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Algebraic derivations on a lattice
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To all of them I dedicate this humble work and I ask ALLAH Almighty to guide us to that which is good.

Introduction

The concept of derivation was first introduced on function spaces as a linear function d with the additional property $d(f \cdot g) = d(f) \cdot g + f \cdot d(g)$. In this context, several authors apply this notion of derivation to other mathematical domains based on this property. First, Posner [8] introduced the concept of derivation on a prime ring $(R, *, +)$ as a mapping $(d : R \longrightarrow R)$ satisfies the following two conditions listed below:

$$d(x * y) = (d(x) * y) + (x * d(y)) \text{ and } d(x + y) = d(x) + d(y), \text{ for any } x, y \in R.$$

This concept of derivation on ring structures has played an important role in many fields of mathematics and it has many applications (see, e.g. [2]).

Later on, Szasz [10] extended the concept of derivation to the lattice structures based on the algebraic meet $(\wedge)$ and join $(\vee)$ operations. A $(\wedge, \vee)$-derivation (for short, derivation) on a given lattice $(L, \leq, \wedge, \vee)$ with respect to the meet $(\wedge)$ and join $(\vee)$ operations is a mapping $d : L \longrightarrow L$ satisfying the two conditions listed below:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \text{ and } d(x \vee y) = d(x) \vee d(y), \text{ for any } x, y \in L.$$

Following that, Ferrari [5] investigated some properties of this concept of derivation on lattices and provided some interesting examples in specific classes of lattices. Then, Xin et al. [12] extended the concept of derivation. They demonstrated this on a lattice by considering only the first condition, and they showed that the second condition clearly applies to isotone derivations on lattices. Also, they characterized distributive and modular lattices in terms of their isotone derivations. Later, Xin [11] focused his attention on the structure of the set of the fixed points of a derivation on a lattice and he demonstrated some relationships between this collection of fixed points as well as the lattice ideals. Recently, this concept of derivations on lattice structures is witnessing increasing attention [1, 13, 14, 15].

This thesis contains three chapters are organized as follows:

- In the first chapter, we recall the necessary basic concepts and properties of binary relations on a set and partially ordered sets.
- In the second chapter, we recall the necessary basic concepts and general information on lattices, algebraic properties of some lattice classes.
- In the third chapter, we give the notion of an algebraic derivation on a lattice and discuss some related properties. We analyze some classes of derivations on a lattice, like principal and isotone derivations. Moreover, we recall the notion of fixed point of a derivation on a lattice and we present the algebraic structure of the set of these fixed points.

Chapter 1

Preliminaries on ordered sets

In this chapter, we recall the necessary basic concepts and properties of binary relations on a set and partially ordered sets that will be needed throughout this thesis. Further information on binary relations and posets can be found in [3, 4, 6, 13].

1.1 Notion of binary relations on a set

In this section, we recall the notion of binary relations on a set and analyze several associated properties.

1.1.1 Binary relations on a set

Definition 1.1. *If X and Y are two nonempty sets, we define the **cartesian product** $X \times Y$ as the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. Thus*

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

Example 1.1. *Let $X = \{1, 2, 3\}$ and $Y = \{r, s\}$. Then*

$$X \times Y = \{(1, r), (1, s), (2, r), (2, s), (3, r), (3, s)\} \text{ and}$$
$$Y \times X = \{(r, 1), (r, 2), (r, 3), (s, 1), (s, 2), (s, 3)\}.$$

Remark 1.1. *As a remark, the cartesian product does not necessary commutative. As can seen from Example 1.1 that $X \times Y \neq Y \times X$.*

Theorem 1.1. *For any two finite nonempty sets X and Y, we have*

$$card(X \times Y) = card(X) \times card(Y).$$

Definition 1.2. *If X is a nonempty set, we define the cartesian product $X \times X$ as the set of all ordered pairs (x, y) with $x \in X$ and $y \in X$. Thus*

$$X \times X = X^2 = \{(x, y) \mid x \in X \text{ and } y \in X\}.$$

Example 1.2. *Let $X = \{1, 2, 3\}$, then*

$$X \times X = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}.$$

Definition 1.3. *Let X be a nonempty set. A* ***binary relation*** *R on a set X is a subset of the cartesian product $X \times X$. If $(x, y) \in R$, we write xRy and we say that "x is related to y with respect to R".*

Example 1.3. *(1) Let $X = \mathbb{R}$ and $R = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$.*

(2) Let $X = P(E)$ be the power set of E and $R = \{(A, B) \in P(E) \times P(E) \mid A \subseteq B\}$.

Definition 1.4. *We can represent a binary relation on a finite set with a matrix as follows, if $X = \{x_1, x_2, ..., x_m\}$ is a finite set containing elements and R is a binary relation on X. We represent R by the $m \times m$ matrix $M_R = [m_{ij}]$ which is defined by*

$$m_{ij} = \begin{cases} 1 & \text{if } (x_i, x_j) \in R, \\ 0 & \text{if } (x_i, x_j) \notin R. \end{cases}$$

The matrix M_R is called the matrix of R. Often, the matrix M_R provides an easy way to check whether R has a given property.

Example 1.4. *Let $X = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)\}$ be a binary relation on X. Then the associated matrix M_R of R is presented as follows*

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Definition 1.5. *If X is a finite set and R is a binary relation on X, we can also represent R pictorially as follows. Draw a small circle for each element of X and label the circle with the corresponding element of X. These circles are called vertexes. Draw an arrow, called an edge, form vertex x_i to vertex x_j if and only if $x_i R x_j$. The resulting pictorial representation of R is called a directed graph or digraph of R.*
Thus, if R is a binary relation on X, the edges in the digraph of R correspond exactly to the pair in R, and the vertexes correspond exactly to the elements of the set X. Sometimes, when we want to emphasize the geometric nature of some property of R, we may refer to the pairs of R themselves as edges and the elements X as vertexes.

Example 1.5. *The directed graph of the binary relation in Example 1.4 is shown in*

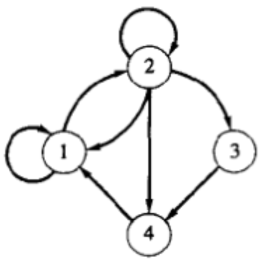

Figure 1.1: The directed graph of a binary relation.

1.1.2 Properties of binary relations on a set

Definition 1.6. *A binary relation R on a set X is called*

(i) reflexive if $(x, x) \in R$ for all $x \in X$, that is, $(\forall x \in X : xRx)$.

(ii) irreflexive if $(x, x) \notin R$ for all $x \in X$, that is, $(\forall x \in X : x \not R x)$.

Example 1.6. *(1) Let X be a nonempty set and $\Delta = \{(x, x) \mid x \in X\}$ be a binary relation. So Δ is the equality binary relation on the set X. Since $(x, x) \in \Delta$ for all $x \in X$, then Δ is reflexive.*

(2) Let $R = \{(x, y) \in X \mid x \neq y\}$, so that R is the inequality binary relation on the set X. Since $(x, x) \notin R$ for all $x \in X$, then R is irreflexive.

(3) *Let* $X = \{1, 2, 3\}$ *and* $R = \{(1, 1), (1, 2)\}$*, then* R *is not reflexive (i.e.,* $(2, 2) \notin R$ *and* $(3, 3) \notin R$*). Also,* R *is not irreflexive (i.e.,* $(1, 1) \in R$*).*

Definition 1.7. *A binary relation* R *on a set* X *is called*

- *symmetric, If all* $x, y \in X$

$$xRy \Rightarrow yRx.$$

- *asymmetric, If all* $x, y \in X$

$$xRy \Rightarrow y \not R x.$$

- *anti symmetric, if all* $x, y \in X$

$$xRy \text{ and } yRx \Rightarrow x = y.$$

Example 1.7. (1) *Let* $X = \mathbb{Z}$ *the set of integers and* $R = \{(x, y) \in \mathbb{Z}^2 \mid x < y\}$*. So that* R *is the binary relation less than. We have:*

(a) *if* $x < y$*, then it is not true that* $y < x$*, so* R *is not symmetric.*

(b) *if* $x < y$*, then* $y \not< x$*, so* R *is asymmetric.*

(c) *if* $x \neq y$*,* $x < y$ *or* $y < x$*, so that* R *is anti symmetric.*

(2) *Let* $X = \{1, 2, 3, 4\}$ *and* $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$ *a binary relation on* X*. Then* R *is not symmetric (*$(1, 2) \in R$ *but* $(2, 1) \notin R$*). Also,* R *is asymmetric and antisymetric.*

Definition 1.8. *A binary relation* R *on a set* X *is transitive if whenever* xRy *and* yRz*, then* xRz *(*$\forall x, y, z \in X : xRy \wedge yRz \Rightarrow xRz$*).*
The binary relation R *is not transitive if there exists* $x, y, z \in X$ *such that* xRy *and* yRx*, but* $x \not R z$*.*

If x, y, z *do not exist, then* R *is transitive.*

Example 1.8. (1) *Let* $X = \mathbb{N}^*$ *and* $R = \{(x, y) \in \mathbb{N}^2 \mid x \text{ divides } y\}$*. Then* R *is transitive. Indeed, if* xRy *and* yRz*, so that* $x|y$ *and* $y|z$*. It holds that* $x|z$*, for any* $x, y, z \in \mathbb{N}$*.*

(2) *Let* $X = \{1, 2, 3, 4\}$ *and* $R = \{(1, 2), (1, 3), (4, 2)\}$*. Since there are no elements* $x, y, z \in X$ *such that* xRy *and* yRz*, but* $x \not R z$*, we conclude that* R *is transitive.*

A binary relation R on a set X is called an equivalence binary relation if it is reflexive, symmetric and transitive.

Example 1.9. *Let* $X = \{1, 2, 3, 4\}$ *and* $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$. *It is easy to verify that* R *is an equivalence binary relation on the set* X.

1.2 Partially ordered sets

In this section, we give the notion of a partially ordered set and its particular elements.

1.2.1 Definitions and examples

Definition 1.9. *[4] Let* X *be a non empty set. An order (a partial order) on* X *is a binary relation* $\leq$ *on* X *such that, for all* $x, y, z \in X$,

(i) reflexive ($x \leq x$)*;*

(ii) antisymmetric ($x \leq y$ *and* $y \leq x \Rightarrow x = y$)*;*

(iii) transitive ($x \leq y$ *and* $y \leq z \Rightarrow x \leq z$)*.*

Example 1.10. *Let* $X = \mathbb{N}$ *be the set of positive integers. The usual relation* $\leq$ *(less than or equal to) is a partial order on* $\mathbb{N}$.

Example 1.11. *The divisibility relation (*aRb *if and only if* $a \mid b$*) is a partial order on* $\mathbb{N}^*$.

Definition 1.10. *A set* P *equipped with an order relation* $\leq$ *is called a partially ordered set (poset, for short) and denoted by* $(P, \leq)$.

