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وزارة التعليم العالي والبحث العلمي

جامعة المسيلة
كلية التكنولوجيا
قسم الهندسة الميكانيكية

جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

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مستخلص اللجنة العلمية لقسم الهندسة الميكانيكية
بخصوص تقييم مطبوعة جامعية

بناءا على التقارير الإيجابية الواردة من السادة أعضاء لجنة دراسة وتقييم مطبوعة جامعية , والآتية أسماؤهم :

• الأستاذ : زرقان السعيد	أستاذ محاضر أ	جامعة محمد بوضياف - المسيلة
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صادق أعضاء اللجنة العلمية على قبول المطبوعة المنجزة باللغة الإنجليزية من طرف

الأستاذ ة : بختي فاطمة الزهراء

أستاذة محاضرة - أ - قسم الهندسة الميكانيكية , كلية التكنولوجيا , جامعة محمد بوضياف بالمسيلة.

- تحت عنوان **Mass and Heat Transfer**

وفق البرنامج المقترح لطلبة السنة الثالثة ليسانس هندسة الميكانيكية فرع علوم المواد.

رئيس اللجنة العلمية للقسم

رئيس اللجنة العلمية لقسم الهندسة الميكانيكية

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Abstract

Heat and mass transfer are fundamental subjects that underpin many engineering and scientific applications. These phenomena are critical for understanding and designing systems in industries such as energy, environment, materials, and manufacturing.

This handout for the mass and heat transfer course is intended for students in the Bachelor's program in Mechanical Engineering with a specialization in Materials Engineering. It is in accordance with the official program issued by the Ministry of Higher Education and Scientific Research This document is organized into two key parts to provide an in-depth treatment of both heat and mass transfer.

Part A: Heat Transfer

This section introduces the foundational principles of heat transfer, examining the mechanisms through which thermal energy is transported in different media and conditions.

Part B: Mass Transfer

Building on the understanding of heat transfer, this part transitions to the study of mass transfer, with a strong focus on diffusion processes and their applications.

Keywords : Heat transfer, mass transfer, conduction, convection, radiation , diffusion

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Faculty of Technology
Department of Mechanical Engineering

Course Handout

Intended for students of the 3rd year Bachelor's Degree in Mechanical Engineering, Materials Engineering option

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Mass and Heat Transfer

Prepared by

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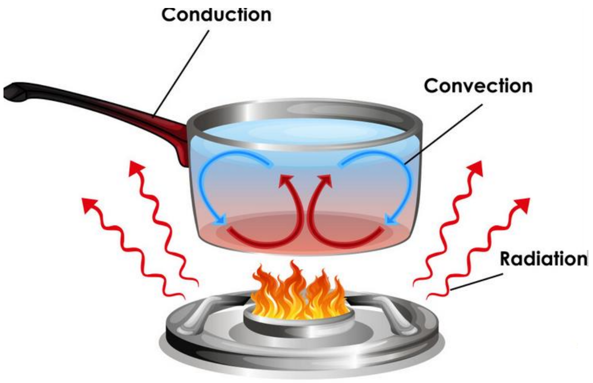


Contents

Chapiter III Heat Transfer by Conduction in a Variable Regime

Chapiter IV Heat Transfer by Convection

Chapiter V Heat Transfer by Radiation

Part B: Mass Transfer

Chapiter VI Applications of Diffusion 144

Appendix

References

Preface

Heat and mass transfer are fundamental subjects that underpin many engineering and scientific applications. These phenomena are critical for understanding and designing systems in industries such as energy, environment, materials, and manufacturing.

This handout for the mass and heat transfer course is intended for students in the Bachelor's program in Mechanical Engineering with a specialization in Materials Engineering. It is in accordance with the official program issued by the Ministry of Higher Education and Scientific Research

This document is organized into two key parts to provide an in-depth treatment of both heat and mass transfer.

Part A: Heat Transfer

This section introduces the foundational principles of heat transfer, examining the mechanisms through which thermal energy is transported in different media and conditions.

Chapter I sets the stage by explaining the general principles of heat transfer, describing the three primary modes: conduction, convection, and radiation, along with the governing laws such as Fourier's law, Newton's law, and the Stefan-Boltzmann law.

Chapter II delves into one-dimensional, steady-state conduction, covering diverse scenarios like composite walls, hollow cylinders, and concentric spheres, emphasizing analytical methods for solving heat transfer problems.

Chapter III focuses on conduction in a variable regime, including transient heat transfer and methodologies like the thermal capacity approach and semi-infinite medium analysis.

Chapter IV explores heat transfer by convection, highlighting the physical mechanisms, classifications of fluid flow, boundary layer theory, and practical correlations for calculating convection coefficients in natural and forced flows.

Chapter V addresses the complex phenomena of heat transfer by radiation, discussing black body radiation, the laws of Planck, Wien, and Stefan-Boltzmann, and radiative exchanges between surfaces, integrating theoretical and practical perspectives.

Each chapter concludes with exercises designed to reinforce the theoretical concepts through applied problem-solving.

Part B: Mass Transfer

Building on the understanding of heat transfer, this part transitions to the study of mass transfer, with a strong focus on diffusion processes and their applications.

Chapter I introduces mass transfer concepts, including types, mechanisms, and comparisons with heat and momentum transfer. The discussion extends to solid-state diffusion mechanisms such as interstitial, substitutional, and vacancy diffusion.

Chapter II presents Fick's laws, examining steady-state and non-stationary diffusion and their applications, along with factors influencing the diffusion coefficient.

Chapter III provides a phenomenological theory of diffusion, emphasizing the relationship between flux and concentration gradients and modeling diffusion in various mediums like alloys, porous media, and polymers.

Chapter IV investigates diffusion in metals and alloys, particularly in the absence of chemical gradients, with applications in metallurgical and industrial contexts.

Chapter V focuses on surface diffusion, contrasting it with volume diffusion and exploring its significance in thin-film technologies and surface engineering.

Chapter VI highlights applications of diffusion in fields such as semiconductors, catalysis, energy systems, and materials science, showcasing the broad relevance of mass transfer principles.

This course is designed to serve a learning resource for students. Its structured approach ensures a gradual and coherent progression through fundamental principles to advanced applications. By integrating theoretical knowledge with practical exercises and real-world examples, the document provides a robust foundation for mastering heat and mass transfer.

PART A

Heat Transfer

Chapiter I

General Principles of Heat Transfer

I.1. Introduction

From the study of thermodynamics, we have learned that energy can be transferred by interactions of a system with its surroundings. These interactions are called work and heat. However, thermodynamics deals with the end states of the process during which an interaction occurs and provides no information concerning the nature of the interaction or the time rate at which it occurs. The objective of this course is to extend thermodynamic analysis through the study of the modes of heat transfer and through the development of relations to calculate heat transfer rates.

Heat transfer within a system only occurs if there are temperature gradients between different parts of the system, which implies that it is not in thermodynamic equilibrium (the temperature is not uniform throughout the system). During the transformation of the system towards a final equilibrium state, the temperature will evolve both in time and space. The purpose of heat transfer analysis is to identify the modes of transfer involved during the transformation and to quantitatively determine how the temperature varies at each point in the system over time.

I.2.The three modes of heat transfer

Heat can be transferred in three different modes: conduction, convection, and radiation. All modes of heat transfer require the existence of a temperature difference and all modes are from the high temperature medium to lower temperature one.

There are three modes of heat transfer: conduction, convection, and radiation. Any energy exchange between bodies occurs through one of these modes or a combination of them. Conduction is the transfer of heat through solids or stationery fluids. Convection uses the movement of fluids to transfer heat. Radiation does not require a medium for transferring heat; this mode uses the electromagnetic radiation emitted by an object for exchanging heat.

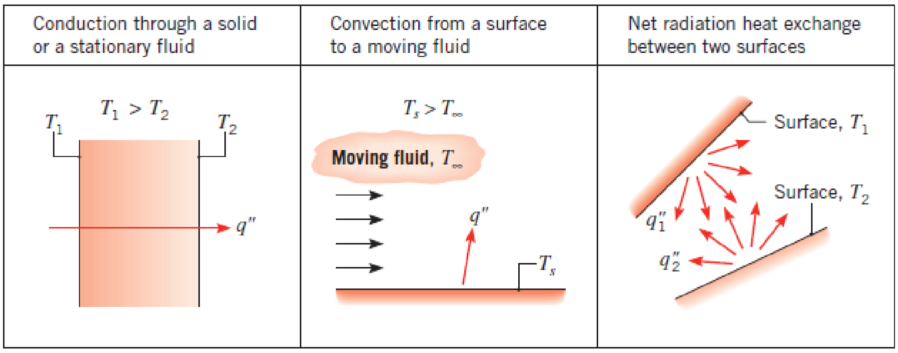


Figure I.1. Conduction, convection, and radiation heat transfer modes.

I.2.1. Conduction

It is the transfer of heat within an opaque medium, without the movement of matter, under the influence of a temperature difference. The propagation of heat by conduction within a body is due to microscopic physical phenomena (agitation of atoms or molecules, flow of free electrons...). It can be seen as a transfer of energy from the most energetic particles (the hot particles with high vibrational energy) to the less energetic particles (the cold particles with lower vibrational energy), due to collisions between particles. In solids, energy transfer can also occur due to the movement of free electrons within the crystal lattice (for example, in metals). Thus, good conductors of electricity are generally also good conductors of heat.

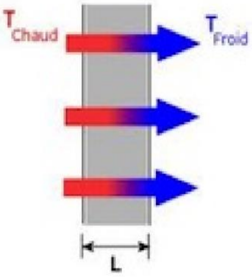


Figure I.2. Phenomenon of conduction in a solid.

I.2.2. Heat transfer by convection

Thermal convection is the transfer of energy between two media, at least one of which is a fluid, through molecular movement. It involves the combined effects of conduction and fluid movement. In the absence of fluid movement, heat transfer is ensured exclusively by pure conduction.

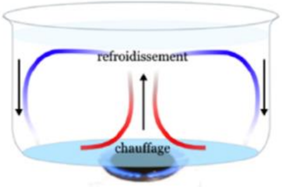


Figure I.3. Phenomenon of convection in a liquid.

We distinguish between two types of convection:

- ***Natural convection***: the movements are due to variations in density within a fluid subjected to the gravitational field. Density variations can be generated by temperature gradients (warm air is lighter than cold air) and/or by composition gradients.

- *Forced convection*: the movement of the fluid is caused by external mechanical actions (pump, fan...).

- We will talk about ***mixed convection*** when both types of convection coexist in a system.

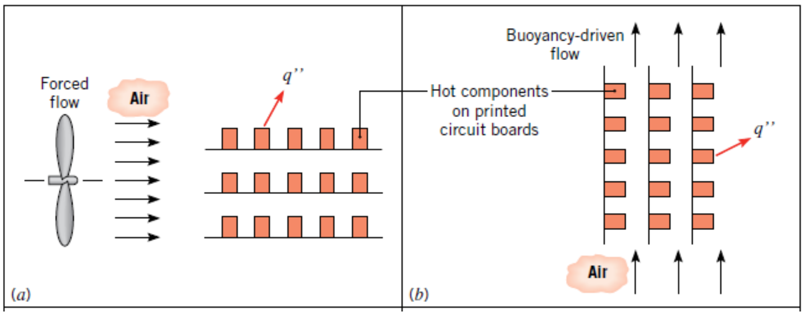


Figure 1.4. Convection heat transfer processes.(*a*) Forced convection. (*b*) Natural convection.

I.2.3. Heat transfer by radiation

Every material body emits and absorbs energy in the form of electromagnetic radiation.

The transfer of heat by radiation between two bodies separated by a vacuum or a semi-transparent medium occurs through electromagnetic waves, thus without a material medium.

The phenomenon of a body's emission corresponds to the conversion of material energy (the agitation of the electrons constituting the matter, the intensity of which depends on the temperature) into radiative energy. The phenomenon of absorption is the reverse conversion.

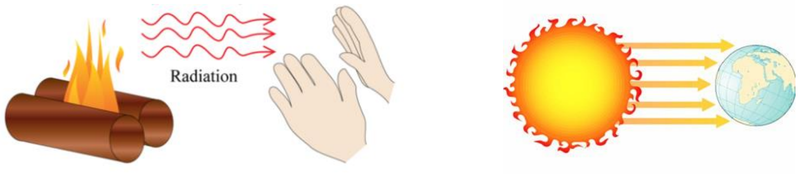


Figure I.5. Examples of thermal radiation.

I.3. Definitions

I.3.1. Temperature Field

Energy transfers are determined based on the evolution of temperature in space and time: T= f(x, y, z, t). The instantaneous value of the temperature at every point in space is a scalar called the temperature field. We distinguish two cases:

-*Temperature field independent of time:* the regime is said to be ***permanent or stationary.***

- *Evolution of the temperature field over time:* the regime is said to be ***variable or unsteady.***

I.3.2. Temperature Gradient

If we gather all the points in space that have the same temperature, we obtain a surface called an isothermal surface. The variation in temperature per unit length is maximum along the normal to the isothermal surface. This variation is characterised by the temperature gradient:

$$\overrightarrow{\mathrm{grad}}(\mathrm{T}) = \vec{\mathrm{n}}.\frac{\partial \mathrm{T}}{\partial \mathrm{n}}, \quad \vec{\nabla}\mathrm{T} = \overrightarrow{\mathrm{grad}}\mathrm{T} = \left| \begin{array}{l} \frac{\partial \mathrm{T}}{\partial \mathrm{x}} \\ \frac{\partial \mathrm{T}}{\partial \mathrm{y}} \\ \frac{\partial \mathrm{T}}{\partial \mathrm{z}} \end{array} \right. \qquad (\mathrm{I}.1)$$

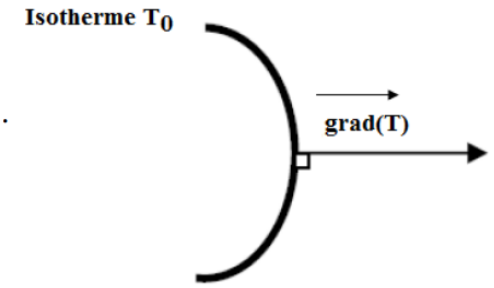


Figure I.6. Isothermal surface and thermal gradient.

I.3.3. Heat Flux

A heat flux is an amount of energy transferred in the form of heat per unit of time. It is therefore a power, expressed in Watts (J/s):

$$\varphi = \frac{\mathrm{dQ}}{\mathrm{dt}} \quad [\mathrm{W}] \qquad (\mathrm{I}.2)$$

I.3.4. Heat flux density

In general, the flux exchanged through a surface is not uniform across the entire surface. We then define a heat flux density ϕ, which corresponds to a heat flux per unit area.

$$\phi = \frac{1}{S}\frac{dQ}{dt} \quad \left[\frac{W}{m^2}\right] \qquad (I.3)$$

I.4. Basic laws of heat transfer

For each mode of heat transfer, there is a law that provides the expression for the heat flux.

I.4.1. Heat flux exchanged by conduction – Fourier's law

This transfer mechanism is governed by a phenomenological law established by Joseph Fourier in 1822, stipulating that the flux exchanged by conduction is proportional to the temperature gradient. This law, called Fourier's law, is written as:

$$\vec{\varphi} = -\lambda . S . \overrightarrow{grad}(T) \qquad (I.5)$$

In general

$$\varphi = -\lambda . S . \left(\frac{\partial T}{\partial X}\right) \qquad (I.6)$$

With:

φ: Heat flux transmitted by conduction (W)

λ: Thermal conductivity of the medium (W. m^{-1}.°C^{-1})

x: Spatial variable in the direction of the flow (m)

S: Area of the heat flow passage section (m^2)

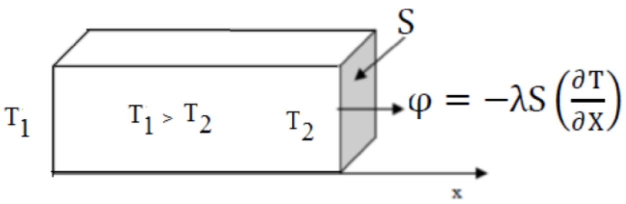


Figure I.7. Diagram of conductive heat transfer.

The sign (-) in Fourier's equation reflects the second law of thermodynamics: the heat flow goes in the opposite direction of the temperature gradient, meaning that the flow moves from the highest temperature to the lowest. So, this sign indicates that a negative temperature gradient (decrease in temperature in the positive x direction) multiplied by the sign (–)gives a positive heat flux in the positive direction of x.

The exchange surface S is the surface that is crossed by the heat flow. Figure 1.7 shows how to calculate this area in different cases.

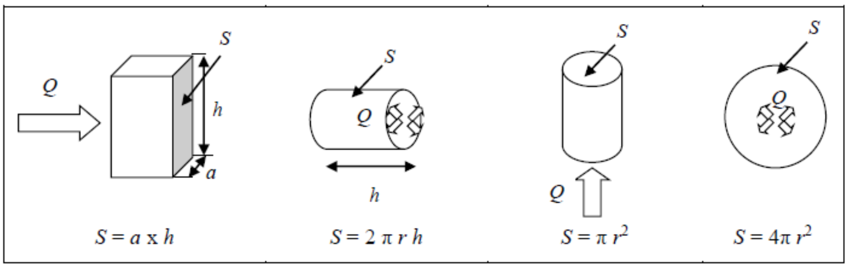
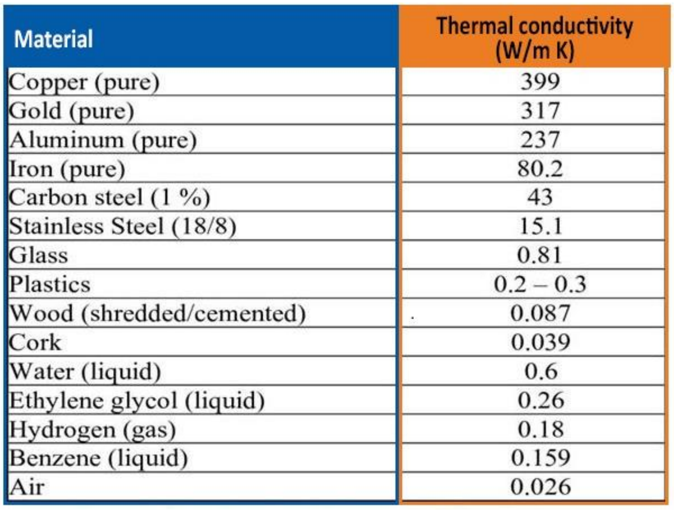


Figure I.8. Calculation of the exchange surface.

Thermal conductivity λ depends on the nature of the material in question and generally depends on the temperature. It translates the ability of a material to conduct heat. Thus, for a given temperature gradient, the heat flux will be greater the higher the conductivity. For heat-conducting materials, λ will be high, and conversely, it will be low for insulators.

In Table I.1, you will find the thermal conductivity values λ of some of the most common materials.

Table I.1. Thermal conductivity of certain materials.



Material	Thermal conductivity (W/m K)
Copper (pure)	399
Gold (pure)	317
Aluminum (pure)	237
Iron (pure)	80.2
Carbon steel (1 %)	43
Stainless Steel (18/8)	15.1
Glass	0.81
Plastics	0.2 – 0.3
Wood (shredded/cemented)	0.087
Cork	0.039
Water (liquid)	0.6
Ethylene glycol (liquid)	0.26
Hydrogen (gas)	0.18
Benzene (liquid)	0.159
Air	0.026

I.4.2. Heat flux exchanged by convection – Newton's law

This transfer mechanism is governed by Newton's law, which states that the heat flux exchanged between a solid wall and a flowing fluid is proportional to the temperature difference that gave rise to it.

The heat flux entering the fluid or leaving the solid if $T_p > T_\infty$ is expressed by Newton's law:

$$\vec{\varphi} = h.S.(T_P - T_\infty).\vec{n} \qquad (I.7)$$

$$\varphi = h.S.(T_P - T_\infty) \qquad (I.8)$$

With:

φ: Heat flux transmitted by convection (W)

h: Coefficient of heat transfer by convection (W/m² °C)

T_p: Surface temperature of the solid (°C)

T_∞: Temperature of the fluid far from the solid surface (°C)

S: Area of the solid/liquid contact surface (m^2)

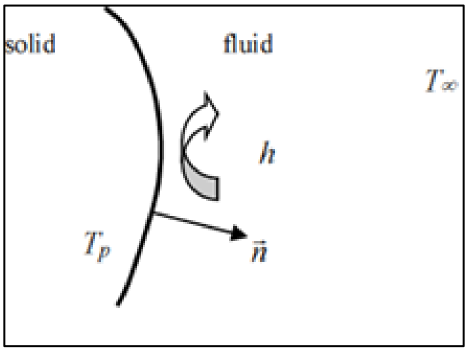
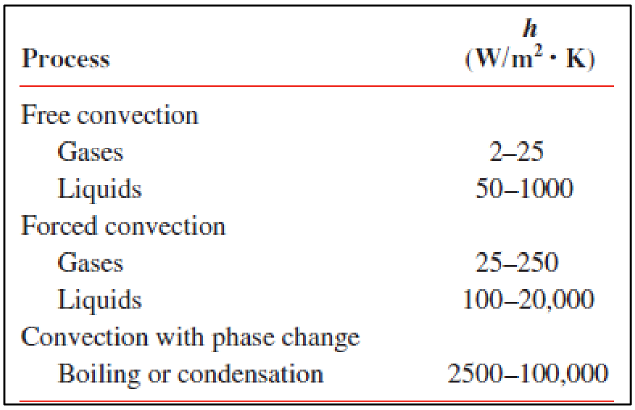


Figure I.9. Diagram of convective heat transfer

h is a positive quantity called the convective heat transfer coefficient, in ($W.m^{-2}.K^{-1}$). This coefficient depends on numerous parameters (fluid, type of flow, surface condition...) and is therefore extremely difficult to quantify precisely.

Table I.2. Typical values of the convection heat transfer coefficient.



Process	*h* ($W/m^2 \cdot K$)
Free convection	
Gases	2–25
Liquids	50–1000
Forced convection	
Gases	25–250
Liquids	100–20,000
Convection with phase change	
Boiling or condensation	2500–100,000

I.4.3. Heat flux exchanged by radiation – Stefan-Boltzmann law

All surfaces emit energy in the form of electromagnetic waves. In the absence of an intermediate medium, the net and total heat transfer occurs between two surfaces at different temperatures. The maximum value of the heat flux emitted by a surface is given by the Stefan-Boltzmann law:

$$\varphi = \sigma . S . T_P^4 \qquad (I.9)$$

Such a body is called an ideal body or a black body. In this expression, σ is the Stefan-Boltzmann constant:

$\sigma = 5.67 \times 10^{-8}$ ($W\ m^{-2}\ K^{-4}$);

T p: Surface temperature (K).

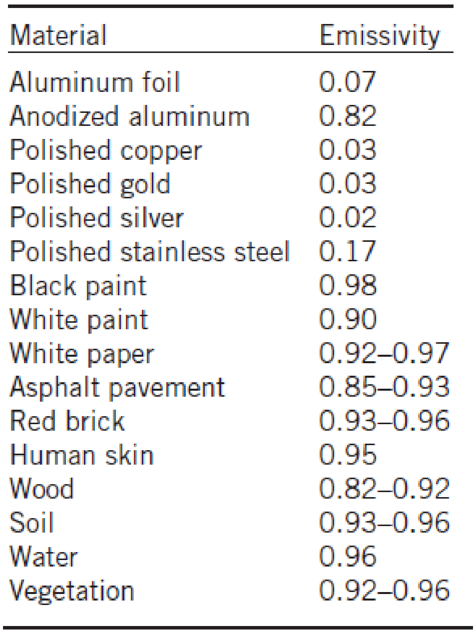
The heat flux emitted by a real surface is less than that of a blackbody at the same temperature and is given by

$$\varphi = \sigma . \varepsilon . S . T_P^4 \qquad (I.10)$$

Where ε is a radiative property of the surface termed the ***emissivity***. With values in the range $0 \leq \varepsilon \leq 1$, this property provides a measure of how efficiently a surface emits energy relative to a blackbody. It depends strongly on the surface material and finish.

The emissivities of some surfaces are given in Table I.3.

Table I.3. Emissivities of some materials at 300 K.



Material	Emissivity
Aluminum foil	0.07
Anodized aluminum	0.82
Polished copper	0.03
Polished gold	0.03
Polished silver	0.02
Polished stainless steel	0.17
Black paint	0.98
White paint	0.90
White paper	0.92–0.97
Asphalt pavement	0.85–0.93
Red brick	0.93–0.96
Human skin	0.95
Wood	0.82–0.92
Soil	0.93–0.96
Water	0.96
Vegetation	0.92–0.96

Another important radiation property of a surface is its **absorptivity** a, which is the fraction of the radiation energy incident on a surface that is absorbed by the surface. Like emissivity, its value is in the range $0 \leq \alpha \leq 1$. A blackbody absorbs the entire radiation incident on it. That is, a blackbody is a perfect absorber ($\alpha = 1$) as it is a perfect emitter.

In general, both ε and α of a surface depend on the temperature and the wave length of the radiation. Kirchhoff's law of radiation states that the emissivity and the absorptivity of a surface at a given temperature and wave length are equal. In many practical applications, the surface temperature and the temperature of the source of incident radiation are of the same order of magnitude, and the average absorptivity of a surface is taken to be equal to its average emissivity.

$$\varphi_{\text{absorbed}} = \alpha.\ \varphi_{\text{incident}} \quad (\text{W}) \qquad (\text{I}.11)$$

Where $\varphi_{\text{incident}}$ is the rate at which radiation is incident on the surface and a is the absorptivity of the surface. For opaque (nontransparent) surfaces, the portion of incident radiation not absorbed by the surface is reflected back.

When a surface of emissivity ε and surface area S at a thermodynamic temperature T_p is completely enclosed by a much larger (or black) surface at thermodynamic temperature T_∞ separated by a gas (such as air) that does not intervene with radiation, the net rate of radiation heat transfer between these two surfaces is given by

$$\varphi = \sigma.\varepsilon.S.(T_P^4 - T_\infty^4) \qquad (I.12)$$

In this special case, the emissivity and the surface area of the surrounding surface do not have any effect on the net radiation heat transfer.

Exercises

Exercise 01:

Calculate the thermal flux as well as the thermal flux density through a flat and homogeneous plate with a thickness of 50mm if it is:

- In stainless steel ($\lambda a=16W/m.K$) with dimensions 3m x 2m.
- In concrete ($\lambda_b=0.92W/m.K$) with dimensions 30m x 20m.

In both cases, the temperatures at the surfaces of the plate are maintained constant and equal to: $Tp_1=100°C$, $Tp_2=90°C$.

Solution:

-In stainless steel ($\lambda a=16W/m.K$) with dimensions 3m x 2m.

$$\varphi_a = \frac{\lambda_a}{e}.S.(T_1 - T_2) = \frac{16}{50.10^{-3}}.(3.2)(100-90) = \mathbf{19200W}$$

The heat flux density

$$\phi_a = \frac{\varphi_a}{S} = \frac{19200}{3.2} = 3200W/m^2$$

-In concrete ($\lambda_b=0.92W/m.K$) with dimensions 30m x 20m.

$$\varphi_b = \frac{\lambda_b}{e}(T_1 - T_2) = \frac{0,92}{50.10^{-3}}(30.20)(100-90) = \mathbf{110400W}$$

The heat flux density

$$\phi_b = \frac{\varphi_b}{S} = \frac{110400}{30.20} = \mathbf{184W/m^2}$$

Exercise 02:

-Determine the heat flux through a flat wall with a thickness of e=10cm and a conductivity of $\lambda=8.5W/m.K$. The temperatures on the limiting faces of the wall are respectively equal to 100°C and 30°C. The surface area of the wall $S=3m^2$.

- Find the temperature gradient in the direction of flow.

-Calculate the depth of the wall where the temperature is 60°C.

Solution:

$$\varphi = \frac{\lambda}{e}.S.(T_1 - T_2) = \frac{8{,}5}{10.10^{-2}}.(3)(100 - 30) = \mathbf{17850W}$$

$$\varphi = -\lambda.S.\frac{dT}{dx}$$

$$\Rightarrow \frac{dT}{dx} = \frac{-\varphi}{\lambda.S} = \frac{-17850}{8{,}5.3} = \mathbf{-700K/m}$$

$$T(x) = \frac{(T_2 - T_1)}{e}x + T_1 \Rightarrow T(x) = -700.x + 100$$

$$x = \frac{100 - T(x)}{700} = \frac{100 - 60}{700} = 0{,}057m = \mathbf{5,7cm}$$

Exercise 03:

The heat flux density Φ is 5000 W.m^{-2} at the surface of an electric heating element. The temperature of this same element is 110°C when it is cooled by forced convection in air with a temperature of 60°C.

-What is the average exchange coefficient h?

-What will be the temperature of the heating element if the flux density is reduced to 2000 W.m^{-2}?

Solution:

Using the expression for the total exchanged flux density, one can determine the average exchange coefficient:

$$\bar{h} = \frac{\varphi}{\Delta T} = \frac{5000}{110 - 60} = \mathbf{100W.m^{-2}.K^{-1}}$$

If the flow is reduced, the forced convection coefficient remains the same, so the temperature difference will be:

$$\Delta T = T_{element} - 60 = \frac{\varphi}{h} = \frac{2000}{100} = \mathbf{20°C}$$

So the temperature

$T_{element} = \Delta T + 60 = 20 + 60 = \mathbf{80°C}$

Exercise 04:

A pipe transports steam through a large room where the air and walls are at a temperature of 25°C. The external diameter of the pipe is 70mm and the surface temperature is 200°C. Its emissivity is 0.8.

1. What is the power emission flux density of the surface of the duct?

2. What is the density of the radiation power flux from this surface?

3. The convection coefficient from the surface to the ambient air is 15W/m²·K and the surface is considered gray.

-Calculate the heat transfer flux density of this surface per unit length of the pipe?

Solution:

1.Emission power flux density:

$$\mathrm{q_{surface}} = \varepsilon.\sigma.\mathrm{T_p^4} = 0.8\mathrm{x}5.67\mathrm{x}10^{-8}\mathrm{x}(200+273.15)^4 = 2270\mathrm{W/m^2}$$

2. Radiation power flux density:

$$\mathrm{q_{max}} = \sigma.T^4 = 5.67\mathrm{x}10^{-8}\mathrm{x}(25+273.15)^4 = 447\mathrm{W/m^2}$$

3. Heat transfer flux from this surface:

$$\mathrm{Q}_{tot} = \mathrm{Q}_{ray} + \mathrm{Q}_{conv} = \mathrm{h.S.}(T_P - T_\infty) + \varepsilon.\mathrm{S}.\sigma.\left(\mathrm{T_p^4} - \mathrm{T_\infty^4}\right)$$

$$S = \pi x D x L = \pi x 70 x 10^{-3} x L$$

$$\mathrm{Q}_{tot} = \pi x D x L x\left[\mathrm{h.}(T_P - T_\infty) + \varepsilon.\sigma.\left(\mathrm{T_p^4} - \mathrm{T_\infty^4}\right)\right]$$

Heat transfer flux density of this surface per unit length of the pipe:

$$\mathrm{q_{tot}}\frac{\mathrm{Q_{tot}}}{\mathrm{L}} = \pi\mathrm{xDxLx}\left[\mathrm{h.}(\mathrm{T_P} - \mathrm{T_\infty}) + \varepsilon.\sigma.\left(\mathrm{T_p^4} - \mathrm{T_\infty^4}\right)\right]$$

$$= \pi\mathrm{x}70\mathrm{x}10^{-3}\mathrm{x}[15.(200-25) + 0.8\mathrm{x}5.67\mathrm{x}10^{-8}((200+273.15)^4 - (25+273.15)^4)]$$

$$= 998\mathrm{W/m}$$

Chapiter II

One-Dimensional, Steady-State Conduction

II.1.General equation of conduction

In its one-dimensional form, it describes the unidirectional heat transfer through a flat wall

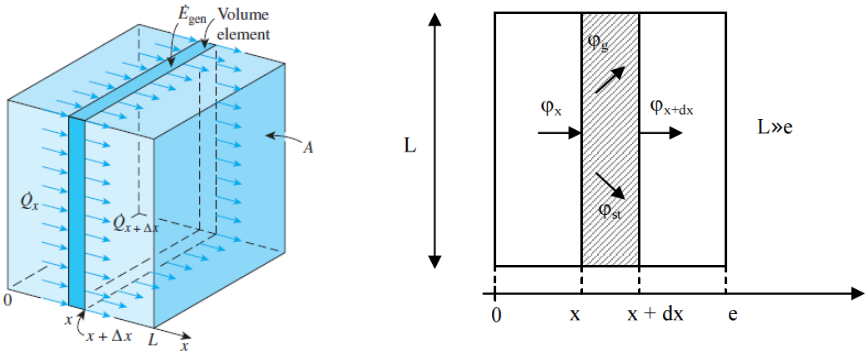


Figure II.1. One-dimensional heat conduction through a volume element in a large plane wall.

