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Existence and Stability of Some Nonlinear Boundary Value Problems

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DEDICATION

To my dear parents

To my brothers and sisters

To my family

To my friends

To anyone interested in this work

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All praise and glory to Almighty Allah (Subhanahu Wa Taalaa) who gave me courage and patience to carry out this work. Peace and blessings of Allah be upon the last Prophet Muhammad (Peace Be Upon Him).

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LIST OF SYMBOLS

$\mathbb{R}$	: the set of real numbers.
$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$	: the extended real line.
$\mathbb{R}^d$	: the d -dimension Euclidean space .
$\mathbb{R}_+$	: the set of nonnegative real numbers.
$\mathbb{S}^d$	: the space of second-order symmetric tensors on $\mathbb{R}^d$.
Ω	: an open bounded and connected set in the space $\mathbb{R}^d$
	with Lipschitz boundary.
$\Gamma = \partial\Omega$	: the boundary of Ω is decomposed as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where Γ_1, Γ_2 and Γ_3
	are three disjoint measurable parts with $\text{meas } \Gamma_1 > 0$.
Γ_i	: a part of Γ which is the boundary of Ω .
$f', f'', f^{(i)}$	: the derivatives of the function denoted by f with respect to the time .
f_t, f_{tt}	: are partial derivative with respect to the time .
$u \cdot v$	: the canonical inner product of the vectors $u, v \in \mathbb{R}^d$.
$\sigma \cdot \tau$	: the canonical inner product of the tensors $\sigma, \tau \in \mathbb{S}^d$.
ν	: the unit outward normal to the boundary Γ .
Div	: divergence operator.
∇	: gradient operator.
I	: identity operator.
$L^p(\Omega)$	: the Lebesgue space of p -integrable functions
	for $1 \leq p < \infty$.
$L^\infty(\Omega)$	: the space of Lebesgue measurable functions u in Ω such that
	$\exists c > 0 : u < c, a.e. \text{ in } \Omega$.
$C^m(\overline{\Omega})$	: the space of functions whose derivatives up to and including
	order m are continuous in $\overline{\Omega}$.
$C_0^\infty(\Omega)$	: the space of infinitely differentiable functions with compact support in Ω .
$W^{k,p}(\Omega)$	: the Sobolev space of functions whose weak derivatives of
	orders less than or equal to k are p -integrable on Ω .

$H^k(\Omega)$	$\equiv W^{k,2}(\Omega).$
$W_0^{k,p}(\Omega)$	: the closure of the space $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega).$
$H_0^k(\Omega)$	$\equiv W_0^{k,2}(\Omega).$
H	$= \left\{ v = (v_1, \dots, v_d)^T : v_i \in L^2(\Omega), 1 \leq i \leq d \right\} = L^2(\Omega)^d.$
$\mathcal{H}$	$= \left\{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega), 1 \leq i, j \leq d \right\}.$
H_1	$= \left\{ v = (v_1, \dots, v_d)^T : v_i \in H^1(\Omega), 1 \leq i \leq d \right\}.$
$\mathcal{H}_1$	$= \left\{ \tau \in \mathcal{H} : \text{Div } \tau \in H \right\}.$
V	$= \left\{ v \in H^1 \mid v = 0 \quad \text{on} \quad \Gamma_i \right\}.$
X	: denotes a Hilbert space with the inner product $(\cdot, \cdot)_X$
	or a Banach space with norm $\ \cdot\ _X.$
X'	: denotes the dual of $X.$
$\langle u, v \rangle_{X \times X'}$	: the duality paring between the spaces X and $X'.$
V'	: the dual of $V.$
$\langle u, v \rangle_{V \times V'}$	: the duality paring between the spaces V and $V'.$
$\langle u, v \rangle_H$	: inner product of $H.$
$C^m([0, T]; X)$	$= \left\{ v \in C([0, T]; X) : v^{(j)} \in C([0, T]; X), j = 1, \dots, m \right\}.$
$L^p(0, T; X)$	$= \left\{ v : [0, T] \rightarrow X \text{ measurable} : \ v\ _{L^p(0, T; X)} < \infty \right\}.$
$W^{k,p}(0, T; X)$	$= \left\{ v \in L^p(0, T; X) : \ v^{(j)}\ _{L^p(0, T; X)} < \infty, \forall j \leq k \right\}.$
$H^k(0, T; X)$	$\equiv W^{k,2}(0, T; X) .$
r_+	$= \max\{r, 0\} : \text{positive part of } r.$
a.e.	: almost everywhere.
$\forall$	: for all.
$\mathcal{L}(t) \sim \mathcal{E}(t)$	: $\mathcal{L}(t)$ and $\mathcal{E}(t)$ are equivalent.

LIST OF PAPERS

Chapter 1

- Khireddine, H., Benabderrahmane, B., Ouakil, I., Feng, B. Stability of a dynamic frictionless contact problem for viscoelastic materials with rigid-plastic foundation and long-term memory. *Z Angew Math Mech*, **105**, e70260 (2025). <https://doi.org/10.1002/zamm.70260>.

Chapter 2

- Khireddine, H., Benabderrahmane, B., Boukhatem, Y., Feng, B. A fully dynamic frictional contact problem for viscoelastic materials with long-term memory and simplified coulomb's friction law. *submitted for publication*.

Chapter 3

- Khireddine, H., Benabderrahmane, B., Abdelaziz, L., Feng, B. Semigroup Well-posedness and Exponential Decay of a Viscoelastic Bilateral Contact Model with Tresca's Law and Infinite Memory. *submitted for publication*.

Chapter 4

- Khireddine, H., Benabderrahmane, B., Feng, B. Global Well-posedness and Stability for a type III thermo-viscoelastic frictional contact problem with infinite memory and simplified coulomb's friction law. *submitted for publication*.

GENERAL INTRODUCTION

Contact mechanics has numerous applications in both industry and daily systems, including metalworking, braking mechanisms, coupling devices, and internal combustion engines. These processes play a crucial role in mechanical and structural engineering and are typically described by nonlinear initial-boundary value problems characterized by complex surface interactions.

In recent years, variational inequalities grew to become a robust device in the mathematical learning of many nonlinear problems of physics and mechanics, as the complexity of the boundary conditions and the range of the constitutive equations lead to variational formulations of inequality type. The principles of virtual work and power, first put out via Fourier in 1823, can be defined through the concept of variational inequalities. The prototypes are the Signorini-Fichera and frictional contact in elasticity issues, which lead to a class of variational inequalities. The discipline of variational inequality used to be established concurrently with the answer to the Signorini Problem. For extra statistics on the preliminary developments of the elasticity principle and variational inequality, see, for example, [68]. As a generalization of variational inequalities, Panagiotopoulos [70] developed the hemivariational inequalities. Numerous monographs devoted to resolving a variety of complicated phenomena in contact problems with various bodies and foundations have been posted for the reason that the theory of variational or hemivariational inequalities has progressively improved. For instance, Han, Migórski and Zeng [72] considered a frictionless viscoelastic unilateral contact problem with normal damped response condition and the existence of a unique weak solution was proved. In [73], the authors considered a viscoelastic contact problem with nonmonotone normal damped response. One can find more results on contact problems with normal damped response condition in [76,77]. Matei and Micu [78] considered a nonlinear antiplane problem which models the deformation of an elastic cylindrical body in frictional contact with a rigid foundation and studied an optimal control problem which consists of leading the stress tensor as close as possible to a given target, by acting with a control on the boundary of the body. And then they studied the boundary optimal control of a frictional contact problem with normal compliance, see [97]. For more results of optimal control for contact problems, one can find in [79,80,88]. We refer to [83–87,89] on more dynamic contact problems.

By the pioneering works of Boltzmann [90] and Maxwell [91], the theory of viscoelasticity includes the description of materials that exhibit elastic and viscous characteristics. The quantitative description of such substances involves a strain stress relationship that depends on time. The classical linearization model generates an integral differential equation that enhances the relevant elastic stress tensor with appropriate memory terms. Regarding the contact problems for viscoelastic materials with memory terms, by the surjectivity of multivalued pseudomon-

tone operators, Zeng and Migórski [95] proved the unique solvability of a dynamic frictional history-dependent viscoelastic contact problem with adhesion. Then they established the solvability and regularity of solutions for a quasistatic frictional contact problem with a viscoelastic constitutive law, see [92]. Oultou et al. [93, 94] studied two kinds of memory-type viscoelastic frictional contact problems with wear and proved the unique solvability by history-dependent quasivariational inequality/history-dependent variational-hemivariational inequality. In [50] and [51], the authors investigated a dynamic frictional contact problem for viscoelastic materials with normal damped response and finite/infinite memory and established the existence and exponential stability of solutions. The stability of a dynamic frictionless contact problem for memory-type viscoelastic materials with rigid-plastic foundation was established in [49].

This thesis focuses on dynamic viscoelastic contact problems between a deformable body and a rigid foundation. The study establishes the local and global existence of weak solutions and investigates the exponential stability of energy solutions. The contact is modeled with Tresca's friction law, simplified Coulomb's friction law and rigid-plastic foundation, incorporating long-term memory effects. The analysis relies on nonlinear operator theory, semigroup method, fixed point arguments, Galerkin approximation, and Lyapunov stability. In addition to the theoretical framework, the thesis provides a detailed modeling of the physical setting, material behavior, and contact conditions, contributing to the mathematical understanding of complex viscoelastic phenomena.

This thesis is divided into four chapters. A brief description follows

- ✎ The first chapter deals with a dynamic frictionless contact problem in viscoelasticity with a rigid-plastic foundation. The constitutive law is assumed to involve a long-term memory. We prove the local existence of a weak solution under a specific assumption on the source term, using a fixed-point argument combined with a classical existence and uniqueness results for nonlinear first-order evolution inequalities. Then, the global existence result is proved via energy methods. Finally, we establish the exponential stability by constructing a suitable Lyapunov functional and applying the energy method. The obtained results are also the subject of an article that has been published in [49].
- ✎ In Chapter 2, we analyze a dynamic frictional contact problem for viscoelastic materials with long-term memory, where the contact conditions are governed by a simplified Coulomb's friction law. We establish a rigorous variational formulation and prove the local existence and uniqueness of a weak solution via a Galerkin scheme combined with compactness and semicontinuity arguments. By deriving suitable a priori estimates, we extend the analysis to demonstrate the global existence of solutions through energy methods. Furthermore, by constructing an appropriate Lyapunov functional, we show that the system enjoys exponential stability, thereby ensuring that the energy decays at an exponential rate over time. The outcomes of the current chapter have been submitted for publication.
- ✎ In Chapter 3, we consider a dynamic bilateral frictional contact problem for viscoelastic

materials. The friction is modeled by Tresca's law, while the constitutive relation is assumed to be nonlinear with an infinite memory term. Using semigroup theory, we establish the global well-posedness of the problem. Moreover, we apply the multiplier method to prove the exponential stability of the system. The obtained results are also the subject of an article that has been submitted for publication.

-  In Chapter 4, we investigate a thermo-viscoelastic contact problem with friction, where a type III thermoelasticity model is adopted. The constitutive law incorporates thermal effects together with long-term memory. The global existence of weak solutions is established by employing nonlinear semigroup theory in combination with arguments from nonlinear operator analysis. Furthermore, by applying the energy method, we prove the exponential decay of the solution's energy. The results obtained in this chapter have also been submitted for publication.

We end our thesis with a general conclusion discussing the achieved results and perspectives for future work. This is followed by an appendix presenting the basic equations and boundary conditions, preliminaries of functional analysis including function spaces, elements of nonlinear analysis, and some elementary inequalities, which together form the mathematical framework of the contact problem.

A DYNAMIC CONTACT PROBLEM WITH RIGID-PLASTIC FOUNDATION

This chapter investigates the local and global existence of a solution to a dynamic contact problem with a rigid-plastic foundation and establishes the exponential decay of the system. The analysis is based on evolutionary variational inequalities and employs the Banach fixed point theorem to prove existence and uniqueness. An appropriate Lyapunov functional is then constructed to demonstrate the exponential decay.

1.1 Problem Statement and Variational Formulation

In this section, we present a dynamic frictionless contact problem between a viscoelastic body and a rigid-plastic foundation. We list the assumptions imposed on the problem data and derive a variational formulation for the contact problem. The governing equations are given by

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^t g(t-s)B\varepsilon(u(s))ds \quad \text{in } \Omega \times (0, \infty), \quad (1.1)$$

$$\rho\ddot{u} = \text{Div}\sigma(t) + f_0(t) \quad \text{in } \Omega \times (0, \infty), \quad (1.2)$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \quad (1.3)$$

$$\sigma(t)\nu = f_2 \quad \text{on } \Gamma_2 \times (0, \infty), \quad (1.4)$$

$$\left. \begin{array}{ll} \sigma_v(t) = 0 & \text{if } u_v(t) < 0, \\ -F \leq \sigma_v(t) \leq 0 & \text{if } u_v(t) = 0, \\ \sigma_v(t) = -F & \text{if } u_v(t) > 0, \end{array} \right\} \quad \text{on } \Gamma_3 \times (0, \infty), \quad (1.5)$$

$$\sigma_\tau(t) = 0 \quad \text{on } \Gamma_3 \times (0, \infty), \quad (1.6)$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \quad \text{in } \Omega. \quad (1.7)$$

To derive the variational formulation of problem (1.1)-(1.7), we introduce some additional notation. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 \mid v = 0 \quad \text{on } \Gamma_1, \quad v_\nu = 0 \quad \text{on } \Gamma_3\}.$$

In the study of the mechanical problem (1.1)-(1.7), we assume that the viscosity operator A satisfies (A.6), the elasticity operator G satisfies (A.7), the relaxation tensor B satisfies (A.9), and the mass density ρ satisfies (A.4).

We suppose that the yield limit F satisfies

$$F \in L^2(\Gamma_3), \quad F(x) \geq 0, \quad \text{on } \Gamma_3, \quad (1.8)$$

and we assume that the body forces and surface tractions possess the following regularity

$$f_0(t) \in W^{1,1}(\mathbb{R}_+; H), \quad f_2(t) \in W^{1,1}(\mathbb{R}_+; L^2(\Gamma_2)^d). \quad (1.9)$$

Finally, the initial data satisfy

$$u_0 \in V, \quad u_1 \in H. \quad (1.10)$$

Now, by the Riesz representation theorem, we define the mapping $f : \mathbb{R}_+ \rightarrow V'$ by

$$\langle f(t), v \rangle_{V' \times V} = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v da, \quad (1.11)$$

for every $v \in V$ and for almost every $t \in \mathbb{R}_+$, we deduce from (1.9) that

$$f \in W^{1,1}(\mathbb{R}_+; V'). \quad (1.12)$$

With respect to the kernel function $g(t)$, we assume that

(H-1) $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nonincreasing function of C^1 class and satisfies that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = l > 0. \quad (1.13)$$

(H-2) There exists a constant $\alpha > 0$ such that

$$g'(t) \leq -\alpha g(t), \quad \forall t \geq 0. \quad (1.14)$$

(H-3) There exists a positive constant ζ such that

$$\|f'(t)\|_{V'} \leq \zeta \|f(t)\|_{V'}, \quad \forall t \geq 0. \quad (1.15)$$

Remark 1.1. It is straightforward to verify that the function $f(t) = ce^{-\zeta t}h$ belongs to $W^{1,1}(\mathbb{R}_+; V')$ and satisfies condition (1.15), where $h \in V'$ and c is a constant.

Next, the functional $j : V \rightarrow \mathbb{R}$ is defined by

$$j(v) = \int_{\Gamma_3} Fv_v^+ da, \quad \forall v \in V. \quad (1.16)$$

Using a standard procedure based on Green's formula (A.27), we derive the following variational formulation of the contact problem (1.1)-(1.7).

Problem PV Find a displacement $u : \mathbb{R}_+ \rightarrow V$ such that

$$\begin{aligned} & \langle \ddot{u}(t), w - \dot{u}(t) \rangle_{V' \times V} + \langle A\varepsilon(\dot{u}(t)), \varepsilon(w - \dot{u}(t)) \rangle_{\mathcal{H}} + \langle G\varepsilon(u(t)), \varepsilon(w - \dot{u}(t)) \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^t g(t-s)B\varepsilon(u(s))ds, \varepsilon(w - \dot{u}(t)) \right\rangle_{\mathcal{H}} + j(w) - j(\dot{u}(t)) \\ & \geq \langle f(t), w - \dot{u}(t) \rangle_{V' \times V} \quad \forall w \in V, \quad \forall t > 0, \end{aligned} \quad (1.17)$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1. \quad (1.18)$$

1.2 Local Existence

The following theorem establishes the local existence and uniqueness of the solution.

Theorem 1.1. *Let the assumptions (A.4),(A.6),(A.7),(A.9) and (1.8)-(1.10) hold. Then, system (1.17)-(1.18) admits a unique local solution u on $[0, T]$ for some $T > 0$ satisfying*

$$u \in W^{1,2}([0, T]; V) \cap C^1([0, T]; H) \cap W^{2,2}([0, T]; V'). \quad (1.19)$$

Consequently, under assumptions (A.4),(A.6),(A.7),(A.9) and (1.8)-(1.10), the mechanical problem (1.1)-(1.7) admits a unique weak solution with regularity stated in (1.19).

We prove Theorem 1.1 by employing several technical lemmas stated below. The proof is based on arguments from evolutionary variational inequalities, classical existence and uniqueness results for nonlinear first-order evolution inequalities, and the Banach fixed-point theorem. Let $\eta \in L^2(0, T; V')$ be given. In the first step, we consider the following variational problem

Problem PV _{η} Find a displacement $u_\eta : [0, T] \rightarrow V$ such that

$$\begin{aligned} & \langle \ddot{u}_\eta(t), w - \dot{u}_\eta(t) \rangle_{V' \times V} + \langle A\varepsilon(\dot{u}_\eta(t)), \varepsilon(w - \dot{u}_\eta(t)) \rangle_{\mathcal{H}} + \langle \eta(t), w - \dot{u}_\eta(t) \rangle_{V' \times V} + j(w) - j(\dot{u}_\eta(t)) \\ & \geq \langle f(t), w - \dot{u}_\eta(t) \rangle_{V' \times V} \quad \forall w \in V, \text{ a.e. } t \in (0, T), \end{aligned} \quad (1.20)$$

$$u_\eta(0) = u_0, \dot{u}_\eta(0) = u_1. \quad (1.21)$$

We define the function $f_\eta(t) \in V'$ for a.e. $t \in (0, T)$ by

$$\langle f_\eta(t), w - \dot{u}_\eta(t) \rangle_{V' \times V} = \langle f(t) - \eta(t), w - \dot{u}_\eta(t) \rangle_{V' \times V} \quad \forall w \in V. \quad (1.22)$$

It follows from (1.12) that

$$f_\eta \in L^2(0, T; V') \text{ and } u_1 \in H. \quad (1.23)$$

Let us define the operator $A : V \rightarrow V'$ as follows

$$\langle Av, w - \dot{u}_\eta(t) \rangle_{V' \times V} = \langle A\varepsilon(v), \varepsilon(w - \dot{u}_\eta(t)) \rangle_{\mathcal{H}}, \quad \forall v, w \in V. \quad (1.24)$$

The function $u_\eta : [0, T] \rightarrow V$ is then defined by

$$u_\eta = \int_0^t v_\eta(s) ds + u_0, \quad \forall t \in [0, T]. \quad (1.25)$$

The following variational inequality is considered.

Problem QV _{η} Find a displacement $v_\eta : [0, T] \rightarrow V$ such that

$$\begin{aligned} & \langle \dot{v}_\eta(t), w - v_\eta(t) \rangle_{V' \times V} + \langle Av_\eta(t), w - v_\eta(t) \rangle_{\mathcal{H}} + j(w) - j(v_\eta(t)) \\ & \geq \langle f_\eta(t), w - v_\eta(t) \rangle_{V' \times V} \quad \forall w \in V, \text{ a.e. } t \in (0, T), \end{aligned} \quad (1.26)$$

$$v_\eta(0) = v_0. \quad (1.27)$$

The first significant result in this section is presented below

Lemma 1.1. For all $\eta \in L^2(0, T, V')$, **Problem QV $_{\eta}$** admits a unique solution

$$v_{\eta} \in L^2(0, T, V) \cap C(0, T, H) \cap W^{1,2}(0, T, V'). \quad (1.28)$$

Proof. Existence: Using the continuous embedding of V into $L^2(\Gamma_3)^d$, we deduce that j is convex and continuous. We define the sequence by

$$j_{\varepsilon}(w) = \int_{\Gamma_3} F \sqrt{|w_v|^2 - \varepsilon^2} da, \quad \forall w \in V.$$

For all $\varepsilon > 0$, j_{ε} is convex and C^1 class on V , and its Frechet derivative satisfies

$$\exists C > 0 \quad \forall w \in V \quad \|j'_{\varepsilon}(w)\|_{V'} \leq C \|F\|_{L^2(\Gamma_3)}. \quad (1.29)$$

From (c) of (A.6) and the monotonicity of j'_{ε} it follows that $A + j'_{\varepsilon}$ is monotone. Moreover, assumption (b) of (A.6) implies that $A : V \rightarrow V$ is a Lipschitz operator, and therefore A is a continuous. In particular, the mapping $t \rightarrow A(u + tv)$ is continuous, which ensures the semicontinuity of A . Since j itself is continuous, we conclude that operator $A + j$ is also semicontinuous.

Now, exploiting (c) of (A.6) and the monotonicity of j'_{ε} that $A + j'_{\varepsilon}$, we conclude that $A + j'_{\varepsilon}$ is monotone, hence

$$((A + j'_{\varepsilon})u - (A + j'_{\varepsilon})v, u - v)_{V' \times V} \geq m_A \|u - v\|_V^2. \quad (1.30)$$

Choosing $v = 0_v$ in (1.30) and applying the inequality $2ab \leq m_A a^2 + \frac{1}{m_A} b^2$, it is obtained that

$$((A + j'_{\varepsilon})u, u)_{V' \times V} \geq m_A \|u\|_V^2 - \|A0_v\|_{V'}^2, \quad \|u\|_V \geq \frac{1}{2} m_A \|u\|_V^2 - \frac{1}{2m_A} \|A0_v\|_{V'}^2.$$

From (b) of (A.6) and (1.29), we have

$$((A + j'_{\varepsilon})u - (A + j'_{\varepsilon})v, u - v)_{V' \times V} \leq L_A \|u - v\|_V^2 + C.$$

Choosing $v = 0_v$ we get

$$((A + j'_{\varepsilon})u, u)_{V' \times V} \leq C(\|u\|_V^2 + 1).$$

For all $\frac{1}{2}m_A > 0, \frac{1}{2m_A} \|A0_v\|_{V'}^2 \in \mathbb{R}$ and $C > 0$, it follows from (1.10) and (1.23) that there exists a unique function v_{η} that $\forall \varepsilon > 0, v_{\eta}^{\varepsilon} \in L^2(0, T; V) \cap W^{1,2}(0, T; V')$ such that

$$\begin{cases} \dot{v}_{\eta}^{\varepsilon}(t) + A v_{\eta}^{\varepsilon}(t) + j'_{\varepsilon}(v_{\eta}^{\varepsilon}(t)) = f(t) & a.e. t \in (0, T), \\ v_{\eta}^{\varepsilon}(0) = v_0. \end{cases} \quad (1.31)$$

Then, we have

$$\begin{aligned} & (\dot{v}_{\eta}^{\varepsilon}(t), w - v_{\eta}^{\varepsilon}(t))_{V' \times V} + (A v_{\eta}^{\varepsilon}(t), w - v_{\eta}^{\varepsilon}(t))_{V' \times V} + j_{\varepsilon}(w) - j'_{\varepsilon}(v_{\eta}^{\varepsilon}(t)) \\ & \geq (f_{\eta}(t), w - v_{\eta}^{\varepsilon}(t))_{V' \times V} \quad \forall w \in V, \quad a.e. t \in (0, T). \end{aligned} \quad (1.32)$$

From (1.31), it follows that

$$\begin{aligned} & (\dot{v}_\eta^\varepsilon(t), v_\eta^\varepsilon(t))_{V' \times V} + (Av_\eta^\varepsilon(t), v_\eta^\varepsilon(t))_{V' \times V} + (j'(v_\eta^\varepsilon(t)), v_\eta^\varepsilon(t))_{V' \times V} \\ &= \langle f_\eta(t), v_\eta^\varepsilon(t) \rangle_{V' \times V} \quad \forall v_\eta^\varepsilon \in V, \text{ a.e. } t \in (0, T). \end{aligned}$$

From (A.6), the monotonicity of j_ε and (1.31), we concluded that

$$\exists C > 0, \quad \forall t \in [0, T], \quad \|v_\eta^\varepsilon(t)\|_H \leq C, \quad \int_0^T \|v_\eta^\varepsilon(t)\|_V dt \leq C, \quad \int_0^T \|\dot{v}_\eta^\varepsilon(t)\|_V dt \leq C.$$

Then, one can extract a subsequence, still denoted by $v_\eta^\varepsilon(t)$, such that

$$\begin{cases} v_\eta^\varepsilon(t) \rightharpoonup v_\eta(t) & \text{in } L^2(0, T; V), \text{ and star weakly in } L^\infty(0, T; H), \\ \dot{v}_\eta^\varepsilon(t) \rightharpoonup \dot{v}_\eta(t) & \text{in } L^2(0, T; V'). \end{cases} \quad (1.33)$$

We thus obtain that

$$v_\eta \in C(0, T, H) \text{ and } v_\eta^\varepsilon(t) \rightharpoonup v_\eta(t) \text{ weakly in } H, \quad \forall t \in [0, T]. \quad (1.34)$$

By integrating (1.32), it follows that for all $w \in L^2(0, T, V)$,

$$\begin{aligned} & \int_0^T (\dot{v}_\eta^\varepsilon(t), w)_{V' \times V} dt + \int_0^T (Av_\eta^\varepsilon(t), w)_{V' \times V} dt + \int_0^T j_\varepsilon(w) dt \\ & \geq \int_0^T (\dot{v}_\eta^\varepsilon(t), v_\eta^\varepsilon(t))_{V' \times V} dt + \int_0^T (Av_\eta^\varepsilon(t), v_\eta^\varepsilon(t))_{V' \times V} dt \\ & \quad + \int_0^T j'_\varepsilon(v_\eta^\varepsilon(t)) + \int_0^T \langle f_\eta(t), w - v_\eta^\varepsilon(t) \rangle_{V' \times V} dt \\ & \geq \frac{1}{2} \|v_\eta^\varepsilon(T)\|_H^2 - \frac{1}{2} \|v_\eta^\varepsilon(0)\|_H^2 + \int_0^T (Av_\eta^\varepsilon(t), v_\eta^\varepsilon(t))_{V' \times V} dt \\ & \quad + \int_0^T j'_\varepsilon(v_\eta^\varepsilon(t)) + \int_0^T \langle f_\eta(t), w - v_\eta^\varepsilon(t) \rangle_{V' \times V} dt. \end{aligned}$$

By combining (1.33) and (1.34) with the weak lower semicontinuity, it follows that for all $w \in L^2(0, T, V)$,

$$\begin{aligned} & \int_0^T (\dot{v}_\eta(t), w - v_\eta(t))_{V' \times V} dt + \int_0^T (Av_\eta(t), w - v_\eta(t))_{V' \times V} dt + \int_0^T j(w) - j(v_\eta(t)) dt \\ & \geq \int_0^T \langle f_\eta(t), w - v_\eta(t) \rangle_{V' \times V} dt, \end{aligned}$$

which implies that

$$\begin{aligned} & (\dot{v}_\eta(t), w - v_\eta(t))_{V' \times V} + (Av_\eta(t), w - v_\eta(t))_{V' \times V} + j(w) - j(v_\eta(t)) \\ & \geq \langle f_\eta(t), w - v_\eta(t) \rangle_{V' \times V} \quad \forall w \in V, \text{ a.e. } t \in (0, T). \end{aligned}$$

It follows that **Problem QV** $_\eta$ admits at least one solution $v_\eta \in L^2(0, T, V) \cap C(0, T, H) \cap W^{1,2}(0, T, V')$.

Uniqueness: Let v_η^1, v_η^2 be two solutions of QV_η . From (1.26), we obtain for almost every $t \in (0, T)$,

$$\langle \dot{v}_\eta^1(t) - \dot{v}_\eta^2(t), v_\eta^1(t) - v_\eta^2(t) \rangle_{V' \times V} + \langle Av(t)_\eta^1 - Av(t)_\eta^2, v_\eta^1(t) - v_\eta^2(t) \rangle_{\mathcal{H}} \leq 0.$$

By integrating with respect to t and using (A.6) and (1.24), we derive

$$\frac{1}{2} \|v_\eta^1(t) - v_\eta^2(t)\|_H^2 + m_A \int_0^t \|v_\eta^1(s) - v_\eta^2(s)\|_V^2 ds \leq 0, \quad \forall t \in [0, T].$$

Hence,

$$v_\eta^1(t) = v_\eta^2(t).$$

This completes the proof. $\square$

Let $\Psi : L^2(0, T; V') \rightarrow L^2(0, T; V')$ be the operator defined as follows

$$\langle \Psi \eta(t), w \rangle_{V' \times V} = - \left\langle \int_0^t g(t-s) B \varepsilon(u_\eta(s)) ds, \varepsilon(w) \right\rangle_{\mathcal{H}} + \langle G \varepsilon(u_\eta(t)), \varepsilon(w) \rangle_{\mathcal{H}}. \quad (1.35)$$

The second step is given by the following result.

Lemma 1.2. *The operator $\Psi : L^2(0, T; V') \rightarrow L^2(0, T; V')$ has a unique fixed point $\eta^* \in L^2(0, T; V')$.*

Proof. Let $\eta_1, \eta_2 \in L^2(0, T; V')$ and $t \in [0, T]$. Denote $u_i = u_{\eta_i}$ for $i = 1, 2$.

Using (1.35), (A.7) and (A.9), we obtain the estimate

$$\begin{aligned} & | \langle \Psi \eta_1(t) - \Psi \eta_2(t), w \rangle_{V' \times V} | \\ & \leq \left(L_B \int_0^t g(t-s) \|u_1(s) - u_2(s)\|_V ds + L_G \|u_1(t) - u_2(t)\|_V \right). \end{aligned} \quad (1.36)$$

From (1.26), we get

$$\langle \dot{v}_1 - \dot{v}_2, v_1 - v_2 \rangle_{V' \times V} + \langle Av_1 - Av_2, v_1 - v_2 \rangle_{\mathcal{H}} \leq - \langle \eta_1 - \eta_2, v_1 - v_2 \rangle_{V' \times V}. \quad (1.37)$$

By integrating (1.37) over t and applying the initial conditions $v_1(0) = v_2(0) = u_1$ together with (A.6), we obtain

$$m_A \int_0^t \|v_1(s) - v_2(s)\|_V^2 ds \leq \int_0^t |\eta_1(s) - \eta_2(s)|_{V'} |v_1(s) - v_2(s)|_V ds. \quad (1.38)$$

By Young's inequality, we obtain

$$\begin{aligned} & \int_0^t \langle \eta_1(s) - \eta_2(s), v_1(s) - v_2(s) \rangle_{V' \times V} ds \leq \int_0^t |\eta_1(s) - \eta_2(s)|_{V'} |v_1(s) - v_2(s)|_V ds \\ & \leq \frac{1}{m_A} \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 ds + \frac{m_A}{4} \int_0^t \|v_1(s) - v_2(s)\|_V^2 ds. \end{aligned} \quad (1.39)$$

Combining (1.38) and (1.39), we get

$$\int_0^t \|v_1(s) - v_2(s)\|_V^2 ds \leq \frac{4}{3m_A^2} \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 ds.$$

From (1.25), using the fact that $u_1(0) = u_2(0) = u_0$, we obtain

$$\|u_1(t) - u_2(t)\|_V^2 \leq c \int_0^t \|v_1(s) - v_2(s)\|_V^2 ds. \quad (1.40)$$

By employing the previous inequality along with (1.36), we obtain

$$\|\Psi\eta_1(t) - \Psi\eta_2(t)\|_{V'}^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_{V'}^2 ds \quad \forall t \in [0, T]. \quad (1.41)$$

After m iterations of the inequality, it follows that

$$\|\Psi^m\eta_1 - \Psi^m\eta_2\|_{L^2(0,T;V')}^2 \leq \frac{c^m T^m}{m!} \|\eta_1 - \eta_2\|_{L^2(0,T;V')}^2. \quad (1.42)$$

For sufficiently large m , the operator Ψ^m defines a contraction on the Hilbert space $L^2(0, T; V')$. Consequently, Ψ admits a unique fixed point $\eta^* \in L^2(0, T; V')$. $\square$

Proof of Theorem 1.1. Let $\eta^* \in L^2(0, T; V')$ be the fixed point of the operator Ψ defined in (1.35). We denote

$$u^* = u_{\eta^*}, \quad (1.43)$$

to obtain

$$\langle \eta^*(t), w \rangle_{V' \times V} = - \left\langle \int_0^t g(t-s) B\varepsilon(u^*(s)) ds, \varepsilon(w) \right\rangle_{\mathcal{H}} + \langle G\varepsilon(u^*(t)), \varepsilon(w) \rangle_{\mathcal{H}}. \quad (1.44)$$

We show that u^* satisfies (1.17)-(1.18) for $t \in [0, T]$.

We write (1.20) for $\eta = \eta^*$ and using (1.43), (1.44) to get that (1.17) is satisfied for $t \in (0, T)$. And (1.18) follows from (1.21). From Lemma 1.1 we can get the regularity (1.19).

By using the fixed point of the operator Φ we can obtain the uniqueness. $\square$

1.3 Global Existence

The following theorem establishes the global in time existence of the solution.

Theorem 1.2. Assume that $u(x, t)$ is a weak solution of problem (1.1)-(1.7) belonging to the class $u \in W^{1,2}(0, T; V) \cap C^1(0, T, H)$, assumptions (H-1), (H-3), (A.4), (A.6), (A.7), (A.9) and (1.8)-(1.10) hold. Then, $u(x, t)$ is global in time.

