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Title

Study of some boundary value problems using variational methods

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Dedications

• To my dear father: Dilmi kamal

To the man of my life, my example of a respectful, honest father, eternal source of joy and happiness, the flame of my heart, no dedication can express the love and esteem I have always had for you, I love you dad and I implore the almighty that God welcomes you into his paradise amen

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To all my family for their support throughout my university career, May this work be the fulfillment of your much-alleged wishes, and the result of your unfailing support, Thank you for always being there for me

Contents

Notations

- $\bar{\Omega}$ denotes the closing of Ω.
- $C(\Omega)$ the space of continuous functions.
- $C^k(\Omega)$ the space of continuous k functions differentiable in Ω.
- $D(\Omega) = C_c^\infty(\Omega)$ the space of test functions.
- $\partial\Omega = \Gamma$ border of Ω.
- div denotes the divergence operator.
- V denotes the Hilbert space.
- a.e. denotes almost every where.
- $\langle ., . \rangle$ Scalar product.
- $\frac{\partial u}{\partial x}$ weak partial derivative of u .
- ∂ the partial derivative operator.
- Δ the laplace operator.
- ∇ the gradient operator .
- III denotes dirac comb .
- H^1, H_0^1, H^m denotes Sobolev spaces.
- $|.|$ denotes the norm in space $H^1(\Omega)$.
- For $a \in \mathbb{R}^n$, $\mid x \mid$ denotes the Euclidean norm in $\mathbb{R}^n$.
- n_i is the $i-th$ component of the outward unit normal of Ω .
- If $f : \mathbb{R}^n \to \mathbb{C}$, the support of f is denotedé by $suppf$.
- $\mathcal{D}(\mathbb{R}^n)$ is the space of functions $\mathcal{C}^\infty(\mathbb{R}^n)$ with compact support , $\mathcal{D}'(\mathbb{R}^n)$ is the dual space of $\mathcal{D}(\mathbb{R}^n)$, is called 'e also the space of distributions on $\mathbb{R}^n$.
- $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space, of functions $\mathcal{C}^\infty(\mathbb{R}^n)$ with rapid decay on $\mathbb{R}^n$, the dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

- p' is the conjugate exponenté of p, $\frac{1}{p}+\frac{1}{p'}=1$ where $p\in[1,+\infty]$.
- $L^p(\mathbb{R}^n)$ is the space of measurable functions f on $\mathbb{R}^n$ such that

$$\| f \|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} | f(x) |^p \, dx\right)^{\frac{1}{p}} < \infty.$$

Introduction

The partial differential equations, abbreviated as "PDEs" hereafter, constitute an important branch of applied mathematics. They express, in the form of equalities, relationships that the partial derivatives of a certain unknown function u of several variables must satisfy in order to describe a physical phenomenon and to fulfill a prescribed property.

In mathematical analysis, Sobolev spaces are functional spaces particularly well-suited for solving partial differential equations.

Sobolev spaces are a fundamental tool for studying partial differential equations. Indeed, the solutions to these equations belong to a space of continuously partially differentiable functions in the classical sense. Elliptic partial differential equations correspond to stationary physical models, meaning they are independent of time.

In this work, we study several boundary value problems using variational methods, i.e., if we have the following boundary value problems:

$$\begin{cases} -\Delta u = f \ in \ \ \Omega \\ u = 0 \ \ on \ \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\partial\Omega$ is its boundary, and f is a given function.

The classical formulation of (1) is to seek $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, a solution to this problem. In other words, a function $u : \Omega \to \mathbb{R}$ is said to be a classical solution to this problem if $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and u satisfies (1).

Our dissertation is organized into three chapters:

• In the first chapter , we provide a basic introduction to measured spaces and distributions. Then, we recall some essential notions about Sobolev spaces, many of the properties of these concepts will be used in the next chapters.

• In the second chapter, we present some integration by parts formulas, known as Green's formulas, and then we define what a variational formulation is. Finally, we delve into the Lax-Milgram theorem, which will be the essential tool for proving existence and uniqueness results of the variational formulation solution.

• In The last chapter, we will study some mathematical problems using variational methods, i.e., we seek the existence and uniqueness of the solution to these problems.

• Finally, general conclusions and respectives are drawn.

CHAPTER 1

SOBOLEV SPACES

The aim of this first chapter is to give a basic introduction to measured spaces and distributions. Then, we recall some essential notions about Sobolev spaces, many of the properties of these concepts will be used in the next chapters.

1.1 Some essential notions about measurement theory

In this section, we will recall some definitions on measured spaces and some theorems concerning integration.

Definition 1.1. [4] Let Ω be a set, $\mathcal{M}$ be a family of parts of Ω (i.e., $\mathcal{M} \subset \mathcal{P}(\Omega)$).
$\mathcal{M}$ is a σ-algebra in Ω , if it satisfies the following conditions

(a) $\phi \in \mathcal{M}$.

(b) $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$.

(c) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ whenever $A_n \in \mathcal{M} \ \ \forall n$.

Definition 1.2. [4] Let $(\Omega, \mathcal{M})$be a measurable space, μ is a measure, i.e., $\mu : \mathcal{M} \longrightarrow [0;\infty]$ satisfies

(a) $\mu(\phi) = 0$.

(b) $\begin{cases} \mu(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \mu(A_n) \; whenever (A_n) is \; a \; disjoint \\ countable \; family \; of \; members \; of \mathcal{M}. \end{cases}$

The members of $\mathcal{M}$ are called the measurable sets.

Definition 1.3. [4] $(\Omega, \mathcal{M}, \mu)$ is $\sigma - finite$, i.e., there exists a countable family (Ω_n) in $\mathcal{M}$ such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty \; \forall n$.

Theorem 1.1. *[4]**(monotone convergence theorem, Beppo Levi)** Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) $f_1 \leq f_2 \leq ... \leq f_n \leq f_{n+1} \leq ...$*a.e. on* Ω.

(b) $\sup_n \int_\Omega f_n < \infty$.

Then $f_n(x)$ converges a.e.on Ω to a finite limit, which we denote by $f(x)$, the function f belongs to L^1 and $\|f_n - f\|_1 \to 0$.

Theorem 1.2. *[4]**(dominated convergence theorem, Lebesgue)** Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) $f_n(x) \to f(x)$ *a.e.on* Ω.

(b) there is a function $g \in L^1$ such that for all n, $|f_n(x)| \leq g(x)$ a.e. on Ω.

Then $f \in L^1$ and $\|f_n - f\|_1 \to 0$.

Lemma 1.1. *[4]**(Fatou's Lemma)** Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) for all n,$f_n \geq 0$ a.e.

(b) $\sup_n \int_\Omega f_n < \infty$.

For almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to+\infty} f_n(x) \leq +\infty$. Then $f \in L^1$ and

$$\int_\Omega f \leq \liminf_{n\to\infty} \int_\Omega f_n.$$

Theorem 1.3. *[4]**(Tonelli)** Let $F(x,y) : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function satisfying*

(a) $\int_{\Omega_2} |F(x,y)|\, d\mu_2 < \infty$ *for a.e.* $x \in \Omega_1$.
and

(b) $\int_{\Omega_1} d\mu_1 \int_{\Omega_2} |F(x,y)|\, d\mu_2 < \infty$.

Then $F \in L^1(\Omega_1 \times \Omega_2)$.

Theorem 1.4. *[4]**(Fubini)** Assume that $F \in L^1(\Omega_1 \times \Omega_2)$.Then for a.e. $x \in \Omega_1$,
$F(x,y) \in L^1_y(\Omega_2)$ and $\int_{\Omega_2} F(x,y)d\mu_2 \in L^1_x(\Omega_1)$. Similarly, for a.e. $y \in \Omega_2$,
$F(x,y) \in L^1_x(\Omega_1)$ and $\int_{\Omega_1} F(x,y)d\mu_1 \in L^1_y(\Omega_2)$.
Moreover , one has*
$\int_{\Omega_1} d\mu_1 \int_{\Omega_2} F(x,y)d\mu_2 = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} F(x,y)d\mu_1 + \int\int_{\Omega_1\times\Omega_2} F(x,y)d\mu_1 d\mu_2$.

1.2 L^p Spaces

This section containing some definitions and elementary properties on L^p spaces.

Definition 1.4. [4] Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}, f \ is\ measurable\ and\ |f|^p \in L^1(\Omega) \right\}.$$

with

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

Definition 1.5. [4] We set

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R}, f \text{ is measurable and there is a constant } C \text{ such that } |f(x)| \leq C \ a.e.\ on\ \Omega \}.$$

With

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ C > 0 : \ |f(x)| \leq C \ a.e.\ on\ \Omega \}.$$

Theorem 1.5. *[4]**(Hölder's inequality)** Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then $fg \in L^1$ and*

$$\int_\Omega |fg| \leq \|f\|_p \, \|g\|_{p'}. \tag{1.1}$$

Proof. See [4] □

Theorem 1.6. *[4] L^p is a vector space and $\|\|_p$ is a norm for any p , $1 \leq p \leq \infty$.*

Proof. See [4] □

Theorem 1.7. *[4]**(Fisher-Riesz)** L^p is a Banach space for any p, $1 \leq p \leq \infty$.*

Proof. See [4] □

Theorem 1.8. *[4]**(Riesz representation theorem)** Let $1 < p < \infty$ and let $\phi \in (L^p)^\star$. Then there exists a unique function $u \in L^{p'}$ such that*

$$\langle \phi, f \rangle = \int_\Omega uf \ \ \forall f \in L^p.$$

Moreover,

$$\|u\|_{p'} = \|\phi\|_{(L^p)^\star}.$$

Remark 1.1. Theorem 1.8 is very important. It says that every continuous linear func-tional on L^p with $1 < p < \infty$ can be represented "concretely" as an integral. The mapping $\phi \longmapsto u$, which

is a linear surjective isometry, allows us to identify the “abstract” space $(L^p)^\star$ with $L^{p'}$.
In what follows, we shall systematically make the identification

$$(L^p)^\star = L^{p'}.$$

Proof. See [4] □

Theorem 1.9. *[4] The space $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for any p ,$1 \leq p < \infty$, where $C_c(\mathbb{R}^n)$ is the space of all continuous functions on $\mathbb{R}^n$ with compact support.*

Proof. See [4] □

1.3 Distributions

In this section, we will recall some definitions and elementary properties about distributions.

