 3D strains.

Example:

As a result of measurements made on the surface of a machine component with strain gages oriented in various ways, it has been established that the principal strains on the free surface are: $\varepsilon_I = +400 \times 10^{-6}$ and $\varepsilon_{II} = -50 \times 10^{-6}$, knowing that Poisson's ratio for the given material is ν=0.3, determine (a) the maximum in-plane shearing strain, (b) the maximum shearing strain near the surface of the component?

(a) Maximum in-plane shearing strain:

We draw Mohr's circle through the points A and B corresponding to the given principal strain (Figure.IV.33.). The maximum in-plane shearing strain is defined by points D and E and is equal to the diameter of Mohr's circle;

$$\gamma_{\max(in-plane)} = 400 \times 10^{-6} + 50 \times 10^{-6} = 450 \times 10^{-6}\, rad$$

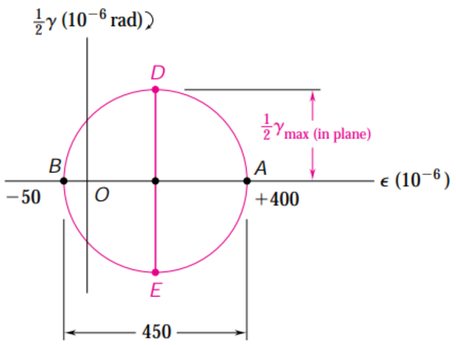


Figure.IV.33. The maximum in-plane shearing strain.

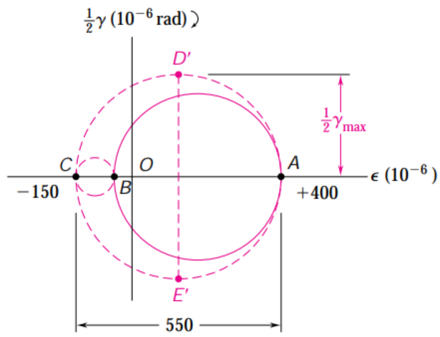


Figure.IV.34. The maximum shearing strain near the surface of the component.

(b) Maximum shearing strain:

We first determine the third principal strain ε_{III}. Since we have a state of plane stress on the surface of the machine component, we use the equation:

$$\varepsilon_{III} = -\frac{\nu}{1-\nu}(\varepsilon_I + \varepsilon_{II}) = -\frac{0.30}{0.70}(400 \times 10^{-6} - 50 \times 10^{-6}) = -150 \times 10^{-6};$$

Drawing Mohr's circles through A and C and through B and C (Figure.IV.33.), we find that the maximum shearing strain is equal to the diameter of the circle CA:

$$\gamma_{max} = 400 \times 10^{-6} + 150 \times 10^{-6} = 550 \times 10^{-6}\, rad\,.$$

IV.4. The strain vector for a given direction from the Mohr's strains tricircles:

The Figure.IV.35 shows the reference angles φ and β for intersection point Q on surface of body octant.

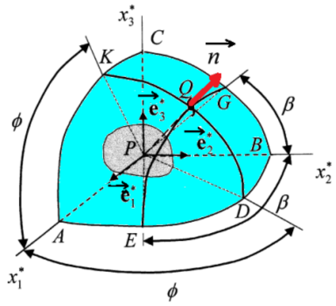


Figure.IV.35. The reference angles φ and β for intersection point Q on surface of body octant.

The Mohr's semicircle for the precedent stress state displayed in Figure.IV.36.

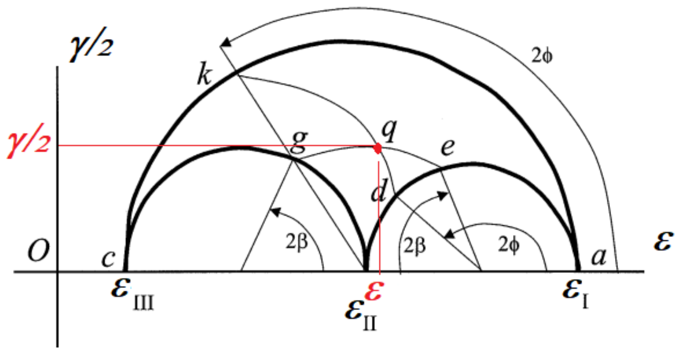


Figure.IV.36. The Mohr's stress semicircles for octant of small spherical portion.

In summary, for a specific at point P in the body, point Q, where the line of action of intersects the spherical octant of the body (Figure.IV.35.), is located at the common point of circle arcs KD and EG, and at the same time, the corresponding stress point q (having coordinates ε and γ is located at the intersection of circle arcs kd and eg, in the stress plane of Figure.IV.36.

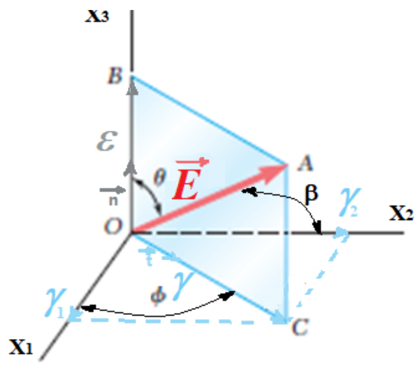


Figure.IV.37. Construction of the strain vector from a given vector on the Mohr's strains tricircles.

k. Analysis of a rosette using extensometry [2]:

To know the deformations in the plane of a deforming surface, we can stick a rosette on this surface. A rosette, made up of three strain gauges, measures the relative elongations in three different directions of the plane, i.e. at 45°, for the so-called 45° rosettes or at 60°, for the so-called 60° rosettes, see Figure.IV.38.

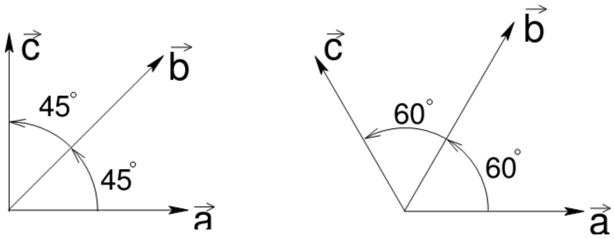


Figure.IV.38. Strains rosette.

A strain gauge, Figure.IV.40., can be compared to a metallic resistor made up of a very fine straight wire, which is glued to the surface of the structure studied. We thus transmit the deformations of the structure to the wire, hence a variation in its length, which produces a variation in its resistance. This variation is measured using a Wheatstone bridge (Figure.IV.39.).

We can thus precisely obtain the relative elongation εx in the x direction of the gauge. From εa, εand εc, it is possible to find the eigendeformations and eigenvectors in the plane. Let us note , the angle that the gauge makes in the direction $\vec{a}$ with respect to the (unknown) eigenvector $\vec{e}$I: the relative elongation, εa in the direction $\vec{a}$ is given by:

$$\varepsilon_a = \frac{\varepsilon_I + \varepsilon_{II}}{2} + \frac{\varepsilon_I - \varepsilon_{II}}{2}\cos(2a)$$

Similarly, if is α the angle of the rosette, we have:

$$\begin{cases} \varepsilon_b = \dfrac{\varepsilon_I + \varepsilon_{II}}{2} + \dfrac{\varepsilon_I - \varepsilon_{II}}{2}\cos 2(a+\alpha) \\ \varepsilon_c = \dfrac{\varepsilon_I + \varepsilon_{II}}{2} + \dfrac{\varepsilon_I - \varepsilon_{II}}{2}\cos 2(a+2\alpha) \end{cases}$$

to solve these three equations with three unknowns, we will set $d = \frac{\varepsilon_I+\varepsilon_{II}}{2}$ and $r = \frac{\varepsilon_I-\varepsilon_{II}}{2}$.
For the rosette at 45°, we have:

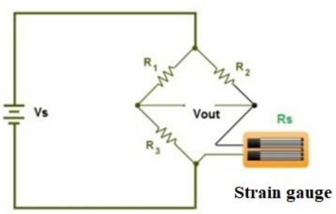
$$\begin{cases} \varepsilon_a = d + r\cos(2a) \\ \varepsilon_b = d + r\cos(2a + \dfrac{\pi}{2}) \\ \varepsilon_c = d + r\cos(2a + \pi) \end{cases}$$

where do we get from : $d = \frac{\varepsilon_a+\varepsilon_c}{2}$,$r = \frac{1}{2}\sqrt{(\varepsilon_c - \varepsilon_a)^2 + (\varepsilon_a + \varepsilon_c - 2\varepsilon_b)^2}$ and $tg(2a) = \frac{\varepsilon_b - d}{\varepsilon_a - d}$.
For the rosette at 60°, we have:

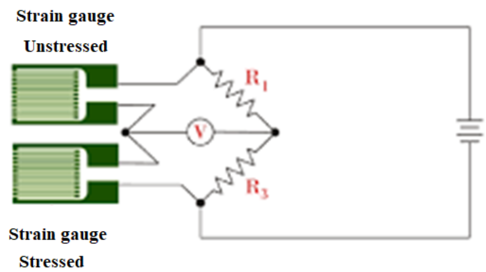
$$\begin{cases} \varepsilon_a = d + r\cos(2a) \\ \varepsilon_b = d + r\cos(2a + \dfrac{2\pi}{3}) \\ \varepsilon_c = d + r\cos(2a + \dfrac{4\pi}{3}) \end{cases}$$

where do we get from : $d = \frac{\varepsilon_a+\varepsilon_b+\varepsilon_c}{3}$,$r = \frac{1}{3}\sqrt{(2\varepsilon_a - \varepsilon_b - \varepsilon_c)^2 + 3(\varepsilon_c - \varepsilon_b)^2}$ and $tg(2a) = \frac{\sqrt{3}}{3}\frac{\varepsilon_c - \varepsilon_b}{\varepsilon_a - d}$.
finally, from d and r, we obtain: $\varepsilon_I = d + r, \varepsilon_{II} = d - r$.

I. Wheatstone Bridge:



(a) Weatstone bridge with one gauge.



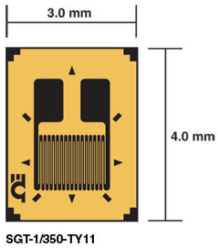
(b) Weatstone bridge with two gauges (by comparaison).

Figure.IV.39. Types of Weatstone bridges gauges [8].

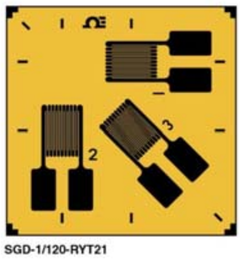
m. Types and dimensions of strain gauges:

The strain gauge is a resistive sensor that changes its resistance when pressure is applied. This change in resistance is proportional to the stress placed on it due to the change in pressure. So basically it uses a property of materials (wire, sheet or film) that tends to change their electrical resistance when pressure is applied.

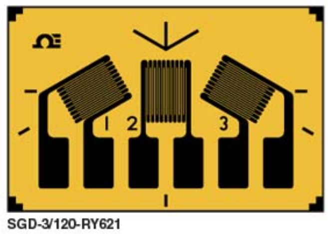
The electrical resistance of a material is directly proportional to its length and inversely to its section. This implies that when tensile deformation or stretching of the material occurs, its electrical resistance increases as its length increases and the cross-sectional area decreases simultaneously. Consequently, if it is compressed, its electrical resistance decreases. This change in resistance is very small. A Wheatstone bridge can be used to detect its resistance change.



(a) Strain gauge.



(b) 45° strain gauge



(c) 60° strain gauge

Figure.IV.40. Types of strains gauges.

Strain gauges, their sizes differ depending on the manufacturer and their uses (example of dimensions given in a catalog)

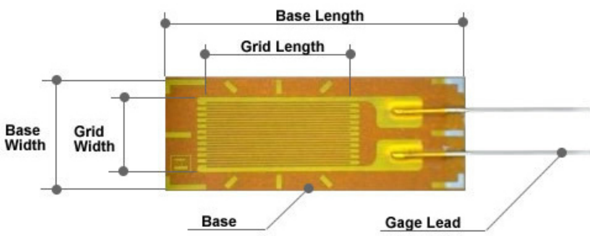


Figure.IV.41. Constituents of the strain gauge.

The thickness of strain gauges is approximately 0.2mm.

n. Areas of application of strain measurement gauges ;

Figure.IV.42. Control of deformations in a clamping screw and a polish under full load.

Figure.IV.43. Railway monitoring with strain gauges.

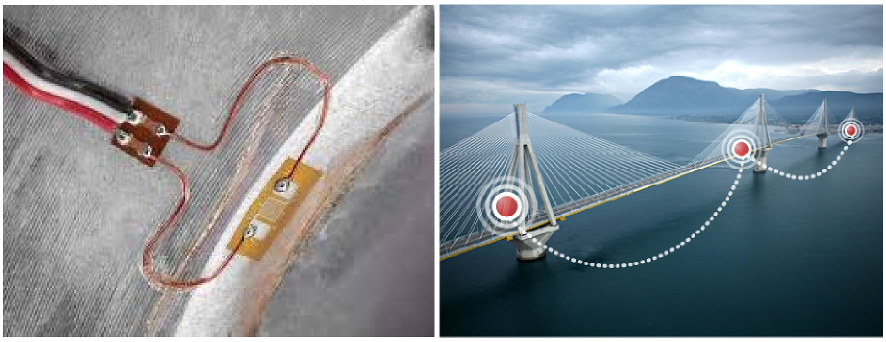


Figure.IV.44. Checking bridge cables with strain gauges.

Figure.IV.45. Strain Gauges in Aerospace Applications [9].

o. Strain tensors transformation laws:

The different deformation tensors: L_{ij} , E_{ij} , l_{ij}, ϵ_{ij} are second order tensors. for another rotated material reference system X'_i having the transformation matrix $[b_{ij}]$ (the direction cosines), with respect to the local reference system we will have:

$L'_{ij} = b_{ip}b_{jq}\, L_{pq}$ or : $L'_G = B.L_G.B^t$ and $l'_{ij} = b_{ip}b_{jq}\, l_{pq}$ or : $L' = B.L.B^t$

The same thing for a rotated spatial reference, x'_i : in possession of the transformation matrix $[a_{ij}]$ (the direction cosines), therefore:

$E'_{ij} = a_{ip}ab_{jq}\, E_{pq}$ or : $E'_A = A.E_A.A^t$ and $\epsilon'_{ij} = a_{ip}a_{jq}\, \epsilon_{pq}$ or : $E' = A.E.A^t$

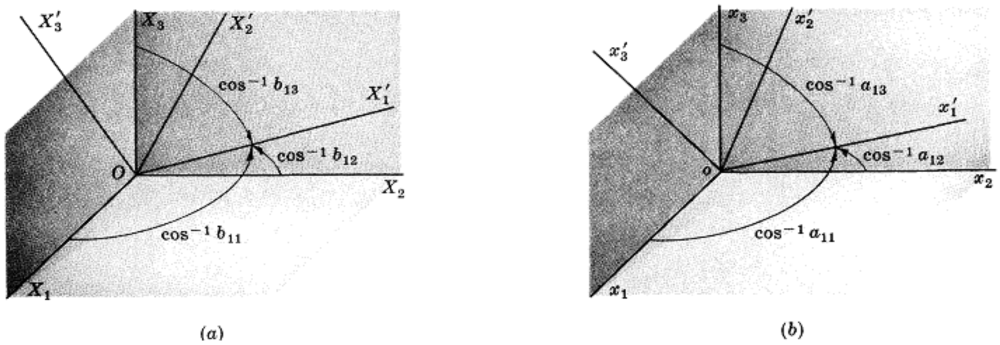


Figure.IV.46. Strain tensors transformation laws.