Example 1.12. *The two structures* $(\mathbb{N}, \leq)$ *and* $(\mathbb{N}^*, |)$ *are partially ordered sets (posets).*

Example 1.13. *The inclusion relation* $\subseteq$ *is a partial order on the power set of a set* S*. Indeed,* $A \subseteq A$ *whenever* A *is a subset of* S*, then* $\subseteq$ *is reflexive. It is antisymmetric because if* $A \subseteq B$ *and* $B \subseteq A$ *imply that* $A = B$*. Finally,* $\subseteq$ *is transitive, because if* $A \subseteq B$ *and* $B \subseteq C$ *imply that* $A \subseteq C$*. Hence,* $\subseteq$ *is a partial order on* $P(S)$*. Consequently,* $(P(S), \subseteq)$ *is a poset.*

Definition 1.11. *Let* $(P, \leq)$ *be a poset. An element* $y \in P$ *covers an element* $x \in P$*, if* $x < y$ *and there is no element* $z \in P$ *such* $x \leq z$ *and* $z \leq y$*. The set of pairs* (x, y) *such that* y *covers* x *is called the covering relation of* $(P, \leq)$.

Definition 1.12. *The Hasse diagram of a finite poset* $(P, \leq)$ *is a picture of the digraph whose vertexes are the elements of* P *and which has line segments between of their vertexes. If an element* $y \in P$ *covers an element* $x \in P$*, we get the vertex* y *is higher up than the vertex* x *and they are connected with a line segment.*

Example 1.14. *Let* $D(20)$ *be the set of positive divisors of* 20 *and* $|$ *be the divisibility order. The Figure 3.3 presents the Hasse diagram of the poset* $(D(20), |)$.

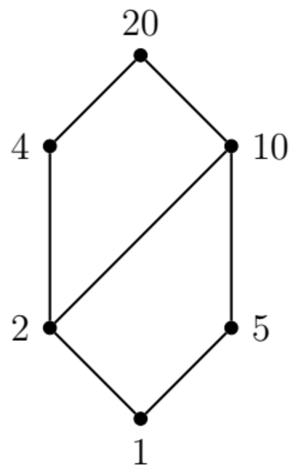

Figure 1.2: The Hasse diagram of the poset $(D(20), |)$.

1.2.2 Particular elements of an ordered set

Definition 1.13. *Let* $(P, \leq)$ *be a poset and* A *be a subset of* P*. An element* $x_0 \in P$ *is called a lower bound of* A *if* $x_0 \leq x$*, for any* $x \in P$*.* x_0 *is called the greatest lower bound (or the infimum) of* A *if* x_0 *is a lower bound of* A *and* $m \leq x_0$*, for any lower bound* m *of* A*. Upper bound and least upper bound (or supremum) are defined dually.*

Example 1.15. *Let* $(D(20), |)$ *be the poset given in Example 1.14. The greatest lower bound of the subset* $\{1, 2, 5\}$ *is* 1 *and* 10 *is its least upper bound.*

Definition 1.14. *A poset* $(P, \leq)$ *is called bounded, if it has a least and a greatest element respectively denoted by* 0 *and* 1*, i.e.,* $0 \leq x \leq 1$*, for any* $x \in P$*. Usually, the notation* $(P, \leq, 0, 1)$ *is used to describe a bounded poset.*

Example 1.16. *Let* $(D(20), |)$ *be the poset given in Example 1.14. This poset has* 1 *as the least element and* 20 *as the greatest element. Indeed,* 1 *divides all the elements of* $D(20)$ *and any element of* $D(20)$ *divides* 20*. Thus, the structure* $(D(20), |, 1, 20)$ *is a bounded poset.*

1.2.3 Morphisms of ordered sets

Definition 1.15. *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$ *be posets and* $f : P_1 \longrightarrow P_2$ *be a mapping between* P_1 *and* P_2*. The mapping* f *is called an order-preserving (order-morphism) form* $(P_1, \leq_1)$ *to* $(P_2, \leq_2)$ *if, for any* $x, y \in P_1$*:*

$$x \leq_1 y \Rightarrow f(x) \leq_2 f(y).$$

If $P_1 = P_2$*, then the mapping* f *is called an endomorphism.*

Definition 1.16. *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$ *be posets and* $f : P_1 \longrightarrow P_2$ *be a mapping between* P_1 *and* P_2*. The mapping* f *is called an order-isomorphism form* $(P_1, \leq_1)$ *to* $(P_2, \leq_2)$ *if, it is surjective and for any* $x, y \in P_1$*:*

$$x \leq_1 y \Leftrightarrow f(x) \leq_2 f(y).$$

If $P_1 = P_2$*, then the order-isomorphism* f *is called an automorphism.*

Example 1.17. *Let* $(P_1 = \{1, 2, 3, 6\}, \leq_1)$ *and* $(P_2 = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}, \leq_2)$ *be two posets such that* $\leq_1$ *is the divisibility order "*$|$*" and* $\leq_2$ *is the inclusion order "*$\subseteq$*". Their Hasse diagrams are showen in the Figure 1.3. Let* $f : P_1 \longrightarrow P_2$ *be a mapping defined by the following table:*

x	1	2	3	6
$f(x)$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a, b\}$

Thus, it is not difficult to see that f *is an order-isomorphism.*

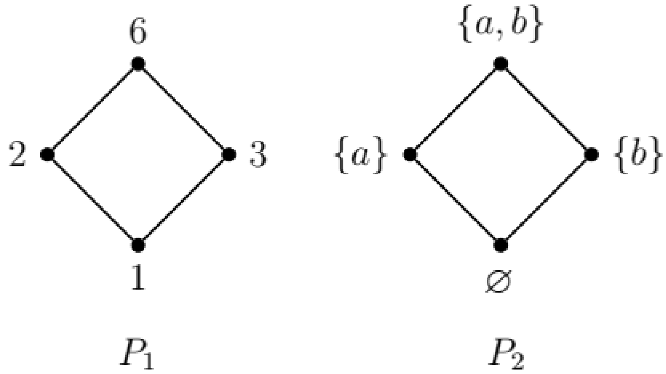

Figure 1.3

Chapter 2

Lattice Basics

In this chapter, we recall the necessary basic concepts and general information on lattices and algebraic properties of some lattice classes that will be needed throughout this thesis. Further information on lattices can be found in [3, 4, 6, 7, 9, 13].

2.1 General information on lattices

In this section, we give the notions of a lattice, ideals and filters in lattices, sublattices and lattice-morphisms.

2.1.1 Algebraic structure of a lattice

Definition 2.1. *A meet-semilattice is a poset* $(L, \leq)$ *in with every subset* $\{x, y\}$ *consisting of two elements have a greatest lower bound denoted by* $x \wedge y$ *and called the meet (infimum) of* x *and* y*, in which* $x \wedge y := inf\{x, y\}$.

Example 2.1. *Let* $(P_1 = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}, \subseteq)$ *and* $(P_2 = \{\{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ $, \subseteq)$ *be two posets ordered by the inclusion order and given by the Hasse diagrams in Figure 2.1. One can easily verify that* $(P_1, \subseteq)$ *is a meet-semilattice, but* $(P_2, \subseteq)$ *is not. Indeed, the two elements* $\{1\}$ *and* $\{2\}$ *have not an infimum in* P_2 *with respect to the inclusion order.*

Definition 2.2. *A join-semilattice is a poset* $(L, \leq)$ *in with every subset* $\{x, y\}$ *consisting of two elements have a least upper bound, denoted by* $x \vee y$ *and called the join (supermum) of* x *and* y*, in which* $x \vee y := sup\{x, y\}$.

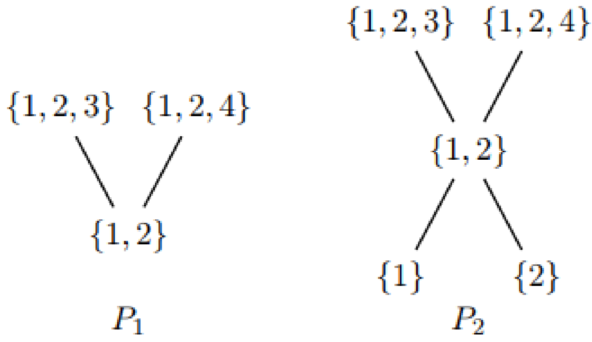

Figure 2.1: The Hasse diagrams of the posets $(P_1, \subseteq)$ and $(P_2, \subseteq)$.

Example 2.2. *Let* $(P_3 = \{[1,2],[2,3],[0,4]\}, \subseteq)$ *and* $(P_4 = \{[1,2],[2,3],[0,4],[2,5]\}, \subseteq)$ *be two posets ordered by the inclusion order and given by the Hasse diagrams in Figure 2.2. It is not difficult to check that* $(P_3, \subseteq)$ *is a join-semilattice, but* $(P_4, \subseteq)$ *is not (the two elements* $[1,2]$ *and* $[2,5]$ *have not an supermum in* P_4 *with respect to the inclusion order).*

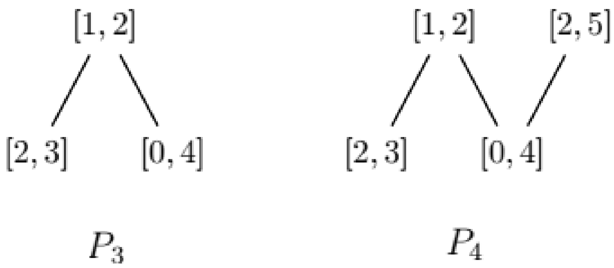

Figure 2.2: The Hasse diagrams of the posets $(P_3, \subseteq)$ and $(P_4, \subseteq)$.

Definition 2.3. *A lattice structure is a poset* $(L, \leq)$ *in which every subset* $\{x, y\}$ *consisting of two elements has a least upper bound and a greatest lower bound. Usually, the structure* $(L, \leq, \wedge, \vee)$ *is used to describe a lattice.*

Example 2.3. *Consider the post* $(\mathbb{N}^*, \leq)$*, where* $\leq$ *is the divisibility order* $|$ *on* $\mathbb{N}^*$*. Then* L *is a lattice in which the join and the meet of* x *and* y *are their least common multiple and greatest common divisor, respectively. Those are,*

$$x \vee y = lcm(x, y) \text{ and } x \wedge y = gcd(x, y), \text{ for any } x, y \in \mathbb{N}^*.$$

Example 2.4. *Let S be a set and $P(S)$ be the power set of S. We know that the structure $(P(S), \subseteq)$ is a poset. Let $A, B \in P(S)$. Then $A \vee B$ is the set $A \cup B$ and $A \wedge B$ is the set $A \cap B$. Thus, $(P(S), \subseteq, \cup, \cap)$ is a lattice.*

Theorem 2.1. *Let L be a lattice, then for every $x, y, z, t \in L$, we have*

(i) $x \vee y = y$ *if only if* $x \leq y$;

(ii) $x \wedge y = x$ *if only if* $x \leq y$;

(iii) $x \wedge y = x$ *if only if* $x \vee y = y$;

(iv) If $x \leq y$, *then* $x \wedge z \leq y \wedge z$ *and* $x \vee z \leq y \vee z$;

(v) If $x \leq y$ *and* $t \leq z$, *then* $x \wedge t \leq y \wedge z$ *and* $x \vee t \leq y \vee z$.