1.3.1 The space of test functions $\mathcal{D}$

Definition 1.6. [6] Let f be a complex-valued function defined on $\mathbb{R}$. The support of f, denoted $Supp(f)$, is the closure of the set $\{x \in \mathbb{R} \mid f(x) \neq 0\}$.

$$Supp(f) = \overline{\{x \in \mathbb{R}; f(x) \neq 0\}}.$$

Definition 1.7. [6] We define the set $\mathcal{D}$ as the space of complex-valued functions defined on $\mathbb{R}$, infinitely differentiable and with bounded support.

Remark 1.2. 1) It is an infinite-dimensional vector space.
2) Since the support is closed by definition, we can replace "bounded support" in the previous definition with "compact support".

Theorem 1.10. *[6] If $\varphi \in \mathcal{D}$ and if f is a summable function with bounded support, then*

$$\psi(x) = \int_\Omega f(t)\varphi(x-t)dt$$

is a function in $\mathcal{D}$.
Now, let's consider the sequence of functions

$$\psi_k(x) = \int_\Omega f(t)\gamma_k(x-t)dt.$$

It can be shown that if f is continuous, then this sequence converges uniformly to f. Hence the following theorem is accepted:

Theorem 1.11. ***(Approximation Theorem)***.*[6] Any continuous function with bounded support can be uniformly approximated by a sequence $(\varphi_n)_{n>0}$ of functions in $\mathcal{D}$.*

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \textit{such that}, \ \forall \ n \geq N, \ \forall \ x, |f(x) - \varphi_n(x)| \leq \epsilon.$$

1.3.2 Topology of $\mathcal{D}$

It will be defined by a convergence criterion for sequences.

Definition 1.8. [6] A sequence $(\varphi_n)n > 0$ of functions from $\mathcal{D}$ converges to a function φ as n tends to infinity if:

1. There exists a bounded set B (independent of n) in $\mathbb{R}$ such that for all $n > 0$, $Supp(\varphi n) \subset B$.
2. For any integer $K \geq 0$, the sequence of derivatives $(\varphi_n^{(k)})_n$ converges uniformly on $\mathbb{R}$ to $\varphi^{(k)}$. It can be shown that the limit φ then belongs to $\mathcal{D}$.

1.4 The Space $\mathcal{D}'$ of Distributions

1.4.1 Definition

Definition 1.9. [6] A distribution is defined as any continuous linear functional on the vector space $\mathcal{D}$. Let T be a distribution. By definition, T is a functional on $\mathcal{D}$, so T associates a complex number, denoted $\langle T, \varphi \rangle$ (or sometimes $T(\varphi)$), to every function $\varphi \in \mathcal{D}$. The definition of a distribution implies the following two points:

1. Linearity
   - $\langle T, \varphi_1 + \varphi_2 \rangle = \langle T, \varphi_1 \rangle + \langle T, \varphi_2 \rangle$,
   - $\langle T, \lambda\varphi_1 \rangle = \lambda \langle T, \varphi_1 \rangle$.
2. If $(\varphi_k)_{k>0}$ converges in $\mathcal{D}$ to φ, then the sequence $(\langle T, \varphi k \rangle)_{k>0}$ converges in the usual sense to $\langle T, \varphi \rangle$,i.e.,
   $$\forall \epsilon > 0, \exists N \in \mathbb{N} tel\ que, \forall k > N, |\langle T, \varphi \rangle - \langle T, \varphi_k \rangle| \leq \epsilon.$$

   The set of distributions is a vector space denoted by $\mathcal{D}'$. The sum of two distributions and the product of a distribution by a scalar are defined as follows:
   - $\langle S + T, \varphi \rangle = \langle S, \varphi \rangle + \langle T, \varphi \rangle$,
   - $\langle \lambda T, \varphi \rangle = \lambda \langle T, \varphi \rangle$.

1.4.2 Examples, regular and singular distributions

Definition 1.10. [6] A function $f : \mathbb{R} \to \mathbb{C}$ is said to be locally integrable if it is integrable over any bounded interval. For any locally integrable function f, we associate the distribution T_f defined by

$$\forall \varphi \in \mathcal{D}, \langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx.$$

Such a distribution is called regular. The others (those not of the form T_f for f locally integrable) are called singular.

Proposition 1.1. *[6] Two locally integrable functions define the same distribution if and only if they are equal almost everywhere.*

Example 1.1. An example of a regular distribution is the Cauchy principal value distribution of $\frac{1}{x}$ denoted by $vp\frac{1}{x}$ and defined by:

$$\forall \varphi \in \mathcal{D}, \left\langle vp\frac{1}{x}, \varphi \right\rangle = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x} dx.$$

1.4.3 Support of a distribution

Definition 1.11. [6] We say that two distributions S and T are equal if $\langle S, \varphi \rangle = \langle T, \varphi \rangle$ for all $\varphi \in \mathcal{D}$. We say they are equal on an open set $\Omega \subset \mathbb{R}$ if $\langle S, \varphi \rangle = \langle T, \varphi \rangle$ for all $\varphi \in \mathcal{D}$ with support in Ω.

Examples 1.1. The regular distributions T_1 and H are equal on $]0, +\infty[$.
The distributions δ and III are equal on $]\frac{-1}{2}, \frac{1}{2}[$.

Definition 1.12. [6] Consider the union of all open sets on which a distribution T is zero. This set is then the largest open set on which T is zero.
Its complement (which is closed) is called the support of the distribution T; it is denoted by $Supp(T)$.

Definition 1.13. [6] The space $\mathcal{D}'+$ (resp. $\mathcal{D}'-$) of distributions with support in $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) is called the space of distributions with left-bounded support (resp. right-bounded support).

Examples 1.2. $Supp(\delta_a) = \{a\}$ and $Supp(III) = \mathbb{Z}$.

1.5 Operations on distributions

1.5.1 Translation

If f is locally integrable and if $a \in \mathbb{R}$, then the translated function f_a of f is the function given by $f_a(x) = f(x-a)$. The regular distribution associated with f_a therefore satisfies:

$$\forall \varphi \in \mathcal{D}, \langle T_{f_a}, \varphi \rangle = \int_\Omega f(x-a)\varphi(x)dx = \int_\Omega f(y)\varphi(y+a)dy = \langle T_f, \varphi_{-a} \rangle .$$

Definition 1.14. [6] The translated distribution of a distribution T, denoted by T_a, is the distribution defined by:

$$\forall \varphi \in \mathcal{D}, \langle T_a, \varphi \rangle = \langle T, \varphi_{-a} \rangle .$$

Example 1.2. The translated Cauchy principal value of $1/x$ is the Cauchy principal value of $1/(x-a)$.

1.5.2 Transpose

Let f be a locally integrable function, and let's find the distribution associated with the function $\check{f}$ which maps x to $f(-x)$. We have:

$$\forall \varphi \in \mathcal{D}, \left\langle T_{\check{f}}, \varphi \right\rangle = \int_\Omega f(-x)\varphi(x)dx = \int_\Omega f(x)\varphi(-x)dx = \langle T_f, \check{\varphi} \rangle .$$

Definition 1.15. [6] The transpose of a distribution T, denoted by $\check{T}$, is the distribution defined by:

$$\forall \varphi \in \mathcal{D}, \left\langle \check{T}, \varphi \right\rangle = \langle T, \check{\varphi} \rangle .$$

Remark 1.3. This allows us to define even and odd distributions, just as for functions.

1.5.3 Dilation (homothety or change of scale)

If f is locally integrable and $a \in \mathbb{R}^*$, then the dilated function of f is defined by $x \to f(ax)$. Its associated regular distribution satisfies

$$\forall \varphi \in \mathcal{D}, \left\langle T_{f_{(ax)}}, \varphi \right\rangle = \int_\Omega f(ax)\varphi(x)dx = \int_\Omega f(y)\varphi(\frac{y}{a})\frac{dy}{|a|} = \frac{1}{|a|}\left\langle T_f, \varphi(\frac{x}{a}) \right\rangle .$$

Definition 1.16. [6] The dilated distribution of a distribution T is the distribution defined by:

$$\forall \varphi \in \mathcal{D}, \langle T(ax), \varphi \rangle = \frac{1}{|a|}\left\langle T, \varphi(\frac{x}{a}) \right\rangle .$$

Example 1.3. :$\delta(ax) = \frac{1}{|a|}\delta$.

1.5.4 Multiplication of distributions

There is no general way to multiply arbitrary distributions together. Furthermore, if f and g are two locally integrable functions, then their product is not necessarily locally integrable (for example, if $f(x) = g(x) = 1/\sqrt{x}$). However, if ψ is an indefinitely differentiable function, then the product by ψ of a test function in $\mathcal{D}$ is still in $\mathcal{D}$. Let f be a locally integrable function and ψ an indefinitely differentiable function. Then we have

$$\forall\varphi \in \mathcal{D}, \left\langle T_{\psi_f}, \varphi\right\rangle = \int_\Omega (\psi(x)f(x))\varphi(x)dx = \int_\Omega f(x)(\psi(x)\varphi(x))dx = \langle T_f, \psi\varphi\rangle\,.$$

Definition 1.17. [6] Let ψ be an indefinitely differentiable function. The product ψT of a distribution T by ψ is the distribution defined by:

$$\forall\varphi \in \mathcal{D}, \langle \psi T, \varphi\rangle = \langle T, \psi\varphi\rangle\,.$$

From this definition, we can define the product of any distribution T by a regular distribution T_ψ associated with an indefinitely differentiable function ψ as follows:

$$\forall\varphi \in \mathcal{D}, \langle \psi T, \varphi\rangle = \langle T, \psi\varphi\rangle\,.$$

Lemma 1.2. *[6] Let ψ be an indefinitely differentiable function. We have*

$$\psi\delta = \psi(0)\delta.$$

Proof. We have

$$\forall\varphi \in \mathcal{D}, < \psi\delta, \varphi >=< \delta, \psi\varphi >= \psi(0)\varphi(0) = \psi(0) < \delta, \varphi >=< \psi(0)\delta, \varphi > .$$

□

1.5.5 Differentiation of distributions

Let f be a locally integrable function that we additionally assume to be differentiable. In this case, f' is locally integrable, and its associated regular distribution satisfies

$$\forall\varphi \in \mathcal{D}, \langle T_{f'}, \varphi\rangle = \int_\Omega f'(x)\varphi(x)dx = -\int_\Omega f(x)\varphi'(x)dx = -\langle T_f, \varphi'\rangle\,.$$

Note that this is obtained by integration by parts using the fact that φ has bounded support. This is the main reason for the restrictive choice of test functions, i.e., from the space $\mathcal{D}$.