The orientation of plane ABC may be defined in terms of the angles between a unit normal $\vec{n}$ to the plane and the X_1, X_2, and X_3 directions. The direction cosines associated with these angles are:

$$\cos(\phi) = \cos(\vec{n}, \acute{X}_1) = l$$
$$\cos(\beta) = \cos(\vec{n}, \acute{X}_2) = m$$
$$\cos(\theta) = \cos(\vec{n}, \acute{X}_3) = n$$

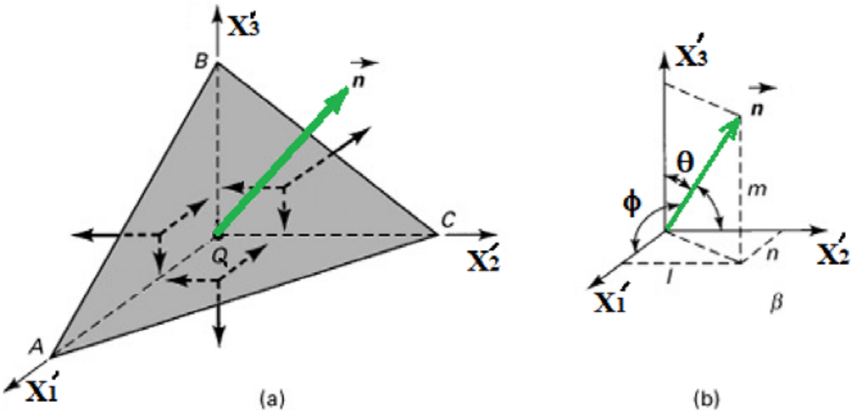


Figure.IV.47. Strains components on a tetrahedron.

The three direction cosines for the $\vec{n}$ direction are related by: $l^2 + m^2 + n^2 = 1$ Known as the Lamé Formula in 3dimensions.

Notation for Direction Cosines:

	X_1	X_2	X_3
X'_1	l_1	m_1	n_1
X'_2	l_2	m_2	n_2
X'_3	l_3	m_3	n_3

The six transformation equations for the normal and shear stresses in a new coordinate system are as follows:

$$\begin{cases} \varepsilon'_{X1'} = \varepsilon_{X1} l_1^2 + \varepsilon_{X2} m_1^2 + \varepsilon_{X3} n_1^2 + 2(\varepsilon_{X1X2} l_1 m_1 + \varepsilon_{X2X3} m_1 n_1 + \varepsilon_{X1X3} l_1 n_1 \\ \varepsilon'_{X2'} = \varepsilon_{X1} l_2^2 + \varepsilon_{X2} m_2^2 + \varepsilon_{X3} n_2^2 + 2(\varepsilon_{X1X2} l_2 m_2 + \varepsilon_{X2X3} m_2 n_2 + \varepsilon_{X1X3} l_2 n_2 \\ \varepsilon'_{X3'} = \varepsilon_{X1} l_3^2 + \varepsilon_{X2} m_3^2 + \varepsilon_{X3} n_3^2 + 2(\varepsilon_{X1X2} l_3 m_3 + \varepsilon_{X2X3} m_3 n_3 + \varepsilon_{X1X3} l_3 n_3 \\ \gamma'_{X1'X2'} = \varepsilon_{X1} l_1 l_2 + \varepsilon_{X2} m_1 m_2 + \varepsilon_{X3} n_1 n_2 + \gamma_{X1X2}(l_1 m_2 + m_1 l_2) + \gamma_{X2X3}(m_1 n_2 + n_1 m_2) + \gamma_{X1X3}(n_1 l_2 + l_1 n_2) \\ \gamma'_{X1'X3'} = \varepsilon_{X1} l_1 l_3 + \varepsilon_{X2} m_1 m_2 + \varepsilon_{X3} n_1 n_3 + \gamma_{X1X2}(l_1 m_3 + m_1 l_3) + \gamma_{X2X3}(m_1 n_3 + n_1 m_3) + \gamma_{X1X3}(n_1 l_3 + l_1 n_3) \\ \gamma'_{X1'X3'} = \varepsilon_{X1} l_2 l_3 + \varepsilon_{X2} m_2 m_3 + \varepsilon_{X3} n_2 n_3 + \gamma_{X1X2}(m_2 l_3 + l_2 m_3) + \gamma_{X2X3}(n_2 m_3 + m_2 n_3) + \gamma_{X1X3}(l_2 n_3 + n_2 l_3) \end{cases}$$ [10].

p. Spherical and deviator strain tensors:

The Lagrange and Euler linear deformation tensors can be decomposed as in the case of stress tensors, into deviator and spherical tensors, by:

$l_{ij} = d_{ij} + \delta_{ij} \frac{l_{kk}}{3}$ or $L = L_D + \frac{I(tr\, L)}{3}$ and $e_{ij} = e_{ij} + \delta_{ij} \frac{e_{kk}}{3}$ or $E = E_D + \frac{I(tr\, E)}{3}$

q. Compatibility equations for linear strain:

If the strain components ε_{ij} are given explicity as functions of the coordinates, the six independent equations:

$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ may be viewed as a system of six partial differential equations for determining the three displacement components u_i. The system is over-determined and will not, in general, possess a solution for an arbitrary choice of the strain components ε_{ij}. Therefore if the displacement components u_i to be single-valued and continuous, some conditions must be imposed upon the strain components. The necessary and sufficient conditions for such a displacement field are expressed by the equations:

$$\frac{\partial^2\varepsilon_{ij}}{\partial x_k\partial x_m}+\frac{\partial^2\varepsilon_{km}}{\partial x_i\partial x_j}-\frac{\partial^2\varepsilon_{ik}}{\partial x_j\partial x_m}-\frac{\partial^2\varepsilon_{jm}}{\partial x_i\partial x_k}=0;$$

There are 81 equations in all but only six are distinct. These six writtn in explicit and symbolic form appear as:

$$\begin{cases}\dfrac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\dfrac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\dfrac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}\\ \dfrac{\partial^2\varepsilon_{22}}{\partial x_3^2}+\dfrac{\partial^2\varepsilon_{33}}{\partial x_2^2}=2\dfrac{\partial^2\varepsilon_{23}}{\partial x_2\partial x_3}\\ \dfrac{\partial^2\varepsilon_{33}}{\partial x_1^2}+\dfrac{\partial^2\varepsilon_{11}}{\partial x_3^2}=2\dfrac{\partial^2\varepsilon_{31}}{\partial x_3\partial x_1}\\ \dfrac{\partial}{\partial x_1}(-\dfrac{\partial\varepsilon_{23}}{\partial x_1}+\dfrac{\partial\varepsilon_{31}}{\partial x_2}+\dfrac{\partial\varepsilon_{12}}{\partial x_3})=\dfrac{\partial^2\varepsilon_{11}}{\partial x_2\partial x_3}\\ \dfrac{\partial}{\partial x_2}(\dfrac{\partial\varepsilon_{23}}{\partial x_1}-\dfrac{\partial\varepsilon_{31}}{\partial x_2}+\dfrac{\partial\varepsilon_{12}}{\partial x_3})=\dfrac{\partial^2\varepsilon_{22}}{\partial x_3\partial x_1}\\ \dfrac{\partial}{\partial x_3}(\dfrac{\partial\varepsilon_{23}}{\partial x_1}+\dfrac{\partial\varepsilon_{31}}{\partial x_2}-\dfrac{\partial\varepsilon_{12}}{\partial x_3})=\dfrac{\partial^2\varepsilon_{33}}{\partial x_1\partial x_2}\end{cases}$$

Compatibility equations in terms of Lagrangian linear strain tensor may also be written down by an obvious correspondance to the Eulerian form given above. For plane strain parallel to the x_1x_2 plane, the six equations are reduced to the single equation $\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$ [1].

Example :

A centric axial force P and a horizontal force Qx are both applied at point C of the rectangular bar shown. A 45° strain rosette on the surface of the bar at point A indicates the following strains:

$\varepsilon_1=-75\times10^{-6}\ mm/mm$; $\varepsilon_2=+300\times10^{-6}\ mm/mm$;

$\varepsilon_3=+250\times10^{-6}\ mm/mm$

Knowing that $E=200\ GPa$ and ν=0.30, determine the magnitudes of the applied forces P and Qx?

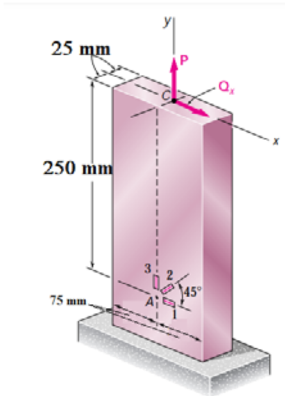


Figure.IV.48. A 45° strain rosette on the surface a centric axial bar.

The steps to solve this exercise are: 1- find strains, 2- find the stresses and finally 3- evaluate the forces

$$\varepsilon_x=\varepsilon_1=-75\times10^{-6}\ ;\ \varepsilon_y=\varepsilon_2=+250\times10^{-6}$$
$$\gamma_{xy}=2\varepsilon_2-\varepsilon_1-\varepsilon_3=425\times10^{-6}$$

$$\sigma_x=\frac{E}{1-\nu^2}(\varepsilon_x+\nu\varepsilon_y)=\frac{200\times10^9}{1-0.3^2}(-75+0.3*250)\times10^{-6}=0\ MPa$$

$$\sigma_y=\frac{E}{1-\nu^2}(\varepsilon_y+\nu\varepsilon_x)=\frac{200\times10^9}{1-0.3^2}(250+0.3*-75)\times10^{-6}=50\ MPa$$

$\sigma_y=\frac{P}{A}$ ➔

$$P=A\sigma_y=50*150*50=375\ kN$$

$$G=\frac{E}{2(1+\nu)}=\frac{200\times10^9}{2*1.3}=76.92\,GPa$$

$$\tau_{xy}=G*\gamma_{xy}=76.92\times10^9*425\times10^{-6}=32.69\,MPa$$

$$I=\frac{1}{12}bh^3=\frac{1}{12}50*150^3=14062500\,mm^4$$

Area torque $Q=A.\bar{y}=75*50*\frac{75}{2}=140625\,mm^3$; t=50 mm,

The shear stress: $\tau_{xy}=\frac{VQ}{It}\rightarrow V=\frac{It}{Q}\tau_{xy}=\frac{14062500*50}{140625}*32.69=163.45\,kN$

The applied force: $Q_x=V=163.45\,kN$

Bibliographic references:

[1] George E. Mase, 1970. Continum mechanics, Schaum's outline series in engineering, Mc graw hill book company.
[2] Samir DEGHBOUDJ,2016, POLYCOPIE DE COURS « MECANIQUE DES MILIEUX CONTINUS COURS ET APPLICATIONS ».
[3] https://fiveable.me/statics-strength-materials/unit-7/normal-shear-strain/study-guide/gZR6S4eAovm9NDS4
[4] R.C. Hibbeler, 2011, "Mechanics of materials, 8th edition".
[5]R.R.Archer, N.H. Cook,S.H.Crandall,N.C.Dahl, T.J.Lardner, F.A.McClintock,E.Rabinowicz, G.S.Reichenbach,1978,"An introduction to the mechanics of solids-2nd edition",
[6] http://www.engineeringcorecourses.com/solidmechanics1/C8-strain-transformation/C8.1-equations-of-strain-transformation/theory/
[7] https://fr.scribd.com/doc/202732829/Chapter-7-98
[8] X. Oliver, C. Agelet de Saracibar ,2017, "CONTINUUM MECHANICS FOR ENGINEERS THEORY AND PROBLEMS-2nd Edition", doi:10.13140/RG.2.2.25821.20961.
[9] https://www.sensing-systems.com/blog/strain-gauge-technology-in-field-testing
[10] https://www.eng.auburn.edu/~htippur/mech7330-17/elasticity%20review%20vol%202.pdf

Chapitre V:

Description

of the continuous medium's kinematics

Chapter V: Description of the continuous medium's kinematics

- Introduction
- Kinds of Motion
- Equation of motion
- Descriptions of motion
- Velocity and Acceleration
- The rate of deformation, the spin tensors and the (vorticity) rotation vector
- Stationarity, Streamline, Streamtubes, Trajectory or Pathline, Streakline.

1. Introduction :

Kinematics is the study of the movement of bodies without taking into account the causes that cause it. The movement of a deformable body can be described in two ways: By following the particles in their movements or by placing oneself at a fixed point in order to see them scroll.

The essential problems of continuous media mechanics are:
1. The study of the movements of deformable solids.
2. The study of the movements of floating and submerged bodies.
3. The study of non-stationary movements of gases.

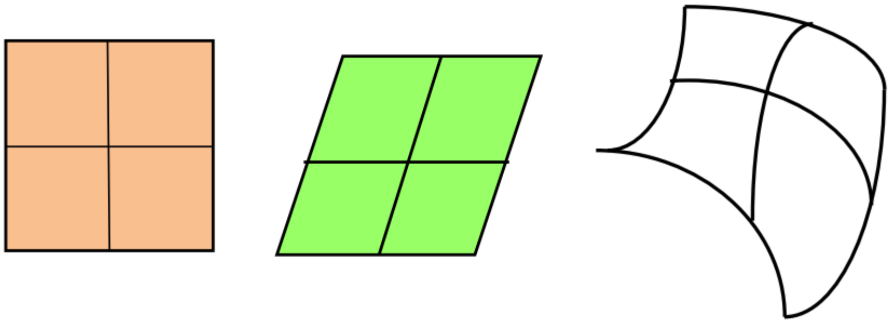


Figure .V.1. Deformations example of a continuous medium.

In mechanics, to characterize the movements of deformable bodies, we use an Eulerian or Lagrangian description.

2. Kinds of Motion

Motion of a continuous medium, also denoted by deformation, is characterized by the following types: Rigid Body Motion: Characterized by maintaining the original shape of the body after motion, i.e. it is characterized by preserving the distance between particles. The rigid body motion can be classified by: translation and/or rotation. Motion with Deformation: Characterized by changes of distance between particles. In general, motion is characterized by deformation and rigid body motion simultaneously [1].

3. Equation of motion:

The most basic description of the motion of a continuous medium can be achieved by means of mathematical functions that describe the position of each particle along time. In general, these functions and their derivatives are required to be continuous.

4. Descriptions of motion:

The mathematical description of the properties of the particles of the continuous medium can be addressed in two alternative ways: the material description (typically used in solid mechanics) and the spatial description (typically used in fluid mechanics).both descriptions essentially differ in the type of argument (material coordinates or spatial coordinates) that appears in the mathematical functions that describe the properties of the continuous medium.

Generally, each material point is given its initial coordinates, denoted by $\vec{X}$.
These so-called material coordinates are constant over time, so they are intrinsic information of the particle. On the other hand, the spatial coordinates of the particle, $\vec{x}$, evolve over time: $\vec{x} = \phi(\vec{X}, t)$[2].

Example :

A 2D domain that deforms according to a parallelogram. The reference configurations and those at time t = 1 are presented in Figure.V.2, the transformation $\vec{x} = \phi(\vec{X}, t)$ is given by:

$$\begin{cases} x_1 = \dfrac{1}{4}(18t + 4X_1 + 6tX_2) \\ x_2 = \dfrac{1}{4}(14t + (4 + 2t)X_2) \end{cases}$$

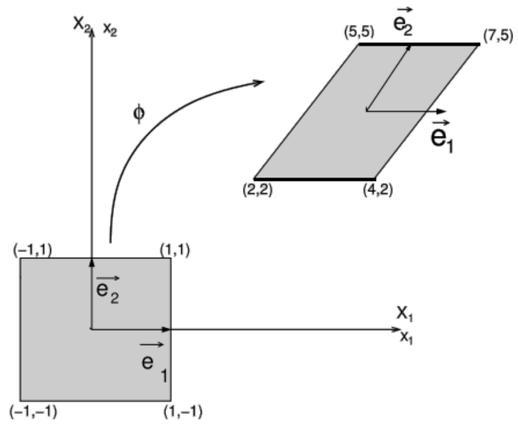


Figure.V.2. Deformation of a square.