Theorem 2.2. *Let $(L, \leq, \wedge, \vee)$ be a lattice. Then for any x, y and z in L, we have*

1. *Idempotent properties*
   - $x \vee x = x$;
   - $x \wedge x = x$.
2. *Commutative properties*
   - $x \vee y = y \vee x$;
   - $x \wedge y = y \wedge x$.
3. *Associative properties*
   - $x \vee (y \vee z) = (x \vee y) \vee z$;
   - $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.
4. *Absorption properties*
   - $x \vee (x \wedge y) = x$;
   - $x \wedge (x \vee y) = x$.

Definition 2.4. *A lattice* $(L, \leq, \wedge, \vee)$ *is called bounded if it has a least and a greatest element denoted by* 0 *and* 1*, respectively. Usually, the structure* $(L, \leq, \wedge, \vee, 0, 1)$ *is used to describe a bounded lattice.*

Example 2.5. *(1) The lattice* $(P(S), \subseteq, \cap, \cup)$ *is bounded (*S *is its greatest element and* $\varnothing$ *its least element).*

(2) The lattice $(\mathbb{Z}, \leq, min, max)$ *is not bounded (*$\mathbb{Z}$ *has neither a greatest nor a least element).*

(3) Any finite lattice is bounded.

2.1.2 Ideals and filters in lattices

Definition 2.5. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice. A non-empty subset* I *of* L *is called an ideal if*

(i) for every $x, y \in I$ *implies* $x \vee y \in I$ *(closed under the join operation "*$\vee$*");*

(ii) if $x \in L$ *and* $y \in I$ *such that* $x \leq y$ *imply* $x \in I$ *(*I *is a down-set).*

Example 2.6. *Let* $(D(30), |, gcd, lcm)$ *be the lattice of positive divisors of* 30 *given by the Hasse diagram in Figure 2.3. Let* $I_1 = \{1, 2, 3, 6\}$ *and* $I_2 = \{2, 6, 10, 30\}$ *be two subsets of* $D(30)$*. Then*

- $I_1 = \{1, 2, 3, 6\}$ *is an ideal of* $D(30)$*.*
- $I_2 = \{2, 6, 10, 30\}$ *is not an ideal of* $D(30)$*. Indeed, we have* $6 \in I_2$ *and* $3 \leq 6$*, but* $3 \notin I_2$*.*

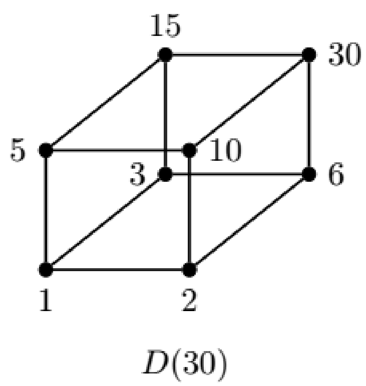

Figure 2.3: The Hasse diagram of the lattice $(D(30), |, lcm, gcd)$.

Definition 2.6. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice. A non-empty subset* F *of* L *is called a filter if* *(i)* *for every* $x, y \in F$ *implies* $x \wedge y \in F$ *(closed under the meet operation "$\wedge$");*

(ii) *if* $x \in L$ *and* $y \in F$ *such that* $x \leq y$ *imply* $x \in F$ *(F is an up-set).*

Example 2.7. *Let* $(D(30), |, gcd, lcm)$ *be the lattice given in Example 2.3 and*

$$F_1 = \{2, 6, 10, 30\}, F_2 = \{1, 2, 3, 6\}$$

be two subsets of $D(30)$. *Then*

- F_1 *is a filter of* $D(30)$.
- F_2 *is not a filter of* $D(30)$. *Indeed, we have* $2 \in F_2$ *and* $2 \leq 10$, *but* $10 \notin F_2$.

Remark 2.1. *Let* $(L, \leq, \wedge, \vee)$ *and* α *be a fixed element of* L. *We can define two associated sets to this element* α *as follows:*

$$I_\alpha = \downarrow \alpha = \{x \in L \mid x \leq \alpha\} \text{ and } F_\alpha = \uparrow \alpha = \{x \in L \mid \alpha \leq x\}.$$

The set $I_\alpha = \downarrow \alpha$ *(resp.* $F_\alpha = \uparrow \alpha$*) is an ideal (resp. a filter) of* L *called the principal ideal (resp. filter) generated by* α. *Indeed, if* $x, y \in I_\alpha$, *so* $x \leq \alpha$ *and* $y \leq \alpha$. *Using Theorem 2.1, we get* $x \vee y \leq \alpha$. *Thus,* $x \vee y \in I_\alpha$. *Hence,* I_α *is closed under the join operation "$\vee$" of* L. *Also, if* $x \in L$ *and* $y \in I_\alpha$ *such that* $x \leq y$, *then the transitivity property of* $\leq$ *guarantees that* $x \leq \alpha$. *Hence,* $x \in I_\alpha$. *Therefore,* I_α *is a down-set. Consequently,* I_α *an ideal of* L. *By the same way, we can prove that* F_α *is a filter of* L.

In the following examples, we present some principal ideals and filters.

Example 2.8. *All the following* I_n *(resp.* F_n*) subsets of the lattice* $(D(30), |, gcd, lcm)$ *are principal ideals (resp. filters).*

- $I_1 = \{x \in D(30) \mid x \leq 1\} = \{1\}$;
- $I_2 = \{x \in D(30) \mid x \leq 2\} = \{1, 2\}$;
- $I_3 = \{x \in D(30) \mid x \leq 3\} = \{1, 3\}$;
- $I_5 = \{x \in D(30) \mid x \leq 5\} = \{1, 5\}$;
- $I_6 = \{x \in D(30) \mid x \leq 6\} = \{1, 2, 3, 6\}$;

- $I_{10} = \{x \in D(30) \mid x \leq 10\} = \{1, 2, 5, 10\}$;

- $I_{15} = \{x \in D(30) \mid x \leq 15\} = \{1, 3, 5, 15\}$;

- $I_{30} = \{x \in D(30) \mid x \leq 30\} = \{1, 2, 3, 5, 6, 10, 15, 30\}$;

- $F_1 = \{x \in D(30) \mid 1 \leq x\} = \{1, 2, 3, 5, 6, 10, 15, 30\} = D(30)$;

- $F_2 = \{x \in D(30) \mid 2 \leq x\} = \{2, 6, 10, 30\}$;

- $F_3 = \{x \in D(30) \mid 3 \leq x\} = \{3, 6, 15, 30\}$;

- $F_5 = \{x \in D(30) \mid 5 \leq x\} = \{5, 10, 15, 30\}$;

- $F_6 = \{x \in D(30) \mid 6 \leq x\} = \{6, 30\}$;

- $F_{10} = \{x \in D(30) \mid 10 \leq x\} = \{10, 30\}$;

- $F_{15} = \{x \in D(30) \mid 15 \leq x\} = \{15, 30\}$;

- $F_{30} = \{x \in D(30) \mid 30 \leq x\} = \{30\}$.

An ideal or a filter is called proper if it does not coincide with L. It is a very easy exercise to show that an ideal I of a lattice with 1 is proper if and only if $1 \notin I$, and dually, a filter F of a lattice with 0 is proper if and only if $0 \notin F$.

Example 2.9. *Let $(D(60), |, gcd, lcm)$ be the lattice of positive divisors of 60 given by the Hasse diagram in Figure 2.4. It is easy to see that all its ideals and filters are principal.*

- *the subset $I_6 = \{1, 2, 3, 6\}$ is a proper ideal of $D(60)$;*

- *the subset $F_6 = \{6, 12, 30, 60\}$ is a proper filter of $D(60)$.*

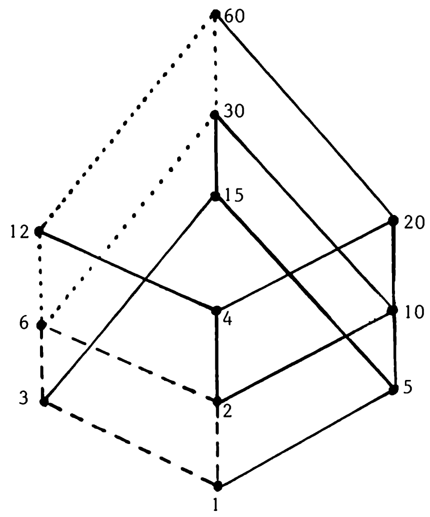

Figure 2.4: The Hasse diagram of the lattice $(D(60), |, lcm, gcd)$.

Definition 2.7. *An ideal I of a lattice $(L, \leq, \wedge, \vee)$ is called a prime ideal if it satisfies the following condition:*

$$x \wedge y \in I \text{ implies } x \in I \text{ or } y \in I, \text{ for all } x, y \in L.$$

Example 2.10. *Let $(D(6), |, gcd, lcm)$ be the lattice of positive divisors of 6 given by the Hasse diagram in Figure 2.5. Let $I_1 = \{1, 2\}$ and $I_2 = \{1\}$ be two subsets of $D(6)$.*

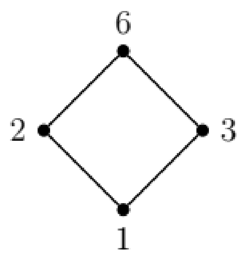

Figure 2.5: the Hasse diagram of $D(6)$.

(1) I_1 is a prime ideal of $D(6)$. Indeed,

- $1 \wedge 2 = 1 \in I_1, 1 \in I_1$ *and* $2 \in I_1$*;*
- $1 \wedge 3 = 1 \in I_1$ *and* $1 \in I_1$*;*
- $2 \wedge 3 = 1 \in I_1$ *and* $2 \in I_1$*;*

- $2 \wedge 6 = 2 \in I_1$ *and* $2 \in I_1$.

(2) I_2 *is an ideal but it is not prime. Indeed,* $2 \wedge 3 = 1 \in I_2$ *but* $2 \notin I_2$ *and* $3 \notin I_2$.

2.1.3 Sublattices and lattice-morphisms

Definition 2.8. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice. A non empty subset* S *of* L $(S \neq \varnothing$ *and* $S \subset L)$ *is called a sublattice of* L *if and only if*

$$\forall (x, y) \in S^2 \Longrightarrow x \wedge y \in S \text{ and } x \vee y \in S.$$

Example 2.11. *Let* $(D(20), |)$ *be the lattice given in Figure 3.3 and* $S = \{1, 2, 5, 10\}, T = \{2, 4, 5, 10\}$ *be two subsets of* $D(20)$ *given by the Hasse diagrams in Figures 2.6 and 2.7. Then, the subset* S *is a sublattice of* $D(20)$ *(in which every subset* $\{a, b\}$ *of* S *consisting of two elements has a least upper bound and a greatest lower bound on* S*). But,* T *is not a sublattice of* $D(20)$ *(the least upper bound of* 4 *and* 10 *and the greatest lower bound of* 2 *and* 5 *do not exist in* T*).*

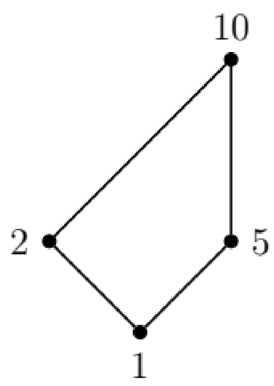

Figure 2.6: The Hasse diagram of S.