Definition 1.18. [12] The derivative T' of a distribution T from $\mathcal{D}'$ is the distribution defined by

$$\forall \varphi \in \mathcal{D}, \langle T', \varphi \rangle = - \langle T, \varphi' \rangle .$$

Similarly, we can define the successive derivatives $T^{(m)}$ by

$$\forall \varphi \in \mathcal{D}, \left\langle T^{(m)}, \varphi \right\rangle = (-1)^m \left\langle T, \varphi^{(m)} \right\rangle .$$

Example 1.4. :$H' = \delta$. Where $H(x)$ is the Heaviside function

$$\forall \varphi \in \mathcal{D}, \langle H', \varphi \rangle = - \langle H, \varphi' \rangle = - \int_0^{+\infty} \varphi'(x) dx = - [\varphi(x)]_0^{+\infty} = \varphi(0) = \langle \delta, \varphi \rangle .$$

Lemma 1.3. *[12] Let T be an arbitrary distribution and ψ an indefinitely differentiable function. We then have the following Leibniz rule:*

$$(T\psi)' = T'\psi + T\psi'.$$

1.5.6 Derivative of a discontinuous function

Theorem 1.12. *[12] Let f be a piecewise $\mathcal{C}^\infty$ function, then we have*

$$(T_f)' = T_{f'} + \sum_i \sigma_i^{(0)} \delta_{a_i}.$$

We will simply denote

$$(T_f)' = T_{f'} + \sigma^{(0)} \delta.$$

Similarly, let f be a piecewise $\mathcal{C}^\infty$ function. If we denote by $\sigma^{(j)}$ the jumps of discontinuity of $f^{(j)}$, we have

$$(T_f)^{(m)} = T_{f^{(m)}} + \sigma^{(m-1)} \delta + \sigma^{(m-2)} \delta' + ... + \sigma^{(0)} \delta^{(m-1)}.$$

1.6 The space $H^1(\Omega)$

Definition 1.19. [14] Let Ω be an open set in $\mathbb{R}^n$. We say that $u \in H^1(\Omega)$ if and only if $u \in L^2(\Omega)$ and for all $1 \leq i \leq n$, $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$, that is:

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \;\; such \;\; that \; \forall i \in \{1, ..., n\} \, , \frac{\partial u}{\partial x_i} \in L^2(\Omega) \right\} ,$$

where $\frac{\partial u}{\partial x_i}$ is the distributional derivative.

Example 1.5. Let u be defined on $]a, b[$ by:

$$u(x) = \begin{cases} 1 \ \ si \ \ x \in]a, b[\\ 0 \ \ si \ \ x \notin]a, b[\end{cases}$$

Then $u \notin H^1(]a, b[)$, indeed $u \in L^2(]a, b[)$ but $u' = (1)\delta_a + (-1)\delta_b = (\delta_a - \delta_b)$ which cannot be identified with a function from $L^2(]a, b[)$.

Theorem 1.13. *(of density)[14] If Ω is a bounded, regularly open set of class C^1, or if $\Omega = \mathbb{R}^n$, then $C_c^\infty(\bar{\Omega}) = \mathcal{D}(\bar{\Omega})$ is dense in $H^1(\Omega)$.*

Theorem 1.14. *[14] The space $H^1(\Omega)$ is a vector space equipped with the scalar product,*

$$\begin{aligned} \langle u, v \rangle_{H^1} &= \int_\Omega u(x)v(x)dx + \sum_{i=1}^n \int_\Omega \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx, \\ &= \int_\Omega (u(x)v(x) + \nabla u(x).\nabla v(x))dx. \end{aligned}$$

It is a Hilbert space. Its norm is denoted $\|.\|_{H^1}$:

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \sum_{i=1}^n \left\| \frac{\partial u(x)}{\partial x_i} \right\|_{L^2}^2.$$

Remark 1.4. For any open set Ω in $\mathbb{R}^n$, we have:

$$\mathcal{D}(\Omega) \subset H^1(\Omega) \subset L^2(\Omega).$$

Proposition 1.2. *[14] The space $H^1(\Omega)$ equipped with the norm $|.|_{H^1}$ is a Hilbert space..*

Proof. We just need to show that the space $H^1(\Omega)$ is complete. Consider a sequence $(u_n)_n$ that is Cauchy in $H^1(\Omega)$. By definition of the norm in $H^1(\Omega)$, this means that the sequences $(u_n)_n$ and $(\frac{\partial u_n}{\partial x_i})_n$ for i = 1, . . . , n are all Cauchy in $L^2(\Omega)$ which is complete, implying that all these sequences converge in $L^2(\Omega)$. We denote $u \in L^2(\Omega)$ the limit of the sequence $(u_n)_n$ and $v_i \in L^2(\Omega)$ the limit of $(\frac{\partial u_n}{\partial x_i})_n$ for all $i = 1, ..., n$. The continuity of the injection $L^2(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ ensures that if a sequence converges in $L^2(\Omega)$, it also converges in the sense of distributions. For any $\varphi \in \mathcal{D}(\Omega)$, we can therefore take the limit as $n \to \infty$ in the equalities:

$$\int_\Omega u_n \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi \frac{\partial u_n}{\partial x_i} dx, \ \ \forall i = 1, ..., n,$$

to obtain that

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}dx = -\int_\Omega \varphi v_i dx, \ \ \forall i = 1, ..., n,$$

and thus $v_i = \partial u/\partial x_i$ for every i= 1,...,n. □

1.7 $H_0^1(\Omega)$ Space

Definition 1.20. [14] We define $H_0^1(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ (with respect to the norm of $H^1(\Omega)$). In other words:

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega), \exists v_n \in \mathcal{D}(\Omega) \ such \ \ that \ v_n \to u \ \ inH^1(\Omega)\right\}.$$

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}.$$

Remark 1.5. 1. $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$.

2. If $\Omega \neq \mathbb{R}^n$ then $H_0^1(\Omega) \subset H^1(\Omega)$ with strict inclusion.

3. $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ for the norm $H^1(\Omega)$.

4. The space $H_0^1(\Omega)$ is a closed subset of $H^1(\Omega)$.

Theorem 1.15. *[14] The space $H_0^1(\Omega)$ is a Hilbert space with respect to the scalar product of $H^1(\Omega)$.*

Proposition 1.3. *[14] The space $H_0^1(\Omega)$ is the set of functions in $H^1(\Omega)$ that vanish on the boundary of Ω (when this boundary is defined).*

Proposition 1.4. ***(Poincare Inequality)****[14] Let Ω be a bounded open set in $\mathbb{R}^n$. There exists a constant $C > 0$ such that, for any function $u \in H_0^1(\Omega)$:*

$$\int_\Omega |u(x)|^2 \, dx \leq C \int_\Omega |\nabla u(x)|^2 \, dx.$$

Proof. For functions $u \in \mathcal{D}(\Omega)$, we have already proven Poincaré's inequality using a density argument. The result remains true for any function $u \in H_0^1(\Omega)$. Indeed, since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that:

$$\lim_{n\to\infty} \|u - u_n\|^2_{H^1(\Omega)} = \lim_{n\to\infty} \int_\Omega (|u - u_n|^2 + |\nabla(u - u_n)|^2)dx = 0.$$

In particular, we deduce that:

$$\lim \int_\Omega |u|^2 \, dx \ \ et \ \ \lim \int_\Omega |\nabla u_n|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx.$$

$$\int_\Omega |u_n(x)|^2\,dx \le C\int_\Omega |\nabla u_n(x)|^2\,dx.$$

we then take the limit as $n\to\infty$ in each of the two terms of the inequality to obtain the desired result. □

Corollary 1.1. *[14] Let Ω be a bounded open set in $\mathbb{R}^n$, then the semi-norm:*

$$\|v\|_{H_0^1(\Omega)} = \|\nabla v(x)\|_{L^2(\Omega)}\,,$$

and a norm on $H_0^1(\Omega)$ equivalent to the usual norm induced by that of $H^1(\Omega)$.