Let us consider the energy functional defined as

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2} \left(\|\dot{u}(t)\|_H^2 + \left(1 - \int_0^t g(s) ds\right) \|u(t)\|_V^2 + \frac{1}{2} (g \diamond u)(t) \right) \\ &\quad + \frac{1}{2} \left(e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 + \|f(t)\|_{V'}^2 \right), \quad \forall t \geq 0.\end{aligned}\quad (1.45)$$

Lemma 1.3. *Assume that (H-1),(H-3),(A.4),(A.6),(A.7),(A.9) and (1.8)-(1.10) hold. Then the energy functional defined by (1.45) satisfies, for $\delta > 0$ and for all $t \geq 0$*

$$\begin{aligned}\mathcal{E}'(t) &\leq -(m_A - \mathcal{C}_1) \|\dot{u}(t)\|_V^2 + \delta \mathcal{C}_2 \|u(t)\|_V^2 + (\zeta + \delta) \|f(t)\|_{V'}^2, \\ &\quad + \delta \mathcal{C}_3 (g \diamond u)(t) + \frac{1}{2} (g' \diamond u)(t) + \delta e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2,\end{aligned}\quad (1.46)$$

where $\mathcal{C}_1 = (4\delta)^{-1}(7 + C_0^2)$, $\mathcal{C}_2 = (1 - l)^2(1 + L_B^2) + L_G^2 + 1$, $\mathcal{C}_3 = (1 + L_B^2)(1 - l)$.

Proof. By using equations (1.1)-(1.2) together with Green's formula (A.27), and noting that $w = 0_v$ and $w = 2u'(t)$, we see that

$$\begin{aligned}\mathcal{E}'(t) &= \langle \ddot{u}(t), \dot{u}(t) \rangle_H - \frac{1}{2} g(t) \|u(t)\|_V^2 + \langle \dot{u}(t), u(t) \rangle_V \left(1 - \int_0^t g(s) ds\right) + \frac{1}{2} (g' \diamond u)(t) \\ &\quad + \int_0^t g(t-s) \langle \dot{u}(t), u(t) - u(s) \rangle_V ds - \frac{\zeta}{2} e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 \\ &\quad + e^{-\zeta t} \langle Fu(t), \dot{u}(t) \rangle_{L^2(\Gamma_3)} + \langle f'(t), f(t) \rangle_{V'} \\ &\leq -\langle G\varepsilon(u(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} - \langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} F \dot{u}_v^+(a) da \\ &\quad + \langle f(t), \dot{u}(t) \rangle_{V' \times V} - \frac{1}{2} g(t) \|u(t)\|_V^2 + \left(1 - \int_0^t g(s) ds\right) \langle \dot{u}(t), u(t) \rangle_V \\ &\quad + \int_0^t g(t-s) \langle \dot{u}(t), u(t) - u(s) \rangle_V ds + \frac{1}{2} (g' \diamond u)(t) ds \\ &\quad + e^{-\zeta t} \langle Fu(t), \dot{u}(t) \rangle_{L^2(\Gamma_3)} + \langle f'(t), f(t) \rangle_{V'}.\end{aligned}$$

It follows from (A.6) that

$$-\langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \leq -m_A \|\dot{u}(t)\|_V^2. \quad (1.47)$$

Using Cauchy-Schwarz's together with Young's inequality and (A.7), we conclude that

$$-\langle G\varepsilon(u(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta L_G^2 \|u(t)\|_V^2. \quad (1.48)$$

Exploiting Young's inequality and (A.9) together with (H-1), we have

$$\begin{aligned}
& \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \\
& \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta L_B^2 \left(\int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)} \|u(t) - u(s)\|_V ds \right)^2 \\
& \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta L_B^2 (1-l)(g \diamond u)(t).
\end{aligned} \tag{1.49}$$

Using (H-1), we obtain

$$\left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \leq L_B^2 (1-l)^2 \delta \|u(t)\|_V^2 + \frac{1}{4\delta} \|\dot{u}(t)\|_V^2. \tag{1.50}$$

Through the application of Young's inequality, we infer that

$$\langle f(t), \dot{u}(t) \rangle_{V' \times V} \leq \delta \|f(t)\|_{V'}^2 + \frac{1}{4\delta} \|\dot{u}(t)\|_V^2. \tag{1.51}$$

Using (1.8) and (A.23), we get

$$\begin{aligned}
e^{-\zeta t} \langle Fu(t), \dot{u}(t) \rangle_{L^2(\Gamma_3)} & \leq \delta e^{-2\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 + \frac{1}{4\delta} \|\dot{u}(t)\|_{L^2(\Gamma_3)}^2 \\
& \leq \delta e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 + \frac{C_0^2}{4\delta} \|\dot{u}(t)\|_V^2.
\end{aligned} \tag{1.52}$$

By employing Cauchy-Schwarz's and Young inequalities along with (H-1), it follows that

$$\left(1 - \int_0^t g(s) ds\right) \langle \dot{u}(t), u(t) \rangle_V \leq \frac{1}{2\delta} \|\dot{u}(t)\|_V^2 + \delta \left(1 + (1-l)^2\right) \|u(t)\|_V^2, \tag{1.53}$$

and

$$\int_0^t g(t-s) \langle \dot{u}(t), u(t) - u(s) \rangle_V ds \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + (1-l)\delta(g \diamond u)(t). \tag{1.54}$$

From (H-3), we infer that

$$\langle f'(t), f(t) \rangle_{V'} \leq \zeta \|f(t)\|_{V'}^2. \tag{1.55}$$

Hence, (1.46) results from the estimates (1.47)-(1.55) and (H-1). $\square$

Remark 1.2. From (1.46), we get that the energy functional is not decreasing in general.

Proof of Theorem 1.2. From (1.46), we have

$$\begin{aligned}
\mathcal{E}'(t) & \leq -(m_A - \mathcal{C}_1) \|\dot{u}(t)\|_V^2 + \delta \mathcal{C}_2 \|u(t)\|_V^2 + \delta \mathcal{C}_3 \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds(t) \\
& \quad + \delta e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 + (\zeta + \delta) \|f(t)\|_{V'}^2.
\end{aligned} \tag{1.56}$$

Choosing δ large enough such that

$$m_A - \frac{7 + C_0^2}{4\delta} > 0,$$

we deduce

$$\mathcal{E}'(t) \leq \mathcal{C}_4 \left(\|u(t)\|_V^2 + \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds + \|f(t)\|_{V'}^2 + e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 \right), \quad (1.57)$$

where $\mathcal{C}_4 = \max \{\delta \mathcal{C}_2, \zeta + \delta, \delta \mathcal{C}_3\}$.

From (1.45), we also have

$$\mathcal{E}(t) \geq \mathcal{C}_5 \left(\|u(t)\|_V^2 + \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds + \|f(t)\|_{V'}^2 + e^{-\beta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 \right), \quad (1.58)$$

where $\mathcal{C}_5 = \min \left\{ \frac{1}{2}, \frac{l}{2} \right\}$.

Therefore, combining (1.57) and (1.58), we obtain

$$\mathcal{E}'(t) \leq \mathcal{C}_6 \mathcal{E}(t), \quad (1.59)$$

where $\mathcal{C}_6 = \frac{\mathcal{C}_4}{\mathcal{C}_5}$. Therefore,

$$\mathcal{E}(t) \leq \mathcal{E}(0) e^{\mathcal{C}_6 t}. \quad (1.60)$$

Hence, by the continuation principle and inequality (1.60), the proof of global existence is complete. $\square$

1.4 Exponential Stability

The aim of this section is to establish the exponential stability of the solution. To this end, we introduce the Lyapunov functional $\mathcal{L}(t)$ defined as follows

$$\mathcal{L}(t) = \mathcal{E}(t) + \epsilon(N\mathcal{F}_1(t) + \mathcal{F}_2(t)), \quad (1.61)$$

where ϵ, N are positive constants to be chosen later and

$$\mathcal{F}_1(t) = \left(-\int_0^t g(s) ds \right) \|f(t)\|_{V'}^2 + e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 + \langle \dot{u}(t), u(t) \rangle_H, \quad \forall t \geq 0, \quad (1.62)$$

$$\mathcal{F}_2(t) = -\left\langle \dot{u}(t), \int_0^t g(t-s) (u(t) - u(s)) ds \right\rangle_H, \quad \forall t \geq 0. \quad (1.63)$$

Lemma 1.4. Assume that (H-1) holds. Then, there exist two positive constants α_1 and α_2 such that

$$\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t). \quad (1.64)$$

Proof. From Young's inequality and (A.26), we get

$$\begin{aligned} |\mathcal{F}_1(t)| &\leq \left| \left(- \int_0^t g(s) ds \right) \|f(t)\|_{V'}^2 + e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 + \langle \dot{u}(t), u(t) \rangle_H \right| \\ &\leq \max \{ d^2 l^{-1}, 2(1-l), 2 \} \mathcal{E}(t), \end{aligned} \quad (1.65)$$

similarly

$$\begin{aligned} |\mathcal{F}_2(t)| &= \left| - \left\langle \dot{u}(t), \int_0^t g(t-s) (u(t) - u(s)) ds \right\rangle_H \right| \\ &\leq \max \{ 1, (1-l)d^2 \} \mathcal{E}(t). \end{aligned} \quad (1.66)$$

From (1.45), (1.65) and (1.66), we obtain

$$|\mathcal{L}(t) - \mathcal{E}(t)| \leq \epsilon \alpha_3 \mathcal{E}(t),$$

where

$$\alpha_3 = N \max \{ d^2 l^{-1}, 2(1-l), 2 \} + \max \{ 1, (1-l)d^2 \}.$$

If

$$\epsilon < \frac{1}{\alpha_3}, \quad (1.67)$$

then $\mathcal{L}$ is equivalent to $\mathcal{E}$ with $\alpha_1 = 1 - \epsilon \alpha_3$ and $\alpha_2 = 1 + \epsilon \alpha_3$. $\square$

Lemma 1.5. Assume that (H-1), (H-3), (A.4), (A.6), (A.7), (A.9) and (1.8)-(1.10) hold. Then, the functional $\mathcal{F}_1$ defined by (1.62) satisfies for $\delta_1 > 0$,

$$\begin{aligned} \mathcal{F}_1'(t) &\leq (-k_0 + 3\delta_1) \|u(t)\|_V^2 + \mathcal{C}_7 (g \diamond u)(t) + \mathcal{C}_8 \|\dot{u}(t)\|_V^2 \\ &\quad + \left(\frac{1}{4\delta_1} - \gamma \right) \|f(t)\|_{V'}^2 + \left(\frac{1}{2\delta_1} - \zeta \right) e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2, \text{ for all } t \geq 0, \end{aligned} \quad (1.68)$$

where $k_0 = m_G - L_B(1-l)$, $\mathcal{C}_7 = \frac{(1-l)L_B^2}{4\delta_1}$, $\mathcal{C}_8 = \frac{L_A^2}{4\delta_1} + 2C_0^2\delta_1 + d^2$.

Proof. We obtain

$$\begin{aligned} \mathcal{F}_1'(t) &= -g(t) \|f(t)\|_{V'}^2 + 2 \left(- \int_0^t g(s) ds \right) \langle f'(t), f(t) \rangle_{V'} - \zeta e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 \\ &\quad + 2e^{-\zeta t} \langle Fu(t), \dot{u}(t) \rangle_{L^2(\Gamma_3)} + \|\dot{u}(t)\|_H^2 + \langle \ddot{u}(t), u(t) \rangle_H. \end{aligned}$$

By multiplying (1.2) by $w - u(t)$ and using (1.1) with $w = 0_v$ and $w = 2u(t)$, we deduce from

assumption (A.7) that

$$\begin{aligned}
\mathcal{F}'_1(t) \leq & \|\dot{u}(t)\|_H^2 - \langle A\varepsilon(\dot{u}(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 \\
& + \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\
& + \left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} F u_{\nu}^+(t) da \\
& + \langle f(t), u(t) \rangle_{V' \times V} - g(t) \|f(t)\|_{V'}^2 + 2 \left(- \int_0^t g(s) ds \right) \langle f'(t), f(t) \rangle_{V'} \\
& - \zeta e^{-\zeta t} \|F u(t)\|_{L^2(\Gamma_3)}^2 + 2e^{-\zeta t} \langle F u(t), \dot{u}(t) \rangle_{L^2(\Gamma_3)}.
\end{aligned} \tag{1.69}$$

From (A.6), we obtain

$$\begin{aligned}
\langle A\varepsilon(\dot{u}(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} & \leq \|A\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\
& \leq \frac{L_A^2}{4\delta_1} \|\dot{u}(t)\|_V^2 + \delta_1 \|u(t)\|_V^2.
\end{aligned}$$

By Young's inequality, (A.9) and (H-1), it holds that

$$\begin{aligned}
& \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\
& \leq \frac{L_B^2}{4\delta_1} \left(\int_0^t g(t-s) \|u(s) - u(t)\|_V ds \right)^2 + \delta_1 \|u(t)\|_V^2 \\
& \leq \frac{(1-l)L_B^2}{4\delta_1} (g \diamond u)(t) + \delta_1 \|u(t)\|_V^2.
\end{aligned}$$

From (A.9) and the hypothesis (H-1), it follows that

$$\left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \leq L_B(1-l) \|u(t)\|_V^2.$$

From (1.8) and (A.23), we conclude that

$$\begin{aligned}
2e^{-\zeta t} \langle F u(t), \dot{u}(t) \rangle_{L^2(\Gamma_3)} & \leq \frac{e^{-\zeta t}}{2\delta_1} \|F u(t)\|_{L^2(\Gamma_3)}^2 + 2e^{-\zeta t} \delta_1 \|\dot{u}(t)\|_{L^2(\Gamma_3)}^2 \\
& \leq \frac{e^{-\zeta t}}{2\delta_1} \|F u(t)\|_{L^2(\Gamma_3)}^2 + 2C_0^2 \delta_1 \|\dot{u}(t)\|_V^2.
\end{aligned}$$

Using (H-3), we derive

$$\begin{aligned}
2 \left(- \int_0^t g(s) ds \right) \langle f'(t), f(t) \rangle_{V'} & \leq 2 \left(\int_0^t g(s) ds \right) \|f'(t)\|_{V'} \|f(t)\|_{V'} \\
& \leq 2(1-l)\zeta \|f(t)\|_{V'}^2,
\end{aligned}$$

and

$$\langle f(t), u(t) \rangle_{V' \times V} \leq \frac{1}{4\delta_1} \|f(t)\|_{V'}^2 + \delta_1 \|u(t)\|_V^2.$$

Inserting the above estimates into (1.69) and using (A.26), it is inferred that

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-k_0 + 3\delta_1) \|u(t)\|_V^2 + \frac{(1-l)L_B^2}{4\delta_1} (g \diamond u)(t) \\ & + \left(\frac{L_A^2}{4\delta_1} + 2C_0^2\delta_1 + d^2 \right) \|\dot{u}(t)\|_V^2 + \left(\frac{1}{4\delta_1} - \gamma \right) \|f(t)\|_{V'}^2 + \left(\frac{1}{2\delta_1} - \zeta \right) e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2, \end{aligned}$$

where $\gamma = \min g(t) - 2(1-l)\zeta$. □

Lemma 1.6. Assume that (H-1), (A.4), (A.6), (A.7), (A.9) and (1.8)-(1.10) hold. Then the functional $\mathcal{F}_2(t)$ given by (1.63) satisfies for $\delta_1 > 0$,

$$\begin{aligned} \mathcal{F}'_2(t) \leq & - \int_0^t g(s) ds \|\dot{u}(t)\|_H^2 + \mathcal{C}_9 \delta_1 \|\dot{u}(t)\|_V^2 + \mathcal{C}_{10} \delta_1 \|u(t)\|_V^2 + \mathcal{C}_{11} (g \diamond u)(t) \\ & - \mathcal{C}_{12} (g' \diamond u)(t) + \delta_1 \|f(t)\|_{V'}^2 + \delta_1 C_0^2 \mathcal{E}(t), \text{ for all } t \geq 0, \end{aligned} \quad (1.70)$$

where

$$\mathcal{C}_9 = L_A^2 + d^2, \quad \mathcal{C}_{10} = L_G^2 + L_B^2,$$

and

$$\mathcal{C}_{11} = \frac{4(1-l)}{4\delta_1} + \frac{(1-l)^3}{4\delta_1} + L_B(1-l), \quad \mathcal{C}_{12} = \frac{g(0)d^2}{4\delta_1}.$$

Proof. First, we have

$$\begin{aligned} \mathcal{F}'_2(t) = & - \left\langle \ddot{u}(t), \int_0^t g(t-s)(u(t) - u(s)) ds \right\rangle_H \\ & - \left\langle \dot{u}(t), \int_0^t g'(t-s)(u(t) - u(s)) ds \right\rangle_H \\ & - \left\langle \dot{u}(t), \int_0^t g(t-s) \dot{u}(t) ds \right\rangle_H. \end{aligned}$$

From equations (1.1)-(1.2) and the Green's formula (A.27), it follows that

$$\begin{aligned} \mathcal{F}'_2(t) = & \left\langle A\varepsilon(\dot{u}(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s))) ds \right\rangle_{\mathcal{H}} \\ & + \left\langle G\varepsilon(u(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s))) ds \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^t g(t-s) (B\varepsilon(u(t)) - B\varepsilon(u(s))) ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s))) ds \right\rangle_{\mathcal{H}} \\ & - \left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s))) ds \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} F \int_0^t g(t-s)(u_v^+(t) - u_v^+(s)) ds da - \left\langle f(t), \int_0^t g(t-s)(u(t) - u(s)) ds \right\rangle_{V' \times V} \\ & - \left\langle \dot{u}(t), \int_0^t g'(t-s)(u(t) - u(s)) ds \right\rangle_H - \int_0^t g(s) ds \|\dot{u}(t)\|_H^2. \end{aligned} \quad (1.71)$$

From assumptions (H-1) and (A.6), we obtain

$$\begin{aligned} & \left\langle A\varepsilon(\dot{u}(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\ & \leq \delta_1 L_A^2 \|\dot{u}(t)\|_V^2 + \frac{(1-l)}{4\delta_1} (g \diamond u)(t). \end{aligned} \quad (1.72)$$

From (H-1) and (A.7), we derive

$$\begin{aligned} & \left\langle G\varepsilon(u(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\ & \leq \delta_1 L_G^2 \|u(t)\|_V^2 + \frac{(1-l)}{4\delta_1} (g \diamond u)(t). \end{aligned} \quad (1.73)$$

Using Cauchy–Schwarz inequality together with (H-1) and (A.9), we deduce

$$\begin{aligned} & \left\langle \int_0^t g(t-s)(B\varepsilon(u(t)) - B\varepsilon(u(s)))ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\ & \leq L_B(1-l)(g \diamond u)(t). \end{aligned} \quad (1.74)$$

By applying Young's inequality together with (H-1) and (A.9), we derive

$$\begin{aligned} & - \left\langle \int_0^t g(t-s)B\varepsilon(u(t))ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\ & \leq \delta_1 L_B^2 \|u(t)\|_V^2 + \frac{(1-l)^3}{4\delta_1} (g \diamond u)(t). \end{aligned} \quad (1.75)$$

Taking into account (1.8), (H-1), (A.23) and (1.45), we conclude that

$$\begin{aligned} & \int_{\Gamma_3} F \left(\int_0^t g(t-s)(u_v^+(t) - u_v^+(s))ds \right) da \\ & \leq \delta_1 C_0^2 \mathcal{E}(t) + \frac{(1-l)}{4\delta_1} (g \diamond u)(t). \end{aligned} \quad (1.76)$$

Using the same approach, it follows that

$$- \left\langle f(t), \int_0^t g(t-s)(u(t) - u(s))ds \right\rangle_{V' \times V} \leq \delta_1 \|f(t)\|_{V'}^2 + \frac{(1-l)}{4\delta_1} (g \diamond u)(t), \quad (1.77)$$

and

$$\begin{aligned} & - \left\langle \dot{u}(t), \int_0^t g'(t-s)(u(t) - u(s))ds \right\rangle_H \\ & \leq \|\dot{u}(t)\|_H \left\| \int_0^t g'(t-s)(u(t) - u(s))ds \right\|_H \\ & \leq d^2 \delta_1 \|\dot{u}(t)\|_V^2 - \frac{g(0)d^2}{4\delta_1} (g' \diamond u)(t). \end{aligned} \quad (1.78)$$

Substituting estimates (1.72)-(1.78) into (1.71), inequality (1.70) follows. $\square$

The following theorem establishes the exponential stability of the solution

Theorem 1.3. Assume that (H-1),(H-3),(A.4),(A.6),(A.7),(A.9) and (1.8)-(1.10) hold. Assume that m_G and m_A satisfies

$$m_G > 3\delta_1 + L_B(1 - l). \quad (1.79)$$

$$m_A > \lambda_0, \quad (1.80)$$

where $\lambda_0 > 0$ is introduced in (1.87), and the kernel of the memory term $g(t)$ satisfies

$$g(t) > 2(1 - l)\zeta. \quad (1.81)$$

Then, for any $t_0 > 0$ the energy functional $\mathcal{E}(t)$ of problem (1.1)-(1.7) satisfies

$$\mathcal{E}(t) \leq \lambda_2 e^{-\lambda_1 t}, \quad \forall t \geq t_0, \quad (1.82)$$

where λ_1 and λ_2 are positive constants.

Proof. Since g is positive, it follows that

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds := g_0 > 0, \quad t \geq t_0.$$

It follows from (1.46), (1.68), and (1.70), that for all $t \geq t_0$

$$\begin{aligned} \mathcal{L}'(t) &= \mathcal{E}'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\ &\leq -\epsilon \left((g_0 - \delta_1 C_0^2) \|\dot{u}(t)\|_H^2 + \left(N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0)) - \frac{\mathcal{C}_2 \delta}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ &\quad -\epsilon \left(\mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) - \frac{\mathcal{C}_3 \delta}{\epsilon} \right) (g \diamond u)(t) \\ &\quad + (\mathcal{C}_1 - m_A + \epsilon(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)) \|\dot{u}(t)\|_V^2 \\ &\quad + \left(\frac{1}{2} - \epsilon \mathcal{C}_{12} \right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds + \epsilon \mathcal{C}_{13} (g \diamond u)(t) \\ &\quad -\epsilon \left(N \left(\zeta - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon} \right) e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 \\ &\quad -\epsilon \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right) \|f(t)\|_{V'}^2, \end{aligned} \quad (1.83)$$

where $\mathcal{C}_{13} = N\mathcal{C}_7 + \mathcal{C}_{11} + \mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) + \delta_1 C_0^2$.

From (1.79), δ_1 may be chosen such that

$$\min \left\{ \frac{1}{2\zeta}, \frac{1}{4\gamma} \right\} < \delta_1 < \max \left\{ \frac{k_0}{3}, \frac{g_0}{C_0^2} \right\}. \quad (1.84)$$

Accordingly, we can choose N large enough to ensure

$$\begin{aligned} N(k_0 - 3\delta_1) - \delta_1(\mathcal{C}_{10} + C_0^2(1 - g_0)) &> 0, \\ N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2) &> 0. \end{aligned} \quad (1.85)$$

From (H-2), we obtain

$$\begin{aligned} &\left(\frac{1}{2} - \epsilon\mathcal{C}_{12}\right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds + \epsilon\mathcal{C}_{13} \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds \\ &\leq \left(\frac{1}{2} - \epsilon\mathcal{C}_{14}\right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds, \end{aligned}$$

where $\mathcal{C}_{14} = \mathcal{C}_{12} + \frac{\mathcal{C}_{13}}{\alpha}$.

From (1.83), we deduce that

$$\begin{aligned} \mathcal{L}'(t) &= \mathcal{E}'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\ &\leq -\epsilon \left((g_0 - \delta_1 C_0^2) \|\dot{u}(t)\|_H^2 + \left(N(k_0 - 3\delta_1) - \delta_1(\mathcal{C}_{10} + C_0^2(1 - g_0)) - \frac{\mathcal{C}_2\delta}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ &\quad -\epsilon \left(\mathcal{C}_3 \left(N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2) \right) - \frac{\mathcal{C}_3\delta}{\epsilon} \right) (g \diamond u)(t) \\ &\quad + (\mathcal{C}_1 - m_A + \epsilon(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)) \|\dot{u}(t)\|_V^2 \\ &\quad + \left(\frac{1}{2} - \epsilon\mathcal{C}_{14}\right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds \\ &\quad -\epsilon \left(N\left(\zeta - \frac{1}{2\delta_1}\right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon} \right) e^{-\zeta t} \|Fu(t)\|_{L^2(\Gamma_3)}^2 \\ &\quad -\epsilon \left(N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right) \|f(t)\|_{V'}^2. \end{aligned} \quad (1.86)$$

We assume that m_A satisfies condition (1.80) by choosing λ_0 as

$$\begin{aligned} \lambda_0 &= \max \left\{ \frac{\delta(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)}{N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2)}, \frac{\mathcal{C}_2\delta(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)}{N(k_0 - 3\delta_1) - \delta_1(\mathcal{C}_{10} + C_0^2(1 - g_0))}, \right. \\ &\quad \left. \frac{\delta(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)}{N\left(\zeta - \frac{1}{2\delta_1}\right) - \delta_1 C_0^2}, \frac{(\delta + \zeta)(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)}{N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2)} \right\} + \mathcal{C}_1. \end{aligned} \quad (1.87)$$

Moreover, δ choosing such that

$$\delta < \min \left\{ \frac{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}{\alpha_3}, \frac{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}{2\mathcal{C}_{14}}, \right. \\ \frac{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0))}{\mathcal{C}_2\alpha_3}, \frac{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0))}{2\mathcal{C}_2\mathcal{C}_{14}}, \\ \frac{N\left(\zeta - \frac{1}{2\delta_1}\right) - \delta_1 C_0^2}{\alpha_3}, \frac{N\left(\zeta - \frac{1}{2\delta_1}\right) - \delta_1 C_0^2}{2C_{14}}, \\ \left. \frac{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}{\alpha_3} - \zeta, \frac{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}{2C_{14}} - \zeta \right\}, \quad (1.88)$$

we get

$$\frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{14}} \right\}, \quad (1.89)$$

and

$$\frac{\mathcal{C}_2\delta}{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0))} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{14}} \right\}, \quad (1.90)$$

and

$$\frac{\delta}{\left(\zeta - \frac{1}{2\delta_1}\right) N - \delta_1 C_0^2} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{14}} \right\}, \quad (1.91)$$

and

$$\frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{14}} \right\}. \quad (1.92)$$

From (1.87), we also obtain

$$\frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}, \quad (1.93)$$

and

$$\frac{\mathcal{C}_2\delta}{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0))} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}, \quad (1.94)$$

and

$$\frac{\delta}{\left(\zeta - \frac{1}{2\delta_1}\right) N - \delta_1 C_0^2} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}, \quad (1.95)$$

and

$$\frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}. \quad (1.96)$$

Therefore, by combining (1.67) with (1.89)-(1.96), one may choose ϵ such that

$$\begin{aligned} & \max \left\{ \frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}, \frac{\mathcal{C}_2 \delta}{N (k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0))}, \right. \\ & \quad \left. \frac{\delta}{\left(\zeta - \frac{1}{2\delta_1}\right) N - \delta_1 C_0^2}, \frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} \right\} \\ & < \epsilon < \min \left\{ \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}, \frac{1}{2\mathcal{C}_{14}}, \frac{1}{\alpha_3} \right\}. \end{aligned} \quad (1.97)$$

Thus, we get

$$\mathcal{L}'(t) \leq -\epsilon \mathcal{C}_{15} \mathcal{E}(t) + \left(\frac{1}{2} - \epsilon \mathcal{C}_{14}\right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds,$$

where

$$\begin{aligned} \mathcal{C}_{15} = & 2 \min \left\{ g_0 - \delta_1 C_0^2, \frac{N (k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0)) - \frac{\mathcal{C}_2 \delta}{\epsilon}}{1 - g_0}, \right. \\ & \mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) - \frac{\mathcal{C}_3 \delta}{\epsilon}, \\ & \left. N \left(\zeta - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon}, N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right\} > 0. \end{aligned}$$

By (H-1), (1.97) and (1.64), we get

$$\mathcal{L}'(t) \leq \frac{-\epsilon \mathcal{C}_{15}}{1 + \epsilon \alpha_3} \mathcal{L}(t),$$

which implies

$$\mathcal{L}(t) \leq \mathcal{L}(t_0) e^{-\lambda_1(t-t_0)}, \quad \forall t \geq t_0, \quad (1.98)$$

where

$$\lambda_1 = \frac{\epsilon \mathcal{C}_{15}}{1 + \epsilon \alpha_3}.$$

Finally, from (1.64) and (1.98) it follows that

$$\mathcal{E}(t) \leq \frac{1}{1 - \epsilon \alpha_3} \mathcal{L}(t) \leq \frac{1}{1 - \epsilon \alpha_3} \mathcal{L}(t_0) e^{\lambda_1 t_0} e^{-\lambda_1 t}.$$

Consequently, inequality (1.82) holds with

$$\lambda_2 = \frac{1}{1 - \epsilon \alpha_3} \mathcal{L}(t_0) e^{\lambda_1 t_0}.$$

This completes the proof of theorem 1.3. □

A DYNAMIC CONTACT PROBLEM WITH SIMPLIFIED COULOMB'S FRICTION LAW

This chapter is devoted to a rigorous analysis of a contact problem governed by a simplified Coulomb's friction law. We first establish the local existence and uniqueness of the solution via a Galerkin approximation, then extend the results to demonstrate global existence using energy methods. Finally, the exponential stability of the energy solutions is rigorously established using the multiplier technique.

2.1 Problem Statement and Variational Formulation

Consider the following nonlinear contact problem with long-term memory and a simplified Coulomb's friction law

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^t g(t-s)B\varepsilon(u(s))ds \quad \text{in } \Omega \times (0, \infty), \quad (2.1)$$

$$\rho \ddot{u} = \text{Div} \sigma(t) + f_0(t) \quad \text{in } \Omega \times (0, \infty), \quad (2.2)$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \quad (2.3)$$

$$\sigma(t)\nu = f_2 \quad \text{on } \Gamma_2 \times (0, \infty), \quad (2.4)$$

$$\begin{cases} \sigma_\nu = S, & |\sigma_\tau| \leq \mu|\sigma_\nu| \\ |\sigma_\tau| < \mu|\sigma_\nu| \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu|\sigma_\nu| \Rightarrow \exists \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0, \infty), \quad (2.5)$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \quad \text{in } \Omega. \quad (2.6)$$

Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 \mid v = 0 \quad \text{on } \Gamma_1\}.$$

We assume that the viscosity operator A satisfies (A.6), the elasticity operator G satisfies (A.7), the relaxation tensors B satisfies (A.9), and the mass density ρ satisfies (A.4).

Let $S : \Gamma_3 \rightarrow \mathbb{R}_+$ be given such that

$$S \in L^\infty(\Gamma_3). \quad (2.7)$$

We assume that the body forces and tractions satisfy the following regularity conditions

$$f_0(t) \in W^{1,2}(\mathbb{R}_+; H), \quad f_2(t) \in W^{1,2}(\mathbb{R}_+; L^2(\Gamma_2)^d). \quad (2.8)$$

The initial data satisfy

$$u_0, u_1 \in V. \quad (2.9)$$

The operators A and G are Fréchet differentiable on $\mathcal{H}$.

There exist positive constants m'_A, m'_G and L'_G such that

$$\langle DA(\sigma)\tau, \tau \rangle \geq m'_A \|\tau\|^2, \quad \forall \sigma, \tau \in \mathcal{H}, \quad (2.10)$$

$$\langle DG(\sigma)\tau, \tau \rangle \geq m'_G \|\tau\|^2, \quad \forall \sigma, \tau \in \mathcal{H}. \quad (2.11)$$

and

$$\|DG(\sigma_1) - DG(\sigma_2)\| \leq L'_G \|\sigma_1 - \sigma_2\|, \quad \forall \sigma_1, \sigma_2 \in \mathcal{H}. \quad (2.12)$$

By the Riesz representation theorem, the mapping $f : \mathbb{R}_+ \rightarrow V'$ is defined as follows

$$\langle f(t), v \rangle_{V' \times V} = \int_{\Omega} f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v da + \int_{\Gamma_3} S \cdot v_\nu da. \quad (2.13)$$

For all $v \in V$ and almost every $t \in \mathbb{R}_+$, it follows from assumption (2.8) that

$$f \in W^{1,2}(\mathbb{R}_+; V'). \quad (2.14)$$

(H-4) $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is assumed to be a non-increasing function of class C^1 , satisfying

$$g'(t) \geq -\beta g(t), \quad g(0) > 0, \quad 1 - \int_0^\infty g(s) ds = l > 0. \quad (2.15)$$

We now introduce the functional $j : V \rightarrow \mathbb{R}$, defined by

$$j(v) = \int_{\Gamma_3} \mu |S| |v_\tau| da, \quad \forall v \in V. \quad (2.16)$$

Applying Green's formula (A.27), and using the boundary conditions together with the constitutive law (2.1), we obtain the following variational formulation of the system (2.1)-(2.6), expressed in terms of displacement.

Problem PV Find a displacement $u : \mathbb{R}_+ \rightarrow V$ such that

$$\begin{aligned} & \langle \ddot{u}(t), w - \dot{u}(t) \rangle_{V' \times V} + \langle A\varepsilon(\dot{u}(t)), \varepsilon(w - \dot{u}(t)) \rangle_{\mathcal{H}} + \langle G\varepsilon(u(t)), \varepsilon(w - \dot{u}(t)) \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(t-s) B\varepsilon(u(s)) ds, \varepsilon(w - \dot{u}(t)) \right\rangle_{\mathcal{H}} + j(w) - j(\dot{u}(t)) \\ & \geq \langle f(t), w - \dot{u}(t) \rangle_{V' \times V} \quad \forall w \in V, \quad \forall t > 0, \end{aligned} \quad (2.17)$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1. \quad (2.18)$$

2.2 Local Existence

The existence of a local solution to the variational problem (2.17)-(2.18) is established by the following theorem

Theorem 2.1. *Let assumptions (A.4),(A.6),(A.7),(A.9) and (2.7)-(2.12) be satisfied. Then there exists a unique solution u of (2.17)-(2.18) in $[0, T)$ for $T > 0$ satisfying*

$$u \in W^{1,\infty}([0, T]; V) \cap W^{2,\infty}([0, T]; V'). \quad (2.19)$$

Consequently, under the assumptions (A.4),(A.6),(A.7),(A.9) and (2.7)-(2.12), the mechanical problem (2.1)-(2.6) admits a unique local solution u satisfying (2.19).

The proof of Theorem 2.1 is divided into several steps. We begin by proving the existence result.