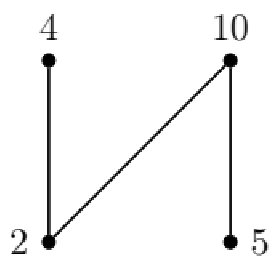

Figure 2.7: The Hasse diagram of T.

Definition 2.9. *Let* $(L_1, \leq_1, \wedge_1, \vee_1)$ *and* $(L_2, \leq_2, \wedge_2, \vee_2)$ *be two lattices. A mapping* $f : L_1 \to L_2$ *is called a meet-morphism (resp. join-morphism) from* L_1 *into* L_2*, if for all* $x, y \in L_1$*,*

$$f(x \wedge_1 y) = f(x) \wedge_2 f(y) \text{ (resp. } f(x \vee_1 y) = f(x) \vee_2 f(y)).$$

We can say also "f is both a meet-preserving (resp. join-preserving).

Definition 2.10. *A lattice-morphism is a meet-morphism and a join-morphism.*

Example 2.12. *Let* $D(6) = \{1, 2, 3, 6\}$ *and* $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *be two lattices ordered by the divisibility order, and* $f : D(6) \longrightarrow D(30)$ *be a mapping defined in the following table.*

x	1	2	3	6
$f(x)$	1	2	5	10

The mapping f *is a lattice-morphism. Indeed, let* $x, y \in L$*, we have:*

(i) if $(x, y) = (1, 1)$*, then*

$$f(1 \wedge 1) = f(1) = 1 = f(1) \wedge f(1) = 1 \wedge 1 \text{ and } f(1 \vee 1) = f(1) = 1 = f(1) \vee f(1) = 1 \vee 1;$$

(ii) if $(x, y) = (1, 2)$*, then*

$$f(1 \wedge 2) = f(1) = 1 = f(1) \wedge f(2) = 1 \wedge 2 \text{ and } f(1 \vee 2) = f(2) = 2 = f(1) \vee f(2) = 1 \vee 2;$$

(iii) if $(x, y) = (1, 3)$*, then*

$$f(1 \wedge 3) = f(1) = 1 = f(1) \wedge f(3) = 1 \wedge 5 \text{ and } f(1 \vee 3) = f(3) = 5 = f(1) \vee f(3) = 1 \vee 5;$$

(iv) if $(x, y) = (1, 6)$*, then*

$$f(1 \wedge 6) = f(1) = 1 = f(1) \wedge f(6) = 1 \wedge 10 \text{ and}$$
$$f(1 \vee 6) = f(6) = 10 = f(1) \vee f(6) = 1 \vee 10;$$

(v) if $(x, y) = (2, 2)$*, then*

$$f(2 \wedge 2) = f(2) = 2 = f(2) \wedge f(2) = 2 \wedge 2 \text{ and } f(2 \vee 2) = f(2) = 2 = f(2) \vee f(2) = 2 \vee 2;$$

(vi) *if* $(x, y) = (2, 3)$, *then*

$$f(2 \wedge 3) = f(1) = 1 = f(2) \wedge f(3) = 2 \wedge 5 \text{ and}$$
$$f(2 \vee 3) = f(6) = 10 = f(2) \vee f(3) = 2 \vee 5;$$

(vii) *if* $(x, y) = (2, 6)$, *then*

$$f(2 \wedge 6) = f(2) = 2 = f(2) \wedge f(6) = 2 \wedge 10 = 2 \text{ and}$$
$$f(2 \vee 6) = f(6) = 10 = f(2) \vee f(6) = 2 \vee 10;$$

(viii) *if* $(x, y) = (3, 3)$, *then*

$$f(3 \wedge 3) = f(3) = 5 = f(3) \wedge f(3) = 5 \wedge 5 = 5 \text{ and}$$
$$f(3 \vee 3) = f(3) = 5 = f(3) \vee f(3) = 5 \vee 5;$$

(ix) *if* $(x, y) = (3, 6)$, *then*

$$f(3 \wedge 6) = f(3) = 5 = f(3) \wedge f(6) = 5 \wedge 10 = 5 \text{ and}$$
$$f(3 \vee 6) = f(6) = 10 = f(3) \vee f(6) = 5 \vee 10.$$

Definition 2.11. *Let* $(L_1, \wedge_1, \vee_1)$ *and* $(L_2, \wedge_2, \vee_2)$ *be two lattices. A mapping* $f : L_1 \to L_2$ *is called*

(i) *A meet-monomorphism (resp. join-monomorphism) if it is an injective meet-morphism (resp. injective join-morphism).*

(ii) *A meet-epimorphism (resp. join-epimorphism) if it is a surjective meet-morphism (resp. join-morphism).*

(iii) *A lattice isomorphism if it is a bijective lattice-morphism. If* $L_1 = L_2$, *a lattice isomorphism* $f : L_1 \to L_2$ *is called a lattice-automorphism.*

Example 2.13. *Let* $(P(\{a, b\}), \subseteq, \cap, \cup)$ *and* $(D(10), |, gcd, lcm)$ *be two lattices. The mapping* $f : P(\{a, b\}) \longrightarrow D(10)$ *defined by the following table is a lattice-isomorphism.*

x	$\varnothing$	$\{a\}$	$\{b\}$	A
$f(x)$	1	2	5	10

Theorem 2.3. *Let $(L_1, \wedge_1, \vee_1)$, $(L_2, \wedge_2, \vee_2)$ be tow lattices and $f : L_1 \to L_2$ be a mapping. Then the following assertions are equivalents:*

(i) f is a lattice-isomorphism;

(ii) f is an order-isomorphism;

(iii) f is a join-isomorphism;

(iv) f is a meet-isomorphism.

Proof. $(i) \Leftrightarrow (ii)$: We have f is a lattice isomorphism $\Leftrightarrow$ f bijective and $\forall (x, y) \in L_1$: $f(x \wedge y) = f(x) \wedge f(y)$ and $\forall (x, y) \in L_1 : f(x \vee y) = f(x) \vee f(y)$; f order morphism $\Leftrightarrow \forall (x, y) \in L_1 : x \leq y \Leftrightarrow f(x) \leq f(y)$.

(1) $(i) \Rightarrow (ii)$: Assume that f is a lattice-isomorphism and let $x, y \in L_1$. We only show that

$$x \leq_1 y \Leftrightarrow f(x) \leq_1 f(y).$$

Then

$$\begin{aligned} x \leq_1 y &\Leftrightarrow x \vee_1 y = y \\ &\Leftrightarrow f(x \vee_1 y) = f(y) \\ &\Leftrightarrow f(x) \vee_2 f(y) = f(y) \\ &\Leftrightarrow f(x) \leq_2 f(y). \end{aligned}$$

Hence, f is an order-isomorphism.

(2) $(ii) \Rightarrow (i)$: Consider that f is an order-isomorphism and let $x, y \in L_1$. Then $\begin{cases} x \leq x \vee y \\ y \leq x \vee y \end{cases} \Longrightarrow \begin{cases} f(x) \leq f(x \vee y) \\ f(y) \leq f(x \vee y) \end{cases}$ (f is an order-morphism). $\Longrightarrow f(x) \vee f(y) \leq f(x \vee y)$. Since f is surjective, it holds that there exists an element $t \in L_1$ such that $f(t) = f(x) \vee f(y)$. So $\begin{cases} f(x) \leq f(x) \vee f(y) = f(t) \\ f(y) \leq f(x) \vee f(y) = f(t) \\ f(t) \leq f(x \vee y) \end{cases} \Longrightarrow \begin{cases} f(x) \leq f(t) \\ f(y) \leq f(t) \\ f(t) \leq f(x \vee y) \end{cases} \Longrightarrow \begin{cases} x \leq t \\ y \leq t \\ t \leq x \vee y \end{cases}$ (f an order-morphism). Thus, $t = x \vee y$. Therefore, $t = x \vee y$. Then $f(t) = f(x \vee y)$. Hence,

$f(x) \vee f(y) = f(x \vee y)$. Ultimately, f is a join-isomorphism. By a similar way, we can show that f is a meet-isomorphism. Consequently, f is a lattice-isomorphism.

(3) $(iii) \Rightarrow (iv)$: Assume that f is a join-isomorphism and let $x, y \in L_1$. Then

$$\begin{cases} x \wedge y \leq x \\ x \wedge y \leq y \end{cases} \Longrightarrow \begin{cases} (x \wedge y) \vee x = x \\ (x \wedge y) \vee y = y \end{cases} \Longrightarrow \begin{cases} f((x \wedge y) \vee x) = f(x) \\ f((x \wedge y) \vee y) = f(y) \end{cases}$$

$$\Longrightarrow \begin{cases} f(x \wedge y) \vee f(x) = f(x) \\ f(x \wedge y) \vee f(y) = f(y) \end{cases} \quad (f \text{ is a join-morphism}). \text{ Then}$$

$$\Longrightarrow \begin{cases} f(x \wedge y) \leq f(x) \\ f(x \wedge y) \leq f(y) \end{cases} \Longrightarrow f(x \wedge y) \leq f(x) \wedge f(y).$$ ($f(x) \wedge f(y)$ is the greatest lower bound). Since f is surjective, it holds that there exists an element $s \in L_1$ such that $f(s) = f(x) \wedge f(y)$. So

$$\begin{cases} f(s) = f(x) \wedge f(y) \leq f(x) \\ f(s) = f(x) \wedge f(y) \leq f(y) \\ f(x \wedge y) \leq f(x) \wedge f(y) = f(s) \end{cases} \Leftrightarrow \begin{cases} f(s) \leq f(x) \\ f(s) \leq f(y) \\ f(x \wedge y) \leq f(s) \end{cases} \Leftrightarrow \begin{cases} f(s) \vee f(x) = f(x) \\ f(s) \vee f(y) = f(y) \\ f(x \wedge y) \vee f(s) = f(s) \end{cases}$$

$$\Leftrightarrow \begin{cases} f(s \vee x) = f(x) \\ f(s \vee y) = f(y) \\ f((x \wedge y) \vee s) = f(s) \end{cases} \Leftrightarrow \begin{cases} s \vee y = x \\ s \vee y = y \\ (x \wedge y) \vee s = s \end{cases} \quad (f \text{ is injective}). \text{ Then}$$

$$\Leftrightarrow \begin{cases} s \leq x \\ s \leq y \\ x \wedge y \leq s \end{cases}$$. So ultimately, $x \wedge y = s$. Thus, $f(x \wedge y) = f(s) = f(x) \wedge f(y)$. Then f is bijective and a meet-morphism. Therefore, f is a meet-isomorphism.