Proof. Let's show that

$$\|v\|_{H_0^1(\Omega)} = \left(\sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Represents a norm on $H_0^1(\Omega)$

$$\|v\|_{H_0^1(\Omega)} = 0 \;\Rightarrow\; \left(\sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|^2_{L^2(\Omega)}\right) = 0,$$

$$\Rightarrow\; \left\|\frac{\partial v}{\partial x_i}\right\|_{L^2(\Omega)} = 0,\quad \forall i = 1,...,n.$$

He comes for everything $v\in H_0^1(\Omega)$,we have

$$\|v\|_{L^2(\Omega)} \le C(\Omega)\left(\sum_{i=1}^n \left\|\frac{\partial v}{\partial x_n}\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Hence

$$\|v\|_{L^2(\Omega)} = 0 \;\Rightarrow\; v = 0$$

Let's show that there exist two constants C_1 and C_2 such that

$$C_1\,|v|_{H_0^1(\Omega)} \le \|v\|_{H^1(\Omega)} \le C_2\,|v|_{1,\Omega}\,.$$

Indeed,

$$|v|^2_{H_0^1(\Omega)} \;=\; \sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|^2_{L^2(\Omega)},$$

$$\leq \|v\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)} = \|v\|^2_{H^1(\Omega)} .$$

Hence

$$|v|^2_{H^1_0(\Omega)} \leq \|v\|_{H^1(\Omega)} .$$

Conversely

$$\begin{aligned}
\|v\|^2_{H^1(\Omega)} &= \|v\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)} ,\\
&\leq C_\Omega(\sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)}) + \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)} ,\\
&\leq (C_\Omega + 1)(\sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|^2_{L^2(\Omega)}),\\
&\leq C_\Omega \, |v|^2_{H^1_0} \ \ avec \ \ C_\Omega = C_\Omega + 1.
\end{aligned}$$

Hence the equivalence of norms

$$\|v\|^2_{H^1(\Omega)} \leq C_\Omega \, |v|^2_{H^1_0(\Omega)} ,$$

with

$$C_1 = 1 \ \ and \ \ C_1 = C_\Omega.$$

□

1.8 $H^m(\Omega)$ space

Definition 1.21. [13] Let Ω be a bounded open set of $\mathbb{R}^n$, $m \in \mathbb{N}^\star$. We call $H^m(\Omega)$ the space of functions u from $L^2(\Omega)$ whose derivatives, in the sense of distributions, up to order m, belong to $L^2(\Omega)$.
$H^m = \{u \in L^2(\Omega) \text{ such that } \forall \alpha \in \mathbb{N}^n, |\alpha| \leq m, \partial^\alpha u \in L^2(\Omega)\}$ in the weak sense.
On $H^m(\Omega)$, we introduce the scalar product:

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \langle \partial^\alpha u, \partial^\alpha v \rangle .$$

and the associated norm

$$\|u\|_{H^m(\Omega)} = \sqrt{\langle u, u\rangle}. \tag{1.2}$$

Theorem 1.16. *[13] Let Ω be a bounded open set of $\mathbb{R}^n$, and let $m \in \mathbb{N}^\star$. The space $H^m(\Omega)$ equipped with the scalar product $\langle u, v\rangle$ is a separable Hilbert space.*

In the case where $\Omega = \mathbb{R}^n$, it is possible to define Sobolev spaces using the Fourier transform. If $u \in \mathcal{S}'(\mathbb{R}^n)$, we denote $\widehat{u}$ its Fourier transform, and we have

Proposition 1.5. *[13] The space $H^m(\mathbb{R}^n)(m \in \mathbb{N}^\star)$ can be defined by*

$$H^m(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{m/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\right\}, \tag{1.3}$$

and the norm defined by 1.2 is equivalent to (keeping the same notation):

$$\|u\|_{H^m} = \left(\int_\Omega (1 + |\xi|^2)^m |\hat{u}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}. \tag{1.4}$$

Proposition 1.6. *[13] The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for every positive integer n and for every $s \in \mathbb{R}$.*

Proof. We recall the following injections:

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow_{continuous} \mathcal{S}(\mathbb{R}^n) \hookrightarrow_{continuous} H^s(\mathbb{R}^n)$$

The injection $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ is furthermore dense. So, it suffices to show that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. We use the Fourier transform. Let $u \in H^s(\mathbb{R}^n)$, then $(1 + |\xi|^2)^{\frac{s}{2}}\hat{u} \in L^2(\mathbb{R}^n)$. Since the space $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$,There exists a sequence of functions $(\varphi_n)_n \subset \mathcal{S}(\mathbb{R}^n)$ such that $\varphi_n \to (1+|\xi|^2)^{\frac{s}{2}}\hat{u}$ as $n \to \infty$ in $L^2(\mathbb{R}^n)$. By definition of $\mathcal{S}(\mathbb{R}^n)$, $\psi_n = (1+|\xi|^2)^{\frac{s}{2}}\varphi_n$ is still in $\mathcal{S}(\mathbb{R}^n)$. Since the Fourier transform is a bijective and continuous mapping from $\mathcal{S}(\mathbb{R}^n)$ onto itself, we can consider the sequence $(u_n)_n \subset \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{u_n} = \psi_n$. We then have:

$$\|u_n - u\|_{H^s(\mathbb{R}^n)} \leq C \left\|(1 + |\xi|^2)^{\frac{s}{2}}(\widehat{u_n} - \widehat{u})\right\|_{L^2(\mathbb{R}^n)} = \left\|\varphi_n - (1 + |\xi|^2)^{\frac{s}{2}}\widehat{u}\right\|_{L^2(\mathbb{R}^n)},$$

and this latter quantity tends to 0 as $n \to \infty$. □

CHAPTER 2

VARIATIONAL FORMULATION OF SOME MATHEMATICAL PROBLEMS

In this chapter, we present some integration by parts formulas, known as Green's formulas, and then we define what a variational formulation is. Finally, we delve into the Lax-Milgram theorem, which will be the essential tool for proving existence and uniqueness results of the variational formulation solution.

2.1 Generalities on Partial Differential Equations

Definition 2.1. *(PDE)*[17]. A partial differential equation, is a relation involving an unknown function u from $\mathbb{R}^n$ to $\mathbb{R}$, the variables $x_1, x_2, ...,$ its partial derivatives, $\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, ... \frac{\partial^m u}{\partial x_n^m}\right)$. It is generally written as :

$$F(x_1, x_2, ..., x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, ... \frac{\partial^m u}{\partial x_n^m}) = 0. \tag{2.1}$$

The equation (2.1) is considered in the domain Ω of $\mathbb{R}^n$.
The solutions of the partial differential equation (2.1) are the functions that satisfy this equation within Ω.

2.1.1 Second order partial differential equations

Definition 2.2. [17] We call linear P.D.E of order less than or equal to 2 in a domain $\Omega \subset \mathbb{R}^n$ and of unknown

$$u : \Omega \to \mathbb{R}$$

a type equation

$$\sum_{i,j=1} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i(x)\frac{\partial u(x)}{\partial x_i} + g(x)u(x) = h(x).$$

By convention, we will assume that $a_{ij}(x) = a_{ji}(x)$.
With $A(x) = (a_{ij})_{1\leq i,j\leq n}$ the symmetric $n \times n$ matrix of coefficients in front of the terms of order 2.

2.1.2 Classification of partial differential equations in $\mathbb{R}^n$

Partial differential equation of elliptic type

Definition 2.3. [17] A second-order linear PDE is said to be elliptic at $x \in \Omega$ if the matrix $A(x)$ has only non-zero eigenvalues and they are all of the same sign.

Example 2.1. (Laplace's equation). Let $u(x, y, ...)$ be a function defined on a domain Ω in $\mathbb{R}^n$, and satisfying in this domain the Laplace's equation:

$$\Delta u = 0.$$

Functions that satisfy this equation are said to be harmonic in Ω.

Partial differential equation hyperbolic

Definition 2.4. [17] A PDE is said to be hyperbolic at $x \in \Omega$ if $A(x)$ has only *non-zero* eigenvalues, and they are **all of the same sign except one of opposite sign**.

Example 2.2. (Wave Equation). Let $u(x, y, ..., t)$ be a function of space variables $(x, y, ...)$ and time t, defined on a domain Ω in $\mathbb{R}^n$ and for positive t. The wave equation for the function u is written as:

$$\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = 0.$$

Partial differential equation parabolic

Definition 2.5. [17] It is said that the PDE is parabolic at $x \in \Omega$ if $A(x)$ has $n - 1$ *non-zero* eigenvalues of the same sign and **one zero eigenvalue**.

Example 2.3. (Heat Equation). Let $u(x, y, ...)$ be a function of space variables $(x, y, ...)$ and time t, defined on a domain Ω in $\mathbb{R}^n$ and for positive t. The heat equation for the function u is written as:

$$\frac{\partial u}{\partial t} - \Delta u = f \quad with\ f\ given.$$

This equation governs, among other things, the evolution of the temperature u at a point (x, y, z) in a domain Ω aover time t in the presence of a volumetric heat source defined by f. It is this same equation that describes the evolution of the concentration of a product in a solvent (hence the name "diffusion equation").

2.2 Classification of partial differential equations in $\mathbb{R}^2$

Definition 2.6. a, b, c being three functions defined in an open of $\mathbb{R}^2$, and F a function defined in an open of $\mathbb{R}^5$, we call **second-order semi-linear partial differential equation**, with unknown u, an equation of the form :

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}). \tag{2.2}$$

Remark 2.1. Following the 2.2 before the matrix A is defined in this form:

$$A = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}$$

then the characteristic polynomial of this matrix :

$$\det(A - \lambda I) = \lambda^2 - (a+c)\lambda + ac - b^2,$$

so there are two eigenvalues λ_1 and λ_2 with:

$$\lambda_1 \times \lambda_2 = \frac{ac - b^2}{1} \;\; and \;\; \lambda_1 + \lambda_2 = \frac{a+c}{2}.$$

2.2.1 Hyperbolic partial differential equation

Definition 2.7. An equation such that, in a domain Ω:

$$b^2(x,y) - a(x,y)c(x,y) > 0. \tag{2.3}$$

is called **hyperbolic** in this domain.

Remark 2.2. In the 2.7, we say that the hyperbolic equation if the eigenvalues are non-zero and of the same sign except one then:

$$\lambda_1 \times \lambda_2 < 0 \Rightarrow ac - b^2 < 0,$$

$$and \; what \; gives :\Rightarrow b^2 - ac > 0.$$

Example 2.4. (Wave Equation). The study of the vibrations of an infinite string, free from any request, consists of studying the variations of the transverse displacement u during time. We give ourselves the position u and the speed of the transverse movement u at time zero. The corresponding model is written:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \;\; \forall x \in \mathbb{R}, \forall t \in \mathbb{R}^+. \tag{2.4}$$

$$u(x,0) \;\; = \;\; f(x) data.$$

$$\frac{\partial u}{\partial t} u(x,0) \;\; = \;\; g(x) \;\; data.$$

2.2.2 Parabolic partial differential equation

Definition 2.8. . An equation such that, in a domain Ω :

$$b^2(x,y) - a(x,y)c(x,y) = 0, \tag{2.5}$$

is called **parabolic** in this domain.