5. Velocity and Acceleration

Lagrange velocity:

It is obtained by deriving the position vector ($\vec{u}$) versus time (t).

Lagrange acceleration:

We define the Lagrange acceleration by differentiating twice versus time (t), the position vector ($\vec{u}$).

The two Lagrangian and Eulerian descriptions are equivalent. The Lagrange description is adopted rather in the study of deformable solids. In the case of fluids, the Euler variables are preferred. The Lagrange velocity is equal to that of Euler.

Example:

Let the motion be described by the following equations of motion:

$$\begin{cases} x_1 = \dfrac{X_1}{1 + tX_1} \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

Calculate the components of the velocity vector in terms of the Lagrangian and then Eulerian variables.

- Lagrange velocity:

$$\begin{cases} V_1 = \dfrac{\partial x_1}{\partial t} = \dfrac{-X_1^2}{(1+tX_1)^2} \\ V_2 = 0 \\ V_3 = 0 \end{cases}$$

- Euler velocity:

$$\begin{cases} x_1 = \dfrac{X_1}{1+tX_1} \to X_1 = \dfrac{x_1}{1-tx_1} \to v_1 = V_1 = \dfrac{-\left(\dfrac{x_1}{1-tx_1}\right)^2}{\left(1+t\left(\dfrac{x_1}{1-tx_1}\right)\right)^2} \\ V_2 = 0 \\ V_3 = 0 \end{cases}$$

6. The rate of deformation, the spin tensors and the (vorticity) rotation vector:

We consider two particles P and P' in a continuum in motion.the coordinate of P is X_R initially, and xi at time t. for the sake of simplicity, the origins and axes of the material and spatial coordinate systems coincide. The coordinat of P' is X_R+ X_R initially and xi at time t. the line element between the two particles is dX_R initially and xi at time t. the initial and deformed line element are related by:

$$dx_i = \frac{\partial x_i}{\partial X_R} dX_R$$

The relative velocity of P' with respect to P is :

$$dv_i = v_i(P',t) - v_i(P,t) = v_i(X_R + dX_R, t) - v_i(X_R, t) = \tilde{v}_i(x_j + dx_j, t) - \tilde{v}_i(x_j, t)$$

In which the particle velocity is either expressed in the material description by the use of function $v_i(X_R, t)$ or in the spatial description by the use of function $\tilde{v}_i(x_i, t)$. From the material description:

$$dv_i = \frac{D}{Dt}[x_i(X_R + dX_R, t) - x_i(X_R, t)]$$

Observing that dxi denotes the line element connecting two adjacent material particle,

$$dv_i = \frac{D}{Dt}(dx_i)$$

It is seen that the order of differentiation between d and D/Dt is interchangeable.
Using the spatial description,

$$dv_i = \tilde{v}_i(x_j + dx_j, t) - \tilde{v}_i(x_j, t)$$

The first term on the right hand side may be expanded in Taylor's series and, by retaining only the first-order term,

$$dv_i = \tilde{v}_i(x_j, t) + \frac{\partial \tilde{v}_i}{\partial x_k} dx_k - \tilde{v}_i(x_j, t) = \frac{\partial \tilde{v}_i}{\partial x_k} dx_k = v_{i,j} dx_j$$

By combining we obtain:

$$dv_i = \frac{D}{Dt}(dx_i) = v_{i,j} dx_j$$

Where $v_{i,j} = L_{ij}$ is known as the velocity gradient tensor. This tensor is generally not symmetric and so it can be split into symmetric and antisymmetric parts.
Therefore, we write:

$$L_{ij} = v_{i,j} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) + \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) = D_{ij} + W_{ij}$$

Where $D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$ or $\mathrm{D} = \frac{1}{2}(\mathrm{L} + \mathrm{L}^T)$

And

$$W_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) \text{ or } \mathrm{W} = \frac{1}{2}(\mathrm{L} - \mathrm{L}^T)$$

In which D_{ij} is the rate of deformation tensor and W_{ij} is the spin tensor.

Therefore,

$$dv_i = D_{ij}dx_j + W_{ij}dx_j$$

The vorticity tensor is the antisymmetric (or-skew-symmetric) part of the velocity gradient tensor, also known as the spin tensor.

Because W is skew-symmetric ($W^T = -W$), it has only three independent components,

$$w_{ij} = \begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$ where: $w_1 \equiv -w_{23} = w_{32} = \frac{1}{2}(v_{3,2} - v_{2,3})$

$$w_2 \equiv -w_{31} = w_{13} = \frac{1}{2}(v_{1,3} - v_{3,1})$$

$$w_3 \equiv -w_{12} = w_{21} = \frac{1}{2}(v_{2,1} - v_{1,2})$$

These components can be used to define a vector $\vec{w}$ called the axial vector of w_{ij} as follows:

$w_i = -\frac{1}{2}\varepsilon_{ijk}w_{jk} = \frac{1}{2}\varepsilon_{ijk}\, v_{k,j}$

The vector $\vec{w}$ is also called the rotation vector, and twice the rotation vector $\vec{\xi} = 2\vec{w}$ is called the vorticity vector. The flow is said to be irrotational if the vorticity vector is zero $\vec{\xi} = \vec{0}$.

Or:

$$w_1 = -w_{23} = w_{32} = \frac{1}{2}(v_{2,3} - v_{3,2})$$

$$w_2 = -w_{31} = w_{13} = \frac{1}{2}(v_{1,3} - v_{3,1})$$

$$w_3 = -w_{12} = w_{21} = \frac{1}{2}(v_{2,1} - v_{1,2})$$

Expressions for L in Terms of F:

The expressions of L in terms of the deformation gradient F. We do this because F has been used extensively in the description of deformation. The velocity gradient L may be written as

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$$ [3].

Example :

The motion of a body is given as

$$\begin{cases} x_1 = X_1 \\ x_2 = X_2 + X_1 t/t_0 \\ x_3 = X_3 + X_2(1 - e^{-t/t_0}) \end{cases}$$ where t_0 is a constant [s].

Determine:

-the deformation gradient F_{ij}?

-the Green-Lagrange strain E_{ij}?

-the velocity in Euler description $v(x)$?

Solution:

The deformation gradient F_{ij} is:

$$F_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ t/t_0 & 1 & 0 \\ 0 & 1 - e^{-t/t_0} & 1 \end{bmatrix}$$

and the Green Lagrange strain E_{ij} is:

$E = \frac{1}{2}(F^T F - I);$

$$E_{ij}=\begin{bmatrix}(t/t_0)^2/2 & (t/t_0)/2 & 0\\ (t/t_0)/2 & 1/2\left(1-e^{-t/t_0}\right)^2 & 1/2\left(1-e^{-t/t_0}\right)\\ 0 & 1/2\left(1-e^{-t/t_0}\right) & 1\end{bmatrix}$$

The velocity in Euler description:

$$\begin{cases} v_1=0\\ v_2=x_1/t_0\\ v_3=1/t_0(x_2-x_1t/t_0)e^{-t/t_0}\end{cases}$$

The velocity v s given in Lagrangian and Eulerian coordinates as follows:

$$\begin{cases} v_1=0\\ v_2=X_1/t_0=x_1/t_0\\ v_3=1/t_0\,X_2e^{-t/t_0}=1/t_0(x_2-x_1t/t_0)e^{-t/t_0}\end{cases}$$

The acceleration can be obtained from the Lagrangian form directly as:

$$\begin{cases} a_1=0\\ a_2=0\\ a_3=-1/t_0^2\,X_2e^{-t/t_0}\end{cases}$$

And the Eulerian form as:

$$a_i=\frac{\partial v_i(x,t)}{\partial x_j}\frac{\partial x_j(X,t)}{\partial t}+\left.\frac{\partial v_i(x,t)}{\partial t}\right|_x$$

$\frac{\partial v_i(x,t)}{\partial t}$ is called as the local derivative

$\frac{\partial v_i(x,t)}{\partial x_j}\frac{\partial x_j(X,t)}{\partial t}$ is called as the convective derivative.

Which in matrix form becomes

$$a_i=\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ \dfrac{1}{t_0} & 0 & 0\\ -\dfrac{t}{t_0^2e^{-\frac{t}{t_0}}} & \dfrac{1}{t_0e^{-\frac{t}{t_0}}} & 0\end{bmatrix}\begin{Bmatrix}0\\ \dfrac{x_1}{t_0}\\ \dfrac{1}{t_0\left(x_2-\dfrac{x_1t}{t_0}\right)e^{-\frac{t}{t_0}}}\end{Bmatrix}+\begin{Bmatrix}0\\0\\ \dfrac{1}{t_0\left(-\dfrac{x_2}{t_0}+\dfrac{x_1t}{t_0^2}-\dfrac{x_1}{t_0}\right)e^{-\frac{t}{t_0}}}\end{Bmatrix}$$

$$=\begin{Bmatrix}0\\0\\-1/t_0^2(x_2-x_1t/t_0)e^{-t/t_0}\end{Bmatrix}\text{[4].}$$

Example :

The motion of a continuous medium is given by:

$$\begin{cases}x_1=X_1\cosh t+X_2\sinh t\\ x_2=X_1\sinh t+X_2\cosh t\\ x_3=X_3\end{cases}$$

Determine :

- The velocity components in the material description,
- The velocity components in the spatial description,
- The components of the rate of deformation and vorticity tensors.

Solution:

The given mapping is:

$$\begin{cases} x_1 = X_1 \cosh t + X_2 \sinh t \\ x_2 = X_1 \sinh t + X_2 \cosh t \\ x_3 = X_3 \end{cases}$$

The inverse of the mapping is given by :

$$\begin{cases} X_1 = x_1 \cosh t - x_2 \sinh t \\ X_2 = -x_1 \sinh t + x_2 \cosh t \\ X_3 = x_3 \end{cases}$$

- The velocity components in the material description are:

$$\begin{cases} v_1 = \left(\dfrac{\partial x_1}{\partial t}\right)_{X\,fixed} = X_1 \sinh t + X_2 \cosh t \\ v_2 = \left(\dfrac{\partial x_2}{\partial t}\right)_{X\,fixed} = X_1 \cosh t + X_2 \sinh t \\ v_3 = \left(\dfrac{\partial x_3}{\partial t}\right)_{X\,fixed} = 0 \end{cases}$$

- The velocity components in the spatial description can be obtained by rewriting them from part1 in terms of the spatial coordinates. We obtain

$$\begin{cases} v_1 = X_1 \sinh t + X_2 \cosh t = (x_1 \cosh t - x_2 \sinh t) \sinh t + (-x_1 \sinh t + x_2 \cosh t) \cosh t = x_2 \\ v_2 = X_1 \cosh t + X_2 \sinh t = (x_1 \cosh t - x_2 \sinh t) \cosh t + (-x_1 \sinh t + x_2 \cosh t) \sinh t = x_1 \\ v_3 = 0 \end{cases}$$

- The components of the deformation gradients tensor $F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right)$ are:

$$F_{11} = \frac{\partial x_1}{\partial X_1} = \cosh t,\ F_{12} = \frac{\partial x_1}{\partial X_2} = \sinh t,\ \ F_{13} = \frac{\partial x_1}{\partial X_3} = 0,$$
$$F_{21} = \frac{\partial x_2}{\partial X_1} = \sinh t,\ F_{22} = \frac{\partial x_2}{\partial X_2} = \cosh t,\ \ F_{23} = \frac{\partial x_2}{\partial X_3} = 0,$$
$$F_{31} = \frac{\partial x_3}{\partial X_1} = \cosh t,\ F_{32} = \frac{\partial x_3}{\partial X_2} = \sinh t,\ \ F_{33} = \frac{\partial x_3}{\partial X_3} = 1,$$

$$F_{ij} = \begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The components of the velocity gradient tensor $L_{ij} = \left(\frac{\partial v_j}{\partial x_i}\right)$ are:

$$L_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The components of the rate of deformation tensor $D_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)$are:

$$D_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The components of the vorticity tensor $w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$are:

$$w_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example :

Given the following velocity field in certain flow, determine the components of D_{ij} , w_{ij} and the rotation vector w:

$$v_1 = x_1^2 + x_2^2 + x_3^2$$
$$v_2 = x_1 x_2 + x_2 x_3 + x_3^2$$
$$v_3 = 4 - 3x_1 x_3 - \frac{x_3^2}{2}$$

***Solution*:**

$D_{11} = \frac{\partial v_1}{\partial x_1} = 2x_1$; $D_{22} = \frac{\partial v_2}{\partial x_2} = 2x_2$; $D_{33} = \frac{\partial v_3}{\partial x_3} = 2x_3$

$D_{12} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) = 1.5\, x_2$, $D_{13} = \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) = -0.5\, x_3$, $D_{23} = \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) = 0.5\, x_2 + x_3$

$$D_{ij} = \begin{bmatrix} 2x_1 & 1.5\, x_2 & -0.5\, x_3 \\ 1.5\, x_2 & 2x_2 & 0.5\, x_2 + x_3 \\ -0.5\, x_3 & 0.5\, x_2 + x_3 & 2x_3 \end{bmatrix}$$

$w_{11} = 0$; $w_{22} = 0$; $w_{33} = 0$

$w_{12} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) = 0.5x_2$; $w_{13} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) = 2.5x_3$; $w_{23} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) = 0.5x_2 + x_3$;

$$w_{ij} = \begin{bmatrix} 0 & 0.5\, x_2 & 2.5\, x_3 \\ 0.5\, x_2 & 0 & 0.5\, x_2 + x_3 \\ 2.5\, x_3 & -0.5\, x_2 - x_3 & 0 \end{bmatrix}$$

The rotation vector *w* :

$$w_1 = -w_{23} = w_{32} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) = -0.5x_2 - x_3$$
$$w_2 = -w_{31} = w_{13} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) = 2.5x_3$$
$$w_3 = -w_{12} = w_{21} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) = -0.5x_2$$

7. Stationarity:

A property is stationary when its spatial description does not depend on time.
This does not imply that, for a same particle, such property does not vary along time (the material description may depend on time).

Example:

Consider the solid in the figure below, which rotates at a constant angular velocity ω and has the expression:

$$\begin{cases} x_1 = R\sin(\omega t + \phi) \\ x_2 = R\cos(\omega t + \phi) \end{cases}$$

as its equation of motion.
Find the velocity and acceleration of the motion described both in material and spatial forms.

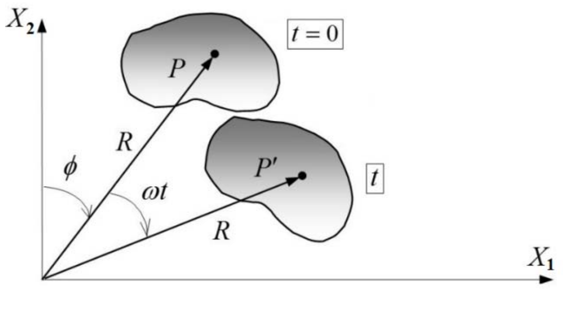


Figure.V.3. The centripetal acceleration.

