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*Study of a fractional non-local problem with a singular nonlinearity and
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Dedication

Nothing compares to the joy of graduation, as it is one of the most beautiful moments in our lives. The tiredness of the years, the sleepless nights and the prayers of the parents have been harvested, and the psychological pressures we have gone through have disappeared and we have forgotten them as soon as we feel the joy of graduation.

First, I have mercy on my dear mother, then I dedicate my graduation to everyone who supported me and stood with me from my childhood to my old age, so my sister Ndia, who supported me with all my requirements and my brothers and father, and I say to them thank you for this beautiful situation. And I really thank my friends Nabila and Widad for making my years at university better.

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Contents

1	Introduction	v
2	PRELIMINAIRES AND SOME BASIC TOOLS	7
2.1	Functional spaces	7
2.1.1	$L^p(\Omega)$ spaces	7
2.2	Fractional Sobolev spaces	10
2.2.1	Embedding properties	11
2.3	The fractional Laplacian operator	13
2.4	Derivatives of a functional and critical points	17
3	A uniqueness result for kirchhoff type problem with singularity	20
3.1	Presentation of Problem	20
3.2	Main result	21
3.3	Proof of main result	22
3.3.1	Global minimizer of I	22
3.3.2	Minimizer of I is a positive solution of (3.1)	24
3.3.3	Uniqueness of solution	26
4	A fractional Kirchhoff problem involving a singular term and critical nonlin-	
	earity	28
4.1	Main result	28
4.2	Variational setting	29
4.3	Proof of Theorem (4.1)	32
4.3.1	Geometric condition	32
4.3.2	Existence of a local minimizer	33

4.3.3 The local minimizer is a weak solution	35
4.3.4 Positivity of solution	39
4.4 More general result	39

INTRODUCTION

Kirchhoff's rules, two statements about multi-loop electric circuits that embody the laws of conservation of electric charge and energy and that are used to determine the value of the electric current in each branch of the circuit. They were first described in 1845 by German physicist Gustav Kirchhoff. This generalized the work of Georg Ohm and preceded the work of James Clerk Maxwell. Widely used in electrical engineering, they are also called Kirchhoff's rules or simply Kirchhoff's laws. These laws can be applied in time and frequency domains and form the basis for network analysis.

Both of Kirchhoff's laws can be understood as corollaries of Maxwell's equations in the low-frequency limit. They are accurate for DC circuits, and for AC circuits at frequencies where the wavelengths of electromagnetic radiation are very large compared to the circuits. The first rule, the junction theorem, states that the sum of the currents into a specific junction in the circuit equals the sum of the currents out of the same junction. Electric charge is conserved: it does not suddenly appear or disappear; it does not pile up at one point and thin out at another. The second rule, the loop equation, states that around each loop in an electric circuit the sum of the emf's (electromotive forces, or voltages, of energy sources such as batteries and generators) is equal to the sum of the potential drops, or voltages across each of the resistances, in the same loop. All the energy imparted by the energy sources to the charged particles that carry the current is just equivalent to that lost by the charge carriers in useful work and heat dissipation around each loop of the circuit.

On the basis of Kirchhoff's two rules, a sufficient number of equations can be writ-

ten involving each of the currents so that their values may be determined by an algebraic solution.

Kirchhoff's rules are also applicable to complex alternating-current circuits and with modifications to complex magnetic circuits.

In the first chapter, we are going to give some definitions and reminders of the results necessary for the continuation of this work like Holder's inequality, Young's inequality and Fatou's lemma. And we are going to cite in particular certain elementary results of the functional spaces.

In the second chapter, we are going to talk about the singular Kirchhoff type problem with the Dirichlet boundary value condition in the simple case and try to prove the uniqueness of its solution.

In the last chapter we are going to present Kirchhoff type problem driven by a nonlocal fractional operator and involving singular term and a critical non linearity and we are going to prove existence of the solution.

PRELIMINAIRES AND SOME BASIC TOOLS

In this chapter, we present some definitions and reminders of the results necessary for the continuation of this work. We will cite in particular certain elementary results of the functional spaces.

2.1 Functional spaces

2.1.1 $L^p(\Omega)$ spaces

In what follows, Ω denotes an open of \mathbb{R}^N equipped with the Lebesgue measure dx .

Definition 2.1. Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, we pose

$$L^p(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable and } \int_{\Omega} |f(x)|^p dx < \infty \right\},$$

we note

$$\|f\|_{L^p} = \left[\int_{\Omega} |f(x)|^p dx \right]^{1/p}.$$

It can easily be checked that $\|\cdot\|_{L^p}$ defines a norm on the vector space $L^p(\Omega)$ what show that $L^p(\Omega)$ is a normed space.

Definition 2.2. We pose

$$L^\infty(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \text{ } f \text{ is measurable. } \exists C > 0 \text{ such that } |f(x)| \leq C \text{ p.p sur } \Omega\},$$

$L^\infty(\Omega)$ is a normed space when it is endowed by

$$\|f\|_{L^\infty} = \inf \{C, |f(x)| \leq C \text{ on } \Omega\}.$$

Theorem 2.1. [3] we have

1. The space $L^p(\Omega)$ is a Banach for $1 \leq p \leq \infty$,
2. The space $L^p(\Omega)$ is separable for $1 \leq p < \infty$,
3. The space $L^p(\Omega)$ is reflexif for $1 < p < \infty$.

Let $1 \leq p \leq \infty$, we denote by q the conjugate exponent of p , i.e.

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \text{so } q = \frac{p}{p-1}.$$

Proposition 2.1 (Hölder's inequality). Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $p \in [1, \infty[$ so $f.g \in L^1(\Omega)$ and

$$\int_{\Omega} |fg| dx \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proposition 2.2 (Young's inequality). $\forall a \geq 0, \forall b \geq 0$, suppose that $1 < p < \infty$

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Writing $f_n(x) \rightarrow f(x)$ a.e on Ω , means that the sequence $f_n(x)$ converges to $f(x)$ almost everywhere on Ω .

Theorem 2.2 (Lebesgue dominated convergence). Let (f_n) a sequence of functions $L^1(\Omega)$. We suppose that

1. $f_n(x) \rightarrow f(x)$ a.e. on Ω ,
2. There exists a function $g \in L^1(\Omega)$ such as for each $n, |f_n(x)| \leq g(x)$ a.e. on Ω so, $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1} \rightarrow 0$.

Theorem 2.3 (Inverse Lebesgue dominated convergence). Let (f_n) a sequence of $L^p(\Omega)$ such that $\|f_n - f\|_{L^1} \rightarrow 0$. So, there is an extracted subsequence (f_{n_k}) such that

1. $f_{n_k}(x) \rightarrow f(x)$ a.e on Ω ,

2. $|f_{n_k}(x)| \leq h(x) \forall k$ and a.e on Ω with $h \in L^p(\Omega)$.

The following result introduces some topological properties of Lebesgue spaces.

Definition 2.3 (weak and strong convergence). Let $(x_n)_{n \geq 1}$ a sequence of the normed vector space $(E, \|\cdot\|_E)$ and let E' its topological dual. We say that (x_n) converges weakly in E if there exists an element $x \in E$ such that

$$\forall f \in E', \lim_{n \rightarrow \infty} f(x_n) = f(x) \quad \text{so} \quad (\langle f, x_n \rangle \rightarrow \langle f, x \rangle).$$

We will note $x_n \rightharpoonup x$ in E , the weak convergence in E . We will note likewise $x_n \rightarrow x$ in E , i.e. the strong convergence in E (convergence in norm).

The Sobolev Space $W^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 2.4. The Sobolev space $W^{1,p}(\Omega)$ is defined to be

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega); \exists g \in L^p(\Omega) \text{ such that } \int_{\Omega} u \nabla \varphi = \int_{\Omega} g \varphi \quad \forall \varphi \in C_c^1(\Omega) \right\}.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

Notation

The space $W^{1,p}$ is equipped with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}$$

or sometimes, if $1 < p < \infty$, with the equivalent norm $(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{1/p}$. The space H^1 is equipped with the scalar product

$$(u, v)_{H^1} = (u, v)_{L^2} + (\nabla u, \nabla v)_{L^2} = \int_{\Omega} (uv + \nabla u \nabla v),$$

and with the associated norm

$$\|u\|_{H^1} = (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{1/2}$$

The Sobolev Space $W_0^{1,p}(\Omega)$

Definition 2.5. Given $1 \leq p < \infty$, denoted by $W_0^{1,p}(\Omega)$ the clouser of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$. Set

$$H_0^1(\Omega) = W_0^{1,p}(\Omega).$$

The space $W_0^{1,p}(\Omega)$ is equipped with the norm of $W^{1,p}(\Omega)$, and the space H_0^1 is equipped with the scalar product of H^1 . The space $W_0^{1,p}(\Omega)$ is a separable Banach space. Moreover, it is reflexive for $p > 1$. The space H_0^1 is a separable Hilbert space.

Remark 2.1. When $\Omega = \mathbb{R}^N$ we know that $C_c^1(\mathbb{R}^N)$ is dense in $W_0^{1,p}(\mathbb{R}^N)$ and therefore $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$

2.2 Fractional Sobolev spaces

Generalities

Let Ω be a possibly nonsmooth, open set of the Euclidean space \mathbb{R}^n and $p \in [1, +\infty)$. for any $0 < s < 1$, we would define the fractional Sobolev space $W^{s,p}(\Omega)$. In the literature, fractional Sobolev-type spaces are also called Aronszajn, Gagliardo, or Slobodeckij spaces, by the names of the ones who introduced them, almost simultaneously. For more details see [Z] [1].