Remark 2.3. .In the definition 2.8, We say that the parabolic equation if a eigenvalues ??is zero.

$$\lambda_1 \times \lambda_2 = 0 \Rightarrow b^2 - ac = 0.$$

Example 2.5. (Heat Equation). Consider the following physical problem: we want to know the temperature in an infinite bar, without heat input and whose temperature is initially given. The corresponding model is written:

$$find \;\; u : (x,y) \to u(x,y) \;\;\; suchthat :$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \;\; = \;\; 0 \;\; \forall x \in \mathbb{R}, \forall t \in \mathbb{R}^+. \tag{2.6}$$

$$u(x,0) = f(x) \;\;\; data.$$

2.2.3 Elliptic partial differential equation

Definition 2.9. An equation such that, in a domain Ω:

$$b^2(x,y) - a(x,y)c(x,y) < 0, \tag{2.7}$$

is called **elliptical** in this domain.

Remark 2.4. .In the Definition 2.9 , We say that the elliptic equation if the values own are non-zero and of the same sign then:

$$\lambda_1 \times \lambda_2 > 0 \quad \Rightarrow \quad ac - b^2 > 0, \tag{2.8}$$

$$\Rightarrow \quad b^2 - ac < 0.$$

Example 2.6. (Laplace equation). Consider the following physical problem: we wants to know the temperature in a half plane knowing the temperature on the edge, knowing that this temperature tends towards 0 moving away from this edge and that there is no heat supply. The corresponding mathematical model is written:

$$find \quad u : (x,y) \to u(x,y) \quad suchthat :$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \;\; \forall x \in \mathbb{R}, \forall y \in \mathbb{R}^+, \tag{2.9}$$

$$u(x,0) = f(x) \quad data,$$

$$u(x,+\infty) = 0.$$

2.3 Variational Approach

The principle of the variational approach for solving partial differential equations is to replace the equation with an equivalent formulation, called a **variational** formulation, obtained by integrating the equation multiplied by an arbitrary function, called a test function. As it is necessary to carry out integrations by parts in establishing the **variational formulation**.

2.3.1 Green's Formulas

We assume that Ω is a bounded open set in $\mathbb{R}^n$ $(n \geq 1)$, with its boundary denoted by $\Gamma = \partial\Omega$. We then define the **outward normal** to the boundary Γ as the unit vector $n = (n_i)_{1\leq i\leq n}$ normal at every point to the tangent plane of Ω and pointing outward from Ω. In $\Omega \subset \mathbb{R}^n$, we denote dx as the **volume measure**, or **Lebesgue measure** of dimension n on Γ, and we denote ds as the **surface measure**, or **Lebesgue measure** of dimension $(n-1)$ on Γ.
The Green's formula is a fundamental tool for solving PDE. It coincides, in dimension 1, with the integration by parts formula.

Theorem 2.1. *(**Ostrogradsky's formula**) Let Ω be a bounded open set of class C^1 and Γ its boundary. Let w be a function from $C^1(\bar{\Omega})$ to $\mathbb{R}^n$ (a vector field). Then:*

$$\int_\Omega div(w(x))dx = \int_\Gamma w(x).n(x)d\Gamma. \tag{2.10}$$

Theorem 2.2. *Let Ω be a bounded open set of class C^1. Let w be a function from $C^1(\bar{\Omega})$ with bounded support in the closure $\bar{\Omega}$. Then:*

$$\int_\Omega \frac{\partial w}{\partial x_i}(x)dx = \int_\Gamma w(x)n_i(x)ds,$$

where
n_i is the $i-th$ component of the outward unit normal of Ω .

Remark 2.5. In this formula, $n(x)$ is the unit normal vector to Γ at point x , directed outward from Ω.

Corollary 2.1. *(**Integration by parts formula**) Let Ω be a bounded open set of class C^1. Let u and v be two functions from $C^1(\bar{\Omega})$ with bounded support in the closure $\bar{\Omega}$. Then they satisfy the integration by parts formula:*

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)dx + \int_\Gamma u(x)v(x)n_i(x)ds. \tag{2.11}$$

Proof. In theorem (2.2), we take $w = uv$:

$$\int_\Omega \frac{\partial(uv)}{\partial x_i}dx = \int_\Gamma u(x)v(x)n_i(x)ds. \tag{2.12}$$

We have:

$$\int_\Omega \frac{\partial(uv)}{\partial x_i}dx = \int_\Omega \frac{\partial u(x)}{\partial x_i}v(x)dx + \int_\Omega u(x)\frac{\partial v(x)}{\partial x_i}dx,$$

so

$$\int_\Omega u(x)\frac{\partial v(x)}{\partial x_i}dx = -\int_\Omega v(x)\frac{\partial u(x)}{\partial x_i}dx + \int_\Omega \frac{\partial (uv)}{\partial x_i}dx. \tag{2.13}$$

According to (2.12) and (2.13) we have:

$$\int_\Omega u(x)\frac{\partial v(x)}{\partial x_i}dx = -\int_\Omega v(x)\frac{\partial u(x)}{\partial x_i}dx + \int_\Gamma u(x)v(x)n_i(x)ds.$$

□

Corollary 2.2. *(**Green's formula**) Let Ω be a bounded open set of class C^1. Let u be a function from $C^2(\bar{\Omega})$ and v be a function from $C^1(\bar{\Omega})$, we have :*

$$\int_\Omega \Delta u(x)v(x)dx = -\int_\Omega \nabla u(x).\nabla v(x)dx + \int_\Gamma \frac{\partial u}{\partial n}(x)v(x)ds, \tag{2.14}$$

where
$\nabla u = (\frac{\partial u}{\partial x_i})_{1\leq i\leq n}$ *is the gradient vector* u *, and* $\frac{\partial u}{\partial n}(x) = \nabla u(x).n(x)$.

Proof. We just need to apply theorem (2.1) with: $w(x) = v(x)\nabla u(x)$ and notice that:

$$div(v(x)\nabla u(x)) = v(x)\Delta u(x) + \nabla u(x).\nabla v(x),$$

so:

$$\int_\Omega div(v(x)\Delta u(x))dx = \int_\Omega v(x)\Delta u(x)dx + \int_\Omega \nabla u(x).\nabla v(x)dx.$$

and

$$\int_\Omega div(v(x)\nabla u(x)) = \int_\Gamma v(x)\nabla u(x).n(x)ds.$$

which implies:

$$\begin{aligned}\int_\Omega \Delta u(x)v(x)dx &= -\int_\Omega \nabla u(x).\nabla v(x)dx + \int_\Gamma \nabla u(x).n(x)v(x)ds,\\ &= -\int_\Omega v(x)\nabla u(x).\nabla v(x)dx + \int_\Gamma \frac{\partial u}{\partial n}(x)v(x)ds.\end{aligned}$$

□

2.3.2 Variational formulation

Consider the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta u = f \ in \ \Omega \\ u = 0 \ \ on \ \partial\Omega \end{cases} \tag{2.15}$$

Ω is a bounded open set in the space $\mathbb{R}^n$ and the right-hand side f is continuous on $\bar{\Omega}$, and u is the unknown.

Lemma 2.1. *Let Ω be an open set in $\mathbb{R}^n$. Let g be a continuous function on Ω. If for every v in $C^\infty(\Omega)$ with compact support in Ω, we have:*

$$\int_\Omega g(x)v(x)dx = 0.$$

Then the function g is zero in Ω.

Proof. Let's suppose there exists a point $x_0 \in \Omega$ such that $g(x_0) \neq 0$. Without loss of generality, we can assume that $g(x_0) > 0$((otherwise we take -g).By continuity, there exists a small open neighborhood $w \subset \Omega$ of x_0 such that $g(x) > 0$ for all $x \in w$. Let v be a positive, non-zero test function with support included in w, then we have:

$$\int_\Omega g(x)v(x)dx = \int_w g(x)v(x) = 0.$$

which contradicts the assumption about g . Therefore $g(x) = 0$ for all $x \in \Omega$. □

Proposition 2.1. *Let u be a function from $C^2(\bar{\Omega})$. Let X be the space defined by:*

$$X = \left\{\phi \in C^1(\bar{\Omega}) \ \ suchthat \ \ \phi = 0 \ on \ \Gamma\right\}. \tag{2.16}$$

Then u is a solution to the boundary value problem (2.15) if and only if u belongs to X and satisfies the equality:

$$\int_\Omega \nabla u(x).\nabla v(x)dx = \int_\Omega f(x).v(x)dx. \tag{2.17}$$

For any function $v \in X$
The equality (2.17) is called the variational formulation of the boundary value problem (2.15).

Remark 2.6. An immediate advantage of the variational formulation is that it **makes sense** even if the solution u is only a function of $C^1(\bar{\Omega})$, unlike the classical formulation which requires u to belong to $C^2(\bar{\Omega})$.

Proof. (proposition (2.1))

1. $\Rightarrow$) If u is the solution to the boundary problem (2.15), we multiply the equation by $v \in X$ and we use the Green formula from the Corollary (2.2)

$$-\int_\Omega \Delta u(x)v(x)dx = -(-\int_\Omega \nabla u(x).\nabla v(x)dx + \int_\Gamma \frac{\partial u}{\partial n}(x)v(x)ds),$$

$$= \int_\Omega \nabla u(x).\nabla v(x)dx - \int_\Gamma \frac{\partial u}{\partial n}(x)v(x)ds.$$

But $v = 0$ on Γ since $v \in X$, therefore:

$$\int_\Omega f(x)v(x)dx = \int_\Omega \nabla u(x).\nabla v(x)dx.$$

which is nothing other than the formula (2.17).