***Solution*:**

The equation of motion can be written as:

$$\begin{cases} x_1 = R\sin(\omega t + \phi) = R\sin\omega t\cos\phi + R\cos\omega t\ sin\phi \\ x_2 = R\cos(\omega t + \phi) = R\cos\omega t\cos\phi - R\sin\omega t\ sin\phi \end{cases}$$

And, since t=0 , $X_1 = Rsin\phi$ and $X_2 = Rcos\phi$, the canonical form of the equation of motion and its inverse equation result in

$$\begin{cases} x_1 = X_1\sin\omega t + X_2\cos\omega t \\ x_2 = -X_1\sin\omega t + X_2\cos\omega t \end{cases} \text{ and } \begin{cases} X_1 = x_1\cos\omega t - x_2\sin\omega t \\ X_2 = x_1\sin\omega t + x_2\cos\omega t \end{cases}$$

Velocityy in spatial description:

Replacing the canonical form of the equation of motion into the material description of the velocity results in :

$$v(x,t) = V(X(x,t),t) = \begin{Bmatrix} \omega x_2 \\ -\omega x_1 \end{Bmatrix};$$

Acceleration in material description:

$$A(X,t) = \frac{\partial V(X,t)}{\partial t}$$

$$A(X,t) = \begin{Bmatrix} \dfrac{\partial v_{x1}}{\partial t} = -X_1\omega^2\cos(\omega t) - X_2\omega^2\sin(\omega t) \\ \dfrac{\partial v_{x2}}{\partial t} = X_1\omega^2\sin(\omega t) - X_2\omega^2\cos(\omega t) \end{Bmatrix} = -\omega^2 \begin{Bmatrix} X_1\cos(\omega t) + X_2\sin(\omega t) \\ -X_1\sin(\omega t) + X_2\cos(\omega t) \end{Bmatrix}$$

Acceleration in spatial description:
Replacing the canonical form of the equation of motion into the material description of the acceleration results in

$$a(x,t) = A(X(x,t),t) = \begin{Bmatrix} -\omega^2 x_1 \\ -\omega^2 x_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

This same expression can be obtained if the expression for the velocity *v(x,t)* and the definition of the material derivative are taken into account,

$$a(x,t) = \frac{dv(x,t)}{dt} = \frac{\partial}{\partial t}\begin{Bmatrix} \omega x_2 \\ -\omega x_1 \end{Bmatrix} + [\omega x_2 \quad -\omega x_1] \begin{Bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{Bmatrix} [\omega x_2 \quad -\omega x_1]$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + [\omega x_2 \quad -\omega x_1] \begin{Bmatrix} \dfrac{\partial}{\partial x_1}(\omega x_2) & \dfrac{\partial}{\partial x_1}(-\omega x_1) \\ \dfrac{\partial}{\partial x_2}(\omega x_2) & \dfrac{\partial}{\partial x_2}(-\omega x_1) \end{Bmatrix} = -\omega^2 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Note that the result obtained using both procedures are identical.
The direction of the velocity vector for a same particle is tangent to its circular trajectory and changes along time.
The acceleration (material derivative of the velocity), appears due to the change in direction of the velocity vector of the particles and is known as the centripetal acceleration [5].

8. Streamline:

A streamline at time t is a curve whose tangent at every point has the direction of the instantaneous velocity vector of the particle at the point. Experimentally, streamlines on the surface of a fluid are often obtained by sprinkling it with reflecting particles and making a short-time exposure photograph of the surface. Each reflecting particle produces a short line on the photograph approximating the tangent to a streamline.

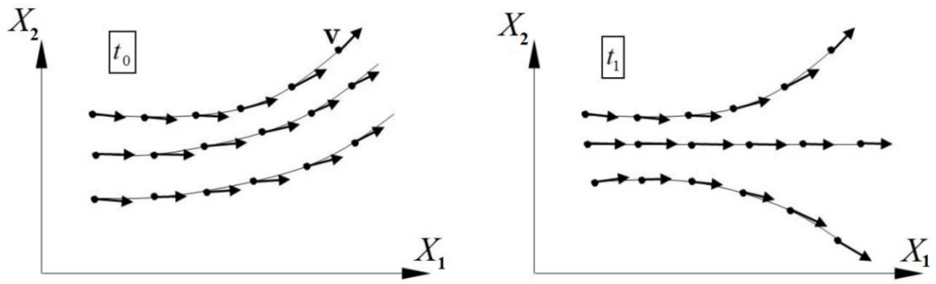


Figure.V.4. Streamlines at two different instants of time.

Mathematically, streamlines can be obtained from the velocity field v(x,f) as follows:

Let x = x(s) be the parametric equation for the streamline at time t, which passes through a given point x_0. Then an s can always be chosen such that

$$\frac{dx}{ds} = v(x,t)$$
$$x(0) = x_0$$

Example :

Given the velocity field in dimensionless form

$$v_1 = \frac{x_1}{1+t},\ v_2 = x_2,\ v_3 = 0$$

find the streamline which passes through the point $(\alpha_1, \alpha_2, \alpha_3)$ at time t

Solution:

From $\frac{dx_1}{ds} = \frac{x_1}{1+t}$

We have $\int_{\alpha_1}^{x_1} \frac{dx_1}{x_1} = \int_0^s \frac{1}{1+t} ds$

Thus, $\ln x_1 - \ln \alpha_1 = \frac{s}{1+t} \Rightarrow$ $x_1 = \alpha_1 e^{\left[\frac{s}{1+t}\right]}$

Similarly, from $\frac{dx_2}{ds} = x_2$, we have : $\int_{\alpha_2}^{x_2} \frac{dx_2}{x_2} = \int_0^s ds$

Thus, $x_2 = \alpha_2 e^s$.

obviously , $x_3 = \alpha_3$ [6]

9. Streamtubes:

A Streamtube is a surface formed by a bundle of streamlines that occupy the points of a closed line, fixed in space, and that does not constitute a streamline.

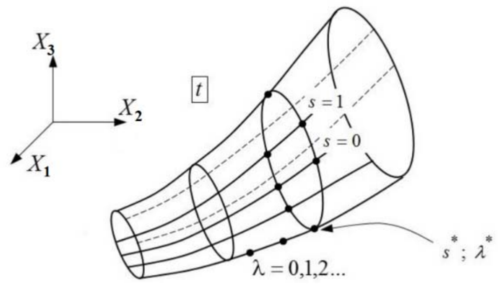


Figure.V.5. Streamtube at a given time t.

Streamlines constitute a family of curves of the type

$$x = f(C_1, C_2, C_3, \lambda, t).$$

where (C_1, C_2, C_3 are the curves)

The problem consists in determining, for each instant of time, which curves of the family of curves of the streamlines cross a closed line, which is fixed in the space, To this aim, one imposes, in terms of the parameters $\lambda^*$and s^*, that a same point belong to both curves,

$$g(s^*) = f(C_1, C_2, C_3, \lambda^*, t)$$

A system of three equations is obtained from which s*, λ* and C_3 can be isolated,

$$s^* = s^*(C_1, C_2, t)$$
$$\lambda^* = \lambda^*(C_1, C_2, t)$$
$$C_3 = C_3(C_1, C_2, t)$$

Introducing them into previous equation yields to :

$$x = f(C_1, C_2, C_3(C_1, C_2, t), \lambda^*(C_1, C_2, t), t).$$

which constitutes the parameterized expression (in terms of the parameters C_1 and C_2) of the streamtube for each time t (see Figure.V.5.) [5].

10. Trajectory or Pathline:

A pathline is the path traversed by a fluid particle. To photograph a pathline, it is necessary to use long time exposure of a reflecting particle.

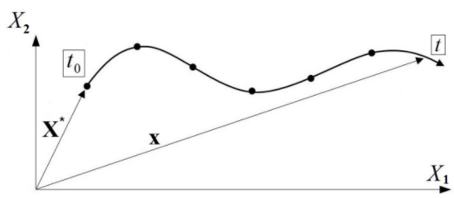


Figure.V.6. Trajectory or pathline of a particle.

Mathematically, the pathline of a particle which was at X at time t_0 can be obtained from the velocity field v(x,*t*) as follows:

Let x = x(t) be the pathline, then

$$\frac{dx}{dt} = v(x,t)$$
$$x(t_0) = X$$

Example:

For the velocity field of the previous example, find the pathline for a particle which was at (X_1,X_2,X_3)at time t_0

***Solution*:** From

$$\frac{dx_1}{dt} = \frac{x_1}{1+t}$$

we have

$$\int_{X_1}^{x_1} \frac{dx_1}{x_1} = \int_{t_0}^{t} \frac{dt}{1+t}$$

Thus,

$\ln x_1 - \ln X_1 = \ln(1+t) - \ln(1+t_0)$➔ $x_1 = X_1 \frac{1+t}{1+t_0}$

Similarly from $\frac{dx_2}{ds} = x_2$, we have $\int_{X_2}^{x_2} \frac{dx_2}{x_2} = \int_{t_0}^{t} dt$ ➔ $x_2 = X_2 e^{t-t_0}$

and obviously, $x_3 = X_3$

11. Streakline:

A streakline through a fixed point x_0, is the line at time *t* formed by all the particles which passed through x_0 at $\tau < t$.

Let X = X(x, *t*) denote the inverse of x = x(X, t), then the particle which was at x_0, at time τ, has the material coordinates given by X = X(x_0,τ); this same particle is then at x = x (X(x_0, τ), *t*) at time *t*. Thus, the streakline at time *t* is given by

x = x (X(x_0, τ), *t*) for fixed time *t* and variable τ.

Example:

Given the dimensionless velocity field

$$v_1 = \frac{x_1}{1+t}, v_2 = x_2, v_3 = 0.$$

Find the streakline formed by the particles which passed though the spatial position $(\alpha_1, \alpha_2, \alpha_3)$.

***Solution*:** The pathline equations for this velocity field was obtained in precedent Example to be :

$$x_1 = X_1 \frac{1+t}{1+t_0}, x_2 = X_2 e^{t-t_0}, x_3 = X_3 .$$

From which we obtain the inverse equations

$$X_1 = x_1 \frac{1+t_0}{1+t}, X_2 = x_2 e^{-t+t_0}, X_3 = x_3 .$$

Thus, the particle which passes through $\alpha_1, \alpha_2, \alpha_3$at time τ is given by

$$X_1 = \alpha_1 \frac{1+t_0}{1+\tau}, X_2 = \alpha_2 e^{-\tau+t_0}, X_3 = \alpha_3 .$$

Substituting the last equation into the spatial equation , we obtain the parametric equations for the streakline to be

$$x_1 = \alpha_1 \frac{1+t}{1+\tau}, x_2 = \alpha_2 e^{t-\tau}, x_3 = \alpha_3 .$$

Example:

Given the two dimensional problem

$$v_1 = kx_2, v_2 = 0.$$

Obtain

(a) the streamline passing through the point (α_1, α_2),

(b) the pathline for the particle (X_1, X_2),

(c) the streakline for the particles which passed through the point (α_1, α_2).

Solution:

(a) the streamline passing through the point (α_1, α_2) is

$$\frac{dx_1}{ds} = kx_2 \,, \frac{dx_2}{ds} = 0$$

thus,

$$x_2 = \alpha_2 \,, x_1 = \alpha_1 + k\alpha_2 s \ldots..0 \leq s < \infty$$

This is obviously a straight line parallel to the x_1 axis,

(b) the pathline for the particle (X_1, X_2) is

$$\frac{dx_1}{dt} = kx_2 \,, \frac{dx_2}{dt} = 0$$

thus,

$$x_2 = X_2 \,, x_1 = X_1 + kX_2 t \quad \ldots..0 \leq t < \infty$$

Again, this is a straight line parallel to the x_1 axis,

(c) the streakline for the particles, we have:

$$X_1 = x_1 - kx_2 t, X_2 = x_2$$

Therefore,

$$X_1 = \alpha_1 - k\alpha_2 \tau, X_2 = \alpha_2$$

By substitution, we obtain:

$$x_1 = \alpha_1 - k\alpha_2(t - \tau) \; x_2 = \alpha_2 \quad -\infty \leq \tau < t$$

Again, this is a straight line parallel to the x_1 axis [6].

Bibliographic references:

[1] Eduardo W.V. Chaves ,"Notes on Continuum Mechanics".

[2] Nicolas MOËS, Mécanique des milieux continus

[3] Han-Chin Wu, Continuum Mechanics and Plasticity;

[4] Magnus Ekh, 2019, Mechanics of solids and fluids- Introduction to continuum mechanics.

[5] X. Oliver, C. Agelet de Saracibar-CONTINUUM MECHANICS FOR ENGINEERS THEORY AND PROBLEMS-2nd Edition

[6] W.M.Lai, D. Ruban, E Kremple , Introduction to Continuum Mechanics 3rd

Chapitre VI:

Constitutive equations

Chapter VI: Constitutive equations

- Introduction
- Elastic solids
- Linear elasticity
- Generalized Hooke's law
- Generalized Hooke's law for isotropic materials
- Solving a problem in the mechanics of deformable solids

I. Introduction :

Constitutive equations are mathematical models of the behavior of materials that are validated against experimental results. The differences between theoretical predictions and experimental findings are often attributed to inaccurate representation of the constitutive behavior.
A material body is said to be homogeneous if the material properties are the same throughout the body (i.e., independent of position). In a heterogeneous body, the material properties are a function of position. An isotropic material is one for which every material property is the same in all directions at a point. An isotropic material can be homogeneous or heterogeneous.
Materials for which the constitutive behavior is only a function of the current state of deformation are known as elastic.
Constitutive equations are often postulated directly from experimental observations. The mathematical model involves undetermined parameters that characterize certain responses of the material being studied.

II. Elastic solids

The constitutive equations to be developed here for the stress tensor σ do not include creep at constant stress and stress relaxation at constant strain. Thus, the material coefficients that specify the constitutive relationship between the stress and strain components are assumed to be constant during the deformation.
Materials for which the constitutive behavior is only a function of the current state of deformation are known as elastic. For an elastic material, all of the deformation is recoverable on removal of loads causing the deformation and there is no loss of energy, that is, loading and unloading is along the same stress-strain path, as shown in following Figure:

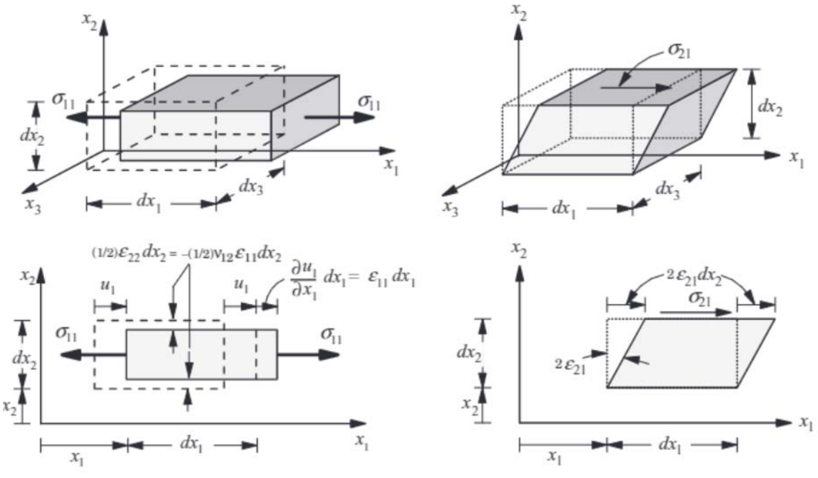


Figure.VI.1. Strains produced by stresses in a cube of material.

A linearly elastic material is one for which the relationship between the stress and strain is linear. A nonlinearly elastic material is one that has a nonlinear relationship between stress and strain.

III. Linear elasticity:

The linear theory of elasticity can be considered a simplification of the general theory of elasticity, but a close enough approximation for most engineering applications.