Proof. Existence: In order to prove the existence of the solution u to the system (2.17)-(2.18), we shall use the Faedo-Galerkin approach and the compactness method. To this, let us consider $(\varphi_i)_{i \in \mathbb{N}} \in V$ a sequence of linear independent functions such that $V = \overline{U_{m=1}^\infty V^m}$ where $V^m = \text{Span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$. Since $u_0, u_1 \in V$ let $u_0^m, u_1^m \in V^m$ such that

$$u_0^m \rightarrow u_0, \quad u_1^m \rightarrow u_1 \quad \text{strongly in } V. \quad (2.20)$$

We now define the approximated solution

$$u_\epsilon^m(x, t) = \sum_{i=1}^m \alpha_{im}(t) \varphi_i(x). \quad (2.21)$$

For all $\epsilon > 0$, we consider the family of convex and differentiable functions $\psi_\epsilon: \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$\psi_\epsilon(v) = \sqrt{|v|^2 + \epsilon^2} - \epsilon, \quad v \in \mathbb{R}^d.$$

It follows that such a family of functions satisfies

$$0 \leq \psi_\epsilon(v) \leq |v|, \quad \forall v \in \mathbb{R}^d, \quad (2.22)$$

$$|\psi_\epsilon(v) - |v|| \leq \epsilon, \quad \forall v \in \mathbb{R}^d, \quad (2.23)$$

$$\psi'_\epsilon(v)(w) = \frac{v \cdot w}{\sqrt{|v|^2 + \epsilon^2}}, \quad \forall v, w \in \mathbb{R}^d. \quad (2.24)$$

From equation (2.24), we deduce that

$$|\psi'_\epsilon(v)(w)| \leq |w|, \quad \forall v, w \in \mathbb{R}^d. \quad (2.25)$$

Now, we define a family of regularized frictional functional $j_\epsilon: V \rightarrow \mathbb{R}$ by

$$j_\epsilon(v) = \int_{\Gamma_3} |S(s)| \mu(s) \psi_\epsilon(v_\tau(s)) ds, \quad \forall v \in V. \quad (2.26)$$

The functionals j_ϵ are Gateaux-differentiable with respect to v and represent an approximation of j .i.e there exists a constant C such that

$$|j_\epsilon(v) - j(v)| \leq C\epsilon, \quad \forall v \in V. \quad (2.27)$$

Let $J_\epsilon : V \rightarrow V'$ denote the derivative of the functional j_ϵ , which is given by

$$\langle J_\epsilon(v), w \rangle_{V',V} = \int_{\Gamma_3} |S(s)| \mu(s) \psi'_\epsilon(v_\tau(s)) (w_\tau(s)) ds, \quad \forall v, w \in V. \quad (2.28)$$

We now introduce the following variational problem with regularized friction in the finite-dimensional space V^m :

Problem PV $_\epsilon^m$ Find a displacement $u_\epsilon^m : [0, T] \rightarrow V^m$ such that

$$\begin{aligned} & \langle \ddot{u}_\epsilon^m(t), w \rangle_{V' \times V} + \langle A\varepsilon(\dot{u}_\epsilon^m(t)), \varepsilon(w) \rangle_{\mathcal{H}} + \langle G\varepsilon(u_\epsilon^m(t)), \varepsilon(w) \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^t g(t-s) B\varepsilon(u_\epsilon^m(s)) ds, \varepsilon(w) \right\rangle_{\mathcal{H}} + \langle J_\epsilon \dot{u}_\epsilon^m(t), w \rangle_{V',V} \\ & = \langle f(t), w \rangle_{V' \times V} \quad \forall w \in V^m, \quad \forall t > 0, \end{aligned} \quad (2.29)$$

$$u_\epsilon^m(0) = u_0^m, \dot{u}_\epsilon^m(0) = u_1^m. \quad (2.30)$$

As the mapping $v \rightarrow J_\epsilon(v)$ is Lipschitz continuous on V^m , it follows that problem (2.29)-(2.30) has a unique global solution $u_\epsilon^m \in C^2([0, T_\epsilon^m]; V^m)$.

The continuation of the proof is divided into three parts. We begin by proving that, for each $\epsilon > 0$ and all $m \in \mathbb{N}$ the approximate problem admits a unique solution u_ϵ^m . To this end, we derive a set of a priori estimates, which will be established in the first two parts of the proof. In the third part, we prove that as $\epsilon \rightarrow 0$ and $m \rightarrow \infty$ the sequence u_ϵ^m converges (in an appropriate sense) to a solution of the problem (2.17)-(2.18).

To simplify the exposition, we omit the indices m and ϵ in the first two parts of the proof

(I) The first estimate.

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\dot{u}(t)\|_V^2 + \langle G\varepsilon(u(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} + \langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \\ & \leq \left\langle \int_0^t g(t-s) B\varepsilon(u(s)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} + \langle f(t), \dot{u}(t) \rangle_{V',V}. \end{aligned} \quad (2.31)$$

From (A.6) and (A.7), we obtain

$$\langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} d\phi \geq m_A \|\dot{u}(t)\|_V^2. \quad (2.32)$$

and

$$\begin{aligned} |\langle G(\varepsilon(u(t))), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}}| & \leq \|G(\varepsilon(u(t)))\|_{\mathcal{H}} \|\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \\ & \leq L_G (1 + \|u(t)\|_V) \|\dot{u}(t)\|_V \\ & \leq \frac{L_G^2}{4\delta} (1 + \|u(t)\|_V)^2 + \delta \|\dot{u}(t)\|_V. \end{aligned} \quad (2.33)$$

By exploiting Young's inequality together (A.9), we get

$$\begin{aligned} & \left\langle \int_0^t g(t-s) B\varepsilon(u(s)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \\ & \leq \delta \|\dot{u}(t)\|_V^2 + \frac{1}{4\delta} \left(\int_0^t g(t-s) B\varepsilon(u(s)) ds \right)^2 \\ & \leq \delta \|\dot{u}(t)\|_V^2 + \frac{\|g\|_{L^1} \|g\|_{L^\infty} L_B^2}{4\delta} \int_0^t \|u(s)\|_V^2 ds. \end{aligned} \quad (2.34)$$

Using Cauchy-Schwarz's and Young's inequalities leads to

$$\langle f(t), \dot{u}(t) \rangle_{V' \times V} \leq \frac{1}{4\delta} \|f(t)\|_{V'}^2 + \delta \|\dot{u}(t)\|_V^2. \quad (2.35)$$

Inserting (3.48)-(2.35) into (2.31), we derive that

$$\frac{1}{2} \frac{d}{dt} \|\dot{u}(t)\|_V^2 + K_0 \|\dot{u}(t)\|_V^2 \leq \frac{1}{4\delta} \|f(\phi)\|_{V'}^2 + K_1 (1 + \|u(t)\|_V^2) + K_2 \int_0^t \|u(s)\|_V^2 ds. \quad (2.36)$$

where $K_0 = (m_A - 3\delta)$, $K_1 = \frac{L_G^2}{2\delta}$, $K_2 = \frac{\|g\|_{L^1} \|g\|_{L^\infty} L_B^2}{4\delta}$.

Note that

$$\|u(t)\|_V \leq \|u_0\|_V + \int_0^t \|\dot{u}(s)\|_V ds \quad (2.37)$$

we obtain

$$\|u(t)\|_V^2 \leq K_3 + K_3 \int_0^t \|\dot{u}(s)\|_V^2 ds \quad (2.38)$$

where $K_3 = \max \{2\|u_0\|_V^2, 2T\}$.

Substituting the last inequalities into (2.36) and integrate over $(0, t)$, we have,

$$\begin{aligned} & \frac{1}{2} \|\dot{u}(t)\|_V^2 + K_0 \int_0^t \|\dot{u}(\phi)\|_V^2 d\phi \leq \frac{1}{2} \|u_1\|_V^2 + K_4 T + \frac{1}{4\delta} \int_0^t \|f(\phi)\|_{V'}^2 d\phi \\ & + K_5 T \int_0^t \|\dot{u}(s)\|_V^2 ds \end{aligned} \quad (2.39)$$

where $K_4 = K_1 + K_1 K_3 + K_2 K_3 T$, $K_5 = K_1 K_3 + K_2 K_3 T$.

Now, from (2.14), we deduce that $\int_0^t \|f(\phi)\|_{V'}^2 d\phi \leq K$. By using the assumptions on the data and applying Gronwall's inequality, we obtain the following estimates.

$$\begin{cases} \{u_\epsilon^m\}_{m,\epsilon} & \text{is bounded in } L^\infty(0, T; V), \\ \{\dot{u}_\epsilon^m\}_{m,\epsilon} & \text{is bounded in } L^\infty(0, T; V) \cap L^2(0, T; V). \end{cases} \quad (2.40)$$

(II) The second estimate.

Differentiating (2.29) with respect to t and setting $w = \ddot{u}(t)$, we have

$$\begin{aligned} & \langle \ddot{u}_\epsilon^m(t), \ddot{u}(t) \rangle_{V' \times V} + \langle DG(\varepsilon(u(t))) \varepsilon(\dot{u}(t)), \varepsilon(\ddot{u}(t)) \rangle_{\mathcal{H}} \\ & + \langle DA(\varepsilon(\dot{u}(t))) \varepsilon(\ddot{u}(t)), \varepsilon(\ddot{u}(t)) \rangle_{\mathcal{H}} + \left\langle \frac{d}{dt} J(\dot{u}(t)), \ddot{u}(t) \right\rangle_{V', V} \\ & \leq \left\langle g(0)B\varepsilon(u(t)) + \int_0^t g'(t-s)B\varepsilon(u(s)) ds, \varepsilon(\ddot{u}(t)) \right\rangle_{\mathcal{H}} + \langle f'(t), \ddot{u}(t) \rangle_{V', V}. \end{aligned} \quad (2.41)$$

Since

$$\left\langle \frac{d}{dt} J_\epsilon(\dot{u}_\epsilon^m(t)), \ddot{u}(t) \right\rangle_{V', V} = \int_{\Gamma_3} \frac{|S(s)|\mu(s)}{(|\dot{u}_\tau|^2 + \epsilon^2)^{\frac{3}{2}}} [|\ddot{u}_\tau|^2 |\dot{u}_\tau|^2 - (\ddot{u}_\tau^2 \cdot \dot{u}_\tau^2) + \epsilon^2 |\ddot{u}_\tau|^2] \geq 0,$$

Note that

$$\begin{aligned} & \langle DG(\varepsilon(\dot{u}_m(t))) \varepsilon(\dot{u}_m(t)), \varepsilon(\ddot{u}_m(t)) \rangle_H = \frac{d}{dt} \langle DG(\varepsilon(\dot{u}_m(t))) \varepsilon(\dot{u}_m(t)), \varepsilon(\dot{u}_m(t)) \rangle_H \\ & - \langle D^2 G(\varepsilon(\dot{u}_m(t))) [\varepsilon(\ddot{u}_m(t))] \varepsilon(\dot{u}_m(t)), \varepsilon(\dot{u}_m(t)) \rangle_H - \langle DG(\varepsilon(\dot{u}_m(t))) \varepsilon(\ddot{u}_m(t)), \varepsilon(\dot{u}_m(t)) \rangle_H \end{aligned} \quad (2.42)$$

It follows together with the above equality that

$$\begin{aligned} & \frac{1}{2} \|\ddot{u}(t)\|_V^2 + \langle DG(\varepsilon(\dot{u}(t))) \varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} + m'_A \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi \leq \frac{1}{2} \|\ddot{u}(0)\|_V^2 \\ & + \langle DG(\varepsilon(u_1)) \varepsilon(u_1), \varepsilon(u_1) \rangle_{\mathcal{H}} + \int_0^t \langle D^2 G(\varepsilon(\dot{u}(\phi))) [\varepsilon(\ddot{u}(\phi))] \varepsilon(\dot{u}(\phi)), \varepsilon(\dot{u}(\phi)) \rangle_{\mathcal{H}} d\phi \\ & + \int_0^t \langle DG(\varepsilon(\dot{u}(\phi))) \varepsilon(\ddot{u}(\phi)), \varepsilon(\dot{u}(\phi)) \rangle_{\mathcal{H}} d\phi + 2 \int_0^t \langle f'(\phi), \ddot{u}(\phi) \rangle_{V', V} d\phi \\ & + \int_0^t \left\langle g(0)B\varepsilon(u(\phi)) + \int_0^\phi g'(\phi-s)B\varepsilon(u(s)) ds, \varepsilon(\ddot{u}(\phi)) \right\rangle_{\mathcal{H}} d\phi \end{aligned} \quad (2.43)$$

In order to obtain a second a priori estimate, we give the estimate $\ddot{u}(0)$, For this purpose, considering $t = 0$ in the equation (2.29) and use (2.30), we get

$$\langle \ddot{u}(0), w \rangle_{V' \times V} + \langle A\varepsilon(u_1), \varepsilon(w) \rangle_{\mathcal{H}} + \langle G\varepsilon(u_0), \varepsilon(w) \rangle_{\mathcal{H}} + \langle J_\epsilon(u_1), w \rangle_{V', V} = \langle f(0), w \rangle_{V' \times V}.$$

Since $\langle J_\epsilon(u_1), w \rangle_{V', V} = 0$ for all $w \in V^m$ we obtain

$$\langle \ddot{u}(0), w \rangle_{V' \times V} = \langle \text{Div} \sigma_0, w \rangle_{\mathcal{H}} + \langle f_0(0), w \rangle_{V' \times V}.$$

which implies

$$\rho \ddot{u}(0) = \text{Div} \sigma_0 + f_0(t).$$

It follows that there exists a constant $C > 0$, which does not depend on m , such that

$$\ddot{u}(0) \leq C. \quad (2.44)$$

Using Cauchy-Schwarz's, Young's inequalities and (2.12), leads to

$$\begin{aligned}
& \int_0^t \langle D^2 G(\varepsilon(\dot{u}(\phi))) [\varepsilon(\ddot{u}(\phi))] \varepsilon(\dot{u}(\phi)), \varepsilon(\dot{u}(\phi)) \rangle_{\mathcal{H}} d\phi \\
& \leq \int_0^t \|D^2 G(\varepsilon(\dot{u}(\phi)))\| \|\varepsilon(\ddot{u}(\phi))\|_{\mathcal{H}} \|\varepsilon(\dot{u}(\phi))\|_{\mathcal{H}}^2 d\phi \\
& \leq \int_0^t L'_G \|\varepsilon(\ddot{u}(\phi))\|_{\mathcal{H}} \|\varepsilon(\dot{u}(\phi))\|_{\mathcal{H}}^2 d\phi \\
& \leq \delta \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi + \frac{L_G'^2}{4\delta} \int_0^t \|\dot{u}(\phi)\|_V^4 d\phi. \\
& \leq \delta \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi + K_6(\delta).
\end{aligned} \tag{2.45}$$

and

$$\begin{aligned}
& \int_0^t \langle DG(\varepsilon(\dot{u}(\phi))) \varepsilon(\ddot{u}(\phi)), \varepsilon(\dot{u}(\phi)) \rangle_{\mathcal{H}} d\phi \\
& \leq \int_0^t \|DG(\varepsilon(\dot{u}(\phi)))\| \|\varepsilon(\ddot{u}(\phi))\|_{\mathcal{H}} \|\varepsilon(\dot{u}(\phi))\|_{\mathcal{H}} d\phi \\
& \leq \int_0^t L'_G \|\varepsilon(\ddot{u}(\phi))\|_{\mathcal{H}} \|\varepsilon(\dot{u}(\phi))\|_{\mathcal{H}}^2 d\phi \\
& \leq \delta \int_0^t \|\ddot{u}(t)\|_V^2 d\phi + K_7(\delta).
\end{aligned} \tag{2.46}$$

By exploiting Young's inequality together (A.9) and (H-4), we get

$$\begin{aligned}
& \int_0^t \langle g(0) B \varepsilon(u(\phi)), \varepsilon(\ddot{u}(\phi)) \rangle_{\mathcal{H}} d\phi \\
& \leq \int_0^t \frac{g(0)^2 L_B^2}{\delta} \|u(\phi)\|_V^2 d\phi + \delta \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi.
\end{aligned} \tag{2.47}$$

and

$$\begin{aligned}
& \int_0^t \left\langle \int_0^\phi g'(\phi-s) B \varepsilon(u(s)) ds, \varepsilon(\ddot{u}(\phi)) \right\rangle_{\mathcal{H}} d\phi \\
& \leq \delta \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi + \frac{1}{4\delta} \int_0^t \left(\int_0^\phi g'(\phi-s) B \varepsilon(u(s)) ds \right)^2 d\phi \\
& \leq \delta \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi + \frac{\beta^2}{4\delta} \|g\|_{L^1} \|g\|_{L^\infty} L_B^2 \int_0^t \|u(s)\|_V^2 ds.
\end{aligned} \tag{2.48}$$

Using Cauchy-Schwarz's and Young's inequalities, we have

$$\int_0^t \langle f'(\phi), \ddot{u}(\phi) \rangle_{V' \times V} d\phi \leq \frac{1}{4\delta} \int_0^t \|f'(\phi)\|_{V'}^2 d\phi + \delta \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi \tag{2.49}$$

Combining (2.44)-(2.49), choosing δ small enough, we conclude

$$\begin{aligned} & \frac{1}{2} \|\ddot{u}(t)\|_V^2 + \langle DG(\varepsilon(\dot{u}(t))) \varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} + m'_A \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi \leq \frac{1}{2} \|\ddot{u}(0)\|_V^2 \\ & + \langle DG(\varepsilon(u_1)) \varepsilon(u_1), \varepsilon(u_1) \rangle_{\mathcal{H}} + \frac{1}{4\delta} \int_0^t \|f'(\phi)\|_{V'}^2 d\phi + 5\delta \int_0^t \|\ddot{u}(\phi)\|_V^2 d\phi \\ & + \frac{\beta^2}{\delta} \|g\|_{L^1} \|g\|_{L^\infty} L_B^2 \int_0^t \|u(s)\|_V^2 ds + \int_0^t \frac{g(0)^2 L_B^2}{\delta} \|u(\phi)\|_V^2 d\phi. \end{aligned} \quad (2.50)$$

Now, by using the assumptions on the data and Gronwall's inequality, we have the following estimates

$$\begin{cases} \{\dot{u}_\epsilon^m\}_{m,\epsilon} & \text{is bounded in } L^\infty(0, T; V), \\ \{\ddot{u}_\epsilon^m\}_{m,\epsilon} & \text{is bounded in } L^\infty(0, T; V) \cap L^2(0, T; V). \end{cases} \quad (2.51)$$

(III) Passage to the limit in m and ϵ .

From (2.40) and (2.51), we deduce that there exists a subsequence of $\{u_\epsilon^m\}_{m,\epsilon}$ such that for $m \rightarrow \infty$ and $\epsilon \rightarrow 0$ we have

$$\begin{aligned} u_\epsilon^m &\rightharpoonup u \quad \text{weak}^* \quad \text{in } L^\infty(0, T; V), \\ \dot{u}_\epsilon^m &\rightharpoonup \dot{u} \quad \text{weak}^* \quad \text{in } L^\infty(0, T; V), \\ \ddot{u}_\epsilon^m &\rightharpoonup \ddot{u} \quad \text{weak}^* \quad \text{in } L^\infty(0, T; H). \end{aligned}$$

If we denote $\Omega_T = \Omega \times (0, T)$, we obtain that

$$\{u_\epsilon^m\}_{\epsilon,m}, \{\dot{u}_\epsilon^m\}_{\epsilon,m} \text{ are bounded in } [H^1(\Omega_T)]^d.$$

Since $W^{1,\infty}(0, T; V) \hookrightarrow C([0, T]; V)$ we deduce that there exists a subsequence of $\{u_\epsilon^m\}_{m,\epsilon}$ such that

$$u_\epsilon^m \rightarrow u \quad \text{strongly in } L^2(0, T; H), \quad (2.52)$$

$$\dot{u}_\epsilon^m \rightarrow \dot{u} \quad \text{strongly in } L^2(0, T; H). \quad (2.53)$$

Moreover, since the trace map from $H^1(\Omega_T)$ to $L^2(\partial\Omega_T)$ is a compact operator and $\partial\Omega_T = (\partial\Omega \times]0, T[) \cup (\Omega \times \{0\}) \cup (\Omega \times \{T\})$, we find

$$\begin{cases} u_\epsilon^m(T) \rightarrow u(T), \text{ strongly in } H, \\ \dot{u}_\epsilon^m(T) \rightarrow \dot{u}(T), \text{ strongly in } H. \end{cases} \quad (2.54)$$

we have

$$u_\epsilon^m(T) \rightarrow u(T), \text{ weakly in } V. \quad (2.55)$$

We only have to verify that u is the solution of (2.17)-(2.18), let $w \in L^2(0, T; V)$ and $\{w^m\} \in L^2(0, T; V^m)$ be a sequence such that $w^m \rightarrow w$ strongly in $L^2(0, T; V)$. Let us remark that the

convexity of j_ϵ and the inequality (2.27) imply that

$$\langle J_\epsilon(z), w - z \rangle_{V', V} \leq j_\epsilon(w) - j_\epsilon(z) \leq j(w) - j(z) + 2C\epsilon, \forall z, w \in V. \quad (2.56)$$

If we use the inequality (2.56), we take $v = w^m(t) - \dot{u}_\epsilon^m(t)$ in (2.29) and integrate from 0 to T , we get

$$\begin{aligned} & 2\epsilon CT + \int_0^T ((\ddot{u}_\epsilon^m(t), w^m(t) - \dot{u}_\epsilon^m(t)) + \langle A\varepsilon(\dot{u}_\epsilon^m(t)), \varepsilon(w^m(t) - \dot{u}_\epsilon^m(t)) \rangle_{\mathcal{H}}) dt \\ & + \int_0^T \langle G\varepsilon(u_\epsilon^m(t)), \varepsilon(w^m(t) - \dot{u}_\epsilon^m(t)) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s)B\varepsilon(u_\epsilon^m(s)) ds, \varepsilon(w^m(t) - \dot{u}_\epsilon^m(t)) \right\rangle_{\mathcal{H}} dt \\ & + \int_0^T (j(w^m(t)) - j(\dot{u}_\epsilon^m(t))) dt \\ & \geq \int_0^T \langle f(t), w^m(t) - \dot{u}_\epsilon^m(t) \rangle_{V' \times V} dt. \end{aligned}$$

The last inequality implies

$$\begin{aligned} & 2\epsilon CT + \int_0^T ((\ddot{u}_\epsilon^m(t), w^m(t)) + \langle A\varepsilon(\dot{u}_\epsilon^m(t)), \varepsilon(w^m(t)) \rangle_{\mathcal{H}}) dt \\ & + \int_0^T \langle G\varepsilon(u_\epsilon^m(t)), \varepsilon(w^m(t)) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s)B\varepsilon(u_\epsilon^m(s)) ds, \varepsilon(w^m(t)) \right\rangle_{\mathcal{H}} dt \\ & + \frac{1}{2} \left(\|u_1\|^2 - \|\dot{u}_\epsilon^m(T)\|^2 + \langle G\varepsilon(u_0), \varepsilon(u_0) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s)B\varepsilon(u_0) ds, \varepsilon(u_0) \right\rangle_{\mathcal{H}} \right) \\ & + \int_0^T (j(w^m(t)) - j(\dot{u}_\epsilon^m(t))) dt - \int_0^T \langle f(t), w^m(t) - \dot{u}_\epsilon^m(t) \rangle_{V', V} dt \\ & \geq \frac{1}{2} \left(\langle G\varepsilon(u_\epsilon^m(T)), \varepsilon(u_\epsilon^m(T)) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s)B\varepsilon(u_\epsilon^m(T)) ds, \varepsilon(u_\epsilon^m(T)) \right\rangle_{\mathcal{H}} \right) \\ & + \int_0^T \langle A\varepsilon(\dot{u}_\epsilon^m(t)), \varepsilon(\dot{u}_\epsilon^m(t)) \rangle_{\mathcal{H}} dt. \end{aligned} \quad (2.57)$$

Since $w^m \rightarrow w$ strongly in $L^2(0, T; V)$ then having in mind (2.52)-(2.53) and (2.54) we find

$$\int_0^T (j(w^m(t)) - j(\dot{u}_\epsilon^m(t))) dt \rightarrow \int_0^T (j(w(t)) - j(\dot{u}(t))) dt.$$

Therefore, passing to the lower limit in inequality (2.57), first as $m \rightarrow \infty$ and then as $\epsilon \rightarrow 0$, and

using the weak lower semicontinuity, we obtain

$$\begin{aligned}
& \int_0^T ((\ddot{u}(t), w(t)) + \langle A\varepsilon(\dot{u}(t)), \varepsilon(w(t)) \rangle_{\mathcal{H}}) dt \\
& + \int_0^T \langle G\varepsilon(u(t)), \varepsilon(w(t)) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s) B\varepsilon(u(s)) ds, \varepsilon(w(t)) \right\rangle_{\mathcal{H}} dt \\
& + \frac{1}{2} \left(\|u_1\|^2 - \|\dot{u}(T)\|^2 + \langle G\varepsilon(u_0), \varepsilon(u_0) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s) B\varepsilon(u_0) ds, \varepsilon(u_0) \right\rangle_{\mathcal{H}} \right) \\
& + \int_0^T (j(w(t)) - j(\dot{u}(t))) dt - \int_0^T \langle f(t), w(t) - \dot{u}(t) \rangle_{V', V} dt \\
& \geq \liminf_{m \rightarrow +\infty, \epsilon \rightarrow 0} \frac{1}{2} \left(\langle G\varepsilon(u_\epsilon^m(T)), \varepsilon(u_\epsilon^m(T)) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s) B\varepsilon(u_\epsilon^m(T)) ds, \varepsilon(u_\epsilon^m(T)) \right\rangle_{\mathcal{H}} \right) \\
& + \liminf_{m \rightarrow +\infty, \epsilon \rightarrow 0} \int_0^T \langle A\varepsilon(\dot{u}_\epsilon^m(t)), \varepsilon(\dot{u}_\epsilon^m(t)) \rangle_{\mathcal{H}} dt \\
& \geq \frac{1}{2} \left(\langle G\varepsilon(u(T)), \varepsilon(u(T)) \rangle_{\mathcal{H}} - \left\langle \int_0^t g(t-s) B\varepsilon(u(T)) ds, \varepsilon(u(T)) \right\rangle_{\mathcal{H}} \right) \\
& + \int_0^T \langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} dt.
\end{aligned} \tag{2.58}$$

From this inequality, we have

$$\begin{aligned}
& \int_0^T (\langle \ddot{u}(t), w - \dot{u}(t) \rangle_{V' \times V} + \langle A\varepsilon(\dot{u}(t)), \varepsilon(w - \dot{u}(t)) \rangle_{\mathcal{H}} + \langle G\varepsilon(u(t)), \varepsilon(w - \dot{u}(t)) \rangle_{\mathcal{H}}) \\
& - \int_0^T \left(\left\langle \int_0^t g(t-s) B\varepsilon(u(s)) ds, \varepsilon(w - \dot{u}(t)) \right\rangle_{\mathcal{H}} + j(w) - j(\dot{u}(t)) \right) dt \\
& \geq \int_0^T \langle f(t), w - \dot{u}(t) \rangle_{V' \times V} dt.
\end{aligned} \tag{2.59}$$

Uniqueness: Let u_1, u_2 be two solutions of PV with the regularity (2.19) and write $v = u_1 - u_2$, if we write the variational inequality (2.17) successively for u_1 and u_2 , taking $w = \dot{u}_2(t)$ in the first inequality and $w = \dot{u}_1(t)$ in the second one, and add the resulting inequalities, we obtain

$$\begin{aligned}
& \langle \ddot{v}(t), \dot{v}(t) \rangle_{V' \times V} + \langle A\varepsilon(\dot{v}(t)), \varepsilon(\dot{v}(t)) \rangle_{\mathcal{H}} + \langle G\varepsilon(v(t)), \varepsilon(\dot{v}(t)) \rangle_{\mathcal{H}} \\
& \leq \left\langle \int_0^t g(t-s) B\varepsilon(v(s)) ds, \varepsilon(\dot{v}(t)) \right\rangle_{\mathcal{H}}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\dot{v}(t)\|_V^2 + \langle G\varepsilon(v(t)), \varepsilon(\dot{v}(t)) \rangle_{\mathcal{H}} + \langle A\varepsilon(\dot{v}(t)), \varepsilon(\dot{v}(t)) \rangle_{\mathcal{H}} \\
& \leq \left\langle \int_0^t g(t-s) B\varepsilon(v(s)) ds, \varepsilon(\dot{v}(t)) \right\rangle_{\mathcal{H}}.
\end{aligned}$$

By integrating the preceding inequality over time and applying condition (c) of assumptions (A.6) and (A.7), along with the initial conditions $v(0) = \dot{v}(0) = 0$, we conclude that the solution

is unique

$$\frac{1}{2}\|\dot{v}(t)\|_V^2 + (m_A - 2\delta) \int_0^t \|\dot{v}(\phi)\|_V^2 d\phi \leq T \left(\frac{L_G^2}{4\delta} + \frac{\|g\|_{L^1}\|g\|_{L^\infty}L_B^2}{4\delta} \right) \int_0^t \|\dot{v}(s)\|_V^2 ds \quad (2.60)$$

Hence, the last inequality implies that

$$u_1(t) = u_2(t),$$

which completes the proof. $\square$

2.3 Global Existence

In this section, we prove the global existence of solution to system (2.1)-(2.6) using the energy method under assumption (H-3) and (H-4).

Let $\mathcal{E}$ denote the energy functional corresponding to problem (2.1)-(2.6), given by

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \left(\|\dot{u}(t)\|_H^2 + \left(1 - \int_0^t g(s) ds \right) \|u(t)\|_V^2 \right) \\ &\quad + \frac{1}{2} \left((g \diamond u)(t) + e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \|f(t)\|_{V'}^2 \right), \quad \forall t \geq 0. \end{aligned} \quad (2.61)$$

Lemma 2.1. *Assume that (H-3),(H-4),(A.4),(A.6),(A.7),(A.9) and (2.7)-(2.9) hold. Then the energy functional is not decreasing in general satisfies for $\delta > 0$, for all $t \geq 0$*

$$\begin{aligned} \mathcal{E}'(t) &\leq - (m_A - \mathcal{C}_1) \|\dot{u}(t)\|_V^2 + \delta \mathcal{C}_2 \|u(t)\|_V^2 + (\zeta + \delta) \|f(t)\|_{V'}^2, \\ &\quad + \delta \mathcal{C}_3 (g \diamond u)(t) + \frac{1}{2} (g' \diamond u)(t) + \delta e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2, \end{aligned} \quad (2.62)$$

where $\mathcal{C}_1 = (4\delta)^{-1}(7 + C_0^2)$, $\mathcal{C}_2 = (1 - l)^2(1 + L_B^2) + L_G^2 + 1$, $\mathcal{C}_3 = (1 + L_B^2)(1 - l)$.

Proof. Differentiating the equation (2.61), using (2.1)-(2.2) and Green's formula (A.27) and noting

that $w = 0_v$ and $w = 2\dot{u}(t)$, we arrive at

$$\begin{aligned}
\mathcal{E}'(t) &= \langle \ddot{u}(t), \dot{u}(t) \rangle_H - \frac{1}{2}g(t)\|u(t)\|_V^2 + \langle \dot{u}(t), u(t) \rangle_V \left(1 - \int_0^t g(s)ds\right) \\
&\quad + \int_0^t g(t-s) \langle \dot{u}(t), u(t) - u(s) \rangle_V ds + \frac{1}{2}(g' \diamond u)(t) \\
&\quad + e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} - \frac{1}{2}e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \langle f'(t), f(t) \rangle_{V'} \\
&\leq -\langle G\varepsilon(u(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} - \langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \\
&\quad + \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \\
&\quad + \left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} \mu |S| |\dot{u}_\tau| da \\
&\quad + \langle f(t), \dot{u}(t) \rangle_{V' \times V} - \frac{1}{2}g(t)\|u(t)\|_V^2 + \left(1 - \int_0^t g(s)ds\right) \langle \dot{u}(t), u(t) \rangle_V \\
&\quad + \int_0^t g(t-s) \langle \dot{u}(t), u(t) - u(s) \rangle_V ds + \frac{1}{2}(g' \diamond u)(t) \\
&\quad + e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} + \langle f'(t), f(t) \rangle_{V'}.
\end{aligned}$$

From (A.6), we obtain

$$-\langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \leq -m_A \|\dot{u}(t)\|_V^2. \quad (2.63)$$

Using Cauchy–Schwarz’s inequalities, Young’s inequalities along with (A.7), we deduce that

$$-\langle G\varepsilon(u(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta L_G^2 \|u(t)\|_V^2. \quad (2.64)$$

By exploiting Young’s inequality together with (A.9) and assumption (H-4), we get

$$\begin{aligned}
&\left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \\
&\leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta L_B^2 \left(\int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)} \|u(t) - u(s)\|_V ds \right)^2 \\
&\leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta(1-l) L_B^2 (g \diamond u)(t).
\end{aligned} \quad (2.65)$$

From (H-4), it follows that

$$\left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \leq (1-l)^2 L_B^2 \delta \|u(t)\|_V^2 + \frac{1}{4\delta} \|\dot{u}(t)\|_V^2. \quad (2.66)$$

By Young’s inequalities, we have

$$\langle f(t), \dot{u}(t) \rangle_{V' \times V} \leq \delta \|f(t)\|_{V'}^2 + \frac{1}{4\delta} \|\dot{u}(t)\|_V^2. \quad (2.67)$$

From (A.23) together with (2.7), we obtain

$$\begin{aligned} e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} &\leq \delta e^{-2t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \frac{1}{4\delta} \|\dot{u}(t)\|_{L^\infty(\Gamma_3)}^2 \\ &\leq \delta e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \frac{C_0^2}{4\delta} \|\dot{u}(t)\|_V^2. \end{aligned} \quad (2.68)$$

By Cauchy-Schwarz's and Young's inequalities along with (H-4), it follows that

$$\left(1 - \int_0^t g(s) ds\right) \langle \dot{u}(t), u(t) \rangle_V \leq \frac{1}{2\delta} \|\dot{u}(t)\|_V^2 + \delta \left(1 + (1-l)^2\right) \|u(t)\|_V^2, \quad (2.69)$$

and

$$\int_0^t g(t-s) \langle \dot{u}(t), u(t) - u(s) \rangle_V ds \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + (1-l)\delta(g \diamond u)(t). \quad (2.70)$$

From (H-3), we infer that

$$\langle f'(t), f(t) \rangle_{V'} \leq \zeta \|f(t)\|_{V'}^2. \quad (2.71)$$

By combining estimates (2.63) through (2.71), and applying assumption (H-3) and (H-4), we finally obtain the desired estimate (2.62). $\square$

The following theorem establishes global existence.