Consider a thin element of thickness Δx in a large plane wall, as shown in figure I.10. Assume the density of the wall is ρ, the specific heat is c_p, and the area of the wall normal to the direction of heat transfer is S. An energy balance on this thin element during a small time interval dt can be expressed as

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

$$\varphi_x - \varphi_{x+dx} + \varphi_g = \varphi_{st} \qquad \text{(II.1)}$$

With

$$\varphi_x = -\left(\lambda.S.\frac{\partial T}{\partial x}\right)_x \qquad \text{(II.2)}$$

and $\varphi_{x+dx} = -\left(\lambda.S.\frac{\partial T}{\partial x}\right)_{x+dx}$ (II.3)

$$\varphi_g = \dot{q}_s . S . dx \qquad \text{(II.4)}$$

$$\varphi_{st} = \rho . c_p . S . dx . \frac{\partial T}{\partial t} \qquad \text{(II.5)}$$

By substituting into the energy balance and dividing by dx, we obtain:

$$\frac{\left(\lambda . S . \frac{\partial T}{\partial x}\right)_{x+dx} - \left(\lambda . S . \frac{\partial T}{\partial x}\right)_x}{dx} + \dot{q}_s . S = \rho . c_p . S . \frac{\partial T}{\partial t} \Rightarrow \frac{\partial}{\partial x}\left(\lambda . S . \frac{\partial T}{\partial x}\right) + \dot{q}_s . S = \rho . c_p . S . \frac{\partial T}{\partial t} \qquad \text{(II.6)}$$

And in the three-dimensional case, we obtain the heat equation in the most general case:

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + \dot{q}_s = \rho . c_p . \frac{\partial T}{\partial t} \qquad \text{(II. 7)}$$

$$\rho . c_p . \frac{\partial T}{\partial t} = \mathrm{div}\left[\lambda . \overrightarrow{\mathrm{grad}} T\right] + \dot{q}_s \qquad \text{(II. 8)}$$

The thermal conductivity λ of a material, in general, depends on the temperature T (and therefore x), and thus it cannot be taken out of the derivative. However, the thermal conductivity in most practical applications can be assumed to remain constant at some average value. The equation above in that case reduces to

$$\nabla^2 T + \frac{\dot{q}_s}{\lambda} = \frac{1}{a} . \frac{\partial T}{\partial t} \qquad \text{(II. 9)}$$

The ratio $a=\lambda/(\rho.c_p)$ is called thermal diffusivity ($m^2.s^{-1}$), which characterises the speed of heat flow propagation through a material.

-In steady state, we obtain the Laplace equation

$$\nabla^2 T = 0 \qquad \text{(II. 10)}$$

This equation can be written in cylindrical or spherical coordinates as follows,

-Heat equation in cylindrical coordinates:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}_s}{\lambda} = \frac{1}{a}\frac{\partial T}{\partial t} \qquad \text{(II. 11)}$$

In the case of a problem with cylindrical symmetry where the temperature depends only on r and t, equation (II.11) can be written in a simplified form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{q}_s}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t} \qquad (II.12)$$

-Heat equation in spherical coordinates:

$$\frac{1}{r}\frac{\partial^2(rT)}{\partial r^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta.\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2.\sin^2\theta}\frac{\partial^2 T}{\partial\varphi^2}+\frac{\dot{q}_s}{\lambda}=\frac{1}{a}\frac{\partial T}{\partial t} \qquad (II.13)$$

II.2. Spatiotemporal boundary conditions for the solution of the heat equation

The general heat equation expresses a relationship between the temperature function T and the variables x, y, z, and t. The mathematical solution to this second-order linear partial differential equation theoretically admits an infinite number of solutions. Also, its resolution requires knowledge, on the one hand, of the initial condition, that is to say, the initial distribution of temperatures at every point in the medium T(x, y, z, 0), and on the other hand, the law of variation over time of the temperature or its derivative on the surface S. These are the spatiotemporal boundary conditions.

- **Initial condition**

It is the temperature distribution at the moment t=0, that is T_0=f(x, y, z, 0). Generally, this condition is known.

- **Boundary conditions**

At the boundaries of a material, different types of boundary conditions can appear in problems commonly encountered in heat transfer.

a) Dirichlet conditions (1er type)

The temperature distribution Tp at the considered boundary surface is given as a function of time and for all points on the surface. T=f(x,y,z). The most common case is when Tp depends neither on t nor on space (uniform in space).

Tp = constant

b) Neumann boundary conditions (2nd type)

We impose the flux density on the surface, for all points on the surface as a function of time: $\varphi_s(x, y, z, t)$

$$\phi_s = -\lambda\left(\frac{dT}{dn}\right)_S = f(M_S, t) \qquad (II.14)$$

Where $\left(\frac{dT}{dn}\right)_S$ is the normal derivative at the surface.

Particular case of the adiabatic boundary: in this case, the heat flux crossing the boundary is zero.

$$\phi_s = -\lambda\left(\frac{dT}{dn}\right)_S = 0 \qquad (II.15)$$

c) Robin boundary conditions (3rd type)

-Convective exchanges on a wall

When a wall exchanges heat by convection with the outside, the density of the flux it exchanges is proportional to the temperature difference between the wall and the external fluid medium, multiplied by an exchange coefficient h in W/m².K that takes into account the different physical and kinematic properties of the fluid in contact with the wall. We impose a temperature on the ambient fluid, which we will denote as T_∞, that will be known.

$$\phi_s = -\lambda\left(\frac{dT}{dn}\right)_S = h(T_P - T_\infty) \qquad (II.16)$$

-Radiative exchanges on a wall

The radiation from a wall can be a mode of exchange to consider, especially if the temperature is high (above approximately 100 degrees). The flux density of a wall at temperature T_p exchanges by radiation with the external medium at temperature T_∞ is:

$$\phi = \sigma.\varepsilon.\left(T_p^4 - T_\infty^4\right) \qquad (II.17)$$

In such a case, the boundary condition at the border will therefore become:

$$-\lambda\left(\frac{\partial T}{\partial n}\right)_{wall} = \sigma.\varepsilon.\left(T_p^4 - T_\infty^4\right) \qquad (II.18)$$

d) The solid under study is in contact with another material

-Perfect contact

At the interface S of the two media with different conductivities λ_1 and λ_2, the conservation of flux is written as:

$$\lambda_1 . \overrightarrow{\mathrm{grad}} T_1 = \lambda_2 . \overrightarrow{\mathrm{grad}} T_2 \qquad \text{(II.19)}$$

A second condition is obtained in the case of perfect contact. It concerns the temperatures on S:

$T_1 = T_2$

-Contact imperfect

In reality, this condition is not met: the two surfaces are not strictly in contact due to the air present between the mediums. The temperatures are no longer equal. We consider a contact thermal resistance Rc [$m^2.K.W^{-1}$].

The condition obtained on the interface is then written as:

$$\lambda_1 . \overrightarrow{\mathrm{grad}} T_1 = \lambda_2 . \overrightarrow{\mathrm{grad}} T_2 \qquad \text{(II.20)}$$

$$\Phi = \frac{1}{R_C}(T_1 - T_2) \qquad \text{(II.21)}$$

II.3. Steady-state conduction without internal heat dissipation

II.3.1. A simple wall without a source

We consider a homogeneous and undeformable solid (or a fluid at rest) of thickness e and large transverse dimensions, whose extreme faces are at temperatures T_1 and T_2. We assume that the thermal conductivity of the material is constant. Let's revisit the heat equation established previously:

$$\rho . c_p . \frac{\partial T}{\partial t} = \lambda \nabla^2 T + \dot{q}_s \qquad \text{(II. 22)}$$

In steady-state (permanent) conditions and without internal heat dissipation

$$\dot{q}_s = 0 \quad \text{et} \quad \frac{\partial T}{\partial t} = 0$$

Where ;

$$\Delta T = 0$$

Boundary conditions of the problem: The two faces of the wall are maintained at fixed temperatures over time (Dirichlet conditions)

We will take x=0 for one of the faces and x=e for the other.

$T=T_1$ if $x=0$ and $T=T_2$ if $x=e$

The problem addressed is a one-dimensional problem. The temperature is solely a function of the variable x.

$$\Delta T = 0 \Rightarrow \frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{dT}{dx} = A$$

$$\Rightarrow T(x) = A.x + B$$

If $x=0$, $T=T_1 \Rightarrow T(0)= A.(0)+B=T_1$

$\Rightarrow B=T_1$

If $x=e$, $T=T_2 \Rightarrow T(e)=A.e+T_1= T_2$

Where $A = \frac{T_2-T_1}{e}$

$$T(x) = \frac{T_2-T_1}{e} x + T_1 \qquad (II.23)$$

$$\varphi = -\lambda.S.\frac{dT}{dx}$$

Where ; $\varphi = -\lambda.S.\frac{T_2-T_1}{e}$

$$\varphi = \lambda.S.\frac{T_1-T_2}{e} \qquad (II.24)$$

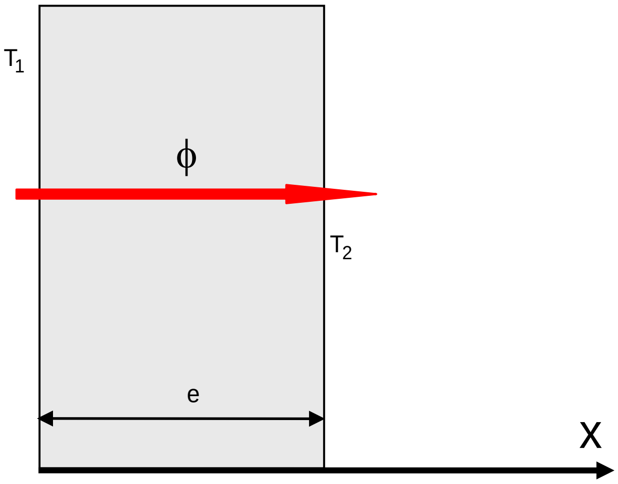


Figure II.2. One-dimensional conduction in a simple wall.

The relation (II.24) can also be expressed as: $\varphi = \frac{(T_1-T_2)}{\frac{e}{\lambda s}}$, this relation is analogous to Ohm's law in electricity, which defines the current intensity as the ratio of the difference in electric potential to the electrical resistance ($V_1-V_2=R.I \Rightarrow I = \frac{V_1-V_2}{R}$). The temperature thus appears as a thermal potential and the term $\frac{e}{\lambda S}$ appears as the thermal resistance of a flat wall of thickness e, thermal conductivity λ, and lateral surface S. We therefore refer to the equivalent diagram shown in Figure II.3.

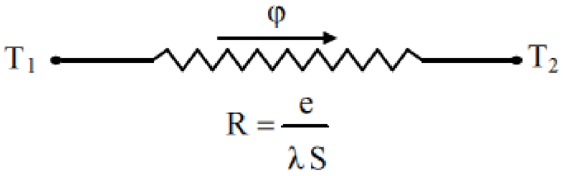


Figure II.3. Equivalent electrical circuit of a simple wall.

II.3.2. Simple wall with conductivity that varies with temperature

If the range of temperatures encountered in a conduction problem is such that the values of λ are different from one end to the other of this range, we can no longer assume λ is constant.

In this case, we can make the approximation that the thermal conductivity varies linearly with temperature, i.e., $\lambda=\lambda_0(1+bT)$ with λ_0 being the conductivity at T=0, and b depending on the material.

For a wall, a one-dimensional problem, we must then return to the general equation of conduction in the case of non-uniform thermal conductivity:

$\frac{d}{dx}\left(\lambda\frac{dT}{dx}\right) = 0$ (Without heat source $\dot{q}=0$, in steady state $\frac{dT}{dt} = 0$)

$$\frac{d}{dx}\left(\lambda_0(1+bT)\frac{dT}{dx}\right) = 0$$

$$\lambda_0(1+bT)\frac{dT}{dx} = E$$

$$\lambda_0 T + \frac{\lambda_0 b T^2}{2} = EX + D \qquad (II.25)$$

x=0 $T=T_1$ $T_1>T_2$

x=e $T=T_2$

The distribution of temperatures is therefore parabolic within the wall.

We can solve the problem by considering two Dirichlet conditions:

Which lead to

$$D = \lambda_0\left(T_1 + \frac{bT_1^2}{2}\right)$$

And

$$E = \frac{\lambda_0}{e}\left[\frac{b}{2}(T_2^2 - T_1^2) + (T_2 - T_1)\right]$$

By substituting into (II.25) and expressing T(x), we derive:

$$T(x) = -\frac{1}{b} + \sqrt{\left(\frac{1}{b} + T_1\right)^2 + \frac{2.E.x}{b.\lambda_0}} \qquad (II.26)$$

Three cases need to be considered. :

b>0, b=0 et b<0

b=0 :$\lambda=\lambda_0$: linear case between T_1 and T_2

b>0 : concavity upwards
b<0 : concavity downwards.
(see Figure II.4)

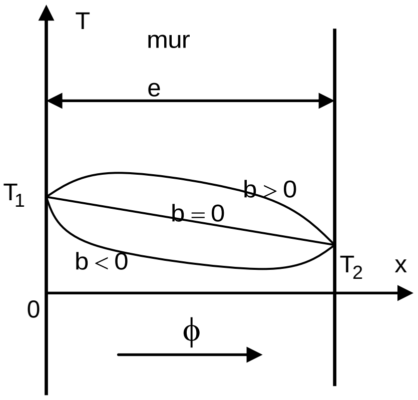


Figure II.4. Simple wall with conductivity varying with temperature (case already handled)

II.3.3. Multi-layer wall

This is the case for real walls (schematized in Figure II.5) composed of several layers of different materials, where only the temperatures T_{f1} and T_{f2} of the fluids in contact with the two faces of the wall with lateral surface S are known.

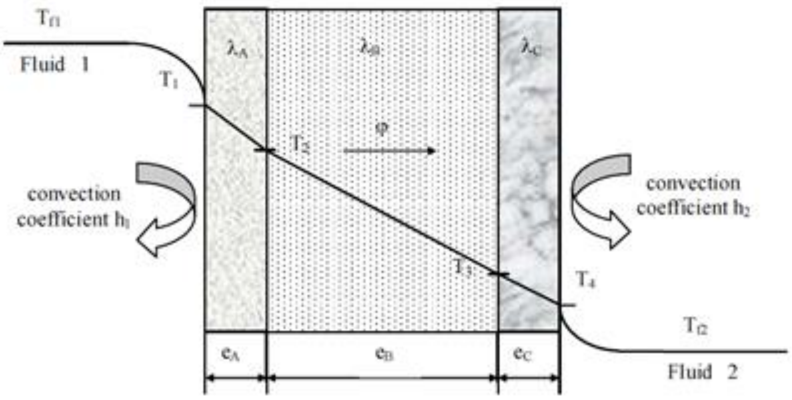


Figure II.5. Schematic representation of the flows and temperatures in a multilayer wall.

In steady state, the heat flux is conserved as it crosses the wall and is expressed as:

$$\varphi = h_1.S.(T_{f1} - T_1) = \frac{\lambda_A S(T_1 - T_2)}{e_A} = \frac{\lambda_B S(T_2 - T_3)}{e_B}$$
$$= \frac{\lambda_C S(T_3 - T_4)}{e_C} = h_2.S.(T_4 - T_{f2}) \qquad (II.27)$$

Where ;

$$\varphi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 S} + \frac{e_A}{\lambda_A S} + \frac{e_B}{\lambda_B S} + \frac{e_C}{\lambda_C S} + \frac{1}{h_2 S}} \qquad (II.28)$$

It was assumed that the contacts between layers of different natures were perfect and that there was no temperature discontinuity at the interfaces. In reality, given the roughness of the surfaces, a micro-layer of air exists between the hollows of the opposing surfaces, which contributes to the creation of a thermal resistance (air is an insulator) called contact thermal resistance. The previous formula is then written as:

$$\varphi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 S} + \frac{e_A}{\lambda_A S} + R_{AB} + \frac{e_B}{\lambda_B S} + R_{BC} + \frac{e_C}{\lambda_C S} + \frac{1}{h_2 S}} \qquad (II.29)$$

The equivalent electrical diagram is shown in Figure II.6.

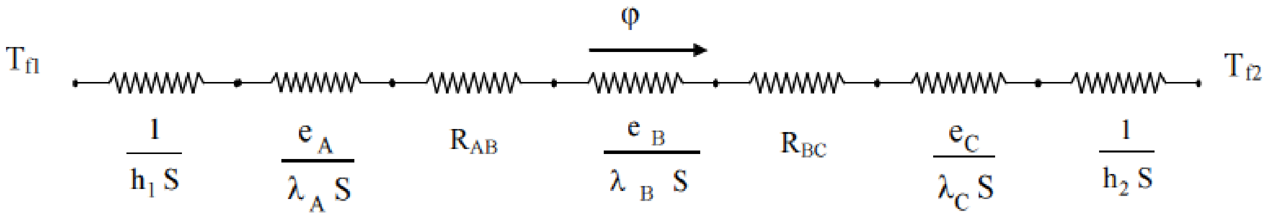


Figure II.6. Equivalent electrical diagram of a multilayer wall.

II.3.4. Composite wall

This is the most commonly encountered case in reality where the walls are not homogeneous. Let's consider, for example, a wall of width L made of hollow blocks. (Figure II.7).

Assuming unidirectional transfer and taking into account the axes of symmetry, we can reduce the calculation of the flux through the isolated element on the right side of the figure and calculate the equivalent thermal resistance R of a portion of the wall with a width L and a height $l = l_1 + l_2 + l_3$ using the laws of association of resistances in series and parallel by the relation:

$$R = R_1 + R_2 + \frac{1}{\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}} + R_6 + R_7 \qquad (II.30)$$

Where ;

$$R_1 = \frac{1}{h_1.l.L};\ R_2 = \frac{e_1}{\lambda_1.l.L};\ R_3 = \frac{e_2}{\lambda_2.l_1.L};\ R_4 = \frac{e_2}{\lambda_1.l_2.L};\ R_5 = \frac{e_2}{\lambda_2.l_3.L};\ R_6 = \frac{e_3}{\lambda_1.l.L};\ R_7 = \frac{1}{h_2.l.L}$$

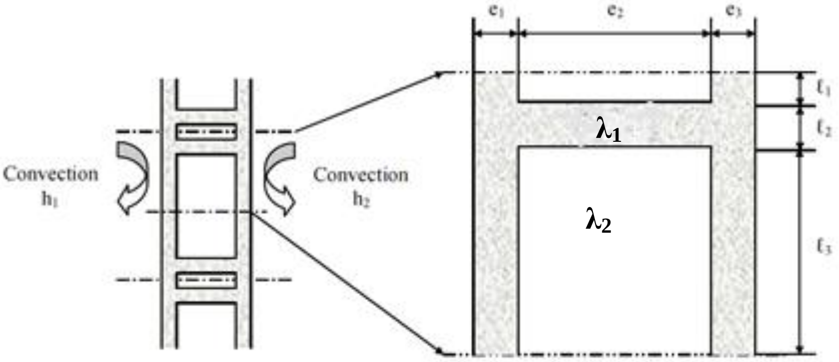


Figure II.7. Schematic of a composite wall.

This can be represented by the equivalent electrical diagram shown in Figure II.8.

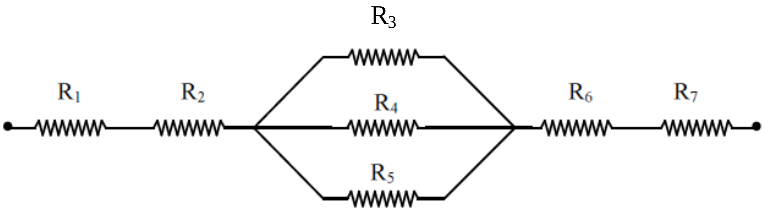


Figure II.8. Equivalent electrical diagram of the composite wall.

II.3.5. Long hollow cylinder

We consider a hollow cylinder with thermal conductivity λ, an inner radius r_1, an outer radius r_2, and a length L, with the temperatures of the inner and outer surfaces being T_1 and T_2, respectively. (cf. figure II.9). It is assumed that the longitudinal temperature gradient is negligible compared to the radial gradient.

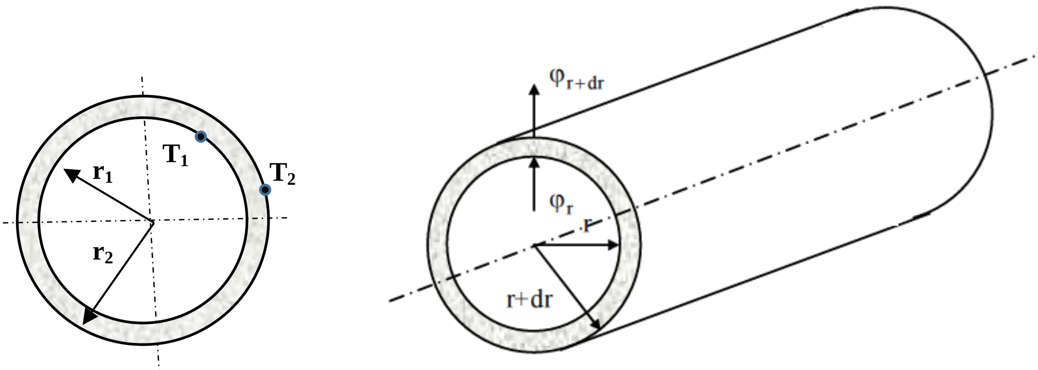


Figure II.9. Diagram of transfers in a hollow cylinder.

The conduction equation is:

$$\lambda.\Delta T + \dot{q}_s = \rho.c_p.\frac{\partial T}{\partial t} \qquad (II.31)$$

In the case of conduction without an internal source ($\dot{q}_s$=0 in steady state $\left(\frac{\partial T}{\partial t} = 0\right)$):

$$\Delta T = 0$$

In the case of the cylinder, the problem is radial and depends only on r. We recall that the Laplacian in cylindrical coordinates (independent of θ and z) is written as:

$$\Delta T = \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}$$

Let: $r\frac{d^2T}{dr^2} + \frac{dT}{dr} = 0$ or again $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$

After an initial integration:

$$r\frac{dT}{dr} = A$$

$$\frac{dT}{dr} = \frac{A}{r}$$

Where from :

$$T(r) = A.\ln r + B \qquad (II.32)$$

Let's consider Dirichlet boundary conditions:

$T=T_1$ if $r=r_1$
$T=T_2$ if $r=r_2$

We obtain the system: $\begin{cases} T_1 = A.\ln r_1 + B \\ T_2 = A.\ln r_2 + B \end{cases}$

Let

$$A = \frac{T_1 - T_2}{\ln\left(\frac{r_1}{r_2}\right)} \quad \text{et } B = \frac{(T_2 \ln r_1 - T_1 \ln r_2)}{\ln \frac{r_1}{r_2}}$$

Where from

$$T(r) = \frac{T_1 - T_2}{\ln\left(\frac{r_1}{r_2}\right)} \ln r + \frac{(T_2 \ln r_1 - T_1 \ln r_2)}{\ln \frac{r_1}{r_2}} \qquad (II.33)$$

And by applying Fourier's law: $\varphi = -\lambda.S.\frac{dT}{dr}$

Let:

$\varphi = -\lambda.(2.\pi.L.r).\frac{dT}{dr}$ if we consider a length L of the cylinder.

$$\frac{dT}{dr} = \left[\frac{T_1 - T_2}{\ln\left(\frac{r_1}{r_2}\right)}\right] \frac{1}{r}$$

Where from :

$$\varphi = -\frac{\lambda.2.\pi.L}{\ln\left(\frac{r_1}{r_2}\right)} (T_1 - T_2)$$

$$\varphi = \frac{\lambda.2.\pi.L}{\ln\left(\frac{r_2}{r_1}\right)} (T_1 - T_2) \qquad (II.34)$$

Thermal resistance is defined as:

$R_{12} = \frac{T_1 - T_2}{\varphi}$

Let $\qquad R_{12} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{\lambda.2.\pi.L} \qquad (II.35)$

We have thus defined a new thermal resistance for a cylindrical pipe.

This relationship is represented by the equivalent electrical diagram:

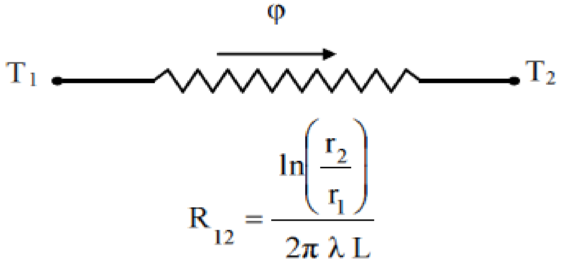


Figure II.10. Equivalent electrical circuit of a hollow cylinder.

II.3.6. Multi-layer hollow cylinder

This is the practical case of a tube covered with one or more layers of different materials, where only the temperatures T_{f1} and T_{f2} of the fluids in contact with the inner and outer faces of the cylinder are known; h_1 and h_2 are the heat transfer coefficients by convection between the fluids and the inner and outer faces. (see figure II.11)

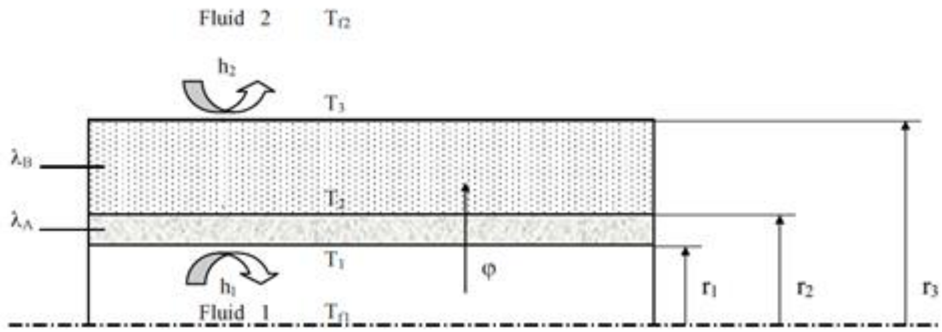


Figure II.11. Diagram of transfers in a multilayer hollow cylinder.

In steady state, the heat flux φ is conserved as it passes through the different layers and is expressed as:

$$\varphi = h_1.2\pi.r_1.L(T_{f1} - T_1) = \frac{2\pi\lambda_A L(T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)} = \frac{2\pi\lambda_B L(T_2 - T_3)}{\ln\left(\frac{r_3}{r_2}\right)} = h_2.2\pi r_3 L(T_3 - T_{f2})$$

Where from :

$$\varphi = \frac{T_{f1} - T_{f2}}{\frac{1}{h_1 2\pi r_1 L} + \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi\lambda_A L} + \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\pi\lambda_B L} + \frac{1}{h_2 2\pi r_3 L}} \qquad (II.36)$$

Which can be represented by the equivalent electrical diagram in Figure II.12.

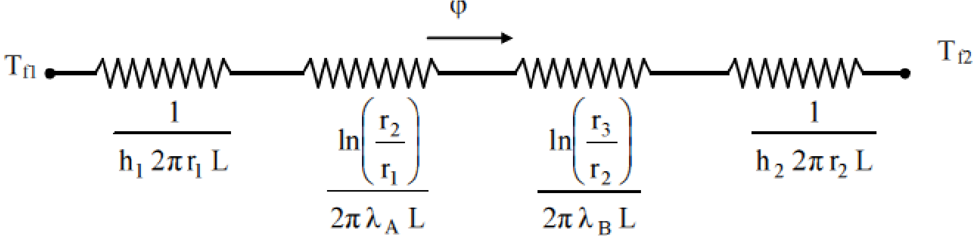


Figure II.13. Equivalent electrical diagram of a multilayer hollow cylinder.

In the case of a composite circular tube, composed for example of n superimposed materials limited by cylinders r_0, r_1… r_n with respective conductivities λ_1, λ_2, ……λ_n.

The thermal resistance of each cylinder is:

$$R_i = \frac{\ln\left(\frac{r_i}{r_{i-1}}\right)}{2\pi\lambda_i L}$$

The total thermal resistance of the tube is: $R_T = \sum_{i=1}^{n} R_i$ (series resistors) .

If the extreme temperatures T_1 and T_2 are imposed, the flux can be calculated using the relation:

$$\varphi = \frac{T_1 - T_2}{R_T}$$

Where R_T is the total thermal resistance.

II.3.7. Concentric Spheres

In the case of conduction without an internal source $\dot{q}_s=0$ in steady state $\left(\frac{\partial T}{\partial t} = 0\right)$:

$\Delta T=0$

Let's consider a hollow sphere with an outer radius r_e and an inner radius r_i. The problem is radial (r). In spherical coordinates, we have:

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

Either

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

Which leads to

$$T(r) = \frac{-A}{r} + B \qquad \text{(II.37)}$$

Considering Dirichlet boundary conditions:

$T=T_e$ if $r=r_e$

$T=T_i$ if $r=r_i$

We shoot

$$\begin{cases} T_i = \dfrac{-A}{r_i} + B \\ T_e = \dfrac{-A}{r_e} + B \end{cases}$$

$$\Rightarrow \begin{cases} A = \dfrac{(T_e - T_i)}{\dfrac{1}{r_i} - \dfrac{1}{r_e}} \\ B = T_i + \dfrac{1}{r_i} . \dfrac{(T_e - T_i)}{\dfrac{1}{r_i} - \dfrac{1}{r_e}} \end{cases}$$

$$\mathrm{T(r)} = \mathrm{T_i} + (\mathrm{T_i} - \mathrm{T_e}) \frac{\dfrac{1}{\mathrm{r}} - \dfrac{1}{\mathrm{r_i}}}{\dfrac{1}{\mathrm{r_i}} - \dfrac{1}{\mathrm{r_e}}} \qquad (\mathrm{II}.38)$$

And by applying Fourier's law:

$$\varphi = \frac{4\pi\lambda}{\dfrac{1}{\mathrm{r_i}} - \dfrac{1}{\mathrm{r_e}}} (\mathrm{T_i} - \mathrm{T_e}) \qquad (\mathrm{II}.39)$$

This relationship can also be expressed as:

$$\varphi = \frac{\mathrm{T_i} - \mathrm{T_e}}{\mathrm{R_{12}}}$$

With:

$$\mathrm{R_{12}} = \frac{\dfrac{1}{\mathrm{r_i}} - \dfrac{1}{\mathrm{r_e}}}{4\pi\lambda} \qquad (\mathrm{II}.40)$$

Exercises

Exercise 01:

A 15 cm thick concrete wall separates a room at a temperature of Ti= 20 °C from the outside where the temperature is Te= 5 °C.

We are given: hi = 9.1 W.m-2.K^{-1}, he = 16.7 W.m-2.K^{-1}, λ = 1.74 W.m^{-1}.K^{-1}.

Calculate:

- The total thermal resistance.
- The flux density.
- The internal and external temperatures of the wall.

Solution:

$$\varphi = h_i\left(T_i - T_{ip}\right) = \frac{\lambda}{e}\left(T_{ip} - T_{ep}\right) = h_e\left(T_{ep} - T_e\right) = \frac{(T_i - T_e)}{R}$$

With

$$R = \frac{1}{h_i} + \frac{e}{\lambda} + \frac{1}{h_e} = \frac{1}{9{,}1} + \frac{0{,}15}{1{,}74} + \frac{1}{0{,}06} = \mathbf{0,2562m^2.K.W^{-1}}$$

$$\varphi = \frac{(T_i - T_e)}{R} = \frac{20 - 5}{0{,}2562} = \mathbf{58,546W.m^2}$$

From the first equality:

$$T_{ip} = T_i - \frac{\varphi}{h_i} = 20 - 58{,}546\,.\,0{,}11 = \mathbf{13,6°C}$$

In the same way

$$T_{ep} = T_e + \frac{\varphi}{h_e} = 5 + 58{,}546\,.\,0{,}06 = \mathbf{8,5°C}$$

Exercise 02:

Calculate the flux passing through a glass pane with an area of 1 m² and a thickness of 3.5 mm. The temperature of the inner surface of the glass is 10°C, while that of the outer surface is 5°C.

1-Deduce the thermal resistance of the glass, knowing that the thermal conductivity of the glass is: λg= 0.7 W.m^{-1}.K^{-1}

2-For the same wall temperatures, calculate the flux passing through 1 m^2 of a 26 cm thick brick wall.