For a fixed $s \in (0, 1)$, the sobolev space $W^{s,p}(\Omega)$ is defined as follows:

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{n/p+s}} \in L^p(\Omega \times \Omega) \right\}.$$

It is endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_{\Omega} |u(x)|^p dx + \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{1/p}$$

, where the term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{1/p}$$

is the Gagliardo seminorm of u .

The space $W^{s,p}(\Omega)$ is well defined and is a Banach space for every $0 < s < 1$. As in the classical case (i.e., $s = 1$), any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support. Indeed, for any $0 < s < 1$,

$$\overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^n)}} = W^{s,p}(\mathbb{R}^n),$$

that is the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.

In general if $\Omega \subset \mathbb{R}^n$, the space $C_0^\infty(\Omega)$ is not dense in $W_0^{s,p}(\Omega)$. Hence, we denote by $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\Omega)}$; that is,

$$W_0^{s,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}}.$$

2.2.1 Embedding properties

This subsection is devoted to the embeddings of fractional Sobolev spaces into Lebesgue spaces. We point out that Sobolev inequalities and continuous (compact) embeddings of the spaces $W^{s,p}$ into the classical Lebesgue spaces L^q are exhaustively treated in [Z1]. Here we recall briefly some basic facts.

Now we are ready to discuss the embedding properties of $W^{s,p}$. For this purpose, we distinguish three different cases, that is, $sp < n$, $sp = n$, and $sp > n$. We refer to [Z] for a proof of these results.

Case 1: $sp < n$

Theorem 2.4. *Let $s \in (0, 1)$ and $p \in [1, +\infty)$ such that $sp < n$. Let $\Omega \subset \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C := C(n, p, s, \Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)},$$

for any $q \in [p, p_s^*]$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any

$q \in [p, p_s^*]$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, p_s^*]$.

Case 2: $sp = n$

For an extension domain Ω , we have the following embedding result:

Theorem 2.5. *Let $s \in (0, 1)$ and $p \in [1, +\infty)$ such that $sp = n$. Let $\Omega \subset \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C := C(n, p, s, \Omega)$ such that, for any $u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)},$$

for any $q \in [p, +\infty)$; that is, the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, +\infty)$. If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, +\infty)$.

Case 3: $sp > n$. Here $C^{0,\alpha}(\Omega)$ denotes the space of Hölder continuous functions, endowed with the standard norm

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_{L^\infty(\Omega)} + \sup_{x,y \in \Omega, \|x-y\| \leq 1} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha}.$$

Corollary 2.1. *Let $s \in (0, 1)$ and $p \in [1, +\infty)$ such that $sp > n$. Let Ω be a $C^{0,1}$ bounded domain of \mathbb{R}^n . Then the embedding*

$$W^{s,p}(\Omega) \rightarrow C^{0,\beta}(\Omega),$$

is compact for every $\beta < \alpha$, with $\alpha := (sp - n)/p$.

The Sobolev space $H^s(\Omega)$

In this subsection, we focus our attention on the Hilbert case $p = 2$, dealing with its relation with the fractional Laplacian. Let Ω be an open subset in \mathbb{R}^n , and denote

$$H^s(\Omega) := W^{s,2}(\Omega),$$

for any $s \in (0, 1)$. This is an important case because the preceding fractional Sobolev space turns out to be a Hilbert space. Indeed, the inner product on $H^s(\Omega)$ defined by

$$\langle u, v \rangle_{H^s(\Omega)} := \int_{\Omega} u(x)u(y)dx + \int_{\Omega \times \Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy.$$

Clearly, for every $s \in (0, 1)$, one has

$$H^s(\mathbb{R}^n) := W^{s,p}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : [u]_{W^{s,2}(\mathbb{R}^n)} < +\infty\}.$$

2.3 The fractional Laplacian operator

Nonlocal equations have attracted much attention in recent decades. The basic operator involved in this kind of problems is the so-called fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$. This operator and its generalization appear in many areas of mathematics, such as, for instance, harmonic analysis, probability theory, potential theory, quantum mechanics, statistical physics, and cosmology, as well as in many applications, as we highlighted at the beginning of this chapter. This section is devoted to the definition of this operator and to its properties.

Let $s \in (0, 1)$, and define the operator $(-\Delta)^s: \zeta \rightarrow L^2(\mathbb{R}^n)$ by

$$(-\Delta)^s u(x) := C(n, s) \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B(x, \varepsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \quad x \in \mathbb{R}^n, \quad (2.1)$$

Where $B(x, \varepsilon)$ is the ball centered at $x \in \mathbb{R}^n$ with radius ε , and $C(n, s)$ is the following (positive) normalization constant

$$C(n, s) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right)^{-1},$$

with $\zeta = (\zeta_1, \zeta')$, $\zeta' \in \mathbb{R}^{n-1}$.

The operator defined in (2.1) is the fractional Laplacian. Commonly, in defining $(-\Delta)^s$, the abbreviation for “in the principal-value sense” is adopted. Precisely, setting

$$P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = \lim_{\varepsilon \rightarrow \infty} \int_{\mathbb{R}^n \setminus B(x, \varepsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy.$$

We can write

$$(-\Delta)^s u(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \quad x \in \mathbb{R}^n. \quad (2.2)$$

The singular integral given in (2.2) can be written as a weighted second-order differential

Proposition 2.3. *Let $s \in (0, 1)$. Then, for any $u \in \zeta$.*

$$(-\Delta)^s u(x) = -\frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy \quad x \in \mathbb{R}^n. \quad (2.3)$$

Proof. by (2.2) we have that

$$(-\Delta)^s u(x) = -C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x - y|^{n+2s}} dy, \quad (2.4)$$

for every $x \in \mathbb{R}^n$. Hence, substituting $z = y - x$ in (2.4), it follows that

$$(-\Delta)^s u(x) = -C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}} dz, \quad (2.5)$$

for every $x \in \mathbb{R}^n$. However, by putting $z^* = -z$, one has

$$P.V. \int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}} dz = P.V. \int_{\mathbb{R}^n} \frac{u(x-z^*) - u(x)}{|z^*|^{n+2s}} dz^*.$$

So, after relabeling z^* as z , the following equalities hold

$$\begin{aligned} 2.P.V. \int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}} dz &= P.V. \int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}} dz + P.V. \int_{\mathbb{R}^n} \frac{u(x-z) - u(x)}{|z|^{n+2s}} dz \\ &= P.V. \int_{\mathbb{R}^n} \frac{u(x+y) - u(x-y) - 2u(x)}{|y|^{n+2s}} dy. \end{aligned}$$

Finally, a second order Taylor expansion yields

$$\frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \leq \frac{\|D^2 u\|_{L^\infty(\mathbb{R}^n)}}{|y|^{n+2s-2}},$$

and since $s \in (0, 1)$, one has

$$\frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \in L^1(\mathbb{R}^n).$$

Thus , for any $u \in \zeta$, we have that

$$P.V. \int_{\mathbb{R}^n} \frac{u(x+y) - u(x-y) - 2u(x)}{|y|^{n+2s}} dy = \int_{\mathbb{R}^n} \frac{u(x+y) - u(x-y) - 2u(x)}{|y|^{n+2s}} dy.$$

□

Remark 2.2. Let $s \in (0, 1/2)$. Observe that for any $u \in \zeta$ and for a fixed $x \in \mathbb{R}^n$, we have that

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy &\leq C \int_{B(x, \mathbb{R})} \frac{|x-y|}{|x-y|^{n+2s}} dy + \|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(x, \mathbb{R})} \frac{1}{|x-y|^{n+2s}} dy \\ &\leq C \left(\int_0^R \frac{1}{\rho^{2s}} d\rho + \int_R^\infty \frac{1}{\rho^{2s+1}} d\rho \right) < +\infty, \end{aligned}$$

where C is a positive constant depending only on the dimension n and the L^∞ norm of the function u . Hence, in the case $s \in (0, 1/2)$, the integral

$$\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy,$$

is not singular near the point x , so one can get rid of the P.V. Some recent results on fractional Laplacian equations can be found in [4, 7] and references therein. Moreover, very recently, a new nonlocal and nonlinear operator (the fractional p -Laplacian $(-\Delta)_p^s$) was considered. Namely, for $p \in (1, +\infty)$, $s \in (0, 1)$, and u smooth enough, it is defined as

$$(-\Delta)_p^s u(x) = P.V. \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x-y|^{n+sp}} dy \quad x \in \mathbb{R}^n.$$

Up to some normalization constant depending on n , p , and s , this definition is consistent with the one of the fractional Laplacian $(-\Delta)^s$ in the case $p = 2$. For the motivations that lead to the study of such operators, we refer the reader to the seminal paper of Caffarelli [2, 5]

Theorem 2.6 (Lebesgue's dominated convergence theorem). Let (f_n) be a sequence of complex-valued measurable functions on a measure space (S, Σ, μ) . Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x),$$

for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable (in the Lebesgue sense) and

$$\lim_{n \rightarrow \infty} \int_S |f_n - f| d\mu = 0,$$

which also implies

$$\lim_{n \rightarrow \infty} \int_S f_n d\mu = \int_S f d\mu.$$

Theorem 2.7 (The Monotone Convergence Theorem(Beppo-Levi)). Let (X, Σ, μ) be a measure space. Let $\langle f_n \rangle_{n \in \mathbb{N}} \in \mathcal{M}_{\mathbb{R}}^+$, be an increasing sequence of positive Σ -measurable functions. Let $\sup_{n \in \mathbb{N}} f_n : X \rightarrow \bar{\mathbb{R}}$, be the pointwise supremum of $\langle f_n \rangle_{n \in \mathbb{N}}$, where $\bar{\mathbb{R}}$ denotes the extended real numbers. Then

$$\int \sup_{n \in \mathbb{N}} f_n d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu.$$