IV. Generalized Hooke's law:

Hooke's law for unidimensional problems establishes the proportionality between the stress, σ, and the strain, ε, by means of the constant named elastic modulus, E,

$$\sigma = E\varepsilon \, .$$

In the theory of elasticity, this proportionality is generalized to the multidimensional case by assuming the linearity of the relation between the components of the stress tensor σ and those of the strain tensor ε in the expression known as generalized Hooke's law,

$$\sigma_{ij} = C_{ijhk}\varepsilon_{hk}; \text{ or } \{\sigma\} = [C]\{\varepsilon\}$$
$$\varepsilon_{ij} = S_{ijhk}\sigma_{hk}; \text{ or } \{\varepsilon\} = [S]\{\sigma\}$$

$[C]$ is the stiffness matrix,
$[S]$ is the flexibility matrix.

V. Generalized Hooke's law for isotropic materials:

Isotropic materials are those for which the material properties are independent of the direction, and we have:

$$E_1 = E_2 = E_3 = E, G_{12} = G_{13} = G_{23} = G = \frac{E}{2(1+\nu)}; \nu_{12} = \nu_{23} = \nu_{13} = \nu.$$

The strain-stress relations are:

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix}$$

The stress-strain relations are:

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{Bmatrix}$$

The stiffness coefficients become with : [$\Delta = (1+\nu)^2(1-2\nu)$

$$C_{11} = C_{22} = C_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)},$$
$$C_{12} = C_{13} = C_{23} = \frac{\nu E}{(1+\nu)(1-2\nu)};$$
$$C_{44} = C_{55} = C_{66} = G.$$

The stress-strain relations can be written in the index-notation form:

$$\sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}.$$

An isotropic elastic behavior is completely described by two scalar material parameters: the two Lamé coefficients λ and μ. the isotropic elastic behavior is written:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

This law is called Hooke's law. Rather than using Lamé coefficients, we can describe the behavior in terms of Young's modulus, E, and Poisson's ratio ν:

$$\sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}.$$

The Lamé coefficients are related to Young's modulus and Poisson's ratio by the following relations:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

and vice versa:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

The inverse relations are:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

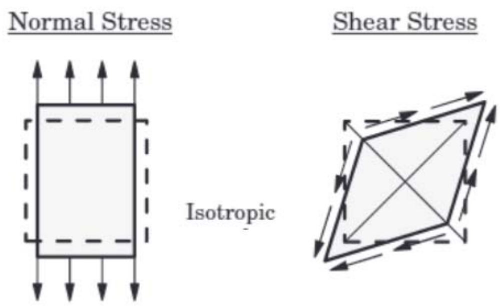


Figure.VI.2. Deformation of isotropic rectangular block under uniaxial tension[1].

VI. Solving a problem in the mechanics of deformable solids:

A general problem in the mechanics of deformable solids is posed as follows: let a deformable solid (D) be surrounded by a surface (S). This system is subjected to volume forces $\vec{F}$ (M, t) of components ($\vec{F}$i) and to conditions on the boundary (S) called boundary conditions or border conditions (Figure.VI.3.).

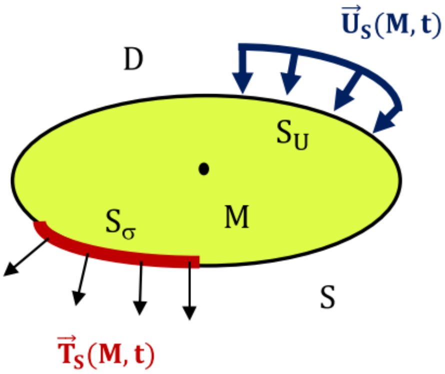


Figure.VI.3. Deformable solid (D) be surrounded by a surface (S).

To solve any an isotropic linear elasticity problem , we summarizes the equations available to us in index notation:

Kinematic compatibility equations:

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ with boundary conditions on displacement Su

dynamic equation: $\sigma_{ij,j} + f_i = \rho\frac{dx_i^2}{dt^2}$ [2].

Static equilibrium equation:

$$\sigma_{ij,j} + f_i = 0$$

Surface equilibrium (Stress boundary conditions) : on A_T : $\sigma_{ij}n_j = T_i$

on V stress state symmetry $\sigma_{ij} = \sigma_{ji}$

Behavior equations :…. on V: (isotrpic material) $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

Let us count the number of equations and the number of unknowns in V. The number of unknowns in V is:

– 3 for displacement $\vec{u}$;
– 9 for stress σ ;
– 9 for deformations ε ;
– a total of 21 unknowns.

Concerning the equations on V, we have:

– 9 compatibility equations,
– 3 static equilibrium equations,
– 3 stress tensor symmetry conditions;
– 9 behavior equations;
– a total of 24 equations. In reality, only 21 of these equations are independent because given the symmetry of the stress tensor, only 6 constitutive (behavior) equations are necessary [3].

Bibliographic references:

[1] J. N. Reddy, 2010,"Principles of Continuum Mechanics A Study of Conservation Principles with Applications".
[2] DEGHBOUDJ Samir, 2016, POLYCOPIE DE COURS « MECANIQUE DES MILIEUX CONTINUS COURS ET APPLICATIONS ».
[3] Nicolas MOËS, 2003, « Mécanique des milieux continus ».

Chapitre VII:

Applications in solid Mechanics

Chapter VII: Applications in solid Mechanics

1. Introduction :

In solid and structural mechanics, not all problems are solved by resorting to the equations of elasticity. Due to the nature of the stress and stress fields experienced, certain problems can be reduced to two-or-one dimensional problems by certain simplifying assumptions.

In this study, we will primarily study the problems of bars, beams, and some simple two-dimensional elasticity problems.

II. Analysis of bars:

If the body force per unit length is *f(x)*.
The equation governing the equilibrium of a bar is:

$$\frac{\partial}{\partial x}(A\sigma) + f = 0 \quad \Rightarrow 0<x<L.$$

Let the axial stress in the bar at *x* be $\sigma(x)$.
Here *A* denotes the area of cross section,
N(x) is the axial force: $N(x) = \int_A \sigma_{X1X1} dA,$

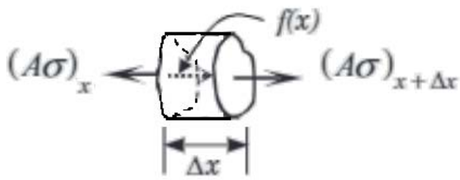


Figure.VII.1. Element of a bar subjected to axial load.

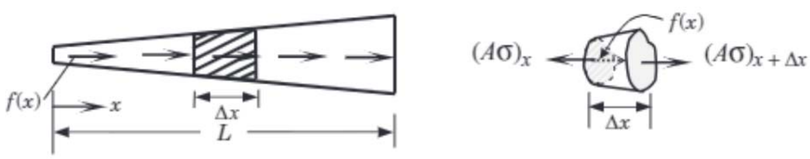


Figure.VII.2. A bar of variable cross section subjected to axial load.

The axial stress is related to the axial strain by:

$$\sigma_{X1X1} dA = \int_A E \cdot \varepsilon_{X1X1} dA = \int_A E \frac{du_{X1}}{dX1} dA = EA \frac{du_{X1}}{dX1}$$

The equilibrium equation reduces to: $\frac{d}{dX1}\left(EA\frac{du_{X1}}{dX1}\right) + f = 0, \quad 0<x<L.$

The second-order differential equation in *u* requires 2 boundary conditions to completely determine the solution *u(x)*.
The boundary conditions are provided by the known geometric constraints and forces applied to the bar.
The general solution of equation is obtained by successive integrations:

$$EA\frac{du_{X1}}{dX1} = -\left(\int f(x)dx + C_1\right), \qquad \text{eq (1)}$$

$$u(x) = \left\{\int \left[\frac{1}{EA}\left(\int f(x)dx + C_1\right)\right] dx + C_2\right\}; \qquad \text{eq(2)}$$

Where C_1 and C_2 are constants of integration that are to be evaluated using known conditions on *u* and $EA(du_{X1}/dX_1)$. Such conditions are called boundary conditions.

From the eqs (1) and (2), it should be clear that the solution can be obtained only if the following integrals can be evaluated analytically:

$$\int f(x)dx \,, \int \left[\frac{1}{EA}\left(\int f(x)dx + C_1\right)\right] dx.$$

Example:

Consider the composite bar consisting of a tapered steel bar fastened to an aluminum rod of uniform cross section and subjected to loads, as shown in Figure.VII.3.
Determine the displacement field in the composite bar?
Take the following values of the data:

E_1=200 GPa, $A_1 = \left(1.5 - \frac{x}{1.92}\right)^2 \times 10^{-4} m^2$,
E_2=73 GPa,
A_2=10^{-4} m^2, *a*=0.96 m, *L*=2.16 m, P_1=2.000 N,
P_2=1.000 N.

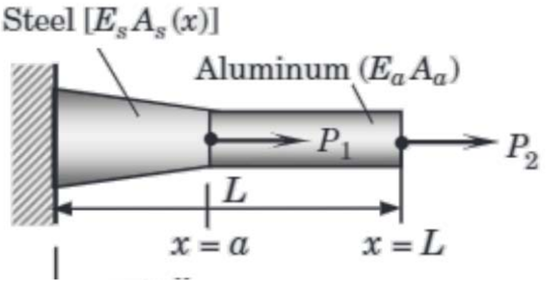


Figure.VII.3. Axial deformation of a composite member.

Solution:
The governing equations in each part of the composite bar are given by:

$$\frac{d}{dx}\left(E_1A_1\frac{du_1}{dx}\right) = 0, \qquad 0 < x < a,$$

$$\frac{d}{dx}\left(E_2A_2\frac{du_2}{dx}\right) = 0, \qquad 0 < x < L,$$

Where the subscript 1 refers to steel and 2 to aluminum.
The solutions of the previous two equations are:

$$u_1(x) = \frac{C_1}{(2.88 - x)} + C_2, \qquad 0 < x < a,$$

$$u_2(x) = C_3x + C_4, \qquad 0 < x < L,$$

Where C_1, C_2, C_3 and C_4 are constants of integration, which are to be determined using the boundary and interface conditions. The boundary conditions are:

$$u_1(0) = 0, \left(E_2A_2\frac{du_2}{dx}\right)_{x=2.16} = P_2.$$

The interface conditions include the continuity of the displacement at x=*a*,

$$u_1(a) = u_2(a),$$

And the balance of forces at x=*a*,

$$\left[\left(E_1A_1\frac{du_1}{dx}\right)_{x=0.96+} - \left(E_2A_2\frac{du_2}{dx}\right)_{x=0.96-}\right] = P_1.$$

The condition $u_1(0) = 0$ gives C_1=-2.88 C_2 and the second condition $C_3 = \frac{P_2}{E_2A_2} = \frac{10^{-2}}{73}$.

From the condition $u_1(a) = u_2(a)$, we obtain the equation:

$$\left(1 - \frac{2.88}{1.92}\right)C_2 = 0.96\frac{P_2}{E_2A_2} + C_4;$$

And the other continuity condition gives:

$$\left(E_1C_1\frac{1}{(1.92)^2} - E_2C_3\right)10^{-4} = P_1 \text{ or : } -2.88E_1C_2\frac{10^{-4}}{(1.92)^2} = P_1 + P_2$$

Solving for the constants, we obtain:

$$C_1=5.5296\times10^{-4},\ C_2=-1.92\times10^{-4},\ C_3=1.36986\times10^{-4},\ C_4=-0.94685\times10^{-4}.$$

The displacement field in the composite bar is:

$$\begin{cases} u_1(x) = \left(\dfrac{1.92x}{2.88-x}\right)10^{-4}\ \text{m}, & 0\le x\le 0.96, \\ u_2(x) = (1.36986x - 0.94685)10^{-4} & 0.96 \le x \le 2.16\,. \end{cases}$$

The strain field will be:

$$\begin{cases} \varepsilon_1(x) = \dfrac{5.5296}{(2.88-x)^2}10^{-4}\ m/m\,, & 0 \le x \le 0.96, \\ \varepsilon_2(x) = 1.36986\times10^{-4}\quad m/m, & 0.96 \le x \le 2.16\,. \end{cases}$$

The stress field will be:

$$\begin{cases} \sigma_{X1} = E\cdot\varepsilon_1 = \dfrac{11.0592}{(2.88-x)^2}10^{7}\ N/m, & 0 \le x \le 0.96, \\ \sigma_{X2} = E\cdot\varepsilon_2 = 10^{7}\ \ N/m, & 0.96 \le x \le 2.16\,. \end{cases}$$

It can be verified that the force at $x=L$ is equal to $P2=10^3$ N and that at $x=0$ is equal to $P_1+P_2=3\times10^3$ N.

III. Analysis of beams:

Another elasticity problem whose governing equation cannot be deduced directly from the equations of elasticity is a beam. A beam is a solid whose cross-sectional dimensions are much smaller than the longitudinal dimension (as in case of bars) and subjected to loads that are transverse to the length, and hence tend to bend the solid about an axis (X_2 or X_3) perpendicular to the beam axis (X_1). Thus, a beam differs from a bar only in the way forces are applied. In the case of a bar, the forces applied are axial and tend to elongate or shorten the bar. On the other hand, forces applied on a beam are in a direction transverse to the length and tend to bend the beam.

The Euler-Bernoulli beam theory: (pure bending of beams):

Let we consider a thin strip of material of length L, height h and thickness t in the undeformed configuration. If the strip is bent into an arc of a circle, as shown in Figure.VII.4., determine the axial strain ε_{xx} induced in the strip?
Assume that the changes in the height h and thickness t are negligible.

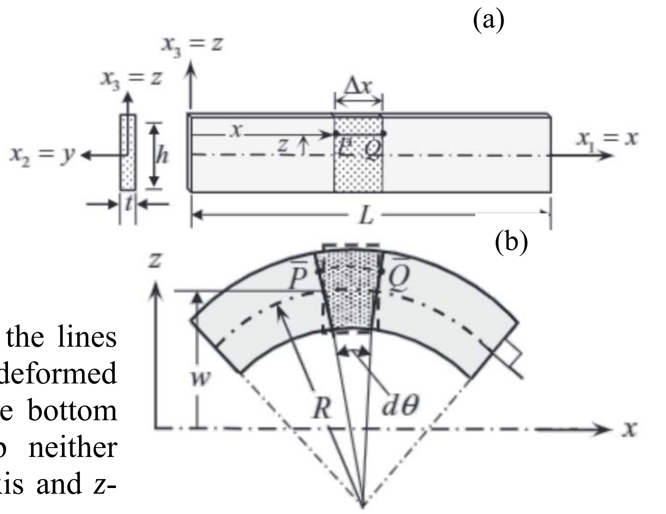


Figure.VII.4. Undeformed and deformed configurations of a strip.

From the deformed configuration, it is clear that the lines parallel to the length of the strip have deformed proportionately, with the top line elongating and the bottom line shortening. Thus, the centerline of the strip neither elongated nor shortened. All lines parallel to th y-axis and z-axis have remained the same length, and they remained perpendicular to the axis of the strip at every point along the length, implying that all shear strains are zero. Hence, the only nonzero is ε_{xx}.

To compute the infinitesimal strain ε_{xx}, we use the definition of change in length divided by the original length. We have:

$$\varepsilon_{xx} = \frac{\overline{P}\,\overline{Q} - PQ}{PQ} = \frac{(R+z)\theta - R\theta}{R\theta} = \frac{z}{R}.$$

From a course on geometry, the *radius of curvature* $\boldsymbol{R}$(1/$\boldsymbol{R}$ is called *curvature*) is related to the transverse displacement *w* of the center line of the strip by:

$$\frac{1}{R} = -\frac{\frac{d^2w}{dx^2}}{\left[1+\left(\frac{dw}{dx}\right)^2\right]^{3/2}} \approx -\frac{d^2w}{dx^2}.$$

Thus, the infinitesimal strain is $\varepsilon_{xx} = -z\frac{d^2w}{dx^2}$.