3-Deduce the thermal resistance, knowing that the thermal conductivity of the bricks is: $\lambda_b = 0.52\ W.m^{-1}.K^{-1}$.

Solution:

-Flux passing through 1m^2 of glass:

$$\varphi = \frac{\lambda.}{e}.S.(T_1 - T_2) = \frac{0,7}{3,5\,.\,10^{-3}}.1.(10-5) = \mathbf{1000\ W}$$

-Thermal resistance of 1m^2 of glass:

$$R_{glass} = \frac{(T_1 - T_2)}{\varphi} = \frac{10-5}{1000} = \mathbf{5.10^{-3}}\ \frac{\mathbf{°C}}{\mathbf{W}}$$

Or

$$R_{verre} = \frac{e}{\lambda.S} = \frac{3,5\,.\,10^{-3}}{0,7\,.\,1} = \mathbf{5\,.\,10^{-3}}\ \frac{\mathbf{°C}}{\mathbf{W}}$$

-Flux passing through 1m^2 of brick wall:

$$\varphi = \frac{\lambda.}{e}.S.(T_1 - T_2) = \frac{0,52}{0,26}.1.(10-5) = \mathbf{10\ W}$$

-Thermal resistance of 1m^2 of brick wall:

$$R_{brick} = \frac{(T_1 - T_2)}{\varphi} = \frac{10-5}{100} = \mathbf{0,5}\ \frac{\mathbf{°C}}{\mathbf{W}}$$

Or

$$R_{brick} = \frac{e}{\lambda.S} = \frac{0,26}{0,52\,.\,1} = \mathbf{0,5}\ \frac{\mathbf{°C}}{\mathbf{W}}$$

Exercise 03

A double glazing consists of two glass panes separated by a layer of still dry air. The thickness of each glass pane is 3.5 mm and that of the air layer is 12 mm.

The thermal conductivity of glass is equal to 0.7 $W.m^{-1}.°C^{-1}$ and that of air is 0.024 $W.m^{-1}.°C^{-1}$ in the studied temperature range. For a temperature drop of 5°C between the two extreme faces of the double glazing, calculate the heat losses for a 1m^2 window.

Compare these thermal losses to those that would be obtained with a single pane of glass with a thickness of 3.5 mm.

Solution:

-Double glazing consists of three thermal resistances in series.

$$\mathrm{R_{tot}} = \mathrm{R_v} + \mathrm{R_a} + \mathrm{R_v}$$

The flux passing through this double glazing is given by:

$$\varphi_{\text{double glazing}} = \frac{\mathrm{T_{int}} - \mathrm{T_{ext}}}{\mathrm{R_{tot}}} = \frac{\mathrm{T_{int}} - \mathrm{T_{ext}}}{\mathrm{R_g} + \mathrm{R_a} + \mathrm{R_g}} = \frac{\mathrm{T_{int}} - \mathrm{T_{ext}}}{\frac{\mathrm{e_g}}{\lambda_{\mathrm{g}}\mathrm{S}} + \frac{\mathrm{e_a}}{\lambda_{\mathrm{a}}.\mathrm{S}} + \frac{\mathrm{e_g}}{\lambda_{\mathrm{g}}.\mathrm{S}}} = \frac{(\mathrm{T_{int}} - \mathrm{T_{ext}}).S}{2\frac{\mathrm{e_g}}{\lambda_{\mathrm{g}}} + \frac{\mathrm{e_a}}{\lambda_{\mathrm{a}}.}}$$

$$\varphi_{\text{double glazing}} = \frac{5\,.\,1}{2.\frac{3{,}5\,.\,10^{-3}}{0{,}7} + \frac{12\,.\,10^{-3}}{0{,}024}} = \mathbf{9{,}8W}$$

Let's compare the flux passing through double glazing to that passing through a single glass pane for the same surface area and the same temperature difference.

$$\varphi_{1\text{ single pane}} = \frac{\mathrm{T_{int}} - \mathrm{T_{ext}}}{\mathrm{R_g}} = \frac{\lambda_g.S}{e_g}(\mathrm{T_{int}} - \mathrm{T_{ext}})$$

$$\varphi_{1\text{ single pane}} = \frac{0{,}7\,.\,1}{3{,}5\,.\,10^{-3}}\,.\,5 = \boldsymbol{1000\ W}$$

Exercise 04:

Calculate the flux passing through the 50 m² facade of a house. The wall is made of bricks 26 cm thick. The facade is pierced with 4 windows of 2 m² in surface area and 3.5 mm in thickness and a wooden door of 2 m² and 42 mm in thickness.

It is assumed that the internal wall temperature is equal to 10°C for all the materials constituting the facade, likewise, the external wall temperature is 5°C.

Thermal conductivity of glass: $\lambda g = 0.7\ \mathrm{W.m^{-1}.K^{-1}}$

Thermal conductivity of bricks: $\lambda b = 0.52\ \mathrm{W.m^{-1}.K^{-1}}$

Thermal conductivity of wood: $\lambda_{\mathrm{wood}} = 0.21\ \mathrm{W.m^{-1}.K^{-1}}$

Solution:

The facade can be likened to 3 resistances in parallel: that of the windows, that of the door, and that of the brick wall. We therefore calculate each of these resistances to deduce the equivalent resistance and finally to calculate the flux passing through the facade.

-Thermal resistance of windows:

$$\mathrm{R_{glass}} = \frac{\mathrm{e_{glass}}}{\lambda_\mathrm{g}.\,\mathrm{S_{glass}}} = \frac{3{,}5\,.10^{-3}}{0{,}7.\ 4.\ 2} = \mathbf{0,625\ .10^{-3}\ °C/W}$$

-Thermal resistance of the door:

$$\mathrm{R_{door}} = \frac{\mathrm{e_{door}}}{\lambda_\mathrm{wood}.\,\mathrm{S_{door}}} = \frac{0{,}042}{0{,}21\,.2} = \mathbf{0,1\ °C/W}$$

-Thermal resistance of the wall:

$$\mathrm{R_{wall}} = \frac{e_{wall}}{\lambda_\mathrm{b}.\,\mathrm{S_{wall}}} = \frac{0{,}26}{0{,}52.\,(50-4.\,2-2)} = \mathbf{0,0125\ °C/W}$$

-Equivalent resistance of the facade:

$$\frac{1}{\mathrm{R}} = \frac{1}{\mathrm{R_{vitre}}} + \frac{1}{\mathrm{R_{porte}}} + \frac{1}{\mathrm{R_{mur}}} = \frac{1}{0{,}625\,.10^{-3}} + \frac{1}{0{,}1} + \frac{1}{0{,}0125} = 1690\mathrm{W/°C}$$

$$\mathrm{R} = \frac{1}{1690} = \mathbf{0,592\,.10^{-3}\ °C/W}$$

So the flow passing through the facade is:

$$\varphi_{facade} = \frac{(\mathrm{T_1}-\mathrm{T_2})}{\mathrm{R}} = 5.\ 1690 = \mathbf{8450\ W}$$

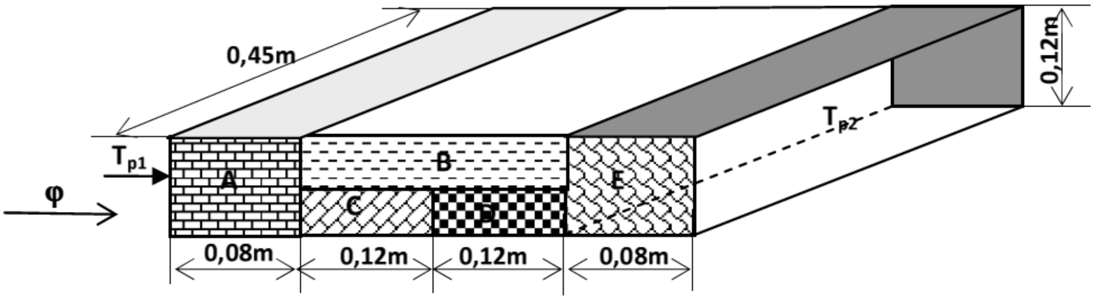
Exercise 05:

Let's consider a wall composed of several layers of different materials illustrated below with the dimensions in the three directions. Assuming one-dimensional conduction and knowing the temperatures of the left and right surfaces, respectively, Tp_1 and Tp^2, as well as the thermal conductivities of these different layers.

-Calculate the heat flux per unit area through this wall.

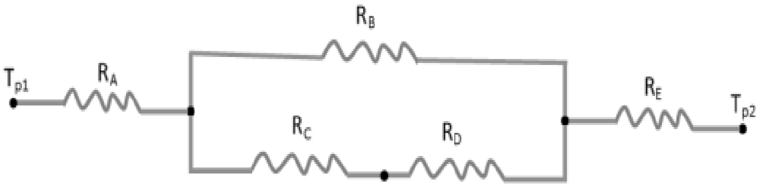
Data:

T_{p1} =200°C, T_{p2} =50°C, k_A =70 W/m.K, k_B =60 W/m.K, k_C =40 W/m.K, k_D =30 W/m.K, k_E =20 W/m.K



Solution:

To simplify the calculation, we must represent the equivalent diagram of the wall in question, of course based on the analogy existing between thermal and electrical quantities.



Equivalent electrical diagram

The heat flux is therefore given by:

$$\varphi = \frac{T_{P1} - T_{p2}}{R_A + R_{éq} + R_E}$$

such as;

$$\begin{cases} R_A = \dfrac{L_A}{k_A.S_A} = \dfrac{0{,}08}{70\,.\,0{,}12\,.\,0{,}45} = 0{,}02116 K/W \\ R_B = \dfrac{L_B}{k_B.S_B} = \dfrac{0{,}24}{60\,.\,0{,}06\,.\,0{,}45} = 0{,}1481 K/W \\ R_C = \dfrac{L_C}{k_C.S_C} = \dfrac{0{,}12}{40\,.\,0{,}06\,.\,0{,}45} = 0{,}1111 K/W \\ R_D = \dfrac{L_D}{k_D.S_D} = \dfrac{0{,}12}{30\,.\,0{,}06\,.\,0{,}45} = 0{,}1481 K/W \\ R_E = \dfrac{L_E}{k_E.S_E} = \dfrac{0{,}08}{20\,.\,0{,}12\,.\,0{,}45} = 0{,}0740 K/W \end{cases}$$

$$\frac{1}{R_{éq}} = \frac{1}{R_B} + \frac{1}{R_C + R_D} = \frac{R_B + R_C + R_D}{R_B.(R_C + R_D)}$$

$$\Rightarrow R_{éq} = \frac{R_B.(R_C + R_D)}{R_B + R_C + R_D}$$

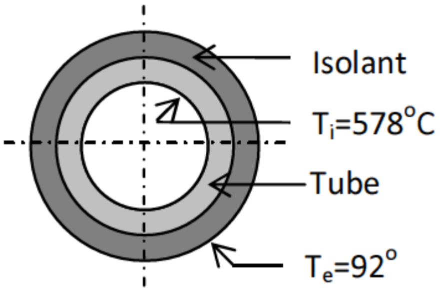
$$R_{éq} = \frac{0{,}1481.(0{,}1111 + 0{,}1481)}{0{,}1481 + 0{,}1111 + 0{,}1481} = \mathbf{0{,}09425K/W}$$

$$\varphi = \frac{(T_{P1} - T_{P2})}{R_A + R_{éq} + R_E} = \frac{(200 - 50)}{0{,}02116 + 0{,}09425 + 0{,}0740} = \mathbf{791{,}94W}$$

Exercise 6:

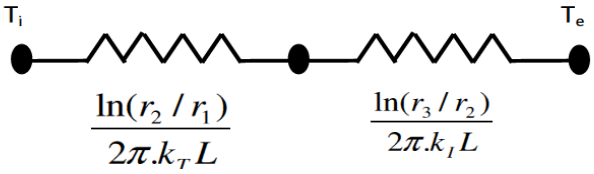
A stainless steel tube (k=19W/m.°C) with a 3cm inner diameter and a 5cm outer diameter is insulated by a layer of asbestos (k=0.2 W/m.°C) with a thickness of 2.5cm. Knowing that the temperature of the inner wall of the tube is maintained at 578°C and that of the outer wall of the insulation is at 92°C.

1. Provide the equivalent electrical diagram.
2. Calculate the heat loss per meter of length.



Solution:

1. Equivalent electrical diagram:



2. The heat flux lost per meter of length:

$$\varphi = \frac{\Delta T}{\sum R_i} = \frac{T_i - T_e}{R_T + R_I}$$

$$R_T = \frac{\ln(r_2/r_1)}{2\pi.k_T.L}$$

$$R_I = \frac{\ln(r_3/r_2)}{2\pi.k_I.L}$$

$$\varphi = \frac{2\pi.L(T_i - T_e)}{\frac{\ln(r_2/r_1)}{k_T} + \frac{\ln(r_3/r_2)}{k_I}}$$

$$\Rightarrow \frac{\varphi}{L} = \frac{2\pi(T_i - T_e)}{\frac{\ln(r_2/r_1)}{k_T} + \frac{\ln(r_3/r_2)}{k_I}}$$

$$\frac{\varphi}{L} = \frac{2\pi(578 - 92)}{\frac{\ln(2,5/1,5)}{19} + \frac{\ln(5/2,5)}{k_I}} = \mathbf{874,3087W/m}$$

Chapitre III
Heat Transfer by Conduction in a Variable Regime

III.1. Introduction

When the temperature of a medium undergoes a sudden change, it will take some time to reach a new equilibrium value. It evolves over time for most industrial applications (heating, cooling, etc.).

To analyze heat transfer by conduction in a transient regime, one necessarily proceeds with the resolution of the heat equation in the form:

$$\frac{1}{a}\frac{\partial T}{\partial t} = \Delta T + \frac{\dot{q}}{\lambda} \qquad (\mathrm{III.1})$$

Before delving into the methods for solving this equation in detail, we first examine a particular case where the temperature distribution is solely temporal (uniform temperature throughout the medium).

III.2. Method of Thermal Capacity

We consider a small metallic ball initially at temperature Ti and placed in a bath at constant temperature T_0. If we assume that the temperature inside the ball is uniform.

We can model this solid as an "input resistance" (convection) followed by a "capacity." Indeed, the solid receives a flow that has the effect of modifying its temperature. Moreover, the entire flow is used to raise its temperature. There is therefore no "output resistance."

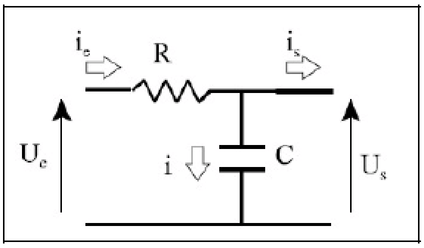


Figure III.1. Electrical Analogy

So we can write the following equations:

a) Electrical equations:

$$\begin{cases} i = i_e - i_s = C.\dfrac{dU}{dt} \\ U = U_e - U_s = R.i_e \end{cases}$$

If we recall that according to the electrical-thermal analogy

$$i \rightarrow \dot{Q}$$
$$U \rightarrow T$$
$$C \rightarrow m.c_p$$
$$R_{convection} \rightarrow {}^{1}/_{(h.S)}$$

b) Thermal equations:

$$\begin{cases} \dot{Q} = \dot{Q}_e - \dot{Q}_s = m.c_p.\dfrac{dT}{dt} \\ T = T_e - T_s = \dfrac{1}{h.S}.\dot{Q}_e \end{cases}$$

And if we replace $\dot{Q}_e$ in the first equation, we get:

$$(T_e - T_s).h.S - \dot{Q}_s = m.c_p.\frac{dT}{dt} \qquad (III.2)$$

With :

T_e : T_0 (Ambient temperature)

T_s: variable $T(t)$

$\dot{Q}_s = 0$: everything stays in the solid

We write:

$$h.S\big(T_0 - T(t)\big) = m.c_p.\frac{dT}{dt} \Rightarrow (T(t) - T_0) = -\frac{m.c_p}{h.S}\frac{dT}{dt}$$
$$\Rightarrow \frac{dT}{(T(t) - T_0)} = -\frac{h.S}{\rho.c_p.V}dt$$

$$(T(t) - T_0) = C.\exp\left(-\frac{h.S}{\rho.c_p.V}t\right) \qquad (III.3)$$

We determine the constant of the integral C from the initial condition:

At t=0 $T(0)= T_i$

$(III.3) \Rightarrow T_i - T_0 = C$

$$\Rightarrow \frac{T(t) - T_0}{T_i - T_0} = \exp\left(-\frac{h.S}{\rho.c_p.V}t\right) \qquad (III.4)$$

Where S and V represent respectively the surface area and the volume of the ball.

We notice that the grouping $\frac{\rho.c_p.V}{h.S}$ is homogeneous to a time, we will call it τ the time constant of the system:

$$\tau = \frac{\rho.c_p.V}{h.S} \qquad (III.5)$$

This quantity is fundamental in that it provides the order of magnitude of time for the physical phenomenon; indeed, we have:

$$\frac{T - T_0}{T_i - T_0} = \exp\left(-\frac{t}{\tau}\right) \qquad (III.6)$$

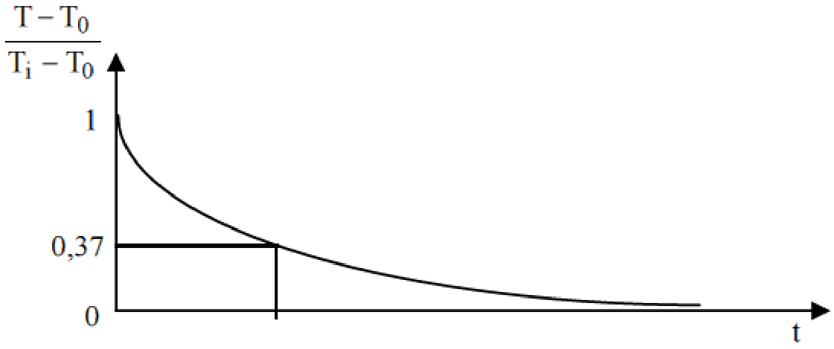


Figure III.2. Evolution of the temperature of a medium at uniform temperature

In equation (III.4), the exponential term can be expressed in the form:

$$\frac{h.S}{\rho.c_p.V}t = \left(\frac{h.V}{\lambda.S}\right)\left(\frac{S^2.\lambda.t}{\rho.c_p.V^2}\right) = \left(\frac{h.V}{\lambda.S}\right)\left(\frac{a.t}{\left(V/S\right)^2}\right) = B_i.F_0$$

It is always interesting in physics to present the results in a dimensionless form, two dimensionless numbers are particularly important in variable regimes:

-***The Biot number:*** represents the ratio of conductive thermal resistance $\left(R_{cd} = \frac{L}{\lambda.S}\right)$ to convective thermal resistance $\left(R_{cv} = \frac{1}{h.S}\right)$

L is the characteristic dimension of the medium, L=r for a sphere.

Let:

$$B_i = \frac{h.L}{\lambda} \qquad (III.7)$$

The assumption of uniform temperature is justified when Bi < 0.1.

- ***The Fourier number:*** the Fourier number characterizes the penetration of heat in a variable regime.

$$F_0 = \frac{a.t}{L^2} \qquad (III.8)$$

The definition of these two numbers allows us to write the expression for the temperature of the ball in the form:

$$\frac{T - T_0}{T_i - T_0} = \exp(-B_i.F_0) \qquad (III.9)$$

The knowledge of the product of the Biot and Fourier numbers allows for determining the temperature evolution of the sphere. It is generally considered that a system such that Bi < 0.1 can be regarded as having a uniform temperature; the criterion Bi < 0.1 is called the "thermal accommodation" criterion.

III.3. Semi-infinite medium

A semi-infinite medium is a wall of sufficient thickness so that the disturbance applied to one face is not felt by the other face. Such a system represents the evolution of a wall of finite thickness for a sufficiently short time so that the disturbance created on one face has not reached the other face.

III.3.1.Constant temperature imposed on the surface(Dirichlet condition)

Method: *Laplace integral transform over time and inversion using tables*.

The semi-infinite medium is initially at the uniform temperature Ti. We abruptly impose the temperature T0 on its surface, this boundary condition is called the Dirichlet condition:

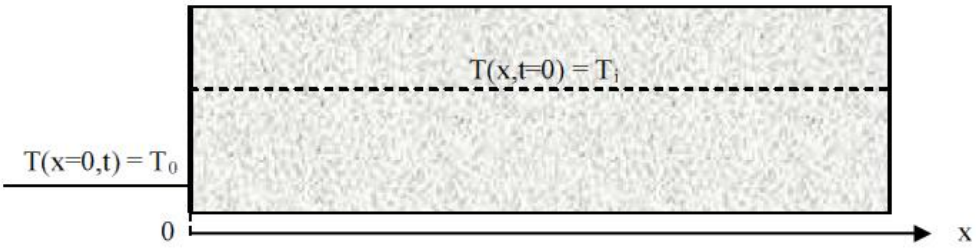


Figure III.3. Diagram of the semi-infinite medium with imposed surface temperature.

The heat equation is written as:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t} \qquad (\mathrm{III.10})$$

With the boundary conditions:

$$\begin{cases} T(x,0) = T_i & (\mathrm{III.11}) \\ T(x=0,t) = T_0 & (\mathrm{III.12}) \\ \lim\limits_{x\to\infty} T(x,t) = T_i & (\mathrm{III.13}) \end{cases}$$

We perform the following change of variable:

$$\overline{T} = T - T_i$$

Hence:

$$\frac{\partial \overline{T}}{\partial x} = \frac{\partial T}{\partial x}, \frac{\partial^2 \overline{T}}{\partial x^2} = \frac{\partial^2 T}{\partial x^2} \text{ et } \frac{\partial \overline{T}}{\partial t} = \frac{\partial T}{\partial t}$$

The equation (III.10) can then be written as:

$$\frac{\partial^2 \overline{T}}{\partial x^2} = \frac{1}{a}\frac{\partial \overline{T}}{\partial t} \qquad (\mathrm{III.14})$$

And the boundary conditions become:

$$\begin{cases} \overline{\mathrm{T}}(\mathrm{x},0) = 0 & \text{(III.15)} \\ \overline{\mathrm{T}}(\mathrm{x}=0,\mathrm{t}) = \mathrm{T}_0 - T_i & \text{(III.16)} \\ \lim\limits_{\mathrm{x}\to\infty} \overline{\mathrm{T}}(\mathrm{x},\mathrm{t}) = 0 & \text{(III.17)} \end{cases}$$

The Laplace transform of $\overline{\mathrm{T}}$(x,t) with respect to time is written (see Appendix C.1 on integral transformations):

$$\theta(\mathrm{x},\mathrm{p}) = \mathrm{L}\{\overline{\mathrm{T}}(\mathrm{t})\} = \int_0^\infty \exp(-\mathrm{pt}).\overline{\mathrm{T}}(\mathrm{x},\mathrm{t})\mathrm{dt}$$

The Laplace transform of equation (III.14) leads to:

$$\frac{\mathrm{d}^2\theta}{\mathrm{dx}} - \frac{1}{\mathrm{a}}[\mathrm{p}\theta - \overline{\mathrm{T}}(\mathrm{x},0)] = 0 \ \text{ avec } \ \overline{\mathrm{T}}(\mathrm{x},0) = 0$$

This equation is therefore of the form:

$$\frac{\mathrm{d}^2\theta}{\mathrm{dx}} - q^2\theta = 0 \qquad \text{(III.18)}$$

With: $\mathrm{q}^2 = \frac{\mathrm{p}}{\mathrm{a}}$

Hence:

$$\theta(\mathrm{x},\mathrm{p}) = \mathrm{A}.\mathrm{e}^{-\mathrm{qx}} + \mathrm{B}.\mathrm{e}^{+\mathrm{qx}} \qquad \text{(III.19)}$$

The temperature remains finite as x →∞, so B = 0, we deduce that:

$$\theta(\mathrm{x},\mathrm{p}) = \mathrm{A}.\mathrm{e}^{-\mathrm{qx}}$$

The Laplace transform of equation (III.16) leads to

$$\theta(0,\mathrm{p}) = \frac{\mathrm{T}_0 - \mathrm{T}_\mathrm{i}}{\mathrm{p}} \ \text{ d'où } \ \mathrm{A} = \frac{\mathrm{T}_0 - \mathrm{T}_\mathrm{i}}{\mathrm{p}}$$

$$\Rightarrow \theta = (\mathrm{T}_0 - \mathrm{T}_\mathrm{i})\frac{\mathrm{e}^{-\mathrm{qx}}}{\mathrm{p}} \qquad \text{(III.20)}$$

The use of the inverse Laplace transform tables presented in Appendix C.2 leads to the following result:

$$\frac{T(x,t)-T_0}{T_i-T_0}=\operatorname{erfc}\left(\frac{x}{2\sqrt{at}}\right) \qquad (III.21)$$

With : $\operatorname{erfc}(u)=\frac{2}{\sqrt{\pi}}\int_0^u \exp(-t^2)\,dt$

The erf function is called the error function. (cf. valeurs en annexe C.3)

III.3. 2. Imposed Flux (Condition de Neumann)

Method: Laplace integral transform over time and inversion using tables.

Let's consider the same configuration but by abruptly imposing a heat flux density on the surface of the semi-infinite medium; this boundary condition is called the Neumann condition.

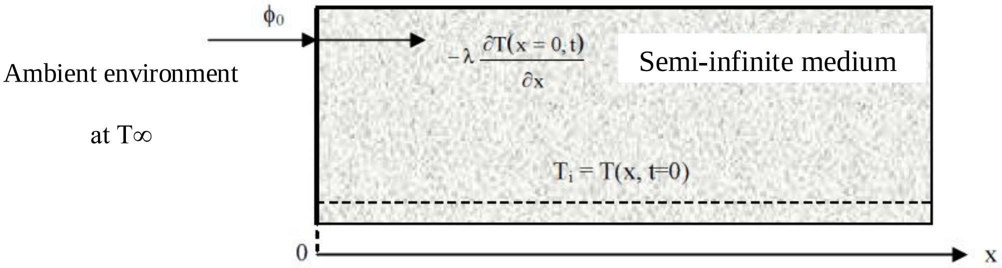


Figure III.4. Diagram of the semi-infinite medium with an imposed surface flux**.**

The heat equation is written as:

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{a}\frac{\partial T}{\partial t} \qquad (III.10)$$

With the boundary conditions:

$$\begin{cases} T(x,0)=T_i & (III.22)\\ T(\infty,t)=T_i & (III.23)\\ -\lambda\dfrac{\partial T(0,t)}{\partial x}=\phi_0 & (III.24)\end{cases}$$

The solution of the differential equation with these initial conditions using the inverse Laplace transform tables presented in Appendix C.2 gives the temperature distribution:

$$\overline{T}(x,t) = T(x,t) - T_i = \frac{2\phi_0}{\lambda}\sqrt{a.t}\ \text{ierfc}\left(\frac{x}{2\sqrt{a.t}}\right) \qquad (III.25)$$

With :

$$\text{ierfc}(u) = \frac{1}{\sqrt{\pi}}\exp(-u^2) - u[1 - \text{erf}(u)],$$

This function is tabulated in Appendix C.3

III.3.3. Imposed transfer coefficient

Method: *Laplace integral transform over time and inversion using tables.*

We consider the case where the convective heat transfer coefficient h between the semi-infinite medium and the ambient medium is imposed; this boundary condition is called the Newton condition.

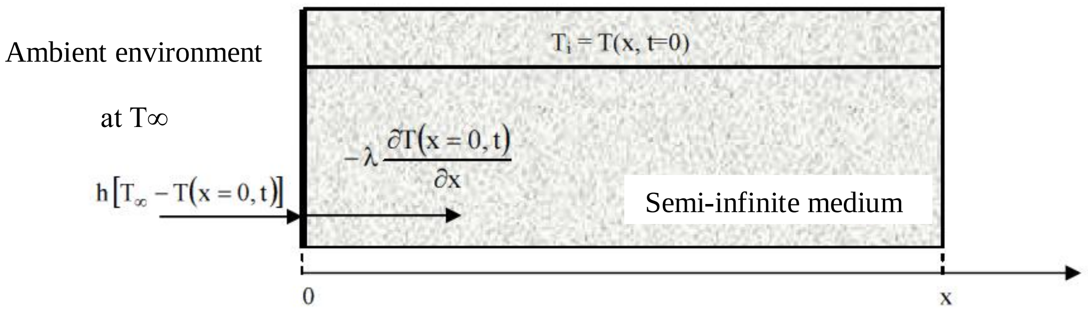


Figure III.4. Diagram of the semi-infinite medium with an imposed convective transfer coefficient.

The heat equation is written as:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t} \qquad (III.26)$$

With the boundary conditions:

$$\begin{cases} T(x,0) = T_i & (III.27) \\ T(\infty,t) = T_i & (III.28) \\ -\lambda \dfrac{\partial T(0,t)}{\partial x} = h[T_\infty - T(x=0,t)] & (III.29) \end{cases}$$

The solution of the differential equation with these initial conditions using the inverse Laplace transform tables presented in Appendix C.2 gives the temperature distribution:

$$\frac{T - T_\infty}{T_i - T_\infty} = \mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right) + \exp\left(\frac{h.x}{\lambda} + \frac{ah^2t}{\lambda^2}\right)\mathrm{erfc}\left(\frac{x}{2\sqrt{at}} + \frac{h\sqrt{at}}{\lambda}\right) \qquad (III.30)$$

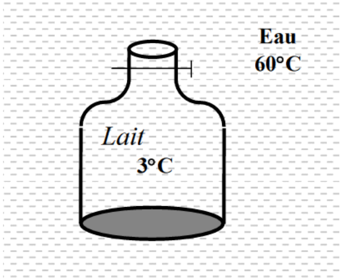
Exercises

Exercise 1

To warm up baby milk, the mother pours the milk into a thin glass wall with a diameter of 6 cm. The height of the milk in the glass is 7 cm. She then places the glass in a large pot filled with water at 60°C. The milk is constantly stirred, so its temperature is uniform at all times. If the heat transfer coefficient between the water and the glass is 120W/m².°C, determine the time it takes for the milk to warm up from 3°C to 38°C. Assuming that the properties of the milk are the same as those of water. Under these conditions, milk can be treated as a thermally thin medium.

Given: The thermal conductivity, density, and specific heat of water at 20°C are:

λ=0.607 W/m.°C, ρ= 998 kg/m3, and Cp = 4.182 kJ/kg.°C.



Solution

Assumptions:

1. The glass container is cylindrical with a radius of r0 = 3 cm.
2. The thermal properties of the milk are taken as those of water.
3. The thermal properties of milk are constant at room temperature.
4. The heat transfer coefficient is constant and uniform over the entire surface.
5. The Biot number, in this case, is important. (beaucoup plus grand que 0,1).

However, the overall analysis system is still applicable since the milk is constantly stirred, ensuring its temperature remains uniform at all times.

Analysis: The characteristic length and the Biot number of the glass of milk are:

$$L_C = \frac{V}{S} = \frac{\pi r_0^2 L}{2\pi r_0 . L + 2\pi r_0^2} = \frac{\pi(0,03)^2.(0,07)}{2\pi(0,03)(0,07) + 2\pi(0,03)^2} = 0,01050\text{m}$$

$$\mathrm{B_i} = \frac{\mathrm{h.L_c}}{\lambda} = \frac{120\ .0{,}0105}{0{,}607} = 2{,}076 > 0{,}1$$

For the reasons mentioned above, we can use the global analysis system to determine the time required for the milk to warm up from 3°C to 38°C.

$$\mathrm{b} = \frac{\mathrm{h.S}}{\rho.\mathrm{C_p.V}} = \frac{\mathrm{h}}{\rho.\mathrm{C_p.L_c}} = \frac{120}{998\,.4182\,.0{,}0105} = 0{,}002738\ \mathrm{s^{-1}}$$

$$\frac{\mathrm{T(t) - T_\infty}}{\mathrm{T_i - T_\infty}} = \mathrm{e^{-bt}} \Rightarrow \frac{38-60}{3-60} = \mathrm{e^{-(0{,}002738).t}} \Rightarrow \mathbf{t = 348s = 5,8min}$$

Therefore, it will take approximately **6 minutes** to heat the milk from 3°C to 38°C.

Exercice 2

A cylindrical bar made of stainless steel, with a diameter of 10 cm, has a uniform temperature of 200°C. The bar is cooled by moving air at 25°C.

$h = 142\ \mathrm{W/m^2°C}$, $\alpha = 4.5 \times 10^{-6}\ \mathrm{m^2/s}$, $k = 15.5\ \mathrm{W/m°C}$

-Estimate the time for the center of the bar to cool to 35°C.

-At this moment, what is the surface temperature?