(4) $(iv) \Rightarrow (i)$: Consider that f is a meet-isomorphism and let $x, y \in L_1$. We prove only that f is a join-isomorphism. Then

$$\begin{cases} x \leq x \vee y \\ y \leq x \vee y \end{cases} \Leftrightarrow \begin{cases} x \wedge (x \vee y) = x \\ y \wedge (x \vee y) = y \end{cases} \Longrightarrow \begin{cases} f(x \wedge (x \vee y)) = f(x) \\ f(y \wedge (x \vee y)) = f(y) \end{cases}$$

$$\Leftrightarrow \begin{cases} f(x) \wedge f(x \vee y) = f(x) \\ f(y) \wedge f(x \vee y) = f(y) \end{cases} \Leftrightarrow \begin{cases} f(x) \leq f(x \vee y) \\ f(y) \leq f(x \vee y) \end{cases} \Longrightarrow f(x) \vee f(y) \leq f(x \vee y)$$

($f(x) \vee f(y)$ is the smallest upper bound). Since f is surjective, it follows that there exists an element $m \in L_1$ such that $f(m) = f(x) \vee f(y)$. So

$$\begin{cases} f(x) \leq f(x) \vee f(y) = f(m) \\ f(y) \leq f(x) \vee f(y) = f(m) \\ f(m) = f(x) \vee f(y) \leq f(x \vee y) \end{cases} \Leftrightarrow \begin{cases} f(x) \leq f(m) \\ f(y) \leq f(m) \\ f(m) \leq f(x \vee y) \end{cases} \Leftrightarrow \begin{cases} f(x) \wedge f(m) = f(x) \\ f(y) \wedge f(m) = f(y) \\ f(m) \wedge f(x \vee y) = f(m) \end{cases}$$

$$\Leftrightarrow \begin{cases} f(x \wedge m) = f(x) \\ f(y \wedge m) = f(y) \\ f(m \wedge (x \vee y)) = f(m) \end{cases}$$ (f is a meet-morphism). Then

$$\begin{cases} x \wedge m = x \\ y \wedge m = y \\ m \wedge (x \vee y) = m \end{cases}$$ (f is injective). Thus, $$\begin{cases} x \leq m \\ y \leq m \\ m \leq x \vee y \end{cases}$$. Hence, $m = x \vee y$. Then $f(m) = f(x \vee y)$, i.e., $f(x) \vee f(y) = f(x \vee y)$. Ultimately, f is bijective, a join-morphism and a meet-morphism then, f lattice-isomorphism

(5) $(i) \Rightarrow (iii)$: The prove is very easy.

□

2.2 Algebraic properties of some lattice classes

In this section, we recall some algebraic properties of some lattice classes as distributive, modular and complementary lattices.

2.2.1 Distributive lattice

Definition 2.12. *A lattice $(L, \leq, \wedge, \vee)$ is called distributive, if one of the following two equivalent conditions hold:*

(a) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, *for any* $a, b, c \in L$;

(a^d) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, *for any* $a, b, c \in L$.

Example 2.14. *(1) The lattice $(D(20), |, gcd, lcm, 1, 20)$ given by the Hasse diagram in Figure 3.3 is distributive;*

(2) The lattice $(P(S), \subseteq, \cap, \cup, \varnothing, S)$ given in Example 2.4 is also distributive.

Remark 2.2. *(i) Any totally ordered set (a chain) is a distributive lattice.*

(ii) Any sublattice of a distributive lattice is itself a distributive lattice.

Example 2.15. *Let $M_3 = \{0, a_1, a_2, a_3, 1\}$ be the diamond lattice showen by the Hasse diagram in Figure 2.8. Then M_3 is not distributive. Indeed,*

$$a_1 \wedge (a_2 \vee a_3) = a_1 \wedge 1 = a_1, \text{ but } (a_1 \wedge a_2) \vee (a_1 \wedge a_3) = 0 \vee 0 = 0.$$

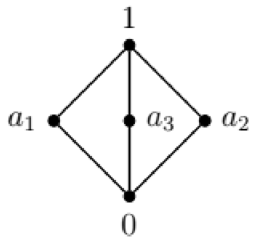

Figure 2.8: The Hasse diagram of the M_3 diamond lattice.

2.2.2 Modular lattice

Definition 2.13. *A lattice structure $(L, \leq, \wedge, \vee)$ is called modular if it satisfies the following modular law, for any $a, b, c \in L$:*

$$\text{if } a \leq c \text{ then } a \vee (b \wedge c) = (a \vee b) \wedge c.$$

Example 2.16. *(1) The diamond lattice M_3 given in Figure 2.8 is modular.*

(2) There are many lattice structures don not modular, for example, let $L = \{1, 2, 4, 5, 20\}$ ordered by the divisibility order. We have $2 \leq 4$, but $2 \vee (5 \wedge 4) = 2 \vee 1 = 2$ and $(2 \vee 5) \wedge 4 = 20 \wedge 4 = 4$. Then $2 \vee (5 \wedge 4) \neq (2 \vee 5) \wedge 4$. Thus, L is not modular.

Example 2.17. *Let $N_5 = \{0, b_1, b_2, b_3, 1\}$ be the pentagon lattice given by the Hasse diagram in Figure 2.9. It is not difficult to see that N_5 is not distributive. Further, it is not modular. Indeed, we have $b_1 \leq b_2$, but $b_1 \vee (b_3 \wedge b_2) = b_1 \vee 0 = b_1$ and $(b_1 \vee b_3) \wedge b_2 = 1 \wedge b_2 = b_2$. Then $b_1 \vee (b_3 \wedge b_2) \neq (b_1 \vee b_3) \wedge b_2$. Thus, N_5 is not modular.*

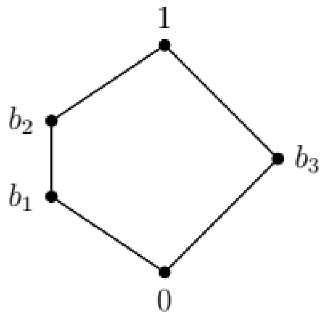

Figure 2.9: The Hasse diagram of the N_5 pentagon lattice.

Remark 2.3. *Any distributive lattice $(L, \leq, \wedge, \vee)$ is modular. Indeed, let $a, b, c \in L$ such that $a \leq c$. The fact that L is distributive implies that*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \wedge c) = (a \vee b) \wedge c.$$

The converse does not necessary correct. Indeed, Example 2.15 guarantees that M_3 is modular, bit it is not distributive.

Now, we present a fundamental characterization theorem for distributive and modular lattices.

Theorem 2.4. *Let $(L, \leq, \wedge, \vee)$ be a lattice. The following equivalences hold:*

(i) L is distributive if and only if L does not contain a sublattice isomorphic to M_3;

(ii) L is modular if and only if L does not contain a sublattice isomorphic to N_5.

2.2.3 Complementary lattices

Definition 2.14. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and a be fixed element of L. An element $a' \in L$ is called a complement of a if*

$$a \wedge a' = 0 \text{ and } a \vee a' = 1.$$

Remark 2.4. *In any bounded lattice, we have 0 and 1 are always complementary.*

Theorem 2.5. *A lattice L is called complemented if it is bounded and every element in L has a complement.*

Example 2.18. *The lattice* $(P(S), \subseteq, \cap, \cup, \varnothing, S)$ *given in Example 2.4 is complemented. Observe that in this case each element of* L *has a unique complement.*

Theorem 2.6. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded distributive lattice. If a complement exists, it is unique.*

Proof. Let a' and a'' be two complements of the element $a \in L$. Then

$$a \vee a' = 1, a \vee a'' = 1 \text{ and}$$

$$a \wedge a' = 0, a \wedge a'' = 0.$$

Using the distributive lows, we obtain

$$a' = a' \vee 0 = a' \vee (a \wedge a'') = (a' \vee a) \wedge (a' \vee a'') = 1 \wedge (a' \vee a'') = a' \vee a''.$$

Also,

$$a'' = a'' \vee 0 = a'' \vee (a \wedge a') = (a'' \vee a) \wedge (a'' \vee a') = 1 \wedge (a' \vee a'') = a' \vee a''.$$

Thus, $a'' \leq a'$ and $a' \leq a''$. Hence, $a' = a''$. □

Definition 2.15. *A lattice* L *is said to be a complete lattice, if each of its non-empty subsets* A *of* L *has both a greatest lower bound denoted by* $\wedge A$ *and called the infimum of* A*, and a least upper bound denoted by* $\vee A$ *and called the supremum of* A*.*

Theorem 2.7. *Every finite lattice is complete.*

Chapter 3

Algebraic derivations on a lattice

In this chapter, we give the notion of derivation on a lattice and discuss some related properties. We analyze some classes of derivations on a lattice, like principal and isotone derivations. Moreover, we recall the notion of fixed point of a derivation on a lattice and we present the algebraic structure of the set of these fixed points. Further information can be found in [11, 12].

3.1 Notion of derivations on a lattice

3.1.1 Definitions and examples

Definition 3.1. *Let $(L, \leq, \wedge, \vee)$ be a lattice. A mapping $d : L \longrightarrow L$ is called a derivation on L, if it satisfies the following condition for any $x, y \in L$:*

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)).$$

Next, we illustrate some examples of derivations on lattices.

Example 3.1. *Let $(L, \leq, \wedge, \vee, 0)$ be a lattice with a least element $0 \in L$.*

(1) If d is the identity mapping defined by $d(x) = x$, for any $x \in L$, then d is a derivation;

(2) If d is the null mapping defined by $dx = 0$, for each $x \in L$, then d is a derivation.

Example 3.2. *Let $(L_1 = \{0, a, b, 1\}, \leq_1, \wedge_1, \vee_1)$ and $(L_2 = \{0, a, b, 1\}, \leq_2, \wedge_2, \vee_2)$ be two lattices given by the Hasse diagrams in Figure 3.1. Let $d_1 : L_1 \longrightarrow L_1$ and $d_2 : L_2 \longrightarrow L_2$ be two* *mappings defined as follow:*

$$d_1(x) = \begin{cases} 0 & ,x = 0 \ or \ 1; \\ b & ,x = a; \\ b & ,x = b. \end{cases}$$

and

$$d_2(x) = \begin{cases} x & ,x = 0 \ or \ 1; \\ b & ,x = a; \\ a & ,x = b. \end{cases}$$

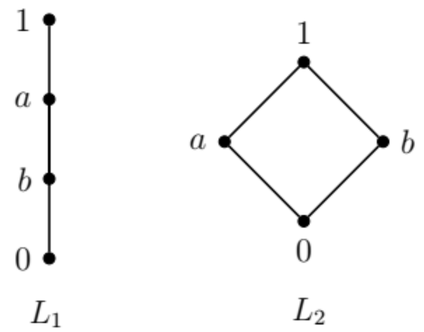

Figure 3.1: The Hasse diagrams of L_1 and L_2.