2. $\Leftarrow$) if $u \in X$ vérifie (2.17), using the previous Green formula 'in reverse' we obtain:

$$\int_\Omega \nabla u(x)v(x)dx = -\int_\Omega \Delta u(x)v(x)dx + \int_\Gamma \frac{\partial u}{\partial n}(x)v(x)dx,$$

$$= \int_\Omega f(x)v(x)dx.$$

Which implies:

$$\int_\Omega (\Delta u(x) + f(x))v(x)dx = 0, \ \ \forall v \in X.$$

As $(\Delta u + f)$ is a continuous function, thanks to the lemma (2.1) we conclude that $-\Delta u(x) = f(x)$ for all $x \in \Omega$.
moreover, like $u \in X$, We find the boundary condition $u = 0$ on $\Gamma = \partial\Omega$, that is, u is solution of the boundary problem (2.17)

□

Theorem 2.3. ***(Existence of the trace operator)*** *[11] Let Ω be an open (bounded or unbounded) of $\mathbb{R}^n$, $n \geq 1$, with boundary $\partial\Omega$ Lipschtzian, then the space $C_c^\infty(\bar{\Omega})$ of class functions C^∞ and with compact support in $\bar{\Omega}$ is dense in $H^1(\Omega)$. We can therefore define by continuity the map "trace", which is linear continuous from $H^1(\Omega)$ into $L^2(\partial\Omega)$, defined by:*

$$\gamma(u) = u_{|\partial\Omega} \quad if\ u \in C_c^\infty(\bar{\Omega}).$$

and

$$\gamma(u) = \lim_{n\to+\infty} \gamma(u_n) \;\; if \; u \in H^1(\Omega), u = \lim_{n\to+\infty} u_n, or (u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\bar{\Omega}).$$

Saying that the (linear) map γ is continuous is equivalent to saying that there exists $C \in \mathbb{R}_+$ such that

$$\|\gamma(u)\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)} \;\; forall u \in H^1(\Omega).$$

Definition 2.10. Let ℓ be a linear form defined on V (V Hilbert space), we say that ℓ is continuous if there exists a constant $C > 0$ such that:

$$\forall v \in V \; : \; |\ell(v)| \leq C \|v\|_V . \tag{2.18}$$

Definition 2.11. Let a be a bilinear form defined on $V \times V$, We say that a is continuous and coercive if it satisfies the following two conditions:

1. a is continuous on $V \times V$. That is, there exists a constant $M > 0$ such that:

$$\forall (u, v) \in V \times V \; : \; |a(u, v)| \leq M \|u\|_V \|v\|_V . \tag{2.19}$$

2. a is coercive, meaning there exists a constant $\alpha > 0$ such that ,

$$\forall v \in V : a(v, v) \geq \alpha \|v\|_V^2 . \tag{2.20}$$

Theorem 2.4. *[4](**Stampacchia theorem**). Assume that $a(u, v)$ is a continuous coercive bilinear form on V. Let $K \subset V$ be a nonempty closed and convex subset. Then, given any $\ell \in V^*$, there exists a unique element $u \in K$ such that*

$$a(u, v - u) \geq \langle \ell, v - u\rangle \;\; \forall v \in K.$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in K \;\; and \;\; \frac{1}{2}a(u, u) - \langle \ell, u\rangle = \min_{v\in K} \left\{ \frac{1}{2}a(v, v) - \langle \ell, v\rangle \right\}.$$

Theorem 2.5. *[4](**Lax-Milgram theorem**) Let V be a real Hilbert space, a be a bilinear, continuous and coercive form on V and ℓ be a continuous linear form on V. Then, there exists a unique element u of V solution of the problem*

$$\ell(v) = a(u, v) \quad\;\; \forall v \in V. \tag{2.21}$$

Moreover if a is symmetric, u is the unique solution of the following minimization problem,

$$J(u) \leq J(v) \quad\;\; \forall v \in V. \tag{2.22}$$

Or J is set to V by;

$$J(v) = \frac{1}{2}a(u,v) - \ell(v) \quad \forall v \in V. \tag{2.23}$$

Proof. We first notice that due to the coercivity of a, if such a solution exists, it is unique. Let u_1 and u_2 be two solutions of the problem (2.21), then by subtraction we have,

$$\forall v \in V \quad a(u_1 - u_2, v) = 0, \tag{2.24}$$

in particular for $v = u_1 - u_2$ we have,

$$a(u_1 - u_2, u_1 - u_2) = 0.$$

Now, as $\alpha > 0$, we deduce from the reduction (2.20) that $u_1 = u_2$.
Let us now show the existence of such a solution. For all $u \in V$, the linear form $a(u,.)$ being continuous on V, there exists, according to the Riesz representation theorem, a unique element $Au \in V$ such that ,

$$\forall v \in V \quad a(u,v) = (Au, v)_V. \tag{2.25}$$

Similarly, there exists a unique $f \in V$ such that,

$$\forall u \in V \;\; \ell(v) = (f, v)_V,$$

the equation (2.21) is then equivalent to finding aV such that $Au = f$. To do this, it is enough to show that the operator A is linear and surjective.
Let $u_1, u_2 \in V$ and $\lambda \in \mathbb{R}$.We have:

$$\begin{aligned}
(A(u_1 + \lambda u_2), v) &= a(u_1 + \lambda u_2, v). \\
&= a(u_1, v) + a(\lambda u_2, v) \\
&= a(u_1, v) + \lambda a(u_2, v) \\
&= (Au_1, v) + \lambda(Au_2, v).
\end{aligned}$$

We also notice that A is continuous, because we have, according to (2.19), for $v = Au$ in (2.25) we have,

$$\|Au\|_V^2 = a(u, Au) \leq M \|u\|_V \|Au\|_V ,$$

Which gives us,

$$\|Au\|_V \leq M \|u\|_V .$$

Let us show that A is surjective: We first show that the image of A, denoted $Im(A)$, is closed. Let (v_p) be a cauchy sequence in $Im(A)$:
$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such as if $q \geq p \geq n_0$, SO $\|v_q - v_p\| \leq \epsilon$.
Since v_p and v_q are elements of $Im(A)$, there exist elements u_p and u_q in V such that $Au_p = v_p$ and $Au_q = v_q$. We have:

$$\begin{aligned} \alpha \|u_q - u_p\|_V^2 &\leq \|a(u_q - u_p, u_q - u_p)\| \\ &= \|(A(u_q - u_p), u_q - u_p)\| \\ &\leq \|A(u_q - u_p)\|_V \|(u_q - u_p)\|_V . \end{aligned}$$

It follows that,

$$\|(u_q - u_p)\|_V \leq \frac{1}{\alpha} \|Au_q - Au_p\|_V .$$

The space V being complete, the sequence $(u_p)_{p\in\mathbb{N}}$ converges to an element $u \in V$, and the operator A being continuous , we have ,Au_p converges to Au, from where $v = Au$.
We have thus shown that the image of A is closed. This result being established, consequently, We have

$$V = Im(A) \oplus [Im(A)]^{\perp} .$$

Suppose that $Im(A) \neq V$; then $[Im(A)]^{\perp} \neq \{0\}_V$ and there exists a non-zero element $u \in V$ such that , $\forall v \in V, (Av, u)_V = 0$. This relation being in particular true for $v = u$, we obtain,

$$0 = a(u, u) \geq \alpha \|u\|_V^2 .$$

Which leads to $u = 0$, we thus end up with a contradiction, which means that $Im(A) = V$.
To conclude the proof of the theorem, it remains to show that if a is symmetric, the minimization problem (2.22) is equivalent to the problem (2.21).
Let $u \in V$ be the unique solution of (2.21) and let us show that u is the solution of (2.22). Let $w \in V$, we will show that $J(u + w) \geq J(u)$

$$\begin{aligned} J(u + w) &= \frac{1}{2}a(u + w, u + w) - \ell(u + w) \\ &= \frac{1}{2}a(u, u) + \frac{1}{2}[a(u, w) + a(w, u)] + \frac{1}{2}a(w, w) - \ell(u) - \ell(w) \end{aligned}$$

$$= \frac{1}{2}a(u,u) - \ell(u) + [a(u,w) - \ell(w)] + \frac{1}{2}a(w,w)$$

$$= J(u) + \frac{1}{2}a(w,w) \geq J(u) + \frac{\alpha}{2}\|w\|_V^2 .$$

So $J(u+w) > J(u)$, for all $w \in V, w \neq 0_V$.
Conversely, let us now suppose that u is the solution of the minimization problem (2.22) and show that u is the solution (2.21).
Let $w \in V$ and $t > 0$, We have:

$$J(u+tw) - J(u) \geq 0$$

and

$$J(u-tw) - J(u) \geq 0.$$

Because u minimizes J . We deduce that,

$$t\,[a(u,w) - \ell(w)] + \frac{1}{2}t^2 a(w,w) \geq 0$$

and

$$-t\,[a(u,w) - \ell(w)] + \frac{1}{2}t^2 a(w,w) \geq 0 \;\; \forall w \in V.$$

As t is strictly positive, we can divide these two inequalities by t to obtain

$$[a(u,w) - \ell(w)] + \frac{1}{2}ta(w,w) \geq 0$$

and

$$-\,[a(u,w) - \ell(w)] + \frac{1}{2}ta(w,w) \geq 0, \forall w \in V.$$

we then make t tend towards 0, we obtain

$$a(u,w) - \ell(w) \geq 0$$

and

$$a(u,w) - \ell(w) \leq 0, \;\; \forall w \in V.$$

Which ultimately gives the equality $a(u,w) = \ell(w), \; \forall w \in V$. this means that u is the solution of the problem (2.21). □

CHAPTER 3

STUDY OF SOME MATHEMATICAL PROBLEMS

In this chapter, we will study some mathematical problems using variational methods, i.e., we seek the existence and uniqueness of the solution to these problems.

3.1 Laplace's equation with Dirichlet boundary conditions

3.1.1 Dirichlet problem (homogeneous)

A Dirichlet problem refers to a Laplace's equation with Dirichlet boundary conditions. For any bounded open set Ω in $\mathbb{R}^n$ and $\Gamma = \partial\Omega$, the Dirichlet problem is formulated as follows :
Determine a function u in a certain functional space such that

$$\begin{cases} -\Delta u = f \quad in \ \Omega, \\ u = 0 \quad on \ \Gamma. \end{cases} \tag{3.1}$$

Where f is a second member belonging to the space $L^2(\Omega)$.