Solution

$$\mathrm{B_i} = \frac{\mathrm{h.r}}{\lambda} = \frac{142.0{,}05}{15{,}5} = 0{,}45$$

$$\frac{\mathrm{T - T_f}}{\mathrm{T_i - T_f}} = \frac{35-25}{200-25} = \frac{10}{175} = 0{,}057$$

$$F_0 = \frac{\alpha.t}{r^2} \cong 4 \Rightarrow t = \frac{4r^2}{\alpha} = \frac{4\,.0{,}05^2}{4{,}5.\ 10^{-6}} = \boldsymbol{2222,22\ s}$$

$$\frac{T-25}{200-25} = 0{,}05$$

With

$F_0 = 4, B_i = 0{,}45$

T=0,05 . 175 + 25 =**33,75°C**

Exercice 3

A steel ball with a diameter of 8cm and initially at a uniform temperature of 288°C is suddenly immersed in an environment where the temperature is 90°C with a convection coefficient of 10 W/m²°C. Determine the time required for the ball to reach a temperature of 144°C.

The characteristics of the ball are:

ρ=7850kg/m³, C=0.46 kJ/kg.°C, k=46 W/m.°C.

Solution

In this case, the heat losses by conduction are negligible, those by convection will obviously be represented by a decrease in the internal energy of the considered ball.

The heat flow exchanged by convection = decrease in the internal energy of the ball.

Fluid:

$\varphi_c = h.S.(T - T_1);$ $T_1 = 92°C$

The ball:

$$Q = -m.c.\frac{dT}{dt} = \rho.V.c.\frac{dT}{dt}$$

$$h.S.(T - T_1) = -\rho.V.c.\frac{dT}{dt} \Rightarrow \int \frac{dT}{T - T_1} = -\frac{h.S}{\rho.V.c}.\int dt \Leftrightarrow \ln(T - T_1) = -\frac{h.S}{\rho.V.c}.t + \ln C_1$$

$$\Rightarrow T - T_1 = C_1.e^{\frac{h.S}{\rho.V.c}.t}$$

A t=0 : T= T_i=288°C. (of the ball) :

$T-T_i=C_1.\ e^0$ => $C_1=T_i-T$.

$$\frac{T-T_1}{T_i-T_1}=e^{\frac{-h.S}{\rho.V.c}.t}=e^{-\alpha.t},$$

$$\alpha=\frac{h.S}{\rho.V.c},$$

$$S=4.\pi.r^2, V=\frac{4}{3}.\pi.r^3$$

$$\alpha=\frac{h.4\pi.r^2}{\rho.(4/3)\pi.r^3.c}=\frac{3h}{\rho.r.c}=\frac{3\ .10}{7850.\ \ 0{,}04\ \ .460}=2{,}077.10^{-4}\ s^{-1}$$

$$\frac{T-T_1}{T_i-T_1}=e^{-2{,}077\ .10^{-4}.t},$$

at $T=144°C\ ;t=?$

$$e^{-2{,}077\ .10^{-4}.t}=\frac{144-92}{288-92}=0{,}265\ \Leftrightarrow -2{,}077\ .10^{-4}.t=\ln(0{,}265)\Rightarrow t=\frac{\ln(0{,}265)}{-2{,}077.10^{-4}}$$

$$t=6394s\approx 106{,}567\min\approx \mathbf{1\ h\ 46,567min}$$

Chapiter IV

Heat Transfer by Convection

IV.1. Introduction

Thus far we have focused on heat transfer by conduction and have considered convection only to the extent that it provides a possible boundary condition for conduction problems. We used the term convection to describe energy transfer between a surface and a fluid moving over the surface. Convection includes energy transfer by both the bulk fluid motion (advection) and the random motion of fluid molecules (conduction or diffusion). In our treatment of convection, we have two major objectives. In addition to obtaining an understanding of the physical mechanisms that underlie convection transfer, we wish to develop the means to perform convection transfer calculations.

This chapter is developed primarily to achieving the former objective. Physical origins are discussed, and relevant dimensionless parameters, as well as important analogies, are developed.

IV.2. Definitions

Convection is defined as the mode of heat transfer between a solid surface and a fluid (liquid or gas) at different temperatures. It is therefore a transfer of heat accompanied by speed.

In convection, the transfer of heat at the solid surface occurs only through conduction. But, in the parts of the fluid that surround the surface, two simultaneous phenomena occur: conduction and mass diffusion through movement at both the molecular and macroscopic levels. Thanks to this movement, the transferred heat flow is greater. The higher the speed, the greater the heat transfer. We can summarize the mechanism of heat transfer by convection as follows: the fluid in contact with the solid surface receives heat from it through conduction and then transfers it to the rest of the fluid that is not in direct contact with the surface through diffusion, thanks to the movement of the fluid.

We can consider two types of convection, depending on the causes that produce the movement of the fluid: forced convection and free or natural convection.

•***Forced convection***: occurs when the movement of the fluid is a consequence of imposed external actions (pump, fan, wind, etc.). In this case, the temperature field is convected by an imposed external flow.

• ***Free or natural convection***: is produced by a movement of the fluid, caused by a difference in density between the cold and hot parts of the fluid. The distribution of temperature generates its own movement by creating rotational Archimedean forces.

• When both types of convection exist simultaneously without one being negligible compared to the other, the convection is said to be ***mixed.***

IV.3. Physical mechanism of convection

Convection heat transfer is complicated by the fact that it involves fluid motion as well as heat conduction. The fluid motion enhances heat transfer, since it brings warmer and cooler chunks of fluid into contact, initiating higher rates of conduction at a greater number of sites in a fluid. Therefore, the rate of heat transfer through a fluid is much higher by convection than it is by conduction. In fact, the higher the fluid velocity, the higher the rates of heat transfer.

Experience shows that convection heat transfer strongly depends on the fluid properties dynamic viscosity m, thermal conductivity λ, density ρ, and specific heat cp, as well as the fluid velocity U. It also depends on the geometry and the roughness of the solid surface, in addition to the type of fluid flow (such as being streamlined or turbulent). Thus, we expect the convection heat transfer relations to be rather complex because of the dependence of convection on so many variables. This is not surprising, since convection is the most complex mechanism of heat transfer.

Despite the complexity of convection, the rate of convection heat transfer is observed to be proportional to the temperature difference and is conveniently expressed by Newton's law of cooling as

$$\varphi = h.S.(T_S - T_\infty) \qquad (IV.1)$$

Where
h : convection heat transfer coefficient, W/m^2·K
S : heat transfer surface area, m^2
T_s: temperature of the surface, °C
T_∞: temperature of the fluid sufficiently far from the surface, °C.

Fluid flow is often confined by solid surfaces, and it is important to understand how the presence of solid surfaces affects fluid flow.

All experimental observations indicate that a fluid in motion comes to a complete stop at the surface and assumes a zero velocity relative to the surface. That is, a fluid in direct contact with a solid “sticks” to the surface due to viscous effects, and there is no slip. This is known as the no-slip condition. A fluid layer adjacent to a moving surface has the same velocity as the surface.

A consequence of the no-slip condition is that all velocity profiles must have zero values with respect to the surface at the points of contact between a fluid and a solid surface(Figure IV.1).

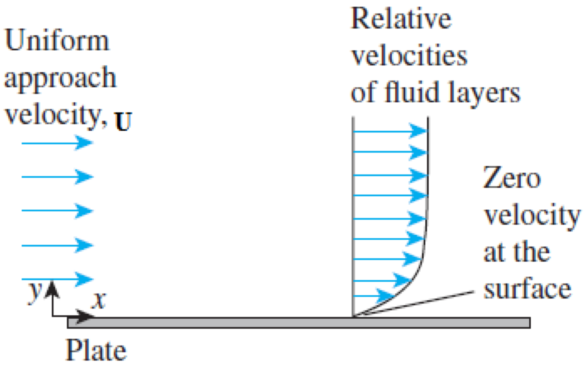


Figure IV.1. A fluid flowing over a stationary surface comes to a complete stop at the surface because of the no-slip condition

An implication of the no-slip condition is that heat transfer from the solid surface to the fluid layer adjacent to the surface is by pure conduction, since the fluid layer is motionless, and can be expressed as

$$\varphi_{conv} = \varphi_{cond} = -\lambda_{fluid} . S . \left.\frac{\partial T}{\partial y}\right]_{y=0} \qquad (IV.2)$$

Where T represents the temperature distribution in the fluid and $\left.\frac{\partial T}{\partial y}\right]_{y=0}$ is the *temperature gradient* at the surface. Heat is then *convected away* from the surface as a result of fluid motion. Note that convection heat transfer from a solid surface to a fluid is merely the conduction heat transfer from the solid surface to the fluid layer adjacent to the surface. Therefore, we can equate Eqs. IV.1 and IV.2 for the heat flux to obtain

$$h = \frac{-\lambda_{\text{fluid}} \cdot \left.\frac{\partial T}{\partial y}\right|_{y=0}}{(T_S - T_\infty)} \qquad (IV.3)$$

IV.4. Classification of fluid flows

Convection heat transfer is closely tied with fluid mechanics, which is the science that deals with the behavior of fluids at rest or in motion, and the interaction of fluids with solids or other fluids at the boundaries. There is a wide variety of fluid flow problems encountered in practice, and it is usually convenient to classify them on the basis of some common characteristics to make it feasible to study them in groups. There are many ways to classify fluid flow problems, and here we present some general categories.

IV.4.1.Viscous versus Inviscid Regions of Flow

When two fluid layers move relative to each other, a friction force develops between them and the slower layer tries to slow down the faster layer. This internal resistance to flow is quantified by the fluid property *viscosity*, which is a measure of internal stickiness of the fluid. Viscosity is caused by cohesive forces between the molecules in liquids and by molecular collisions in gases. There is no fluid with zero viscosity, and thus all fluid flows involve viscous effects to some degree. Flows in which the frictional effects are significant are called ***viscous flows***. However, in many flows of practical interest, there are *regions* (typically regions not close to solid surfaces) where viscous forces are negligibly small compared to inertial or pressure forces. Neglecting the viscous terms in such ***inviscid flow regions*** greatly simplifies the analysis without much loss in accuracy.

IV.4.2. Internal versus External Flow

A fluid flow is classified as being internal or external, depending on whether the fluid is forced to flow in a confined channel or over a surface. The flow of an unbounded fluid over a surface such as a plate, a wire, or a pipe is ***external flow***. The flow in a pipe or duct is ***internal flow*** if the fluid is completely bounded by solid surfaces. Water flow in a pipe, for example, is internal flow, and airflow over a ball or over an exposed pipe during a windy day is external flow. The flow of liquids in a duct is called *open-channel flow* if the duct is only partially filled with the liquid and there is a free surface. The flows of water in rivers and irrigation ditches are examples of such flows. Internal flows are dominated by the influence of viscosity throughout the flow field. In external flows the viscous effects are limited to boundary layers near solid surfaces and to wake regions downstream of bodies.

IV.4.3. Compressible versus Incompressible Flow

A flow is classified as being compressible or incompressible, depending on the level of variation of density during flow. Incompressibility is an approximation, and a flow is said to be incompressible if the density remains nearly constant throughout. Therefore, the volume of every portion of fluid remains unchanged over the course of its motion when the flow (or the fluid) is incompressible.

IV.4.4. Laminar versus Turbulent Flow

To address a convection problem, it is important to determine the flow regime of the fluid. The convective heat transfer coefficient depends heavily on the flow regime. Some flows are smooth and orderly while others are rather chaotic. The highly ordered fluid motion characterized by smooth layers of fluid is called ***laminar***. The word laminar comes from the movement of adjacent fluid particles together in “laminates.” The flow of high-viscosity fluids such as oils at low velocities is typically laminar. The highly disordered fluid motion that typically occurs at high velocities and is characterized by velocity fluctuations is called ***turbulent*** (Figure IV.2). The flow of low-viscosity fluids such as air at high velocities is typically turbulent. The flow regime greatly influences the required power for pumping. A flow that alternates between being laminar and turbulent is called transitional.

The transition between laminar and turbulent flow can be determined by calculating the critical Reynolds number. In the case of a flat wall:

$$\mathrm{Re_c} = \frac{\mathrm{U_\infty . x}}{\mathrm{\nu}} \qquad \text{(IV. 4)}$$

U_∞ is the free stream velocity far from the solid obstacle and ν is the kinematic viscosity in m^2/s:

$$\nu = \frac{\mu}{\rho}$$

μ is the dynamic viscosity.

In the case of flow in a cylindrical tube of diameter D, the Reynolds number is:

$$\mathrm{Re_c} = \frac{\mathrm{U_\infty . D}}{\nu} = \frac{\rho.\mathrm{U_\infty . D}}{\mu} \qquad \text{(IV. 5)}$$

The Reynolds number represents the ratio of inertial forces to viscous forces.

The critical Reynolds number depends on the surface roughness and the level of turbulence of the free flow. It is generally on the order of 10^5 to $3\text{x}10^6$. In the cylinders, it is generally equal to 2300. It characterizes the transition from laminar flow to turbulent flow, that is to say:

If $\mathrm{Re} < \mathrm{Re_c}$ the flow is laminar

If $\mathrm{Re} > \mathrm{Re_c}$ the flow is turbulent.

In general, the representative value of the critical Reynolds number is:

$\mathrm{Re_c} = 5\text{x}10^5$ for flat plates

$\mathrm{Re_c} = 2300$ for cylinders

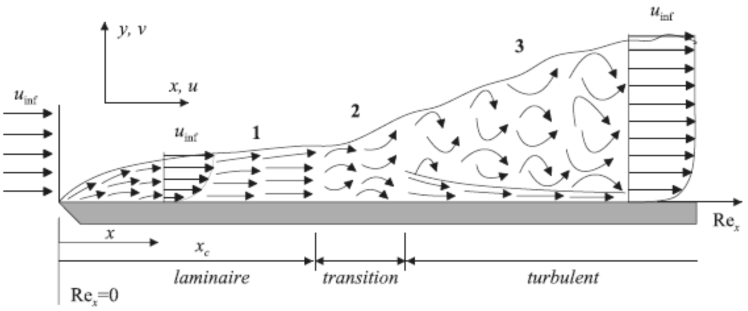


Figure IV.2. Development of boundary layer on a long flat surface
Three flow regimes are visible: laminar (1), transitional (2), and turbulent. (3).

IV.4.5. Natural (or Unforced) versus Forced Flow

A fluid flow is said to be natural or forced, depending on how the fluid motion is initiated. In forced flow, a fluid is forced to flow over a surface or in a pipe by external means such as a pump or a fan. In natural flows, any fluid motion is due to natural means such as the buoyancy effect, which manifests itself as the rise of the warmer (and thus lighter) fluid and the fall of cooler (and thus denser) fluid.

IV.4.6. Steady versus Unsteady Flow

The term ***steady*** implies no change at a point with time. The opposite of steady is **unsteady**.

The term uniform implies no change with location over a specified region. These meanings are consistent with their everyday use (steady girlfriend, uniform distribution, etc.).

The terms unsteady and transient are often used interchangeably, but these terms are not synonyms. In fluid mechanics, unsteady is the most general term that applies to any flow that is not steady, but transient is typically used for developing flows. The term periodic refers to the kind of unsteady flow in which the flow oscillates about a steady mean.

IV.5. Significance of the Boundary Layers

IV.5.1. Velocity Boundary Layer

To introduce the concept of a boundary layer, consider flow over the flat plate of Figure IV.5. When fluid particles make contact with the surface, their velocity is reduced significantly relative to the fluid velocity upstream of the plate, and for most situations it is valid to assume that the particle velocity is zero at the wall.

These particles then act to retard the motion of particles in the adjoining fluid layer, which act to retard the motion of particles in the next layer, and so on until, at a distance $y=\delta$ from the surface, the effect becomes negligible. This retardation of fluid motion is associated with *shear stresses* τ acting in planes that are parallel to the fluid velocity (Figure IV.3). With increasing distance y from the surface, the x velocity component of the fluid, u, must then increase until it approaches the free stream value u_∞. The subscript ∞ is used to designate conditions in the *free stream* outside the boundary layer.

The quantity δ is termed the *boundary layer thickness*, and it is typically defined as the value of y for which $u = 0.99.u_\infty$. The *boundary layer velocity profile* refers to the manner in which u varies with y through the boundary layer. Accordingly, the fluid flow is characterized by two distinct regions, a thin fluid layer (the boundary layer) in which velocity gradients and shear stresses are large and a region outside the boundary layer in which velocity gradients and shear stresses are negligible. With increasing distance from the leading edge, the effects of viscosity penetrate farther into the free stream and the boundary layer grows (δ increases with x).

The significance of velocity boundary layer stems from its relation to the surface shear stress τ_s, and hence to surface frictional effects. For external flows it provides the basis for determining the local *friction coefficient.*

$$C_f = \frac{\tau_s}{\rho . u_\infty^2/2} \qquad (IV.6)$$

Assuming a *Newtonian fluid*, the surface shear stress may be evaluated from knowledge of the velocity gradient at the surface.

$$\tau_s = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \qquad (IV.7)$$

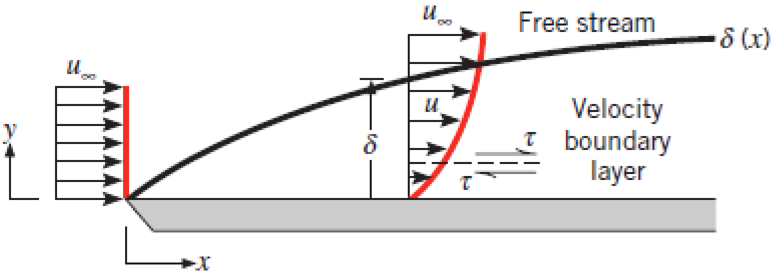


Figure IV.3. Velocity boundary layer development on a flat plate.

IV.5.2. Thermal Boundary Layer

Just as a velocity boundary layer develops when there is fluid flow over a surface, a thermal boundary layer must develop if the fluid free stream and surface temperatures differ. Consider flow over an isothermal flat plate (Figure IV.4). At the leading edge the temperature profile is uniform, with $T(y) = T_\infty$. However, fluid particles that come into contact with the plate achieve thermal equilibrium at the plate's surface temperature. In turn, these particles exchange energy with those in the adjoining fluid layer, and temperature gradients develop in the fluid. The region of the fluid in which these temperature gradients exist is the thermal boundary layer, and its thickness δ_t is typically defined as the value of y for which the ratio $[(T_s - T)/(T_s - T_\infty)] = 0.99$. With increasing distance from the leading edge, the effects of heat transfer penetrate farther into the free stream and the thermal boundary layer grows.

The relation between conditions in this boundary layer and the convection heat transfer coefficient may readily be demonstrated. At any distance x from the leading edge, the local surface heat flux may be obtained by applying Fourier's law to the fluid at y=0. That is,

$$\Phi = -\lambda_f . \frac{\partial T}{\partial y}\bigg|_{y=0} \qquad (IV.8)$$

This expression is appropriate because, at the surface, there is no fluid motion and energy transfer occurs only by conduction. Recalling Newton's law of cooling, we see that

$$\Phi = h.(T_S - T_\infty) \qquad (IV.9)$$

And combining this with Equation (IV.8), we obtain

$$h = \frac{-\lambda_f . \frac{\partial T}{\partial y}\Big|_{y=0}}{(T_S - T_\infty)} \qquad (IV.10)$$

Hence, conditions in the thermal boundary layer, which strongly influence the wall temperature gradient $\partial T/\partial y|_{y=0}$, determine the rate of heat transfer across the boundary layer.

Since (T_s-T_∞) is a constant, independent of *x*, while δ_t increases with increasing *x*, temperature gradients in the boundary layer must decrease with increasing *x*. Accordingly,

The magnitude of $\partial T/\partial y|_{y=0}$ decreases with increasing *x*, and it follows that ϕ and *h* decrease with increasing *x*.

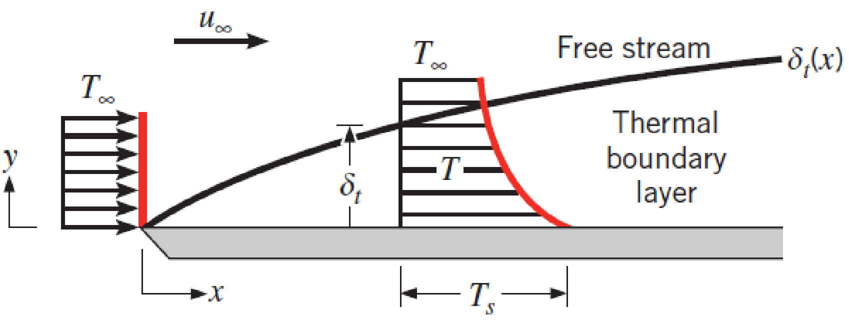


Figure IV.4. Thermal boundary layer development on an isothermal flat plate.

IV.6. Dimensional analysis

We can express physical quantities in terms of a limited number of fundamental dimensions.

Examples: Speed: $L.T^{-1}$; dynamic viscosity: M.L-1.T^{-1}; force: $M.L.T^{-2}$

In these examples, we see that the number of fundamental dimensions is 3: Mass M, Length L, Time T.

These three fundamental dimensions are not always sufficient. For heat transfer problems, it is necessary to add a 4th dimension: temperature θ and, when the exchange of energy between mechanical quantities and thermal quantities is not measurable, we will add the amount of heat Q which will be considered a 5th dimension.

The method of dimensional analysis, which is based on the principle of dimensional homogeneity of the terms in an equation, is known as the Vaschy-Buckingham theorem or the π-group theorem.

So all physical quantities are expressed in the following fundamental dimensions:

- The mass (kg) M
- Length (m) L
- Time (s) T
- Temperature (K) θ

The basic variables of heat transfer problems and their dimensions are given in Table IV.1.

Table IV.1. Units and dimensions of variables in heat transfer by convection.

Quantities	**Symbol**	**S.I. Unit**	**Dimensional Equation**
Speed	U	m/s	$L.T^{-3}$
Characteristic length	D	m	L.
Fluid density	ρ	Kg/m^3	$M.L^{-3}$
Dynamic viscosity of the fluid	μ	Kg/(m.s)	$M.L^{-1}.T^{-1}$
Specific heat capacity	C	J/(kg.K)	$L^2.T^2.\theta^{-1}$
Thermal conductivity of the fluid	λ	W/(m.K)	$M.L.T^{-3}.\theta^{-1}$
Convective heat transfer coefficient	h	$W/(m^2.K)$	$M.T^{-3}.\theta^{-1}$
Temperature difference	T_P-T_∞	K	θ

The VASCHY-BUCKINGHAM theorem allows us to predict that the most general form of the physical law describing the studied phenomenon will be written as:

$$F(\pi_1, \pi_2, \pi_3, \pi_4) = 0$$

Where the πi are dimensionless groups of the form:

$$\pi_i = D^a . \lambda^b . U^c . \rho^d . \mu^e . C^f . h^g . (T_P - T_\infty)^i \qquad (IV.11)$$

The number of independent variables in the convection heat transfer problem (D, λ, U, ρ, μ, C, h, Tp-T∞) is n = 8.

According to the BUCKINGHAM theorem, we can formulate N = n - r dimensionless variables, where (r) represents the rank of the matrix defined by the exponents of the dimensions of the variables given in Table IV.2.

Table IV.2. Equations with dimensions of the 8 quantities.

	D	U	P	μ	λ	C	h	Tp-T∞
Length L	1	1	-3	-1	1	2	0	0
Mass M	0	0	1	1	1	0	1	0
Time T	0	-1	0	-1	-3	-2	-3	0
Temperature θ	0	0	0	0	-1	-1	-1	1

The matrix defined in Table IV.1 has rank r = 4. According to BUCKINGHAM, we can therefore formulate

N = 8 - 4 = 4 dimensionless parameters for heat transfer problems by convection.

Each parameter (πi) will be formulated by potential expressions of the basic variables.

$$\pi = (L)^a . (L^{+1} . M^{+1} . T^{-3} . \theta^{-1})^b . (L^{+1} . T^{-1})^c . (L^{-3} . M^{+1})^d . (L^{-1} . M^{+1} . T^{-1})^e . (L^{+2} . T^{-2} . \theta^{-1})^f . (M^{+1} . T^{-3} . \theta^{-1})^g . (\theta)^i$$

Or with the grouped dimensions

$$[\pi] = [M]^{(b+d+e+g)} . [L]^{(a+b+c-3d+e+2f)} . [T]^{(-3b-c-e-2f-3g)} . [\theta]^{(-b-f-g+i)} \qquad (IV.12)$$

For π to remain dimensionless, it is necessary to eliminate the dimensions; the exponents in parentheses must have a value of zero, which defines the following system of linear equations.

$$\begin{cases} b+d+e+g=0 \\ a+b+c-3d-e+2f=0 \\ -3b-c-e-2f-3g=0 \\ -b-f-g+i=0 \end{cases} \qquad (IV.13)$$

In order to determine the dimensionless parameters πi, we must solve the system of linear equations (IV.1). For the definition of each parameter πi, we can choose four coefficients arbitrarily (usually set to zero) and calculate the other four coefficients using the equation (IV.1).

We obtain the following dimensionless parameters for thermal convection using the dimensional analysis method:

- **Nusselt number**

g = 1 To obtain a law of the form h = f (. . .)

c = d = 0 The group π found will not depend on the kinetic energy of the fluid ρU^2

i = 0 The found π group will not depend on the temperature difference T_p - T_∞.

Resolution of the system determining the first dimensionless group π

With g=1 and c=d=i=0

$$\begin{cases} b+d+e+g=0 \\ a+b+c-3d-e+2f=0 \\ -3b-c-e-2f-3g=0 \\ -b-f-g+i=0 \end{cases} \Rightarrow \begin{cases} b+e=-1 \\ a+b-e+2f=0 \\ -3b-e-2f-3g=3 \\ -b-f=1 \end{cases} \qquad (IV.14)$$

$\Rightarrow a=1, b=-1, e=0, f=0$

$\pi_i = D^a.\lambda^b.U^c.\rho^d.\mu^e.C^f.h^g.(T_P-T_\infty)^i$

With: a=1, b=-1, c=0, d=0, e=0, f=0, g=1, i=0

$$\Rightarrow \pi_1 = Nu = \frac{h.D}{\lambda} \qquad (IV.15)$$

- **Reynolds number**

$$\pi_i = D^a.\lambda^b.U^c.\rho^d.\mu^e.C^f.h^g.(T_P-T_\infty)^i$$

4 out of the 8 parameters can be chosen arbitrarily

b=0, f=0, g=0, i=0

$$\begin{cases} b+d+e+g=0 \\ a+b+c-3d-e+2f=0 \\ -3b-c-e-2f-3g=0 \\ -b-f-g+i=0 \end{cases} \Rightarrow \begin{cases} d+e=0 \\ a+c-3d-e=0 \\ -c-e=0 \\ i=0 \end{cases} \qquad (IV.16)$$

⇒ d=1, c=1, a=1, e=-1
Then

a=1, b=0, c=1, d=1, e=-1, f=0, g=0, i=0

$$\Rightarrow \pi_2 = Re = \frac{\rho.U.D}{\mu} \qquad (IV.17)$$

- **Prandtl number Pr**

$$\pi_i = D^a.\lambda^b.U^c.\rho^d.\mu^e.C^f.h^g.(T_P - T_\infty)^i$$

4 out of the 8 parameters can be chosen arbitrarily

a=0, c=0, g=0, i=0

In such a way as to retain only the characteristics of the fluid: ρ, μ, λ, C

With a=c=g=i=0

$$\begin{cases} b+d+e+g=0 \\ a+b+c-3d-e+2f=0 \\ -3b-c-e-2f-3g=0 \\ -b-f-g+i=0 \end{cases} \Longrightarrow \begin{matrix} b+d+e=0 \\ b-3d-e+2f=0 \\ -3b-e-2f=0 \\ -b-f=0 \end{matrix} \qquad (IV.18)$$

⇒b=-1, d=0, e=1, f=1

$$\Rightarrow \pi_3 = Pr = \frac{\mu.C}{\lambda} \qquad (IV.19)$$

IV.7. Interpretation of Dimensionless Numbers

- **Reynolds number**: The ratio of inertial forces to viscous forces characterizes the type of flow in a pipeline.

$$Re = \frac{\rho.U.D}{\mu} \qquad (IV.20)$$

ρ: density of the fluid [kg/m³],

U: average velocity of the fluid [m/s],

D: smallest geometric dimension of the problem, diameter Dh for a pipe in [m], width L for a plate,

μ: dynamic viscosity of the fluid [Pa.s].

D_h: hydraulic diameter, $D_h = \frac{4.S}{P}$ (S: surface, P: perimeter)

-Rectangular tube:

$$D_h = \frac{4.a.b}{2.(a+b)} = \frac{2.a.b}{(a+b)}$$

-Annular space:

$$D_h = \frac{4.[\tau.(D_2^2 - D_1^2]}{4.(\tau.D_1 + \tau.D_2)} = D_2 - D_1$$

-Space between two planes: $D_h = 2.b$

Nusselt number:

The ratio of the amount of heat exchanged by convection to the amount of heat exchanged by conduction.

$$Nu = \frac{h.D}{\lambda} \qquad (IV.21)$$

h: convective heat transfer coefficient in [W/m².K],

λ: thermal conductivity of the fluid in [W/m.K].

Prandtl number:

This number compares the fluid's ability to diffuse momentum through its viscosity to its ability to diffuse heat through its thermal diffusivity.

$$Pr = \frac{\mu.C_p}{\lambda} = \frac{\nu}{a} \qquad (IV.22)$$

Cp: specific heat capacity of the fluid in [J/kg.K].

Stanton or Margoulis number:

Ratio of heat flux to a reference heat flux by convection.

$$St = Ma = \frac{h}{\rho.U.Cp} = \frac{Nu}{Re.Pr} \qquad (IV.23)$$

- **Grashof number:**

Characterizes the flow in natural convection (replaces Re), an increase in Gr indicates an increase in the intensity of natural convection.

$$Gr = \frac{\beta . g . \Delta T . \rho^2 . D^3}{\mu^2} \qquad (IV.24)$$

β: fluid expansibility in [K^{-1}],

ΔT: temperature difference between fluid and wall: $\Delta T = T_{wall} - T_{fluid}$

- **Rayleigh number:**

Characterizes the flow in natural convection (remplace Re)

$$Ra = Pr . Gr = \frac{g . \beta . \Delta T . D^3}{a . \nu} \qquad (IV.25)$$

$a = \lambda / (\rho . Cp)$: thermal diffusivity [m^2/s],

$\nu = \mu / \rho$: kinematic viscosity of the fluid [m^2/s].

IV.8. Practical correlations for calculating the thermal convection coefficient

The knowledge of the convection coefficient h is mediated by the Nusselt number, whose expression, yet to be determined, is a function only of the Reynolds number and the Prandtl number (forced convection) or the Grashof number and the Prandtl number. (Free convection).

Nu = f(Re, Pr) Forced convection

Nu = f(Gr, Pr) Free convection(natural)

Due to the difference in flow characteristics for laminar and turbulent regimes, we present correlations specific to each case.

IV.8.1. Natural Convection

a)- Vertical flat plate

The relations accounting for experimental studies of heat transfer in natural convection are generally of the form:

$$Nu_L = C(Gr_L . Pr)^n = C(Ra_L)^n \qquad (IV.26)$$

The physical quantities involved in the Grashof and Prandtl numbers must be calculated for the average temperature $\frac{T_P+T_\infty}{2}$

The exponent n will take the following values:

n=1/4 when the convection is laminar

n=1/3 when the convection is turbulent.

The value of the coefficient C depends on the convection regime as well as the geometry and inclination of the wall. This value is given in Table IV.3.

Table IV.3: Constants C and n.