Theorem 2.8 (Vitali-Caratheodory). Suppose $f \in L^1(\mu)$, f is real-valued, and $\varepsilon > 0$. Then there exist functions u and v on X such that $u \leq f \leq v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below, and

$$\int_X (v - u) d\mu < \varepsilon.$$

Lemma 2.1 (Fatou's lemma). Let $(f_n)_n$ a sequence of functions $L^1(\Omega)$ such that

1. For all n , $f_n \geq 0$ a.e. on Ω ,
2. $\sup \int_{\Omega} f_n(x) dx \leq +\infty$.

for all $x \in \Omega$ we pose

$$f(x) = \liminf_{n \rightarrow \infty} \int_{\Omega} f_n(x) dx,$$

so $f \in L^1(\Omega)$ and

$$\int_{\Omega} f(x) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n(x) dx,$$

Lemma 2.2. Let (X, μ) be a measure space and let f_n be a sequence of measurable complex-valued functions on X which converge almost everywhere to a function f . The limiting function f is automatically measurable. The Brezis–Lieb lemma asserts that if p is a positive number,

then

$$\lim_{n \rightarrow \infty} \int_X \left| |f|^p - |f_n|^p + |f - f_n|^p \right| d\mu = 0,$$

provided that the sequence f_n is uniformly bounded in $L^p(X, \mu)$. A significant consequence, which sharpens Fatou's lemma as applied to the sequence $|f_n|^p$, is that

$$\int_X |f|^p d\mu = \lim_{n \rightarrow \infty} \left(\int_X |f_n|^p d\mu - \int_X |f - f_n|^p d\mu \right).$$

2.4 Derivatives of a functional and critical points

Gateaux derivative

Definition 2.6. Let be Ω part of a Banach space E and $I : \Omega \rightarrow \mathbb{R}$ a real-valued function. If $u \in \Omega$ and $v \in E$ are such as for $t > 0$ quite small $u + tv \in \Omega$, we say that I admits (on point u) directional derivative v if the limit

$$\lim_{t \rightarrow 0^+} \frac{I(u + tv) - I(u)}{t},$$

exists. Note this limit $I'_v(u)$.

We note that function I can have a derivative directional in any direction $v \in E$, without being continuous. when derivative direction I exists for some $v \in E$ we introduce the notion of derivative in the sens of Gateaux.

Definition 2.7. Let be Ω a set of a Banach space E and $I : \Omega \rightarrow \mathbb{R}$.

If $u \in \Omega$, we say that I is derivable in the sens de Gateaux (where G -derivable or G -differentiable) in u , if there exists $A \in E'$ such as in any direction $v \in E$ where $I(u + tv)$ exists for $t > 0$ quite small, directional derivative $I'_v(u)$ exists and we have :

$$\lim_{t \rightarrow 0^+} \frac{I(u + tv) - I(u)}{t} = \langle A, v \rangle,$$

we pose $I'(u) := A$.

Example 2.1. Let Ω be an open of \mathbb{R} and let be $2 < p < +\infty$, the functionals

$$\varphi(u) = \int_{\Omega} |u(t)|^p dt, \quad \psi(u) = \int_{\Omega} |u^+(t)|^p dt,$$

where u^+ is the positive part of u , are class $C^1(L^p(\Omega), \mathbb{R})$ and we have

$$\langle \varphi'(u), h \rangle = p \int_{\Omega} |u(t)|^{p-2} u(t) h(t) dt, \quad \langle \psi'(u), h \rangle = p \int_{\Omega} \{u^+(t)\}^{p-1} h(t) dt.$$

Frechet derivative

Definition 2.8. Let E be a Banach space, $\Omega \subseteq E$ an open set and $I : \Omega \rightarrow \mathbb{R}$ a functional.

we say that I is differentiable in the sens of Frechet in $u \in \Omega$, if it exists $A \in E'$ such that

$$\lim_{\|v\| \rightarrow 0} \frac{I(u+v) - I(u) - Av}{\|v\|} = 0.$$

So, for I a differentiable functional we have

$$I(u+v) - I(u) = Av + o(\|v\|)$$

if I is differentiable, A is unique and we note $I'(u) := A$.

The set of differentiable functions of $\Omega \rightarrow \mathbb{R}$ will be noted $C^1(\Omega, \mathbb{R})$ which implies that if I is differentiable en u , then I is continuous in u .

Proposition 2.4. Suppose that $\Omega \subseteq E$ an open, such that I Gateau differentiable of Gateau in Ω and I'_G is continuous in $u \in \Omega$.

So I is differentiable too in u , and of course

$$I'_G(u) = I'(u).$$

Remark 2.3. The importance of the proposition (2.4) resides in the fact that it is often technically easier to calculate the derivative in the sens of Gateau and then to prove that is continuous, rather than directly proving differentiability in the sense of Frechet.

Critical points

Definition 2.9. Let Ω be an open set of a Banach space E . Suppose that $I \in C^1(\Omega, \mathbb{R})$. we say that $u \in \Omega$ is a critical point of I , if

$$I'(u) = 0.$$

If u is not a critical point, then we say u est is a regular point of I . If $c \in \mathbb{R}$, then we say c is a critical value of I , if there exists $u \in \Omega$ such that

$$I(u) = c \text{ and } I'(u) = 0,$$

if c is not a critical value, then we say it is a regular value of I .

A UNIQUENESS RESULT FOR KIRCHHOFF TYPE PROBLEM WITH SINGULARITY

In this chapter, we are going to immerse ourselves in a singular Kirchoff type problem with the Dirichlet boundary value condition in the simple case and try to prove the uniqueness of its solution.

3.1 Presentation of Problem

Let consider the following kirchhoff type problem

$$\begin{cases} - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x)u^{-\gamma} - \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \quad (3.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain, $0 < \gamma < 1$, $\lambda \geq 0$, $0 < p \leq 2^* - 1$ and $a, b \geq 0$, $a + b > 0$ are parameters. The coefficient $f \in L^{\frac{2^*}{2^* + \gamma - 1}}(\Omega)$ with $f(x) > 0$ for any $x \in \Omega$, and $2^* = \frac{2N}{N-2}$ denotes the critical Sobolev exponent for the embedding $H_0^1(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \frac{2N}{N-2}]$.

The Kirchoff type problems have been widely investigated, and a lot of classical results have been obtained on a bounded domain or unbounded domain see [9], [10], [11], [12], [13], [14], [15], [16], [17].

Definition 3.1 (weak solution). A function u is called a weak solution of problem (3.1) if $u \in H_0^1(\Omega)$ such that $u > 0$ in Ω and

$$\left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \int_{\Omega} (\nabla u, \nabla \phi) dx + \lambda \int_{\Omega} u^p \phi dx - \int_{\Omega} f(x) u^{-\gamma} \phi dx = 0, \quad (3.2)$$

for all $\phi \in H_0^1(\Omega)$.

The energy functional corresponding to problem (3.1) is defined by

$$I(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + \frac{\lambda}{p+1} \int_{\Omega} |u|^{p+1} dx - \frac{1}{1-\gamma} \int_{\Omega} f(x) |u|^{1-\gamma} dx.$$

Because of the singular term the functional I can not be Frechet differentiable, so we can not apply the critical point theory to obtain the existence of solutions directly.

3.2 Main result

The main result in this section is the following

Theorem 3.1. Suppose $a, b \geq 0$, $a + b > 0$, $\lambda \geq 0$, $0 < \gamma < 1$, $0 < p \leq 2^* - 1$ and $f \in L^{\frac{2^*}{2^*+\gamma-1}}(\Omega)$ with $f(x) > 0$ for almost every $x \in \Omega$. Then problem (3.1) possesses a unique positive solution. Moreover, this solution is a global minimizer solution.

Remark 3.1. From one side, to the best of our information and knowledge, the studies of the existence and uniqueness of solutions for problem (3.1) are rare. The result that we get in Theorem (3.1) holds not only for the degenerate case, but also for the non-degenerate case. On the other side, in Refs. [13, 10, 12], problem (3.1) was considered only in dimension $N = 3$. However, we get the existence and uniqueness of solutions for problem (3.1) in high dimensions, i.e. $N \geq 3$.

Remark 3.2. When $a = 1$, $b = 0$, problem (3.1) reduces to the classic semilinear singular equation. Theorem (3.1) is also true. Moreover, when $\lambda = 0$, our Theorem is the corresponding result of [6]. We point out that the condition that $f \in L^{\frac{2^*}{2^*+\gamma-1}}(\Omega)$ is more general than the condition that $f \in L^\infty(\Omega)$ in [6].