The variational approach to study (3.1) consists of two steps which we detail :
Step 1: Establishing a variational formulation
The purpose of this first step is to propose the variational formulation of the boundary value problem (3.1), i.e., finding a bilinear form $a(.,.)$, a linear form $\ell(.)$, and a Hilbert space V such that (3.1) is equivalent to finding :

$$\forall u \in V \ \ such\ that \ \ a(u,v) = \ell(v), \ \forall v \in V.$$

To find the variational formulation :

- We multiply equation (3.1) by a test function v, such that $v \in V$ a Hilbert space and choice $v = 0$ on Γ :

$$-\Delta u.v = fv.$$

- And we integrate by parts :

$$\int_\Omega -\Delta u.v dx = \int_\Omega fv dx.$$

- According to the corollary, we have

$$-\int_\Gamma v\frac{\partial u}{\partial n}dx + \int_\Omega \nabla u.\nabla v dx \quad = \quad \int_\Omega fv dx.$$

$$-\int_\Gamma v\nabla u.n dx + \int_\Omega \nabla u.\nabla v dx \quad = \quad \int_\Omega fv dx.$$

$$-\int_\Gamma v\frac{\partial u}{\partial x_i}.n dx + \int_\Omega \nabla u.\nabla v dx = \int_\Omega fv dx. \tag{3.2}$$

- Then we use the Green's formula :

$$\int_\Omega \frac{\partial u}{\partial x_i}dx = \int_\Gamma u(x)n_i(x)ds.$$

So :

$$\int_\Omega v\frac{\partial u}{\partial x_i}n_i dx = \int_\Gamma u(x)n_i(x)ds - \int_{\partial\Omega} u\frac{\partial v}{\partial x_i}dx. \tag{3.3}$$

- Replacing (3.3) into (3.2), we obtain :

$$-\int_\Gamma vun_i ds + \int_{\partial\Omega} u\frac{\partial v}{\partial x_i}dx + \int_\Omega \nabla u.\nabla v dx = \int_\Omega fv dx.$$

Since u must satisfy a boundary condition, $u = 0$ on Γ, then $v = 0$ on Γ, so :

$$\int_\Omega fv dx = \int_\Omega \nabla u(x).\nabla v(x) dx,$$

with $\nabla u, \nabla v$ and $v \in L^2(\Omega)$, and we assume that $f \in L^2(\Omega)$.

- Therefore, a reasonable choice for the Hilbert space is $V = H_0^1(\Omega)$, the subspace of $H^1(\Omega)$ whose elements vanish on the boundary Γ.

- So, the proposed variational formulation for (3.1) is:
to find

$$u \in H_0^1(\Omega).$$

such that

$$\int_\Omega \nabla u.\nabla v dx = \int_\Omega f v dx, \forall v \in H_0^1(\Omega). \tag{3.4}$$

With:

$$a(u,v) = \int_\Omega \nabla u.\nabla v dx.$$

$$\ell(v) = \int_\Omega f v dx.$$

Step 2: Solving the variational formulation

The purpose of the second step is solely to verify that the variational formulation (3.4) admits a unique solution. In other words, we will demonstrate that a is a continuous and coercive bilinear form, and ℓ is a continuous linear form on $H_0^1(\Omega)$.

For this purpose, we utilize the Lax-Milgram theorem (2.5).

- **a is a continuous bilinear form**

$$a(u,v) = \int_\Omega \nabla u.\nabla v dx.$$

$$a \text{ is continuous} \Leftrightarrow \exists c \in \mathbb{R}^* \text{ such that } |a(u,v)| \leq c \|u\| \|v\|.$$

$$|a(u,v)| = \left| \int_\Omega \nabla u.\nabla v dx \right|.$$

According to the Cauchy-Schwarz inequality :

$$\begin{aligned} |a(u,v)| &\leq (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}} (\int_\Omega |\nabla v|^2 dx)^{\frac{1}{2}}, \\ &\leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \text{ avec } c = 1, \\ &\leq \|u\|_{H_0^1} \|v\|_{H_0^1} \text{ avec } c = 1. \end{aligned}$$

So a is a continuous bilinear form.

- **a is coercive**

Let's see if a is coercive, we need to show that there exists $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|^2_{H^1(\Omega)} \quad for\ all \ \ u \in H^1_0(\Omega).$$

By assumption on a, we have :

$$a(u,u) = \int_\Omega |\nabla u|^2 \, dx = \|\nabla u\|^2_{L^2(\Omega)}, \; u \in H^1_0(\Omega).$$

We then apply the Poincaré inequality (1.4)

$$\int_\Omega |u(x)|^2 \, dx \leq C \int_\Omega |\nabla u(x)|^2 \, dx,$$

so:

$$\int_\Omega |\nabla u(x)|^2 \, dx \geq \frac{1}{C} \int_\Omega |u(x)|^2 \, dx.$$

Which implies that :

$$a(u,u) = \int_\Omega |\nabla u(x)|^2 \, dx \geq \frac{1}{C+1} (\int_\Omega |u(x)|^2 \, dx + \int_\Omega |\nabla u(x)|^2 \, dx),$$

and we have:

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega |u(x)|^2 \, dx + \int_\Omega |\nabla u(x)|^2 \, dx.$$

This implies:

$$a(u,u) \geq \frac{1}{C+1} \|u\|^2_{H^1(\Omega)}.$$

This demonstrates the coerciveness of a.

- **ℓ is a continuous linear form**

$$\ell \; continuous \Leftrightarrow |\ell(v)| \leq C \|v\| \quad \forall C \in \mathbb{R}^*.$$

We have:

$$|\ell(v)| = \left| \int_\Omega f(x) v(x) dx \right|,$$

according to the Cauchy-Schwarz inequality :

$$|\ell(v)| \quad \leq \quad (\int_\Omega |f(x)|^2 \, dx)^{\frac{1}{2}} (\int_\Omega |v(x)|^2 \, dx)^{\frac{1}{2}},$$

$$\leq \|f\|_{L^2} \|v\|_{L^2},$$

We then apply the Poincaré inequality (1.4)

$$\leq c \|f\|_{L^2} \|\nabla v\|_{L^2},$$

$$\leq c \|f\|_{L^2} \|v\|_{H_0^1},$$

$$\leq C \|v\|_{H_0^1}.$$

With:

$$c(\int_\Omega |f(x)|^2 \, dx)^{\frac{1}{2}} = c \|f\|_{L^2} = C.$$

So:

$$|\ell(v)| \leq C \|v\|.$$

ℓ is a continuous linear form.
Finally, all the assumptions of the Lax-Milgram theorem (2.5) are satisfied, and thus we can conclude that there exists a unique solution $u \in H_0^1(\Omega)$ to the variational formulation (3.1).

3.1.2 Dirichlet problem (non-homogeneous)

$$\begin{cases} -\Delta u = f & in \;\; \Omega, \\ u = g & on \;\; \partial\Omega. \end{cases} \tag{3.5}$$

The variational approach to study (3.5) consists of two steps which we detail :
Step 1: Establishing a variational formulation
where $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$ sont given functions. We show that this problem admits a unique (weak) solution $u \in H^1(\Omega)$. We reason as in the homogeneous case, we multiply the equation $-\Delta u = f$ by a function $v \in H_0^1(\Omega)$ (to cancel the term on the edge),
we obtain :

$$\int_\Omega \nabla u . \nabla v dx = \int_\Omega f . v dx.$$

Via la formule de Green.la formulation variationnelle est

$$find \;\; u \in V, \forall v \in H_0^1(\Omega), \int_\Omega \nabla u . \nabla v dx = \int_\Omega f . v dx.$$

or

$$V = \left\{ u \in H^1(\Omega) : \gamma_0(u) = g \in L^2(\partial\Omega) \right\}.$$

and γ is the trace application. As the solution space is not the same as that of the functions v, then to be able to use the lax milgram theorem, we will 'symmetrize' the problem. As Ω is a bounded open of class C^1 and $g \in H^{\frac{1}{2}}(\partial\Omega)$, then the application traces γ_0 on $\partial\Omega$ of $H^1(\Omega)$ in $L^2(\partial\Omega)$ is such that: $\gamma_0(\zeta) = g, \zeta \in H^1(\Omega)$ .By setting $v = u - \zeta$ (here ζ plays the role of a raising that we noted previously R), the problem above boils down to this one

$$\begin{cases} -\Delta v = f + \Delta\zeta \quad in \ \ \Omega, \\ v = 0 \quad on \ \ \Gamma. \end{cases} \tag{3.6}$$

To establish the variational formulation of this problem and apply the results obtained previously, we will reason a little differently (it should be noted that for $\zeta \in H^1(\Omega)$,$\Delta\zeta \notin L^2(\Omega)$ but $\Delta\zeta \in D'(\Omega)$ where $D'(\Omega)$ is the space of distributions, for any function $\varphi \in D(\Omega)$, we have

$$\langle -\Delta v, \varphi \rangle = \langle f, \varphi \rangle + \langle \Delta\zeta, \varphi \rangle.$$

that is to say (taking into account the derivation in the sense of distributions),

$$\varphi \in D(\Omega), \sum_{i=1}^{n} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial \varphi}{\partial x_i} \right\rangle = \langle f, \varphi \rangle - \sum_{i=1}^{n} \left\langle \frac{\partial \zeta}{\partial x_i}, \frac{\partial \varphi}{\partial x_i} \right\rangle.$$

or again (the terms in the brackets belong to $L^2(\Omega)$),

$$\int_\Omega \nabla v.\nabla\varphi dx = \int_\Omega f.\varphi dx - \int_\Omega \nabla\zeta.\nabla\varphi dx.$$

The variational formulation of the problem above is therefore

$$find \ \ v \in H_0^1(\Omega), \forall w \in H_0^1(\Omega), \int_\Omega \nabla v.\nabla w dx = \int_\Omega f.w dx - \int_\Omega \nabla\zeta.\nabla w dx.$$

then, we apply the lax-Milgram theorem with

$$a(v, w) = \int_\Omega \nabla v.\nabla w dx.$$

$$l(w) = \int_\Omega f.w dx - \int_\Omega \nabla\zeta.\nabla w dx.$$

Step 2: Solving the variational formulation

The purpose of the second step is solely to verify that the variational formulation (3.5) admits

a unique solution. In other words, we will demonstrate that a is a continuous and coercive bilinear form, and ℓ is a continuous linear form on $H^1(\Omega)$.
For this purpose, we utilize the Lax-Milgram theorem (3.5).