Geometry and orientation of the wall	Characteristic dimension L	C	
		Laminar convection n=1/4	Convection turbulente n=1/3
Vertical plate	Height	0.59 $10^4<Ra_L<10^9$	0.13 $10^9<Ra_L<10^{13}$
Cylinder	Outer diameter	0.53 $10^3<Ra_D<10^9$	0.10 $10^9<Ra_D<10^{13}$
Upper surface of a heated plate or lower surface of a cooled plate	Width Or S/P	0.54 $10^5<Ra_L<2x10^7$	0.14 $2x10^4<Ra_L<3x10^{10}$
Lower surface of a heated plate or upper surface of a cooled plate	Width Or S/P	0.27 $3x10^5<Ra_L<3x10^{10}$	0.07 $3x10^{10}<Ra_L<10^{13}$

-**Churchill and Chu** offer for the entire Ra_L range:

$$\overline{Nu_L} = \left(0.825 + \frac{0.387Ra_L^{1/6}}{1 + (0.492/Pr)^{8/27}}\right)^2 \qquad (IV.27)$$

This correlation can be applied to a vertical cylinder if:

$$\frac{D}{L} \geq \frac{35}{Gr_L^{1/4}}$$

Churchill and Chu also propose a more precise correlation in the case of laminar flows:

$$\overline{Nu_L} = 0.68 + \frac{0.67Ra_L^{\frac{1}{4}}}{\left[1 + \left(\frac{0.492}{Pr}\right)^{\frac{9}{16}}\right]^{\frac{4}{9}}} \qquad (IV.28) \qquad Ra_L \leq 10^9$$

b) - Inclined plane

Archimedes' buoyancy has two components: normal and parallel to the surface of the plate. The reduction of the parallel component leads to the reduction of convection. The convection coefficient can be approximately calculated using vertical plate correlations by replacing g with g.cos(θ) for $0 \leq \theta \leq 60°$.

c)- Cylindre horizontal

The average Nusselt number based on the cylinder diameter:

$$\overline{\mathrm{Nu}}_{\mathrm{D}} = \frac{\bar{\mathrm{h}}.\mathrm{D}}{\lambda} = \mathrm{C}.\mathrm{Ra}_{\mathrm{D}}^{\mathrm{n}} \qquad (\mathrm{IV}.29)$$

C and n are constants whose values are given in Table IV.4.

Table IV.4. Constants C and n

Ra_D	C	n
10^{-10}-10^{-2}	0.675	0.058
10^{-2}-10^{2}	1.02	0.148
10^{2}-10^{4}	0.850	0.188
10^{4}-10^{7}	0.480	0.250
10^{7}-10^{12}	0.125	0.333

For a horizontal cylinder, the average Nusselt number for a wide range of Rayleigh numbers has been correlated by **Churchull and Chu** (1975) as follows:

$$\overline{\mathrm{Nu}}_{\mathrm{D}} = \left(0.60 + \frac{0.387\mathrm{Ra}_{\mathrm{D}}^{1/6}}{[1 + (0.559/\mathrm{Pr})^{9/16}]^{8/27}}\right)^2 \qquad (\mathrm{IV}.30) \qquad \mathrm{Ra}_{\mathrm{D}} \leq 10^{12}$$

d)- Spheres

For spheres, by extrapolating the results of Churchill and Chu for cylinders, the expression for the average Nusselt number for $Pr \geq 0.7$ and $Ra_D \leq 10$^11 is given by Churchill (1983):

$$\overline{\mathrm{Nu}}_{\mathrm{D}} = 2 + \frac{0.589\mathrm{Ra}_{\mathrm{D}}^{1/4}}{[1 + (0.469/\mathrm{Pr})^{9/16}]^{4/9}} \qquad (\mathrm{IV}.31)$$

e) Vertical channels

For a vertical channel with constant and identical wall temperatures, Elenbaas determined that:

$$\overline{\mathrm{Nu}_e} = \frac{1}{24}.\mathrm{Ra}_e\left(\frac{e}{L}\right)\left\{1 - \exp\left[-\frac{35}{\mathrm{Ra}_e(e/L)}\right]\right\}^{3/4} \qquad (\mathrm{IV}.32)$$

This correlation is semi-empirical and valid for symmetrically heated channels where:

$$\overline{\mathrm{Nu}_e} = \left(\frac{q/_S}{T_P - T_\infty}\right)\frac{e}{\lambda} \qquad \text{et } \mathrm{Ra}_e = \frac{g\beta(T_P - T_\infty)e^3}{a.\nu}$$

Are the average Nusselt and Rayleigh numbers based on the spacing "e" between the plates.

-For channels with isoflux plates, the local Nusselt numbers are defined as follows:

$$\overline{\mathrm{Nu}_{e,L}} = \left(\frac{\varphi_P}{T_{P,L} - T_\infty}\right)\frac{e}{\lambda} \qquad (\mathrm{IV}.33)$$

L refers to the condition x=L where the temperature of the plate is maximum.

-Pour les canaux avec les plaques isoflux symétriques dans le régime complètement développé :

$$\mathrm{Nu}_{e,L} = 0.144\left[\mathrm{Ra}_e^*\left(\frac{e}{L}\right)\right]^{1/2} \qquad (\mathrm{IV}.34)$$

And the modified Rayleigh number by:

$$\mathrm{Ra}_e^* = \frac{g\beta\varphi_p e^4}{\lambda.a.\nu}$$

And for axisymmetric conditions with an isolated surface at the boundary:

$$\mathrm{Nu}_{e,L} = 0.204\left[\mathrm{Ra}_e^*\left(\frac{e}{L}\right)\right]^{1/2} \qquad (\mathrm{IV}.35)$$

f)- Inclined channels

Azevedo and Sparrow conducted experiments in inclined channels filled with water. For the conditions of isothermal plates and isothermal-insulated plates for $0 \leq \theta \leq 45°$ within the limit of $\mathrm{Ra}_e(e/L) > 200$, the average Nusselt number is given by:

$$\mathrm{Nu}_e = 0.645\left[\mathrm{Ra}_e\left(\frac{e}{L}\right)\right]^{1/4} \qquad (\mathrm{IV}.36)$$

And the properties of the fluid are evaluated at the film temperature. : $T_f = \frac{T_P + T_\infty}{2}$

g). Rectangular cavities

$$\mathrm{Ra}_e = \frac{g\beta(T_1 - T_2)L^3}{a.\nu} > 1708$$

Lorsque les surfaces sont verticales

- For low aspect ratios [Catton]

$$\overline{Nu}_L = 0.22\left[Ra_L\left(\frac{Pr}{0.2+Pr}\right)\right]^{0.28}\left[\frac{H}{L}\right]^{-1/4} \qquad (IV.37) \qquad \begin{cases} 2 < \frac{H}{L} < 10 \\ Pr < 10^5 \\ 10^3 < Ra_L < 10^{10} \end{cases}$$

- For very low aspect ratios [Catton

$$]\overline{Nu}_L = 0.18\left[Ra_L\left(\frac{Pr}{0.2+Pr}\right)\right]^{0.29} \qquad (IV.38) \qquad \begin{cases} 1 < \frac{H}{L} < 2 \\ 10^{-3} < Pr < 10^5 \\ 10^3 < \frac{Ra_L.Pr}{0.2+Pr} \end{cases}$$

- Pour de grands rapports de forme [McGregor]

$$\overline{Nu}_L = 0.42.Ra_L^{1/4}.Pr^{0.012}\left[\frac{H}{L}\right]^{-0.3} \qquad (IV.39) \qquad \begin{cases} 10 < \frac{H}{L} < 40 \\ 1 < Pr < 2x10^4 \\ 10^4 < Ra_L < 10^7 \end{cases}$$

- Pour une large plage de rapports de forme[McGregor]

- $$\overline{Nu}_L = 0.046.Ra_L^{1/3} \qquad (IV.40) \qquad \begin{cases} 1 < \frac{H}{L} < 40 \\ 1 < Pr < 20 \\ 10^6 < Ra_L < 10^9 \end{cases}$$

-For all these correlations $T = \frac{T_1+T_2}{2}$

IV.9.2. Forced Convection

a)- Laminar regime

In the fully developed region, the Nusselt number Nu is characterized by:

$$\begin{cases} Nu_D = 3.66 & \text{uniform temperature } T_p = Cte \\ Nu_D = 4.36 & \text{uniform flux density } \varphi_p = Cte \end{cases}$$

In the entry region, the energy equation is more difficult to solve, and only two analytical solutions have been obtained. In the thermal entry region, the average Nusselt number Nu_D is given by Edwards' correlation.(1979)

$$\mathrm{Nu_D} = 3.66 + \frac{0.065.\mathrm{Re_D}.\mathrm{Pr}.\frac{D}{L}}{1 + 0.04(\mathrm{Re_D}.\mathrm{Pr}.\frac{D}{L})^{2/3}}.\mathrm{T_P} = \mathrm{Cte} \qquad (\mathrm{IV}.41)$$

We can verify that for L→∞, $\mathrm{Nu_D}=3.66$

For a developed laminar flow, Sieder & Tate (1936) propose

$$\mathrm{Nu_D} = 1.86\left(\mathrm{Re_D}.\mathrm{Pr}.\frac{D}{L}\right)^{1/3}.\left(\frac{\mu}{\mu_P}\right)^{0.14} \qquad 0.48 < \mathrm{Pr} < 16700 \qquad (\mathrm{IV}.42)$$

b) - Turbulent regime

Turbulence homogenizes the temperature at the core of the flow, and a strong thermal gradient is observed near the walls. Empirical correlations are quite numerous.

We will only mention:

-Colburn correlations:

$$\mathrm{Nu_D} = 0.023.\mathrm{Re_D^{0.8}}.\mathrm{Pr}^{\frac{1}{3}} \qquad (\mathrm{IV}.43)$$

$$\begin{cases} 0.7 \le \mathrm{Pr} \le 160 \\ \mathrm{Re_D} > 400 \\ x/D \ge 10 \end{cases}$$

-Dittus–Boelter correlation(1933)

$$\mathrm{Nu_D} = 0.023.\mathrm{Re_D^{0.8}}.\mathrm{Pr^n} \qquad (\mathrm{IV}.44)$$

$$\begin{cases} 0.7 \le \mathrm{Pr} \le 160 \\ \mathrm{Re_D} \ge 10000 \\ \frac{x}{D} \ge 10 \end{cases}$$

$$\text{With} \begin{cases} n = 0.3 \quad \text{cooling} \quad T_m > T_P \\ n = 0.4 \quad \text{heating} \quad T_m < T_P \end{cases}$$

In the entrance region of a fully developed turbulent flow (x/D<10), the Colburn correlation must be corrected as follows to account for the variation in the velocity profile:

$$\mathrm{Nu_D} = 0.023.\mathrm{Re_D^{0.8}}.\mathrm{Pr}^{1/3}\left[1 + \left(\frac{D}{x}\right)^{0.7}\right] \qquad (\mathrm{IV}.45)$$

c)- Non-circular tubes

If the cross-section of the tube is not circular, a first approximation can be used by replacing the tube's diameter with its hydraulic diameter (D_h), and it is this diameter that is used to determine $\mathrm{Re_D}$ and $\mathrm{Nu_D}$.

$$D_h = \frac{4.S}{P}$$

S is the passage section and P the wetted perimeter.

Example: rectangular duct with sides a and b

$$D_h = \frac{4.a.b}{2(a+b)} = \frac{2a.b}{(a+b)}$$

Exercises

Exercise 01:

During the flow of air at T∞= 20°C over a plate surface maintained at a constant temperature of Ts= 160°C, the dimensionless temperature profile within the air layer over the plate is determined to be

$$\frac{T(y) - T_\infty}{T_S - T_\infty} = e^{-ay}$$

Where a=3200 m^{-1} and y is the vertical distance measured from the plate surface in m.
-Determine the heat flux on the plate surface and the convection heat transfer coefficient.

Solution:

Airflow over a flat plate has a given temperature profile. The heat flux on the plate surface and the convection heat transfer coefficient are to be determined.

Assumptions

1 The given nondimensional temperature profile is representative of the variation of temperature over the entire plate.

2 Heat transfer by radiation is negligible.

Properties

The thermal conductivity of air at the film temperature of

Tf =(*Ts* + *T*`)/2 = (160°C + 20°C)/2 =90°C is *k* = 0.03024 W/m·K.

Analysis

Noting that heat transfer from the plate to air at the surface is by conduction, heat flux from the solid surface to the fluid layer adjacent to the surface is determined from

$$\dot{q} = \dot{q}_{cond} = -k_{fluid} \cdot \left.\frac{\partial T}{\partial y}\right]_{y=0}$$

Where the temperature gradient at the plate surface is

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right]_{\mathrm{y}=0} = (\mathrm{T_S} - \mathrm{T_\infty})\mathrm{a}[\mathrm{e}^{-\mathrm{ay}}]_{\mathrm{y}=0} = (\mathrm{T_S} - \mathrm{T_\infty})(-\mathrm{a}) = (160 - 20)(-3200)$$

$$= -4.48\mathrm{x}10^{5}{}^\circ\mathrm{C/m}$$

Substituting, the heat flux is determined to be

$$\dot{q} = -0.03024x(-4.48x10^{5}) = \mathbf{1.35x10^{4}W/m^{2}}$$

Then the convection heat transfer coefficient becomes

$$h = \frac{-k_{fluid}\left(\frac{\partial T}{\partial y}\right)_{y=0}}{T_S - T_\infty} = \frac{-0.03024x(-4.48x10^5)}{160-20} = \mathbf{96.8W/m^2.K}$$

Exercise 02:

By following the approach used in forced convection, perform a dimensional analysis to derive the expression for the Grashof number in natural convection.

Solution:

The quantities related to natural convection are:

- Fluid characteristics: λ_f, ρ, μ, C p, β, g, ΔT
- Temperature difference in the boundary layer ΔT
- Geometric characteristic of the wall: length L

We therefore have n=7 physical quantities, which are expressed using k=4 fundamental units: M, L, T, θ (mass, length, time, and temperature).

By applying the Vaschy–Buckingham theorem, with n-k=3 we write the relationship with the dimensionless groups π_1, π_2, π_3:

$$\psi(\pi_1, \pi_2, \pi_3) = 0$$

By choosing 4 base quantities λf, ρ, μ, L, these three groupings are expressed:

$$\begin{cases} \pi_1 = \lambda_f^{a1}.\rho^{b1}.\mu^{c1}.L^{d1}.h \\ \pi_2 = \lambda_f^{a2}.\rho^{b2}.\mu^{c2}.L^{d2}.\beta.g.\Delta T \\ \pi_3 = \lambda_f^{a3}.\rho^{b3}.\mu^{c3}.L^{d3}.Cp \end{cases}$$

a) The dimensional equations of the different parameters are:

$$[L] = L,\ \ [\mu] = Pa.s = M.L^{-1}.T^{-1}, [\lambda_f] = W.m^{-1}.K^{-1} = M.L^{-1}.T^{-3}.\theta^{-1},$$

$$[Cp] = J.kg^{-1}.K^{-1} = L^2.T^{-2}.\theta^{-1}, [h] = W.m^{-2}.K^{-1} = M.T^{-3}.\theta^{-1}$$

$$[\rho] = M.L^{-3}, [g\beta\Delta T] = L.T^{-2}$$

By writing the dimensional equation of the group π2 (the one that involves βgΔT), we obtain:

$$[\pi_2] = (M.L^{-1}.T^{-3}.\theta^{-1})^{a2}(M.L^{-3})^{b2}.(M.L^{-1}.T^{-1})^{c2}.(L)^{d2}.(L.T^{-2})$$

$$\pi_2 = \left[M^{a2+b2+c2}.L^{a2-3b2-c2-d2+1}.T^{-3a2-c2-2}.\theta^{-a2}\right] = 1$$

The exponents are the solution to the system:

$$\begin{cases} a2 + b2 + c2 = 0 \\ a2 - 3b2 - c2 - d2 + 1 = 0 \\ -3a2 - c2 - 2 = 0 \\ a2 = 0 \end{cases}$$

We find: $a_2=0$, $b_2=2$, $c_2=-2$, $d_2=3$. This therefore leads to:

$$\pi_2 = \lambda_f^0 . \rho^2 . \mu^{-2} . L^3 . \beta . g . \Delta\theta = \frac{\beta . g . \Delta T . \rho^2 . L^3}{\mu^2}$$

With the kinematic viscosity of the fluid $\nu=\mu/\rho$:

$$\boldsymbol{\pi_2 = \frac{\beta . g . \Delta T . L^3}{\nu^2}}$$

The grouping π_2 is called the Grashof number:

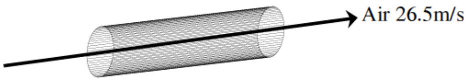
$$\mathbf{Gr} = \frac{\beta . g . \Delta T . L^3}{\nu^2}$$

Exercice 03 :

In a cylinder with a diameter of 4 cm, air flows at an average speed of 26.5 m/s.

-Calculate the heat transfer coefficient h given that:
ρ = 1,2 kg/m^3, Cp = 0,24 kcal/kg°C, η = 1,9 10^{-5}Pa.s,k = 6,2·10^{-6}kcal/m s °C

Nu = 0,023 Re ^{0,8}.Pr^{0,4}



Solution:

$$\mathrm{Pr} = \frac{\mu . C_P}{k} = \frac{1,9 \ . \ 10^{-5} . \ 0,24}{6,2 \ . \ 10^{-6}} = 0,735$$

$$\mathrm{Re} = \frac{\rho . u_m . d}{\mu} = \frac{26,5 \ . \ 0,04 \ . 1,2}{1,9 \ . \ 10^{-5}} = 66947$$

Nu = 0,023 Re ^{0,8}.Pr^{0,4}=147,5

$$\mathrm{Nu} = \frac{h . d}{k}$$

$$\Rightarrow \mathbf{h = 2,286 \ . 10^{-2} \ [kcal . m^{-2} . s^{-1} . {}^{\circ}C^{-1}]}$$

Exercice 04 :

A thin plate with a length of 3m and a width of 1.5m is subjected to an air flow at a speed of 2.0m/s and a temperature of 20°C, in the longitudinal direction. The temperature of the plate surfaces is 84°C. It is requested to calculate:

1. The heat transfer coefficient by convection along the length (for Pr=0.71);
2. The heat flux transmitted by the plate to the air.

The characteristics of the air at 20°C are:

ρ=1.175 kg/m³, $\mu=1.8\times10^{-5}$ kg/m·s, k=0.026 W/m·K, and Cp=1006 J/kg·K.

Solution :

1. The convective heat transfer coefficient h:

-The nature of the flow:

$$\mathrm{Re} = \frac{\rho.\mathrm{L.u}}{\mu} = \frac{1{,}175\,.\,3\,.2}{1{,}80\ \ .\,10^{-5}} = 391666{,}67 < 500.000$$

$$\Rightarrow \mathit{The\ flow\ is\ therefore\ laminar, we\ apply\ in\ this\ case}:$$

$$\mathrm{Nu} = \frac{\mathrm{h.L}}{\mathrm{k}} = 0{,}66.\mathrm{Re}^{0{,}5}.\mathrm{Pr}^{0{,}33} \Rightarrow \mathrm{h} = \frac{0{,}66.\mathrm{Re}^{0{,}5}.\mathrm{Pr}^{0{,}33}.\mathrm{k}}{\mathrm{L}}$$

$$\mathrm{h} = \frac{0{,}66.(391666{,}67)^{0{,}5}.(0{,}71)^{0{,}33}.0{,}026}{3} = 3{,}1972\mathrm{W/m^2.K}$$

1. The heat flux transmitted by the plate (the latter has two walls, lower and upper) to the air is given by:

$$\varphi = 2.\mathrm{h}.S(\mathrm{T_p} - \mathrm{T_a}) = 2\,.3{,}1972.(3\,.1{,}5).(357 - 293) = 1841{,}587\mathrm{W}$$

Exercice 05 :

Calculate the amount of heat transmitted by the flow of water moving in a forced manner through a coil made of a tube with a diameter of 20mm. The water flow rate is 0.28 kg/s and its temperature is 120°C. The temperature of the inner wall of the pipe, which is 4 m long, is considered constant and equal to 95°C.

The characteristics of water at 120°C are:

ρ=945.3 kg/m^3, $\mu=2.34\times10^{-4}$ kg/m.s, k=0.68 W/m.K, and C p =4250 J/kg.K.

Solution

The convective heat transfer coefficient h:

The nature of the flow:

$$\mathrm{Re} = \frac{\mathrm{\rho.u.D}}{\mathrm{\mu}}$$

$$Q = \rho.S.u \Rightarrow u = \frac{Q}{\rho.S} = \frac{0{,}28}{945{,}3\ .\pi.(0{,}01)^2} = 0{,}943 m/s$$

$$Re = \frac{\mathrm{\rho.u.D}}{\mathrm{\mu}} = \frac{945{,}3\ .\ 0{,}943\ .0{,}02}{2{,}34\ .\ 10^{-4}} = 76189{,}56 > 2300$$

The flow regime is therefore turbulent

$$\frac{\mathrm{L}}{\mathrm{D}} > 60\ ,\qquad 1000 < \mathrm{Re} < 120000\ \ \mathrm{et\ Pr} > 0{,}7$$

$$1000 < Re < 120000;\ \frac{L}{D} = \frac{4}{0{,}02} = 200 > 60$$

$$\mathrm{Pr} = \frac{\mathrm{\mu.Cp}}{\mathrm{k}} = \frac{2{,}34.10^{-4}.\ 4250}{0{,}685} = 1{,}45 > 0{,}7$$

$$\mathrm{Nu} = \frac{\mathrm{h.D}}{\mathrm{k}} = 0{,}023.\mathrm{Re}^{0{,}8}.\mathrm{Pr}^{0{,}33} \Rightarrow \mathrm{h} = \frac{0{,}023\ .\mathrm{Re}^{0{,}8}\ .\mathrm{Pr}^{0{,}33}.\mathrm{k}}{\mathrm{D}}$$

$$\mathrm{h} = \frac{0{,}023\ .(76189{,}56)^{0{,}8}.(1{,}45)^{0{,}33}.0{,}685}{0{,}02} = 7164{,}01\mathrm{W/m^2.K}$$

The heat flux transmitted by the water (to the inner wall of the tube) is given by:

$$\varphi = \mathrm{h.S.}(\mathrm{T_e} - \mathrm{T_p}) = \mathrm{h.\pi.D.L.}(\mathrm{T_e} - \mathrm{T_p}) = 7164{,}01\ .\pi.(0{,}024).(393 - 368)$$

$$\boldsymbol{\varphi = 45{,}012\mathrm{kW}}$$

Exercice 06 :

The wall of a building is 6 m high and 10 m long. Under the heating due to the sun, its outside temperature reaches Tm=40°C. The ambient outside temperature is Tair = 20°C.

We provide the following physical properties of air at a temperature of 30°C:

•Density: ρ_{air} = 1.149 kg m–3

•Thermal conductivity: λ_{air} = 0.0258 W $m^{-1}K^{-1}$

•Dynamic viscosity: μ_{air} = 18.4×10−6 Pa s

•Specific heat capacity: Cp_{air} = 1006 J kg−1K−1

Calculate the heat flux exchanged by convection between the wall and the air.

Solution :

In natural convection, such an exchange is calculated by experimental correlation:

$$\mathrm{Nu} = \mathrm{C}(\mathrm{Gr}.\mathrm{Pr})^{\mathrm{n}}$$

The Grashof number is:

$$\mathrm{Gr}_{\mathrm{L}} = \frac{\beta.\mathrm{g}.\Delta\mathrm{T}.\rho_{\mathrm{air}}^{2}.\mathrm{L}^{3}}{\mu_{\mathrm{air}}^{2}}$$

With : $\beta_{\mathrm{air}} = \frac{1}{(273+30)} = 0{,}0029\mathrm{K}^{-1}, \mathrm{g} = 9{,}81\mathrm{m.s}^{-1}, \Delta\mathrm{T} = 20\mathrm{K}, \quad \rho_{\mathrm{air}} = 1{,}149\mathrm{kg.m}^{-3}$,

$\mu_{\mathrm{air}} = 18{,}4\,.\,10^{-6}\ \mathrm{Pa\,.\,s}\,,\mathrm{L} = 6\mathrm{m}$

We obtain:

$$\mathrm{Gr}_{\mathrm{L}} = \frac{0{,}0033\,.\,9{,}81\,.\,20\,.\,(1{,}149)^{2}(6)^{3}}{(18{,}4\,.\,10^{-6})^{2}} = \mathbf{5{,}45\,.\,10^{11}}$$

The Prandtl number

$$\mathrm{Pr} = \frac{\mu_{\mathrm{air}}.\mathrm{Cp}_{\mathrm{air}}}{\lambda_{\mathrm{air}}} = \frac{18{,}4\,.\,10^{6}.\,1006}{0{,}0258} = \mathbf{0{,}718}$$

We then calculate the product Gr_L. Pr=Ra_L which determines the natural convection regime (laminar or turbulent), the critical value being 10^9:

$\mathrm{Gr}_{\mathrm{L}}.\mathrm{Pr} = \mathrm{Ra}_{\mathrm{L}} = 5{,}45\,.\,10^{11}.\,0{,}718 = 3{,}91\,.\,10^{11}$

We are therefore in a regime of turbulent natural convection and we use the coefficients C=0.10 and n=1/3 in the previous correlation. We deduce the value of the Nusselt number:

$$\mathrm{Nu}_{\mathrm{L}} = \frac{\mathrm{h.L}}{\lambda_{\mathrm{air}}} = \mathrm{C}(\mathrm{Gr}_{\mathrm{L}}.\mathrm{Pr})^{\mathrm{n}} = 0{,}10\,.\,(3{,}91\,.\,10^{11})^{1/3} = \mathbf{731}$$

Thus, the convective heat transfer coefficient h is:

$$\mathrm{h} = \frac{\lambda_{\mathrm{air}}.\mathrm{Nu}}{\mathrm{L}} = \frac{0{,}0258.\,731}{6} = \mathbf{3{,}14\ W.m^{-2}.K^{-1}}$$

So the heat flux exchanged over the entire surface S is:

$$\phi = \mathrm{h.S.}(\mathrm{T}_{\mathrm{m}} - \mathrm{T}_{\mathrm{air}})$$

Let

$\phi = 3{,}14\,.\,6.\,10\,.\,(40 - 20) = \mathbf{3768W}$

Exercice 07 :

A horizontal pipe with an outer diameter of 0.1m, used for transporting high-pressure steam, runs through a large room where the air and wall temperatures are 23°C.

The outer surface of the pipe at a temperature of 16.5°C and an emissivity of 0.85.

-Estimate the thermal flux released by the pipe per unit length.

Solution :

Air : $T_f = \frac{165+23}{2} = 94°$, $\lambda = 0{,}0313\ \mathrm{W.m^{-1}}$, $\nu = 22{,}8\ \mathrm{m^2.s^{-1}}, a = 32{,}8\,.\,10^{-6}\mathrm{m^2.s^{-1}}$

and $\mathrm{Pr} = 0{,}697$

$$Ra_D = \frac{g\beta(T_S - T_\infty)D^3}{a.\nu} = \frac{9{,}81\,.\,\frac{1}{367}(165-23)0{,}1^3}{32{,}8\,.\ 10^{-6}\ .\,22{,}8\,.\,10^{-6}} = \mathbf{5{,}073\,.\,10^6}$$

$$Nu_D = \left(0{,}6 + \frac{0{,}387\,.\,(5{,}073\,.\,10^6)^{1/6}}{[1+(0{,}559/Pr)^{9/16}]^{8/27}}\right)^2 = \mathbf{23{,}3}$$

$$h = \frac{Nu_D.\lambda}{D} = \frac{23{,}3\ .\ 0{,}0313}{0{,}1} = \mathbf{7{,}29W.m^{-2}.K^{-1}}$$

$\varphi_{conv} = h.\pi.D.(T_S - T_\infty) = 7{,}29.\pi.0{,}1.(165-23) = \mathbf{325W.m^{-1}}$

$\varphi_{rad} = \varepsilon.\sigma.\pi.D.(T_S^4 - T_\infty^4) = 0{,}85\,.\,5{,}67\,.\,10^{-8}.\pi.0{,}1.((165+273)^4 - (23+273)^4)$

$= \mathbf{441W.m^{-1}}$

$$\varphi = \varphi_{conv} + \varphi_{rad} = 325 + 441 = \mathbf{766\ W}.$$

Chapiter V

Heat Transfer by Radiation

V.1. Introduction

Any heated body emits electromagnetic radiation from its outer surface, the power of which depends on its temperature; this is thermal radiation. The spectral analysis of this radiation shows a majority of short wavelengths at very high temperatures and a majority of long wavelengths for temperatures below 500 K.

V.2. Nature of Radiation

All bodies, regardless of their state: solid, liquid, or gas, emit radiation of an electromagnetic nature. This emission of energy occurs at the expense of the internal energy of the emitting body.

Radiation propagates in a straight line at the speed of light, and it consists of radiations of different wavelengths as demonstrated by William Herschel's experiment:

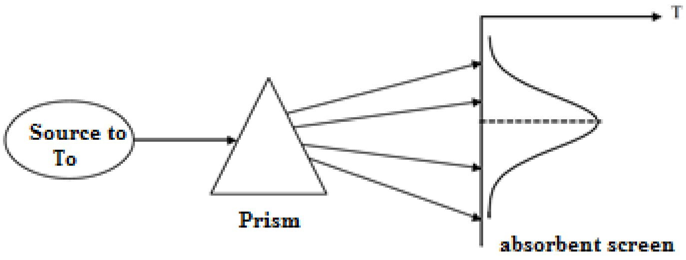


Figure V.1. Principle of William Herschel's experiment.

Passing through a prism, the radiations are more or less deviated according to their wavelength. We therefore send the radiation emitted by a source at temperature T_0 onto a prism and project the deflected beam onto an absorbing (blackened) screen, thus obtaining the decomposition of the total incident radiation into a spectrum of monochromatic radiations..

If a thermometer is moved along the screen, it measures the temperature Te characterizing the energy received by the screen at each wavelength. By constructing the curve $T_e = f(\lambda)$, we obtain the spectral distribution of the radiated energy for the temperature T_0 of the source. It is then observed that:

- The emitted energy is maximum for a certain wavelength λ_m that varies with T_0.

- Energy is emitted only over a wavelength interval $[\lambda_1, \lambda_2]$ characterizing thermal radiation.

The different types of electromagnetic waves and their corresponding wavelengths are represented in Figure V.2. It should be noted that the thermal radiation emitted by bodies ranges between 0.1 and 100 µm. It should also be noted that radiation is perceived by humans:

- By the eye: for 0.38 µm < λ < 0.78 µm visible radiation.
- Through the skin: for 0.78 µm < λ < 314 µm infrared radiation (IR).

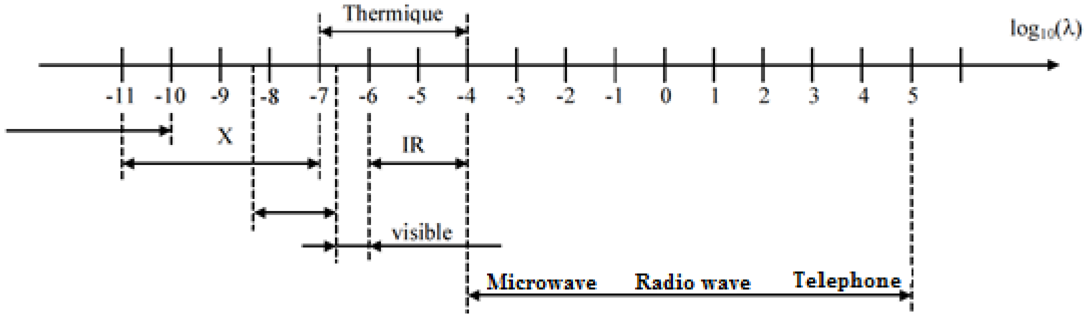


Figure V.2. Spectrum of electromagnetic waves (λ in m).

V.3. Classification of bodies subjected to radiation

Depending on the nature of the body and the wavelength of the incident radiation, one of three phenomena reflection, transmission, and absorption may be predominant.

V.3.1. Transparent Bodies

The propagation of thermal radiation occurs in a vacuum in a straight line, and at the speed of light ($c=3\times10^8$ m/s), without any decrease in the transported energy. It is said, therefore, that a vacuum is a perfectly transparent medium. When radiation does not undergo any attenuation while passing through a medium, it is said that the medium is transparent to that radiation; this is also the case for certain simple gases (O_2, H_2, N_2) in the visible and infrared ranges.

V.3.2. Opaque Bodies

The vast majority of solids and liquids are said to be opaque because they stop the propagation of any radiation right at their surface: these bodies heat up by absorption or radiation.

V.3.3. Semi-transparent bodies

On the other hand, some bodies are partially transparent because the electromagnetic wave can propagate through the medium in question. The propagation is accompanied by electromagnetic absorption that increases the energy of the medium traversed.

V.4. Law of Conservation of Energy

Let ϕ_i be the incident flux, ϕ_r the reflected flux, ϕt the transmitted flux, and ϕa the absorbed flux, the conservation of energy is written as:

$$\phi_i = \phi_r + \phi_a + \phi_t \qquad (V.1)$$

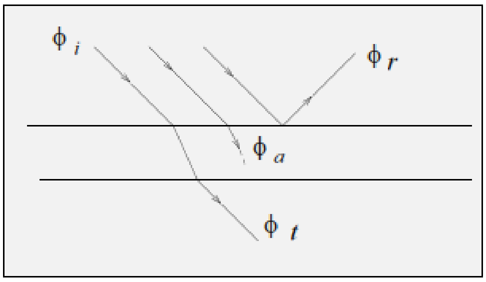


Figure V.3. Decomposition of the incident flux.