Theorem 2.2. *Assume that $u(x, t)$ is a weak solution of problem (2.1)-(2.6) belonging to the class $u \in W^{1,\infty}(0, T; V)$, assumptions (H-3),(H-4),(A.4),(A.6),(A.7),(A.9) and (2.7)-(2.9) hold. Then, $u(x, t)$ is global in time.*

Proof of Theorem 2.2. From (2.62), we have

$$\begin{aligned} \mathcal{E}'(t) &\leq - (m_A - \mathcal{C}_1) \|\dot{u}(t)\|_V^2 + \delta \mathcal{C}_2 \|u(t)\|_V^2 + \delta \mathcal{C}_3 \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds(t) \\ &\quad + \delta e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + (\zeta + \delta) \|f(t)\|_{V'}^2. \end{aligned} \quad (2.72)$$

Together with equation (2.72), for δ large enough, we obtain

$$m_A - \frac{7 + C_0^2}{4\delta} > 0,$$

to derive

$$\mathcal{E}'(t) \leq \mathcal{C}_4 \left(\|u(t)\|_V^2 + \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds + \|f(t)\|_{V'}^2 + e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 \right), \quad (2.73)$$

where $\mathcal{C}_4 = \max \{\delta \mathcal{C}_2, \delta \mathcal{C}_3, \zeta + \delta\}$.

From (2.61), it is inferred that

$$\mathcal{E}(t) \geq \mathcal{C}_5 \left(\|u(t)\|_V^2 + \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds + \|f(t)\|_{V'}^2 + e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 \right), \quad (2.74)$$

where $\mathcal{C}_5 = \min \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\}$. Equations (2.73) and (2.74) imply that

$$\mathcal{E}'(t) \leq \mathcal{C}_5 \mathcal{E}(t), \quad (2.75)$$

where $\mathcal{C}_6 = \frac{\mathcal{C}_4}{\mathcal{C}_5}$. And therefore,

$$\mathcal{E}(t) \leq \mathcal{E}(0) e^{\mathcal{C}_6 t}. \quad (2.76)$$

Since the solution exists locally, and in view of inequality (2.76), the continuation principle [56] ensures that the solution can be extended globally. This completes the proof of global existence. $\square$

2.4 Exponential Stability

In this section, we state and prove the exponential stability result for the solution. We first establish some technical lemmas that will be used to prove our main result.

We define the modified Lyapunov functional $\mathcal{L}(t)$ as

$$\mathcal{L}(t) = \mathcal{E}(t) + \epsilon(N\mathcal{F}_1(t) + \mathcal{F}_2(t)), \quad (2.77)$$

where ϵ, N are positive constants to be chosen later and

$$\mathcal{F}_1(t) = \left(- \int_0^t g(s) ds \right) \|f(t)\|_{V'}^2 + e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \langle \dot{u}(t), u(t) \rangle_H, \quad \forall t \geq 0, \quad (2.78)$$

$$\mathcal{F}_2(t) = - \left\langle \dot{u}(t), \int_0^t g(t-s) (u(t) - u(s)) ds \right\rangle_H, \quad \forall t \geq 0. \quad (2.79)$$

Lemma 2.2. *Assume that (H-4) holds. Then, there exists two positive constants α_1 and α_2 such that*

$$\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t). \quad (2.80)$$

Proof. From Young's inequalities together with (A.26), we obtain

$$\begin{aligned} |\mathcal{F}_1(t)| &= \left| \left(- \int_0^t g(s) ds \right) \|f(t)\|_{V'}^2 + e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \langle \dot{u}(t), u(t) \rangle_H \right| \\ &\leq (1-l) \|f(t)\|_{V'}^2 + e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \frac{1}{2} \|\dot{u}(t)\|_H^2 + \frac{1}{2} \|u(t)\|_H^2 \\ &\leq \max \left\{ d^2 l^{-1}, 2(1-l), 2 \right\} \mathcal{E}(t), \end{aligned} \quad (2.81)$$

and

$$\begin{aligned}
 |\mathcal{F}_2(t)| &= \left| -\left\langle \dot{u}(t), \int_0^t g(t-s)(u(t)-u(s)) ds \right\rangle_H \right| \\
 &\leq \frac{1}{2} \|\dot{u}(t)\|_H^2 + \frac{(1-l)d^2}{2} \|u(t)\|_V^2 \\
 &\leq \max\{1, (1-l)d^2\} \mathcal{E}(t).
 \end{aligned} \tag{2.82}$$

By combining (2.61), (2.81) and (2.82) we obtain

$$|\mathcal{L}(t) - \mathcal{E}(t)| \leq \epsilon \alpha_3 \mathcal{E}(t),$$

where $\alpha_3 > 0$ is chosen such that

$$\alpha_3 = N \max\{d^2 l^{-1}, 2(1-l), 2\} + \max\{1, (1-l)d^2\}.$$

then by choosing

$$\epsilon < \frac{1}{\alpha_3}, \tag{2.83}$$

we have $\mathcal{L} \sim \mathcal{E}$ with $\alpha_1 = 1 - \epsilon \alpha_3$ and $\alpha_2 = 1 + \epsilon \alpha_3$. $\square$

Lemma 2.3. Assume that (H-3),(H-4),(A.4),(A.6),(A.7),(A.9) and (2.7)-(2.9) hold. Then, the functional $\mathcal{F}_1$ defined by (2.78) satisfies for $\delta_1 > 0$,

$$\begin{aligned}
 \mathcal{F}_1'(t) &\leq (-k_1 + 3\delta_1) \|u(t)\|_V^2 + \mathcal{C}_7(g \diamond u)(t) + \mathcal{C}_8 \|\dot{u}(t)\|_V^2 \\
 &\quad + \left(\frac{1}{4\delta_1} - \gamma\right) \|f(t)\|_{V'}^2 + \left(\frac{1}{2\delta_1} - 1\right) e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2, \text{ for all } t \geq 0,
 \end{aligned} \tag{2.84}$$

where $k_1 = m_G - (1-l)L_B$, $\mathcal{C}_7 = \frac{(1-l)L_B^2}{4\delta_1}$, $\mathcal{C}_8 = \frac{L_A^2}{4\delta_1} + 2C_0^2\delta_1 + d^2$.

Proof. Taking the time derivative of (2.78), we derive

$$\begin{aligned}
 \mathcal{F}_1'(t) &= -g(t) \|f(t)\|_{V'}^2 + 2 \left(-\int_0^t g(s) ds \right) \langle f'(t), f(t) \rangle_{V'} - e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 \\
 &\quad + 2e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} + \|\dot{u}(t)\|_H^2 + \langle \ddot{u}(t), u(t) \rangle_H.
 \end{aligned}$$

Multiplying (2.2) by $w - u(t)$ and using (2.1), $w = 0_v$ and $w = 2u(t)$, assumption (A.7) allows us

to deduce that

$$\begin{aligned}
\mathcal{F}'_1(t) &\leq \|\dot{u}(t)\|_H^2 - \langle A\varepsilon(\dot{u}(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 \\
&\quad + \left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\
&\quad + \left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} \mu |S| |u_\tau| da \\
&\quad + \langle f(t), u(t) \rangle_{V' \times V} - g(t) \|f(t)\|_{V'}^2 + 2 \left(- \int_0^t g(s) ds \right) \langle f'(t), f(t) \rangle_{V'} \\
&\quad - e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + 2e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)}.
\end{aligned} \tag{2.85}$$

It follows from (A.6) that

$$\begin{aligned}
\langle A\varepsilon(\dot{u}(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} &\leq \|A\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\
&\leq \frac{L_A^2}{4\delta_1} \|\dot{u}(t)\|_V^2 + \delta_1 \|u(t)\|_V^2.
\end{aligned}$$

Using Young's inequality, (A.9) and (H-4), it follows that

$$\begin{aligned}
&\left\langle \int_0^t g(t-s) (B\varepsilon(u(s)) - B\varepsilon(u(t))) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\
&\leq \frac{L_B^2}{4\delta_1} \left(\int_0^t g(t-s) \|u(s) - u(t)\|_V ds \right)^2 + \delta_1 \|u(t)\|_V^2 \\
&\leq \frac{(1-l)L_B^2}{4\delta_1} (g \diamond u)(t) + \delta_1 \|u(t)\|_V^2.
\end{aligned}$$

From (A.9) and (H-4), it is concluded that

$$\left\langle \int_0^t g(t-s) B\varepsilon(u(t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \leq L_B(1-l) \|u(t)\|_V^2.$$

Recalling (A.23) and (2.7), we have

$$\begin{aligned}
2e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} &\leq \frac{e^{-t}}{2\delta_1} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + 2e^{-t} \delta_1 \|\dot{u}(t)\|_{L^\infty(\Gamma_3)}^2 \\
&\leq \frac{e^{-t}}{2\delta_1} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + 2C_0^2 \delta_1 \|\dot{u}(t)\|_{V'}^2.
\end{aligned}$$

Exploiting Cauchy-Schwarz's inequality and (H-3)), we obtain

$$\begin{aligned}
2 \left(- \int_0^t g(s) ds \right) \langle f'(t), f(t) \rangle_{V'} &\leq 2(1-l) \|f'(t)\|_{V'} \|f(t)\|_{V'} \\
&\leq 2(1-l) \zeta \|f(t)\|_{V'}^2,
\end{aligned}$$

and

$$\langle f(t), u(t) \rangle_{V' \times V} \leq \frac{1}{4\delta_1} \|f(t)\|_{V'}^2 + \delta_1 \|u(t)\|_V^2.$$

Substituting the above (2.85) and using (A.26), we get

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-k_1 + 3\delta_1) \|u(t)\|_V^2 + \frac{(1-l)L_B^2}{4\delta_1} (g \diamond u)(t) \\ & + \left(\frac{L_A^2}{4\delta_1} + 2C_0^2\delta_1 + d^2 \right) \|\dot{u}(t)\|_V^2 + \left(\frac{1}{4\delta_1} - \gamma \right) \|f(t)\|_V^2 + \left(\frac{1}{2\delta_1} - 1 \right) e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2, \end{aligned}$$

where $\gamma = \min g(t) - 2(1-l)\zeta$. $\square$

Lemma 2.4. Assume that (H-4), (A.4), (A.6), (A.7), (A.9) and (2.7)-(2.9) hold. Then the functional $\mathcal{F}_2$ given by (2.79) satisfies for $\delta_1 > 0$,

$$\begin{aligned} \mathcal{F}'_2(t) \leq & - \int_0^t g(s) ds \|\dot{u}(t)\|_H^2 + \mathcal{C}_9 \delta_1 \|\dot{u}(t)\|_V^2 + \mathcal{C}_{10} \delta_1 \|u(t)\|_V^2 + \mathcal{C}_{11} (g \diamond u)(t) \\ & - \mathcal{C}_{12} (g' \diamond u)(t) + \delta_1 \|f(t)\|_V^2 + 2\delta_1 C_0^2 \mathcal{E}(t), \text{ for all } t \geq 0, \end{aligned} \quad (2.86)$$

where

$$\mathcal{C}_9 = L_A^2 + d^2, \quad \mathcal{C}_{10} = L_G^2 + L_B^2,$$

and

$$\mathcal{C}_{11} = \frac{4(1-l)}{4\delta_1} + \frac{(1-l)^3}{4\delta_1} + L_B(1-l), \quad \mathcal{C}_{12} = \frac{g(0)d^2}{4\delta_1}.$$

Proof. Taking the derivative of (2.79) with respect to t , we get

$$\begin{aligned} \mathcal{F}'_2(t) = & - \left\langle \ddot{u}(t), \int_0^t g(t-s)(u(t) - u(s)) ds \right\rangle_H \\ & - \left\langle \dot{u}(t), \int_0^t g'(t-s)(u(t) - u(s)) ds \right\rangle_H \\ & - \left\langle \dot{u}(t), \int_0^t g(t-s)\dot{u}(t) ds \right\rangle_H. \end{aligned}$$

Using (2.1)-(2.2) together with the Green's formula (A.27), we obtain that

$$\begin{aligned} \mathcal{F}'_2(t) = & \left\langle A\varepsilon(\dot{u}(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s))) ds \right\rangle_{\mathcal{H}} \\ & + \left\langle G\varepsilon(u(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s))) ds \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^t g(t-s)(B\varepsilon(u(t)) - B\varepsilon(u(s))) ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s))) ds \right\rangle_{\mathcal{H}} \\ & - \left\langle \int_0^t g(t-s)B\varepsilon(u(t)) ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s))) ds \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} \mu |S| \left| \int_0^t g(t-s)(u_\tau(t) - u_\tau(s)) ds \right| da - \left\langle f(t), \int_0^t g(t-s)(u(t) - u(s)) ds \right\rangle_{V' \times V} \\ & - \left\langle \dot{u}(t), \int_0^t g'(t-s)(u(t) - u(s)) ds \right\rangle_H - \int_0^t g(s) ds \|\dot{u}(t)\|_H^2. \end{aligned} \quad (2.87)$$

Thanks to (H-4) and (A.6), we get

$$\begin{aligned} & \left\langle A\varepsilon(\dot{u}(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\ & \leq \delta_1 L_A^2 \|\dot{u}(t)\|_V^2 + \frac{(1-l)}{4\delta_1} (g \diamond u)(t). \end{aligned} \quad (2.88)$$

Using (H-4) and equation (A.7), we deduc that

$$\begin{aligned} & \left\langle G\varepsilon(u(t)), \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\ & \leq \delta_1 L_G^2 \|u(t)\|_V^2 + \frac{(1-l)}{4\delta_1} (g \diamond u)(t). \end{aligned} \quad (2.89)$$

By applying Cauchy-Schwarz's inequality together with (H-4) and (A.9), we obtain that

$$\begin{aligned} & \left\langle \int_0^t g(t-s)(B\varepsilon(u(t)) - B\varepsilon(u(s)))ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\ & \leq (1-l)L_B(g \diamond u)(t). \end{aligned} \quad (2.90)$$

By Young's inequalities along with (H-4) and (A.9), it follows that

$$\begin{aligned} & - \left\langle \int_0^t g(t-s)B\varepsilon(u(t))ds, \int_0^t g(t-s)(\varepsilon(u(t)) - \varepsilon(u(s)))ds \right\rangle_{\mathcal{H}} \\ & \leq \delta_1 L_B^2 \|u(t)\|_V^2 + \frac{(1-l)^3}{4\delta_1} (g \diamond u)(t). \end{aligned} \quad (2.91)$$

From (2.7), (H-4), (A.23) and (2.61), one obtains

$$\begin{aligned} & \int_{\Gamma_3} \mu |S| \left| \int_0^t g(t-s)(u_\tau(t) - u_\tau(s))ds \right| da \\ & \leq 2\delta_1 C_0^2 \mathcal{E}(t) + \frac{(1-l)}{4\delta_1} (g \diamond u)(t). \end{aligned} \quad (2.92)$$

Similarly, we obtain

$$- \left\langle f(t), \int_0^t g(t-s)(u(t) - u(s))ds \right\rangle_{V' \times V} \leq \delta_1 \|f(t)\|_{V'}^2 + \frac{(1-l)}{4\delta_1} (g \diamond u)(t), \quad (2.93)$$

and

$$\begin{aligned} & - \left\langle \dot{u}(t), \int_0^t g'(t-s)(u(t) - u(s))ds \right\rangle_H \\ & \leq \|\dot{u}(t)\|_H \left\| \int_0^t g'(t-s)(u(t) - u(s))ds \right\|_H \\ & \leq d^2 \delta_1 \|\dot{u}(t)\|_V^2 - \frac{g(0)d^2}{4\delta_1} (g' \diamond u)(t). \end{aligned} \quad (2.94)$$

Substituting the estimates (2.88)-(2.94) in (2.87), we obtain (2.86). $\square$

We now present the main stability result in the following theorem

Theorem 2.3. Assume that (H-3),(H-4),(A.4),(A.6),(A.7),(A.9) and (2.7)-(2.9) hold. Assume that m_G and m_A satisfies

$$m_G > 3\delta_1 + L_B(1 - l). \quad (2.95)$$

$$m_A > \lambda_0, \quad (2.96)$$

where $\lambda_0 > 0$ is introduced in (2.103), and the kernel of the memory term $g(t)$ satisfies

$$g(t) > 2(1 - l)\zeta. \quad (2.97)$$

Then the energy functional $\mathcal{E}(t)$ of problem (2.1)-(2.6) satisfies for any $t_0 > 0$,

$$\mathcal{E}(t) \leq \lambda_2 e^{-\lambda_1 t}, \quad \forall t \geq t_0, \quad (2.98)$$

where λ_1, λ_2 are two positive constants.

Proof. Because g is positive, we have for any $t_0 > 0$

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds := g_0 > 0, \quad t \geq t_0.$$

By differentiating (2.77) and using (2.62), (2.84), (2.86), we obtain, for all $t \geq t_0$

$$\begin{aligned} \mathcal{L}'(t) &= \mathcal{E}'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\ &\leq -\epsilon \left((g_0 - \delta_1 C_0^2) \|\dot{u}(t)\|_H^2 + \left(N(k_1 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0)) - \frac{\mathcal{C}_2 \delta}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ &\quad -\epsilon \left(\mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) - \frac{\mathcal{C}_3 \delta}{\epsilon} \right) (g \diamond u)(t) \\ &\quad + (\mathcal{C}_1 - m_A + \epsilon(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)) \|\dot{u}(t)\|_V^2 \\ &\quad + \left(\frac{1}{2} - \epsilon \mathcal{C}_{12} \right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds + \epsilon \mathcal{C}_{13} (g \diamond u)(t) \\ &\quad -\epsilon \left(N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon} \right) e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 \\ &\quad -\epsilon \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right) \|f(t)\|_{V'}^2, \end{aligned} \quad (2.99)$$

where $\mathcal{C}_{13} = N\mathcal{C}_7 + \mathcal{C}_{11} + \mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) + \delta_1 C_0^2$.

In view of (2.95), we can choose δ_1 such that

$$\min \left\{ \frac{1}{2}, \frac{1}{4\gamma} \right\} < \delta_1 < \max \left\{ \frac{k_1}{3}, \frac{g_0}{C_0^2} \right\}, \quad (2.100)$$

and select N sufficiently large so that

$$\begin{aligned} N(k_1 - 3\delta_1) - \delta_1(\mathcal{C}_{10} + C_0^2(1 - g_0)) &> 0, \\ N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2) &> 0. \end{aligned} \quad (2.101)$$

Under assumption (H-2), it holds that

$$\begin{aligned} &\left(\frac{1}{2} - \epsilon\mathcal{C}_{12}\right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds + \epsilon\mathcal{C}_{13} \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds \\ &\leq \left(\frac{1}{2} - \epsilon\mathcal{C}_{14}\right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds, \end{aligned}$$

where $\mathcal{C}_{14} = \mathcal{C}_{12} + \frac{\mathcal{C}_{13}}{\alpha}$.

Hence, from (2.99), It follows that

$$\begin{aligned} \mathcal{L}'(t) &= \mathcal{E}'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\ &\leq -\epsilon \left((g_0 - \delta_1 C_0^2) \|\dot{u}(t)\|_H^2 + \left(N(k_1 - 3\delta_1) - \delta_1(\mathcal{C}_{10} + C_0^2(1 - g_0)) - \frac{\mathcal{C}_2\delta}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ &\quad -\epsilon \left(\mathcal{C}_3 \left(N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2) \right) - \frac{\mathcal{C}_3\delta}{\epsilon} \right) (g \diamond u)(t) \\ &\quad + (\mathcal{C}_1 - m_A + \epsilon(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)) \|\dot{u}(t)\|_V^2 \\ &\quad + \left(\frac{1}{2} - \epsilon\mathcal{C}_{14}\right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds \\ &\quad -\epsilon \left(N\left(1 - \frac{1}{2\delta_1}\right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon} \right) e^{-t} \|\mu|S|u(t)\|_{L^\infty(\Gamma_3)}^2 \\ &\quad -\epsilon \left(N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right) \|f(t)\|_{V'}^2. \end{aligned} \quad (2.102)$$

Furthermore, suppose m_A satisfies (2.96) if we choose λ_0 as follows

$$\begin{aligned} \lambda_0 &= \max \left\{ \frac{\delta(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)}{N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2)}, \frac{\mathcal{C}_2\delta(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)}{N(k_1 - 3\delta_1) - \delta_1(\mathcal{C}_{10} + C_0^2(1 - g_0))}, \right. \\ &\quad \left. \frac{\delta(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)}{N\left(1 - \frac{1}{2\delta_1}\right) - \delta_1 C_0^2}, \frac{(\zeta + \delta)(\mathcal{C}_8 N + \delta_1 \mathcal{C}_9)}{N\left(\gamma - \frac{1}{4\delta_1}\right) - \delta_1(1 + C_0^2)} \right\} + \mathcal{C}_1, \end{aligned} \quad (2.103)$$

and δ satisfies

$$\delta < \min \left\{ \frac{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}{\alpha_3}, \frac{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}{2\mathcal{C}_{14}}, \frac{N(k_1 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0))}{\mathcal{C}_2\alpha_3}, \frac{N(k_1 - 3\delta_1) - \delta_1 (C_{10} + C_0^2(1 - g_0))}{2\mathcal{C}_2\mathcal{C}_{14}}, \frac{N\left(1 - \frac{1}{2\delta_1}\right) - \delta_1 C_0^2}{\alpha_3}, \frac{N\left(1 - \frac{1}{2\delta_1}\right) - \delta_1 C_0^2}{2\mathcal{C}_{14}}, \frac{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}{\alpha_3} - \zeta, \frac{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}{2\mathcal{C}_{14}} - \zeta \right\}. \quad (2.104)$$

From (2.104), we have

$$\frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{14}} \right\}, \quad (2.105)$$

and

$$\frac{\mathcal{C}_2\delta}{N(k_1 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0))} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{14}} \right\}, \quad (2.106)$$

and

$$\frac{\delta}{\left(1 - \frac{1}{2\delta_1}\right) N - \delta_1 C_0^2} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{14}} \right\}, \quad (2.107)$$

and

$$\frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{14}} \right\}. \quad (2.108)$$

It follows from (2.103) that

$$\frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}, \quad (2.109)$$

and

$$\frac{\mathcal{C}_2\delta}{N(k_1 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2(1 - g_0))} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}, \quad (2.110)$$

and

$$\frac{\delta}{\left(1 - \frac{1}{2\delta_1}\right) N - \delta_1 C_0^2} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}, \quad (2.111)$$

and

$$\frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}. \quad (2.112)$$

Consequently, combining (2.83) and (2.105)-(2.112) we can take ϵ satisfying

$$\begin{aligned} & \max \left\{ \frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)}, \frac{\mathcal{C}_2 \delta}{N (k_1 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2 (1 - g_0))}, \right. \\ & \quad \left. \frac{\delta}{\left(1 - \frac{1}{2\delta_1}\right) N - \delta_1 C_0^2}, \frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right) N - \delta_1 (1 + C_0^2)} \right\} \\ & < \epsilon < \min \left\{ \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8 N + \delta_1 \mathcal{C}_9}, \frac{1}{2\mathcal{C}_{14}}, \frac{1}{\alpha_3} \right\}. \end{aligned} \quad (2.113)$$

Thus, we get

$$\mathcal{L}'(t) \leq -\epsilon \mathcal{C}_{15} \mathcal{E}(t) + \left(\frac{1}{2} - \epsilon \mathcal{C}_{14}\right) \int_0^t g'(t-s) \|u(t) - u(s)\|_V^2 ds, \quad \forall t \geq t_0,$$

where

$$\begin{aligned} \mathcal{C}_{15} = & 2 \min \left\{ g_0 - \delta_1 C_0^2, \frac{N (k_1 - 3\delta_1) - \delta_1 (\mathcal{C}_{10} + C_0^2 (1 - g_0)) - \frac{\mathcal{C}_2 \delta}{\epsilon}}{1 - g_0}, \right. \\ & \mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) - \frac{\mathcal{C}_3 \delta}{\epsilon}, \\ & \left. N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon}, N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right\} > 0. \end{aligned}$$

Using (H-4), (2.113) and (2.80), we get

$$\mathcal{L}'(t) \leq \frac{-\epsilon \mathcal{C}_{15}}{1 + \epsilon \alpha_3} \mathcal{L}(t). \quad (2.114)$$

Integrating (2.114) from t_0 to t , we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(t_0) e^{-\lambda_1 (t-t_0)}, \quad \forall t \geq t_0, \quad (2.115)$$

where

$$\lambda_1 = \frac{\epsilon \mathcal{C}_{15}}{1 + \epsilon \alpha_3}.$$

Using (2.80) and (2.115), we get

$$\begin{aligned} \mathcal{E}(t) & \leq \frac{1}{1 - \epsilon \alpha_3} \mathcal{L}(t) \\ & \leq \frac{1}{1 - \epsilon \alpha_3} \mathcal{L}(t_0) e^{\lambda_1 t_0} e^{-\lambda_1 t}, \end{aligned}$$

which yields (2.98) with $\lambda_2 = \frac{1}{1 - \epsilon \alpha_3} \mathcal{L}(t_0) e^{\lambda_1 t_0}$. □

A DYNAMIC CONTACT PROBLEM WITH TRESCA'S LAW

The primary objective of this chapter is to investigate the global existence of the solution to a dynamic contact problem governed by Tresca's law and to establish their exponential decay. To achieve this, nonlinear semigroup theory is employed to prove the existence of the solution, while exponential stability is studied through the construction of a suitable Lyapunov functional.

3.1 Problem Statement

In this section, we present a concise formulation of a dynamic frictional contact problem between a viscoelastic body and a foundation, together with the assumptions imposed on the problem data. The classical model is given by the following system

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^\infty g(s)B\varepsilon(u(t-s))ds \quad \text{in } \Omega \times (0, \infty), \quad (3.1)$$

$$\rho\ddot{u} = \text{Div}\sigma(t) + f_0(t) \quad \text{in } \Omega \times (0, \infty), \quad (3.2)$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \quad (3.3)$$

$$\sigma(t)\nu = f_2 \quad \text{on } \Gamma_2 \times (0, \infty), \quad (3.4)$$

$$\begin{cases} u_\nu(t) = 0, & |\sigma_\tau(t)| \leq g, \\ |\sigma_\tau(t)| \leq g \Rightarrow \dot{u}_\tau(t) = 0, \\ |\sigma_\tau(t)| = g \Rightarrow \exists \lambda \geq 0 \text{ such that} \\ \sigma_\tau(t) = -\lambda u_\tau(t), \end{cases} \quad \text{on } \Gamma_3 \times (0, \infty), \quad (3.5)$$

$$u(-t) = u_0(t), \quad \dot{u}(x, 0) = u_1(x) := \partial_t u_0(x, t)|_{t=0} \quad \text{in } \Omega \times [0, \infty). \quad (3.6)$$

Let V denote the closed subspace of H_1 defined by

$$V = \{v \in H_1 \mid v = 0 \quad \text{on } \Gamma_1, \quad v_\nu = 0 \quad \text{on } \Gamma_3\}.$$

In the analysis of the mechanical problem (3.1)-(3.6), we assume that the viscosity operator A , the elasticity operator G , the relaxation tensors B , and the mass density ρ satisfy conditions (A.6), (A.7), (A.9), (A.4), respectively.

The yield limit g satisfies

$$g \in L^\infty(\Gamma_3), \quad g \geq 0, \quad \text{on } \Gamma_3. \quad (3.7)$$

Moreover, the body forces and boundary tractions satisfy the following regularity assumptions

$$f_0(t) \in W^{1,1}(\mathbb{R}_+; H), \quad f_2(t) \in W^{1,1}(\mathbb{R}_+; L^2(\Gamma_2)^d). \quad (3.8)$$

For the initial data, we impose

$$u_0 \in V, \quad u_1 \in H. \quad (3.9)$$

By the Riesz representation theorem, the mapping $f : \mathbb{R}_+ \rightarrow V'$ is defined as

$$\langle f(t), v \rangle_{V' \times V} = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v da. \quad (3.10)$$

Assumptions (3.8) implies that, for each $v \in V$ and for almost every $t \in \mathbb{R}_+$

$$f \in W^{1,1}(\mathbb{R}_+; V'). \quad (3.11)$$

(H-5) The kernel function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is class C^1 , nonincreasing function satisfying

$$g(0) > 0, \quad g_0 = \int_0^\infty g(s)ds < \frac{m_G}{L_B}. \quad (3.12)$$

3.2 Global Existence

In this section, we establish the global well-posedness of problem (3.1)-(3.6).

Following a method devised in [19], we introduce a new auxiliary variable η in order to handle the infinite memory and distributed time delay terms,

$$\eta(x, s, t) = u(x, t) - u(x, t - s), \quad x \in \Omega, \quad t, s \in \mathbb{R}_+.$$

Accordingly, system (3.1)-(3.6) can be transformed into

$$\left\{ \begin{array}{ll} \rho \ddot{u}(t) - \text{Div} \sigma = f_0 & \text{in } \Omega \times (0, \infty), \\ \sigma = A\varepsilon(\dot{u}) + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u - \eta(s))ds & \text{in } \Omega \times (0, \infty), \\ \eta_t = \dot{u} - \eta_s & \text{in } \Omega \times (0, \infty) \times (0, \infty), \\ u(x, t) = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \eta(x, s, t) = 0 & \text{on } \Gamma_1 \times (0, \infty) \times (0, \infty), \\ \sigma \nu = f_2 & \text{on } \Gamma_2 \times (0, \infty), \\ u_v(t) = 0, \quad |\sigma_\tau(t)| \leq g, & \\ |\sigma_\tau(t)| \leq g \Rightarrow \dot{u}_\tau(t) = 0, & \text{on } \Gamma_3 \times (0, \infty), \\ |\sigma_\tau(t)| = g \Rightarrow \exists \lambda \geq 0 \text{ such that} & \\ \sigma_\tau(t) = -\lambda u_\tau(t), & \\ u(-t) = u_0(t), \quad \dot{u}(x, 0) = u_1(x) := \partial_t u_0(x, t)|_{t=0} & \text{in } \Omega \times [0, \infty), \\ \eta(s, 0) = u_0(0) - u_0(s) = \eta_0(s) & \text{for } s \in (0, \infty). \end{array} \right. \quad (3.13)$$

We denote by $U = (u, v, \eta)^T$ with $v = \dot{u}$ and assuming that σ is a regular function (say $\sigma \in C^1(\overline{\Omega}; \mathbb{S}^d)$), let consider the following Hilbert space

$$\mathcal{H} = V \times H \times \mathcal{M}. \quad (3.14)$$

The set $\mathcal{M}$ is defined by

$$\mathcal{M} = \left\{ \chi : \mathbb{R}_+ \rightarrow V, \int_0^{+\infty} g(s) \|\chi(s)\|_V^2 ds < +\infty \right\}, \quad (3.15)$$

equipped with the inner product

$$\langle \chi, \phi \rangle_{\mathcal{M}} = \int_0^{+\infty} g(s) \langle \varepsilon(\chi(s)), \varepsilon(\phi(s)) \rangle_{\mathcal{H}} ds. \quad (3.16)$$

The Hilbert space $\mathcal{H}$ is endowed with the following inner product

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \langle u, \tilde{u} \rangle_V + \langle v, \tilde{v} \rangle_H + \langle \eta, \tilde{\eta} \rangle_{\mathcal{M}}, \quad (3.17)$$

for every $U = (u, v, \eta)^T$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta})^T$ in $\mathcal{H}$.

The system (3.13) can be rewritten

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = F(t), & \forall t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0)^T, \end{cases} \quad (3.18)$$

where the operator $\mathcal{A}$ and the mapping $F(t)$ are defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} -v \\ -\rho^{-1} \text{Div}(A\varepsilon(v) + G\varepsilon(u)) + \rho^{-1} \text{Div}(\int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds) \\ \eta_s - v \end{pmatrix}. \quad (3.19)$$

$$F(t) = \begin{pmatrix} 0 \\ \rho^{-1} f_0(t) \\ 0 \end{pmatrix}, \quad (3.20)$$

and the domain $\mathcal{D}(\mathcal{A})$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \eta)^T \in \mathcal{H}; \ v \in V; \ \eta_s \in \mathcal{M}; \\ \sigma\nu = f_2 \text{ on } \Gamma_2; \\ u_v(t) = 0, \quad |\sigma_\tau(t)| \leq g, \\ |\sigma_\tau(t)| \leq g \Rightarrow \dot{u}_\tau(t) = 0, \\ |\sigma_\tau(t)| = g \Rightarrow \exists \lambda \geq 0 \text{ such that} \\ \sigma_\tau(t) = -\lambda u_\tau(t), \end{array} \cdot \text{ on } \Gamma_3; \right\}. \quad (3.21)$$

The following theorem establishes the global well-posedness of problem (3.18).