Now, let we suppose that the beam is subjected to distributed axial force *f(x)* and transverse laod *q(x)*, as shown in Figure.VII.5a. Consider a typical element of length *dx* from the beam acted by area-integrated forces and moments, as shown in Figure.VII.5b.
Using the principal of linear momentum, derive the equations governing the equilibrium of the beam.

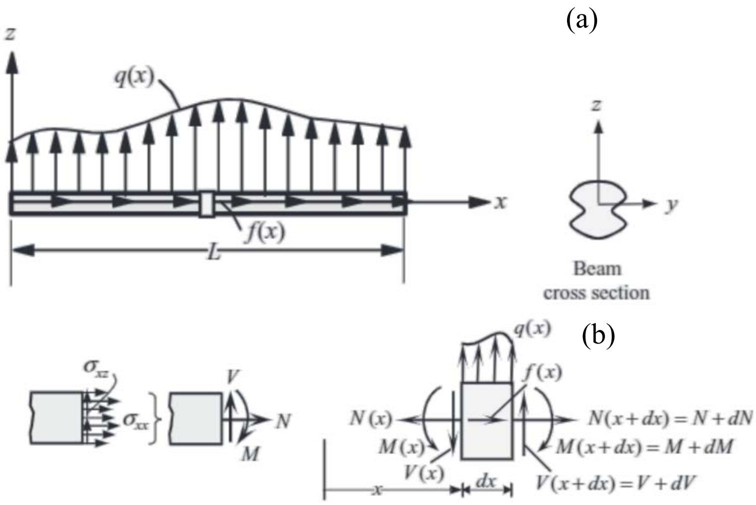


Figure.VII.5. Bending beams (a) Atypical beam with axial and transverse forces, (b) Equilibrium of a beam element.

As shown in Figure.VII.5b., the following area-integrated forces and moment, called the stress resultants, are defined:

$$N(x) = \int_A \sigma_{xx}\, dA,\ M(x) = \int_A \sigma_{xx}\, z\, dA\ ,\ V(x) = \int_A \sigma_{xz}\, dA$$

Here *A* denotes the area of cross section, *N(x)* is called the *axial force*, *V(x)* is called the transverse *shear force* and *M(x)* is called the *bending moment*.

First, sum the forces acting in the x-direction on the element of the beam and obtain:

$$\sum F_x = 0: -N + (N + dN) + f(x)dx = 0,$$

Dividing throughout by *dx*: $\frac{dN}{dx} + f(x) = 0.$

Next, we sum the forces in the z-direction to obtain :

$$\sum F_z = 0: -V + (V + dV) + qdx = 0,$$

Dividing throughout by *dx*: $\frac{dV}{dx} + q = 0.$

Finally, summing the moments of all forces about y-axis using the point on the right end of the beam, we obtain (clockwise moments are taken positive)

$$\sum M_y = 0: -V\,dx - M + (M + dM) - (qdx)(\alpha\, dx) = 0,$$

Where α is a constant, 0 < α < 1. Dividing throughout by *dx* we obtain:

$$-V + \frac{dM}{dx} = 0 \text{ or } V = \frac{dM}{dx}.$$

We can combined the equations by eliminating the shear force V,

$$\frac{d^2M}{dx^2} + q = 0;\ 0 < x < L\ ;$$

The kinematics of deformation of the beam must be studied to determine the strain-displacement relations. Central to the development of the strain-displacement relations of beam is the following three-part hypothesis:

(1) Plane sections before deformation remain plane after deformation, that is, straight lines perpendicular to the axis of the beam before deformation remain straight after deformation.

(2) Plane sections rotate about the y-axis as rigid disks such that they always remain perpendicular to the centroidal axis, which is now bent into an arc of a circle; that is, straight lines perpendicular to the centroidal axis of the beam before deformation remain normal after deformation.

(3) Plane sections do not change their geometric dimensions, that is, straight lines perpendicular to the axis of the beam remain inextensible during deformation. This amounts to neglecting the Poisson effect.

This set of assumptions is known as the Euler-Bernoulli hypothesis, and the resulting beam equations are known as the Euler-Bernoulli beam theory. The consequence of assumptions 1 and 2 is that the transverse shear strain $\gamma_{xz} = 0$. Assumption 3 implies that the transverse normal strain is $\varepsilon_{zz} = 0$.
Strains ε_{yy}, γ_{xy} and γ_{yz} are zero because the Poisson effect is neglected and the bending in the xz-plane.
The only nonzero strain is ε_{xx}, which was to be equal to : $\varepsilon_{xx} = -z\frac{d^2w}{dx^2}$.
Where w is the transverse deflection of the beam. The stresses vary from one section to the other along the length, and they vary linearly along the height.
Because the stresses are not uniform over the cross section, the net moment due to the stress $\sigma_{xx} = E\varepsilon_{xx}$ is defined by:

$$M = \int_A z\sigma_{xx}dA = \int_A z(E\varepsilon_{xx})dA = -E\frac{d^2w}{dx^2}\left(\int_A z^2dA\right) = -EI\frac{d^2w}{dx^2}.$$

Where I is the *second moment of area*.
The shear force is given by:

$$V = \frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right).$$

Combining the equations we arrive at the following fourth-order equation governing the transverse deflection w :

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q(x), 0 < x < L.$$

To solve this equation four boundary conditions are required.
First, the bending moment M can be determined for any given load and independent of the material and cross-sectional area as

$$M(x) = K_1 + K_2x - \int(\int q(x)dx)dx,$$

Where K_1 and K_2 constants of integration, which are determined using the boundary conditions on M and $dM/dx=V$.
As an example, consider a beam of length L, hinged at both ends (called *simply supported*), and subjected to uniformly distributed transverse load of intensity q0 qs shown in Figure.6. for this case, the bending moment becomes:

$$M(x) = K_1 + K_2x - \frac{q_0x^2}{2}.$$

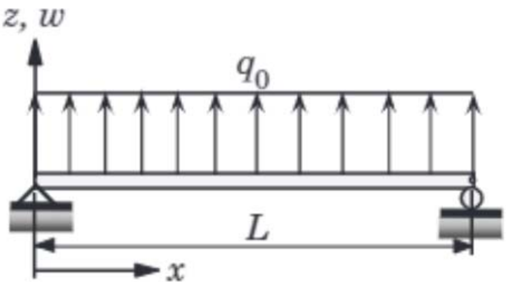


Figure.VII.6. Asimply supported beam under uniformly distributed load.

The boundary conditions on M are $M(0)=0$ and $M(L)=0$.
Hence, we obtain K_1=0 and $K_2=q_0L/2$ and the bending moment at any point x is :

$$M(x) = \frac{q_0}{2}x(L - x), 0 < x < L.$$

The maximum bending moment occurs at the center of the beam:

$$M_{max} = M(L/2) = \frac{q_0L^2}{8}.$$

Next, the transverse displacement w is determined, by integrating the equation successively four times, we obtain:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q(x) \rightarrow \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = \int q(x)\,dx + C_1 \rightarrow EI\frac{d^2w}{dx^2} = \int\left(\int q(x)\,dx\right)dx + C_1x + C_2$$

$$\frac{dw}{dx} = \int\frac{1}{EI}\left[\int\left(\int q(x)\,dx\right)dx\right]dx + C_1\int\frac{x}{EI}dx + C_2\int\frac{1}{EI}dx + C_3$$

$$w = \int\left\{\int\frac{1}{EI}\left[\int\left(\int q(x)\,dx\right)dx\right]dx\right\}dx + C_1\int\left(\int\frac{x}{EI}dx\right)dx + C_2\int\left(\int\frac{1}{EI}dx\right)dx + C_3x + C_4,$$

The constants of integration C_1, C_2, C_3 and C_4 are determined using the boundary conditions on the displacement w, for the simply supported beam under uniform transverse load the boundary conditions are:

$$w(0) = 0, w(L) = 0, M(0) = 0, M(L) = 0.$$

The boundary conditions on the bending moment give: $C_2=0$, $C_1=-q_0L/2$, and :

$$M(x) = -EI\frac{d^2w}{dx^2} = \frac{q_0}{2}x(L-x) \quad , 0 < x < L.$$

If EI is a constant $\rightarrow$ $w(x) = \frac{q_0x^4}{24EI} - \frac{q_0Lx^3}{12\,EI} + C_3x + C_4$.

We find $C_4=0$ and $C_3=(q_0L^3/24\,EI)$. The displacement w becomes:

$$w(x) = \frac{q_0x^4}{24EI} - \frac{q_0Lx^3}{12\,EI} + \frac{q_0xL^3}{24\,EI} = \frac{q_0}{24\,EI}(x^4 - 2x^3L + L^3x).$$

The maximum displacement occurs at the center of the beam and is equal to:

$$w_{max} = w(L/2) = \frac{5q_0L^4}{384\,EI}.$$

Once the deflection w is known, we can compute the strain, stress, bending moment and shear force from the relations:

$$\varepsilon_{xx} = -z\frac{d^2w}{dx^2};\ \sigma_{xx} = \frac{Mz}{I} = -E\frac{d^2w}{dx^2};\quad M = -EI\frac{d^2w}{dx^2}.;\ V = \frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)$$

These equations are valid for beams with any boundary conditions.

IV. Principle of superposition:

For linear boundary value problems, the principle of superposition holds. The principle of superposition is said to hold for a solid body if the displacements obtained under two sets of boundary conditions and forces are equal to the sum of the displacements that would be obtained by applying each set of boundary conditions and forces separately. Suppose that the solution to the two problems is and , respectively. The superposition of the two sets of boundary conditions is: $u=u^{(1)}+u^{(2)}$; or $f=f^{(1)}+f^{(2)}$.

Because of the linearity of the elasticity equations, the solution of the boundary value problem with the superposed data is $u(x)=u^{(1)}(x)+u^{(2)}(x)$ in Ω . this is known as the superposition principle. It can be used to represent a linear problem with complicated boundary conditions or loads as a combination of linear problems that are equivalent to the original problem.

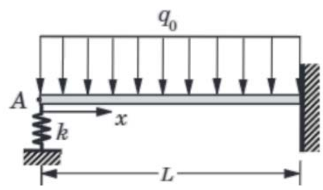


Figure.VII.7. The indeterminate beam.

Example:

Consider the indeterminate beam shown in Figure.VII.7. Determine the deflection of point A using the superposition principle.

Solution:

The solution to the problem can be viewed as the superposition of the solutions of the two beam problems shown there. Within the restrictions of the linear Euler-Bernoulli beam theory, the deflections are linear functions of the loads. Therefore, the principle of superposition holds. In particular, the deflection w_A at point A is

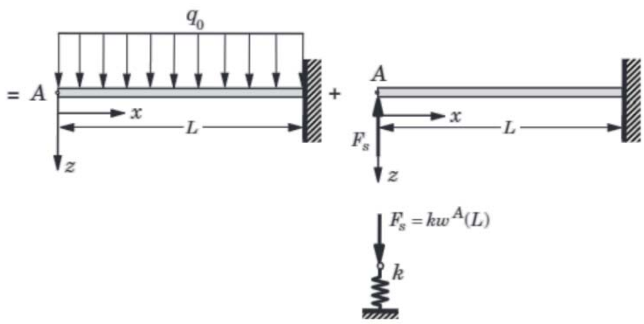


Figure.VII.8. The solution using the superposition principle.

equal to the sum of w_A^q and w_A^{Fs} due to the distributed load q_0 and spring force F_s respectively, at point A:

$$w_A = w_A^q + w_A^{Fs} = \frac{q_0 L^4}{8EI} - \frac{F_s L^3}{3EI}$$

Because the spring force F_s is equal to Kw_A, we can calculate w_A from

$$w_A = \frac{q_0 L^4}{8EI} - \frac{F_s L^3}{3EI} = \frac{q_0 L^4}{8EI} - \frac{(kw_A)L^3}{3EI} \Rightarrow w_A + \frac{Kw_A L^3}{3EI} = \frac{q_0 L^4}{8EI} \Rightarrow w_A\left(1 + \frac{kL^3}{3EI}\right) = \frac{q_0 L^4}{8EI} \Rightarrow w_A = \frac{q_0 L^4}{8EI\left(1+\frac{kL^3}{3EI}\right)}$$ [1].

V. Analysis of plane elasticity problems:

A class of problems in elasticity, due to geometry, boundary conditions and external applied loads, has their solutions (i.e., displacements and stresses) not dependent on one of the coordinates. Such problems are called *plane elasticity* problems. The plane elasticity problems considered here are grouped into *plane strain* and *plane stress* problems. Both classes of problems are described by a set of two coupled partial differential equations expressed in terms of two dependent variables that represent the two components of the displacement vector. The governing equations of plane strain problems differ from those of the plane stress problems only in the coefficients of the differential equations. The discussion here is limited to isotropic materials.

VI. Plane strain problems:

Plane strain problems are characterized by the displacement field,

. $u=u(x_1,x_2)$, $v=v(x_1,x_2)$, $w=0$.

Where (u,v,w) denote the components of the displacement vector $\vec{u}$ in the (x_1,x_2,x_3) coordinate system. An example of plane strain is provided by the long cylindrical member under external loads that are independent of x_3, as shown in Figure .VII.9. for cross sections sufficiently far from the ends, it is clear that the displacement u_{x3} is zero and that x_1 and x_2 are independent of x_3, that is a plane state of plane strain exists.

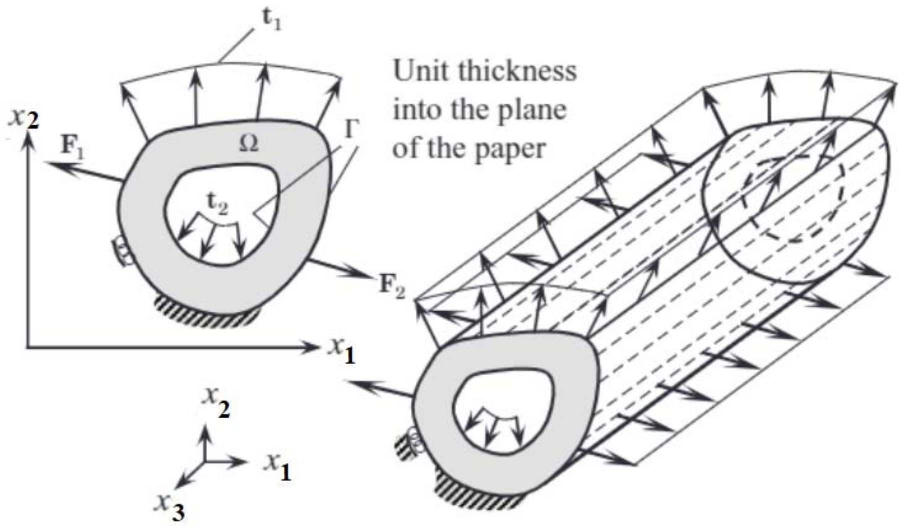


Figure.VII.9. Example of plane strain problem.

The displacement field of equation, results in the following strain field:

$$\varepsilon_{x1x3} = \varepsilon_{x2x3} = \varepsilon_{x3x3} = 0,$$

$$\varepsilon_{x1x1} = \frac{\partial u}{\partial x1}, 2\varepsilon_{x1x2} = \frac{\partial u}{\partial x2} + \frac{\partial v}{\partial x1}, \varepsilon_{x2x2} = \frac{\partial v}{\partial x2}.$$

Clearly, the body is in a state of plane strain.