-Thermal absorption coefficient

Let's ask:

$$\rho = \frac{\phi_r}{\phi_i} \quad \text{reflection factor}$$

$$\alpha = \frac{\phi_a}{\phi_i} \quad \text{absorption factor}$$

$$\tau = \frac{\phi_t}{\phi_i} \quad \text{transmission factor}$$

The conservation of energy is written as: $\rho + \alpha + \tau = 1$. These parameters characterize the behavior of a body in relation to the received radiation. The coefficient is important in thermodynamics: it measures the proportion of conversion of incident electromagnetic radiation into thermal energy.

The coefficient α is low for polished and non-oxidized metal surfaces. It increases for bodies that appear black but always remains less than one.

V.5. Definition of Energy Quantities

V.5.1. Energy Flux

The flux of a source S is called the power radiated, denoted φ by S, in all the space surrounding it, across all wavelengths. The flux φ is expressed in W.

- The flux sent by a surface element dS into an elementary solid angle dΩ is denoted $d^2\varphi$
- The flux sent throughout space by an elementary surface dS is denoted dφ.
- The flux sent by a surface S in the solid angle dΩ surrounding the Ox direction is denoted $d\varphi_x$.

We therefore have the following relationships:

$$d\varphi = \int d^2\varphi \quad \text{et} \quad \varphi = \int_S d\varphi = \int_\Omega d\varphi_x \qquad (V.2)$$

V.5.2. Energy Emittance

It is the total flux emitted per unit area of the source:

$$M = \frac{d\phi}{dS}\left(\frac{W}{m^2}\right) \qquad (V.3)$$

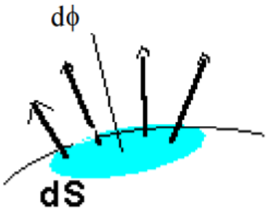


Figure V.4. Emittance of a source, in W/m².

-Emittance Monochromatic:

A surface element dS emits a certain flux of energy by radiation in all directions of the half-space. This flux is distributed over a range of wavelengths. If we consider the energy flux $d\varphi_\lambda^{\lambda+d\lambda}$ emitted between the two wavelengths λ and λ+dλ, the monochromatic emissivity of a source at temperature T is defined by:

$$M_{\lambda T} = \frac{d\phi^{2}{}_{\lambda}^{\lambda+d\lambda}}{dS.\,d\lambda} \qquad (V.4)$$

-Total Emittance:

It is the heat flux density emitted by radiation by dS over the entire spectrum of wavelengths. It is no longer a function of the temperature T and the nature of the source:

$$M_T = \int_{\lambda=0}^{\lambda=\infty} M_{\lambda T}\, d\lambda = \frac{d\varphi}{dS} \qquad (V.5)$$

V.5.3.Solid angle

The solid angle dΩ characterizes directions originating from a point and contained within a portion of space. He is to space what the angle is to the plane. The solid angle under which a surface Σ is seen from a given point O is equal to the area cut out on a unit sphere by the cone with vertex O surrounding the surface Σ.

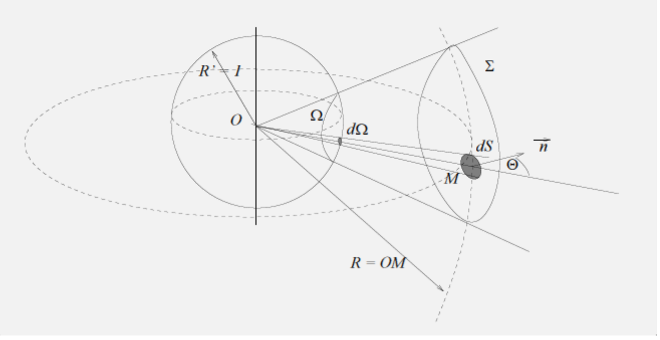


Figure V.5. Construction of the solid angle Dω.

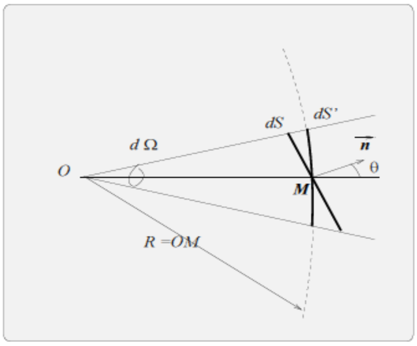


Figure V.6. Projection of any surface onto a sphere of radius R.

To evaluate dΩ, let's construct the sphere centered at O with a radius of R=OM. The projection of the surface element dS onto the sphere of radius R=OM cuts out a spherical cap dS’. The solid angle is equal to the surface projected from the surface dS onto the unit sphere.

$$d\Omega = \frac{dS'}{R^2} = \frac{dS.\cos\theta}{R^2} \quad (\text{Stéradian Sr}) \qquad (V.6)$$

V.5.4. Intensity of a source in a direction

Let there be a direction $\vec{n}$ that makes an angle θ with the normal to the surface of an emissive body. If dϕn is the fraction of flux radiated in the elemental angle dΩ, the total energy intensity of a source in the direction $\vec{n}$ is called the flux radiated per unit solid angle in that direction. It is expressed in Watts per Steradian. ($W.Sr^{-1}$).

$$I_n = \frac{d\phi_n}{d\Omega} \qquad (V.7)$$

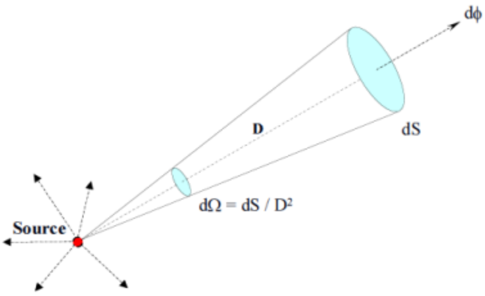


Figure V.7. Intensity of a thermal source

V.5.5. Luminance of a source in a direction

We define the luminance L_n of a surface area dS source, in the direction $\vec{n}$, as the quotient of the intensity I_n of the source in that direction, by the apparent surface area dS' of the source in the same direction.

$$L_n = \frac{I_n}{dS'} = \frac{I_n}{dS.\cos\theta} \qquad (V.8)$$

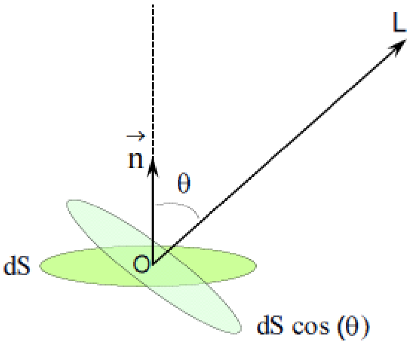


Figure V.8. Luminance of a source in a direction

En effet, vu de la direction $\vec{n}$, tout se passe comme si le flux était émis par la surface projetée dS'=dS.cosθ. En introduisant la définition de l'intensité (I_n), la luminance s'exprime par la relation :

Indeed, from the direction $\vec{n}$, everything happens as if the flux were emitted by the projected surface dS'=dS.cosθ. By introducing the definition of intensity (In), luminance is expressed by the relation:

$$L_n = \frac{\frac{d\phi_n}{d\Omega}}{dS.\cos\theta} = \frac{d^2\phi_n}{d\Omega.dS.\cos\theta} \qquad (V.9)$$

V.5.6.Illuminance (Irradiation)

a)-Spectral Illuminance (Irradiation)

The concept of emittance is replaced, for incident radiation, by the illumination of the receiving surface. The spectral illuminance E_λ ($W.m^{-2}.\mu m^{-1}$) is the rate of transfer of radiant energy received at the wavelength λ in all directions per unit of receiving surface area, per unit of wavelength dλ around λ.

b)-Eclairement(Irradiation) total

This is how we refer to the total flux received by the receiving surface unit:

$$E = \int_0^{\infty} E_\lambda(\lambda)d\lambda = \frac{d\phi}{dS} \qquad (V.10)$$

Illuminance E is expressed in $W.m^{-2}$.

It is therefore the flux density arriving on the unit receiving surface dS, coming from the half-space visible from this surface.

V.6. Lambert's Law

It is said that a source follows Lambert's law (or that it is diffusely emitting) if its luminance does not depend on the emission direction. Most emissive bodies verify this property.

$$L_n=L$$

V.7.Thermal radiation of a black body

V.7.1.Definition of a black body

A black body is an object that will absorb all incident radiation striking it, without reflecting or allowing any fraction to escape, regardless of the wavelengths and directions of propagation. Similarly, a black body will be capable of radiating at each wavelength the maximum amount of thermal energy theoretically storable in this frequency band at a determined temperature level T. Such a body, thermally ideal, does not exist in nature.

In all that follows, the quantities related to the black body will be assigned a superscript "0". Thus, for example, the monochromatic emittance of the black body at the wavelength λ will be denoted M_0^λ.

V.7.2.Laws of thermal radiation of a black body

V.7.2.1.Planck's Law

This law relates the monochromatic emittance of the black body M_0^λ to the wavelength λ and its absolute temperature T. She expresses herself in the form:

$$M_\lambda^0 = \frac{2\pi.h.C^2.\lambda^{-5}}{e^{\frac{hC}{k\lambda T}} - 1} \qquad (V.11)$$

With:

$C = C_0/n$: n is the refractive index of the medium, and $C_0 = 2.9979 \times 10^{-8}$ m/s

- h: is Planck's constant, $h = 6.6255 \times 10^{-34}$ J.s
- k: is Boltzmann's constant, $k = 1.3805 \times 10^{-23}$ J/K.

When radiation propagates in a medium whose refractive index is equal to one, which is strictly the case for a vacuum, and for air in the first approximation, PLANCK's law can be expressed in the following simplified form, which is the one that will be used in everyday practice:

$$M_\lambda^0 = \frac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1} \qquad (V.12)$$

C_1 and C_2 are two physical constants whose values are given, in SI units, in the following table V.1:

Table V.1. The values of physical constants.

T	**λ**	$\mathbf{C_1 = 2\pi h C_0^2}$	$\mathbf{C_2 = \frac{hC_0}{k}}$	$\mathbf{M_\lambda^0}$
K	m	$3.741 \cdot 10^{-16}$ W.m^2	0.014388 m.K	W/m^3
K	µm	$3.741 \cdot 10^{8}$ W.µm^4/ m^2	14.388 µm.K	W/(m^2.µm)

Figure V.9 below represents the curves of variation of the monochromatic emissivity of a black body, calculated using relation (V.12), for various values of the absolute temperature of this black body. Each curve shows a maximum at a certain abscissa λm, which is more pronounced the higher the temperature T.

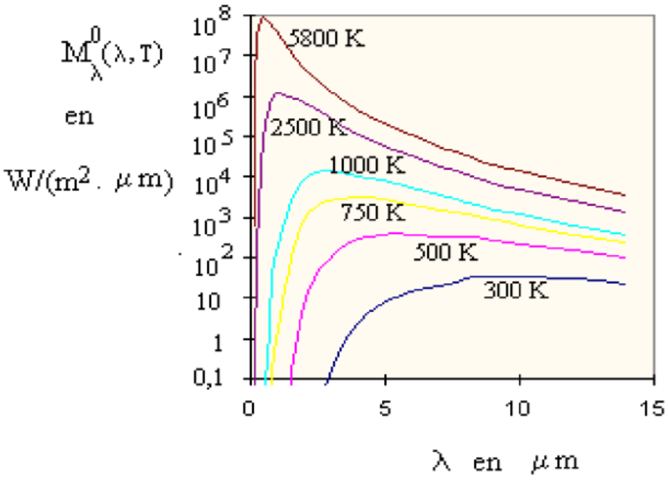


Figure V.9. Monochromatic Emittance of the Black Body.

V.7.2.2. Wien's Laws

- 1st Wien's Law or laws of displacement

The abscissa λm of the maximum of M_0^λ shifts towards shorter wavelengths as the temperature increases. This is what the "displacement law" of WIEN expresses:

$$\lambda_m . T = 2898\ \mu m.K \qquad (V.13)$$

- 2nd Law of WIEN

This law provides the value of the maximum $M_{\lambda m}^0$ as a function of T. She expresses herself in the following way:

$$M_{\lambda m}^0 = B.T^5 \qquad (V.14)$$

The constant B is given in the following table V.2:

Table V.2. The values of constant B

T	**λ**	**B**	**$M_{\lambda m}^0$**
K	m	$1.287 .10^{-5}$ W/(m^3.K^5)	W/m^3
K	µm	$1.287 .10^{-11}$ W/(m^2.µm.K^5)	W/(m^2.µm)

V.7.2.3. Stefan-Boltzmann Law

This law provides the total emissivity of black body radiation in a vacuum, as a function of its absolute temperature. We can establish it by integrating over the entire spectrum, the relation (V.11) expressing Planck's law.

The Stefan-Boltzmann law is expressed by the relation:

$$M^0 = \sigma . T^4 \qquad (V.15)$$

σ is the Stefan-Boltzmann constant, whose expression and numerical value are given in the following table V.3:

Table V.3. The value of the Stefan-Boltzmann constant.

$\boldsymbol{\sigma = \frac{2\pi^5}{15} \frac{k^4}{C_0^2 h^3}}$	**T**	**M^0**
$5.67.10^{-8}$ W/(m^2.K^4)	K	W/m^2

V.8. Thermal Radiation of Real Bodies

The evaluation of the emissive properties of real substances is done in relation to those of a black body placed under the same temperature and wavelength conditions, using coefficients called emissivities, total or monochromatic, hemispherical or directional. Thus, the total and monochromatic emittance of a given real surface will be provided by relations of the type:

$$M = \varepsilon . M^0 \quad \text{et} \quad M_\lambda = \varepsilon_\lambda . M_\lambda^0$$

Relations in which ε is the hemispherical emissivity of the body, and ε_λ its monochromatic emissivity at the wavelength λ.

As for the luminances L and L_λ of the body, they will be related to those of the black body by the relations:

$$L_{Ox} = \varepsilon_{Ox} . L^0 = \varepsilon_{Ox} . \frac{M^0}{\pi} \quad \text{et} \quad L_{Ox,\lambda} = \varepsilon_{Ox,\lambda} . \frac{M_\lambda^0}{\pi}$$

In which ε_{Ox} is the total directional emissivity of the body, and $\varepsilon_{Ox,\lambda}$ its monochromatic directional emissivity.

V.8.1. Notion of gray body

Simplifying assumption: all surfaces present will have emissivity independent of the emission direction and wavelength, just like a black body, but at lower energy levels. Such bodies can then be classified as gray, and will be characterized in terms of their radiative properties by the following relations: $\varepsilon_{Ox} = \varepsilon_{Ox,\lambda} = \varepsilon =$ Constant.

The total emissivity of a gray body will then be deduced from that of a black body given by the Stefan-Boltzmann law:

$$M = \varepsilon . \sigma . T^4 \quad \text{en} \frac{W}{m^2} \qquad (V.16)$$

Table V.4 below provides some values of emissivity ε.

Table V.4. The emissivity values.

Nature of the substance	**Emissivity ε**
Refractory	0.8
Reflective paint	0.3
Polished Iron	0.15 (à20°C) à 0.35 (à900°C)
Polished Aluminum	0.05
Oxidized Aluminum	0.15

V.8.2.Kirchhoff's Law

This law establishes a relationship between the emissive and absorptive properties of a body. To demonstrate this, we consider a body placed in a closed enclosure whose walls possess the properties of a black body. The entire system is in thermal equilibrium at a uniform temperature T, and the body in question therefore neither gains nor loses heat.

Under these conditions, a surface element dS of the body emits in a solid angle element dΩ surrounding a direction Ox inclined at an angle β to the normal a monochromatic flux:

$$\left\lfloor d^2\varphi_{Ox,\lambda}\right\rfloor_{émis} = \varepsilon_{Ox,\lambda}\ L_\lambda^0\ dS\cos\theta\ d\Omega \qquad (V.17)$$

Simultaneously, the surface dS receives, in the same solid angle dΩ and at the same wavelength λ, a flux emitted by the black body which has the value :

$$L_\lambda^0\ dS\cos\theta\ d\Omega \qquad (V.18)$$

The body in question will therefore absorb part of it:

$$\left\lfloor d^2\Phi_{Ox,\lambda}\right\rfloor_{absorbed} = \alpha_{Ox,\lambda}\ L_\lambda^0\ dS\cos\theta\ d\Omega \qquad (V.19)$$

The thermal equilibrium of the body implies the equality of emitted and absorbed fluxes, hence the following relationship that expresses Kirchhoff's Law:

$$\varepsilon_{Ox,\lambda} = \alpha_{Ox,\lambda}$$

In the case where the radiations emitted and received by the body are perfectly distributed over all directions of the hemispherical space (case of diffuse emission and illumination), the previous Kirchhoff law is also applicable to the hemispherical monochromatic properties:

$$\varepsilon_\lambda = \alpha_\lambda$$

It is generally not possible to extend Kirchhoff's law to the total radiation emitted and absorbed by any body, and therefore to consider that α = ε.

Indeed, the total emissivity ε(T) of a body is defined by the relation:

$$\varepsilon(T) = \frac{M(T)}{M^0(T)} = \frac{\int_0^\infty \varepsilon_\lambda\ M_\lambda^0(T)\ d\lambda}{\sigma\ T^4} \qquad (V.20)$$

This function ε(T) is a characteristic property of a single emitting body, depending on its monochromatic emissivity ε_λ, and varying with its temperature T. On the other hand, the total absorption coefficient of the same body is the fraction α absorbed by the body, over all the incident wavelengths. If E_λ is the monochromatic illumination falling on the body, we have:

$$\alpha = \frac{\int_0^\infty \alpha_\lambda\, E_\lambda\, d\lambda}{\int_0^\infty E_\lambda\, d\lambda} = \frac{\int_0^\infty \alpha_\lambda\, E_\lambda\, d\lambda}{E} \qquad (V.21)$$

The absorption coefficient α therefore also depends on the body in question, through the α_λ, but also on the spectral composition E_λ of the received radiation, and thus ultimately on the nature and temperature of the body that emitted the absorbed radiation.

That is why the total absorption coefficient α cannot be an intrinsic characteristic of a body, as its total emissivity ε is.(T).

Generally speaking, we will therefore have: $\alpha \neq \varepsilon$. There are two important exceptions to this situation:

-The black body, defined by the properties: $\varepsilon_\lambda = 1$ and $\alpha_\lambda = 1$ for all λ
It immediately follows that: $\varepsilon = \alpha$
-Grey bodies, defined by the property: $\varepsilon_\lambda = \varepsilon$ for any λ
The relation: $\varepsilon_\lambda = \alpha_\lambda$ therefore leads to: $\alpha_\lambda = \varepsilon$ for any λ,
That is to say: $\varepsilon = \alpha$.

V.9.Radiative exchanges between surfaces

In general, in a real case, several bodies are placed in mutual interaction. Each body emits radiation in all directions of space and receives complex radiation that is the result of direct emissions from the bodies surrounding it and a large number of reflections depending on the considered geometry.

Although an exact and precise calculation is generally difficult, standard geometries can be considered to analyze radiative exchanges in order to determine the net exchange flux. These particular cases also require having information on the behavior of the surfaces and volumes considered with respect to radiation: black body, opaque, gray, or transparent.

V.9.1. Radiative exchanges between black surfaces

The equation:

$$d^2\phi_{1\to 2} = \mathrm{L}^0_{\mathrm{T}1}\, \mathrm{dS}_1 \cos\beta_1\, \mathrm{d}\Omega_{1\to 2} \qquad (\mathrm{V.22})$$

describes the total flux emitted by a surface element dS_1 of a black body in the solid angle $d\Omega_{1\to 2}$(direction D).

With:

$$\mathrm{d}\Omega_{1\to 2} = \frac{\mathrm{dS}_2 \cos\beta_2}{\mathrm{d}^2} \qquad (\mathrm{V.23})$$

So

$$d^2\phi_{1\to 2} = \frac{\mathrm{M}^0_{\mathrm{T}1}}{\pi} \frac{\mathrm{dS}_1 \cos\beta_1\ \mathrm{dS}_2 \cos\beta_2}{d^2} \qquad (\mathrm{V.24})$$

Let there be a second black body whose surface element dS_2 intercepts the radiation emitted by $d\Omega_2 \to 1$. So when body number 2 is a black body, this flux is completely absorbed. Simultaneously, dS_2 (at the temperature $2T_2$) emits towards dS_1:

$$\mathrm{d}^2\Phi_{2\to 1} = \frac{\mathrm{M}^0_{\mathrm{T}2}}{\pi} \frac{\mathrm{dS}_1 \cos\beta_1\ \mathrm{dS}_2 \cos\beta_2}{\mathrm{d}^2} \qquad (\mathrm{V.25})$$

Le bilan de l’échange est :

$$\begin{aligned} \mathrm{d}^2\Phi_{12} = \mathrm{d}^2\Phi_{21} &= \mathrm{d}^2\Phi_{1\to 2} - \mathrm{d}^2\Phi_{2\to 1} \\ &= \sigma(\mathrm{T}_1^4 - \mathrm{T}_2^4) \frac{\mathrm{dS}_1 \cos\beta_1\ \mathrm{dS}_2 \cos\beta_2}{\pi.\mathrm{d}^2} \end{aligned} \qquad (\mathrm{V.26})$$

By integration, one obtains The total flow exchanged between S_1 and S_2:

$$\Phi_{12} = \sigma(\mathrm{T}_1^4 - \mathrm{T}_2^4) \iint_{S_1\ S_2} \frac{\mathrm{dS}_1 \cos\beta_1\ \mathrm{dS}_2 \cos\beta_2}{\pi.\mathrm{d}^2} \qquad (\mathrm{V.27})$$

$$\Phi_{12} = \sigma(\mathrm{T}_1^4 - \mathrm{T}_2^4).S_1F_{12} = \sigma(\mathrm{T}_1^4 - \mathrm{T}_2^4).S_2F_{21} \qquad (\mathrm{V.28})$$

With

$$S_1F_{12} = S_2F_{21} = \iint_{S_1\ S_2} \frac{\mathrm{dS}_1 \cos\beta_1\ \mathrm{dS}_2 \cos\beta_2}{\pi.\mathrm{d}^2} \qquad (\mathrm{V.29})$$

F_{12} and F_{21} are purely geometric and dimensionless quantities.

F_{12}: Form factor under which S_1 sees S_2

F_{21}: Form factor under which S_2 sees S_1

As a result, the problem of calculating exchanges is reduced solely to the calculation of these shape factors.

V.9.1.1. The Form Factors

We define form factors by:

$$\mathrm{S_iF_{ij} = S_jF_{ji}} = \iint\limits_{\mathrm{S_i\ S_j}} \frac{\mathrm{dS_i\cos\beta_i\ \ dS_j\cos\beta_j}}{\mathrm{\pi.d^2}} \qquad (\mathrm{V.30})$$

The form factor Fij is also the fraction of the hemispherical flux that reaches Sj from Si:

$$\mathrm{F_{i,j}} = \frac{\phi_{\mathrm{i\to j}}}{\phi_{\mathrm{i}}} \qquad (\mathrm{V.31})$$

II.9.1.2. Relationship between form factors

- *Addition relation*

Let a surface Sj be decomposable into two surfaces (S_{j1} and S_{j2}), the integral operator being a linear operator, we obtain the relation:

$$\mathrm{S_j = S_{j1} + S_{j2} \Rightarrow F_{ij} = F_{ij1} + F_{ij2}} \qquad (\mathrm{V.32})$$

-Reciprocal relationship

The symmetry of the expressions of F_{12} and F_{21} leads to the reciprocity of the form factors.

$$\mathrm{S_1.F_{12} = S_2.F_{21}} \qquad (\mathrm{V.33})$$

-Relation of enclosure (additivity relation or complementarity relation)

In a closed enclosure, the flux emitted ϕ_i by a surface i is likely to be received by the n surfaces that constitute the surface (the surface i understand).

$$\Phi_{\mathrm{i}} = \Phi_{\mathrm{i1}} + \Phi_{\mathrm{i2}} + \cdots \ldots\ldots\ldots\ldots\ldots. + \Phi_{\mathrm{ii}} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots. + \Phi_{\mathrm{in}} \qquad (\mathrm{V.34})$$

$$\Rightarrow \Phi_i = \sum_{j=1}^{n} \Phi_{ij} = \sum_{j=1}^{n} F_{ij}\Phi_i = \phi_i \sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{F_{ij}} \qquad (\mathrm{V.35})$$

From where, $\sum_{j=1}^{n} F_{ij} = 1$

V.9.1.3. Estimation of form factors

There are several calculation methods to estimate the desired form factor. For example, we cite:

- Use of reciprocity and enclosure relationships. This method is only possible for simple cases.

- Use of formulas and charts(see Appendix R.3 , R.4, R5).

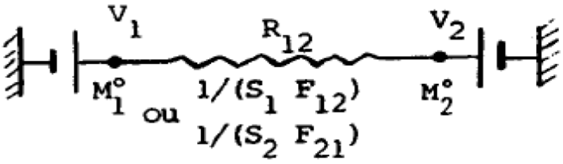
V.9.1.4. Representation of radiative exchanges between black surfaces by electrical analogy

The relationship expressing the net flux exchanged between two black surfaces

$$\Phi_{1,2} = \sigma(T_1^4 - T_2^4)S_1.F_{12} = (M_1^0 - M_2^0)S_1.F_{12} \qquad (V.36)$$

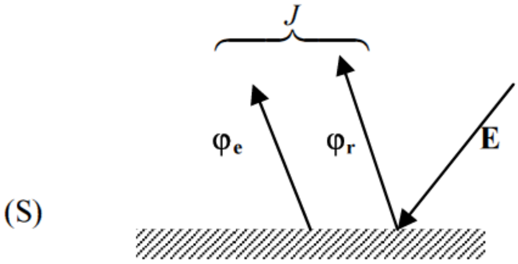
From where:

$$(M_1^0 - M_2^0) = \frac{\phi_{1,2}}{S_1F_{12}} \qquad (V.37)$$



V.9.2. Radiative exchanges between opaque gray surfaces separated by a perfectly transparent medium.

Radiative exchanges between gray surfaces are more complex because the surfaces reflect by radiation, leading to multiple reflections in a closed enclosure. A new concept is being used: it is the radiosity J.



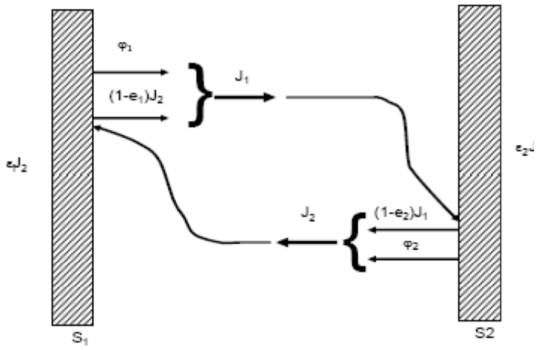
The radiosity of the surface is the sum of the emitted flux and the reflected flux per unit area.

$$J = \varphi_e + \varphi_r \qquad (V.38)$$

From where:

$$\varphi_r = \rho E = (1-\alpha)E = (1-\varepsilon)E \qquad \text{(For the gray body: } \alpha = \varepsilon)$$

Case of two infinite parallel planes (total influence):



For S_1 we have:

$$\varphi_e = \varphi_1 = \varepsilon_1 . \sigma . T_1^4 \qquad (V.39)$$

$$\varphi_r = (1 - \varepsilon_1) . J_2$$

J_2 constitutes the illumination of S_1, J_1 constitutes the illumination of S_2:

$$J_1 = \varphi_1 + J_2 . (1 - \varepsilon_1) \qquad (V.40)$$

$$J_2 = \varphi_2 + J_1 . (1 - \varepsilon_2) \qquad (V.41)$$

The final expression of the illuminances is:

$$J_1 = \frac{\varphi_1 + (1 - \varepsilon_1)\varphi_2}{1 - (1 - \varepsilon_1)(1 - \varepsilon_2)} \qquad (V.42)$$

$$J_2 = \frac{\varphi_2 + (1 - \varepsilon_2)\varphi_1}{1 - (1 - \varepsilon_1)(1 - \varepsilon_2)} \qquad (V.43)$$

The flux φ_{12} is written as:

$$\phi_{12} = S_1 \varphi_{12} = S_1(J_1 - J_2) = S_1 \frac{\varepsilon_1 \varepsilon_2}{1 - (1 - \varepsilon_1)(1 - \varepsilon_2)} \sigma(T_1^4 - T_2^4) \qquad (V.44)$$

$$\frac{\phi_{12}}{S} = (J_1 - J_2) = \frac{\varepsilon_1 \varepsilon_2}{1 - (1 - \varepsilon_1)(1 - \varepsilon_2)} \sigma(T_1^4 - T_2^4) \qquad (V.45)$$

With: $S_1 = S_2 = S$

V.9.3. Radiation of Partially Transparent Bodies

For this type of body, we must introduce the transmission τ, the balance becomes:

$$\alpha + \rho + \tau = 1 \qquad (V.46)$$

When a radiation φ_0 penetrates a semi-transparent medium, it gradually diminishes through absorption; between x and x + dx, it will have lost a certain percentage of its value at x. If this percentage is proportional to dx, we will have:

$$\frac{d\phi}{\phi(x)} = -k dx \qquad (V.47)$$

And after integration

$$\phi(x) = \phi_0 e^{-kx} \qquad (V.48) \qquad \text{(Beer-Lambert law)}$$

In fact, the problem can be decomposed into a sum $\sum \phi_\lambda(x)$ with a coefficient K_λ for each wavelength:

$$\phi_\lambda(x) = \phi_{0\lambda} e^{-k_\lambda x} \qquad (V.49)$$

-Gases as radiation receivers

A gas layer of thickness L will therefore absorb (not to be confused with L the luminance):

$$\phi_{abs} = \phi_0 - \phi_L = \phi_0\left[1 - e^{-kL}\right] = \phi_0 . \alpha_L \qquad (V.50)$$

$\alpha_L = 1 - e^{-KL}$ is therefore the absorption coefficient of the L layer. The transmission factor will therefore be:

$$\tau_L = e^{-kL} = 1 - \alpha_L \qquad (V.51)$$

-Gases as radiation emitters

The extension of Kirchhoff's law for gases of thickness L is written as

$$\varepsilon_{\lambda L} = \alpha_{\lambda L} = 1 - e^{k_\lambda L} \qquad (V.52)$$

Gases are not gray bodies: $\varepsilon_L \neq \alpha_L$

Indeed, ε_L depends on the temperature of the gas, whereas α_L depends on the temperature of the absorbed radiation.

Exercises

Exercise 01:

For a black body maintained at 115°C, determine:

1. The total emissive power
2. The wavelength at which the maximum spectral emissive power occurs.
3. The maximum spectral emissive power.

Solution:

$$\mathrm{M}^0 = \sigma.\mathrm{T}^4 = 5{,}67.10^{-8}.388^4 = 1285\mathrm{W.m}^{-2}$$

$$\lambda_{\max}.\mathrm{T} = 2897{,}6 \Rightarrow \lambda_{\max} = \frac{2897{,}6}{388} = 7{,}47\mu\mathrm{m}$$

$$\left(\mathrm{M}_\lambda^0\right)_{\max} = \frac{3{,}742\,.\,10^8.\lambda^{-5}}{\exp\left(\frac{1{,}439\,.\,10^4}{\lambda\mathrm{T}}\right)-1} = \frac{3{,}742\,.\,10^8.7{,}47^{-5}}{\exp\left(\frac{1{,}439\,.\,10^4}{7{,}47\,.\,388}\right)-1} = 113{,}06\ \mathrm{W.m}^{-2}.\mu\mathrm{m}^{-1}$$

Exercise 2:

A 100 W lamp is powered at 220 V. It consists of a tungsten filament placed at the center of a spherical bulb with a diameter of 8 cm, inside which a vacuum is created. For the light to be sufficiently white, the temperature of the filament must be 2600 K.

1) Determine the diameter and length of the filament if the total hemispherical emissivity factor of tungsten is 0.3 (tungsten resistivity 88 μΩcm).

2) Determine the power radiated in the visible range (between 0.4 and 0.7 μm) if the hemispherical spectral emission factor is 0.45 in this range.

3) What is the power absorbed by the bulb assuming the glass is perfectly transparent up to 2.7 μm and behaves like a black body beyond that, given that the hemispherical spectral emission factor of tungsten is 0.2 in the entire range beyond 2.7 μm?