Theorem 3.1. *Under the assumptions (H-5),(A.4),(A.6),(A.7),(A.9) and (3.7)-(3.9), for any $U_0 \in \overline{\mathcal{D}(\mathcal{A})}$ and $F \in L^1(\mathbb{R}_+; \mathcal{H})$, system (3.18) admits a unique mild solution satisfying*

$$U \in C(\mathbb{R}_+; \mathcal{H}). \quad (3.22)$$

Proof. To prove Theorem 3.1, it suffices to show that the operator $\mathcal{A} + \omega I$ is m-accretive for some positive ω . The which can be divided into two steps

Step 1. $\mathcal{A} + \omega I$ is accretive operator for some positive constant ω

Let $U_i = (u_i, v_i, \eta_i) \in \mathcal{D}(\mathcal{A})$, $i = 1, 2$. Using (3.17), we obtain

$$\begin{aligned}
 & \langle (\mathcal{A} + \omega I) U_1 - (\mathcal{A} + \omega I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\
 &= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
 &\quad - \left\langle \rho^{-1} \operatorname{Div} (A\varepsilon(v_1) - A\varepsilon(v_2)), v_1 - v_2 \right\rangle_H \\
 &\quad - \left\langle \rho^{-1} \operatorname{Div} (G\varepsilon(u_1) - G\varepsilon(u_2)), v_1 - v_2 \right\rangle_H \\
 &\quad + \left\langle \rho^{-1} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, v_1 - v_2 \right\rangle_H \\
 &\quad - \left\langle \rho^{-1} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, v_1 - v_2 \right\rangle_H \\
 &\quad - \langle -\eta_{1s} + \eta_{2s} + v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\
 &\quad + \omega \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}.
 \end{aligned} \tag{3.23}$$

Thanks to Green's formula (A.27) and boundary conditions, we have

$$\begin{aligned}
 & \langle (\mathcal{A} + \omega I) U_1 - (\mathcal{A} + \omega I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\
 &= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
 &\quad + \langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
 &\quad + \langle G\varepsilon(u_1) - G\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
 &\quad - \left\langle \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
 &\quad + \left\langle \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
 &\quad + \int_{\Gamma_3} (\sigma_{1\tau} - \sigma_{2\tau}) \cdot (u_{1\tau} - u_{2\tau}) da \\
 &\quad - \frac{1}{2} \int_0^\infty g'(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds - \langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\
 &\quad + \omega \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}.
 \end{aligned}$$

Using Cauchy-Schwarz's inequality in combination with Young's inequality, we obtain

$$-\langle v_1 - v_2, u_1 - u_2 \rangle_V \geq -\frac{1}{4\mu^*} \|v_1 - v_2\|_V^2 - \mu^* \|u_1 - u_2\|_V^2. \tag{3.24}$$

From (A.6), it follows that

$$\langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_H \geq m_A \|v_1 - v_2\|_V^2. \tag{3.25}$$

Using Cauchy-Schwarz's inequality together with Young's inequality and (A.7), we conclude

that

$$\langle G\varepsilon(u_1) - G\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_H \geq -\frac{1}{4\mu^*} \|v_1 - v_2\|_V^2 - \mu^* L_G^2 \|u_1 - u_2\|_V^2. \quad (3.26)$$

By applying Young's inequality together with (A.9) and assumption (H-5), we obtain

$$\begin{aligned} & - \left\langle \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds - \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_H \\ & \geq -L_B \int_0^\infty g(s) \|u_1 - u_2\|_V ds \|v_1 - v_2\|_V - L_B \int_0^\infty g(s) \|\eta_1(s) - \eta_2(s)\|_V ds \|v_1 - v_2\|_V \\ & \geq -\frac{1}{2\mu^*} \|v_1 - v_2\|_V^2 - \mu^* L_B^2 g_0^2 \|u_1 - u_2\|_V^2 - \mu^* L_B^2 g_0 \int_0^\infty g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds. \end{aligned} \quad (3.27)$$

From (3.7), we have

$$\int_{\Gamma_3} (\sigma_{1\tau} - \sigma_{2\tau}) \cdot (u_{1\tau} - u_{2\tau}) da \geq 0. \quad (3.28)$$

Moreover, by applying Cauchy-Schwarz's inequality together with Young's inequality and assumption (H-5), we deduce

$$\begin{aligned} -\langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} & \geq -\|v_1 - v_2\|_V \int_0^{+\infty} g(s) \|\eta_1(s) - \eta_2(s)\|_V ds \\ & \geq -\frac{1}{4\mu^*} \|v_1 - v_2\|_V^2 - \mu^* g_0 \int_0^{+\infty} g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds. \end{aligned} \quad (3.29)$$

Combining estimates (3.24)-(3.29), we obtain

$$\begin{aligned} & \langle (\mathcal{A} + \omega I) U_1 - (\mathcal{A} + \omega I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\ & \geq \left(m_A - \frac{5}{4\mu^*} \right) \|v_1 - v_2\|_V^2 \\ & \quad - \mu^* (1 + L_G^2 + g_0^2 L_B^2) \|u_1 - u_2\|_V^2 \\ & \quad - \mu^* (L_B^2 g_0 + g_0) \int_0^\infty g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\ & \quad + \omega \|U_1 - U_2\|_{\mathcal{H}}^2. \end{aligned}$$

Consequently, by taking $\mu^* > \frac{5}{4m_A}$, we obtain

$$\langle (\mathcal{A} + \omega I) U_1 - (\mathcal{A} + \omega I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \geq (\omega - L) \|U_1 - U_2\|_{\mathcal{H}}^2,$$

where $L = \max \{ \mu^* (1 + L_G^2 + g_0^2 L_B^2), \mu^* (L_B^2 g_0 + g_0) \}$. Thus, the operator $\mathcal{A} + \omega I$ is accretive when $\omega > L$.

Step 2. $\lambda I + \mathcal{A}$ is surjective for some positive λ .

It suffices to show that the range of $\lambda I + \mathcal{A}$ is $\mathcal{H}$ for some $\lambda > 0$. We need to show that for any $(I_1, I_2, I_3)^T \in \mathcal{H}$, there exists $U = (u, v, \eta)^T \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I + \mathcal{A})(u, v, \eta)^T = (I_1, I_2, I_3)^T,$$

that is

$$\begin{aligned}\lambda u - v &= I_1 \\ \lambda v - \frac{1}{\rho} \operatorname{Div} (G\varepsilon(u) + A\varepsilon(v)) + \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds \right) &= I_2 \\ \lambda \eta + \eta_s - v &= I_3.\end{aligned}\tag{3.30}$$

Since $v \in V$, the first equation in (3.30) yields

$$u = \frac{I_1 + v}{\lambda}.\tag{3.31}$$

Therefore,

$$u \in V.$$

Integrating the third equation in (3.30) and using the condition $\eta(0) = 0$, we obtain

$$\eta(s) = e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v) d\tau, \quad s \in \mathbb{R}_+.\tag{3.32}$$

Hence, system (3.30) is equivalent to

$$\begin{aligned}& \lambda v - \frac{1}{\rho} \operatorname{Div} \left(G\varepsilon \left(\frac{I_1 + v}{\lambda} \right) + A\varepsilon(v) \right) \\ & + \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B\varepsilon \left(\frac{I_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v) d\tau \right) ds \right) \\ & = I_2.\end{aligned}\tag{3.33}$$

Let $\Phi : V \rightarrow V'$ be the operator defined by

$$\begin{aligned}& \langle \Phi(v), v \rangle_{V' \times V} \\ & = \langle \lambda v, v \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A\varepsilon(v), \varepsilon(v) \right\rangle_{\mathcal{H}} \\ & \quad - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) \right\rangle_{\mathcal{H}} \\ & \quad + \int_{\Gamma_3} \sigma_\tau(v) \cdot v_\tau da - \int_{\Gamma_2} f_2 v da.\end{aligned}\tag{3.34}$$

Therefore, it suffices to prove that the operator $\Phi : V \rightarrow V'$ is surjective. By (Corollary 1.2 page 45 in [7]) it is enough to show that Φ is maximal monotone and coercive.

We first show that $\Phi : V \rightarrow V'$ is maximal monotone. Using Theorem 1.3 in ([7], p.45), it suffices to verify that Φ is monotone and hemicontinuous.

Let us check the monotonicity of Φ , for any $v, \tilde{v} \in V$, we have

$$\begin{aligned}
& \langle \Phi v - \Phi \tilde{v}, v - \tilde{v} \rangle_{V' \times V} \\
&= \langle \lambda v - \lambda \tilde{v}, v - \tilde{v} \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&+ \langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\
&- \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&+ \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&+ \int_{\Gamma_3} (\sigma_\tau(v) - \sigma_\tau(\tilde{v})) \cdot (v_\tau - \tilde{v}_\tau) da.
\end{aligned}$$

Using condition (c) of (A.6) and (A.7), we obtain

$$\langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \geq m_A \|v - \tilde{v}\|_V^2 \quad (3.35)$$

and

$$\left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \geq \frac{1}{\lambda} m_G \|v - \tilde{v}\|_V^2. \quad (3.36)$$

By Cauchy-Schwarz's inequality, condition (b) of (A.9) and assumption (H-5), we have

$$\begin{aligned}
& - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& \geq -\frac{g_0 L_B}{\lambda} \|v - \tilde{v}\|_V^2.
\end{aligned} \quad (3.37)$$

From (3.7), we obtain

$$\int_{\Gamma_3} (\sigma_\tau(v) - \sigma_\tau(\tilde{v})) \cdot (v_\tau - \tilde{v}_\tau) da \geq 0. \quad (3.38)$$

Combining estimates (3.35)-(3.38), we obtain

$$\langle \Phi v - \Phi \tilde{v}, v - \tilde{v} \rangle_{V \times V'} \geq \left(\frac{m_G - g_0 L_B}{\lambda} \right) \|v - \tilde{v}\|_V^2.$$

It then follows from (3.12) that Φ is strongly monotone for some $\lambda > 0$. Consequently, Φ is monotone and coercive.

Next, we prove that Φ is hemicontinuous (see definition 2.2 in [8]).

Let $\tilde{v} \in V$ and $\lambda > 0$. Then

$$\begin{aligned}
& \langle \Phi(v + \theta v_1), \tilde{v} \rangle_{V' \times V} - \langle \Phi(v), \tilde{v} \rangle_{V' \times V} \\
&= \theta \langle \lambda v_1, \tilde{v} \rangle_H + \left\langle G\varepsilon \left(\frac{I_1 + v + \theta v_1}{\lambda} \right) - G\varepsilon \left(\frac{I_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&\quad + \langle A\varepsilon(v + \theta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\
&\quad - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{I_1 + v + \theta v_1}{\lambda} \right) \right. \\
&\quad \left. - e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v + \theta v_1) d\tau \right\rangle_{\mathcal{H}} ds, \varepsilon(\tilde{v}) \Big\rangle_{\mathcal{H}} \\
&\quad + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{I_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} (\sigma_\tau(v_\tau + \theta v_{1\tau}) - \sigma_\tau(v_\tau)) \cdot \tilde{v}_\tau da.
\end{aligned} \tag{3.39}$$

By Cauchy–Schwarz’s inequality and relation (A.26), we obtain

$$\begin{aligned}
|\theta \langle \lambda v_1, \tilde{v} \rangle_H| &\leq |\theta| \lambda \|v_1\|_H \|\tilde{v}\|_H \\
&\leq |\theta| \lambda d^2 \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0.
\end{aligned} \tag{3.40}$$

Moreover, using Cauchy–Schwarz’s inequality and condition (b) of (A.7), we have

$$\left| \left\langle G\varepsilon \left(\frac{I_1 + v + \theta v_1}{\lambda} \right) - G\varepsilon \left(\frac{I_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| \leq \frac{|\theta|}{\lambda} L_G \|v_1\|_V \|\tilde{v}\|_V,$$

$$\xrightarrow{\theta \rightarrow 0} 0. \tag{3.41}$$

Similarly, using (A.6), we obtain

$$|\langle A\varepsilon(v + \theta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}}| \leq |\theta| L_A \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0. \tag{3.42}$$

Using Cauchy–Schwarz’s inequality, assumption (H-5), and condition (b) of (A.9), we have

$$\begin{aligned}
& \left| - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{I_1 + v + \theta v_1}{\lambda} \right) \right. \right. \\
& \quad \left. \left. - e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v + \theta v_1) d\tau \right\rangle_{\mathcal{H}} ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& \quad + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{I_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \Big| \\
& \leq \frac{|\theta| L_B g_0}{\lambda} \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0.
\end{aligned} \tag{3.43}$$

Finally, by Cauchy–Schwarz’s inequality and (A.23), we obtain

$$\left| \int_{\Gamma_3} (\sigma_\tau(v_\tau + \theta v_{1\tau}) - \sigma_\tau(v_\tau)) \cdot \tilde{v}_\tau da \right| \leq C_0^2 |\theta| \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0. \quad (3.44)$$

Combining (3.40)–(3.44), we conclude that

$$|\langle \Phi(v + \theta v_1), \tilde{v} \rangle_{V' \times V} - \langle \Phi(v), \tilde{v} \rangle_{V' \times V}| \xrightarrow{\theta \rightarrow 0} 0. \quad (3.45)$$

Hence, the operator $\Phi : V \rightarrow V'$ is hemicontinuous.

Since Φ is monotone, hemicontinuous, and coercive, it follows that Φ is maximal monotone and coercive, and therefore surjective. Consequently, for any $(I_1, I_2, I_3)^T \in \mathcal{H}$, there exists $v \in V$ satisfying equation (3.33).

Moreover, from (3.32), there exists a constant $C > 0$ such that

$$\|\eta\|_{\mathcal{M}}^2 \leq C \left(\|I_3\|_{\mathcal{M}}^2 + \|v\|_V^2 \right),$$

which implies $\eta \in \mathcal{M}$.

From the third equation of (3.30), there exists $C_* > 0$ such that

$$\|\eta_s\|_{\mathcal{M}}^2 \leq C_* \left(\|I_3\|_{\mathcal{M}}^2 + \|v\|_V^2 + \|\eta\|_{\mathcal{M}}^2 \right),$$

and hence $\eta_s \in \mathcal{M}$.

Therefore, $U = (u, v, \eta)^T \in \mathcal{D}(\mathcal{A})$ satisfies (3.30), and $\lambda I + \mathcal{A}$ is surjective for some $\lambda > 0$.

Finally, by Theorem A.29 in ([6], p.236) and the fact that $F \in L^1(\mathbb{R}_+; \mathcal{H})$, system (3.18) admits a unique mild solution

$$U \in C(\mathbb{R}_+; \mathcal{H}) \quad \text{for every } U_0 \in \overline{\mathcal{D}(\mathcal{A})},$$

which completes the proof of Theorem 3.1. □

3.3 Exponential Stability

In this section, we investigate the exponential stability of the solution to the contact problem governed by Tresca’s law. Assuming that σ is a smooth, we construct an appropriate Lyapunov functional to establish the desired result.

Let $\mathcal{E}$ denote the energy functional defined by

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2 = \frac{1}{2}\left(\|\dot{u}(t)\|_H^2 + \|u(t)\|_V^2 + \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds\right) \\ &\quad + \frac{1}{2}\left(e^{-t}\|gu(t)\|_{L^\infty(\Gamma_3)}^2 + \|f(t)\|_{V'}^2\right), \quad \forall t \geq 0.\end{aligned}\quad (3.46)$$

Lemma 3.1. Assume that (H-3), (H-5), (A.4), (A.6), (A.7), (A.9) and (3.7)-(3.9) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the energy functional (3.46) satisfies for $\delta > 0$, for all $t \geq 0$

$$\begin{aligned}\mathcal{E}'(t) &\leq -(m_A - \mathcal{C}_1) \|\dot{u}(t)\|_V^2 + \delta \mathcal{C}_2 \|u(t)\|_V^2 + \delta \mathcal{C}_3 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \delta e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + (\zeta + \delta) \|f(t)\|_{V'}^2,\end{aligned}\quad (3.47)$$

where $\mathcal{C}_1 = (4\delta)^{-1}(6 + C_0^2)$, $\mathcal{C}_2 = L_G^2 + g_0^2 L_B^2 + 1$, $\mathcal{C}_3 = g_0 + g_0 L_B^2$.

Proof. Differentiating (3.46) with respect to t and using (3.13)₁ and (3.13)₃, together with Green's formula (A.27) taking $w = 0_v$ and $w = 2\dot{u}(t)$, we obtain

$$\begin{aligned}\mathcal{E}'(t) &= \langle \ddot{u}(t), \dot{u}(t) \rangle_H + \langle \dot{u}(t), u(t) \rangle_V + \int_0^{+\infty} g(s) \langle \eta_t(s, t), \eta(s, t) \rangle_V ds \\ &\quad + e^{-t} \langle gu(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} - \frac{1}{2} e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + \langle f_t(t), f(t) \rangle_{V'} \\ &= -\langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} - \langle G\varepsilon(u(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \\ &\quad - \int_{\Gamma_3} g |u_\tau| da + \langle f(t), \dot{u}(t) \rangle_{V' \times V} + \langle \dot{u}(t), u(t) \rangle_V \\ &\quad + \int_0^\infty g(s) \langle \dot{u}(t), \eta(s, t) \rangle_V ds + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + e^{-t} \langle gu(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} + \langle f_t(t), f(t) \rangle_{V'}.\end{aligned}$$

From (A.6), we obtain

$$-\langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \leq -m_A \|\dot{u}(t)\|_V^2. \quad (3.48)$$

Using Cauchy-Schwarz's and Young's inequalities together with (A.7), we have

$$-\langle G\varepsilon(u(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta L_G^2 \|u(t)\|_V^2. \quad (3.49)$$

Applying Young's inequality together with (A.9) and assumption (H-5), we obtain

$$\begin{aligned}&\left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \\ &\leq \left\| \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds \right\|_{\mathcal{H}} \|\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \\ &\leq \frac{1}{2\delta} \|\dot{u}(t)\|_V^2 + \delta g_0^2 L_B^2 \|u(t)\|_V^2 + \delta g_0 L_B^2 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds,\end{aligned}\quad (3.50)$$

and

$$\langle f(t), \dot{u}(t) \rangle_{V' \times V} \leq \delta \|f(t)\|_{V'}^2 + \frac{1}{4\delta} \|\dot{u}(t)\|_V^2. \quad (3.51)$$

From Cauchy-Schwarz's inequalities together with Young's inequalities and (H-5), we have

$$\langle \dot{u}(t), u(t) \rangle_V \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta \|u(t)\|_V^2, \quad (3.52)$$

and

$$\int_0^\infty g(s) \langle \dot{u}(t), \eta(s, t) \rangle_V ds \leq \frac{1}{4\delta} \|\dot{u}(t)\|_V^2 + \delta g_0 \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds, \quad (3.53)$$

From (3.7) and (A.23), we deduce

$$\begin{aligned} e^{-t} \langle gu(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} &\leq \delta e^{-2t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + \frac{1}{4\delta} \|\dot{u}(t)\|_{L^\infty(\Gamma_3)}^2 \\ &\leq \delta e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + \frac{C_0^2}{4\delta} \|\dot{u}(t)\|_V^2. \end{aligned} \quad (3.54)$$

Finally, by assumption (H-3), we have

$$\langle f_t(t), f(t) \rangle_{V'} \leq \zeta \|f(t)\|_{V'}^2. \quad (3.55)$$

Combining estimates (3.48)–(3.55) yields inequality (3.47), which completes the proof. $\square$

Remark 3.1. Inequality (3.47) shows that the energy functional is not decreasing in general.

We now construct a Lyapunov functional $\mathcal{L}(t)$ equivalent to the energy functional by employing the auxiliary functions $(\mathcal{F}_i, i = 1, 2)$, through which the desired result can be established.

Lemma 3.2. Assume that (H-3), (H-5), (A.4), (A.6), (A.7), (A.9) and (3.7)–(3.9) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the functional

$$\mathcal{F}_1(t) = \left(-\int_0^t g(s) ds \right) \|f(t)\|_{V'}^2 + e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + \langle \dot{u}(t), u(t) \rangle_H, \quad \forall t \geq 0, \quad (3.56)$$

satisfies for $\delta_1 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}_1'(t) &\leq (-k_0 + 3\delta_1) \|u(t)\|_V^2 + \mathcal{C}_4 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + \mathcal{C}_5 \|\dot{u}(t)\|_V^2 \\ &\quad + \left(\frac{1}{4\delta_1} - \gamma \right) \|f(t)\|_{V'}^2 + \left(\frac{1}{2\delta_1} - 1 \right) e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2, \end{aligned} \quad (3.57)$$

Where $k_0 = m_G - g_0 L_B$, $\mathcal{C}_4 = \frac{g_0 L_B^2}{4\delta_1}$, $\mathcal{C}_5 = \frac{L_A^2}{4\delta_1} + 2C_0^2 \delta_1 + d^2$.

Proof. Applying the time derivative to (3.56) leads to

$$\begin{aligned} \mathcal{F}_1'(t) &= -g(t) \|f(t)\|_{V'}^2 + 2 \left(-\int_0^t g(s) ds \right) \langle f_t(t), f(t) \rangle_{V'} - e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 \\ &\quad + 2e^{-t} \langle gu(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} + \|\dot{u}(t)\|_H^2 + \langle \ddot{u}(t), u(t) \rangle_H. \end{aligned}$$

Multiplying the first equation of (3.13) by $w - u(t)$, integrating by parts over Ω and recalling Green's formula (A.27) with $w = 0_v$ and $w = 2u(t)$, together with assumption (A.7), we infer

$$\begin{aligned} \mathcal{F}'_1(t) &\leq \|\dot{u}(t)\|_H^2 - \langle A\varepsilon(\dot{u}(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 \\ &\quad + \left\langle \int_0^\infty g(s) B \varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} g |u_\tau| da \\ &\quad + \langle f(t), u(t) \rangle_{V' \times V} - g(t) \|f(t)\|_{V'}^2 + 2 \left(- \int_0^t g(s) ds \right) \langle f_t(t), f(t) \rangle_{V'} \\ &\quad - e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + 2e^{-t} \langle gu(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)}. \end{aligned} \quad (3.58)$$

From (A.6), we infer that

$$\begin{aligned} \langle A\varepsilon(\dot{u}(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} &\leq \|A\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ &\leq \frac{L_A^2}{4\delta_1} \|\dot{u}(t)\|_V^2 + \delta_1 \|u(t)\|_V^2. \end{aligned}$$

By Young's inequality together with (A.9) and (H-5), we get

$$\begin{aligned} &\left\langle \int_0^\infty g(s) B \varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\ &\leq \left\| \int_0^\infty g(s) B \varepsilon(u(t) - \eta(s, t)) ds \right\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ &\leq L_B \int_0^\infty g(s) (\|u(t)\|_V + \|\eta(s, t)\|_V) ds \|u(t)\|_V \\ &\leq (g_0 L_B + \delta_1) \|u(t)\|_V^2 + \frac{L_B^2 g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \end{aligned}$$

It follows from (A.23) and (3.7) that

$$\begin{aligned} 2e^{-\omega t} \langle gu(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} &\leq \frac{e^{-t}}{2\delta_1} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + 2e^{-t} \delta_1 \|\dot{u}(t)\|_{L^\infty(\Gamma_3)}^2 \\ &\leq \frac{e^{-t}}{2\delta_1} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + 2C_0^2 \delta_1 \|\dot{u}(t)\|_V^2. \end{aligned}$$

By (H-3), the following estimates are obtained

$$\begin{aligned} 2 \left(- \int_0^t g(s) ds \right) \langle f_t(t), f(t) \rangle_{V'} &\leq 2g_0 \|f_t(t)\|_{V'} \|f(t)\|_{V'} \\ &\leq 2g_0 \zeta \|f(t)\|_{V'}^2, \end{aligned}$$

and

$$\langle f(t), u(t) \rangle_{V' \times V} \leq \frac{1}{4\delta_1} \|f(t)\|_{V'}^2 + \delta_1 \|u(t)\|_V^2.$$

Substituting the above estimates into (3.58) and using (A.26), we get

$$\begin{aligned} \mathcal{F}'_1(t) \leq & (-k_0 + 3\delta_1) \|u(t)\|_V^2 + \frac{g_0 L_B^2}{4\delta_1} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & + \left(\frac{L_A^2}{4\delta_1} + 2C_0^2 \delta_1 + d^2 \right) \|\dot{u}(t)\|_V^2 + \left(\frac{1}{4\delta_1} - \gamma \right) \|f(t)\|_{V'}^2 + \left(\frac{1}{2\delta_1} - 1 \right) e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2, \end{aligned}$$

where $\gamma = \min g(t) - 2g_0\zeta$. □

Lemma 3.3. Assume that (H-5), (A.4), (A.6), (A.7), (A.9) and (3.7)-(3.9) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional

$$\mathcal{F}_2(t) = - \left\langle \dot{u}(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H, \quad \forall t \geq 0, \quad (3.59)$$

satisfies for $\delta_1 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}'_2(t) \leq & -g_0 \|\dot{u}(t)\|_H^2 + \mathcal{C}_6 \delta_1 \|\dot{u}(t)\|_V^2 + \mathcal{C}_7 \delta_1 \|u(t)\|_V^2 \\ & + \mathcal{C}_8 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ & - \mathcal{C}_9 \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \delta_1 \|f(t)\|_{V'}^2 + 2\delta_1 C_0^2 \mathcal{E}(t), \end{aligned} \quad (3.60)$$

where $\mathcal{C}_6 = L_A^2 + d^2$, $\mathcal{C}_7 = L_G^2 + L_B^2$, $\mathcal{C}_8 = \frac{4g_0}{4\delta_1} + \frac{g_0^3}{4\delta_1} + L_B g_0$, $\mathcal{C}_9 = \frac{g(0)d^2}{4\delta_1}$.

Proof. Taking the derivative of (3.59) with respect to t and using the third equation of (3.13), we get

$$\begin{aligned} \mathcal{F}'_2(t) = & - \left\langle \ddot{u}(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_H - g_0 \|\dot{u}(t)\|_H^2 \\ & + \left\langle \dot{u}(t), \int_0^\infty g(s) \eta_s(s, t) ds \right\rangle_H. \end{aligned}$$

Integrating by parts with respect to s and using $\lim_{s \rightarrow +\infty} g(s) = 0$ and $\eta(0, t) = 0$, we obtain

$$\begin{aligned} \mathcal{F}'_2(t) = & - \left\langle \ddot{u}(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_H - g_0 \|\dot{u}(t)\|_H^2 \\ & - \left\langle \dot{u}(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H. \end{aligned}$$

Using the first equation of (3.13), integrating over Ω and employing Green's formula (A.27)

together with the boundary conditions, we obtain

$$\begin{aligned}
\mathcal{F}'_2(t) &= \left\langle A\varepsilon(\dot{u}(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
&+ \left\langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
&- \left\langle \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
&+ \int_{\Gamma_3} g \left| \int_0^\infty g(s)\eta_\tau(s,t)ds \right| da - \left\langle f(t), \int_0^\infty g(s)\eta(s,t)ds \right\rangle_{V' \times V} \\
&- \left\langle \dot{u}(t), \int_0^\infty g'(s)\eta(s,t)ds \right\rangle_H - g_0 \|\dot{u}(t)\|_H^2.
\end{aligned} \tag{3.61}$$

Thanks to (H-5) and (A.6), we get

$$\begin{aligned}
&\left\langle A\varepsilon(\dot{u}(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
&\leq \|A\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
&\leq \delta_1 L_A^2 \|\dot{u}(t)\|_V^2 + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{3.62}$$

And from (H-5) and (A.7), we infer that

$$\begin{aligned}
&\left\langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
&\leq \|G\varepsilon(u(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
&\leq \delta_1 L_G^2 \|u(t)\|_V^2 + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{3.63}$$

From (H-5) and (A.9), it follows that

$$\begin{aligned}
&- \left\langle \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
&\leq \left\| \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds \right\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
&\leq \delta_1 L_B^2 \|u(t)\|_V^2 + \left(\frac{g_0^3}{4\delta_1} + L_B g_0 \right) \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{3.64}$$

Taking into account (A.23), (H-5), (3.7), and (3.46), it follows that

$$\begin{aligned}
&\int_{\Gamma_3} g \left| \int_0^\infty g(s)\eta_\tau(s,t)ds \right| da \\
&\leq 2\delta_1 C_0^2 \mathcal{E}(t) + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{3.65}$$

Similarly, we get

$$-\left\langle f(t), \int_0^\infty g(s)\eta(s, t)ds \right\rangle_{V' \times V} \leq \delta_1 \|f(t)\|_{V'}^2 + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds, \quad (3.66)$$

and

$$\begin{aligned} & -\left\langle \dot{u}(t), \int_0^\infty g'(s)\eta(s, t)ds \right\rangle_H \\ & \leq \|\dot{u}(t)\|_H \left\| \int_0^\infty g'(s)\eta(s, t)ds \right\|_H \\ & \leq \delta_1 \|\dot{u}(t)\|_H^2 - \frac{g(0)}{4\delta_1} \left(\int_0^\infty g'(s) \|\eta(s, t)\|_H^2 ds \right) \\ & \leq d^2 \delta_1 \|\dot{u}(t)\|_V^2 - \frac{g(0)d^2}{4\delta_1} \left(\int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \right). \end{aligned} \quad (3.67)$$

By substituting the estimates (3.62)-(3.67) into (3.61), we obtain

$$\begin{aligned} \mathcal{F}_2'(t) & \leq -g_0 \|\dot{u}(t)\|_H^2 + \delta_1 (L_A^2 + d^2) \|\dot{u}(t)\|_V^2 \\ & \quad + \delta_1 (L_G^2 + L_B^2) \|u(t)\|_V^2 + \delta_1 \|f(t)\|_{V'}^2 \\ & \quad - \frac{g(0)d^2}{4\delta_1} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds \\ & \quad + \left(\frac{4g_0}{4\delta_1} + \frac{g_0^3}{4\delta_1} + L_B g_0 \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds + 2\delta_1 C_0^2 \mathcal{E}(t). \end{aligned}$$

□

We define the Lyapunov functional as follows

$$\mathcal{L}(t) = \mathcal{E}(t) + \epsilon(N\mathcal{F}_1(t) + \mathcal{F}_2(t)), \quad (3.68)$$

where ϵ, N are positive constants to be chosen later.

Lemma 3.4. *Assume that (H-5) holds. Then, there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t). \quad (3.69)$$

Proof. By Young's inequalities together with (A.26), we obtain

$$\begin{aligned} |\mathcal{F}_1(t)| & = \left| \left(-\int_0^t g(s)ds \right) \|f(t)\|_{V'}^2 + e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 + \langle \dot{u}(t), u(t) \rangle_H \right| \\ & \leq \max(2, 2g_0, d^2) \mathcal{E}(t), \end{aligned} \quad (3.70)$$

and

$$\begin{aligned} |\mathcal{F}_2(t)| &= \left| \left\langle \dot{u}(t), \int_0^{+\infty} g(s)\eta(s,t)ds \right\rangle_H \right| \\ &\leq \max(1, g_0 d^2) \mathcal{E}(t). \end{aligned} \quad (3.71)$$

Combining (3.68), (3.70) and (3.71), we obtain

$$|\mathcal{L}(t) - \mathcal{E}(t)| \leq \epsilon \alpha_3 \mathcal{E}(t),$$

where

$$\alpha_3 = N \max(2, 2g_0, d^2) + \max(1, g_0 d^2).$$

Thus, by choosing

$$\epsilon < \frac{1}{\alpha_3}, \quad (3.72)$$

the functional $\mathcal{L}$ equivalent to $\mathcal{E}$ which completes the proof of Lemma (3.4). $\square$

We are now in a position to state the main exponential stability result.

Theorem 3.2. *Assume that (H-3),(H-5),(A.4),(A.6),(A.7),(A.9) and (3.7)-(3.9) hold. Assume that m_G and m_A satisfies*

$$m_G > 3\delta_1 + g_0 L_B. \quad (3.73)$$

$$m_A > \lambda_0, \quad (3.74)$$

where $\lambda_0 > 0$ is introduced in (3.81), and the kernel of the memory term $g(t)$ satisfies

$$g(t) > 2g_0 \zeta. \quad (3.75)$$

then, for any $U_0 \in \mathcal{D}(\mathcal{A})$, there exist two positive constants λ_1, λ_2 such that the solution of (3.13) satisfies

$$\|U(t)\|_{\mathcal{H}}^2 \leq \lambda_2 e^{-\lambda_1 t}. \quad (3.76)$$

Proof. By differentiating (3.68) and using (3.47), (3.57), (3.60), we obtain

$$\begin{aligned}
\mathcal{L}'(t) &= \mathcal{E}'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\
&\leq -\epsilon \left((g_0 - \delta_1 C_0^2) \|\dot{u}(t)\|_H^2 + \left(N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_7 + C_0^2) - \frac{\mathcal{C}_2 \delta}{\epsilon} \right) \|u(t)\|_V^2 \right) \\
&\quad -\epsilon \left(\mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) - \frac{\mathcal{C}_3 \delta}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
&\quad + (\mathcal{C}_1 - m_A + \epsilon(\mathcal{C}_5 N + \delta_1 \mathcal{C}_6)) \|\dot{u}(t)\|_V^2 \\
&\quad + \left(\frac{1}{2} - \epsilon \mathcal{C}_9 \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon \mathcal{C}_{10} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
&\quad -\epsilon \left(N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon} \right) e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 \\
&\quad -\epsilon \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right) \|f(t)\|_{V'}^2,
\end{aligned} \tag{3.77}$$

where $\mathcal{C}_{10} = N\mathcal{C}_4 + \mathcal{C}_8 + \mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) + \delta_1 C_0^2$.

Using (3.73), we can pick δ_1 such that

$$\min \left\{ \frac{1}{2}, \frac{1}{4\gamma} \right\} < \delta_1 < \max \left\{ \frac{k_0}{3}, \frac{g_0}{C_0^2} \right\}, \tag{3.78}$$

and take N sufficiently large so that

$$\begin{aligned}
N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_7 + C_0^2) &> 0, \\
N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) &> 0.
\end{aligned} \tag{3.79}$$

In view of (H-2), it follows that

$$\begin{aligned}
&\left(\frac{1}{2} - \epsilon \mathcal{C}_9 \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon \mathcal{C}_{10} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
&\leq \left(\frac{1}{2} - \epsilon \mathcal{C}_{11} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,
\end{aligned}$$

where $\mathcal{C}_{11} = \mathcal{C}_9 + \frac{\mathcal{C}_{10}}{\alpha}$.