(1) The mapping d_1 is a derivation on L_1. Indeed, $\forall x, y \in L : d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$. Because:

- *if $(x, y) = (0, 0)$, then*

$$d_1(0 \wedge 0) = d_1(0) = 0 = (d_1 0 \wedge 0) \vee (0 \wedge d_1 0) = (0 \wedge 0) \vee (0 \wedge 0) = 0 \vee 0 = 0;$$

- *if $(x, y) = (0, a)$, then*

$$d_1(0 \wedge a) = d_1(0) = 0 = (d_1 0 \wedge a) \vee (0 \wedge d_1 a) = (0 \wedge a) \vee (0 \wedge b) = 0 \vee 0 = 0;$$

- *if $(x, y) = (0, b)$, then*

$$d_1(0 \wedge b) = d_1(0) = 0 = (d_1 0 \wedge b) \vee (0 \wedge d_1 b) = (0 \wedge b) \vee (0 \wedge b) = 0 \vee 0 = 0;$$

- *if $(x, y) = (0, 1)$, then*

$$d_1(0 \wedge 1) = d_1(0) = 0 = (d_1 0 \wedge 1) \vee (0 \wedge d_1 1) = (0 \wedge 1) \vee (0 \wedge 0) = 0 \vee 0 = 0;$$

- *if* $(x,y)=(a,a)$, *then*

$$d_1(a\wedge a)=d_1(a)=b=(d_1a\wedge a)\vee(a\wedge d_1a)=(b\wedge a)\vee(a\wedge b)=b\vee b=b;$$

- *if* $(x,y)=(a,b)$, *then*

$$d_1(a\wedge b)=d_1(b)=b=(d_1a\wedge b)\vee(a\wedge d_1b)=(b\wedge b)\vee(a\wedge b)=b\vee b=b;$$

- *if* $(x,y)=(b,b)$, *then*

$$d_1(b\wedge b)=d_1(b)=b=(d_1b\wedge b)\vee(b\wedge d_1b)=(b\wedge b)\vee(b\wedge b)=b\vee b=b;$$

- *if* $(x,y)=(1,1)$, *then*

$$d_1(1\wedge 1)=d_1(1)=0=(d_11\wedge 1)\vee(1\wedge d_11)=(0\wedge 1)\vee(1\wedge 0)=0\vee 0=0;$$

- *if* $(x,y)=(1,a)$, *then*

$$d_1(1\wedge a)=d_1(a)=b=(d_11\wedge a)\vee(1\wedge d_1a)=(0\wedge a)\vee(1\wedge b)=0\vee b=b;$$

- *if* $(x,y)=(1,b)$, *then*

$$d_1(1\wedge b)=d_1(b)=b=(d_11\wedge b)\vee(1\wedge d_1b)=(0\wedge b)\vee(1\wedge b)=0\vee b=b.$$

(2) The mapping d_2 is not a derivation on L_2. Indeed, if $(x,y)=(a,a)$ we have

$$d_2(a\wedge a)=b\neq(d_2(a)\wedge a)\vee(a\wedge d_2(a))=(b\wedge a)\vee(a\wedge b))=0.$$

Definition 3.2. *The radical of a positive integer n is denoted $r(n)$. It is the product of the primes of its factorization, i.e.,*

$$r(n)=P_1.P_2....P_t,\ \text{where}\ n=P_1^{\alpha_1}.P_2^{\alpha_2}....P_t^{\alpha_t}.$$

Example 3.3. *Let $(\mathbb{N}^*,|,gcd,lcm)$ be the lattice of positive integers ordered by the divisibility order. Let f be the radical mapping of $\mathbb{N}^*$ (i.e., $f:\mathbb{N}^*\longrightarrow\mathbb{N}^*$ such that $f(n)=r(n)$, for any $n\in\mathbb{N}^*$), then f is a derivation on $\mathbb{N}^*$.*

3.1.2 Properties of derivations on a lattice

Proposition 3.1. *Let $(L, \leq, \wedge, \vee)$ be a lattice and d be a derivation on L. Then the following properties hold for any $x, y \in L$:*

(i) $dx \leq x$ *(the derivative of x is smaller than itself);*

(ii) $d(dx) = dx$ *(the derivative of x is a fixed point of d);*

(iii) $dx \wedge dy \leq d(x \wedge y) \leq dx \vee dy$;

(iv) If L has a least element 0, then $d0 = 0$ (0 is a fixed point of d);

(v) If L has a greatest element 1 and $d(1) = 1$, then $d = id_L$ (d is the identity derivation of L);

(vi) If I is an ideal of L, then $dI \subseteq I$, where $dI = \{dx \mid x \in I\}$.

Remark 3.1. *In Proposition 3.1, we have the interesting property of derivation as $dx \leq x$. This means that any derivation on a lattice is a contraction mapping with respect to the principle of contraction mapping. Also, the second property (ii) guarantees that any derivation on a lattice must has a fixed point.*

Proposition 3.2. *If a lattice $(L, \leq, \wedge, \vee)$ has a greatest element 1 and d is a derivation on L, then*

$$dx = dx \vee (x \wedge d1), \text{ for all } x \in L.$$

Proof. Assume that d is a derivation on L and let $x, y \in L$. Then

$$dx = d(x \wedge 1) = (dx \wedge 1) \vee (x \wedge d1) = dx \vee (x \wedge d1).$$

□

Corollary 3.1. *Let $(L, \leq, \wedge, \vee)$ be a lattice with a greatest element 1 and d be a derivation on L, then we have*

(i) if $d1 \leq x$, then $d1 \leq dx$;

(ii) if $x \leq d1$, then $dx = x$.

Proposition 3.3. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice and* d *be a derivation on* L*. If* $y \leq x$ *and* $dx = x$*, then* $dy = y$*.*

Proof. Suppose that $y \leq x$, then $y = x \wedge y$. Thus

$$dy = d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy) = (x \wedge y) \vee dy = y \vee dy = y.$$

□

Proposition 3.4. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice and* d *be a derivation on* L*. Then*

$$dx = dx \vee (x \wedge d(x \vee y)), \text{ for any } x, y \in L.$$

Proof. Assume that d is a derivation on L and let $x, y \in L$. Then

$$dx = d((x \vee y) \wedge x) = (d(x \vee y) \wedge x) \vee ((x \vee y) \wedge dx) = dx \vee (x \wedge d(x \vee y)).$$

□

Corollary 3.2. *Let* $(L, \leq, \wedge, \vee, 1)$ *be a lattice with a greatest element* $1 \in L$ *and* d *be a derivation on* L*. Then*

$$d1 = 1 \text{ if and only if } d = id_L \text{ the identity derivation of } L \ (dx = x, \text{ for any } x \in L).$$

Proof. Assume that d is a derivation on L such that $d1 = 1$. For the direct implication, on the one hand, Proposition 3.1 (i) guarantees that $dx \leq x$, for all $x \in L$. On the other hand, Proposition 3.2 shows that $dx = dx \vee (x \wedge d1)$, for all $x \in L$. Since $d1 = 1$, it holds that $dx = dx \vee x$. Thus, $dx \leq x$, Hence $dx = x$, for all $x \in L$. The converse implication is obvious. □

Example 3.4. *Let* $(L = \{0, a, b, c, 1\}, \leq, \wedge, \vee)$ *be the lattice given in Figure 3.2 and* d *be a mapping on* L *defined as*

$$d(x) = \begin{cases} x & ,x = 0 \text{ or } a \text{ or } c; \\ a & ,x = b; \\ c & ,x = 1. \end{cases}$$

Note that $dc = c$ *and* $b \leq c$*, but* $db \neq b$*. Then Proposition 3.3 guarantees that* d *is not a derivation on* L*.*

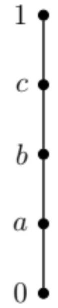

Figure 3.2: The Hasse diagram of L.

3.2 Types of derivations on lattices

In this section, we illustrate some types of derivations on lattices like principal and isotone derivations.

3.2.1 Principal derivations

Definition 3.3. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $\alpha \in L$. Define a mapping d_α on L as*

$$d_\alpha(x) = x \wedge \alpha, \text{ for any } x \in L.$$

This mapping d_α is a derivation on L called a principal derivation.

Remark 3.2. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $\alpha \in L$ such that d_α is the associated principal mapping of α. We show that d_α is a derivation on L. Let $x, y \in L$, then*

$$\begin{aligned} d_\alpha(x \wedge y) &= \alpha \wedge (x \wedge y) \\ &= (\alpha \wedge x \wedge y) \vee (\alpha \wedge x \wedge y) \\ &= (\alpha \wedge x \wedge y) \vee (x \wedge \alpha \wedge y) \\ &= ((\alpha \wedge x) \wedge y) \vee (x \wedge (\alpha \wedge y)) \\ &= (d_\alpha(x) \wedge y) \vee (x \wedge d_\alpha(y)). \end{aligned}$$

Therefore, d_α is a derivation on L.

Example 3.5. *Let $(D(20), |, gcd, lcm)$ be the lattice given in Figure 3.3. All the principal derivations on $D(20)$ are given in the following table*

x	1	2	4	5	10	20
$d_1(x) = gcd(1, x)$	1	1	1	1	1	1
$d_2(x) = gcd(2, x)$	1	2	2	1	2	2
$d_4(x) = gcd(4, x)$	1	2	4	1	2	4
$d_5(x) = gcd(5, x)$	1	1	1	5	5	5
$d_{10}(x) = gcd(10, x)$	1	2	2	5	10	10
$d_{20}(x) = gcd(20, x)$	1	2	4	5	10	20

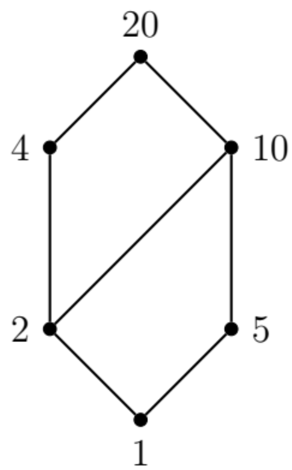

Figure 3.3: The Hasse diagram of the lattice $(D(20), |, gcd, lcm)$.

Proposition 3.5. *Any principal derivation d_α on a lattice L is an order-morphism (i.e., if $x \leq y$, then $d_\alpha(x) \leq d_\alpha(y)$, for any $x, y \in L$).*

Proof. Let d_α is a principal derivation on a lattice L and $x, y \in L$ such that $x \leq y$. Then

$$\begin{aligned} d_\alpha(x) \wedge d_\alpha(y) &= (\alpha \wedge x) \wedge (\alpha \wedge y) \\ &= \alpha \wedge x \wedge y \\ &= \alpha \wedge x \\ &= d_\alpha(x). \end{aligned}$$

Hence, $d_\alpha(x) \leq d_\alpha(y)$. Consequently, d_α is an order-morphism. □

3.2.2 Isotone derivations

Definition 3.4. *Let $(L, \leq, \wedge, \vee)$ be a lattice. A derivation d on L is called isotone, if it satisfies the following condition for any $x, y \in L$:*

$$\text{if } x \leq y, \text{ then } d(x) \leq d(y).$$

Definition 3.5. *A derivation d on a lattice L is said to be isotone, if that is an order-morphism.*

Example 3.6. *Let $(L = \{0, 1, 2, 3\}, \leq, min, max)$ be a chain ordered by the usual order of the natural numbers and represented by the Hasse diagram in Figure 3.4. Let d be a mapping on L defined in the following table:*

x	0	1	2	3
$d(x)$	0	1	2	2

Figure 3.4: The Hasse diagram of the chain $L = \{0, 1, 2, 3\}$.