For an isotropic material, the stress components are given by:

$$\sigma_{x1x3} = \sigma_{x2x3} = 0, \sigma_{x3x3} = \nu(\sigma_{x1x1} + \sigma_{x2x2}),$$

$$\begin{Bmatrix} \sigma_{x1x1} \\ \sigma_{x2x2} \\ \sigma_{x1x2} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x1x1} \\ \varepsilon_{x2x2} \\ 2\varepsilon_{x1x2} \end{Bmatrix}.$$

The equation of equilibrium of three-dimensional linear elasticity, with the body force components : $f_{x3} = 0,\ f_{x1} = f_{x1}(x_1, x_2),\ f_{x2} = f_{x2}(x_1, x_2),$

Reduce to the following two plane strain equations:

$$\frac{\partial \sigma_{x1x1}}{\partial x1} + \frac{\partial \sigma_{x1x2}}{\partial x2} + \rho f_{x1} = 0,$$

$$\frac{\partial \sigma_{x1x2}}{\partial x1} + \frac{\partial \sigma_{x2x2}}{\partial x2} + \rho f_{x2} = 0,$$

The boundary conditions are either the stress type,

$$\begin{cases} T_{x1} = \sigma_{x1x1} n_{x1} + \sigma_{x1x2} n_{x2} = \overrightarrow{t_{x1}} \\ T_{x2} = \sigma_{x1x2} n_{x1} + \sigma_{x2x2} n_{x2} = \overrightarrow{t_{x2}} \end{cases}, \text{ on } \Gamma\sigma$$

Or the displacement type,

$$u = \vec{u}, v = \vec{v} \ \text{ on } \Gamma u,$$

Here (n_{x1}, n_{x2}) denote the components (or directional cosines) of the unit normal vector on the boundary Γ, Γσ and Γu are disjoin portions of the boundary Γ, $\vec{t}_{x1}$and $\vec{t}_{x2}$are the components of the specified traction vector, and $\vec{u}$ and $\vec{v}$ are the components of specified displacement vector. Only one element of each pair, $(u, t_{x1})$and (v, t_{x2})may be specified at a boundary point.

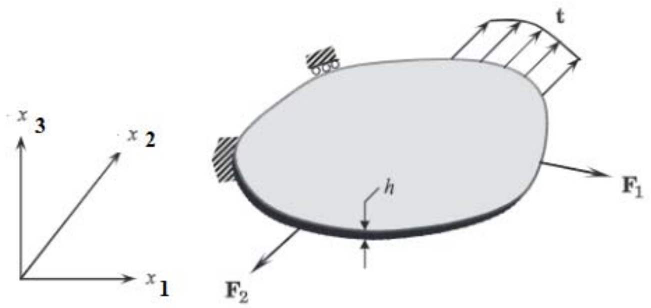


Figure.VII.10. A thin plate in a state of plane stress.

VII. Plane stress problems:

A state of plane stress is defined as one in which the following stress field exists:

$$\sigma_{x1x3} = \sigma_{x2x3} = \sigma_{x3x3} = 0,$$

$$\sigma_{x1x1} = \sigma_{x1x3}(x_1, x_2), \sigma_{x1x2} = \sigma_{x1x2}(x_1, x_2), \sigma_{x2x2} = \sigma_{x2x2}(x_1, x_2).$$

An example of a plane stress problem is provided by a thin plate under external loads applied in the x_1x_2 plane (or parallel to it) that are independent of x_3, as shown in Figure. . the top and bottom surfaces of the plate are assumed to be traction-free, and the specified boundary forces are in the x_1x_2 plane so that $f_{x3} = 0$ and $u_{x3} = 0$,

The stress-strain relations of a plane stress state are:

$$\begin{Bmatrix} \sigma_{x1x1} \\ \sigma_{x2x2} \\ \sigma_{x1x2} \end{Bmatrix} = \frac{E}{(1-\nu)^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1+\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x1x1} \\ \varepsilon_{x2x2} \\ 2\varepsilon_{x1x2} \end{Bmatrix},$$

The equations of equilibrium as well as boundary conditions of a plane stress problem are the same as those listed below:

$$\frac{\partial \sigma_{x1x1}}{\partial x1} + \frac{\partial \sigma_{x1x2}}{\partial x2} + \rho f_{x1} = 0,$$

$$\frac{\partial \sigma_{x1x2}}{\partial x1} + \frac{\partial \sigma_{x2x2}}{\partial x2} + \rho f_{x2} = 0,$$

To unify the formulation for plane strain and plane stress, we introduce the parameter s,

$$s = \begin{cases} \dfrac{1}{1-\nu} \, for\ plane\ strain, \\ 1+\nu \, for\ plane\ stress. \end{cases}$$

Then the constitutive equations of plane stress as well as plane strain can be expressed as:

$$\sigma_{\alpha\beta} = 2\mu\left[\varepsilon_{\alpha\beta} + \left(\frac{s-1}{2-s}\right)\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right],$$

$$\varepsilon_{\alpha\beta} = \frac{1}{2\mu}\left[\sigma_{\alpha\beta} - \left(\frac{s-1}{s}\right)\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right].$$

Where α,β and γ take the values of 1 and 2. The compatibility equations,

$$\frac{\partial^2\varepsilon_{mn}}{\partial x_i\partial x_j} + \frac{\partial^2\varepsilon_{ij}}{\partial x_m\partial x_n} = \frac{\partial^2\varepsilon_{im}}{\partial x_j\partial x_n} + \frac{\partial^2\varepsilon_{jn}}{\partial x_i\partial x_m}.$$

For plane stress and plane strain now take the form:

$$\nabla^2\sigma_{\alpha\alpha} = -s\,\rho\,f_{\alpha,\alpha}.$$

VIII. Solving Plane Elastostatic Problems by using Airy stress function :

The Airy stress function is a potential function introduced to identically satisfy the equations of equilibrium. First, we assume that the body force vector **f** is derivable from a scalar potential *V* such that:

$\rho\mathbf{f} = -\nabla V$ or $f_x = -\frac{\partial V}{\partial x}$, $\rho f_y = -\frac{\partial V}{\partial y}$.

This amounts to assuming that body forces are conservative. Next, we introduce the Airy stress function Φ(x,y) such that:

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} + V,\ \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} + V,\ \sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}.$$

This definition of automatically satisfies the equations of equilibrium.

The stresses derived are subject to the compatibility conditions, we obtain:

$$\nabla^4\Phi + (2-s)\nabla^2 V = 0,$$

Where $\nabla^4 = \nabla^2\nabla^2$ is the *biharmonic operator*, which in two dimensions has the form:

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4};$$

If the body forces are zero, we have $V = 0$, which reduces to the *biharmonic equation*, $\nabla^4\Phi = 0$.

The parameter s is equal to:

$$s = \begin{cases} \dfrac{1}{1-\nu} & \text{for plane strain}, \\ 1+\nu & \text{for plane stress}. \end{cases}$$

In summary, the solution to a plane elastic problem using the Airy stress function involves finding the solution and satisfying the boundary conditions of the problem.

Example:

Consider a beam of length l, height h and thickness e, fixed at its end x_1=0, and subjected to a concentrated load P at its end $x_1 = l$ (neglecting volume forces).

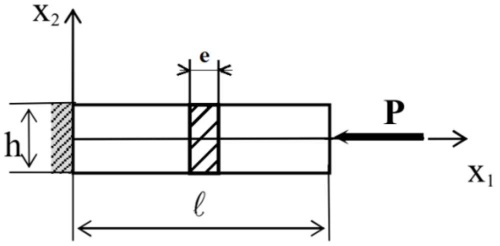


Figure.VII.11. A beam subjected to a concentrated load.

1°) Show that the problem can be solved using a 2nd degree as stress function; then determine σ_{11} , σ_{22} , σ_{12} at a point M(x_1, x_2).

2°) considering the case of plane stress, determine the displacements at a point M(x_1, x_2).

Solution:

1/ the stress function is: $\Phi(x_1, x_2) = A_2 x_1^2 + A_1 x_1 x_2 + A_0 x_2^2$

$\frac{\partial^4 \Phi}{\partial x_1^4} = 0; \frac{\partial^4 \Phi}{\partial x_2^4} = 0; \frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} = 0$; This problem is solved with this stress function,

The stress at a point M

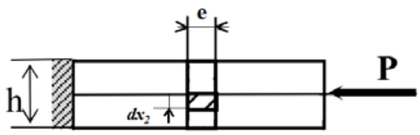


Figure.VII.12. Boundary conditions relating to applied loads.

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2} = 2A_0; \sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2} = 2A_2; \ \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} = -A_1;$$

Boundary conditions relating to applied loads:

For $x_2 = \mp \frac{h}{2}$ ➔ $\sigma_{22} = 0$ ➔ $\sigma_{12} = 0$ ➔ $A_1 = A_2 = 0$

On the edge, we have: $F_1 + P = 0$➔ $F_1 = -P$

$$F_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11} e \, dx_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} (2A_0) e \, dx_2 = 2A_0 e \int_{-\frac{h}{2}}^{\frac{h}{2}} dx_2 = 2A_0 e \, h = -P ➔ A_0 = -\frac{P}{2eh};$$

$$\sigma_{11} = -\frac{P}{eh};$$

2/ in the case of plane stress, the strains are:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{1}{E}(\sigma_{11} - \nu \sigma_{22}) = -\frac{P}{E\,eh}; \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \frac{1}{E}(\sigma_{22} - \nu \sigma_{11}) = \nu \frac{P}{E\,eh}:$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1+\nu}{E}\sigma_{12} = 0 \ ; \varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) = \frac{\nu}{E}\frac{P}{eh}$$

The displacements:

$$u_1 = \int \varepsilon_{11} dx_1 + F(x_2) = \int \left(-\frac{P}{Eeh}\right) dx_1 + F(x_2) = \left(-\frac{P}{Eeh}\right) x_1 + F(x_2);$$

$u_2 = \int \varepsilon_{22} dx_2 + G(x_1) = \int \left(\frac{\nu P}{Eeh}\right) dx_2 + G(x_1) = \left(\frac{\nu P}{Eeh}\right) x_2 + G(x_1);$

$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1+\nu}{E}\sigma_{12} = \frac{dF}{dx_2} + \frac{dG}{dx_1} = 0;$

$\frac{dF}{dx_2} = 0 \Rightarrow F(x_2) = C_1 \Rightarrow u_1 = \left(-\frac{P}{Eeh}\right) x_1 + C_1 \text{ ; } \frac{dG}{dx_1} = 0 \Rightarrow G(x_1) = C_2 \Rightarrow u_2 = \left(\frac{\nu P}{Eeh}\right) x_2 + C_2$

Boundary conditions relating to imposed displacements:

$u_1(0,0) = 0 \Rightarrow C_1 = 0 \Rightarrow u_1 = \left(-\frac{P}{Eeh}\right) x_1$

$u_2(0,0) = 0 \Rightarrow C_2 = 0 \Rightarrow u_2 = \left(\frac{\nu P}{Eeh}\right) x_2$ [2]

IX. Representative Curves of Plane Elasticity:

There is an important tradition in engineering of graphically representing the distribution of plane elasticity. To this aim, certain families of curves are used; whose plotting on the plane of analysis provides useful information of said stress state.

IX.1.1 Isostatics or stress trajectories:

The isostatics or stress trajectories are the envelopes of the vector field determined by the principal stresses.

Considering the envelope of a vector field, isostatics are; at each point, tangent to the two principal directions and, thus, there exist two families of isostatics:

- Isostatics σI, tangent to the direction of the largest principal stress,
- Isostatics σII, tangent to the direction of the smallest principal stress.

In addition, since the principal stress directions are orthogonal to each other, both families of curves are also be orthogonal. The isostatics lines provide information on the mode in which the flux of principal stresses occurs on the plane of analysis.

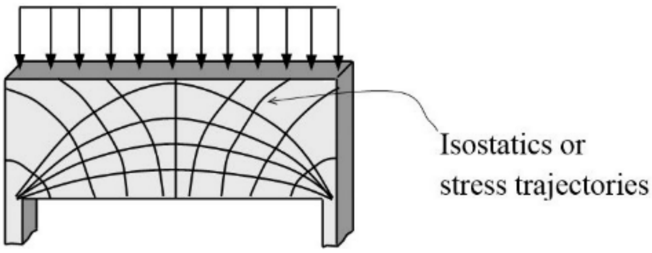


Figure .VII.13. Isostatics or stress trajectories on a beam.

A singular point is a point characterized by the stress state:

$\sigma_{x1x1} = \sigma_{x2x2}$ and $\tau_{x1x2} = 0.$

Its Mohr's circle is a point on the σ axis.

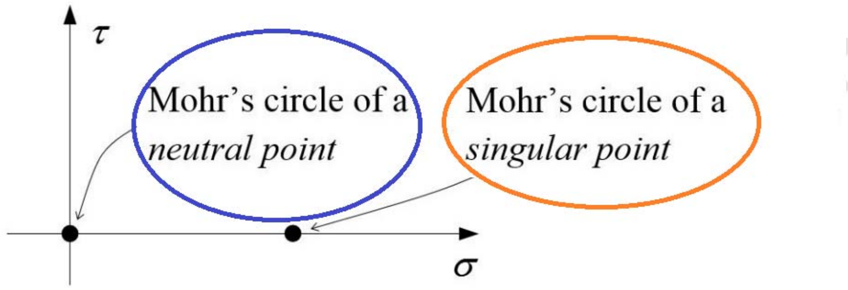


Figure.VII.14. Singular and neutral points.

IX.1.2. Differential equation of the isostatics:

Consider the general equation of an isostatics line $x_2 = f(x_1)$ and the value of the angle formed by the principal stress direction σI with respect to the horizontal direction,

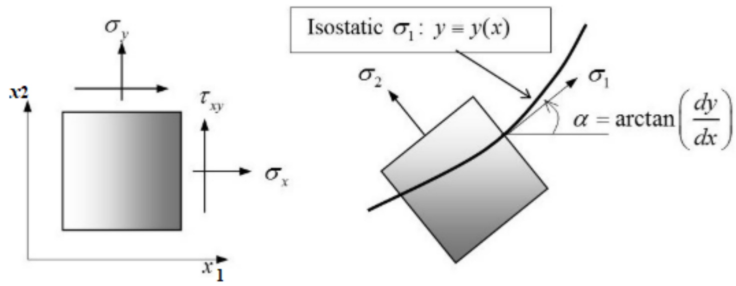


Figure.VII.15. Determination of the differential equation of the isostatics.

$$\begin{cases} tg(2\alpha) = \frac{2\tau_{x1x2}}{\sigma_{x1}-\sigma_{x2}} = \frac{2\,tg(\alpha)}{1-tg(\alpha)^2} \\ tg(\alpha) = \frac{dx_2}{dx1} = x'_2 \end{cases} \Rightarrow \frac{2\tau_{x1x2}}{\sigma_{x1}-\sigma_{x2}} = \frac{2\,x'_2}{1-{x'_2}^2} \Rightarrow {x'_2}^2 + \frac{\sigma_{x1}-\sigma_{x2}}{\tau_{x1x2}}\,x'_2 - 1 = 0$$

And solving the second –order equation for x'$_2$, the differential equation of isostatics is obtained.

$$x'_2 = -\frac{\sigma_{x1}-\sigma_{x2}}{2\tau_{x1x2}} \pm \sqrt{\left(\frac{\sigma_{x1}-\sigma_{x2}}{2\tau_{x1x2}}\right)^2 + 1}$$

The double sign leads to two differential equations corresponding to the two families of isostatics.

Example:

A rectangular plate is subjected to the following stress states.

$\sigma_{x1} = -x_1^3$; $\sigma_{x2} = 2x_1^3 - 3x_1x_2^2$; $\tau_{x1x2} = 3x_1^2x_2$; $\tau_{x1x3} = \tau_{x2x3} = \sigma_{x3x3} = 0$,

Obtain and polt the singular points and distribution of isostatics?