3.3 Proof of main result

3.3.1 Global minimizer of I

$$\int_{\Omega} f(x) |u|^{1-\gamma} dx \leq C |f|_{\frac{2^*}{2^*+\gamma-1}} \|u\|^{1-\gamma},$$

since $0 < \gamma < 1$, $\lambda \geq 0$, with

$$\frac{\lambda}{p+1} \int_{\Omega} |u|^{p+1} dx - \frac{1}{1-\gamma} \int_{\Omega} f(x) |u|^{1-\gamma} dx \geq -C |f|_{\frac{2^*}{2^*+\gamma-1}} \|u\|^{1-\gamma},$$

so by the Hölder inequality and Sobolev inequality, we have got

$$\begin{aligned} I(u) &= \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 + \frac{\lambda}{p+1} \int_{\Omega} |u|^{p+1} dx - \frac{1}{1-\gamma} \int_{\Omega} f(x) |u|^{1-\gamma} dx \\ &\geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - C |f|_{\frac{2^*}{2^*+\gamma-1}} \|u\|^{1-\gamma}, \end{aligned}$$

where $C > 0$ is a constant. This implies that I is coercive and bounded from below on $H_0^1(\Omega)$. Then $m = \inf_{u \in H_0^1(\Omega)} I(u)$ is well defined. Moreover, since $0 < \gamma < 1$ and $f(x) > 0$ for almost every $x \in \Omega$ we can easily get $m < 0$. Before proving Theorem (3.1), we need to see the following useful lemma.

Lemma 3.1. *Assume that $0 < \gamma < 1$, $\lambda \geq 0$, $0 < p \leq 2^* - 1$, $a, b \geq 0$ with $a + b > 0$ and $f \in L^{\frac{2^*}{2^*+\gamma-1}}(\Omega)$ with $\forall x \in \Omega$, $f(x) > 0$. Then I attains the global minimizer in $H_0^1(\Omega)$, that is, there exists $u_* \in H_0^1(\Omega)$ such that $I(u_*) = m < 0$.*

Proof. According to the definition of m , there exists a minimizing sequence $u_n \subset H_0^1(\Omega)$ such that $\lim_{n \rightarrow \infty} I(u_n) = m < 0$. Since $I(|u_n|) = I(u_n)$, then we may assume that $u_n \geq 0$ for almost every $x \in \Omega$. Obviously, u_n is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, still denoted by u_n , there exists $u^* \geq 0$ such that \square

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } H_0^1(\Omega), \\ u_n \rightarrow u_*, & \text{strongly in } L^s(\Omega), \quad 1 \leq s < 2^*, \\ u_n(x) \rightarrow u_*(x), & \text{a.e. in } \Omega, \end{cases} \quad (3.3)$$

as $n \rightarrow \infty$. As usually, letting $w_n = u_n - u_*$, we need prove that $\|w_n\| \rightarrow 0$ as $n \rightarrow \infty$.

So, by the previous theorem, we can claim that

$$\lim_{n \rightarrow \infty} \int_{\Omega} f(x) |u_n|^{1-\gamma} dx = \int_{\Omega} f(x) |u_*|^{1-\gamma} dx. \quad (3.4)$$

And now, we only have to prove that $\{\int_{\Omega} f(x) |u_n|^{1-\gamma} dx, n \in N\}$ is equi-absolutely-continuous. Note that $\{u_n\}$ is bounded, by the Sobolev embedding theorem, there exists a constant $C > 0$, such that $\|u_n\|_{2^*} \leq C$. For every $\varepsilon > 0$, by the absolutely-continuity of $\int_{\Omega} |f(x)|^{\frac{2^*}{2^*+\gamma-1}}$, there exists $\delta > 0$ such that $\int_E |f(x)|^{\frac{2^*}{2^*+\gamma-1}} dx < \varepsilon^{\frac{2^*}{2^*+\gamma-1}}$ for every $E \subset \Omega$ with $\text{meas } E < \delta$. Consequently, by the Holder inequality, we have

$$\int_E f(x) |u_n|^{1-\gamma} dx \leq \|u_n\|_{2^*}^{1-\gamma} \left(\int_E |f(x)|^{\frac{2^*}{2^*+\gamma-1}} dx \right)^{\frac{2^*+\gamma-1}{2^*}} < C^{1-\gamma} \varepsilon.$$

Thus, claim (3.4) is valid. Moreover, by the weak convergence of $\{u_n\}$ in $H_0^1(\Omega)$, we obtain

$$\|u_n\|^2 = \|w_n\|^2 + \|u_*\|^2 + o(1), \quad (3.5)$$

$$\|u_n\|^4 = \|w_n\|^4 + \|u_*\|^4 + 2\|w_n\|^2\|u_*\|^2 + o(1), \quad (3.6)$$

$$\int_{\Omega} |u_n|^{2^*} dx = \int_{\Omega} |w_n|^{2^*} dx + \int_{\Omega} |u_*|^{2^*} dx + o(1), \quad (3.7)$$

where $o(1)$ is an infinitesimal as $n \rightarrow \infty$. Hence, in the case that $0 < p < 2^* - 1$, from (3.3) – (3.6), we deduce that

$$\begin{aligned} m &= \lim_{n \rightarrow \infty} I(u_n) \\ &= I(u_*) + \lim_{n \rightarrow \infty} \left(\frac{a}{2} \|w_n\|^2 + \frac{b}{4} \|w_n\|^4 + \frac{1}{4} \|w_n\|^2 \|u_*\|^2 \right) \\ &\geq I(u_*) \\ &\geq m, \end{aligned}$$

which implies $I(u_*) = m$. In the case that $p = 2^* - 1$, it follows from (3.4) – (3.7) that

$$\begin{aligned} m &= I(u_*) + \lim_{n \rightarrow \infty} \left(\frac{a}{2} \|w_n\|^2 + \frac{b}{4} \|w_n\|^4 + \frac{1}{4} \|w_n\|^2 \|u_*\|^2 + \frac{\lambda}{2^*} \|w_n\|^{2^*} \right) \\ &\geq I(u_*) \\ &\geq m, \end{aligned}$$

which yields $I(u_*) = m$. This completes the proof of the previous lemma.

3.3.2 Minimizer of I is a positive solution of (3.1)

Since $I(u_*) = m < 0$, we have got $u_* \geq 0$ and $u_* \neq 0$. Now, we are going to divide the proof in three steps.

First of all, we prove that $u_*(x) > 0$ for almost every $x \in \Omega$. Since $u_*(x) \geq 0$ for $x \in \Omega$, $\forall \phi \in H_0^1(\Omega)$, $\phi \geq 0$ and $t > 0$, we have

$$\begin{aligned} 0 &\leq \frac{I(u_* + t\phi) - I(u_*)}{t} \\ &= a \int_{\Omega} (\nabla u_*, \nabla \phi) dx + \frac{at}{2} \int_{\Omega} |\nabla \phi|^2 dx + b \|u_*\|^2 \int_{\Omega} (\nabla u_*, \nabla \phi) dx \\ &\quad + \frac{bt}{2} \|u_*\|^2 \|\phi\|^2 + bt \left(\int_{\Omega} (\nabla u_*, \nabla \phi) dx \right)^2 + bt^2 \|\phi\|^2 \int_{\Omega} (\nabla u_*, \nabla \phi) dx \\ &\quad + bt^3 \|\phi\|^4 + \frac{\lambda}{p+1} \int_{\Omega} \frac{(u_* + t\phi)^{p+1} + u_*^{p+1}}{t} dx \\ &\quad - \frac{1}{1-\gamma} \int_{\Omega} f(x) \frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t} dx. \end{aligned}$$

By using the Lebesgue Dominate Convergence Theorem, one has

$$\lim_{t \rightarrow 0^+} \frac{1}{p+1} \int_{\Omega} \frac{(u_* + t\phi)^{p+1} + u_*^{p+1}}{t} dx = \int_{\Omega} u_*^p \phi dx. \quad (3.8)$$

For any $x \in \Omega$, we denote

$$h(t) = f(x) \frac{[u_*(x) + t\phi(x)]^{1-\gamma} - u_*^{1-\gamma}(x)}{(1-\gamma)t}.$$

Then

$$h'(t) = f(x) \frac{u_*^{1-\gamma}(x) - [\gamma t \phi(x) + u_*(x)][u_*(x) + t\phi(x)]^{-\gamma}}{(1-\gamma)t^2} \leq 0,$$

which implies that $h(t)$ is non-increasing for $t > 0$. Moreover, one has

$$\lim_{t \rightarrow 0^+} h(t) = \int_{\Omega} f(x) u_*^{-\gamma}(x) \phi(x) dx,$$

for every $x \in \Omega$, which may be $+\infty$ when $u_*(x) = 0$ and $\phi(x) > 0$. Consequently, by the previous theorem, we have got

$$\lim_{t \rightarrow 0^+} \frac{1}{1-\gamma} \int_{\Omega} f(x) \frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t} dx = \int_{\Omega} f(x) u_*^{-\gamma} \phi dx,$$

which possibly equals to $+\infty$. Combining this with (3.8), let $t \rightarrow 0^+$,

$$\int_{\Omega} f(x) u_*^{-\gamma} \phi dx \leq (a + b \|u_*\|^2) \int_{\Omega} (\nabla u_*, \nabla \phi) dx + \lambda \int_{\Omega} u_*^p \phi dx, \quad (3.9)$$

for all $\phi \in H_0^1(\Omega)$ with $\phi > 0$. Let $e_1 \in H_0^1(\Omega)$ be the first eigenfunction of the operator $-\Delta$ with $e_1 > 0$ and $\|e_1\| = 1$. Particularly, taking $\phi = e_1$ in (3.9), one gets

$$\int_{\Omega} f(x) u_*^{-\gamma} e_1 dx \leq (a + b \|u_*\|^2) \int_{\Omega} (\nabla u_*, \nabla e_1) dx + \lambda \int_{\Omega} u_*^p e_1 dx < \infty,$$

which implies that $u_* > 0$ for almost every $x \in \Omega$. Secondly, we show that u_* is a solution of problem (3.1). We claim that inequality (3.9) is true for all $\phi \in H_0^1(\Omega)$. Indeed, from the definition of u_* , there exists $\delta \in (0, 1)$ such that $u_* + tu_* \in H_0^1$ for all $|t| \leq \delta$. We define that $\varphi : [-\delta, \delta] \rightarrow \mathbb{R}$ by $\varphi(t) = I(u_* + tu_*)$, thus φ attains its minimum at $t = 0$ which implies that

$$\varphi'(0) = a \|u_*\|^2 + b \|u_*\|^4 + \lambda \int_{\Omega} u_*^{p+1} dx - \int_{\Omega} f(x) u_*^{1-\gamma} dx = 0 \quad (3.10)$$