It is not difficult to check that d satisfies the following two properties, for any $x, y \in L$:

(1) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$;

(2) if $x \leq y$, then $d(x) \leq d(y)$.

Consequently, d is an isotone derivation on this chain $L = \{0, 1, 2, 3\}$.

Remark 3.3. *Let $(L, \leq, \wedge, \vee)$ be a lattice. Any principal derivation $d_\alpha(x)$ on L is isotone. Indeed, Proposition 3.5 guarantees that $d_\alpha(x)$ is an order-morphism. Then it is isotone.*

Example 3.7. *Let $L = \mathbb{N} \times \{0, 1\}$ be a lattice ordered by the classical product order, i.e.,*

$$(i_1, j_1) \leq (i_2, j_2) \text{ if and only if } i_1 \leq i_2 \text{ and } j_1 \leq j_2;$$
$$(i_1, j_1) \wedge (i_2, j_2) = (i_1 \wedge i_2, j_1 \wedge j_2);$$
$$(i_1, j_1) \vee (i_2, j_2) = (i_1 \vee i_2, j_1 \vee j_2), \text{ for any } (i_1, j_1), (i_2, j_2) \in L.$$

Define a mapping d on L as follows

$$d(i,j) = (i,0), \text{ for any } (i,j) \in L.$$

Obvious that d is isotone. Moreover, d is a derivation on L. Indeed, let $(i_1,j_1),(i_2,j_2) \in L$. On the one hand, $d((i_1,j_1) \wedge (i_2,j_2)) = d(i_1 \wedge i_2, j_1 \wedge j_2) = (i_1 \wedge i_2, 0)$. On the other hand,

$$\begin{aligned}
(d(i_1,j_1) \wedge (i_2,j_2)) \vee ((i_1,j_1) \wedge d(i_2,j_2)) &= ((i_1,0) \wedge (i_2,j_2)) \vee ((i_1,j_1) \wedge (i_2,0)) \\
&= (i_1 \wedge i_2, 0 \wedge j_2) \vee (i_1 \wedge i_2, j_1 \wedge 0) \\
&= (i_1 \wedge i_2, 0) \vee (i_1 \wedge i_2, 0) \\
&= (i_1 \wedge i_2, 0).
\end{aligned}$$

Therefore,

$$d((i_1,j_1) \wedge (i_2,j_2)) = (d(i_1,j_1) \wedge (i_2,j_2)) \vee ((i_1,j_1) \wedge d(i_2,j_2)).$$

Proposition 3.6. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. Then the following equivalent holds*

$$d \text{ is isotone if and only if } d(x \wedge y) = dx \wedge dy, \text{ for any } x, y \in L.$$

Corollary 3.3. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. Then the following equivalent holds*

$$d \text{ is isotone if and only if } d \text{ is a meet-morhism.}$$

Proposition 3.7. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. If L is distributive and d is isotone, then*

$$d(x \vee y) = dx \vee dy, \text{ for any } x, y \in L.$$

Corollary 3.4. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. If L is distributive and d is isotone, then d is a join-morhism.*

Combining Corollaries 3.3 and 3.4 leads to the following result.

Corollary 3.5. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. If L is distributive and d is isotone, then d is a lattice-morhism.*

Theorem 3.1. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. Then the following are equivalent:*

(i) d is an isotone derivation;

(ii) $d(x \wedge y) = dx \wedge y$.

Proof. (i) $\Longrightarrow$ (ii): Assume that d is an isotone derivation on L. Then

$$d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy). \text{ Thus, } (dx \wedge y) \leq d(x \wedge y).$$

Conversely, since $x \wedge y \leq x$ and $x \wedge y \leq y$, we can get $d(x \wedge y) \leq dx$ and $d(x \wedge y) \leq dy$. Then $d(x \wedge y) \leq dx \wedge dy \leq dx \wedge y$. Therefore, $d(x \wedge y) = dx \wedge y$.
(ii) $\Longrightarrow$ (i): Suppose that $d(x \wedge y) = dx \wedge y$ for all x, y in L. Then $d(x \wedge y) = d(y \wedge x) = dy \wedge x$. Then $(x \wedge dy) \vee (dx \wedge y) = x \wedge dy = d(x \wedge y)$. Furthermore, if $x \leq y$, since $d(x \wedge y) = dx \wedge y = x \wedge dy$, then $dx = x \wedge dy$. Therefore, $dx \vee dy = (x \wedge dy) \vee dy = dy$. We can get $dx \leq dy$. □

Definition 3.6. *$(L, \leq, \wedge, \vee)$ be a lattice. A mapping $f : L \longrightarrow L$ is said to be a $\wedge$-translation on a lattice L, if it satisfies*

$$f(x \wedge y) = x \wedge f(y), \textit{ for any } x, y \in L.$$

Example 3.8. *Theorem 3.1 affirms that any $\wedge$-translation on a given lattice $(L, \leq, \wedge, \vee)$ is an isotone derivation on L.*

In the view of Theorem 3.1, we obtain the following corollary.

Corollary 3.6. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. Then the following are equivalent:*

(i) d is an isotone derivation;

(ii) d is a $\wedge$-translation.

Theorem 3.2. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. Then the following are equivalent:*

(i) $d(x \wedge y) = dx \wedge y$;

(ii) $d(x \wedge y) = dx \wedge dy$.

Proof. (i) $\Longrightarrow$ (ii): On the one hand, Proposition 3.1 (iii) guarantees that $(dx \wedge dy) \leq d(x \wedge y)$. On the other hand, by (i), we have $dx \wedge y = d(x \wedge y) = d(y \wedge x) = dy \wedge x$. Since $dx \wedge y \leq dx$ and $dy \wedge x \leq dy$, we can get $dx \wedge y = dy \wedge x \leq dx \wedge dy$. Hence, $(dx \wedge y) \leq (dx \wedge dy)$. Thus, $d(x \wedge y) \leq (dx \wedge dy)$. Consequently, $(dx \wedge dy) = (dx \wedge y)$.

(ii) $\Longrightarrow$ (i): Assume that $d(x \wedge y) = dx \wedge dy$, for all $x, y \in L$. If $x \leq y$, then $dx = d(x \wedge y) = dx \wedge dy$. We can get $dx \leq dy$. This shows that d is an isotone derivation. Using Theorem 3.1, we obtain that $d(x \wedge y) = dx \wedge y$. $\square$

Theorem 3.3. *Let $(L, \leq, \wedge, \vee)$ be a lattice with a greatest element 1 and $d : L \longrightarrow L$ be a derivation. Then the following are equivalent:*

(i) d is an isotone derivation;

(ii) $d(x) = x \wedge d(1)$ for all $x \in L$;

(iii) $d(x \wedge y) = d(x) \wedge d(y)$ for all $x, y \in L$;

(iv) $d(x) \vee d(y) \leq d(x \vee y)$. for all $x, y \in L$.

Proof. Let $(L, \leq, \wedge, \vee)$ be a lattice with a greatest element 1 and $d : L \longrightarrow L$ be a derivation, and let $x, y \in L$

$(i) \Longrightarrow (ii)$: Since d is isotone, then $dx \leq d1$. Note that $dx \leq (x \wedge d1)$. Using Proposition 3.2 we get $dx = dx \vee (x \wedge d1) = x \wedge d1$.

$(ii) \Longrightarrow (iii)$: Assume that (ii) holds. Then $dx \wedge dy = (x \wedge d1) \wedge (y \wedge d1) = x \wedge y \wedge d1 = d(x \wedge y)$.

$(iii) \Longrightarrow (i)$: Assume that (iii) holds. If $x \leq y$, then $x = x \wedge y$. Hence, $dx = d(x \wedge y) = dx \wedge dy$ (by (iii)). Therefore $dx = dx \wedge dy$, it follows that $dx \leq dy$. This shows that d is an isotone derivation.

$(i) \Longrightarrow (iv)$: The fact that d is isotone derivation implies that $dx \leq d(x \vee y)$ and $dy \leq d(x \vee y)$. Thus, $dx \vee dy \leq d(x \vee y)$.

$(iv) \Longrightarrow (i)$: Let $x \leq y$. Then $dx \vee dy \leq d(x \vee y) = dy$ and then $dx \vee dy = dy$. This means that $dx \leq dy$. Hence, d is isotone derivation. $\square$

Theorem 3.4. *Let $(L, \leq, \wedge, \vee)$ be a modular lattice and $d : L \longrightarrow L$ be a derivation. Then the following are equivalent:*

(i) d is isotone;

(ii) $d(x \wedge y) = dx \wedge dy$;

(iii) $dx = x$ *implies* $d(x \vee y) = dx \vee dy$, *for any* $y \in L$.

Proof. Let $(L, \leq, \wedge, \vee)$ be a modular lattice and $d : L \longrightarrow L$ be a derivation, and let $x, y \in L$ $(i) \Longrightarrow (ii)$: Assume that d is isotone, then $d(x \wedge y) \leq dx$ and $d(x \wedge y) \leq dy$. Thus $d(x \wedge y) \leq dx \wedge dy$. On an other hand, since L is modular and $dx \wedge y \leq x$, we have

$$\begin{aligned} d(x \wedge y) &= (dx \wedge y) \vee (x \wedge dy) \\ &= ((dx \wedge y) \vee dy) \wedge x \\ &= (dy \vee (dx \wedge y)) \wedge x \\ &= ((dy \vee dx) \wedge y) \wedge x \\ &\geq dx \wedge dy \wedge y \wedge x \\ &= dx \wedge dy. \end{aligned}$$

$(ii) \Longrightarrow (i)$: Suppose that $x \leq y$, then $x = x \wedge y$, so $dx = d(x \wedge y) = dx \wedge dy$. It follows that $dx \leq dy$, this shows that d is isotone derivation.

$(i) \Longrightarrow (iii)$: Let $x \in L$ such that $dx = x$. From Proposition 3.4, we have $dy = dy \vee (x \wedge d(x \vee y))$. The fact that L is modular, it holds that $dy = (dy \vee y) \wedge d(x \vee y) = y \wedge d(x \vee y)$. So

$$\begin{aligned} dx \vee dy &= dx \vee (y \wedge d(x \vee y)) \\ &= (dx \vee y) \wedge d(x \vee y) \\ &= (x \vee y) \wedge d(x \vee y) \\ &= d(x \vee y). \end{aligned}$$

$(iii) \Longrightarrow (i)$ Let $x \leq y$. Form proposition 3.1 (ii), we have $d(dx) = dx$. By hypothesis, $d(dx \vee y) = d(dx) \vee dy = dx \vee dy$. On the other hand, $x \leq y$ implies $d(dx \vee y) = dy$ and so $dy = dx \vee dy$. This means that $dx \leq dy$. Thus, d is isotone. □

In general, in a modular lattice L it is not necessary hold that if d is isotone, then $d(x \vee y) = dx \vee dy$.