Hence, (3.77) yields

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\epsilon \left((g_0 - \delta_1 C_0^2) \|\dot{u}(t)\|_H^2 + \left(N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_7 + C_0^2) - \frac{\mathcal{C}_2 \delta}{\epsilon} \right) \|u(t)\|_V^2 \right) \\
& -\epsilon \left(\mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) - \frac{\mathcal{C}_3 \delta}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\
& + (\mathcal{C}_1 - m_A + \epsilon(\mathcal{C}_5 N + \delta_1 \mathcal{C}_6)) \|\dot{u}(t)\|_V^2 \\
& + \left(\frac{1}{2} - \epsilon \mathcal{C}_{11} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\
& -\epsilon \left(N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon} \right) e^{-t} \|gu(t)\|_{L^\infty(\Gamma_3)}^2 \\
& -\epsilon \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right) \|f(t)\|_{V'}^2.
\end{aligned} \tag{3.80}$$

To ensure condition (3.74) holds, we select λ_0 as

$$\begin{aligned}
\lambda_0 = \max & \left\{ \frac{\delta (\mathcal{C}_5 N + \delta_1 \mathcal{C}_6)}{N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2)}, \frac{\mathcal{C}_2 \delta (\mathcal{C}_5 N + \delta_1 \mathcal{C}_6)}{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_7 + C_0^2)}, \right. \\
& \left. \frac{\delta (\mathcal{C}_5 N + \delta_1 \mathcal{C}_6)}{N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2}, \frac{(\zeta + \delta) (\mathcal{C}_5 N + \delta_1 \mathcal{C}_6)}{N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2)} \right\} + C_1,
\end{aligned} \tag{3.81}$$

and δ satisfies

$$\begin{aligned}
\delta < \min & \left\{ \frac{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)}{\alpha_3}, \frac{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)}{2\mathcal{C}_{11}}, \right. \\
& \frac{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_7 + C_0^2)}{\mathcal{C}_2 \alpha_3}, \frac{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_7 + C_0^2)}{2\mathcal{C}_2 \mathcal{C}_{11}}, \\
& \frac{N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2}{\alpha_3}, \frac{N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2}{2\mathcal{C}_{11}}, \\
& \left. \frac{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)}{\alpha_3} - \zeta, \frac{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)}{2\mathcal{C}_{11}} - \zeta \right\},
\end{aligned} \tag{3.82}$$

which leads to

$$\begin{cases} \frac{\delta}{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{11}} \right\}, \\ \frac{\mathcal{C}_2 \delta}{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_7 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{11}} \right\}, \\ \frac{\delta}{\left(1 - \frac{1}{2\delta_1} \right) N - \delta_1 C_0^2} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{11}} \right\}, \\ \frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{11}} \right\}. \end{cases} \tag{3.83}$$

From (3.81), we have

$$\begin{cases} \frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right)N - \delta_1(1+C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8N + \delta_1\mathcal{C}_9}, \\ \frac{\mathcal{C}_2\delta}{N(k_0 - 3\delta_1) - \delta_1(\mathcal{C}_7 + C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8N + \delta_1\mathcal{C}_9}, \\ \frac{\delta}{\left(1 - \frac{1}{2\delta_1}\right)N - \delta_1C_0^2} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8N + \delta_1\mathcal{C}_9}, \\ \frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right)N - \delta_1(1+C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8N + \delta_1\mathcal{C}_9}. \end{cases} \quad (3.84)$$

Therefore, combining (3.72) and (3.83)-(3.84) we can fix ϵ as follows

$$\begin{aligned} & \max \left\{ \frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right)N - \delta_1(1+C_0^2)}, \frac{\mathcal{C}_2\delta}{N(k_0 - 3\delta_1) - \delta_1(\mathcal{C}_7 + C_0^2)}, \right. \\ & \quad \left. \frac{\delta}{\left(1 - \frac{1}{2\delta_1}\right)N - \delta_1C_0^2}, \frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right)N - \delta_1(1+C_0^2)} \right\} \\ & < \epsilon < \min \left\{ \frac{m_A - \mathcal{C}_1}{\mathcal{C}_8N + \delta_1\mathcal{C}_9}, \frac{1}{2\mathcal{C}_{11}}, \frac{1}{\alpha_3} \right\}. \end{aligned} \quad (3.85)$$

Thus, we get

$$\mathcal{L}'(t) \leq -\epsilon\mathcal{C}_{12}\mathcal{E}(t) + \left(\frac{1}{2} - \epsilon\mathcal{C}_{11}\right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,$$

where

$$\mathcal{C}_{12} = 2 \min \begin{pmatrix} g_0 - \delta_1C_0^2 \\ N(k_0 - 3\delta_1) - \delta_1(\mathcal{C}_7 + C_0^2) - \frac{\mathcal{C}_2\delta}{\epsilon} \\ \mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1(1+C_0^2) \right) - \frac{\mathcal{C}_3\delta}{\epsilon} \\ N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1C_0^2 - \frac{\delta}{\epsilon} \\ N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1(1+C_0^2) - \frac{\zeta + \delta}{\epsilon} \end{pmatrix}.$$

Using (H-5), (3.85) and (3.69), we get

$$\mathcal{L}'(t) \leq \frac{-\epsilon\mathcal{C}_{12}}{1 + \epsilon\alpha_3} \mathcal{L}(t). \quad (3.86)$$

Integrating (3.86) from 0 to t , we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\lambda_1 t}, \quad \forall t \in \mathbb{R}_+, \quad (3.87)$$

where $\lambda_1 = \frac{\epsilon\mathcal{C}_{12}}{1 + \epsilon\alpha_3}$.

Using (3.69), (3.85) and (3.87), we get

$$\begin{aligned}\|U(t)\|_{\mathcal{H}}^2 &= 2E(t) \\ &\leq \frac{2}{1-\epsilon\alpha_3}\mathcal{L}(t) \\ &\leq \frac{2}{1-\epsilon\alpha_3}\mathcal{L}(0)e^{-\lambda_1 t},\end{aligned}$$

which yields (3.76) with $\lambda_2 = \frac{2}{1-\epsilon\alpha_3}\mathcal{L}(0)$. □

A DYNAMIC CONTACT PROBLEM WITH THERMO-VISCOELASTIC OF TYPE III

The primary objective of this chapter is to investigate a thermo-viscoelastic contact problem. We establish the global existence of the solution to the system by employing nonlinear semigroup theory combined with arguments of operator theory. Furthermore, under suitable hypotheses on the problem data, we derive an exponential decay estimate for the associated energy functional.

4.1 Problem Statement

In the current chapter, we study the following thermo-viscoelastic contact problem

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^\infty g(s)B\varepsilon(u(t-s)) ds + C\theta \quad \text{in } \Omega \times (0, \infty), \quad (4.1)$$

$$\rho\ddot{u} = \text{Div}\sigma(t) + f_0(t) \quad \text{in } \Omega \times (0, \infty), \quad (4.2)$$

$$\ddot{\theta} - K\Delta\theta - \delta^*\Delta\dot{\theta} + \beta \text{div } \ddot{u} = 0 \quad \text{in } \Omega \times (0, \infty), \quad (4.3)$$

$$u(t) = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \quad (4.4)$$

$$\sigma(t)\nu = f_2 \quad \text{on } \Gamma_2 \times (0, \infty), \quad (4.5)$$

$$\theta = 0 \quad \text{on } \Gamma \times (0, \infty), \quad (4.6)$$

$$\begin{cases} \sigma_\nu = S, & |\sigma_\tau| \leq \mu|\sigma_\nu| \\ |\sigma_\tau| < \mu|\sigma_\nu| \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu|\sigma_\nu| \Rightarrow \exists \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda\dot{u}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0, \infty), \quad (4.7)$$

$$\dot{u}(x, 0) = u_1(x) \quad \text{in } \Omega \times [0, \infty), \quad (4.8)$$

$$u(-t) = u_0(t), \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \theta_1 \quad \text{in } \Omega. \quad (4.9)$$

We introduce the function space V which is the closed subspace of H_1 given by

$$V = \{v \in H_1 \mid v = 0 \quad \text{on } \Gamma_1\}.$$

We assume that the mass density ρ , the viscosity operator A , the elasticity operator G , the relaxation tensors B and the thermal expansion operator C satisfy conditions (A.4), (A.6), (A.7), (A.9), (A.11) respectively.

Let $S : \Gamma_3 \rightarrow \mathbb{R}_+$ be given such that

$$S \in L^\infty(\Gamma_3). \quad (4.10)$$

The body forces and the tractions have the following regularity

$$f_0(t) \in W^{1,1}(\mathbb{R}_+; H), \quad f_2(t) \in W^{1,1}(\mathbb{R}_+; L^2(\Gamma_2)^d). \quad (4.11)$$

Finally, we assume that the initial data satisfy

$$u_0 \in V, \quad u_1 \in H. \quad (4.12)$$

By the Riesz representation theorem, we define the mapping $f : \mathbb{R}_+ \rightarrow V'$ by

$$\langle f(t), v \rangle_{V' \times V} = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v da + \int_{\Gamma_3} S \cdot v_\nu da, \quad (4.13)$$

for all $v \in V$, t a.e. $\in \mathbb{R}_+$. It follows from (4.11) that

$$f \in W^{1,1}(\mathbb{R}_+; V'). \quad (4.14)$$

4.2 Global Existence

In this section, we establish the global existence result for the contact problem (4.1)-(4.9). For this purpose, we introduce the new variable φ defined by

$$\varphi(x, t) = \int_0^t \theta(x, s) ds + \phi(x), \quad \forall x \in \Omega, \forall t \geq 0, \quad (4.15)$$

where $\phi \in E$ solves

$$\begin{cases} -K\Delta\phi = \delta^*\Delta\theta_0 - \theta_1 - \beta \operatorname{div} u_1 & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma. \end{cases} \quad (4.16)$$

Integrating (4.3) with respect to t and taking into account (4.15)-(4.16), we obtain

$$\ddot{\varphi} - K\Delta\varphi - \delta^*\Delta\dot{\varphi} + \beta \operatorname{div} \dot{u} = 0.$$

Moreover, as in [19], we use the new variable

$$\eta(x, s, t) = u(x, t) - u(x, t - s), \quad x \in \Omega, \quad t, s \in \mathbb{R}_+.$$

Then, the system (4.1)-(4.9) takes the equivalent form

$$\left\{ \begin{array}{ll} \rho \ddot{u}(t) - \text{Div} \sigma = f_0 & \text{in } \Omega \times (0, \infty), \\ \sigma = A\varepsilon(\dot{u}) + G\varepsilon(u) - \int_0^\infty g(s)B\varepsilon(u - \eta(s))ds + C\dot{\varphi} & \text{in } \Omega \times (0, \infty), \\ \eta_t = \dot{u} - \eta_s & \text{in } \Omega \times (0, \infty) \times (0, \infty), \\ \ddot{\varphi} - K\Delta\varphi - \delta^*\Delta\dot{\varphi} + \beta \text{div } \dot{u} = 0 & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0 & \text{on } \Gamma_1 \times (0, \infty), \\ \eta(x, s, t) = 0 & \text{on } \Gamma_1 \times (0, \infty) \times (0, \infty), \\ \varphi = 0 & \text{on } \Gamma \times (0, \infty), \\ \sigma\nu = f_2 & \text{on } \Gamma_2 \times (0, \infty), \\ \sigma_\nu = S, \quad |\sigma_\tau| \leq \mu|\sigma_\nu| & \\ |\sigma_\tau| < \mu|\sigma_\nu| \Rightarrow \dot{u}_\tau = 0 & \text{on } \Gamma_3 \times (0, \infty), \\ |\sigma_\tau| = \mu|\sigma_\nu| \Rightarrow \exists \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau & \\ \dot{u}(x, 0) = u_1(x) & \text{in } \Omega, \\ u(-t) = u_0(t) & \text{in } \Omega \times [0, \infty), \\ \eta(s, 0) = u_0(0) - u_0(s) = \eta_0(s) & \text{for } s \in (0, \infty). \\ \varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \varphi_1 & \text{in } \Omega \times [0, \infty). \end{array} \right. \quad (4.17)$$

For $U = (u, v, \eta, \varphi, \psi)^T$ with $v = \dot{u}$ and $\psi = \dot{\varphi}$. Assuming that σ is a regular function, let consider the following Hilbert space

$$\mathcal{H} = V \times H \times \mathcal{M} \times E \times L^2(\Omega). \quad (4.18)$$

The Hilbert space $\mathcal{H}$ is endowed with the following inner product

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \langle u, \tilde{u} \rangle_V + \langle v, \tilde{v} \rangle_H + \langle \eta, \tilde{\eta} \rangle_{\mathcal{M}} + \langle \varphi, \tilde{\varphi} \rangle_E + \langle \psi, \tilde{\psi} \rangle_L^2(\Omega), \quad (4.19)$$

for every $U = (u, v, \eta, \varphi, \psi)^T$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\eta}, \tilde{\varphi}, \tilde{\psi})^T$ in $\mathcal{H}$.

The system (4.17) can be rewritten

$$\begin{cases} U_t(t) + \mathcal{A}U(t) = F(t), & \forall t > 0, \\ U(0) = U_0 = (u_0, u_1, \eta_0, \psi_0, \varphi_1)^T, \end{cases} \quad (4.20)$$

where the operator $\mathcal{A}$ and the mapping $F(t)$ are given by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ \eta \\ \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} -v \\ -\rho^{-1} \text{Div}(A\varepsilon(v) + G\varepsilon(u) + C\psi) + \rho^{-1} \text{Div}(\int_0^\infty g(s)B\varepsilon(u - \eta(s))ds) \\ \eta_s - v \\ -\psi \\ -K\Delta\varphi - \delta^*\Delta\psi + \beta \text{div } v \end{pmatrix}. \quad (4.21)$$

$$F(t) = \begin{pmatrix} 0 \\ \rho^{-1}f_0(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (4.22)$$

and the domain $\mathcal{D}(\mathcal{A})$ given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \eta, \varphi, \psi)^T \in \mathcal{H}; v \in V; \psi \in E, \eta_s \in \mathcal{M}; \\ K\varphi + \delta^*\psi \in H^2(\Omega), \\ \sigma\nu = f_2 \text{ on } \Gamma_2; \\ \sigma_\nu = S, \quad |\sigma_\tau| \leq \mu|\sigma_\nu| \\ |\sigma_\tau| < \mu|\sigma_\nu| \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu|\sigma_\nu| \Rightarrow \exists \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau \end{array} \right\} . \quad (4.23)$$

The well-posedness of the problem (4.20) is ensured by the theorem that follows.

Theorem 4.1. *Under the assumptions (H-5),(A.4),(A.6),(A.7),(A.9),(A.11) and (4.10)-(4.12), for any $U_0 \in \overline{\mathcal{D}(\mathcal{A})}$ and $F \in L^1(\mathbb{R}_+; \mathcal{H})$, system (4.20) admits a unique mild solution satisfying*

$$U \in C(\mathbb{R}_+; \mathcal{H}). \quad (4.24)$$

Proof. To prove Theorem 4.1, we need to show the operator $\mathcal{A} + \omega I$ is m-accretivity for some $\omega > 0$.

First, we prove the accretivity of the operator $\mathcal{A} + \omega I$ for some $\omega > 0$.

For any $U_i = (u_i, v_i, \eta_i \varphi_i, \psi_i) \in \mathcal{D}(\mathcal{A})$, $i = 1, 2$. We have

$$\begin{aligned}
& \langle (\mathcal{A} + \omega I) U_1 - (\mathcal{A} + \omega I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\
&= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
&\quad - \left\langle \rho^{-1} \operatorname{Div} (A\varepsilon(v_1) - A\varepsilon(v_2)), v_1 - v_2 \right\rangle_H \\
&\quad - \left\langle \rho^{-1} \operatorname{Div} (G\varepsilon(u_1) - G\varepsilon(u_2)), v_1 - v_2 \right\rangle_H \\
&\quad + \left\langle \frac{1}{\rho} \operatorname{Div} (C\psi_1 - C\psi_2), v_1 - v_2 \right\rangle_H \\
&\quad + \left\langle \rho^{-1} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, v_1 - v_2 \right\rangle_H \\
&\quad - \left\langle \rho^{-1} \operatorname{Div} \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, v_1 - v_2 \right\rangle_H \\
&\quad - \langle -\eta_{1s} + \eta_{2s} + v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\
&\quad - K \langle \psi_1 - \psi_2, \varphi_1 - \varphi_2 \rangle_E \\
&\quad - K \langle \Delta \varphi_1 - \Delta \varphi_2, \psi_1 - \psi_2 \rangle_{L^2(\Omega)} \\
&\quad - \delta^* \langle \Delta \psi_1 - \Delta \psi_2, \psi_1 - \psi_2 \rangle_{L^2(\Omega)} \\
&\quad + \beta \langle \operatorname{div} v_1 - \operatorname{div} v_2, \psi_1 - \psi_2 \rangle_{L^2(\Omega)} \\
&\quad + \omega \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}.
\end{aligned} \tag{4.25}$$

(4.26)

Using Green's formula (A.27) and boundary conditions, we obtain

$$\begin{aligned}
& \langle (\mathcal{A} + \omega I) U_1 - (\mathcal{A} + \omega I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\
&= -\langle v_1 - v_2, u_1 - u_2 \rangle_V \\
&\quad + \langle A\varepsilon(v_1) - A\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
&\quad + \langle G\varepsilon(u_1) - G\varepsilon(u_2), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\
&\quad + \langle C\psi_1 - C\psi_2, \varepsilon(v_1) - \varepsilon(v_2) \rangle_H \\
&\quad - \left\langle \int_0^\infty g(s) B\varepsilon(u_1 - \eta_1(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
&\quad + \left\langle \int_0^\infty g(s) B\varepsilon(u_2 - \eta_2(s)) ds, \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} (\sigma_{1\tau} - \sigma_{2\tau}) \cdot (u_{1\tau} - u_{2\tau}) da \\
&\quad + \int_{\Gamma_3} (\sigma_{1\nu} - \sigma_{2\nu}) \cdot (u_{1\nu} - u_{2\nu}) da \\
&\quad - \frac{1}{2} \int_0^\infty g'(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds - \langle v_1 - v_2, \eta_1 - \eta_2 \rangle_{\mathcal{M}} \\
&\quad - K \langle \psi_1 - \psi_2, \varphi_1 - \varphi_2 \rangle_E \\
&\quad + K \langle \nabla \varphi_1 - \nabla \varphi_2, \nabla \psi_1 - \nabla \psi_2 \rangle_{L^2(\Omega)} \\
&\quad + \delta^* \langle \nabla \psi_1 - \nabla \psi_2, \nabla \psi_1 - \nabla \psi_2 \rangle_{L^2(\Omega)} \\
&\quad - \beta \langle v_1 - v_2, \nabla \psi_1 - \nabla \psi_2 \rangle_{L^2(\Omega)} \\
&\quad + \omega \langle U_1 - U_2, U_1 - U_2 \rangle_{\mathcal{H}}.
\end{aligned}$$

Using Cauchy-Schwarz's, Young's inequalities and (H-5),(A.6),(A.7),(A.9),(A.11), we obtain

$$\begin{aligned}
& \langle (\mathcal{A} + \omega I) U_1 - (\mathcal{A} + \omega I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \\
&\geq \left(m_A - \frac{6}{4\mu^*} \right) \|v_1 - v_2\|_V^2 \\
&\quad - \mu^* \left(1 + L_G^2 + g_0^2 L_B^2 \right) \|u_1 - u_2\|_V^2 \\
&\quad - \mu^* \left(L_B^2 g_0 + g_0 \right) \int_0^\infty g(s) \|\eta_1(s) - \eta_2(s)\|_V^2 ds \\
&\quad - \left(\frac{1}{4\mu^*} - \delta^* \right) \|\psi_1 - \psi_2\|_E^2 \\
&\quad - \mu^* L_C^2 \|\psi_1 - \psi_2\|_{L^2(\Omega)}^2 \\
&\quad - \frac{\mu^* \beta^2}{\rho_*^2} \|v_1 - v_2\|_H^2 \\
&\quad + \omega \|U_1 - U_2\|_{\mathcal{H}}^2.
\end{aligned}$$

Consequently, by taking $\mu^* > \max\left(\frac{3}{2}m_A, \frac{1}{4\delta^*}\right)$, we obtain

$$\langle (\mathcal{A} + \omega I) U_1 - (\mathcal{A} + \omega I) U_2, U_1 - U_2 \rangle_{\mathcal{H}} \geq (\omega - L) \|U_1 - U_2\|_{\mathcal{H}}^2,$$

where

$$L = \max \left\{ \mu^* \left(1 + L_G^2 + g_0^2 L_B^2 \right), \mu^* \left(L_B^2 g_0 + g_0 \right), \mu^* L_C^2, \frac{\mu^* \beta^2}{\rho_*^2} \right\}.$$

This implies that $\mathcal{A} + \omega I$ is accretive provided $\omega > L$.

Next it suffices to show that the range of $\lambda I + \mathcal{A}$ for some $\lambda > 0$ is $\mathcal{H}$. In other words, we need to prove that for any $(I_1, I_2, I_3, I_4, I_5)^T \in \mathcal{H}$, there exists $U = (u, v, \eta, \varphi, \psi)^T \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I + \mathcal{A})(u, v, \eta, \varphi, \psi)^T = (I_1, I_2, I_3, I_4, I_5)^T.$$

This system can be written as

$$\begin{aligned} \lambda u - v &= I_1, \\ \lambda v - \frac{1}{\rho} \operatorname{Div} (G\varepsilon(u) + A\varepsilon(v) + C\psi) + \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B\varepsilon(u - \eta(s)) ds \right) &= I_2, \\ \lambda \eta + \eta_s - v &= I_3, \\ \lambda \varphi - \psi &= I_4, \\ \lambda \psi - K \Delta \varphi - \delta^* \Delta \psi + \beta \operatorname{div} v &= I_5. \end{aligned} \tag{4.27}$$

Suppose $(v, \psi) \in V \times E$. Then, from (4.27)₁ and (4.27)₄, we obtain

$$u = \frac{I_1 + v}{\lambda}, \quad \varphi = \frac{I_4 + \psi}{\lambda}. \tag{4.28}$$

Thus

$$(u, \varphi) \in V \times E.$$

Integrating (4.27)₃ with $\eta(0) = 0$, we get

$$\eta(s) = e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v) d\tau, \quad s \in \mathbb{R}_+. \tag{4.29}$$

Equation (4.27)₅ is equivalent to

$$\lambda \psi - \frac{K}{\lambda} \Delta \psi - \delta^* \Delta \psi + \beta \operatorname{div} v = I_5 + \frac{K}{\lambda} \Delta I_4. \tag{4.30}$$

Define the operator $T : E \rightarrow E'$ by

$$\begin{aligned} \langle T(\psi), \psi \rangle_{E' \times E} &= \langle \lambda \psi, \psi \rangle_{L^2(\Omega)} + \frac{K}{\lambda} \langle \nabla \psi, \nabla \psi \rangle_{L^2(\Omega)} \\ &\quad + \delta^* \langle \nabla \psi, \nabla \psi \rangle_{L^2(\Omega)} - \beta \langle v, \nabla \psi \rangle_{L^2(\Omega)}. \end{aligned}$$

We aim to establish the surjectivity of $T : E \rightarrow E'$. It suffices to prove that T is monotone, coercive, and hemicontinuous.

Let $\psi, \tilde{\psi} \in E$. Then

$$\begin{aligned} \langle T\psi - T\tilde{\psi}, \psi - \tilde{\psi} \rangle_{E' \times E} &= \langle \lambda(\psi - \tilde{\psi}), \psi - \tilde{\psi} \rangle_{L^2(\Omega)} + \left(\frac{K}{\lambda} + \delta^* \right) \langle \nabla(\psi - \tilde{\psi}), \nabla(\psi - \tilde{\psi}) \rangle_{L^2(\Omega)} \\ &\geq \left(\frac{K}{\lambda} + \delta^* \right) \|\psi - \tilde{\psi}\|_E^2. \end{aligned}$$

Thus, T is strongly monotone for $\lambda > 0$, hence monotone and coercive.

By Definition A.1, we show

$$T(\psi + \zeta\psi_1) \rightarrow T(\psi) \quad \text{as } \zeta \rightarrow 0, \quad \forall \psi, \psi_1 \in E.$$

Let $\tilde{\psi} \in E$ and $\lambda > 0$. Then

$$\langle T(\psi + \zeta\psi_1), \tilde{\psi} \rangle_{E' \times E} - \langle T(\psi), \tilde{\psi} \rangle_{E' \times E} = \zeta \langle \lambda\psi_1, \tilde{\psi} \rangle_{L^2(\Omega)} + \zeta \left(\frac{K}{\lambda} + \delta^* \right) \langle \nabla\psi_1, \nabla\tilde{\psi} \rangle_{L^2(\Omega)}.$$

Using Cauchy-Schwarz's and Poincaré inequalities, we obtain

$$|\zeta| \left| \langle \lambda\psi_1, \tilde{\psi} \rangle_{L^2(\Omega)} \right| \leq |\zeta| \lambda C_*^2 \|\psi_1\|_E \|\tilde{\psi}\|_E \xrightarrow{\zeta \rightarrow 0} 0, \quad (4.31)$$

and

$$|\zeta| \left(\frac{K}{\lambda} + \delta^* \right) \left| \langle \nabla\psi_1, \nabla\tilde{\psi} \rangle_{L^2(\Omega)} \right| \leq |\zeta| \left(\frac{K}{\lambda} + \delta^* \right) \|\psi_1\|_E \|\tilde{\psi}\|_E \xrightarrow{\zeta \rightarrow 0} 0. \quad (4.32)$$

Combining (4.31)–(4.32), we get

$$\langle T(\psi + \zeta\psi_1), \tilde{\psi} \rangle_{E' \times E} - \langle T(\psi), \tilde{\psi} \rangle_{E' \times E} \xrightarrow{\zeta \rightarrow 0} 0,$$

and hence T is hemicontinuous.

Consequently, there exists $\psi \in E$ satisfying (4.27)₅. Moreover,

$$\begin{aligned} &\langle \lambda\psi, \psi \rangle_{L^2(\Omega)} + \frac{K}{\lambda} \langle \nabla\psi, \nabla\psi \rangle_{L^2(\Omega)} + \delta^* \langle \nabla\psi, \nabla\psi \rangle_{L^2(\Omega)} - \beta \langle v, \nabla\psi \rangle_{L^2(\Omega)} \\ &= \langle I_5 + \frac{K}{\lambda} \Delta I_4, \psi \rangle_{L^2(\Omega)}, \quad \forall \psi \in E. \end{aligned}$$

Hence,

$$\frac{K}{\lambda} \Delta\psi + \delta^* \Delta\psi + \frac{K}{\lambda} \Delta I_4 = \lambda\psi - I_5 + \beta \operatorname{div} v \in L^2(\Omega).$$

By the regularity theory of linear elliptic equations, we get

$$\left(\frac{K}{\lambda} + \delta^* \right) \psi + \frac{K}{\lambda} I_4 \in H^2(\Omega).$$

Since $I_4 = \lambda\varphi - \psi$, it follows that

$$K\varphi + \delta^*\psi \in H^2(\Omega).$$

Note that equation (4.27) is equivalent to

$$\begin{aligned} & \lambda v - \frac{1}{\rho} \operatorname{Div} \left(G\varepsilon \left(\frac{I_1 + v}{\lambda} \right) + A\varepsilon(v) \right) + C\psi \\ & + \frac{1}{\rho} \operatorname{Div} \left(\int_0^\infty g(s) B\varepsilon \left(\frac{I_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda\tau} (I_3(\tau) + v) d\tau \right) ds \right) \\ & = I_2. \end{aligned} \quad (4.33)$$

We now introduce the operator $\Phi : V \rightarrow V'$ defined by

$$\begin{aligned} & \langle \Phi(v), v \rangle_{V' \times V} \\ & = \langle \lambda v, v \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) + A\varepsilon(v) + C\psi, \varepsilon(v) \right\rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda\tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} \sigma_\tau(v) \cdot v_\tau da + \int_{\Gamma_3} \sigma_\nu(v) \cdot v_\nu da - \int_{\Gamma_2} f_2 v da. \end{aligned} \quad (4.34)$$

To establish surjectivity of Φ , we proceed by analogy with the arguments used for the operator T .

First, we verify monotonicity. Let $v, \tilde{v} \in V$. Then

$$\begin{aligned} & \langle \Phi v - \Phi \tilde{v}, v - \tilde{v} \rangle_{V' \times V} \\ & = \langle \lambda v - \lambda \tilde{v}, v - \tilde{v} \rangle_H + \left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda\tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda\tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} (\sigma_\tau(v) - \sigma_\tau(\tilde{v})) \cdot (v_\tau - \tilde{v}_\tau) da + \int_{\Gamma_3} (\sigma_\nu(v) - \sigma_\nu(\tilde{v})) \cdot (v_\nu - \tilde{v}_\nu) da. \end{aligned}$$

From (c) of (A.6), (A.7), we obtain

$$\langle A\varepsilon(v) - A\varepsilon(\tilde{v}), \varepsilon(v) - \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \geq m_A \|v - \tilde{v}\|_V^2, \quad (4.35)$$

and

$$\left\langle G\varepsilon \left(\frac{h_1 + v}{\lambda} \right) - G\varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} \right), \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \geq \frac{1}{\lambda} m_G \|v - \tilde{v}\|_V^2. \quad (4.36)$$

From Cauchy-Schwarz's inequality, (b) of (A.9) and (H-5), we have

$$\begin{aligned}
& - \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + v) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& + \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{h_1 + \tilde{v}}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (h_3(\tau) + \tilde{v}) d\tau \right) ds, \varepsilon(v) - \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& \geq - \frac{g_0 L_B}{\lambda} \|v - \tilde{v}\|_V^2.
\end{aligned} \tag{4.37}$$

From (4.10), we obtain

$$\int_{\Gamma_3} (\sigma_\tau(v) - \sigma_\tau(\tilde{v})) \cdot (v_\tau - \tilde{v}_\tau) da \geq 0. \tag{4.38}$$

and

$$\int_{\Gamma_3} (\sigma_\nu(v) - \sigma_\nu(\tilde{v})) \cdot (v_\nu - \tilde{v}_\nu) da \geq 0. \tag{4.39}$$

Combining the estimations (4.36)-(4.38), we obtain

$$\langle \Phi v - \Phi \tilde{v}, v - \tilde{v} \rangle_{V \times V'} \geq \left(\frac{m_G - g_0 L_B}{\lambda} \right) \|v - \tilde{v}\|_V^2,$$

which shows that Φ is strongly monotone for $\lambda > 0$, hence monotone and coercive.

Next, we establish hemicontinuity. For any $v, \tilde{v} \in V$, as $\theta \rightarrow 0$ we have

$$\begin{aligned}
& \langle \Phi(v + \theta v_1), \tilde{v} \rangle_{V' \times V} - \langle \Phi(v), \tilde{v} \rangle_{V' \times V} \\
& = \theta \langle \lambda v_1, \tilde{v} \rangle_H + \left\langle G \varepsilon \left(\frac{I_1 + v + \theta v_1}{\lambda} \right) - G \varepsilon \left(\frac{I_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& \quad + \langle A \varepsilon(v + \theta v_1) - A \varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}} \\
& \quad - \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{I_1 + v + \theta v_1}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v + \theta v_1) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& \quad + \left\langle \int_0^\infty g(s) B \varepsilon \left(\frac{I_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \\
& \quad + \int_{\Gamma_3} (\sigma_\tau(v_\tau + \theta v_{1\tau}) - \sigma_\tau(v_\tau)) \cdot \tilde{v}_\tau da + \int_{\Gamma_3} (\sigma_\nu(v_\nu + \theta v_{1\nu}) - \sigma_\nu(v_\nu)) \cdot \tilde{v}_\nu da.
\end{aligned} \tag{4.40}$$

Applying Cauchy-Schwarz's inequality and (A.26) gives

$$\begin{aligned}
|\theta \langle \lambda v_1, \tilde{v} \rangle_H| & \leq |\theta| \lambda \|v_1\|_H \|\tilde{v}\|_H \\
& \leq |\theta| \lambda d^2 \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0.
\end{aligned} \tag{4.41}$$

Using Cauchy-Schwarz's inequality and (b) of (A.7), we get

$$\left| \left\langle G\varepsilon \left(\frac{I_1 + v + \theta v_1}{\lambda} \right) - G\varepsilon \left(\frac{I_1 + v}{\lambda} \right), \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| \leq \frac{|\theta|}{\lambda} L_G \|v_1\|_V \|\tilde{v}\|_V, \quad (4.42)$$

$$\xrightarrow{\theta \rightarrow 0} 0.$$

Similar to (4.42) and (b) of (A.6) instead of (A.7), we get

$$|\langle A\varepsilon(v + \theta v_1) - A\varepsilon(v), \varepsilon(\tilde{v}) \rangle_{\mathcal{H}}| \leq |\theta| L_A \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0. \quad (4.43)$$

It follows from Cauchy-Schwarz's inequality, (H-5), (b) of (A.9) that

$$\begin{aligned} & \left| - \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{I_1 + v + \theta v_1}{\lambda} \right. \right. \right. \\ & \quad \left. \left. - e^{-\lambda s} \int_0^s e^{\lambda \tau} \left(I_3(\tau) + v + \theta v_1 \right) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right. \\ & \quad \left. + \left\langle \int_0^\infty g(s) B\varepsilon \left(\frac{I_1 + v}{\lambda} - e^{-\lambda s} \int_0^s e^{\lambda \tau} (I_3(\tau) + v) d\tau \right) ds, \varepsilon(\tilde{v}) \right\rangle_{\mathcal{H}} \right| \\ & \leq \frac{|\theta| L_B g_0}{\lambda} \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0. \end{aligned} \quad (4.44)$$

Thanks to Cauchy-Schwarz's inequality, we have

$$\left| \int_{\Gamma_3} (\sigma_\tau(v_\tau + \theta v_{1\tau}) - \sigma_\tau(v_\tau)) \cdot \tilde{v}_\tau da \right| \leq C_0^2 |\theta| \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0. \quad (4.45)$$

and

$$\left| \int_{\Gamma_3} (\sigma_\nu(v_\nu + \theta v_{1\nu}) - \sigma_\nu(v_\nu)) \cdot \tilde{v}_\nu da \right| \leq C_0^2 |\theta| \|v_1\|_V \|\tilde{v}\|_V \xrightarrow{\theta \rightarrow 0} 0. \quad (4.46)$$

Therefore, combining (4.41)-(4.46) yields

$$|\langle \Phi(v + \theta v_1), \tilde{v} \rangle_{V' \times V} - \langle \Phi(v), \tilde{v} \rangle_{V' \times V}| \xrightarrow{\theta \rightarrow 0} 0. \quad (4.47)$$

Consequently, it follows that $\Phi : V \rightarrow V'$ is hemicontinuous.

By exploiting the monotonicity, hemicontinuity and coercivity of Φ , we deduce that Φ is maximal monotone and coercive. This property ensures the surjectivity of the operator. Therefore for any $(I_1, I_2, I_3, I_4, I_5)^T \in \mathcal{H}$, there exists $v \in V$ satisfying equation (4.33).

Finally, from (4.29), one deduces the existence of a constant $C > 0$ such that

$$\|\eta\|_{\mathcal{M}}^2 \leq C (\|I_3\|_{\mathcal{M}}^2 + \|v\|_V^2).$$

Then, it holds that $\eta \in \mathcal{M}$.

It follows from the third equation of (4.27) that there exists $C_* > 0$ such that

$$\|\eta_s\|_{\mathcal{M}}^2 \leq C_* \left(\|I_3\|_{\mathcal{M}}^2 + \|v\|_V^2 + \|\eta\|_{\mathcal{M}}^2 \right).$$

Consequently, the regularity $\eta_s \in \mathcal{M}$ follows.

Therefore, $U = (u, v, \eta)^T \in \mathcal{D}(\mathcal{A})$ which verifies (4.27) and thus $\lambda I + \mathcal{A}$ is surjective for some positive λ .

From Theorem A.29 in ([6], p.236) together with the fact that $F \in L^1(\mathbb{R}_+; \mathcal{H})$, we deduce that the system (4.20) admits a unique mild solution $U \in C(\mathbb{R}_+; \mathcal{H})$ whenever $U_0 \in \overline{\mathcal{D}(\mathcal{A})}$ which completes the proof of Theorem 4.1. $\square$

4.3 Exponential Stability

In the current section, we are concerned with the exponential stability result associated to the energy solution of contact problem (4.20). Assuming that σ is smooth, we are going to establish technical lemmas to prove our result after defining the energy functional $\mathcal{E}$.