Suppose $\phi \in H_0^1$ and $\varepsilon > 0$, define $\psi \in H_0^1$ by $\psi = (u_* + \varepsilon\phi)^+$, where $(u_* + \varepsilon\phi)^+ = \max\{u_* + \varepsilon\phi, 0\}$. Obviously, $\psi \geq 0$. Then replacing ϕ with ψ in (3.9), let $\Omega_1 = \{x \in \Omega : u_* + \varepsilon\phi \leq 0\}$

one has

$$\begin{aligned}
0 &\leq \int_{\Omega} (a + b\|u_*\|^2)(\nabla u_*, \nabla \psi) dx + \lambda \int_{\Omega} u_*^p \psi dx - \int_{\Omega} f(x) u_*^{-\gamma} \psi dx \\
&= \varepsilon \int_{\Omega} \left[(a + b\|u_*\|^2)(\nabla u_*, \nabla \phi) + \lambda \int_{\Omega} u_*^p \phi - f(x) u_*^{-\gamma} \phi \right] dx \\
&\quad - \int_{\Omega_1} (a + b\|u_*\|^2)(\nabla u_*, \nabla(u_* + \varepsilon)) dx - \lambda \int_{\Omega_1} u_*^p (u_* + \varepsilon) dx \\
&\quad + \int_{\Omega_1} f(x) u_*^{-\gamma} (u_* + \varepsilon) dx \\
&< \varepsilon \int_{\Omega} \left[(a + b\|u_*\|^2)(\nabla u_*, \nabla \phi) + \lambda u_*^p \phi - f(x) u_*^{-\gamma} \phi \right] dx \\
&\quad - \varepsilon \int_{\Omega_1} \left[(a + b\|u_*\|^2)(\nabla u_*, \nabla \phi) + \lambda u_*^p \phi \right] dx
\end{aligned}$$

since $\text{meas } \Omega_1 \rightarrow 0$ as $\varepsilon \rightarrow 0^+$, dividing by ε and letting $\varepsilon \rightarrow 0^+$, we have

$$(a + b\|u_*\|^2) \int_{\Omega} (\nabla u_*, \nabla \phi) dx + \lambda \int_{\Omega} u_*^p \phi dx - \int_{\Omega} f(x) u_*^{-\gamma} \phi dx \geq 0,$$

for all $\phi \in H_0^1(\Omega)$. Hence, this inequality also holds equally well for $-\phi$. Then our claim is proved. It follows that u_* is a solution of problem (3.1). Moreover, according to Lemma (3.1), we have $I(u_*) = \inf_{u \in H_0^1} I(u)$. Thus u_* is a global minimizer solution.

3.3.3 Uniqueness of solution

Now, we prove the uniqueness of solutions of problem (3.1). Assume that v_* is another solution of problem (3.1), then it follows from (3.2) that

$$(a + b\|u_*\|^2) \int_{\Omega} (\nabla u_*, \nabla(u_* - v_*)) dx + \lambda \int_{\Omega} u_*^p (u_* - v_*) dx - \int_{\Omega} f(x) u_*^{-\gamma} (u_* - v_*) dx = 0 \quad (3.11)$$

$$(a + b\|v_*\|^2) \int_{\Omega} (\nabla v_*, \nabla(u_* - v_*)) dx + \lambda \int_{\Omega} v_*^p (u_* - v_*) dx - \int_{\Omega} f(x) v_*^{-\gamma} (u_* - v_*) dx = 0 \quad (3.12)$$

From (3.11) and (3.12), we obtain

$$\begin{aligned}
a\|u_* - v_*\|^2 + b \left[\|u_*\|^4 - \|v_*\|^4 \int_{\Omega} (\nabla u_*, \nabla v_*) dx - \|v_*\|^2 \int_{\Omega} (\nabla u_*, \nabla v_*) dx + \|v_*\|^4 \right] \\
+ \lambda \int_{\Omega} (u_*^p - v_*^p)(u_* - v_*) dx - \int_{\Omega} f(x) (u_*^{-\gamma} - v_*^{-\gamma})(u_* - v_*) dx = 0
\end{aligned}$$

Denote

$$J(u_*, v_*) = \|u_*\|^4 - \|u_*\|^2 \int_{\Omega} (\nabla u_*, \nabla v_*) dx - \|v_*\|^2 \int_{\Omega} (\nabla u_*, \nabla v_*) dx + \|v_*\|^4.$$

By, the holder inequality, one has

$$\begin{aligned} J(u_*, v_*) &\geq \|u_*\|^4 - \|u_*\|^3 \|v_*\| - \|v_*\|^3 \|u_*\| + \|v_*\|^4 \\ &= (\|u_*\| - \|v_*\|)^2 (\|u_*\|^2 + \|u_*\| \|v_*\| + \|v_*\|^2) \\ &\geq 0 \end{aligned}$$

Since $0 < \gamma < 1$ and $p > 0$, it is easy to get the following elementary inequalities

$$(m^{-\gamma} - n^{-\gamma})(m - n) \geq 0, \quad \forall m, n > 0.$$

Thus

$$\int_{\Omega} f(x)(u_*^{-\gamma} - v_*^{-\gamma})(u_* - v_*) dx \leq 0, \quad \int_{\Omega} (u_*^p - v_*^p)(u_* - v_*) dx \geq 0.$$

Consequently, if $a > 0$, it follows from that $a\|u_* - v_*\|^2 \leq 0$. Then $\|u_* - v_*\|^2 = 0$. If $a = 0$, one has $\|u_*\| = \|v_*\|$ and $J(u_*, v_*) = 0$. As a result, $J(u_*, v_*) = \|u_*\|^2(2\|u_*\|^2 - 2 \int_{\Omega} (\nabla u_*, \nabla v_*) dx) = \|u_*\|^2 \|u_* - v_*\|^2 = 0$, this implies $\|u_* - v_*\|^2 = 0$. Thus, for every $a \geq 0$, one has $u_* = v_*$. Therefore u_* is the unique positive solution of problem (3.1).

This completes the proof of Theorem (3.1).

A FRACTIONAL KIRCHHOFF PROBLEM INVOLVING A SINGULAR TERM AND CRITICAL NONLINEARITY

4.1 Main result

This chapter is devoted to the study of a class of Kirchhoff-type problems driven by a nonlocal fractional operator and involving a singular term and a critical nonlinearity.

More precisely, we consider

$$\begin{cases} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\theta-1} (-\Delta)^s u = \frac{\lambda}{u^\gamma} + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^{2N} \setminus \Omega. \end{cases} \quad (4.1)$$

Where Ω is an open bounded subset of \mathbb{R}^N with continuous boundary, dimension $N > 2s$ with parameter $s \in (0, 1)$, $2_s^* = 2N/(N - 2s)$ is the fractional critical Sobolev exponent, $\lambda > 0$ is a real parameter, $\theta \in (1, 2_s^*/2)$, while $\gamma \in (0, 1)$. Here $(-\Delta)^s$ is the fractional Laplace operator defined, up to normalization factors, by the Riesz potential as

$$(-\Delta)^s \varphi(x) = \int_{\mathbb{R}^N} \frac{2\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

Then our problem is the fractional version of the following nonlinear problem:

$$\begin{cases} -M \left(\int_{\Omega} |\nabla u(x)|^2 dx \right) \Delta u = \frac{\lambda}{u^\gamma} + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \quad (4.2)$$

Theorem 4.1. *Let $s \in (0, 1)$, $N > 2s$, $\theta \in (1, 2_s^*/2)$, $\gamma \in (0, 1)$ and let Ω be an open bounded subset of \mathbb{R}^N with $\partial\Omega$ continuous. then there exists $\bar{\lambda} > 0$ such that for any $\lambda \in (0, \bar{\lambda})$ problem (4.1) has at least one solution.*

4.2 Variational setting

We suppose without further mentioning that $s \in (0, 1)$, $N > 2s$, $\theta \in (1, 2_s^*/2)$, $\gamma \in (0, 1)$ and Ω is an open bounded subset of \mathbb{R}^N with $\partial\Omega$ continuous. As a matter of notations, we denote with $\varphi^+ = \max\{\varphi, 0\}$ and $\varphi^- = \{-\varphi, 0\}$ respectively the positive and negative part of a function φ . Problem (4.1) has a variational structure, and the natural space where finding solutions is the homogeneous fractional Sobolev space $H_0^s(\Omega)$.