- **a is a continuous and coercive bilinear form**
  We showed previously that the form a is bilinear, continuous and coercive on the space $H_0^1(\Omega) \times H_0^1(\Omega)$.

- **l is a continuous linear form**
  Note that the l is linear and it is continuous because

$$|l(w)| = \left| \int_\Omega f.wdx - \int_\Omega \nabla\zeta.\nabla wdx \right|,$$

$$\leq \|f\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} + \|\nabla\zeta\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)},$$

$$\leq max(\|f\|_{L^2(\Omega)} + \|\nabla\zeta\|_{L^2(\Omega)})(\|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}),$$

$$\leq C\|w\|_{H^1(\Omega)}.$$

  with $C = max\left\{\|f\|_{L^2(\Omega)}, \|\nabla\zeta\|_{L^2(\Omega)}\right\}$.

All the hypotheses of the Lax-milgram theorem being satisfied, we deduce that this problem, i.e.,

$$find\ \ v \in H_0^1(\Omega), \forall w \in H_0^1(\Omega), a(v,w) = l(w).$$

admits a unique solution. As $u = v + \zeta$, then u is a (weak) solution of the proposed problem. Furthermore, this solution is unique (in fact, if u_1,$u_2 \in H^1(\Omega)$ avec $\gamma_0(u_1) = \gamma_0(u_2) = g \in L^2(\partial\Omega)$, and

$$\forall w \in H_0^1(\Omega), \int_\Omega \nabla u_1.\nabla wdx = \int_\Omega \nabla u_2.\nabla wdx = \int_\Omega fwdx,$$

alors , $u_1 - u_2 \in H_0^1(\Omega)$ and we have

$$\forall w \in H_0^1(\Omega), \int_\Omega \nabla(u_1 - u_2).\nabla wdx = \int_\Omega fwdx.$$

The Lax-Milgram theorem states that this problem has a unique solution and since 0 is solution, then $u_1 - u_2 = 0$. We also show that using the Lax-Milgram theorem that the solution v satisfies

$$\|v\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\nabla\zeta\|_{L^2(\Omega)}),$$

$$\leq \quad C(\|f\|_{L^2(\Omega)} + \|\zeta\|_{H^1(\Omega)}),$$

$$\leq \quad C(\|f\|_{L^2(\Omega)} + \|\zeta\|_{H^{\frac{1}{2}}(\Omega)}).$$

Where C only depends on Ω and we have a similar estimate for the solution $u = v + \zeta \in H^1(\Omega)$.

3.2 Neumann problem(homogeneous)

$$\begin{cases} -\Delta u + u = f \quad in \;\; \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad on \;\; \Gamma. \end{cases} \tag{3.7}$$

Where $f \in L^2(\Omega)$ and n is the $i-th$ component of the outward unit normal .

The variational approach to study (3.7) consists of two steps which we detail :
Step 1: Establishing a variational formulation
The purpose of this first step is to propose the variational formulation of the boundary value problem (3.7), i.e., finding a bilinear form $a(.,.)$, a linear form $\ell(.)$, and a Hilbert space V such that (3.7) is equivalent to finding :

$$\forall u \in V \;\; such\ that \;\; a(u,v) = \ell(v), \; \forall v \in V.$$

To find the variational formulation :

- We multiply equation (3.7) by a test function v, such that $v \in H^1(\Omega)$:

$$-\Delta u.v + uv = fv.$$

- And we integrate by parts :

$$\int_\Omega -\Delta u.vdx + \int_\Omega uvdx = \int_\Omega fvdx.$$

- Then we use the Green's formula :

$$\int_\Omega \nabla u.\nabla vdx - \int_\Gamma \frac{\partial u}{\partial n}.vds + \int_\Omega uvdx = \int_\Omega fvdx.$$

  Since satisfy a boundary condition, $\frac{\partial u}{\partial n} = 0$ on Γ, so :

$$\int_\Omega fvdx = \int_\Omega \nabla u(x).\nabla v(x)dx + \int_\Omega u(x)v(x)dx,$$

with $\nabla u, \nabla v$ and $v \in L^2(\Omega)$, and we assume that $f \in L^2(\Omega)$.

- Therefore, a reasonable choice for the Hilbert space is $V = H^1(\Omega)$.
- So, the proposed variational formulation for (3.7) is:

$$u \in H^1(\Omega).$$

Such that

$$\int_\Omega \nabla u.\nabla v dx + \int_\Omega uv dx = \int_\Omega fv dx, \forall v \in H^1(\Omega). \tag{3.8}$$

With:

$$a(u,v) = \int_\Omega \nabla u.\nabla v dx + \int_\Omega uv dx,$$

and

$$\ell(v) = \int_\Omega fv dx.$$

Step 2: Solving the variational formulation
The purpose of the second step is solely to verify that the variational formulation (3.4) admits a unique solution. In other words, we will demonstrate that a is a continuous and coercive bilinear form, and ℓ is a continuous linear form on $H^1(\Omega)$.
For this purpose, we utilize the Lax-Milgram theorem (2.5).

- **a is a continuous bilinear form**

$$a(u,v) = \int_\Omega \nabla u.\nabla v dx + \int_\Omega uv dx.$$

Where

$$a \ is \ continuous \Leftrightarrow \exists c \in \mathbb{R}^* \ \ suchthat \ \ |a(u,v)| \leq c \left\| u \right\| \left\| v \right\|.$$

$$|a(u,v)| = \left| \int_\Omega \nabla u.\nabla v dx + \int_\Omega uv dx \right|.$$

According to the Cauchy-Schwarz inequality :

$$|a(u,v)| \leq (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}} (\int_\Omega |\nabla v|^2 dx)^{\frac{1}{2}} + (\int_\Omega |u|^2 dx)^{\frac{1}{2}} (\int_\Omega |v|^2 dx)^{\frac{1}{2}},$$

$$\leq \quad \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|u\|_{L^2} \|v\|_{L^2} ,$$

$$\leq \quad \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \ with \ \ c = 1.$$

So a is a continuous bilinear form.

- **a is coercive**
  Let's see if a is coercive, we need to show that there exists $\alpha > 0$ such that :

$$a(u,u) \geq \alpha \|u\|_{H^1}^2 (\Omega) \ \ for \ all \ \ u \in H^1(\Omega).$$

By assumption on a, we have :

$$a(u,u) \quad = \quad \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u|^2 \, dx,$$

$$= \quad \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2}^2 (\Omega), \ u \in H^1(\Omega),$$

$$= \quad \|u\|_{H^1(\Omega)}^2 \ \ with \ \ \alpha = 1.$$

So a is coercif.

- **ℓ is a continuous linear form**

$$\ell \ continuous \Leftrightarrow |\ell(v)| \leq M \|v\| \ \ \forall M \in \mathbb{R}^*.$$

We have:

$$|\ell(v)| = \left| \int_\Omega f(x)v(x)dx \right| .$$

According to the Cauchy-Schwarz inequality :

$$|\ell(v)| \quad \leq \quad (\int_\Omega |f(x)|^2 \, dx)^{\frac{1}{2}} (\int_\Omega |v(x)|^2 \, dx)^{\frac{1}{2}},$$

$$\leq \quad \|f\|_{L^2} \|v\|_{L^2} .$$

According to the injection of the space $H^1(\Omega)$ into $L^2(\Omega)$, we have

$$\leq \quad c \|f\|_{L^2} \|v\|_{H^1} .$$

So:

$$|\ell(v)| \leq M \, \|v\| \, .$$

With

$$M = c \, \|f\|_{L^2} \, .$$

Then, ℓ is a continuous linear form.
Finally, all the assumptions of the Lax-Milgram theorem (2.5) are satisfied, and thus we can conclude that there exists a unique solution $u \in H^1(\Omega)$ to the variational formulation (3.7).

General Conclusions and perspectives

The modeling of real problems in mechanics, thermodynamics, electromagnetism chemistry, physics, ... ext is written in the form of ordinary differential equations or by partial differential equations to solve this type of equation there are several methods to use , In this work we studied some boundary problems by variational methods using the Lax-Milgram theorem to show the existence and uniqueness of the solution of these problems.

• In future work, we will study variational formulation on other mathematical problems .

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ملخص:

في هذه المذكرة قمنا بدراسة بعض المسائل الرياضية باستعمال الطريقة التغيرية، اين اعطينا الصيغة التغيرية لهذا الشكل ، ثم استعملنا نظرية لاكس – ميليغرام من اجل برهان الوجود ووحدانية الحل، بمعنى اذا كان لدينا مشكل رياضي فالمطلوب هو البحث عن الدالة الحل لهذا المشكل.

الكلمات المفتاحية: التوزيعات، فضاءات سوبولاف، الصيغة التغيرية، نظرية لاكس – ميليغرام.

Dans ce mémoire, on a étudié quelques problèmes aux limites par les méthodes variationnelles , où on a donnée une formulation variationnelle de ce problème , puis on a utilisé le théorème de Lax-Milgram pour démontrer l'existence et l'unicité de la solution , i.e. , si nous avons un problème mathématique la formulation classique consiste à chercher une fonction solution de ce problème.

Mots-Clés: Les distributions, Espaces de Sobolev , Formulation variationnelle , Théorème de Lax-Milgram.

In this dissertation, we studied some boundary problems using variational methods, where we gave a variational formulation of this problem, then we used the Lax-Milgram theorem to demonstrate the existence and uniqueness of the solution, i.e., if we have a mathematical problem the classic formulation consists to search for a solution function of this problem.

Keywords: Distributions , Sobolev spaces, Variational formulation, Lax-Milgram theorem.