Example 3.9. *Let* $(M_3 = \{0, a_1, a_2, a_3, 1\}, \leq, \wedge, \vee)$ *be a modular lattice given by the Hasse diagram in Figure 3.5 . Consider the derivation induced by* a_3*, that is* $d(x) = x \wedge a_3$*. We know* *that d is isotone. Using Theorem 3.4 we obtain that $d(x \wedge y) = dx \wedge dy$. But, $d(a_1 \vee a_2) = d1 = a_3$ and $d(a_1) \vee d(a_2) = 0$. Thus, $d(a_1 \vee a_2) \neq d(a_1) \vee d(a_2)$.*

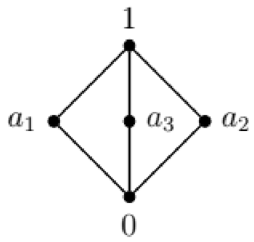

Figure 3.5: The Hasse diagram of the diamond lattice M_3.

However, in a distributive lattice, we can see that a derivation d is isotone if and only if $d(x \vee y) = dx \vee dy$.

Theorem 3.5. *Let $(L, \leq, \wedge, \vee)$ be a distributive lattice and d be a derivation on L. Then the following are equivalent:*

(i) d is isotone;

(ii) $d(x \wedge y) = dx \wedge dy$;

(iii) $d(x \vee y) = dx \vee dy$.

Proof. Suppose that L is a distributive lattice and d is a derivation on L, and let $x, y \in L$;
$(i) \Leftrightarrow (ii)$: are equivalent by Theorem 3.4.
$(i) \implies (iii)$: The fact that d is isotone implies $dx \leq d(x \vee y)$ and $dy \leq d(x \vee y)$. Using Proposition 3.4, we have

$$dx = dx \vee (x \wedge d(x \vee y)).$$

Since L is distributive, it holds $dx = (dx \vee x) \wedge (dx \vee d(x \vee y)) = x \wedge d(x \vee y)$. Similarly we can prove that $dy = y \wedge d(x \vee y)$. Thus

$$\begin{aligned} dx \vee dy &= (x \wedge d(x \vee y)) \vee (y \wedge d(x \vee y)) \\ &= (x \vee y) \wedge d(x \vee y) \\ &= d(x \vee y). \end{aligned}$$

$(iii) \implies (i)$ Let $x, y \in L$ such that $x \leq y$, then $y = x \vee y$. Thus $dy = d(x \vee y)$ by (iii), it follows that $dx \leq dy$. □

Proposition 3.8. *Let $(L, \leq, \wedge, \vee)$ be a lattice. If every isotone derivation d of L satisfies $d(x \vee y) = dx \vee dy$ in L, for any $x, y \in L$, then L is distributive.*

Proof. Assume that d is an isotone derivation on L such that $d(x \vee y) = dx \vee dy$, for any $x, y \in L$. Let $a, b, c \in L$, in order to prove $(a \vee b) \wedge c = (a \wedge c) \vee (a \wedge c)$, we define the principal derivation d_c on L by $d_c(x) = x \wedge c$, for all $x \in L$. We know that d_c is an isotone derivation on L. Then by hypothesis, d_c satisfies $d_c(x \vee y) = d_c(x) \vee d_c(y)$. Thus

$$\begin{aligned}(a \vee b) \wedge c &= d_c(a \vee b) \\ &= d_c(a) \vee d_c(b) \\ &= (a \wedge c) \vee (b \wedge c).\end{aligned}$$

It follows that L is a distributive. □

Combining Theorem 3.5 and Proposition 3.8 leads to the following characterization theorem for distributive lattices based on isotone derivations.

Theorem 3.6. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. Then the following characterizations of a distributive lattice are equivalent:*

(i) L is a distributive lattice;

(ii) every isotone derivation of L satisfies $d(x \vee y) = dx \vee dy$.

In order to provide a characterization theorem for modular lattices by derivations, we need the following proposition.

Proposition 3.9. *Let $(L, \leq, \wedge, \vee)$ be a lattice and d be a derivation on L. If every isotone derivation d of L satisfies that $dx = x$ implies $d(x \vee y) = dx \vee dy$, then L is a modular lattice.*

Proof. Let $x, y, z \in L$ such that $x \leq z$. Consider the principal derivation d_z defined by $d_z(w) = z \wedge w$, for all $w \in L$. Then d is an isotone. Since $x \leq z$, then $d_z(x) = z \wedge x = x$. By hypothesis, $d_z(x \vee y) = d_z(x) \vee d_z(y)$. Notice that $d_z(x \vee y) = z \wedge (x \vee y)$ and $d_z(x) \vee d_z(y) = (z \wedge x) \vee (z \wedge y) = x \vee (y \wedge z)$. Hence $x \vee (y \wedge z) = (x \vee y) \wedge z$, this shows that L is a modular lattice. □

Combining Theorem 3.4 and Proposition 3.9 leads to the following characterization theorem.

Theorem 3.7. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation. Then the following characterizations are equivalent:*

(i) L is a modular lattice;

(ii) every isotone derivation of L satisfies that $dx = x$ implies $d(x \vee y) = dx \vee dy$.

3.2.3 Lattice structure of isotone derivations

Theorem 3.8. *Let $(L, \leq, \wedge, \vee)$ be a distributive lattice and d_1, d_2 be two isotone derivations on L. Define*

$$(d_1 \sqcap d_2)(x) = d_1(x) \wedge d_2(x) \text{ and}$$
$$(d_1 \sqcup d_2)(x) = d_1(x) \vee d_2(x).$$

Then $d_1 \sqcap d_2$ and $d_1 \sqcup d_2$ are also isotone derivations on L.

Theorem 3.9. *Let L be a distributive lattice and $D(L)$ be the set of all isotone derivations on L. Then $(D(L), \sqcap, \sqcup)$ is also a distributive lattice.*

3.3 Algebraic structure for the set of fixed points of a derivation

In this section, we recall the notion of a fixed point of a derivation on a lattice. Also, we present the algebraic structure of the set of these fixed points. Further information can be found in [11].

Definition 3.7. *Let $(L, \leq, \wedge, \vee)$ be a lattice and d be a derivation on L. Define*

$$Fix_d(L) = \{x \in L \mid dx = x\}$$

is the set of all the fixed points of d.

Remark 3.4. *The set $Fix_d(L)$ is a down-set. Moreover, if d is isotone, then $Fix_d(L)$ is an ideal of L. Indeed,*

(i) Proposition 3.3 guarantees that if $x \in Fix_d(L)$ and $y \leq x$, then $y \in Fix_d(L)$. Thus, $Fix_d(L)$ is a down-set;

(ii) The fact that d is isotone implies from Theorem 3.6 that if $x, y \in Fix_d(L)$, then $d(x \vee y) = d(x \vee d(y) = x \vee y$. Therefore, $x \vee y \in Fix_d(L)$.

The above remark leads to the following corollary.

Corollary 3.7. *If d is an isotone derivation on a lattice L, then $Fix_d(L)$ is a lattice.*

Remark 3.5. *Let $(L, \leq, \wedge, \vee)$ be a lattice and d be a derivation on L. Define $d^2x = d(dx)$, for all $x \in L$. Then Proposition 3.1 assures that $d^2 = d$. So, $d(x) \in Fix_d(L)$, for any $x \in L$.*

Proposition 3.10. *Let $(L, \leq, \wedge, \vee)$ be a lattice and d_1, d_2 be two isotone derivations on L. Then*

$$d_1 = d_2 \text{ if and only if } Fix_{d_1}(L) = Fix_{d_2}(L).$$

Theorem 3.10. *Let $(L, \leq, \wedge, \vee)$ be a distributive lattice and $D(L)$ be the set of all the isotone derivations on L. Denote*

$$F = \{Fix_d(L) \mid d \in D(L)\};$$
$$Fix_{d_1}(L) \vee Fix_{d_2}(L) = Fix_{d_1 \vee d_2}(L);$$
$$Fix_{d_1}(L) \wedge Fix_{d_2}(L) = Fix_{d_1 \wedge d_2(L)}.$$

Then $(F, \wedge, \vee)$ is a distributive lattice.

Corollary 3.8. *Let $(L, \leq, \wedge, \vee)$ be a distributive lattice. Then the lattice $(D(L), \wedge, \vee)$ is isomorphic to the lattice $(F, \wedge, \vee)$.*

There are still some open questions between ideals and "fixed sets" for derivations. The question of when an ideal can appear as this "fixed set" for a derivation seems interesting. It is clearly true for principal ideals. Since they are fixed sets of the principal derivation. We present two answers for this question concerning principal and prime ideals.

Theorem 3.11. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $d : L \longrightarrow L$ be a derivation on L*

(i) If d_α is a principal derivation on L, then $Fix_{d_\alpha}(L) = I_\alpha$ is a principal ideal;

(ii) If I is a principal ideal of L, then there exists a unique isotone derivation d such that $Fix_d(L) = I$.

Theorem 3.12. *Let $(L, \leq, \wedge, \vee)$ be a lattice and I be a non-void prime ideal of L. Then there exists a derivation d such that $Fix_d(L) = I$.*

ملخص

في هذه المذكرة بدأنا بعرض مفهوم العلاقات الثنائية على مجموعة وذكرنا تعريف المجموعات المرتبة. ثم قدمنا مفهوم البنية الجبرية لشبكة وعرضنا أهم الخصائص الجبرية لها. بعدها، تطرقنا لمفهوم المشتقات الجبرية على شبكة وضربنا عديدا من الأمثلة التوضيحية حولها. كذلك، قمنا بتحليل بعض أنواع المشتقات الجبرية على شبكة ودرسنا بعض خصائصها. وأخيرا، عرضنا البنية الجبرية لمجموعة النقاط الثابتة لمشتقة جبرية على شبكة.

كلمات مفتاحية:

ترتيب، مجموعة مرتبة، شبكة، مشتقة جبرية، نقطة ثابتة.

Abstract

In the memory, we first have recalled the necessary basic concepts and properties of binary relations on a set and partially ordered sets. Then, we have given general information on lattices and algebraic properties of some lattice classes. Also, we have provided the notion of an algebraic derivation on a lattice and we have discussed some related properties. Further, we have analyzed some classes of derivations on a lattice like principal and isotone derivations. Moreover, we have recalled the notion of a fixed point of a derivation on a lattice and we have presented the algebraic structure of the set of these fixed points.

Key words :

Order, ordered set, lattice, algebraic derivation, fixed point.

Résumé

Dans ce mémoire, nous avons rappelé les concepts de base des relations binaires sur un ensemble et les ensembles ordonnés. De puis, nous avons donné des informations générales sur les treillis et les propriétés algébriques de certaines classes de treillis. Aussi, nous avons fourni la notion de dérivation algébrique sur un treillis et discuté de certaines propriétés connexes. Nous avons analysé certaines classes de dérivations sur un treillis, comme les dérivations principales et isotones. De plus, nous avons rappelé la notion de point fixe d'une dérivation sur un treillis et nous avons présenté la structure algébrique de l'ensemble de ces points fixes.

Mots-clés :

Ordre, ensemble ordonné, treillis, dérivation algébrique, point fixe.

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