In order to study (4.1), we need to encode the ‘‘boundary condition’’ $u = 0$ in $\mathbb{R}^N \setminus \Omega$ in the weak formulation, by considering also that the interaction between Ω and its complementary in \mathbb{R}^N gives a positive contribution in the so-called Gagliardo norm, given as:

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2} \quad (4.3)$$

The functional space that takes into account this boundary condition will be denoted by X_0 and it is defined as:

$$X_0 = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

In X_0 we can consider the norm :

$$\|u\|_{X_0} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}$$

Now, before going on, we need some preliminary results on X and X_0 . In the sequel we denote $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$, where $\mathcal{O} = (C\Omega) \times (C\Omega) \subset \mathbb{R}^{2N}$, and $C\Omega = \mathbb{R}^N \setminus \Omega$. The space X is endowed with the norm defined as

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \left(\int |g(x) - g(y)|^2 K(x - y) dx dy \right)^{1/2}. \quad (4.4)$$

Lemma 4.1. [14] Let $K : \mathbb{R}^N \setminus \{0\} \rightarrow (0, +\infty)$, then

a- there exists a positive constant c , depending only on N and s , such that for any $v \in X_0$

$$\|v\|_{L^{2^*}(\Omega)}^2 = \|v\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq c \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dx dy,$$

b- there exists a constant $C > 1$, depending only on N, s, λ and Ω , such that for any $v \in X_0$

$$\int_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x - y) dx dy \leq \|v\|_{X_0}^2 \leq C \int_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 K(x - y) dx dy,$$

that is

$$\|v\|_{X_0} = \left(\int_Q |v(x) - v(y)|^2 K(x - y) dx dy \right)^{1/2}$$

is a norm in X_0 equivalent to the usual one defined in (4.4)

Proof. Let v be in X_0 and $p = 2$ we obtain

$$\|v\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq c \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} dx dy,$$

where c is a positive constant depending only on N and s . Since $v = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we get assertion a- For part b-, we note that by (4.4) it easily follows that

$$\|v\|_X^2 \geq \int_Q |v(x) - v(y)|^2 K(x - y) dx dy.$$

Moreover, using the fact that $L^{2^*}(\Omega) \hookrightarrow$

$$\begin{aligned}
\|v\|_X^2 &= \left(\|v\|_{L^2(\Omega)} + \left(\int_{\mathbb{Q}} |v(x) - v(y)|^2 K(x-y) dx dy \right)^{1/2} \right)^2 \\
&\leq 2\|v\|_{L^2(\Omega)}^2 + 2 \int_{\mathbb{Q}} |v(x) - v(y)|^2 K(x-y) dx dy \\
&\leq 2|\Omega|^{(2^*-2)/2^*} \|v\|_{L^{2^*}(\Omega)}^2 + 2 \int_{\mathbb{Q}} |v(x) - v(y)|^2 K(x-y) dx dy \\
&\leq 2c|\Omega|^{(2^*-2)/2^*} \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} dx dy + 2 \int_{\mathbb{Q}} |v(x) - v(y)|^2 K(x-y) dx dy \\
&\leq 2 \left(\frac{c|\Omega|^{(2^*-2)/2^*}}{\lambda} + 1 \right) \int_{\mathbb{Q}} |v(x) - v(y)|^2 K(x-y) dx dy.
\end{aligned}$$

□

We also recall that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with the scalar product defined as:

$$\langle u, v \rangle_{X_0} = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dx dy.$$

From now on, in order to simplify the notation, we will denote $\|\cdot\|_{X_0}$ and $\langle \cdot, \cdot \rangle_{X_0}$ by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively, and $\|\cdot\|_{L^q(\Omega)}$ by $\|\cdot\|_q$ for any $q \in [1, \infty]$.

In order to present the weak formulation of the problem (4.1) we will consider the following kirchhoff problem taking into account that we are looking for positive solutions

$$\begin{cases} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{\theta-1} (-\Delta)^s u = \frac{\lambda}{(u^+)^\gamma} + (u^+)^{2^*_s-1} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (4.5)$$

Definition 4.1 (weak solution). *We say that $u \in X_0$ is a (weak) solution of problem (4.5) if u satisfies*

$$\begin{aligned}
&\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{\theta-1} \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dx dy \\
&= \int_{\Omega} \frac{\lambda \varphi}{(u^+)^\gamma} dx + (u^+)^{2^*_s-1} \int_{\Omega} \varphi dx
\end{aligned} \quad (4.6)$$

i.e.

$$\|u\|^{2(\theta-1)} \langle u, \varphi \rangle = \lambda \int_{\Omega} \frac{\varphi}{(u^+)^{\gamma}} dx + (u^+)^{2_s^*-1} \varphi.$$

for any $\varphi \in X_0$.

Problem (4.5) has a variational structure any $J_{\lambda} : X_0 \rightarrow \mathbb{R}$, defined

$$J_{\lambda} = \frac{1}{2\theta} \|u\|^{2\theta} - \frac{\lambda}{1-\gamma} \int_{\Omega} (u^+)^{1-\gamma} dx - \frac{1}{2_s^*} \|u^+\|_{2_s^*}^{2_s^*},$$

is the underlying functional associated to the problem (4.5). Because of the presence of a singular term in (4.5), the functional J_{λ} is not differentiable on X_0 . Therefore, we can not apply directly the usual critical point theory to J_{λ} , in order to solve our problem. If we want to study the compactness property of the functional $J_{n,\lambda}$, we have to recall the best constant of the fractional Sobolev embedding, which will be very useful, and this constant is well defined and strictly positive

$$S = \inf_{V \in H^s(\mathbb{R}^N)} \frac{\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} dx dy}{\left(\int_{\mathbb{R}^N} |v(x)|^{2_s^*} dx \right)^{2/2_s^*}}. \quad (4.7)$$

4.3 Proof of Theorem (4.1)

4.3.1 Geometric condition

In this section we prove the existence of a solution for the problem (4.1) by a local minimization argument. For this, we first study the geometry of functional J_{λ} .

Lemma 4.2. *There exist numbers $\rho \in (1, 0]$, $\lambda_0 = \lambda_0(\rho) > 0$ and $\alpha = \alpha(\rho) > 0$ such that $J_{\lambda} \geq \alpha$ for any $u \in X_0$ with $\|u\| = \rho$ and for any $\lambda \in (0, \lambda_0]$. Furthermore, set*

$$m_{\lambda} = \inf \{ J_{\lambda}(u) : u \in \bar{B}_{\rho} \},$$

Where $\bar{B}_{\rho} = \{u \in X_0 : \|u\| \leq \rho\}$. Then $m_{\lambda} < 0$ for any $\lambda \in (0, \lambda_0]$.

Proof. Let $\lambda > 0$. We know that

$$\int_{\Omega} (u^+)^{1-\gamma} dx \leq \int_{\Omega} |u|^{1-\gamma} dx.$$

$$\left(\int_{\Omega} |u|^{2_s^*} \right)^{\frac{1-\gamma}{2_s^*}} = \|u\|_{2_s^*}^{1-\gamma}.$$

From the Hölder inequality and (4.7) for any $u \in X_0$ we have

$$\int_{\Omega} (u^+)^{1-\gamma} dx \leq |\Omega|^{(2_s^*-1+\gamma)/2_s^*} \|u\|_{2_s^*}^{1-\gamma} \leq |\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2} \|u\|^{1-\gamma}. \quad (4.8)$$

Subsequently, using again (4.7) and (4.8), we get

$$J_{\lambda}(u) \geq \frac{1}{2\theta} \|u\|^{2\theta} - \frac{S^{-2_s^*/2}}{2_s^*} \|u\|^{2_s^*} - \frac{\lambda}{1-\gamma} |\Omega|^{(2_s^*-1+\gamma)/2_s^*} S^{-(1-\gamma)/2} \|u\|^{1-\gamma}. \quad (4.9)$$

Since $1-\gamma < 1 < 2\theta < 2_s^*$, the function

$$\eta(t) = \frac{1}{2\theta} t^{2\theta-1+\gamma} + \frac{S^{-2_s^*/2}}{2_s^*} t^{2_s^*-1+\gamma}, \quad t \in [0, 1],$$

admits a maximum at some $\rho \in (0, 1]$ small enough, that is, $\max_{t \in [0, 1]} \eta(t) = \eta(\rho) >$

0. Thus, let

$$\lambda_0 = \frac{(1-\gamma)S^{(1-\gamma)/2}}{2 |\Omega|^{(2_s^*-1+\gamma)/2_s^*}} \eta(\rho).$$

Then for any $u \in X_0$ with $\|u\| = \rho \leq 1$ and for any $\lambda \leq \lambda_0$, we get

$$J_{\lambda}(u) \geq \rho^{1-\gamma} \eta(\rho) / 2 = \alpha > 0.$$

Furthermore, fixed $v \in X_0$ with $v^+ \neq 0$, for $t \in (0, 1)$ sufficiently small we have

$$J_{\lambda}(tv) = \frac{t^{2\theta}}{2\theta} \|v\|^{2\theta} - t^{1-\gamma} \frac{\lambda}{1-\gamma} \int_{\Omega} (v^+)^{1-\gamma} dx - \frac{t^{2_s^*}}{2_s^*} \|v^+\|_{2_s^*}^{2_s^*} < 0$$

Since $1-\gamma < 1 < 2\theta < 2_s^*$. □

4.3.2 Existence of a local minimizer

We are now ready to prove the existence of solution of (4.1)

Theorem 4.2. Let λ_0 be given as in lemma (4.2). then for any $\lambda \in (0, \lambda_0]$ problem (4.1) has a solution $u_0 \in X_0$ with $J_\lambda(u_0) < 0$.