Solution:

The singular points are defined by:

$$\tau_{x1X2} = 3x_1^2x_2 = 0 \Rightarrow \begin{cases} x_1 = 0 \rightarrow \begin{cases} \sigma_{x1} = -x_1^3 = 0 \\ \sigma_{x2} = 2x_1^3 - 3x_1x_2^2 = 0 \end{cases} \forall x_2 \\ x_2 = 0 \rightarrow \begin{cases} \sigma_{x1} = -x_1^3 \\ \sigma_{x2} = 2x_1^3 - 3x_1x_2^2 = 2x_1^3 \end{cases} \rightarrow x_1 = 0 \end{cases}$$

Therefore, the locus of singular points is the straight line x$_1$=0, these singular points are, in addition, neutral points ($\sigma_{x1} = \sigma_{x2} = 0$).

The isostatics are obtained from:

$$x'_2 = -\frac{\sigma_{x1} - \sigma_{x2}}{2\tau_{x1x2}} \pm \sqrt{\left(\frac{\sigma_{x1} - \sigma_{x2}}{2\tau_{x1x2}}\right)^2 + 1}$$

Which , for the given data of this problem, results in

$$\begin{cases} \frac{dx_2}{dx_1} = \frac{x_2}{x_1} \\ \frac{dx_2}{dx_1} = -\frac{x_2}{x_1} \end{cases} \Rightarrow \text{integrating} \Rightarrow \begin{cases} x_1^2 - x_2^2 = C_1 \\ x_1 x_2 = C_2 \end{cases}$$

Therefore, the isostatics are two families of equilateral hyperboles orthogonal to each other.
To identify the family of isostatics, consider a point in each region.
-point (1,0) : (isostatic in the x_2 direction)
- point (-1,0) (isostatic in the x1 direction),
Finally the distribution of isostatics is as follows:

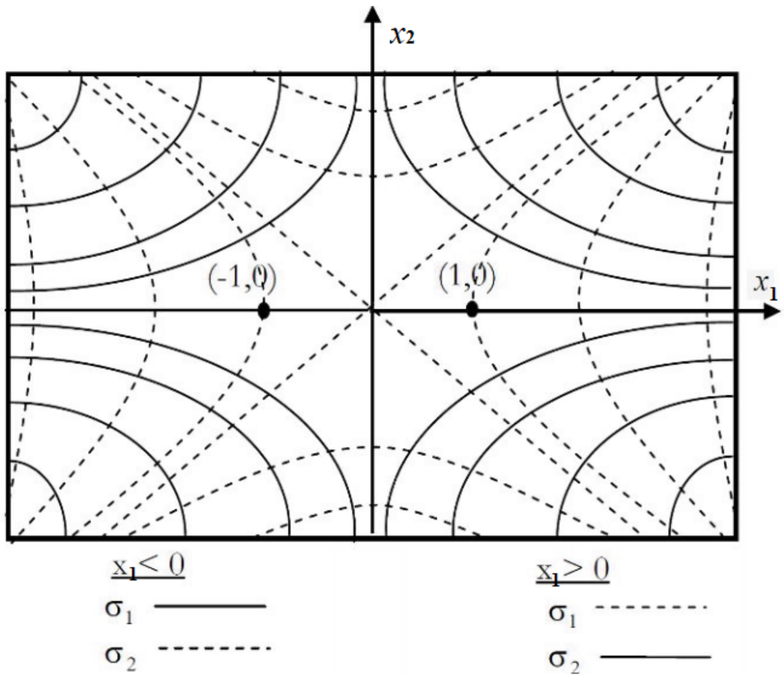


Figure.VII.16. Distribution of the isostatics on a rectangular plate.

IX. 2 Isoclines

Isoclines are the locus of the points in the plane of analysis along which the principal stress directions form a certain angle with the x_1-axis.

It follows from its definition that in all the points of a same isoclines the principal stress directions are parallel to each other, forming a constant angle θ (which characterizes the isoclines) with the x_1-axis.

The algebraic equation of the isoclines is : $tg(2\theta) = \frac{2\tau_{x1x2}}{\sigma_{x1}-\sigma_{x2}}$;
Which constitutes the equation of the family of isoclines parametrized in terms of the angle θ.

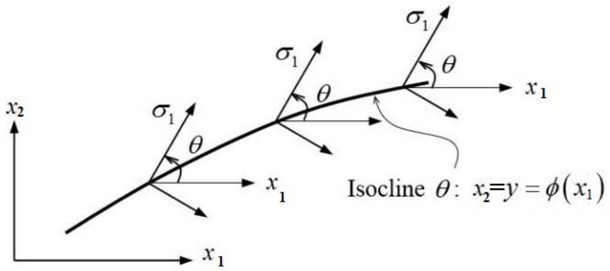


Figure.VII.17. Isocline.

IX. 3.Isobars

Isobars are the locus of points in the plane of analysis with the same value of principal stress σI (or σII).

Two families of isobars will cross at each point of the plane of analysis: one corresponding to σI and another to σII. Note that the isobars depend on the value of σI, but not on its direction.

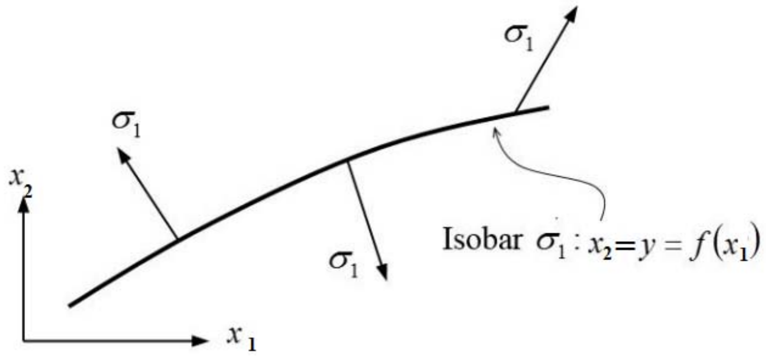


Figure.VII.18. Isobars.

The equation that provides the value of the principal stresses implicitly defines the algebraic equation of the two families of isobars.

Bibliographic references:

[1] J. N. Reddy, 2010, " Principles of Continuum Mechanics A Study of Conservation Principles with Applications"

[2] HECINI Mabrouk, 2016, Ouvrage pédagogique, « Mécanique des Milieux Continus Exercices corrigés avec résumé du cours ».

[3] X. Oliver, C. Agelet de Saracibar, 2017 , « CONTINUUM MECHANICS FOR ENGINEERS THEORY AND PROBLEMS-2nd Edition».

Chapitre VIII:

Failure criteria

Chapter VIII: Failure criteria

- Introduction
- Max.principal stress theory (Rankine theory)
- Maximum shear stress theory (tresca theory)
- Max. elastic strain theory (Saint venant's theory)
- Distortion energy theory (Von Mises theory)

I. Introduction:

Mechanical failure of components is generally identified by fracture or permanent/plastic deformation. Classical failure theories use yield stress and ultimate stresses to predict failure of mechanical

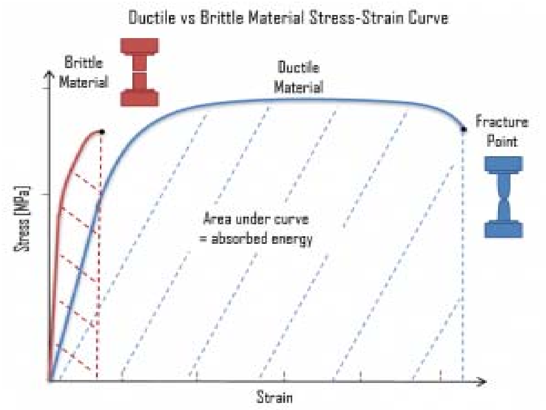


Figure.VIII.1.Ductile vs brittle materials tensile curves.

The quantities used for identifying impending failure include those associated with yield or ultimate stresses.
For example:
Principal stress $\sigma_{x1x1}=\sigma_I$,
Max. shear stress $\tau_{max}=(\sigma_{X1X1}/2)$
Principal strain $\varepsilon_{X1X1}=\sigma_{x1x1}/E$.

II. <u>Max.principal stress theory (Rankine theory):</u>

Here the hypothesis is that failure occurs when max. principal stress reaches the ultimate /yield stress of the material. The theory is relatively well suited for brittle materials.

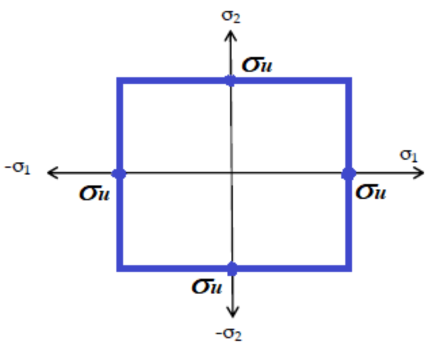


Figure.VIII.2.Rankine failure criterion in 2D.

Within the shaded space, the theory predicts safe performance and failure is imminent as σ_I or σ_{II} reach the boundary of the domain.

Max.principal stress theory is considered satisfactory for components made of brittle materials which do not exhibit significant plastic yielding.

$$\sigma_I \geq \sigma_u \text{ for } \sigma_{uc} \geq \sigma_{ut}$$

III. Maximum shear stress theory (tresca theory):

Given $\sigma_I > \sigma_{II} > \sigma_{III}$ and $\tau_{max} = (\sigma_I - \sigma_{III})/2$, this theory hypothesizes that the failure occurs when τmax reachs a critical value. This theory can be verified using thin uniaxial tensile strips made of ductile materials (mild steel, aluminium) which fail by developing so-called "slip" along ±45° planes relative the edges of the samples.

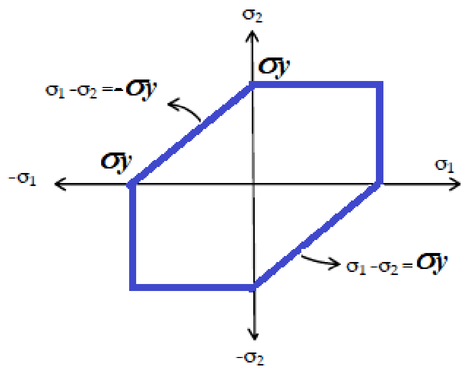


Figure.VIII.3.Tresca failure criterion in 2D.

As before, the material failure is predicted for all points outside the shaded area (if σ_y represents yield stress, then τ_{yield} for the uniaxial case can be evaluated as $\tau_y = \sigma_y/2$).
Some of the exceptions of the theory include failure behavior of brittle materials.
Some of the exceptions of the theory include failure behavior of brittle materials.

IV. Max. elastic strain theory (Saint venant's theory):

In terms of principal stresses $\sigma_I, \sigma_{II}, \sigma_{III}$, the principal strain ε_I is :

$$\varepsilon_I = \frac{1}{E}\left(\sigma_I - \mu(\sigma_{II} + \sigma_{III})\right);$$

The uniaxial yield strain for a material is : $\varepsilon_y = \frac{\sigma_y}{E}$ ($\varepsilon_y \approx \frac{\sigma_u}{E}$ for brittle solids).

Max. elastic strain theory predicts failure when $\varepsilon_y \geq \frac{\sigma_y}{E}$ or ($\varepsilon_y \geq \frac{\sigma_u}{E}$)

Again, this theory is contradictory to the observation that materials can withstand very high triaxial compression. Further, additional contradictions exist under biaxial loading conditions also.

$\varepsilon_I = \frac{1}{E}(\sigma_I - \mu\sigma_{II})$ and ε_I is smaller than $\frac{\sigma_I}{E}$ for $\sigma_2 > 0$ thus, even when σ_I exceeds σ_y , failure should not occur according to this theory. (on the contrary, if $\sigma_2 < 0$, failure would occur befor σ_I reaches σ_y) such observations are not supported by experiments.

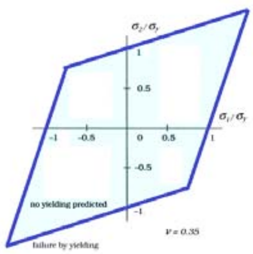


Figure.VIII.4.Saint venant's failure criterion in 2D.

V. Distortion energy theory (Von Mises theory):

The failure is predicted by Von Mises theory when :

$$(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 = 2\sigma_y^2$$

The above theory is well supported by experiments involving ductile materials.
For a 2-D state of stress, $(\sigma_{III} = 0)$ and hence Von Mises theory becomes,

$$(\sigma_I)^2 + (\sigma_{II})^2 - \sigma_I\sigma_{II} = \sigma_y^2$$

An equation of an ellipse in $\sigma_I - \sigma_{II}$ plane, as presented in Figure . .
It should be noted that Von Mises theory compares well with tresca theory as depicted [1].

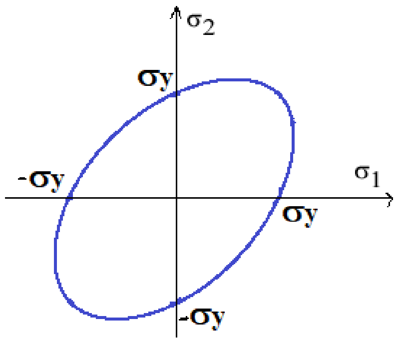


Figure.VIII.5.Von Mises failure criterion in 2D.

Summary of the failure criteria, the Tresca is the most conservative for all materials, the von Mises the most representative for ductile materials, and the Rankine the best fit for brittle materials [2].

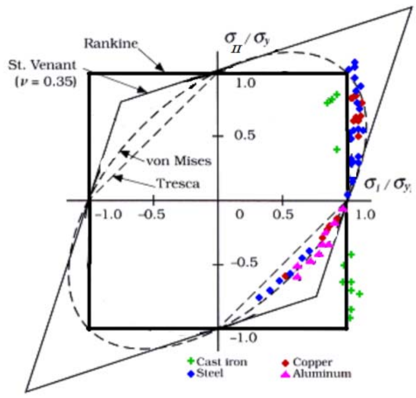


Figure.VIII.6.Failure criteria in 2D.

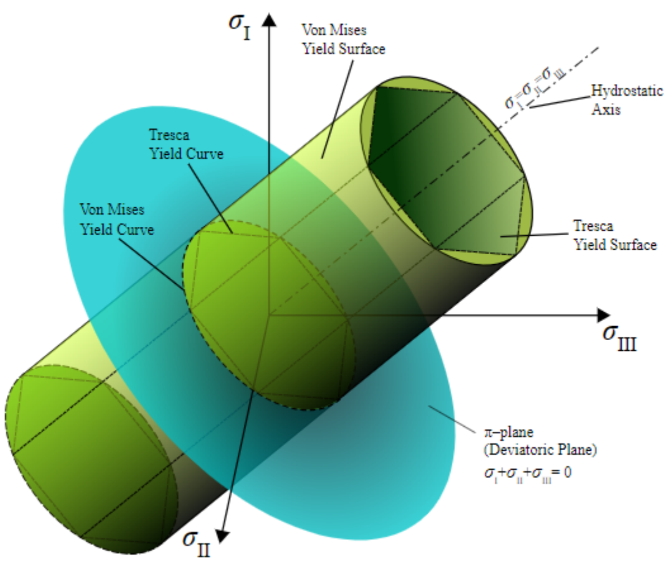


Figure.VIII.7.Failure criteria in 3D.

For more details you can watch the movie on the web site: https://www.youtube.com/watch?v=xkbQnBAOFEg

Biblographic references :

[1] https://www.eng.auburn.edu/~htippur/mech6300/failure%20theories.pdf
[2] https://web.mst.edu/jthomas/classes/2211/lessons/failure/theories/index.html
[3] https://sdcverifier.com/articles/what-is-von-mises-stress/