4) Determine the temperature of the bulb if the total hemispherical emissivity factor of the glass is 0.93, neglecting heat losses due to natural convection.

Solution:

1) Determine the diameter and length of the filament if the total hemispherical emission factor of tungsten is 0.3 (tungsten resistivity 88 µΩcm).
2)

$$M_T^0 = \varepsilon_{Tu}.\sigma.T^4 = 0{,}3\ .\ 5{,}67.10^{-8}.2600^4 = 7{,}77\,.10^5\ \ W/m^2$$

$$\varphi = S.M_T^0 = 100W$$

$$S = \pi.D.L = \frac{100}{M_T^0} = \frac{100}{7{,}77\,.10^5}$$

$$D.L = 4\,.10^{-5}\,m^2$$

The electrical resistance of the wire is:

$$R_e = \frac{V^2}{P} = \frac{220^2}{100} = 484\Omega$$

$$R_e = \rho\frac{L}{\frac{\pi D^2}{4}} = 484\Omega$$

$$\frac{L}{D^2} = 43197\ .10^4\ \ m^{-1}$$

$$\mathbf{L = 0{,}898m\,,D = 4{,}56\ \ 10^{-5}m}$$

2) the power radiated in the visible range (between 0.4 and 0.7 µm) if the hemispherical spectral emission factor is 0.45 in this range:

$$P_{visible} = S.\int_{0{,}4}^{0{,}7} M_{\lambda,T}^0 d\lambda = S.\varepsilon_{visible}\left(\int_0^{0{,}7} M_{\lambda,T}^0 d\lambda - \int_0^{0{,}4} M_{\lambda,T}^0 d\lambda\right)$$

$$= S.\varepsilon_{visible}.\left(F_{0-0{,}7.T} - F_{0-0{,}4.T}\right).\sigma.T^4$$

$$\lambda_1 = 0{,}4\mu m\ \ \text{et}\ \ T = 2600K \rightarrow \lambda_1.T = 0{,}4\ \ .\ 2600 = 1040\mu m.K \rightarrow F_{0-0{,}4T} = 0{,}0005$$

$$\lambda_2 = 0{,}7\mu m\ \ \text{et}\ \ T = 2600K \rightarrow \lambda_2.T = 0{,}7\ \ .\ 2600 = 1820\mu m.K \rightarrow F_{0-0{,}7T} = 0{,}0418$$

$$P_{visible} = \pi.D.L.\varepsilon_{visible}(0{,}0418 - 0{,}0005).5{,}67\,.10^{-8}\ \ .\ 2600^4$$

$$P_{visible} = \pi\ \ .4{,}56\ \ .10^{-5}.0{,}898.\ 0{,}45\,.(0{,}418 - 0{,}0005).5{,}67\,.10^{-8}\ \ .2600^4 = 6{,}2W$$

The calculation can be done differently:

$$P_{Total} = \varepsilon\ .\ S.\sigma.T^4 = P = 100W$$

The total power radiated by the wire is:

$$S\,.\,\sigma\,.\,T^4 = \frac{P}{\varepsilon} = \frac{100}{0,3}$$

$$P_{visible} = S.\varepsilon_{visible}.\left(F_{0-0,7T} - F_{0-0,4T}\right).\sigma.T^4 = P.\frac{\varepsilon_{visible}}{\epsilon}\left(F_{0-0,7T} - F_{0-0,4T}\right)$$

3) the power absorbed by the bulb assuming that the glass is perfectly transparent up to 2.7 µm and behaves like a black body beyond, assuming that the hemispherical spectral emission factor of tungsten is 0.2 throughout the domain beyond 2.7 µm:

$$P_{inf} = S.\varepsilon_{visible}.\left(1 - F_{0-2,7.T}\right).\sigma.T^4 = P.\frac{\varepsilon_{inf}}{\varepsilon}\left(1 - F_{0-2,7.T}\right)$$

$$\lambda_3 = 2{,}7\mu m \quad \text{et} \quad T = 2600K \Rightarrow \lambda_3.T = 2{,}7\ .\ 2600 = 7020\mu m.K$$

$$\Rightarrow F_{0-2,7.T} = \frac{80{,}97}{100} = 0{,}8097$$

$$P_{inf} = P.\frac{\varepsilon_{inf}}{\varepsilon}\left(1 - F_{0-2,7.T}\right) = 100.\frac{0{,}2}{0{,}3}(1 - 0{,}8097) = 12{,}7W$$

4) Steady state: the power absorbed by the glass is equal to the power radiated by the glass to the outside, hence:

$$S_{glass}.\varepsilon_{glass}\,.\sigma.T_{glass}^4 = P_{inf} = \mathbf{12,3W}$$

With ε_{Glass}, the total hemispherical emission factor of the glass.

$$T_{glass}^4 = \frac{12{,}3}{S_{glass}.\varepsilon_{glass}.\sigma} = \frac{12{,}3}{0{,}93\ .\ 5{,}67\,.10^{-8}\ .\ 4\ .\pi.(4\,.10^{-2})^2}$$

$$T_{glass}^4 = 1{,}16\ .10^{10}K\,K \Rightarrow T_{glass} = \mathbf{328K}$$

Exercise 3:

A cylindrical furnace with a diameter of 75 mm and a height of 150 mm is open at the top to an ambient temperature of 27°C. The lateral and bottom surfaces (assumed to be black bodies) are electrically heated and maintained at temperatures of 1350°C and 1650°C, respectively.

- What is the radiative heat flux lost by the furnace upwards?What is the radiative heat flux lost by the furnace upwards?

Solution:

$$\varphi = \varphi_{13} + \varphi_{23} = S_1 . F_{13} . \sigma . (T_1^4 - T_3^4) + S_2 . F_{23} . \sigma . (T_2^4 - T_3^4)$$

$$S_1 = \pi . D . L = \pi . 0{,}075 \,.\, 0{,}15 = 0{,}035 m^2$$

$$S_2 = \frac{\pi . D}{4} = \frac{\pi (0{,}075)^2}{4} = 0{,}0044 m^2$$

$$\left.\begin{aligned} R_1 = R_2 = \frac{r}{h} = \frac{37{,}5}{150} = 0{,}25 \\ X = 1 + \frac{1 + R_2^2}{R_1^2} = 1 + \frac{1 + 0{,}25^2}{0{,}25^2} = 18 \end{aligned}\right\} \rightarrow F_{23} = \frac{1}{2}\left[X - \sqrt{X^2 - 4\left(\frac{R_2}{R_1}\right)^2}\right] = 0{,}056$$

$$F_{21} + F_{23} = 1 \rightarrow F_{21} = 1 - F_{23} = 1 - 0{,}056 = 0{,}944$$

$$F_{12} = \frac{s_2}{s_1} . F_{21} = \frac{0{,}0044}{0{,}035} 0{,}944 = 0{,}118 = F_{13}$$

$$\varphi_{13} = S_1 F_{13} . \sigma . (T_1^4 - T_3^4) = 0{,}035 \,.\, 0{,}118 \,.\, 5{,}67 \,.\, 10^{-8}[(1923)^4 - (300)^4] = 1639 W$$

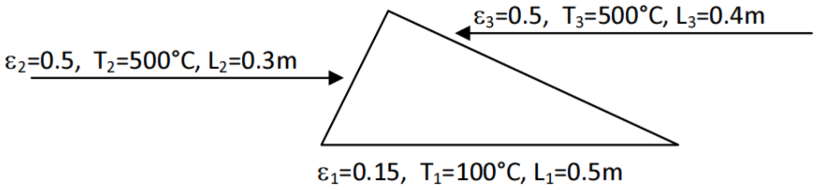
$$\varphi_{23} = S_2 F_{23} . \sigma . (T_2^4 - T_3^4) = 0{,}0044 \,.\, 0{,}056 . 5{,}67 \,.\, 10^{-8}[(1923)^4 - (300)^4] = 205 W$$

$$\varphi = 1639 + 205 = \mathbf{1844W}$$

Exercise 4:

Let there be three infinite gray flat surfaces (1, 2, and 3) with semi-infinite dimensions (see figure):

- Calculate the following shape factors (see figure): F_{11}, F_{12}, F_{13}, F_{21}, F_{22}, F_{23}, F_{31}, F_{32}, F_{33}.
- Provide the radiosity of each surface.
- Provide the definition of the net flux of surface 1.
- Calculate the net flux of surface 1.



Solution:

Calculate the following shape factors (see figure):

F 11 , F 12 , F 13 , F 21 , F 22 , F 23 , F 31 , F 32 , F 33 .

F 11 = F 22 = F 33 =0 flat surfaces.

Application of Hottel's Rule:

$$F_{12} = \frac{S_1 + S_2 - S_3}{2S_1}$$

$$F_{12} = \frac{S_1 + S_2 - S_3}{2S_1} = \frac{L_1 + L_2 - L_3}{2L_1} = \frac{0,5 + 0,3 - 0,4}{2\ .0,5} = \frac{0,4}{1} = 0,4$$

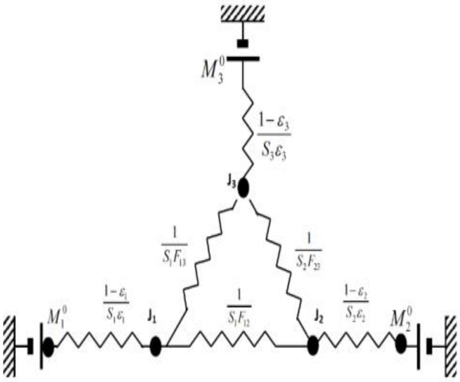
$$F_{13} = \frac{S_1 + S_3 - S_2}{2S_1} = \frac{L_1 + L_3 - L_2}{2L_1} = \frac{0,5 + 0,4 - 0,3}{2\ .0,5} = \frac{0,6}{1} = 0,6$$

$$F_{23} = \frac{S_2 + S_3 - S_1}{2S_2} = \frac{L_2 + L_3 - L_1}{2L_2} = \frac{0,3 + 0,4 - 0,5}{2\ .0,3} = \frac{0,2}{0,6} = \frac{1}{3}$$

$$S_2.F_{23} = S_3.F_{32} \Rightarrow F_{32} = \frac{L_2}{L_3}.F_{23} = \frac{0,3}{0,4}\frac{1}{3} = 0,25$$

$$S_2.F_{21} = S_1.F_{12} \Rightarrow F_{21} = \frac{L_1}{L_2}.F_{12} = \frac{0,5}{0,3}0,4 = \frac{2}{3}$$

$$S_3.F_{31} = S_1.F_{13} \Rightarrow F_{31} = \frac{L_1}{L_3}.F_{13} = \frac{0,5}{0,4}0,6 = 0,75$$



•Radiosity of different surfaces:

$$J_1 = \varphi_1 + (1 - \varepsilon_1)[F_{11}.J_1 + F_{12}.J_2 + F_{13}.J_3]$$

$$J_2 = \varphi_2 + (1 - \varepsilon_2)[F_{21}.J_1 + F_{22}.J_2 + F_{23}.J_3]$$

$$J_3 = \varphi_3 + (1 - \varepsilon_3)[F_{31}.J_1 + F_{32}.J_2 + F_{33}.J_3]$$

•Definition of the net flux of surface 1:

The net flux of surface 1 is equal to the difference between the emitted flux and the absorbed flux.

$$\Phi_{1,nette} = \frac{S_1\varepsilon_1}{1 - \varepsilon_1}(M_1^0 - J_1)$$

•Calculation of the net flux of surface 1.

$$\begin{pmatrix}-\varphi_1\\-\varphi_2\\-\varphi_3\end{pmatrix}=\begin{bmatrix}-1 & (1-\varepsilon_1)F_{12} & (1-\varepsilon_1)F_{13}\\(1-\varepsilon_2)F_{21} & -1 & (1-\varepsilon_2)F_{23}\\(1-\varepsilon_3)F_{31} & (1-\varepsilon_3)F_{32} & -1\end{bmatrix}\begin{pmatrix}J_1\\J_2\\J_3\end{pmatrix}$$

$$\begin{pmatrix}-\varepsilon_1.\sigma.T_1^4\\-\varepsilon_2.\sigma.T_2^4\\-\varepsilon_3.\sigma.T_3^4\end{pmatrix}=\begin{bmatrix}-1 & 0{,}34 & 0{,}51\\\frac{1}{3} & -1 & \frac{1}{6}\\0{,}375 & 0{,}125 & -1\end{bmatrix}\begin{pmatrix}J_1\\J_2\\J_3\end{pmatrix}$$

$$\begin{pmatrix}164\\16466\\16466\end{pmatrix}=\begin{bmatrix}1 & -0{,}34 & -0{,}51\\-\frac{1}{3} & 1 & -\frac{1}{6}\\-0{,}375 & -0{,}125 & 1\end{bmatrix}\begin{pmatrix}J_1\\J_2\\J_3\end{pmatrix}$$

$$[K]=[M].[J]$$

$$[M]^{-1}.[K]=[M]^{-1}.[M].[J]=[J]$$

After inverting the matrix:

$$\begin{bmatrix}J_1=25536\\J_2=29934\\J_3=29785\end{bmatrix}$$

$$\frac{\Phi_{1,nette}}{S_1}=\frac{\varepsilon_1}{1-\varepsilon_1}(M_1^0-J_1)=-4477W.m^{-2}$$

Surface 1 receives heat.

PART B

Mass Transfer

Chapiter I

Phenomena of Solid State Diffusion

I.1. Introduction au transfert de masse

I.1.1. Definition of Mass Transfer

Mass transfer refers to the movement of matter from an area of high concentration to an area of low concentration under the influence of concentration gradients. This can occur in various phases (gaseous, liquid, solid) and may involve a solute or a contaminant moving through a medium. This phenomenon is fundamental in many industrial processes, such as separation, purification, catalysis, and chemical reactions.

I.1.2. Types of Mass Transfer

a)- Diffusion: Movement of molecules or atoms on a microscopic scale, without net movement of the fluid.

b)- Convection: Mass transfer associated with the macroscopic movement of the fluid.

***c)- Migration*:** Effect of external forces, such as the electric or magnetic field, on the movement of particles.

I.1.3. Importance of Mass Transfer

Mass transfer plays a crucial role in industrial processes, particularly in the following areas:

-Séparation des composants : Distillation, filtration, extraction, adsorption.

-Chemical reactions: Particularly in reactors where the reaction rate depends on the diffusion of reactants to the reaction sites.

-Biotechnology: For example, the diffusion of oxygen in a bioreactor.

-Environment: Dispersion of pollutants in the air, water, or soil.

I.1.4. Applications of Mass Transfer

Industrial applications of mass transfer include:

✓ ***Separation processes:*** Such as distillation, absorption, and liquid-liquid extraction, where the different components of a mixture are separated based on their differences in solubility or volatility.
✓ ***Water treatment:*** Absorption of gases such as sulfur dioxide or ammonia.
✓ ***Heterogeneous*** chemical reactions: For example, in solid catalysts where the material diffuses into the solid phase and reacts with the reactants.

I.1.5. Comparison with Other Transfer Mechanisms (Heat and Momentum)

Mass transfer is closely related to the mechanisms of heat transfer (energy) and momentum transfer. (fluid movement). These three phenomena share similar principles, such as:

-The flux is proportional to the gradient (in the case of mass transfer, to the concentration gradient).

-The analogy between the coefficients of diffusion, viscosity, and thermal conductivity.

The basic equations for heat and mass transfer are similar, which allows for the use of analogies between the three processes to simplify calculations and modeling.

I.1.6. Key Units and Parameters in Mass Transfer

In the context of mass transfer, the following parameters and units are important:

- ***Concentration (C):*** Amount of substance (in moles, grams, etc.) per unit volume (m^3 or L).
- ***Mass flux (J):*** Amount of substance transferred per unit area and time (mol/m^2.s, kg/m^2.s).
- ***Diffusion coefficient (D):*** Indicates the ability of a solute to diffuse in a given medium. It depends on the temperature, the nature of the solute, and the solvent. D is expressed in m^2/s.

I.2. Introduction à la diffusion dans les solides

Diffusion in solids refers to the movement of atoms or ions from a region of high concentration to a region of low concentration. It is caused by thermal forces that agitate the particles, allowing their movement within the crystal lattice. In solid materials, particles can move either by interstitial diffusion or by substitutional diffusion.

The phenomena of diffusion in the solid state are influenced by many factors, including:

a)-Temperature: The higher the temperature, the faster the atoms or ions move.
b)-Crystal defects: Vacancies (empty sites in the crystal lattice) and dislocations play an important role in the ease of diffusion.
c)-Nature of the material: The type of material (metals, alloys, ceramics, polymers) influences the diffusion mechanisms.

I.3. Mécanismes de diffusion dans les solides

Diffusion in solids can occur through several mechanisms, which depend on the structure and nature of the material. The two main mechanisms are:

I.3.1. Interstitial diffusion

Interstitial diffusion occurs when atoms diffuse into the empty sites of the crystal lattice, called interstices. This type of diffusion is particularly common in metals, where small atoms, such as carbon or hydrogen, can move into the spaces between the metal atoms. Interstitial atoms are often smaller than the atoms in the crystal lattice, which facilitates their diffusion.

Example: The diffusion of carbon in iron to form alloys like steel.

-Diffusion rate: Interstitial diffusion is generally faster than substitutional diffusion because interstitial atoms are smaller and can move more freely within the crystal lattice.

-Temperature dependence: Interstitial diffusion is highly dependent on temperature. At high temperatures, the movement of atoms becomes easier, which increases the diffusion rate.

I.3.2. Diffusion par substitution

In substitution diffusion, an atom of a chemical element moves to replace another atom in the crystal lattice. This mechanism is more common in metallic alloys, where the atoms of one metal replace those of another metal in the crystal structure.

Example: The diffusion of nickel into copper in a copper-nickel alloy.

-Diffusion rate: Substitutional diffusion is generally slower than interstitial diffusion because it requires the substituting atom to "leave" its site in order to move to another vacant site.

-Temperature dependence: As with interstitial diffusion, substitutional diffusion increases with temperature, but it is slower due to the requirements of moving atoms within the lattice.

I.3.3. Diffusion par vacance

In this mechanism, an atom moves from a crystal site to a vacant site, thereby creating a vacancy. This type of diffusion is also known as vacancy diffusion. The process involves neighboring atoms moving to occupy the vacancy, resulting in a global displacement of the atom within the lattice.

Example: The movement of copper atoms in a pure copper crystal under the effect of vacancy diffusion.

-Mechanism: Vacancy diffusion is generally slower than interstitial diffusion because it relies on the creation and movement of vacancies, which requires more energy.

-Effect of temperature: Vacancy diffusion is also sensitive to temperature, with higher diffusion rates at higher temperatures due to the increased mobility of vacancies.

I.4. Factors Influencing Diffusion in Solids

Diffusion in solids is affected by several essential factors, whether in pure metals, alloys, or other solid materials.

I.4.1. Temperature

Temperature plays a crucial role in solid-state diffusion. The increase in temperature raises the kinetic energy of the particles, facilitating their movement through the crystal lattice. The relationship between the diffusion coefficient D and the temperature T is often described by the Arrhenius law:

$$D = D_0 e^{-\frac{Q}{RT}} \qquad (I.1)$$

D_0 is a pre-exponential factor,

Q is the activation energy for diffusion,

R is the ideal gas constant,

T is the absolute temperature in kelvins.

I.4.2. Structure du matériau

The structure of the material greatly influences the diffusion rate. For example, in a compact crystal lattice (like that of metals), diffusion is faster in directions where interstitial sites are more accessible. The presence of dislocations, vacancies, or other crystal defects can increase the diffusion rate because these defects provide easier pathways for atoms.

I.4.3. Concentration of the diffusing species

A concentration gradient is necessary for diffusion to occur. The higher the concentration gradient, the greater the diffusion flux will be. This means that a large concentration difference between two areas of a material will result in a greater flux of the diffusing species towards the region of lower concentration.

I.4.4. Nature of the material

The chemical and physical nature of the material also influences diffusion. For example:

-In metals, diffusion is often faster than in ceramics or polymers, due to the crystalline structure and the higher mobility of atoms in metals.

-Materials with a well-ordered crystalline structure (for example, perfect crystals) show slower diffusion compared to materials with many crystalline defects.

I.5. Application of diffusion in solids

Diffusion in solids has many applications in various industrial and scientific fields:

-Thermal treatments of metals: Diffusion plays a key role in processes such as layer deposition or carbon diffusion in steel to produce more resistant alloys.

-Alloys and composite materials: The formation of solid solutions, such as brass alloy (copper and zinc alloy), largely depends on the diffusion mechanisms between the different components.

-Semiconductor manufacturing: In the semiconductor industry, elements like gallium arsenide or silicon are doped by diffusion to control their electrical properties.

-Metal corrosion: Corrosion processes involve the diffusion of oxygen or other reactive molecules into the surface layer of the metal, leading to changes in mechanical and chemical properties.

Exercises

Exercise 01:

Given a material where the concentration gradient of a solute is $\frac{dC}{dx} = 2 \text{mol}/\text{m}^3$ and the diffusion coefficient D is 1.5×10^{-9} m^2, calculate the flux J(in mol/m^2).

Solution:

$$J = -D\frac{dC}{dx}$$

Substitute the given values:

$J=-(1.5\times10^{-9})\times2=-3\times10^{-9}$ mol/m^2

Thus, the flux J is −3×10−9 mol/m^2, indicating a flux in the direction of decreasing concentration.

Exercise 02:

Consider a material with an initial uniform concentration of a solute. At t=0, the solute is introduced at one end of the material with a constant concentration of 2 mol/m^3. The diffusion coefficient is 1×10^{-10} m^2.

-Calculate the concentration profile after 2 hours at a distance of 0.5 m.

Solution:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

For this type of problem, the solution requires solving the diffusion equation under the given boundary and initial conditions. However, for simplicity, you can use the approximation for a semi-infinite medium:

$$C(x,t) = C_0 \text{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

Where

C_0 is the concentration at the boundary, and erf is the error function.

$x=0.5$ m, $t=2$ hours$=7200$ s, $D=1\times10^{-10}$ m^2/s

Substitute into the error function equation:

$$C(0.5{,}7200)=\frac{2\mathrm{mol}}{m^3}.\,\mathrm{erf}\left(\frac{0.5}{2\sqrt{1\mathrm{x}10^{-10}\mathrm{x}7200}}\right)$$

Exercise 03:

Given that the diffusion coefficient D of a particular metal follows an Arrhenius form:

$$D=D_0exp\left(\frac{-Q}{RT}\right)$$

Where:

$D_0=5\times10^{-4}$ m^2/s is the pre-exponential factor,

$Q=150$ kJ/mol is the activation energy,

$R=8.314$ J/mol is the gas constant,

T is the temperature in Kelvin.

Calculate the diffusion coefficient at $T=600$ K.

Solution:

Substitute the known values into the Arrhenius equation:

$$\mathrm{D}=5\mathrm{x}10^{-4}.\,\mathrm{exp}\left(\frac{-150\mathrm{x}10^3}{8.314\mathrm{x}600}\right)$$

$$\mathrm{D}=5\mathrm{x}10^{-4}.\,\mathrm{exp}(-30.07)$$

$$\mathrm{D}=9.6\mathrm{x}10^{-17}\mathrm{m}^2/\mathrm{s}$$

Thus, the diffusion coefficient at 600 K is approximately 9.6×10^{-17} m^2/s

Exercise 04:

Consider a situation where the concentration of a diffusing species at one end of a solid is $C_0=5$ mol/m^3, and at the other end $C_1=0$ mol/m^3. The diffusion coefficient $D=1.2\times10^{-9}$ m^2/s. Calculate the steady-state flux of the diffusing species if the length of the solid is 0.1 m.

Solution:

In steady-state diffusion, Fick's first law applies:

$$J = -D\frac{C_1 - C_0}{L}$$

Substitute the given values:

$$J = -(1.2x10^{-9})\frac{0 - 5}{0.1} = -6x10^{-8}mol/m^2.s$$

Thus, the steady-state flux is -6×10^{-8} mol/m^2.s

Exercise 05:

Consider the diffusion of atoms in an alloy. The vacancy formation energy is given as Qv=1.1 eV. Using the formula for vacancy diffusion, calculate the vacancy concentration at 500 K. The vacancy concentration is given by:

$$n_v = N.\exp\left(\frac{-Q_v}{k_B T}\right)$$

Where:

N is the total number of atomic sites,

k_B=8.617×10^{-5} eV/K is the Boltzmann constant,

Qv=1.1 eV,

T=500 K.

Solution:

Substitute the values into the equation:

$$n_v = N.\exp\left(\frac{-1.1}{8.617x10^{-5}x500}\right)$$

$$n_v = N.\exp(-25.47)$$

Using a calculator:

$$n_v \cong N.2.2x10^{-12}$$

Thus, the vacancy concentration will be proportional to N×2.2×10^{-12}.

Chapiter II

Fick's Laws

II.1. Introduction

Fick's laws describe the phenomenon of diffusion, that is, the movement of particles from a region of high concentration to a region of low concentration. These laws are used to model mass flow in different media, whether they are gases, liquids, or solids. They are essential in physics, chemistry, biology, and process engineering, particularly for understanding the transport of matter in materials or during chemical reaction processes.

II.2. First Law of Fick: The Steady-State Diffusion Flux

The first law of Fick describes the relationship between the flux of matter and the concentration gradient in a system in a steady state (i.e., when the diffusion conditions do not vary over time).

$$\mathrm{J} = -\mathrm{D}\frac{\mathrm{dC}}{\mathrm{Dx}} \qquad (\mathrm{II.1})$$

Where:

J is the diffusion flux (in $mol/m^2.s$), which corresponds to the amount of substance diffused per unit area and per unit time.

D is the diffusion coefficient (in m^2/s), which characterizes the ease with which the substance diffuses in the medium.

$\frac{\mathrm{dC}}{\mathrm{Dx}}$ is the concentration gradient of the substance (in $mol/m^3.m$), that is, the rate of change of the concentration C in the x direction.

- **Interpretation of Fick's first law:**

-The flux J is directly proportional to the concentration gradient dC/dx. This means that the greater the concentration difference between two points, the more intense the diffusion will be.

- The sign (-) indicates that particles always diffuse from a region of high concentration to a region of low concentration, in accordance with the law of spontaneous diffusion.

The diffusion coefficient D depends on the type of material, the temperature, the size of the molecules, and the molecular interactions in the medium.

- **Application:**

Fick's first law is used to model diffusion in systems where the concentration of the diffusing substance does not change over time, or under conditions where the diffusion flux is constant. A classic example is the diffusion of gases in the air or solutes in a liquid.

II.3. Second law of Fick: Non-stationary diffusion

The second law of Fick is used to describe diffusion in systems where the concentration of the substance varies over time. It relates the variation of concentration over time to diffusion in the medium.

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \qquad (II.2)$$

Where:

$\frac{\partial C}{\partial t}$ is the variation of the substance concentration over time (in mol/m³.s).

$\frac{\partial^2 C}{\partial x^2}$ is the second derivative of the concentration with respect to position, which measures the divergence of the concentration gradient (in mol/m³.m²).

- **Interpretation of Fick's second law:**

-This law allows for the description of the evolution of concentration over time, which is useful when the system is not in a steady state, that is, when the diffusion conditions change over time.

- This law allows us to describe the evolution of concentration over time, which is useful when the system is not in a steady state, that is, when the diffusion conditions change over time.

- It is particularly important for studying phenomena such as the diffusion of a substance in a material when there is a change in concentration over time, for example, the diffusion of a dye in a liquid or the diffusion of a solute in a gel.

- **Application:**

Fick's second law is used to analyse phenomena such as:

-The diffusion of solutes in a solution when the concentration changes over time (for example, during the dissolution of salt in water).

-Thermal diffusion in solids, where there is a change in temperature over time.

II.4. Diffusion coefficient

The diffusion coefficient D is a key parameter in Fick's laws. It represents the ease with which a substance moves through a given medium. This coefficient is influenced by several factors:

II.4.1. Temperature:

Diffusion is generally faster at higher temperatures because the particles acquire more kinetic energy. The diffusion coefficient follows a thermodynamic relationship with temperature, often described by the Arrhenius law:

$$D = D_0 e^{-\frac{Q}{RT}} \qquad (II.3)$$

Where:

D_0 is the pre-exponential factor (in m²/s),

Q is the activation energy for diffusion (in J/mol),

R is the ideal gas constant (8.314 J/mol·K),

T is the temperature in Kelvin (K).

II.4.2. Nature of the medium:

The diffusion coefficient depends on the medium in which diffusion occurs. For example, it is much higher in gases than in solids or liquids, because particles are freer to move in gases. In solids, the diffusion coefficient is often low, except if crystal defects (vacancies, dislocations) or high temperatures allow for faster diffusion.

II.4.3. Size of molecules:

Small molecules generally diffuse more quickly than large molecules. For example, atoms or small molecules like carbon in steel move more quickly than macromolecules or polymers.

II.5. Cases of diffusion in stationary and non-stationary regimes

II.5.1. Diffusion in steady state

In steady state, the concentration profile of the substance does not vary with time. The diffusion flux J remains constant, which means there is no accumulation of matter. This case is typical when a substance diffuses through a membrane or into a material with a constant concentration gradient.

Example: The diffusion of oxygen in the air or the diffusion of solutes in a stable solution.

II.5.2. Non-stationary diffusion

In a non-stationary regime, the concentration varies with time. This case is more complex and requires Fick's second law to model the change in concentration over time and space. A typical example of non-stationary diffusion is the diffusion of raw materials in a chemical reactor, where the concentration evolves over time.

II.6. Applications of Fick's Laws

Fick's laws are used in many scientific and industrial fields, among which:

1. Chemical engineering and industrial processes:

In chemical reactors, diffusion plays a crucial role in the rate of chemical reactions, especially when a reactant must diffuse to a catalyst or a reaction surface.

2. Pharmacy and medicine:

Diffusion is used to model the transport of drugs in the human body, such as the diffusion of drugs through the skin or through cell membranes.

3. Materials Science:

Diffusion is important in processes such as layer deposition or carbon diffusion in steel (surface hardening), where the modification of the chemical composition of the surface material influences its mechanical properties.

4. Environmental Engineering:

The diffusion of pollutants in water, air, or soil is a key phenomenon for understanding the dispersion of contaminants and their environmental impact.

Exercises

Exercise 01:

A species diffuses across a slab of material with a thickness of L=0.02 m. The concentration of the diffusing species at the two sides of the slab are C_0=2 mol/m^3 at one end and C_L=0.5 mol/m^3 at the other end. The diffusion coefficient D is 1×10^{-9} m^2/s. Calculate the diffusion flux J through the slab.

Solution:

Fick's First Law states:

$$J = -D\frac{C_L - C_0}{L}$$

Substitute the given values:

$$J = -(1x10^{-9})\frac{0.5 - 2}{0.02}$$

$$\mathbf{J = 7.5x10^{-8} mol/m^3.s}$$

Thus, the diffusion flux J is **7.5×10^{-8} mol/m^2**

Exercise 02:

A species is introduced into a semi-infinite medium at time t=0, with an initial concentration of C_0=3 mol/m^3. The diffusion coefficient of the species is D=5×10^{-10} m^2/s. Calculate the concentration C(x,t) at a distance x=0.05 m after t=1000 s.

Solution:

For non-steady-state diffusion in a semi-infinite medium, Fick's Second Law can be solved using the error function:

$$C(x, t) = C_0 \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

Substitute the known values:

$$C(0.05,1000) = 3x\mathrm{erf}\left(\frac{0.05}{2\sqrt{5x10^{-10}x1000}}\right)$$

The error function erf(353.6) is approximately 1, so:

$$C(0.05,1000)=3\times1=\mathbf{3\ mol/m^3}$$

Thus, the concentration at $x=0.05$ m after 1000 seconds is **3 mol/m^3.**

Exercise 03:

The diffusion coefficient D of a species in a material follows the Arrhenius relation:

$$D = D_0 \exp\left(-\frac{Q}{RT}\right)$$

Where:

$D_0=1\times10^{-3}$ m^2/s is the pre-exponential factor,

$Q=1.2\times10^5$ J/mol is the activation energy,

$R=8.314$ J/mol is the universal gas constant,

$T=500$ K is the temperature.

Calculate the diffusion coefficient D at T=500 K.

Solution:

Substitute the given values into the Arrhenius equation:

$$D = 1x\,10^{-3} \exp\left(-\frac{1.2x10^5}{8.314x500}\right)$$

Thus, the equation becomes:

$$D = 1x\,10^{-3}\exp(-28.9) = \mathbf{5.18x10^{-16}m^2/s}$$

So, the diffusion coefficient at 500 K is approximately **5.18×10−16 m^2/s.**

Exercise 04:

A solid has an initial concentration profile given by:

$$C(x,0) = C_0\left(1-\frac{x}{L}\right)$$

Where

$C_0=2$ mol/m^3, $L=0.1$m, and x is the distance from one end of the material. The diffusion coefficient is $D=1\times10^{-9}$ m^2/s.