Proof. Fix $\lambda \in (0, \lambda_0]$ and let ρ be as given in lemma (4.2). We first prove that there exists $u_0 \in \bar{B}_\rho$ such that $J_\lambda(u_0) = m_\lambda < 0$. Let $\{u_k\}_k \subset \bar{B}_\rho$ be a minimizing sequence for m_λ , that is, such that

$$\lim_{k \rightarrow \infty} J_\lambda(u_k) = m_\lambda \quad (4.10)$$

Since $\{u_k\}_k$ is bounded in X_0 , by applying ([14], Lemma 8) and ([3], Theorem 4.9) there exists a subsequence, still denoted by $\{u_k\}_k$, and a function $u_0 \in \bar{B}_\rho$ such that, as $k \rightarrow \infty$, we have X_0 is reflexif and the embedding of X_0 in L^p is compact, $\forall p \in (1, 2_s^*)$, then

$$\begin{cases} u_k \rightharpoonup u_0 & \text{in } X_0, & u_k \rightharpoonup u_0 & \text{in } L^{2_s^*}(\Omega) \\ u_k \rightarrow u_0 & \text{in } L^p(\Omega) \text{ for any } p \in [1, 2_s^*), & u_k \rightarrow u_0 & \text{a.e. in } \Omega. \end{cases} \quad (4.11)$$

Since $\gamma \in (0, 1)$, by the Hölder inequality, for any $k \in \mathbb{N}$ we have

$$\left| \int_{\Omega} (u_k^+)^{1-\gamma} dx - \int_{\Omega} (u_0^+)^{1-\gamma} dx \right| \leq \int_{\Omega} |u_k^+ - u_0^+|^{1-\gamma} dx \leq \|u_k^+ - u_0^+\|_2^{1-\gamma} |\Omega|^{(1+\gamma)/2},$$

which yields, by (4.11)

$$\lim_{k \rightarrow \infty} \int_{\Omega} (u_k^+)^{1-\gamma} dx = \int_{\Omega} (u_0^+)^{1-\gamma} dx. \quad (4.12)$$

Let $w_k = u_k - u_0$ by Lieb-Brezis Lemma it holds true that

$$\|u_k\|^2 = \|w_k\|^2 + \|u_0\|^2 + o(1), \quad \|u_k\|_{2_s^*}^{2_s^*} = \|w_k\|_{2_s^*}^{2_s^*} + \|u_0\|_{2_s^*}^{2_s^*} + o(1) \quad (4.13)$$

as $k \rightarrow \infty$. Since $\{u_k\}_k \subset \bar{B}_\rho$, by (4.13) for k sufficiently large, we have $w_k \in \bar{B}_\rho$. Lemma (4.2) implies that for any $u \in X_0$, with $\|u\| = \rho$, we get

$$\frac{1}{2\theta} \|u\|^{2\theta} - \frac{1}{2_s^*} \|u^+\|_{2_s^*}^{2_s^*} \geq \alpha > 0,$$

and from this, since $\rho \leq 1$, for k sufficiently large we have

$$\frac{1}{2\theta} \|w_k\|^{2\theta} - \frac{1}{2_s^*} \|w_k^+\|_{2_s^*}^{2_s^*} > 0. \quad (4.14)$$

Thus, by (4.10), (4.12) and (4.14) and considering $\theta \geq 1$, it follows that, as $k \rightarrow \infty$,

$$\begin{aligned} m_\lambda &= J_\lambda(u_k) + o(1) \\ &= \frac{1}{2\theta}(\|w_k\|^2 + \|u_0\|^2)^\theta - \frac{\lambda}{1-\gamma} \int_{\Omega} (u_0^+)^{1-\gamma} dx - \frac{1}{2_s^*}(\|w_k^+\|_{2_s^*}^{2_s^*} + \|u_0^+\|_{2_s^*}^{2_s^*}) + o(1) \\ &\geq J_\lambda(u_0) + \frac{1}{2\theta}\|w_k\|^{2\theta} - \frac{1}{2_s^*}\|w_k^+\|_{2_s^*}^{2_s^*} + o(1) \geq J_\lambda(u_0) + o(1) \geq m_\lambda = \inf_{\|u\| \leq \rho} J_\lambda(u) \end{aligned}$$

since $u_0 \in \bar{B}_\rho$. Hence, u_0 is a local minimizer for J_λ , with $J_\lambda(u_0) = m_\lambda < 0$, which implies that u_0 is nontrivial.

4.3.3 The local minimizer is a weak solution

Now, we prove that u_0 is a positive solution of (4.5). For any $\psi \in X_0$, with $\psi \geq 0$ a.e. in \mathbb{R}^N , let us consider a $t > 0$ sufficiently small so that $u_0 + t\psi \in \bar{B}_\rho$. Since u_0 is local minimizer for J_λ , we have

$$J_\lambda(u_0) \leq J_\lambda(u_0 + t\psi)$$

on the other hand

$$\begin{aligned} 0 &\leq J_\lambda(u_0 + t\psi) - J_\lambda(u_0) \\ &= \frac{1}{2\theta}(\|u_0 + t\psi\|^{2\theta} - \|u_0\|^{2\theta}) - \frac{\lambda}{1-\gamma} \int_{\Omega} [((u_0 + t\psi)^+)^{1-\gamma} - (u_0^+)^{1-\gamma}] dx \\ &\quad - \frac{1}{2_s^*}(\|u_0 + t\psi\|_{2_s^*}^{2_s^*} - \|u_0^+\|_{2_s^*}^{2_s^*}). \end{aligned}$$

From this, by dividing by $t > 0$ and passing to the limit as $t \rightarrow 0^+$, it follows that

$$\liminf_{t \rightarrow 0^+} \frac{\lambda}{1-\gamma} \int_{\Omega} \frac{((u_0 + t\psi)^+)^{1-\gamma} - (u_0^+)^{1-\gamma}}{t} dx \leq \|u_0\|^{2(\theta-1)} \langle u_0, \psi \rangle - \int_{\Omega} (u_0^+)^{2_s^*-1} \psi dx. \quad (4.15)$$

We observe that

$$\frac{1}{1-\gamma} \frac{((u_0 + t\psi)^+)^{1-\gamma} - (u_0^+)^{1-\gamma}}{t} = ((u_0 + \xi t\psi)^+)^{-\gamma} \psi \quad \text{a.e. in } \Omega$$

with $\xi \in (0, 1)$ and

$$((u_0 + \xi t\psi)^+)^{-\gamma} \psi \quad \text{a.e. in } \Omega \quad \text{as } t \rightarrow 0^+.$$

Thus, by the Fatou lemma we obtain

$$\lambda \int_{\Omega} (u_0^+)^{-\gamma} \psi dx \leq \liminf_{t \rightarrow 0^+} \frac{\lambda}{1-\gamma} \int_{\Omega} \frac{((u_0 + t\psi)^+)^{1-\gamma} - (u_0^+)^{1-\gamma}}{t} dx. \quad (4.16)$$

Therefore, combining (4.15) and (4.16a), we get

$$\|u_0\|^{2(\theta-1)} \langle u_0, \psi \rangle - \lambda \int_{\Omega} (u_0^+)^{-\gamma} \psi dx - \int_{\Omega} (u_0^+)^{2_s^*-1} \psi dx \geq 0 \quad (4.17)$$

for any $\psi \in X_0$ with $\psi \geq 0$ a.e. in \mathbb{R}^N .

Since $J_{\lambda}(u_0) < 0$ and by lemma (4.2), we have $u_0 \in B_{\rho}$. Hence, there exists $\delta \in (0, 1)$ such that $(1+t)u_0 \in \bar{B}_{\rho}$ for any $t \in [-\delta, \delta]$. Let us define $J_{\lambda}(t) = J_{\lambda}((1+t)u_0)$. Since u_0 is a local minimizer for $J_{\lambda} \in \bar{B}_{\rho}$, the function J_{λ} has a minimum at $t = 0$, that is

$$I'_{\lambda}(0) = \|u_0\|^{2\theta} - \lambda \int_{\Omega} (u_0^+)^{1-\gamma} dx - \int_{\Omega} (u_0^+)^{2_s^*-1} dx = 0. \quad (4.18)$$

For any $\varphi \in X_0$ and any $\varepsilon > 0$, let us define $\psi_{\varepsilon} = u_0^+ + \varepsilon\varphi$. Then by (4.17) we have since

$$\psi_{\varepsilon}^+ = \psi_{\varepsilon} + \psi_{\varepsilon}^-$$

then:

$$\begin{aligned} 0 &\leq \|u_0\|^{2(\theta-1)} \langle u_0, \psi_{\varepsilon}^+ \rangle - \lambda \int_{\Omega} (u_0^+)^{-\gamma} \psi_{\varepsilon}^+ dx - \int_{\Omega} (u_0^+)^{2_s^*-1} \psi_{\varepsilon}^+ dx \\ &= \|u_0\|^{2(\theta-1)} \langle u_0, \psi_{\varepsilon} + \psi_{\varepsilon}^- \rangle - \lambda \int_{\Omega} (u_0^+)^{-\gamma} (\psi_{\varepsilon} + \psi_{\varepsilon}^-) dx - \int_{\Omega} (u_0^+)^{2_s^*-1} (\psi_{\varepsilon} + \psi_{\varepsilon}^-) dx. \end{aligned} \quad (4.19)$$