The energy functional $\mathcal{E}$ is prescribed by

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \|U(t)\|_{\mathcal{H}}^2 = \frac{1}{2} \left(\|\dot{u}(t)\|_H^2 + \|u(t)\|_V^2 + \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \right) \\ &\quad + \frac{1}{2} \left(e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \|f(t)\|_{V'}^2 + K \|\varphi(t)\|_E^2 + \|\dot{\varphi}(t)\|_{L^2(\Omega)}^2 \right), \quad \forall t \geq 0. \end{aligned} \quad (4.48)$$

Lemma 4.1. *Assume that (H-3),(H-5),(A.4),(A.6),(A.7),(A.9),(A.11) and (4.10)-(4.12) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the energy functional (4.48) satisfies for $\delta > 0$, for all $t \geq 0$*

$$\begin{aligned} \mathcal{E}'(t) &\leq - (m_A - \mathcal{C}_1) \|\dot{u}(t)\|_V^2 + \delta \mathcal{C}_2 \|u(t)\|_V^2 + \delta \mathcal{C}_3 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \delta e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + (\zeta + \delta) \|f(t)\|_{V'}^2, \\ &\quad - (\delta^* - \mathcal{C}_4 \delta) \|\dot{\varphi}(t)\|_E^2 + \mathcal{C}_5 \delta \|\varphi(t)\|_E^2, \end{aligned} \quad (4.49)$$

where $\mathcal{C}_1 = (4\delta)^{-1}(7 + C_0^2)$, $\mathcal{C}_2 = L_G^2 + g_0^2 L_B^2 + 1$, $\mathcal{C}_3 = g_0 + g_0 L_B^2$, $\mathcal{C}_4 = L_C^2 C_*^2$, $\mathcal{C}_5 = \beta^2$.

Proof. Differentiating the equation (4.48) with respect to t , using (4.17)₁, (4.17)₃ and (4.17)₄ re-

calling Green's formula (A.27) and noting that $w = 0_v$ and $w = 2\dot{u}(t)$, it holds that

$$\begin{aligned}
\mathcal{E}'(t) &= \langle \ddot{u}(t), \dot{u}(t) \rangle_H + \langle \dot{u}(t), u(t) \rangle_V + \int_0^{+\infty} g(s) \langle \eta_t(s, t), \eta(s, t) \rangle_V ds \\
&\quad + e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} - \frac{1}{2} e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \langle f_t(t), f(t) \rangle_{V'} \\
&\quad + \langle \dot{\varphi}(t), \dot{\varphi}(t) \rangle_{L^2(\Omega)} + K \langle \dot{\varphi}(t), \varphi(t) \rangle_V \\
&= -\langle A\varepsilon(\dot{u}(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} - \langle G\varepsilon(u(t)), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} - \langle C\dot{\varphi}(t), \varepsilon(\dot{u}(t)) \rangle_{\mathcal{H}} \\
&\quad + \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(\dot{u}(t)) \right\rangle_{\mathcal{H}} \\
&\quad - \int_{\Gamma_3} \mu |S| |u_\tau| da + \langle f(t), \dot{u}(t) \rangle_{V' \times V} + \langle \dot{u}(t), u(t) \rangle_V \\
&\quad + \int_0^\infty g(s) \langle \dot{u}(t), \eta(s, t) \rangle_V ds + \frac{1}{2} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds \\
&\quad + e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} + \langle f_t(t), f(t) \rangle_{V'} + K \langle \nabla \varphi, \nabla \dot{\varphi}(t) \rangle_{L^2(\Omega)} \\
&\quad - K \langle \nabla \varphi, \nabla \dot{\varphi}(t) \rangle_{L^2(\Omega)} - \delta^* \|\nabla \dot{\varphi}(t)\|_{L^2(\Omega)}^2 + \beta \langle \dot{u}(t), \nabla \dot{\varphi}(t) \rangle_{L^2(\Omega)}.
\end{aligned}$$

Exploiting (H-3),(H-5),(A.4),(A.6),(A.7),(A.9),(A.11) and (4.10)-(4.12) and Young's inequality, we obtain the estimate (4.49). $\square$

Remark 4.1. From (4.49), shows that the energy functional $\mathcal{E}$ is not decreasing in general.

Now, we proceed to establish the auxiliary lemmas that are necessary for presenting our stability result

Lemma 4.2. Assume that (H-3),(H-5),(A.4),(A.6),(A.7),(A.9),(A.11) and (4.10)-(4.12) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the functional

$$\mathcal{F}_1(t) = \left(- \int_0^t g(s) ds \right) \|f(t)\|_{V'}^2 + e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \langle \dot{u}(t), u(t) \rangle_H, \quad \forall t \geq 0, \quad (4.50)$$

satisfies for $\delta_1 > 0$, for all $t \geq 0$

$$\begin{aligned}
\mathcal{F}_1'(t) &\leq (-k_0 + 4\delta_1) \|u(t)\|_V^2 + \mathcal{C}_6 \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + \mathcal{C}_7 \|\dot{u}(t)\|_V^2 + \mathcal{C}_8 \|\dot{\varphi}(t)\|_E^2 \\
&\quad + \left(\frac{1}{4\delta_1} - \gamma \right) \|f(t)\|_{V'}^2 + \left(\frac{1}{2\delta_1} - 1 \right) e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2,
\end{aligned} \quad (4.51)$$

where $k_0 = m_G - g_0 L_B$, $\mathcal{C}_6 = \frac{g_0 L_B^2}{4\delta_1}$, $\mathcal{C}_7 = \frac{L_A^2}{4\delta_1} + 2C_0^2 \delta_1 + d^2$, $\mathcal{C}_8 = \frac{L_C^2 C_*^2}{4\delta_1}$.

Proof. By differentiating relation (4.50) with respect to t , we obtain

$$\begin{aligned}
\mathcal{F}_1'(t) &= -g(t) \|f(t)\|_{V'}^2 + 2 \left(- \int_0^t g(s) ds \right) \langle f_t(t), f(t) \rangle_{V'} - e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 \\
&\quad + 2e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} + \|\dot{u}(t)\|_H^2 + \langle \ddot{u}(t), u(t) \rangle_H.
\end{aligned}$$

Multiplying the first equation in (4.17) by $w - u(t)$, integrating over Ω and applying Green's formula (A.27) with $w = 0_v$ and $w = 2u(t)$, together with assumption (A.7), we deduce

$$\begin{aligned} \mathcal{F}'_1(t) &\leq \|\dot{u}(t)\|_H^2 - \langle A\varepsilon(\dot{u}(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} - m_G \|u(t)\|_V^2 - \langle C\dot{\varphi}(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \\ &\quad + \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} - \int_{\Gamma_3} \mu |S| |u_\tau| da \\ &\quad + \langle f(t), u(t) \rangle_{V' \times V} - g(t) \|f(t)\|_{V'}^2 + 2 \left(- \int_0^t g(s) ds \right) \langle f_t(t), f(t) \rangle_{V'} \\ &\quad - e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + 2e^{-t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)}. \end{aligned} \quad (4.52)$$

Applying Young's inequality and (A.6), we estimate

$$\begin{aligned} \langle A\varepsilon(\dot{u}(t)), \varepsilon(u(t)) \rangle_{\mathcal{H}} &\leq \|A\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ &\leq \frac{L_A^2}{4\delta_1} \|\dot{u}(t)\|_V^2 + \delta_1 \|u(t)\|_V^2. \end{aligned}$$

Thanks to Cauchy-Schwarz's, Young's, and Poincaré inequalities, and taking into account (A.11), we deduce

$$\begin{aligned} \langle C\dot{\varphi}(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} &\leq \|C\dot{\varphi}(t)\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ &\leq \frac{L_C^2 C_*^2}{4\delta_1} \|\dot{\varphi}(t)\|_E^2 + \delta_1 \|u(t)\|_V^2. \end{aligned}$$

Using Cauchy-Schwarz's inequality, Young's inequality, (A.9) and (H-5), we obtain

$$\begin{aligned} &\left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \varepsilon(u(t)) \right\rangle_{\mathcal{H}} \\ &\leq \left\| \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds \right\|_{\mathcal{H}} \|\varepsilon(u(t))\|_{\mathcal{H}} \\ &\leq L_B \int_0^\infty g(s) (\|u(t)\|_V + \|\eta(s, t)\|_V) ds \|u(t)\|_V \\ &\leq (g_0 L_B + \delta_1) \|u(t)\|_V^2 + \frac{L_B^2 g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds. \end{aligned}$$

Similarly, by Cauchy-Schwarz's and Young's inequalities, together with (A.23) and (4.10), we deduce

$$\begin{aligned} 2e^{-\omega t} \langle \mu |S| u(t), \dot{u}(t) \rangle_{L^\infty(\Gamma_3)} &\leq \frac{e^{-t}}{2\delta_1} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + 2e^{-t} \delta_1 \|\dot{u}(t)\|_{L^\infty(\Gamma_3)}^2 \\ &\leq \frac{e^{-t}}{2\delta_1} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + 2C_0^2 \delta_1 \|\dot{u}(t)\|_{V'}^2. \end{aligned}$$

Moreover, exploiting (H-3),

$$\begin{aligned} 2 \left(- \int_0^t g(s) ds \right) \langle f_t(t), f(t) \rangle_{V'} &\leq 2g_0 \|f_t(t)\|_{V'} \|f(t)\|_{V'} \\ &\leq 2g_0 \zeta \|f(t)\|_{V'}^2, \end{aligned}$$

Finally, Young's inequality yields

$$\langle f(t), u(t) \rangle_{V' \times V} \leq \frac{1}{4\delta_1} \|f(t)\|_{V'}^2 + \delta_1 \|u(t)\|_V^2.$$

Substituting all these bounds into (4.52) and using (A.26), we arrive at

$$\begin{aligned} \mathcal{F}'_1(t) &\leq (-k_0 + 4\delta_1) \|u(t)\|_V^2 + \frac{g_0 L_B^2}{4\delta_1} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad + \left(\frac{L_A^2}{4\delta_1} + 2C_0^2 \delta_1 + d^2 \right) \|\dot{u}(t)\|_V^2 + \left(\frac{1}{4\delta_1} - \gamma \right) \|f(t)\|_{V'}^2, \\ &\quad + \left(\frac{1}{2\delta_1} - 1 \right) e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \frac{L_C^2 C_*^2}{4\delta_1} \|\dot{\varphi}(t)\|_E^2. \end{aligned}$$

where $\gamma = \min g(t) - 2g_0\zeta$. □

Lemma 4.3. *let $U_0 \in \mathcal{D}(\mathcal{A})$ Then the functional*

$$\mathcal{F}_2(t) = \langle \dot{\varphi}(t), \varphi(t) \rangle_{L^2(\Omega)} + \frac{\delta^*}{2} \|\nabla \varphi(t)\|_{L^2(\Omega)}^2, \quad \forall t \geq 0. \quad (4.53)$$

satisfies for $\delta_1 > 0$, for all $t \geq 0$

$$\mathcal{F}'_2(t) \leq \mathcal{C}_9 \|\dot{u}(t)\|_V^2 - (K - \delta_2) \|\varphi(t)\|_E^2 + \mathcal{C}_{10} \|\dot{\varphi}(t)\|_E^2. \quad (4.54)$$

where $\mathcal{C}_9 = \frac{\beta^2 d^2}{4\delta_2 \rho_*^2}$, $\mathcal{C}_{10} = C_*^2$.

Proof. Differentiating (4.53) with respect to t , we obtain

$$\mathcal{F}'_2(t) = \|\dot{\varphi}(t)\|_{L^2(\Omega)}^2 + \langle \ddot{\varphi}(t), \varphi(t) \rangle_{L^2(\Omega)} + \delta^* \langle \nabla \dot{\varphi}(t), \nabla \varphi(t) \rangle_{L^2(\Omega)}.$$

Multiplying (4.17)₄ by $\varphi(t)$, integrating over Ω , and applying Green's formula together with boundary conditions, yields

$$\mathcal{F}'_2(t) \leq \|\dot{\varphi}(t)\|_{L^2(\Omega)}^2 - K \|\nabla \dot{\varphi}(t)\|_E^2 + \beta \langle \dot{u}(t), \nabla \dot{\varphi}(t) \rangle_{L^2(\Omega)}. \quad (4.55)$$

By Cauchy-Schwarz's, Young's inequalities and (A.4), we deduce

$$\beta \langle \dot{u}(t), \nabla \varphi(t) \rangle_{L^2(\Omega)} \leq \frac{\beta^2 d^2}{4\delta_2 \rho_*^2} \|\dot{u}(t)\|_V^2 + \delta_2 \|\varphi(t)\|_E^2.$$

Substituting this estimate into (4.55), we obtain the desired result.

$$\mathcal{F}'_2(t) \leq \frac{\beta^2 d^2}{4\delta_2 \rho_*^2} \|\dot{u}(t)\|_V^2 - (K - \delta_2) \|\varphi(t)\|_E^2 + C_*^2 \|\dot{\varphi}(t)\|_E^2.$$

□

Lemma 4.4. Assume that (H-5), (A.4), (A.6), (A.7), (A.9), (A.11) and (4.10)-(4.12) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then the functional

$$\mathcal{F}_3(t) = - \left\langle \dot{u}(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H, \quad \forall t \geq 0, \quad (4.56)$$

satisfies for $\delta_1 > 0$, for all $t \geq 0$

$$\begin{aligned} \mathcal{F}_3'(t) \leq & -g_0 \|\dot{u}(t)\|_H^2 + \mathcal{C}_{11} \delta_1 \|\dot{u}(t)\|_V^2 + \mathcal{C}_{12} \delta_1 \|u(t)\|_V^2 \\ & + \mathcal{C}_{13} \int_0^{+\infty} g(s) \|\eta(s, t)\|_V^2 ds + \mathcal{C}_{15} \delta_1 \|\dot{\varphi}(t)\|_E^2 \\ & - \mathcal{C}_{14} \int_0^{+\infty} g'(s) \|\eta(s, t)\|_V^2 ds + \delta_1 \|f(t)\|_{V'}^2 + 2\delta_1 C_0^2 \mathcal{E}(t), \end{aligned} \quad (4.57)$$

where $\mathcal{C}_{11} = L_A^2 + d^2$, $\mathcal{C}_{12} = L_G^2 + L_B^2$, $\mathcal{C}_{13} = \frac{5g_0}{4\delta_1} + \frac{g_0^3}{4\delta_1} + L_B g_0$, $\mathcal{C}_{14} = \frac{g(0)d^2}{4\delta_1}$, $\mathcal{C}_{15} = L_C^2 C_*^2$.

Proof. Differentiating (4.56) with respect to t and considering (4.17)₃, we obtain

$$\begin{aligned} \mathcal{F}_3'(t) = & - \left\langle \ddot{u}(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_H - g_0 \|\dot{u}(t)\|_H^2 \\ & + \left\langle \dot{u}(t), \int_0^\infty g(s) \eta_s(s, t) ds \right\rangle_H. \end{aligned}$$

Integrating the term $\langle \dot{u}(t), \int_0^\infty g(s) \eta_s(s, t) ds \rangle_H$ with respect to s , and using $\eta(0, t) = 0$ together with $\lim_{s \rightarrow +\infty} g(s) = 0$, we deduce

$$\begin{aligned} \mathcal{F}_3'(t) = & - \left\langle \ddot{u}(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_H - g_0 \|\dot{u}(t)\|_H^2 \\ & - \left\langle \dot{u}(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H. \end{aligned}$$

By invoking (4.17)₁, integrating over Ω and applying Green's formula, boundary conditions, it holds that

$$\begin{aligned} \mathcal{F}_3'(t) = & \left\langle A\varepsilon(\dot{u}(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ & + \left\langle G\varepsilon(u(t)), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ & + \left\langle C\dot{\varphi}(t), \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ & - \left\langle \int_0^\infty g(s) B\varepsilon(u(t) - \eta(s, t)) ds, \int_0^\infty g(s) \varepsilon(\eta(s, t)) ds \right\rangle_{\mathcal{H}} \\ & + \int_{\Gamma_3} \mu |S| \left| \int_0^\infty g(s) \eta_\tau(s, t) ds \right| da - \left\langle f(t), \int_0^\infty g(s) \eta(s, t) ds \right\rangle_{V' \times V} \\ & - \left\langle \dot{u}(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H - g_0 \|\dot{u}(t)\|_H^2. \end{aligned} \quad (4.58)$$

Thanks to (H-5) and (A.6), we obtain

$$\begin{aligned}
& \left\langle A\varepsilon(\dot{u}(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \|A\varepsilon(\dot{u}(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq \delta_1 L_A^2 \|\dot{u}(t)\|_V^2 + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.59}$$

And from (H-5) and (A.7), we infer that

$$\begin{aligned}
& \left\langle G\varepsilon(u(t)), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \|G\varepsilon(u(t))\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq \delta_1 L_G^2 \|u(t)\|_V^2 + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.60}$$

By Cauchy-Schwarz's inequality, (H-5) and (A.7), it is deduced that

$$\begin{aligned}
& \left\langle C\dot{\varphi}(t), \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \|C\dot{\varphi}(t)\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq \delta_1 L_C^2 C_*^2 \|\dot{\varphi}(t)\|_E^2 + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.61}$$

It follows from (H-5) and (A.9), that

$$\begin{aligned}
& - \left\langle \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s))ds, \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\rangle_{\mathcal{H}} \\
& \leq \left\| \int_0^\infty g(s)B\varepsilon(u(t) - \eta(s,t))ds \right\|_{\mathcal{H}} \left\| \int_0^\infty g(s)\varepsilon(\eta(s,t))ds \right\|_{\mathcal{H}} \\
& \leq \delta_1 L_B^2 \|u(t)\|_V^2 + \left(\frac{g_0^3}{4\delta_1} + L_B g_0 \right) \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.62}$$

Taking into account (A.23), (H-5), (4.10) and (4.48) it follows that

$$\begin{aligned}
& \int_{\Gamma_3} \mu |S| \left| \int_0^\infty g(s)\eta_\tau(s,t)ds \right| da \\
& \leq 2\delta_1 C_0^2 \mathcal{E}(t) + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds.
\end{aligned} \tag{4.63}$$

Similarly we get

$$- \left\langle f(t), \int_0^\infty g(s)\eta(s,t)ds \right\rangle_{V' \times V} \leq \delta_1 \|f(t)\|_{V'}^2 + \frac{g_0}{4\delta_1} \int_0^\infty g(s) \|\eta(s,t)\|_V^2 ds. \tag{4.64}$$

and

$$\begin{aligned}
& - \left\langle \dot{u}(t), \int_0^\infty g'(s) \eta(s, t) ds \right\rangle_H \\
& \leq \|\dot{u}(t)\|_H \left\| \int_0^\infty g'(s) \eta(s, t) ds \right\|_H \\
& \leq \delta_1 \|\dot{u}(t)\|_H^2 - \frac{g(0)}{4\delta_1} \left(\int_0^\infty g'(s) \|\eta(s, t)\|_H^2 ds \right) \\
& \leq d^2 \delta_1 \|\dot{u}(t)\|_V^2 - \frac{g(0)d^2}{4\delta_1} \left(\int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \right).
\end{aligned} \tag{4.65}$$

Substituting (4.59)-(4.65) into (4.58), we can get (4.57). $\square$

We now define the Lyapunov functional as follows

$$\mathcal{L}(t) = \mathcal{E}(t) + \epsilon(N\mathcal{F}_1(t) + \mathcal{F}_2(t) + \mathcal{F}_3(t)), \tag{4.66}$$

where ϵ, N are positive constants to be chosen later.

Lemma 4.5. Assume that (H-5) holds. Then, there exist two positive constants α_1 and α_2 such that

$$\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t). \tag{4.67}$$

Proof. It follows from Cauchy-Schwarz's, Young's inequalities and (A.26), we have

$$\begin{aligned}
|\mathcal{F}_1(t)| &= \left| \left(- \int_0^t g(s) ds \right) \|f(t)\|_{V'}^2 + e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 + \langle \dot{u}(t), u(t) \rangle_H \right| \\
&\leq \max(2, 2g_0, d^2) \mathcal{E}(t),
\end{aligned} \tag{4.68}$$

and

$$\begin{aligned}
|\mathcal{F}_2(t)| &= \left| \langle \dot{\varphi}(t), \varphi(t) \rangle_{L^2(\Omega)} + \frac{\lambda}{2} \|\nabla \varphi(t)\|_{L^2(\Omega)}^2 \right| \\
&\leq \max(1, C_*^2 + \delta^*) \mathcal{E}(t),
\end{aligned} \tag{4.69}$$

and

$$\begin{aligned}
|\mathcal{F}_3(t)| &= \left| \left\langle \dot{u}(t), \int_0^{+\infty} g(s) \eta(s, t) ds \right\rangle_H \right| \\
&\leq \max(1, g_0 d^2) \mathcal{E}(t).
\end{aligned} \tag{4.70}$$

Combining (4.66) and (4.68)-(4.70), we obtain

$$|\mathcal{L}(t) - \mathcal{E}(t)| \leq \epsilon \alpha_3 \mathcal{E}(t),$$

where the positive constant α_3 is defined by

$$\alpha_3 = N \max(2, 2g_0, d^2) + \max(1, C_*^2 + \delta^*) + \max(1, g_0 d^2).$$

Note that, thanks to

$$\epsilon < \frac{1}{\alpha_3}, \quad (4.71)$$

we get $\mathcal{L} \sim \mathcal{E}$ with $\alpha_1 = 1 - \epsilon\alpha_3$ and $\alpha_2 = 1 + \epsilon\alpha_3$. $\square$

The following theorem provides the result concerning exponential stability issue.

Theorem 4.2. *Assume that (H-3),(H-5),(A.4),(A.6),(A.7),(A.9),(A.11) and (4.10)-(4.12) hold. Assume that m_G and m_A satisfies*

$$m_G > 4\delta_1 + g_0 L_B. \quad (4.72)$$

$$m_A > \lambda_0, \quad (4.73)$$

where $\lambda_0 > 0$ is introduced in (4.81), and the kernel of the memory term $g(t)$ satisfies

$$g(t) > 2g_0\zeta, \quad (4.74)$$

then, for any $U_0 \in \mathcal{D}(\mathcal{A})$, there exist two positive constants λ_1, λ_2 such that the solution of (3.13) satisfies

$$\|U(t)\|_{\mathcal{H}}^2 \leq \lambda_2 e^{-\lambda_1 t}. \quad (4.75)$$

Proof. By differentiating (4.66) and using (4.49), (4.51), (4.54), (4.57), we obtain for all $t \geq 0$

$$\begin{aligned} \mathcal{L}'(t) &= \mathcal{E}'(t) + \epsilon(N\mathcal{F}'_1(t) + \mathcal{F}'_2(t)) \\ &\leq -\epsilon \left((g_0 - \delta_1 C_0^2) \|\dot{u}(t)\|_H^2 + \left(N(k_0 - 4\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2) - \frac{\mathcal{C}_2 \delta}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ &\quad -\epsilon \left(\mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) - \frac{\mathcal{C}_3 \delta}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad -\epsilon \left(\frac{\delta^* - \mathcal{C}_4 \delta}{\epsilon} - \mathcal{C}_8 N - \mathcal{C}_{15} \delta_1 - \mathcal{C}_{10} \right) \|\dot{\varphi}(t)\|_E^2 \\ &\quad -\epsilon \left(K - \delta_2 - \frac{\mathcal{C}_5 \delta}{\epsilon} \right) \|\varphi(t)\|_E^2 \\ &\quad + (\mathcal{C}_1 - m_A + \epsilon(\mathcal{C}_7 N + \mathcal{C}_9 + \delta_1 \mathcal{C}_{11})) \|\dot{u}(t)\|_V^2 \\ &\quad + \left(\frac{1}{2} - \epsilon \mathcal{C}_{14} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon \mathcal{C}_{16} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\ &\quad -\epsilon \left(N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon} \right) e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 \\ &\quad -\epsilon \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right) \|f(t)\|_{V'}^2. \end{aligned} \quad (4.76)$$

where $\mathcal{C}_{16} = N\mathcal{C}_6 + \mathcal{C}_{13} + \mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) + \delta_1 C_0^2$.

Using (4.72), we can pick δ_1 such that

$$\min \left\{ \frac{1}{2}, \frac{1}{4\gamma} \right\} < \delta_1 < \max \left\{ \frac{k_0}{4}, \frac{g_0}{C_0^2} \right\}, \quad (4.77)$$

and therefore N large enough such that

$$\begin{aligned} N(k_0 - 4\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2) &> 0, \\ N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) &> 0. \end{aligned} \quad (4.78)$$

Moreover, we have to choose δ_2 such that

$$K > \delta_2. \quad (4.79)$$

Taking into account (H-2), we obtain

$$\begin{aligned} &\left(\frac{1}{2} - \epsilon \mathcal{C}_{14} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds + \epsilon \mathcal{C}_{16} \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\ &\leq \left(\frac{1}{2} - \epsilon \mathcal{C}_{17} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds, \end{aligned}$$

where $\mathcal{C}_{17} = \mathcal{C}_{14} + \frac{\mathcal{C}_{16}}{\alpha}$.

By inserting the previous inequality into (4.76), we obtain the following

$$\begin{aligned} \mathcal{L}'(t) \leq & -\epsilon \left((g_0 - \delta_1 C_0^2) \|\dot{u}(t)\|_H^2 + \left(N(k_0 - 4\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2) - \frac{\mathcal{C}_2 \delta}{\epsilon} \right) \|u(t)\|_V^2 \right) \\ & -\epsilon \left(\mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) \right) - \frac{\mathcal{C}_3 \delta}{\epsilon} \right) \int_0^\infty g(s) \|\eta(s, t)\|_V^2 ds \\ & -\epsilon \left(\frac{\delta^* - \mathcal{C}_4 \delta}{\epsilon} - \mathcal{C}_8 N - \mathcal{C}_{15} \delta_1 - \mathcal{C}_{10} \right) \|\dot{\varphi}(t)\|_E^2 \\ & -\epsilon \left(K - \delta_2 - \frac{\mathcal{C}_5 \delta}{\epsilon} \right) \|\varphi(t)\|_E^2 \\ & + (\mathcal{C}_1 - m_A + \epsilon (\mathcal{C}_7 N + \mathcal{C}_9 + \delta_1 \mathcal{C}_{11})) \|\dot{u}(t)\|_V^2 \\ & + \left(\frac{1}{2} - \epsilon \mathcal{C}_{17} \right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds \\ & -\epsilon \left(N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2 - \frac{\delta}{\epsilon} \right) e^{-t} \|\mu |S| u(t)\|_{L^\infty(\Gamma_3)}^2 \\ & -\epsilon \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \right) \|f(t)\|_{V'}^2. \end{aligned} \quad (4.80)$$

Moreover, we assume that m_A satisfies (4.73) if we choose λ_0 as follows

$$\lambda_0 = \max \left\{ \frac{\delta (\mathcal{C}_7 N + \mathcal{C}_9 + \delta_1 \mathcal{C}_{11})}{N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2)}, \frac{\mathcal{C}_2 \delta (\mathcal{C}_7 N + \mathcal{C}_9 + \delta_1 \mathcal{C}_{11})}{N(k_0 - 4\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2)}, \frac{\mathcal{C}_5 \delta (\mathcal{C}_7 N + \mathcal{C}_9 + \delta_1 \mathcal{C}_{11})}{K - \delta_2}, \right. \\ \left. \frac{\delta (\mathcal{C}_7 N + \mathcal{C}_9 + \delta_1 \mathcal{C}_{11})}{N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2}, \frac{(\zeta + \delta) (\mathcal{C}_7 N + \mathcal{C}_9 + \delta_1 \mathcal{C}_{11})}{N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1 (1 + C_0^2)} \right\} + \mathcal{C}_1, \quad (4.81)$$

and δ satisfies

$$\delta < \min \left\{ \frac{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)}{\alpha_3}, \frac{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)}{2\mathcal{C}_{17}}, \right. \\ \frac{N(k_0 - 4\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2)}{\mathcal{C}_2 \alpha_3}, \frac{N(k_0 - 3\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2)}{2\mathcal{C}_2 \mathcal{C}_{17}}, \\ \frac{N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2}{\alpha_3}, \frac{N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1 C_0^2}{2\mathcal{C}_{17}}, \\ \frac{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)}{\alpha_3} - \zeta, \frac{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)}{2\mathcal{C}_{17}} - \zeta, \\ \frac{K - \delta_2}{\mathcal{C}_5 \alpha_3}, \frac{K - \delta_2}{2\mathcal{C}_5 \mathcal{C}_{17}}, \frac{\delta^*}{\mathcal{C}_4}, \frac{\delta^* (K - \delta_2)}{\mathcal{C}_5 (\mathcal{C}_8 N + \mathcal{C}_{15} \delta_1 + \mathcal{C}_{10}) + \mathcal{C}_4 (K - \delta_2)}, \\ \frac{\delta^* \left(\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2) \right)}{(\mathcal{C}_8 N + \mathcal{C}_{15} \delta_1 + \mathcal{C}_{10}) + \mathcal{C}_4 \left(\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2) \right)}, \\ \left. \frac{\delta^* (N(k_0 - 4\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2))}{\mathcal{C}_2 (\mathcal{C}_8 N + \mathcal{C}_{15} \delta_1 + \mathcal{C}_{10}) + \mathcal{C}_4 (N(k_0 - 4\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2))} \right\}. \quad (4.82)$$

It follows from (4.82), that we have

$$\left\{ \begin{array}{l} \frac{\delta}{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{17}}, \frac{\delta^* - \mathcal{C}_4 \delta}{\mathcal{C}_8 N + \mathcal{C}_{15} \delta_1 + \mathcal{C}_{10}} \right\}, \\ \frac{\mathcal{C}_2 \delta}{N(k_0 - 4\delta_1) - \delta_1 (\mathcal{C}_{12} + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{17}}, \frac{\delta^* - \mathcal{C}_4 \delta}{\mathcal{C}_8 N + \mathcal{C}_{15} \delta_1 + \mathcal{C}_{10}} \right\}, \\ \frac{\delta}{\left(1 - \frac{1}{2\delta_1} \right) N - \delta_1 C_0^2} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{17}}, \frac{\delta^* - \mathcal{C}_4 \delta}{\mathcal{C}_8 N + \mathcal{C}_{15} \delta_1 + \mathcal{C}_{10}} \right\}, \\ \frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1} \right) N - \delta_1 (1 + C_0^2)} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{17}}, \frac{\delta^* - \mathcal{C}_4 \delta}{\mathcal{C}_8 N + \mathcal{C}_{15} \delta_1 + \mathcal{C}_{10}} \right\}, \\ \frac{\mathcal{C}_5 \delta}{K - \delta_2} < \min \left\{ \frac{1}{\alpha_3}, \frac{1}{2\mathcal{C}_{17}}, \frac{\delta^* - \mathcal{C}_4 \delta}{\mathcal{C}_8 N + \mathcal{C}_{15} \delta_1 + \mathcal{C}_{10}} \right\}. \end{array} \right. \quad (4.83)$$

From (4.81), we have

$$\left\{ \begin{array}{l} \frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right)N - \delta_1(1 + C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_7N + \mathcal{C}_9 + \delta_1\mathcal{C}_{11}}, \\ \frac{\mathcal{C}_2\delta}{N(k_0 - 4\delta_1) - \delta_1(\mathcal{C}_{12} + C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_7N + \mathcal{C}_9 + \delta_1\mathcal{C}_{11}}, \\ \frac{\delta}{\left(1 - \frac{1}{2\delta_1}\right)N - \delta_1C_0^2} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_7N + \mathcal{C}_9 + \delta_1\mathcal{C}_{11}}, \\ \frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right)N - \delta_1(1 + C_0^2)} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_7N + \mathcal{C}_9 + \delta_1\mathcal{C}_{11}}, \\ \frac{\mathcal{C}_5\delta}{K - \delta_3} < \frac{m_A - \mathcal{C}_1}{\mathcal{C}_7N + \mathcal{C}_9 + \delta_1\mathcal{C}_{11}}. \end{array} \right. \quad (4.84)$$

Therefore, combining (4.71) and (4.83),(4.84) we can fix ϵ as follows

$$\begin{aligned} & \max \left\{ \frac{\delta}{\left(\gamma - \frac{1}{4\delta_1}\right)N - \delta_1(1 + C_0^2)}, \frac{\mathcal{C}_2\delta}{N(k_0 - 4\delta_1) - \delta_1(\mathcal{C}_{12} + C_0^2)}, \right. \\ & \left. \frac{\delta}{\left(1 - \frac{1}{2\delta_1}\right)N - \delta_1C_0^2}, \frac{\delta + \zeta}{\left(\gamma - \frac{1}{4\delta_1}\right)N - \delta_1(1 + C_0^2)}, \frac{\mathcal{C}_5\delta}{K - \delta_3} \right\} \\ & < \epsilon < \min \left\{ \frac{\delta^* - \mathcal{C}_4\delta}{\mathcal{C}_8N + \mathcal{C}_{15}\delta_1 + \mathcal{C}_{10}}, \frac{m_A - \mathcal{C}_1}{\mathcal{C}_7N + \mathcal{C}_9 + \delta_1\mathcal{C}_{11}}, \frac{1}{2\mathcal{C}_{17}}, \frac{1}{\alpha_3} \right\}. \end{aligned} \quad (4.85)$$

Using (4.85) and inequality of Poincare, we get

$$\begin{aligned} & -\epsilon \left(\frac{\delta^* - \mathcal{C}_4\delta}{\epsilon} - \mathcal{C}_8N - \mathcal{C}_{15} - \mathcal{C}_{10} \right) \|\dot{\varphi}(t)\|_E^2 \\ & \leq -\frac{\epsilon}{C_*^2} \left(\frac{\delta^* - \mathcal{C}_4\delta}{\epsilon} - \mathcal{C}_8N - \mathcal{C}_{15} - \mathcal{C}_{10} \right) \|\dot{\varphi}(t)\|_{L^2(\Omega)}^2, \end{aligned}$$

Thus, we get

$$\mathcal{L}'(t) \leq -\epsilon\mathcal{C}_{18}\mathcal{E}(t) + \left(\frac{1}{2} - \epsilon\mathcal{C}_{17}\right) \int_0^\infty g'(s) \|\eta(s, t)\|_V^2 ds,$$

where

$$\mathcal{C}_{18} = 2 \min \left(\begin{array}{l} g_0 - \delta_1C_0^2 \\ N(k_0 - 4\delta_1) - \delta_1(\mathcal{C}_{12} + C_0^2) - \frac{\mathcal{C}_2\delta}{\epsilon} \\ \mathcal{C}_3 \left(N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1(1 + C_0^2) \right) - \frac{\mathcal{C}_3\delta}{\epsilon} \\ N \left(1 - \frac{1}{2\delta_1} \right) - \delta_1C_0^2 - \frac{\delta}{\epsilon} \\ N \left(\gamma - \frac{1}{4\delta_1} \right) - \delta_1(1 + C_0^2) - \frac{\zeta + \delta}{\epsilon} \\ \frac{1}{K} \left(K - \delta_2 - \frac{\mathcal{C}_5\delta}{\epsilon} \right) \\ \frac{1}{C_*^2} \left(\frac{\delta^* - \mathcal{C}_4\delta}{\epsilon} - \mathcal{C}_8N - \mathcal{C}_{15} - \mathcal{C}_{10} \right) \end{array} \right).$$

Using (H-5), (4.85) and (4.67), we get

$$\mathcal{L}'(t) \leq \frac{-\epsilon \mathcal{C}_{18}}{1 + \epsilon \alpha_3} \mathcal{L}(t). \quad (4.86)$$

Integrating (4.86) from 0 to t , yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\lambda_1 t}, \quad \forall t \in \mathbb{R}_+, \quad (4.87)$$

where $\lambda_1 = \frac{\epsilon \mathcal{C}_{18}}{1 + \epsilon \alpha_3}$.