We observe that, for a.e. $x, y \in \mathbb{R}^N$, we obtain

$$\begin{aligned} (u_0(x) - u_0(y))(u_0^-(x) - u_0^-(y)) &= -u_0^+(x)u_0^-(y) - u_0^-(x)u_0^+(y) - [u_0^-(x) - u_0^-(y)]^2 \\ &\leq -|u_0^-(x) - u_0^-(y)|^2 \end{aligned} \quad (4.20)$$

from which we immediately get

$$(u_0(x) - u_0(y))(u_0^+(x) - u_0^+(y)) \leq |u_0(x) - u_0(y)|^2.$$

From the last inequality we can find that:

$$\begin{aligned}
\langle u_0, \psi_\varepsilon + \psi_\varepsilon^- \rangle &= \iint_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(\psi_\varepsilon(x) + \psi_\varepsilon^-(x) - \psi_\varepsilon(y) - \psi_\varepsilon^-(y))}{|x - y|^{N+2s}} dx dy \\
&\leq \iint_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} dx dy + \varepsilon \iint_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+2s}} dx dy. \tag{4.21}
\end{aligned}$$

Hence, denoting $\Omega_\varepsilon = \{x \in \mathbb{R}^N : u_0^+(x) + \varepsilon\varphi(x) \leq 0\}$ and combining (4.19) with (4.21), we get

$$\begin{aligned}
0 \leq \|u_0\|^{2\theta} &+ \varepsilon \|u_0\|^{2(\theta-1)} \langle u_0, \varphi \rangle + \|u_0\|^{2(\theta-1)} \langle u_0, \psi_\varepsilon^- \rangle - \lambda \int_{\Omega} (u_0^+)^{-\gamma} (u_0^+ + \varepsilon\varphi) dx \\
&- \int_{\Omega} (u_0^+)^{2_s^*-1} (u_0^+ + \varepsilon\varphi) dx + \lambda \int_{\Omega_\varepsilon} (u_0^+)^{-\gamma} (u_0^+ + \varepsilon\varphi) dx + \int_{\Omega_\varepsilon} (u_0^+)^{2_s^*-1} (u_0^+ + \varepsilon\varphi) dx
\end{aligned}$$

From (4.18) we have

$$\begin{aligned}
&\leq \|u_0\|^{2\theta} - \lambda \int_{\Omega} (u_0^+)^{1-\gamma} dx - \int_{\Omega} (u_0^+)^{2_s^*} dx + \|u_0\|^{2(\theta-1)} \langle u_0, \psi_\varepsilon^- \rangle \\
&+ \varepsilon \left[\|u_0\|^{2(\theta-1)} \langle u_0, \varphi \rangle - \lambda \int_{\Omega} (u_0^+)^{-\gamma} \varphi dx - \int_{\Omega} (u_0^+)^{2_s^*-1} \varphi dx \right] \\
&= \|u_0\|^{2(\theta-1)} \langle u_0, \psi_\varepsilon^- \rangle + \varepsilon \left[\|u_0\|^{2(\theta-1)} \langle u_0, \varphi \rangle - \lambda \int_{\Omega} (u_0^+)^{-\gamma} \varphi dx - \int_{\Omega} (u_0^+)^{2_s^*-1} \varphi dx \right], \tag{4.22}
\end{aligned}$$

Where the last equality follows from (4.18). Arguing similarly to (4.20), for a.e. $x, y \in \mathbb{R}^N$ we have

$$(u_0(x) - u_0(y))(u_0^+(x) - u_0^+(y)) \geq |u_0^+(x) - u_0^+(y)|^2. \tag{4.23}$$

Thus denoting

$$\mathcal{U}_\varepsilon(x, y) = \frac{(u_0(x) - u_0(y))(\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+2s}},$$

by the symmetry of the fractional kernel and (4.23), we get

$$\begin{aligned}
\langle u_0, \psi_\varepsilon^- \rangle &= \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \mathcal{U}_\varepsilon(x, y) dx dy + 2 \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \mathcal{U}_\varepsilon(x, y) dx dy \\
&\leq \varepsilon \left(\iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \mathcal{U}(x, y) dx dy + 2 \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \mathcal{U}(x, y) dx dy \right) \\
&\leq 2\varepsilon \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} |\mathcal{U}(x, y)| dx dy,
\end{aligned} \tag{4.24}$$

where we set

$$\mathcal{U}(x, y) = \frac{(u_0(x) - u_0(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}.$$

Clearly $\mathcal{U} \in L^1(\mathbb{R}^{2N})$, so that for any $\sigma > 0$ there exists R_σ sufficiently large such that

$$\iint_{(\text{supp}\varphi) \times (\mathbb{R}^N \setminus B_{R_\sigma})} |\mathcal{U}(x, y)| dx dy < \frac{\sigma}{2}.$$

And, by the definition of Ω_ε , we have $\Omega_\varepsilon \subset \text{supp}\varphi$ and $|\Omega_\varepsilon \times B_{R_\sigma}| \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

Thus, since $\mathcal{U} \in L^1(\mathbb{R}^{2N})$, there exist $\delta_\sigma > 0$ and $\varepsilon_\sigma > 0$ such that for any $\varepsilon \in (0, \varepsilon_\sigma]$,

$$|\Omega_\varepsilon \times B_{R_\sigma}| < \delta_\sigma \quad \text{and} \quad \iint_{\Omega_\varepsilon \times B_{R_\sigma}} |\mathcal{U}(x, y)| dx dy < \frac{\sigma}{2}.$$

Therefore, for any $\varepsilon \in (0, \varepsilon_\sigma]$,

$$\iint_{\Omega_\varepsilon \times \mathbb{R}^N} |\mathcal{U}(x, y)| dx dy < \sigma,$$

from which we get

$$\lim_{\varepsilon \rightarrow 0^+} \iint_{\Omega_\varepsilon \times \mathbb{R}^N} |\mathcal{U}(x, y)| dx dy = 0. \tag{4.25}$$

Combining (4.22) with (4.24), dividing by ε , letting $\varepsilon \rightarrow 0^+$ and using (4.25), we obtain

$$\|u_0\|^{2(\theta-1)} \langle u_0, \varphi \rangle - \lambda \int_{\Omega} (u_0^+)^{-\gamma} \varphi dx - \int_{\Omega} (u_0^+)^{2^*_s-1} \varphi dx \geq 0$$

for any $\varphi \in X_0$. By the arbitrariness of φ , we prove that u_0 verifies 2.3, that is, u_0 is a nontrivial solution of (4.5).

4.3.4 Positivity of solution

Considering $\varphi = u_0^-$ in 2.3 and using (4.20), we see that $\|u_0^-\| = 0$, which implies that u_0 is nonnegative. Moreover, by the maximum principle in the next proposition:

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, and let u be a lower-semicontinuous function in Ω such that $(-\Delta)^\sigma u \geq 0$ in Ω and $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$. Then $u \geq 0$ in \mathbb{R}^N . Moreover, if $u(x) = 0$ for one point x inside Ω , then $u \equiv 0$ in all \mathbb{R}^N .*

We can deduce that u_0 is a positive solution of (4.5), and so also solves problem (4.1). This concludes the proof. \square

4.4 More general result

We end this chapter by observing that the result in Theorem (4.2) can be extended to more general Kirchhoff problems. That is, we can consider the problem:

$$\begin{cases} M \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right) (-\Delta)_P^s u = \frac{\lambda}{u^\gamma} + u^{p_s^* - 1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (4.26)$$

where $p_s^* = pN/(N - ps)$, with $N > ps$ and $p > 1$, while the Kirchhoff coefficient M satisfies the following condition:

(M) $M : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is continuous and nondecreasing. There exist numbers $a > 0$ and ϱ such that for an $t \in \mathbb{R}_0^+$,

$$m(t) := \int_0^t M(\tau) d\tau \geq at^\varrho,$$

with

$$\begin{cases} \varrho \in (1, p_s^*/p) & \text{if } M(0) = 0, \\ \varrho = 1 & \text{if } M(0) > 0. \end{cases}$$

The main operator $(-\Delta)_p^s$ is the fractionel p-Laplacian which may be defined , for any function $\varphi \in C_0^\infty(\Omega)$, as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dy, \quad x \in \mathbb{R}^N.$$

conclusion

In this thesis, we presented a study on some results about a fractional Kirchhoff problem involving a singular term and a critical nonlinearity.

The benefit of studying this topic is to identify new ways to prove the existence of a solution to a nonlinear problem, which is represented by minimizing functional.

this topic is fruitful, we hope that the students of the upcoming batches will immerse more into this type of important topic.

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ملخص

في هذا العمل هدفنا هو دراسة وجود حل مسألة شاذة من نوع كيرشوف مع شرط حدود ديرينخلي، ومسألة كيرشوف الكسرية التي تضم حد شاذ وهي غير خطية حرجة. من خلال الجمع بين الأساليب المتغيرة مع حجة الاقتران المناسبة. **الكلمات المفتاحية:** مسألة كيرشوف، حل ضعيف موجب، حد شاذ، أس حرج، طريقة تصغير.

Abstract

In this work our goal is to study the existence of a solution of a singular kirchhoff type problem with the Dirichlet boundary condition, and a fractional kirchhoff problem involving a singular term and critical non linearity. By combining variational methods with an appropriate truncation argumant. **Keywords:** Kirchhoff problem, positive weak solution, singular term, critical exponant, minimization method.