Using (4.67), (4.85) and (4.87), we get

$$\begin{aligned} \|U(t)\|_{\mathcal{H}}^2 &= 2\mathcal{E}(t) \\ &\leq \frac{2}{1 - \epsilon \alpha_3} \mathcal{L}(t) \\ &\leq \frac{2}{1 - \epsilon \alpha_3} \mathcal{L}(0)e^{-\lambda_1 t}, \end{aligned}$$

which yields (4.75) with $\lambda_2 = \frac{2}{1 - \epsilon \alpha_3} \mathcal{L}(0)$. □

CONCLUSION AND FUTURE WORK

In this thesis, we have investigated several classes of dynamic viscoelastic and thermo-viscoelastic contact problems under different boundary and frictional conditions. By employing a combination of nonlinear semigroup theory, operator methods, Galerkin schemes, fixed-point arguments, and energy techniques, we established rigorous results concerning the existence, uniqueness, and exponential stability of weak solutions.

The main contributions of this work can be summarized as follows

- For frictionless viscoelastic contact problems with long-term memory, we established the local existence of the solution via a fixed-point argument combined with classical existence and uniqueness results for parabolic inequalities, and further proved global existence in time. Using the energy method again and an appropriate Lyapunov functional, we established stability.
- For frictional contact problems with long-term memory governed by a simplified Coulomb's friction law, we developed variational formulations, proved local existence through Galerkin approximations, and extended the analysis to global solvability. Exponential decay was obtained using Lyapunov and multiplier methods.
- For frictional viscoelastic contact problems with infinite memory, we proved the global existence of a unique solution using nonlinear semigroup theory under suitable assumptions on the initial data, and demonstrated the exponential stability via Lyapunov functionals.
- For thermo-viscoelastic contact problems with friction, we incorporated thermal effects into the constitutive law and established global solvability and exponential decay by combining semigroup theory with energy methods.

As perspectives for future research, it seems interesting to pursue the following objectives:

- **Extension of results:** Extend the global existence and stability analysis to electro-viscoelastic contact problems, contact problems with damage, and viscoplastic contact problems.
- **General decay results:** Establish decay rates for the considered contact problems when the relaxation function g satisfies general assumptions beyond exponential decrease.
- **Weaker assumptions on data:** Relax the conditions on problem data, particularly the regularity of the external force f and the positivity of constants m_G and m_A , while still proving global existence and exponential stability.

Appendices



MODELING AND MATHEMATICAL BACKGROUND

In this appendix, we introduce some basic preliminaries from functional analysis that will be required in the subsequent developments. We also present the mathematical modeling of contact processes. First, we review the fundamental equations of continuum mechanics together with the associated boundary conditions. In the second part, we present several essential tools from functional analysis. We begin with the function spaces commonly used in contact mechanics, then discuss some standard results from nonlinear analysis, and finally list several classical inequalities. These elements will be applied repeatedly in the mathematical analysis of the models developed in the subsequent sections of the thesis. For further details on the concepts introduced here, the reader is referred to the references [4, 9, 11, 44, 60].

A.1 Basic Equations and Boundary Conditions

In order to investigate contact problems, we begin by recalling the fundamental equations of continuum mechanics, together with the associated boundary conditions. This section introduces the basic notations, function spaces, and constitutive relations that will be employed throughout the thesis. The presentation is organized into three parts: the physical setting, the governing equations, and the boundary conditions, which together establish the mathematical foundation for the analysis developed in the subsequent sections of the thesis.

A.1.1 Physical Setting

Finite dimensional spaces We first introduce preliminary notations that are needed in this section. Additional notations used in the mathematical analysis of contact problems will be introduced in [Section A.2](#). Let $\mathbb{R}^d$ ($d = 2, 3$) denote the d -dimensional real linear space. The symbol $\mathbb{S}^d$ stands for the space of second-order symmetric tensors on $\mathbb{R}^d$. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$u \cdot v = u_i v_i, \quad \|v\| = (v \cdot v)^{1/2} \quad \text{for all } u = (u_i), v = (v_i) \in \mathbb{R}^d, \quad (\text{A.1})$$

$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{1/2} \quad \text{for all } \sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in \mathbb{S}^d. \quad (\text{A.2})$$

Throughout this work, indices i, j run between 1 and d , and unless otherwise stated, the summation convention over repeated indices is adopted. We will frequently use the following spaces:

$$H = \{u = (u_i) \mid u_i \in L^2(\Omega)\}, \quad \mathcal{H} = \{\sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}.$$

The canonical inner product of the previous real Hilbert spaces $H, \mathcal{H}$, are

$$\begin{aligned} \langle u, v \rangle_{L^2(\Omega)^d} &= \int_{\Omega} u_i v_i \, dx, \\ \langle \sigma, \tau \rangle_{\mathcal{H}} &= \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx, \end{aligned}$$

with associated norms denoted by $\|\cdot\|_H$ and $\|\cdot\|_{\mathcal{H}}$, respectively.

Physical setting. We consider a deformable body occupying a domain $\Omega \subset \mathbb{R}^d$, which may come into contact with a deformable foundation under the action of body forces and boundary tractions.

The boundary Γ of the body is decomposed into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$, with $\text{meas } \Gamma_1 > 0$. The body is fixed on Γ_1 and so the displacement field vanishes there. Surface tractions of density f_2 act on Γ_2 , and a body force of density f_0 is applied in Ω .

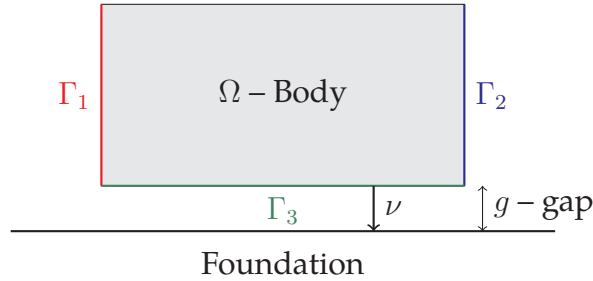


FIGURE A.1.1 The physical setting; Γ_3 is the contact surface.

We are interested in mathematical models that describe the evolution of the mechanical state of the body during the time interval $[0, +\infty)$. To this end, we denote by $u = u(x, t)$ the displacement field, by $\sigma = \sigma(x, t)$ the stress field, and by $\varepsilon = \varepsilon(u)$ the linearized strain tensor. The state of the system is completely determined by (u, σ) ;

in other words, the functions $u : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ and $\sigma : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{S}^d$ will play the role of the unknowns in contact problems.

When temperature effects are taken into account, an additional unknown $\theta : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is introduced, representing the temperature.

We denote partial derivatives and components by subscripts, e.g., the components of the linearized strain tensor $\varepsilon(u)$ are given by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i}).$$

Equations of motion. The equations of motion governing the evolution of the mechanical state of the body are

$$\rho \ddot{u} = \text{Div} \sigma(t) + f_0(t) \text{ in } \Omega \times (0, \infty), \quad (\text{A.3})$$

where ρ denotes the mass density, $\ddot{u}$ represents the acceleration vector and f_0 denotes the density of applied forces. The notation "Div" represents the divergence operator with $\text{Div} \sigma = (\sigma_{ij,j})$. A dot above a quantity denotes the time derivative, thus $\dot{u} = u_t$ and $\ddot{u} = u_{tt}$. We assume that

$$\rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^* \text{ a.e. } x \in \Omega. \quad (\text{A.4})$$

A.1.2 Constitutive Equations

Constitutive equations describe the relationship between stresses and the strains that produce them, thereby characterizing the mechanical behavior of materials. They provide a framework for understanding the deformation of a body under external forces and tractions. Although constitutive laws must satisfy fundamental axioms and invariance principles, their formulation is largely based on experimental observations. For a general overview of diagnostic experiments used to construct constitutive relations, we refer the reader to [20,21].

In this thesis, we focus on nonlinear viscoelastic and thermo-viscoelastic materials. A general viscoelastic constitutive law can be expressed as

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)). \quad (\text{A.5})$$

where A denotes the viscosity operator, G the elasticity operator, and $\dot{u}$ the velocity vector.

For the study of mechanical problems involving viscoelastic materials, we assume that the operators A and G satisfy the following conditions.

$$\left\{ \begin{array}{l} \text{(a) } A : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d; \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \|A(x, \varepsilon_1) - A(x, \varepsilon_2)\| \leq L_A \|\varepsilon_1 - \varepsilon_2\| \quad , \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_A > 0 \text{ such that} \\ (A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_A \|\varepsilon_1 - \varepsilon_2\|^2 \quad , \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto A(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } A(x, 0) = 0 \text{ a.e. } x \in \Omega. \end{array} \right. \quad (\text{A.6})$$

$$\left\{ \begin{array}{l} \text{(a) } G : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d; \\ \text{(b) There exists } L_G > 0 \text{ such that} \\ \|G(x, \varepsilon_1) - G(x, \varepsilon_2)\| \leq L_G \|\varepsilon_1 - \varepsilon_2\| \quad , \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_G > 0 \text{ such that} \\ (G(x, \varepsilon_1) - G(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_G \|\varepsilon_1 - \varepsilon_2\|^2 \quad , \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto G(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } G(x, 0) = 0 \text{ a.e. } x \in \Omega. \end{array} \right. \quad (\text{A.7})$$

A second example is provided by the nonlinear viscoelastic constitutive law with a long memory, involves viscoelastic materials with long memory,

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^t B(t-s)\varepsilon(u(s))ds. \quad (\text{A.8})$$

where A is a fourth-order viscosity operator, G the elasticity operator, and B the relaxation tensor

satisfies the following conditions.

$$\left\{ \begin{array}{l} \text{(a) } B : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d; \\ \text{(b) There exists } L_B > 0 \text{ such that} \\ \|B(x, \varepsilon_1) - B(x, \varepsilon_2)\| \leq L_B \|\varepsilon_1 - \varepsilon_2\| \quad , \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_B > 0 \text{ such that} \\ (B(x, \varepsilon_1) - B(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_B \|\varepsilon_1 - \varepsilon_2\|^2 \quad , \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto B(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } B(x, 0) = 0 \text{ a.e. } x \in \Omega. \end{array} \right. \quad (\text{A.9})$$

When viscoelastic properties depend on temperature, the constitutive law becomes

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^t B(t-s)\varepsilon(u(s)) ds + C\theta. \quad (\text{A.10})$$

where θ denotes the temperature and C is the thermal expansion operator satisfies the following conditions.

$$\left\{ \begin{array}{l} \text{(a) } C : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d; \\ \text{(b) There exists } L_C > 0 \text{ such that} \\ \|C(x, \theta_1) - C(x, \theta_2)\| \leq L_C \|\theta_1 - \theta_2\| \quad , \forall \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega; \\ \text{(c) There exists } m_C > 0 \text{ such that} \\ (C(x, \theta_1) - C(x, \theta_2)) \cdot (\theta_1 - \theta_2) \geq m_C \|\theta_1 - \theta_2\|^2 \quad , \forall \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega; \\ \text{(d) } x \mapsto C(x, \theta) \text{ is Lebesgue measurable on } \Omega; \\ \text{(e) } C(x, 0) = 0 \text{ a.e. } x \in \Omega. \end{array} \right. \quad (\text{A.11})$$

The constitutive laws with hereditary effects used in this thesis are given by

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^t g(t-s)B\varepsilon(u(s)) ds, \quad (\text{A.12})$$

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^\infty g(s)B\varepsilon(u(t-s)) ds, \quad (\text{A.13})$$

$$\sigma(t) = A\varepsilon(\dot{u}(t)) + G\varepsilon(u(t)) - \int_0^\infty g(s)B\varepsilon(u(t-s)) ds + C\theta. \quad (\text{A.14})$$

where The function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the kernel of the memory term.

A.1.3 Boundary Conditions

We now proceed to describe the boundary conditions of the system. The boundary Γ is partitioned into three relatively closed subsets, denoted by Γ_1 , Γ_2 , and Γ_3 , whose interiors are mutually

disjoint. We assume that $\text{meas } \Gamma_1 > 0$. For any vector field v defined on Γ , we decompose it into its normal and tangential components

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

where ν is the outward unit normal to Γ . In what follows, v_ν and v_τ represent the normal and tangential displacements. Similarly, the stress field σ can be decomposed into normal and tangential parts

$$\sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma_\tau = \sigma \nu - \sigma_\nu \nu,$$

where σ_ν and σ_τ represent the normal and the tangential traces of σ .

Displacement and traction boundary conditions. Throughout this thesis, we assume that the body is held fixed on Γ_1 . Therefore, the displacement vanishes on this portion of the boundary, i.e.,

$$u(t) = 0, \quad \text{on } \Gamma_1 \times (0, \infty), \quad (\text{A.15})$$

which represents the displacement boundary condition. Known tractions of density f_2 act on the portion Γ_2 , thus,

$$\sigma(t)\nu = f_2, \quad \text{on } \Gamma_2 \times (0, \infty). \quad (\text{A.16})$$

This relation represents the traction boundary condition, where $\sigma(t)$ denotes the stress field and ν the outward unit normal to the boundary.

Contact boundary conditions. We now turn to a description of various contact conditions on the contact surface Γ_3 , which is where our main interest lies. These are divided naturally into the conditions in the normal direction and those in the tangential directions.

First, consider the so-called bilateral contact, i.e., the contact between the body and the foundation is maintained at all times. This is generally the case in many machines and between moving parts and components in equipment or machinery. Since there is no gap, $g = 0$, and no separation, we have

$$u_\nu(t) = 0 \quad (\text{A.17})$$

The bilateral contact condition (A.17) is applied on Γ_3 in [Chapter 3](#), as expressed in (3.5).

The so-called normal compliance condition describes contact with a rigid-plastic foundation (see, e.g., [69]). It shows that when there is separation (i.e., when $u_\nu(t) < 0$) the reaction of the foundation vanishes (since $\sigma_\nu(t) = 0$); moreover, it shows that penetration arises only if the normal stress reaches the value F , which is interpreted as the yield limit of the foundation

$$\begin{aligned} \sigma_\nu(t) &= 0 & \text{if } u_\nu(t) < 0, \\ -F &\leq \sigma_\nu(t) \leq 0 & \text{if } u_\nu(t) = 0, \\ \sigma_\nu(t) &= -F & \text{if } u_\nu(t) > 0. \end{aligned}$$

We now turn to the conditions in the tangential directions. The simplest one is the so-called frictionless contact condition.

$$\sigma_\tau(t) = 0 \quad (\text{A.18})$$

Simplified Coulomb's friction law. Consider a contact condition modeled by a simplified version of Coulomb's law of dry friction (see, e.g., [71, 82]), i.e.,

$$\begin{aligned} \sigma_\nu &= S, \quad |\sigma_\tau| \leq \mu|\sigma_\nu| \\ |\sigma_\tau| &< \mu|\sigma_\nu| \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| &= \mu|\sigma_\nu| \Rightarrow \exists \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau \end{aligned} \quad (\text{A.19})$$

Here σ_ν denotes the normal stress on the contact boundary, σ_τ is the tangential force on the contact boundary, $\dot{u}_\tau(t)$ denotes the tangential velocity, $S \in L^\infty(\Gamma_3)$ is a given function, and $\mu \geq 0$ is the coefficient of friction.

Bilateral contact with Tresca's friction law. We assume a bilateral contact modeled by Tresca's friction law (see, e.g., [1, 98]), i.e.,

$$\begin{aligned} u_\nu &= 0, \quad |\sigma_\tau| \leq g \\ |\sigma_\tau| &< g \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| &= g \Rightarrow \exists \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau \end{aligned} \quad (\text{A.20})$$

Here u_ν represents the normal displacement, $\dot{u}_\tau$ denotes the tangential velocity, σ_τ is the tangential force on the contact boundary, and $g \geq 0$ is the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins. For simplicity, we assume here that g is a constant but, with minor changes, the nonhomogenous case may also be considered. In (A.20) the strict inequality holds in the stick zone and the equality in the slip zone. The contact is assumed to be bilateral, i.e., there is no loss of the contact during the process.

A.2 Preliminaries on Functional Analysis

This section introduces fundamental tools from functional analysis that are essential for the study of contact problems. We begin by recalling the function spaces commonly employed in contact mechanics, then review basic notions of nonlinear analysis such as semigroups and their properties. Finally, several classical inequalities, including Hölder's, Cauchy-Schwarz and Young's inequalities, are presented as they will be applied repeatedly in the mathematical analysis of the models developed in the subsequent sections of the thesis.

A.2.1 Function Spaces in Contact Mechanics.

The mathematical analysis of mechanical contact problems relies heavily on function spaces associated with the deformation and divergence operators. Throughout this thesis, we assume that $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is open, connected, bounded, and has a Lipschitz boundary Γ . Since the boundary is Lipschitz continuous, the outward unit normal exists almost everywhere on Γ and is denoted by ν .

Displacements are sought in the space

$$H_1 = \left\{ u = (u_i) \mid u_i \in H^1(\Omega) \right\} \quad (\text{A.21})$$

The space H_1 is a Hilbert space with the canonical inner product

$$\langle u, v \rangle_{H_1} = \langle u, v \rangle_{L^2(\Omega)^d} + \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}},$$

where $\varepsilon : H_1 \longrightarrow \mathcal{H}$ denotes the deformation operator defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

The associated norm is denoted by $\|\cdot\|_{H_1}$.

In accordance with the boundary conditions, we introduce some useful functional spaces

$$\begin{aligned} V &= \left\{ v \in H^1 \mid v = 0 \text{ on } \Gamma_i \right\}, \\ E &= \left\{ w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma \right\}. \end{aligned}$$

On this space, Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_K \|v\|_{H_1}, \quad \forall v \in V. \quad (\text{A.22})$$

where $c_K > 0$ depends only on Ω and Γ_1 .

We consider the inner products on V and E defined by

$$\begin{aligned} \langle u, v \rangle_V &= \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}, \\ \langle w, y \rangle_E &= \langle \nabla w, \nabla y \rangle_{L^2(\Omega)}. \end{aligned}$$

It follows from (A.22) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V . Therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. By Sobolev's trace theorem, there exists $C_0 > 0$ depending only on Ω and Γ_1

such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq C_0 \|v\|_V, \quad \forall v \in V. \quad (\text{A.23})$$

We call (V, H, V') a Gelfand triple, with $V \subset H \subset V'$. The duality pairing between V' and V is denoted by $\langle u, v \rangle_{V' \times V}$, we have

$$\langle u, v \rangle_{V' \times V} = \langle u, v \rangle_H = \langle \rho u, v \rangle_{L^2(\Omega)^d}, \quad \forall u \in H, \quad \forall v \in V. \quad (\text{A.24})$$

$$\langle w, y \rangle_{E' \times E} = \langle w, y \rangle_{L^2(\Omega)}, \quad \forall w \in L^2(\Omega), \quad \forall y \in E \quad (\text{A.25})$$

where $(E, L^2(\Omega), E')$ is another Gelfand triple and E' is the dual space of E

Moreover, the inclusion $V \hookrightarrow H$ is continuous and dense, hence there exists $d > 0$ such that

$$\|u\|_H \leq d \|u\|_V, \quad \forall u \in V. \quad (\text{A.26})$$

We now define the space

$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} \mid \text{Div } \sigma \in H\}.$$

where the divergence operator is given by

$$\text{Div } \sigma = (\sigma_{ij,j}), \quad \sigma_{ij,j} = \sum_{j=1}^d \frac{\partial \sigma_{ij}}{\partial x_j}.$$

This is a Hilbert space endowed with the inner product

$$\langle \sigma, \tau \rangle_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\text{Div } \sigma, \text{Div } \tau)_H,$$

and the associated norm $\|\cdot\|_{\mathcal{H}_1}$.

If σ is a regular function, say $\sigma \in C^1(\overline{\Omega}; \mathbb{S}^d)$, then the following Green formula holds

$$\langle \sigma, \varepsilon(v) \rangle_{\mathcal{H}} + \langle \text{Div } \sigma, v \rangle_H = \int_{\Gamma} \sigma \nu \cdot v \, da, \quad \forall v \in H_1. \quad (\text{A.27})$$

Let us introduce the following notation

$$(g \diamond u)(t) = \int_0^t g(t-s) \|u(t) - u(s)\|_V^2 ds, \quad \forall t > 0.$$

This term represents a memory-type integral that arises naturally in the modeling of viscoelastic or history-dependent materials. Physically, it captures the cumulative effect of the material's past deformations on its current behavior. the integrand measures the energy associated with the difference between the current and past strain states, while the kernel serves as a memory function that weights this difference according to the time lag. typically, decays with time,

reflecting the fact that recent deformations have a stronger influence than those in the distant past. such formulations are essential in accurately describing materials that exhibit both elastic and time-dependent dissipative behavior.

A.2.2 Elements of Nonlinear Analysis.

Nonlinear operators. An important class of nonlinear operators defined in Hilbert and Banach spaces is given below.

Definition A.1 (Operators in Hilbert Spaces). Let X be a Hilbert space with inner product $(\cdot, \cdot)_{X' \times X}$ and norm $\|\cdot\|_X$. For an operator $A : D(A) \subset X \rightarrow X'$:

- A is *monotone* if

$$(Au - Av, u - v)_X \geq 0 \quad \forall u, v \in X.$$

- A is *strongly monotone* if there exists $m > 0$ such that

$$(Au - Av, u - v)_X \geq m\|u - v\|_X^p \quad \forall u, v \in X.$$

- *Maximal monotone*, if A is monotone $x \in X$ and $x^* \in X'$,

$$\langle x^* - Av, u - v \rangle_{X' \times X} \geq 0 \quad \forall u, v \in D(A)$$

implies that $u \in D(A)$ and $x^* = Au$.

- A is *hemicontinuous* if $D(A) = X$ and for all $u, v \in X$, we have

$$A(u + \theta v) \rightarrow \langle A(u), w \rangle$$

as $\theta \rightarrow 0$

- A is *Lipschitz continuous* if there exists $L_A > 0$ such that

$$\|Au - Av\|_X \leq L_A\|u - v\|_X \quad \forall u, v \in X.$$

Definition A.2 (Accretive, m -accretive, and ω - m -accretive operators). Let H be a Hilbert space. A single-valued operator $A : D(A) \subset H \rightarrow H$ is called:

- (a) *accretive*, if

$$\langle Au - Av, u - v \rangle_H \geq 0 \quad \forall u, v \in D(A).$$

(b) *m*-accretive, if A is accretive and

$$R(I + A) = H,$$

where $R(I + A)$ denotes the range of the operator $I + A$.

(c) ω -*m*-accretive, for $\omega \in \mathbb{R}$, if $A + \omega I$ is *m*-accretive.

Theorem A.1. *If X is reflexive and $A : X \rightarrow X'$ is monotone and hemicontinuous, then A is maximal monotone.*

Corollary A.1. *If X is reflexive and A is coercive and maximal monotone, then A is surjective.*

Definition A.3 (Lower-semicontinuity). Let $f : X \rightarrow \mathbb{R}$ be a function defined on a normed space X . We say that f is *lower-semicontinuous* (l.s.c.) at $u \in X$ if

$$\liminf_{n \rightarrow \infty} f(u_n) \geq f(u),$$

for every sequence $\{u_n\} \subset X$ converging to u in X . The function f is l.s.c. on a subset of X if it is l.s.c. at each point of the subset, and f is l.s.c. on X if it is l.s.c. at every point of X .

When the inequality above holds for every sequence $\{u_n\} \subset X$ converging weakly to u , the function f is said to be *weakly lower-semicontinuous* (weakly l.s.c.).

It is clear that every continuous function is l.s.c., but the converse does not hold. Since strong convergence in X implies weak convergence, every weakly l.s.c. function is also l.s.c. Moreover, a proper convex function $f : X \rightarrow \mathbb{R}$ is l.s.c. if and only if it is weakly l.s.c.

Banach Fixed-Point Theorem The Banach fixed-point theorem will be used repeatedly in this work in proving existence of solutions to variational problems in Contact Mechanics.

Let X be a Banach space equipped with the norm $\|\cdot\|_X$, and let $K \subset X$ be a nonempty closed subset. Consider an operator $\Lambda : K \rightarrow X$. We are interested in finding $u \in K$ such that

$$\Lambda u = u. \tag{A.28}$$

Any such u is called a *fixed point* of Λ in K .

Theorem A.2 (Banach Fixed-Point Theorem). *Let $\Lambda : K \rightarrow K$ be a contraction mapping with constant $\alpha \in [0, 1)$, i.e.,*

$$\|\Lambda u - \Lambda v\| \leq \alpha \|u - v\| \quad \forall u, v \in K.$$

Then Λ admits a unique fixed point in K .

Theorem A.3. *Let $\Lambda : K \rightarrow K$ and suppose that Λ^m is a contraction for some $m \in \mathbb{N}$. Then Λ has a unique fixed point in K .*

Theorem A.4. Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V . Suppose that $\langle Av, v \rangle : V \times V \rightarrow \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $C_3 > 0$ and $C_2 \in \mathbb{R}$

$$\langle Av, v \rangle + C_2 \|v\|_H^2 \geq C_3 \|v\|_V^2, \quad \forall v \in V. \quad (\text{A.29})$$

Then, for every initial condition $u_0 \in K$ and $f \in L^2(0, T; H)$, there exists a unique function u such that

$$u \in W^{1,2}(0, T; V') \cap L^2(0, T; V), \quad (\text{A.30})$$

$$u(t) \in K, \quad \forall t \in [0, T], \quad (\text{A.31})$$

$$\begin{aligned} & \langle \dot{u}(t), v - u(t) \rangle_{V' \times V} + \langle Au(t), v - u(t) \rangle \\ & \geq \langle f(t), v - u(t) \rangle_H, \quad \forall v \in K, \text{ a.e. } t \in (0, T), \end{aligned} \quad (\text{A.32})$$

$$u(0) = u_0. \quad (\text{A.33})$$

Theorem A.5. Let $V \subset H \subset V'$ be a Gelfand triple. Assume that $A : V \rightarrow V'$ is a hemicontinuous and monotone operator satisfying the following conditions

$$\langle Av, v \rangle_{V' \times V} \geq \omega \|v\|_V^2 + \lambda, \quad \forall v \in V, \quad (\text{A.34})$$

$$\|Av\|_{V'} \leq C_1(\|v\|_V + 1), \quad \forall v \in V. \quad (\text{A.35})$$

where $\omega > 0$, $C_1 > 0$, and $\lambda \in \mathbb{R}$

Then, for every initial value $u_0 \in H$ and $f \in L^2(0, T; V')$, there exists a unique function u such that

$$u \in L^2(0, T; V) \cap C([0, T]; H), \quad \dot{u} \in L^2(0, T; V'),$$

$$\dot{u}(t) + Au(t) = f(t), \quad \text{a.e. } t \in (0, T),$$

$$u(0) = u_0.$$

Definition A.4 (Semigroup). Let C be a subset of a Banach space X . A family $\{S(t)\}_{t \geq 0}$ of mappings $S(t) : C \rightarrow C$ is called a semigroup if:

- $S(0) = I$ (identity),
- $S(t + s)x = S(t)(S(s)x)$ for all $t, s \geq 0$ and $x \in C$.

Definition A.5 (Strong Continuity). A semigroup $\{S(t)\}_{t \geq 0}$ is said to be strongly continuous if the map

$$g : [0, \infty) \times C \rightarrow C, \quad g(t, x) = S(t)x$$

is continuous with respect to the norm topology of x .

Definition A.6 (Contractive Semigroup). A semigroup $\{S(t)\}_{t \geq 0}$ is called contractive if:

$$\|S(t)x - S(t)y\|_X \leq \|x - y\|_X, \quad \forall t \geq 0, x, y \in C.$$

Definition A.7 (Quasi-Contractive Semigroup). A semigroup $\{S(t)\}_{t \geq 0}$ is called quasi-contractive if there exists a constant $\omega \in \mathbb{R}$ such that:

$$\|S(t)x - S(t)y\|_X \leq e^{\omega t} \|x - y\|_X, \quad \forall t \geq 0, x, y \in C.$$

The semigroup framework abstracts the solution behavior of initial value problems of the form:

$$\begin{cases} \frac{\partial u(t)}{\partial t} + Au(t) = 0, & t > 0, \\ u(0) = x, \end{cases}$$

where A is a suitable operator on X and $x \in C$.

Theorem A.6. Let A be a single-valued operator in a reflexive Banach space X , $f \in L^1(\mathbb{R}_+; X)$ and $x_0 \in \overline{D(A)}$. If $A + \omega I$ is m -accretive, then the problem

$$\begin{cases} u'(t) + Au(t) = f(t), & \forall t > 0, \\ u(0) = x_0, \end{cases} \quad (\text{A.36})$$

has a unique mild solution $u \in C(\mathbb{R}_+; X)$.

Remark A.1. This theorem will be used in the particular case where X is a Hilbert space.

A.2.3 Elementary Inequalities

We recall here some classical inequalities that will be used throughout the thesis.

Let $1 \leq r \leq +\infty$. The conjugate exponent s is defined by

$$\frac{1}{r} + \frac{1}{s} = 1.$$

Theorem A.7 (Hölder's Inequality). Let $u \in L^r(D)$ and $v \in L^s(D)$. Then the product uv belongs to $L^1(D)$ and

$$\int_D |u(x)v(x)| dx \leq \|u\|_r \|v\|_s.$$

Corollary A.2 (Cauchy-Schwarz's Inequality). Choosing $r = s = 2$ in Hölder's inequality, we obtain

$$\int_D |u(x)v(x)| dx \leq \|u\|_2 \|v\|_2.$$

Theorem A.8 (Young's Inequality). *Let $\alpha, \beta \in \mathbb{R}^+$. Then, for any $\varepsilon > 0$,*

$$|\alpha\beta| \leq \varepsilon\alpha^2 + \frac{\beta^2}{4\varepsilon}.$$

Lemma A.1. *Let $\alpha, \beta \geq 0$ and $r, s > 1$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$\alpha\beta \leq \frac{\alpha^r}{r} + \frac{\beta^s}{s}.$$

Theorem A.9 (Gronwall's Inequality). *Let $\psi : [0, T] \rightarrow \mathbb{R}$ be a bounded measurable function and let $h : [0, T] \rightarrow \mathbb{R}$ be a nonnegative integrable function. Assume that there exists a constant $\gamma > 0$ such that*

$$\psi(t) \leq \gamma + \int_0^t h(\tau)\psi(\tau) d\tau, \quad \forall t \in [0, T].$$

Then

$$\psi(t) \leq \gamma \exp\left(\int_0^t h(\tau) d\tau\right), \quad \forall t \in [0, T].$$

Theorem A.10 (Poincaré's Inequality). *Let $1 \leq r \leq +\infty$ and let $D \subset \mathbb{R}^m$ be a bounded open domain. Then there exists a constant $C_* > 0$, depending only on D , such that*

$$\|\phi\|_r \leq C_* \|\nabla\phi\|_r, \quad \forall \phi \in W_0^{1,r}(D).$$

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Abstract: In this thesis, we investigate dynamic contact problems, with or without friction, between a deformable body and a rigid foundation. The material behavior is modeled by a nonlinear viscoelastic or thermo-viscoelastic law with long memory effects. Unlike quasi-static models, the system's acceleration is retained to capture the fully dynamic nature of the process. The contact is governed by Tresca's friction law, simplified Coulomb's friction Law and a rigid-plastic foundation. We establish both local and global existence and uniqueness of weak solutions. The analytical approach combines operator theory, Galerkin schemes, fixed-point arguments, and nonlinear semigroup techniques. Under suitable assumptions on the problem data, we also prove the exponential stability of the solutions using energy methods.

Keywords : Dynamic contact problems, Tresca's friction law, Rigid-plastic foundation, Simplified Coulomb's friction Law, Global existence, Fixed-point arguments, Nonlinear semigroup, Exponential stability.

الملخص: في هذه الأطروحة، ندرس بعض مسائل التماس الديناميكي، مع أو بدون احتكاك، بين جسم قابل للتشوه وقاعدة صلبة. يُنمذج سلوك المادة بواسطة قانون لزج-مرن أو لزج-حراري-مرن غير خطي مع تأثيرات ذاكرة طويلة. وعلى خلاف النماذج شبه الساكنة، لم نهمل تسارع النظام من أجل تمثيل الطبيعة الديناميكية الكاملة للعملية. يُحكم التماس بواسطة قانون الاحتكاك لتريسكا و الأساس الصلب-اللدن. لقد أثبتنا الوجود المحلي والعالمي وكذلك تفرد الحلول الضعيفة. يعتمد النهج التحليلي على نظرية المؤثرات، مخططات غاليركين، النقطة الثابتة، وحجة نصف الزمر غير الخطية. وتحت فرضيات مناسبة على معطيات المسألة، أثبتنا أيضًا الاستقرار الأسّي للحلول باستخدام طرق الطاقة.

الكلمات المفتاحية: مسائل التماس الديناميكي، قانون احتكاك تريسكا، الأساس الصلب-اللدن، الوجود العالمي، النقطة الثابتة، نصف الزمر، الاستقرار الأسّي.

Résumé: Dans cette thèse, nous étudions des problèmes de contact dynamique, avec ou sans frottement, entre un corps déformable et une fondation rigide. Le comportement du matériau est modélisé par une loi viscoélastique ou thermo-viscoélastique non linéaire intégrant des effets de mémoire longue. Contrairement aux modèles quasi-statiques, l'accélération du système est conservée afin de représenter pleinement la nature dynamique des processus étudiés. Le contact est modélisé par la loi de frottement de Tresca, la loi de frottement de Coulomb simplifiée et une fondation rigide-plastique. Nous établissons l'existence local et globale ainsi que l'unicité des solutions modérées. Le cadre analytique repose sur la théorie des opérateurs, méthode Galerkin, les arguments de point fixe et les techniques de semi-groupes non linéaires. De plus, sous certaines hypothèses sur les données du problème, nous démontrons également la stabilité exponentielle des solutions à l'aide de la méthode de l'énergie.

Mots clés : Dissipatif, Existence globale, Problèmes de contact dynamique, Point fixe, Semi-groupes non linéaires, Stabilité exponentielle.