-Calculate the concentration C(x,t) at $x=0.05$ m after $t=500$ s.

Solution:

This problem involves solving Fick's Second Law with the initial condition provided. The solution to this type of problem, assuming a linear initial concentration profile, is generally given by:

$$C(x,t) = C_0\left(1-\frac{x}{L}\right)\text{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

Substitute the known values:

$$C(0.05{,}500) = 2\left(1-\frac{0.05}{0.1}\right)\text{erf}\left(\frac{0.05}{2\sqrt{1\text{x}10^{-9}\text{x}500}}\right)$$

$$C(0.05{,}500) = 1\text{xerf}(3.55)$$

Since erf(3.55)≈1, we get:

C(0.05,500)≈1×1=**1 mol/m^3**

Thus, the concentration at x=0.05 m after 500 seconds is approximately **1 mol/m^3.**

Exercise 05:

The vacancy formation energy for a metal is Qv=1.1 eV. Calculate the vacancy concentration at T=800 K, given that N=10^{28} atoms/m^3.

The vacancy concentration is given by the relation:

$$n_v = N.\exp\left(\frac{-Q_v}{k_B T}\right)$$

Where:

Q_v=1.1 eV,

k_B=8.617×10^{-5} eV/K,

T=800K,

N=10^{28} atoms/m^3.

Solution:

First, convert the vacancy formation energy into the correct units, and substitute the values into the equation:

$$n_v = 10^{28}.\exp\left(\frac{-1.1}{8.617x10^{-5}x800}\right)$$

Thus, the vacancy concentration is:

$$n_v = 10^{28}.\exp(-15.98)$$

$$n_v = \mathbf{1.3x10^{21} vacancies/m^3}$$

Thus, the vacancy concentration at T=800 K is approximately **1.3×10^{21} vacancies/m^3.**

Exercise 05:

Consider a cylindrical rod of length L=0.1 m with an initial concentration profile given by:

$$C(x,t) = C_0\left(1-\frac{x}{L}\right), 0 \le x \le L$$

Where C_0=5 mol/m^3. The diffusion coefficient D is 2×10^{-9} m^2/s^2 .

-Find the concentration at a point x=0.05 after t=1000s.

Solution:

For non-steady-state diffusion in a system with an initial linear concentration profile, Fick's second law solution is:

$$C(x,t) = C_0\left(1-\frac{x}{L}\right)\mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

Substitute the given values:

$$C(0.05,1000) = 5\left(1-\frac{0.05}{0.1}\right)\mathrm{erf}\left(\frac{0.05}{2\sqrt{2x10^{-9}x1000}}\right)$$

Simplify:

$$C(0.05,1000) = 2.5x\mathrm{erf}\left(\frac{0.05}{2.82x10^{-3}}\right) = 2.5x\mathrm{erf}(17.7)$$

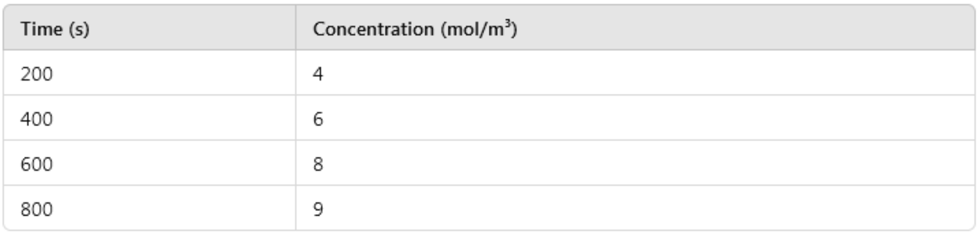
Since erf(17.7)≈1, we get:

$C(0.05,1000) = \mathbf{2.5 mol/m^3}$

Thus, the concentration at x=0.05 m after 1000 seconds is approximately **2.5 mol/m^3**

Exercise 07:

An experiment is conducted to measure the concentration of a diffusing species in a material. The following data is obtained for the concentration at x=0.05 m, after various times:



Time (s)	Concentration (mol/m^3)
200	4
400	6
600	8
800	9

Assume that the diffusion process is governed by Fick's second law, and the concentration profile is of the form:

$$C(x,t) = C_0 \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

-Estimate the diffusion coefficient D from the data.

Solution:

First, choose the data point at t=400 s and x=0.05 m, with concentration C=6 mol/m^3.

The equation is:

$$C(x,t) = C_0 \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

At t=400 s and x=0.05 m, assume C_0=10 mol/m^3 (an estimate based on the general trend of increasing concentration).

$$6 = 10\mathrm{erf}\left(\frac{0.05}{2\sqrt{Dx400}}\right)$$

Substitute the known values:

$$\mathrm{erf}^{-1}\left(\frac{6}{10}\right) = 0.4769$$

$$\frac{0.05}{2\sqrt{Dx400}} = 0.4769$$

$$D = \mathbf{6.85x10^{-6}m^2/s}$$

Thus, the estimated diffusion coefficient is **D=6.85×10^{-6} m^2/s.**

Chapiter III

Phenomenological Theory of Diffusion

III.1. Introduction

The phenomenological theory of diffusion seeks to describe and explain diffusion processes in a way that connects microscopic phenomena to macroscopic observations. In other words, this theory allows us to understand how the random movements of particles at the atomic or molecular level can be associated with a net flow of matter in a given medium, and how these phenomena can be quantified on a large scale.

This approach is particularly important for industrial, scientific, and environmental applications, where it is necessary to understand and predict diffusion behaviors in complex systems.

III.2. Basic Principle of Diffusion: Random movements

Diffusion is often explained through a model of Brownian motion (or random motion) of particles. When particles move in a fluid or solid, their trajectories are influenced by incessant collisions with other molecules in the medium, resulting in a random displacement of the particles. Over large scales of time and space, this results in a net movement of matter from a high concentration area to a low concentration area.

Diffusion phenomena are therefore intrinsically linked to randomness, and phenomenological theory seeks to relate this random behavior to macroscopic and measurable relationships, such as diffusion flux and concentration gradients.

III.3. Relationship between flux and concentration gradient

In a diffusion system, the net flux of particles depends on the concentration gradient, that is, the variation in the concentration of the diffusing substance in space. The phenomenological theory states that this flux is proportional to the concentration gradient, and this relationship is the basis of Fick's first law.

$$J = -D\frac{dC}{dx} \qquad (III.1)$$

Where:

J is the diffusion flux (in $mol/m^2 \cdot s$),

D is the diffusion coefficient (in m^2/s),

$\frac{dC}{dx}$ is the concentration gradient (in $mol/m^3 \cdot m$).

This law reflects the fact that particles spontaneously diffuse from areas of high concentration to areas of low concentration.

- **Phenomenological interpretation:**

-The proportionality between the diffusion flux and the concentration gradient is a fundamental hypothesis of phenomenological theory.

- The negative sign indicates that the flux always occurs in the direction of decreasing concentration, that is, from "high" to "low" in terms of concentration.

- The diffusion coefficient D is a parameter that reflects how easily particles move through the medium.

It depends on the temperature, the nature of the material, and the size of the particles.

III.4. Diffusion in Stationary and Non-Stationary Regimes

Stationary diffusion refers to a state in which the concentration of the diffusing substance remains constant over time at each given position in the medium. The diffusion flux is then constant and uniformly distributed throughout the system. In this case, the concentration C(x) can be modeled by Fick's first law. On the other hand, in a system where the concentration varies with time, we refer to it as non-stationary diffusion. In this case, Fick's second law must be used to describe the temporal evolution of the concentration.

- **Steady state:**

It is a state where the distribution of concentration is stable over time. The particle flux is constant, and there is no net accumulation of material at a given location.

- **Unsteady state:**

In an unsteady state, the concentration varies with time. The second law of Fick governs the evolution of concentration in this case, and it is expressed by:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \qquad \text{(III. 2)}$$

This means that the temporal variations in concentration are directly related to the spatial variations in concentration.

III.5. Modeling Diffusion in a Complex Medium: Phenomenological approaches

Diffusion in complex media (for example, porous media, solids, or polymers) can be more difficult to describe. In these cases, several phenomenological approaches can be used to account for factors such as porosity, obstacles, or interactions between the diffusing molecules.

III.5.1. Diffusion in solids and alloys

In a solid material or an alloy, diffusion can occur either by substitution (where an atom of type A replaces an atom of type B) or by interstitial (where a small atom moves between the sites of a crystal lattice). These two types of diffusion are often modeled using Fick's laws, but factors such as crystal defects (vacancies, dislocations) or temperature can significantly influence diffusion.

III.5.2. Diffusion in porous media

Diffusion in porous materials (such as foams, soils, or filtering materials) can be more complex due to the network structure of the material. Models such as the effective diffusion model can be used to relate diffusion in the pore network to diffusion in the total porous medium.

III.5.3. Diffusion in Polymers and Gels

In polymers and gels, macromolecules can interact with diffusing particles. Diffusion can be influenced by the viscosity of the polymer, the size of the pores, and the interaction forces between the particles and the polymer matrix. Adjusted phenomenological models are often used to predict these effects.

III.6. Relation between phenomenological theory and macroscopic laws

The phenomenological theory is particularly useful for linking microscopic phenomena (movement of individual particles) to the macroscopic laws of diffusion. For example:

-Fick's law allows for the connection between diffusion flux and a concentration gradient, and can thus be used to predict the amount of material moving through a material or fluid.

-In a dynamic system, where the concentration varies over time, Fick's second law can be used to model the temporal evolution of the concentration and predict how concentration distributions propagate in time and space.

III.7. Practical Applications

The phenomenological theory of diffusion is applied in many fields:

1. Diffusion in materials and alloys: To control the properties of materials based on how atoms or molecules diffuse within the material's structure.

2. Control of diffusion in chemical processes: To optimize chemical reactors where reactants must diffuse efficiently through catalysts or membranes.

3. Biological processes: Modeling the diffusion of substances in biological tissues, or the gas exchanges in the lungs.

4. Environmental systems: Study of the diffusion of pollutants in soils, waters, or the atmosphere.

Exercises

Exercise 01:

Consider a system where a species is diffusing in a one-dimensional medium. The mass flux J of the species is given by:

$$\mathrm{J} = -\mathrm{D}.\frac{\mathrm{dC}}{\mathrm{dx}}$$

Where:

D is the diffusion coefficient,

C(x) is the concentration of the species as a function of position.

If the concentration C(x) is a linear function of x, specifically $C(x)=C_0(1-x/L)$, where C_0 is the concentration at x=0 and L is the length of the medium.

- Find the mass flux J of the species.

Solution:

First, calculate the spatial derivative of C(x):

$$\frac{\mathrm{dC}}{\mathrm{dx}} = -\frac{\mathrm{C_0}}{\mathrm{L}}$$

Substitute this into the equation for mass flux:

$$\mathrm{J} = -\mathrm{D}\left(-\frac{\mathrm{C_0}}{\mathrm{L}}\right)$$

$$\mathrm{J} = \mathrm{D}\frac{\mathrm{C_0}}{\mathrm{L}}$$

Thus, the mass flux is:

$$J = D\frac{C_0}{L}$$

Exercise 02:

In a system with a spatially varying diffusion coefficient, the diffusion flux JJJ can be written as:

$$\mathrm{J} = -\mathrm{D}(\mathrm{x})\frac{\mathrm{dC}}{\mathrm{dx}}$$

Where $D(x)=D_0(1+\alpha x)$, D_0 is the reference diffusion coefficient, and α is a constant.

-Given that the concentration profile is linear, $C(x) = C_0 - \frac{C_0}{L}x$,

-Find the flux J.

Solution:

First, compute the spatial derivative of C(x):

$$\frac{\mathrm{dC}}{\mathrm{dx}} = -\frac{\mathrm{C_0}}{\mathrm{L}}$$

Now, substitute this into the equation for J:

$$\mathrm{J} = -\mathrm{D(x)}\frac{\mathrm{C_0}}{\mathrm{L}} = -\mathrm{D_0}(1 + \alpha\mathrm{x})\frac{\mathrm{C_0}}{\mathrm{L}}$$

$$\mathrm{J} = -\mathrm{D_0}\frac{\mathrm{C_0}}{\mathrm{L}}(1 + \alpha\mathrm{x})$$

Thus, the diffusion flux J is:

$$\mathrm{J} = -\mathrm{D_0}\frac{\mathrm{C_0}}{\mathrm{L}}(1 + \alpha\mathrm{x})$$

Exercise 03:

A species is diffusing in a material with a uniform concentration gradient. The measured mass flux J is 2×10^{-6} mol/m^2·s and the concentration gradient $\frac{dC}{dx}$ is 4 mol/m^3·m^{-1}.

-Find the diffusion coefficient D of the species.

Solution:

From Fick's first law:

$$\mathrm{J} = -\mathrm{D}\frac{\mathrm{dC}}{\mathrm{dx}}$$

Rearranging for D:

$$\mathrm{D} = -\frac{\mathrm{J}}{\frac{\mathrm{dC}}{\mathrm{dx}}}$$

Substitute the given values:

$$\mathrm{D} = -\frac{2\mathrm{x}10^{-6}}{4} = \mathbf{-5x10^{-7}m^2/s}$$

Thus, the diffusion coefficient D is **5×10^{-7} m^2/s.**

Exercise 04:

Consider a semi-infinite medium with a species diffusing from an initial concentration of $C_0=10$ mol/m^3 at t=0. The diffusion coefficient is $D=1.5\times10^{-9}$ m^2/s. The concentration at a distance x=0.1 m is measured after t=500 s.

-Find the concentration at x=0.1 m.

Solution:

Using the solution for non-steady-state diffusion in a semi-infinite medium:

$$C(x,t) = C_0 \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

Substitute the given values into the equation:

$$C(0.1,500) = 10\mathrm{erf}\left(\frac{0.1}{2\sqrt{1.5x10^{-9}x500}}\right)$$

First, calculate the argument of the error function:

$$\frac{0.1}{2\sqrt{1.5x10^{-9}x500}} = \frac{0.1}{2x0.0017} = 29.41$$

Thus:

$$C(0.1,500) = 10\mathrm{erf}(29.41) = 10x1 = \mathbf{10mol/m^3}$$

So, the concentration at x=0.1 m after t=500 s is **10 mol/m^3**.

Exercise 05:

A species diffuses through a material consisting of two layers:

Layer 1: Diffusion coefficient $D_1=2\times10^{-9}$ m^2/s, thickness $L_1=0.03$ m,

Layer 2: Diffusion coefficient $D_2=5\times10–^{10}$ m^2/s, thickness $L_2=0.07$ m.

The concentration on the left side of the first layer is $C_0=8$ mol/m^3 and on the right side of the second layer is $C_L=2$ mol/m^3.

-Calculate the steady-state flux J of the species.

Solution:

At steady-state, the flux J is the same through both layers. For each layer, Fick's first law gives:

$$J = -D_1 \frac{c_1 - c_0}{L_1} = -D_2 \frac{c_L - c_1}{L_2}$$

Equating the fluxes:

$$-D_1 \frac{c_1 - c_0}{L_1} = -D_2 \frac{c_L - c_1}{L_2}$$

Substitute the given values:

$$2x10^{-9} \frac{c_1 - 8}{0.03} = 5x10^{-10} \frac{2 - c_1}{0.07}$$

Solve for C_1:

$$\frac{c_1 - 8}{0.03} = \frac{1}{4} \frac{2 - c_1}{0.07}$$

Solve the equation for C_1, and then calculate J using one of the Fick's laws.

Exercise 06:

Consider a binary mixture of species A and B in a two-component system. The diffusion flux of species A can be written as:

$$J_A = -D_{AB} \frac{dC_A}{dx}$$

Where D_{AB} is the diffusion coefficient of species A in species B and $C_A(x)$ is the concentration of A.

If the diffusion coefficient D_{AB} is 2×10^{-9} m2/s and the concentration gradient

$\frac{dC_A}{dx} = 5 mol/m^3.m^{-1}$.

- Calculate the flux J_A.

Solution:

Using Fick’s first law:

$$J_A = -D_{AB} \frac{dC_A}{dx}$$

Substitute the given values:

$$J_A = -(2x10^{-9})x5 = -10x10^{-9} mol/m^2.s$$

$$\mathbf{J_A = -1x10^{-8} mol/m^2.s}$$

Thus, the flux of species A is $\mathbf{J_A = -1\times10^{-8}\ mol/m^2 \cdot s}$

Chapiter IV

Diffusion in Metals and Alloys in the Absence of Chemical Gradients

IV.1. Introduction

Diffusion in metals and alloys plays a crucial role in many industrial and technological processes, particularly in metallurgy, heat treatment of materials, manufacturing of electronic components, and the study of material properties. In this chapter, we will focus on solid-state diffusion in metals and alloys, particularly in cases where there is no external chemical gradient (i.e., in the absence of an initial concentration difference) or where diffusion is influenced by other factors such as temperature or pressure gradients.

This chapter will therefore address diffusion phenomena under specific conditions where the concentration of atoms or ions is homogeneous at the beginning, but where diffusion can still occur due to temperature gradients or other local variations in material properties. Diffusion in solids is a key phenomenon in processes such as decarburization, material alloying, crystal growth, or diffusion reactions in high-temperature materials.

IV.2. Diffusion in Metals and Alloys: Basic concepts

Diffusion in metals refers to the movement of atoms within a metallic matrix, generally due to thermal energy. The atoms move within the crystal lattice, often through interstitial sites or substitutional sites in alloy materials.

The main characteristics of diffusion in metals and alloys are:

1-Movement of atoms: Metallic atoms move due to thermal collisions, which create mobility within the crystal lattice.

2-Diffusion kinetics: The diffusion rate depends on several factors, including temperature, atom size, and the crystalline structure of the metal or alloy.

3-Diffusion mechanisms: In a metal, diffusion can occur through interstitial diffusion or substitutional diffusion:

-*Interstitial diffusion:* Small atoms (such as hydrogen, carbon, or nitrogen) move into the empty spaces of the crystal lattice.

-*Substitution diffusion:* Atoms move by occupying the site of another atom in the crystal lattice.

In metallic alloys, these phenomena can be more complex because there may be interactions between different types of atoms, barriers to diffusion created by solid phases or grains.

IV.3. Diffusion in the absence of chemical gradients

In the absence of a chemical gradient, thermal diffusion is primarily governed by temperature variations rather than differences in chemical concentration. This means that atoms or ions move in response to thermal gradients (i.e., differences in temperature) rather than differences in concentration. This phenomenon is called thermal diffusion or temperature gradient diffusion.

In this case, atoms or molecules diffuse in response to a thermal gradient, rather than a chemical gradient. It is an important phenomenon in the thermal treatments of materials, such as annealing, decarburization, or chemical vapor deposition. (CVD). The process can be modeled by the thermal diffusion equation.

• Temperature gradient and diffusion:

Diffusion in the absence of a chemical gradient but under the influence of a temperature gradient can be formalized using a phenomenological relationship similar to Fick's laws, but adapted to thermal processes.

The equation of thermal diffusion in a solid material can be formulated as follows:

$$J = -D_T . \frac{dT}{dx} \qquad (IV.1)$$

Where:

J is the thermal flux of matter, that is, the amount of material diffused per unit area and per unit time,

D_T is the thermal diffusion coefficient, which depends on the temperature and the material,

$\frac{dT}{dx}$ is the thermal gradient, that is, the variation of temperature in the direction of diffusion.

This equation shows that, even in the absence of a chemical gradient, atoms or ions can still move due to thermal variations in the material.

IV.4. Diffusion by temperature gradient and material flow

The phenomenon of diffusion due to a thermal gradient is often linked to the flux of matter because of thermal expansion and the mobility of atoms in the crystal lattice under the effect of heat. This flux of matter is often associated with a process called the Seebeck effect or the Peltier effect in thermoconductive materials, but in the case of metals and alloys, it specifically refers to the thermal diffusion of the atom within the metal.

The material flux can be expressed by:

$$J = -M \frac{dT}{dx} \qquad (IV.2)$$

Where:

M is the atomic mobility, which depends on the temperature, the material structure, and the presence or absence of crystal defects.

- **Effect of temperature on diffusion:**

Diffusion in metals and alloys is extremely temperature-dependent. As the temperature increases, the atoms in the metal have more energy and move more quickly through the crystal lattice, which increases the diffusion rate. This relationship can be quantified by an Arrhenius law for the diffusion coefficient D, such as:

$$\mathrm{D} = \mathrm{D}_0 . \mathrm{e}^{-\frac{\mathrm{Q}}{\mathrm{RT}}} \qquad (\mathrm{IV}.3)$$

Where:

D_0 is a pre-exponential factor,

Q is the activation energy for diffusion,

R is the ideal gas constant,

T is the temperature in Kelvin.

Thus, even without a chemical gradient, temperature has a significant effect on diffusion by altering atomic mobility and the way atoms interact within the metallic matrix.

IV.5. Diffusion in Metallic Alloys

In metallic alloys, diffusion is often more complex due to the presence of several different types of atoms. (for example, in a copper-silver or steel-carbon alloy). Even in the absence of significant chemical gradients, atoms can diffuse through interstitial sites or substitutional sites, influenced by the crystalline structure of the material and the metallurgical phases present in the alloy.

Interstitial diffusion and substitutional diffusion follow laws similar to those observed in pure metals, but the diffusion rate can be modified by the interactions between the elements present in the alloy.

- **Factors influencing diffusion in alloys:**

-Size and nature of the atoms: Small atoms (like carbon in steel) differ in their ability to move through the crystal lattice compared to larger atoms. (like copper in a copper-zinc alloy).

-*Temperature:* As with pure metals, temperature has a major effect on the diffusion rate in metallic alloys.

-*Solid phases:* Diffusion can be influenced by secondary phases in the alloy, such as precipitates or enriched zones.

IV.6. Industrial and Technological Applications

Diffusion in metals and alloys in the absence of a chemical gradient plays a key role in many industrial processes, particularly in:

1. Thermal treatments of metals: For example, thermal hardening or annealing of alloys to adjust their mechanical properties.

2. The diffusion of carbon in steel: In processes like carburizing, where carbon diffuses from an external source into steel at high temperatures.

3. The growth of crystals in alloys: In solidification or controlled cooling processes, where thermal diffusion influences the structure of the formed crystals.

4. Layer deposition processes: As in the manufacturing of electronic components or metallic coatings where diffusion under thermal gradient can influence surface properties.

Exercises

Exercise 01:

The vacancy formation energy Qv for a metal is 1.5 eV. The universal gas constant R is 8.314 J/mol. Given that the temperature is T=1200 K.

-Calculate the vacancy concentration n_v in the metal, assuming the total atomic concentration is $N=10^{29}$ atoms/m^3.

Use the following equation for vacancy concentration:

$$n_v = N\exp(-\frac{Q_v}{k_B T})$$

Where:

k_B is the Boltzmann constant, $k_B=8.617\times10^{-5}$ eV,

Qv is the vacancy formation energy,

T is the temperature in Kelvin,

N is the atomic concentration.

Solution:

Substitute the known values into the equation:

$$n_v = 10^{29}\exp(-\frac{1.5}{8.617x10^{-5}x1200})$$

$$n_v = \mathbf{4.75x10^{22}vacancies/m^3}$$

Thus, the vacancy concentration is approximately **$n_v=4.75\times10^{22}$ vacancies/m^3.**

Exercise 02:

For a given metal, the self-diffusion coefficient Ds at T=800 K is given by the equation:

$$D_s = D_0\exp\left(-\frac{Q_d}{RT}\right)$$

Where:

$D_0=5\times10^{-3}$ m^2/s is the pre-exponential factor,

Q_d=1.2×105 J/mol is the activation energy for self-diffusion,

R=8.314 J/mol.K is the universal gas constant,

T=800 K.

- Calculate the self-diffusion coefficient Ds at T=800 K.

Solution:

Substitute the known values into the equation:

$$D_s = 5x10^{-3}\exp\left(-\frac{1.2x10^5}{8.314x800}\right)$$

$$\mathbf{D_s = 9.5x10^{-11}m^2/s}$$

Thus, the self-diffusion coefficient at T=800 K is approximately **$Ds=9.5\times10^{-11}$ m^2/s.**

Exercise 03:

In an alloy of metals A and B, the diffusion coefficient of species A in species B is given by:

$$D_{AB} = D_0\exp\left(-\frac{Q_{AB}}{RT}\right)$$

Where:

$D_0=1\times10^{-7}$ m^2/s is the pre-exponential factor,

$Q_{AB}=1.5\times10^5$ J/mol is the activation energy for diffusion of species A in B,

R=8.314 J/mol.K,

T=900 K.

-Calculate the diffusion coefficient D_{AB} at T=900 K.

Solution:

Substitute the known values into the equation:

$$D_{AB} = 1x10^{-7}\exp\left(-\frac{1.5x10^5}{8.314x900}\right)$$

$$D_{AB} = \mathbf{1.9x10^{-16}\ m^2/s}$$

Thus, the diffusion coefficient D_{AB} at T=900 KT is approximately **1.9×10^{-16} m^2/s.**

Exercise 04:

The vacancy diffusion coefficient D_v in a metal can be expressed as:

$$D_v = D_0 \exp\left(-\frac{Q_v}{RT}\right)$$

Where:

$D_0=1.3\times10^{-4}$ m^2/s is the pre-exponential factor,

$Qv=1.1\times10^5$ J/mol is the activation energy for vacancy diffusion,

R=8.314 J/mol.K,

T=1000 K.

-Calculate the vacancy diffusion coefficient D_v at T=1000 K.

Solution:

Substitute the known values into the equation:

$$D_v = D_0 \exp\left(-\frac{Q_v}{RT}\right)$$

$$D_v = 1.3x10^{-4} \exp\left(-\frac{1.1x10^6}{8.314x1000}\right)$$

$$D_v = \mathbf{2.34x10^{-10}m^2/s}$$

Thus, the vacancy diffusion coefficient D_v at T=1000 K is approximately D_v=**2.34×10^{-10} m^2/s.**

Exercise 05:

A metal with a diffusion coefficient $D=2\times10^{-9}$ m^2/s at T=1000 K, experiences a concentration gradient of 1 $mol/m^3\cdot m^{-1}$. Calculate the mass flux J for the species in the metal.

Solution:

Using Fick's first law:

$$J = -D\frac{dC}{dx}$$

Substitute the given values:

$$J = -(2x10^{-9})x1 = \mathbf{-2x10^{-9}mol/m^2.s}$$

Thus, the mass flux J is **-2×10^{-9} $mol/m^2\cdot s$.**

Chapiter V

Surface Diffusion

V.1. Introduction

Surface diffusion refers to the diffusion of atoms or molecules on the surface of a solid material. It differs from bulk diffusion (or within the volume of the material) because it occurs only in the surface layer of the solid, where atoms have properties and behaviors different from those in the volume. This phenomenon is crucial for several industrial and technological processes, such as the decarburization of steels, the diffusion of thin films in electronic devices, or even the chemical reactions catalyzed on the surface of materials.

In this chapter, we will analyze in detail the characteristics of surface diffusion, the mechanisms that underlie it, its differences from volume diffusion, and its practical applications in various industrial fields.

V.2. Principle of Surface Diffusion

Surface diffusion is a phenomenon that mainly concerns the atoms on the surface of a solid material. Unlike internal atoms that occupy sites in the crystal lattice, surface atoms are generally less bound, allowing them to move more freely. This increased mobility facilitates their diffusion on the surface, especially at high temperatures.

Surface diffusion is influenced by several factors:

-The nature of the surface: Crystalline, amorphous, or nanostructured surfaces have different diffusion behaviors.

-Temperature: As with diffusion in the volume, temperature plays a major role in increasing the mobility of surface atoms.

-The interactions between surface atoms: The strength of the interaction between surface atoms and diffusing species can also influence the diffusion rate.

In general, surface atoms have weaker bonds than those inside the material, which allows for greater mobility. However, surface diffusion is often limited by energy barriers (such as surface defects, impurities, or oxides) that can reduce the speed of the process.

V.3. Mechanisms of Surface Diffusion

Surface diffusion can occur through several mechanisms, which are generally influenced by temperature and surface structure. The two main mechanisms are:

1. Diffusion by successive hops (hopping)

In this mechanism, a surface atom "jumps" from one site to another on the surface of the material. Each atom can move by successively occupying neighboring positions on the surface. Hopping diffusion is favored by heat, as it overcomes the activation energy required for atoms to move from one site to another.

2. Diffusion via surface defects (vacancies and surface defects)

Vacancies or surface defects play a major role in surface diffusion. Vacancies are empty sites in the crystalline lattice at the surface of the material, creating places where atoms can infiltrate and move. This phenomenon is similar to diffusion in the bulk but limited to the surface.

Other mechanisms include directed diffusion phenomena where specific interactions between atoms and the surface or molecules can direct diffusion in certain directions, as in catalytic processes.

V.4. Law of surface diffusion

Surface diffusion can be described by an equation similar to that of volumetric diffusion, but adapted to account for differences in surface properties. In the absence of significant chemical gradients in the volume, surface diffusion is generally governed by the following relationship:

$$J = -D_S . \frac{dC}{dx} \qquad (V.1)$$

Where:

J is the diffusion flux of the substance (atom or ion) at the surface ($mol/m^2 \cdot s$),

Ds is the surface diffusion coefficient, which depends on the temperature, the surface structure, and the material characteristics,

$\frac{dC}{dx}$ is the concentration gradient as a function of the position on the surface of the material.

However, unlike bulk diffusion, the surface diffusion coefficient Ds is generally much larger than the diffusion coefficient in the volume. This is explained by the greater mobility of surface atoms, which are less strongly bonded than in the volume.

- **Relation with temperature:**

The surface diffusion coefficient also depends on the temperature, following a relationship similar to that of diffusion in the volume:

$$D_S = D_{S0} e^{-\frac{Q_S}{RT}} \qquad (V.2)$$

Where:

D_{s0} is a pre-exponential factor,

Qs is the activation energy for surface diffusion,

R is the ideal gas constant,

T is the temperature in Kelvin.

Surface diffusion thus becomes much faster at high temperatures, where atoms have more energy to move across the surface.

V.5. Surface diffusion vs Volume diffusion

Although surface diffusion and bulk diffusion are similar processes, they differ in terms of:

1. Diffusion properties: The atoms on the surface are less bound to each other than in the volume, which allows them to move more freely. This result in a much higher surface diffusion coefficient compared to that in the volume.

2. Temperature: Temperature plays a fundamental role in both types of diffusion, but in surface diffusion, the effect of temperature is often more pronounced due to the thermal energy required to overcome the activation barriers at the surface.

3. Concentration and gradient: In surface diffusion, the concentration of atoms at the surface can be affected by the adsorption or desorption of molecules or atoms from the external environment. This can modify the concentration gradient, which is not necessarily uniform on the surface.

4. Adsorbate concentration: In some cases, such as with catalysts, specific molecules can be adsorbed onto the surface, altering the dynamics of surface diffusion.

V.6. Industrial applications of surface diffusion

Surface diffusion is a key phenomenon in many industrial processes and technological applications:

1. Carburizing and nitriding of steels:

Carburizing is a heat treatment where carbon diffuses to the surface of steel to increase its surface hardness. A similar phenomenon, called nitriding, involves the diffusion of nitrogen to the surface to create high-strength nitrocarbide oxide layers.

2. Thin film deposition

Surface diffusion is essential in vapor phase deposition (CVD) and sputtering processes, used to apply metallic or ceramic layers on material surfaces. These layers are formed by the diffusion of precursors or ions on the surface of the substrate.

3. Catalytic reactions:

In solid catalysts, the molecules react on the surface of the catalyst. The surface diffusion of reactants on the catalyst surface is crucial for the reaction rate. This process is essential in applications such as automotive catalysts or chemical reactors.

4. Oxidation and corrosion

The diffusion of oxygen or other reactive elements to the surface of metals is a fundamental process in oxidation and corrosion. For example, in iron alloys, the diffusion of oxygen leads to the formation of rust. (iron oxide).

5. Semiconductor manufacturing

Surface diffusion is used in the manufacturing of thin films and semiconductor solar panels. The doping of semiconductors by the diffusion of elements such as phosphorus or boron is generally done on the surface

Exercises

Exercise 01:

In a certain metal, the activation energy for surface diffusion is Qs=2.5×10^4 J/mol, and the pre-exponential factor for the surface diffusion coefficient is $D_0=1\times10^{-2}$ m^2/s.

Calculate the surface diffusion coefficient Ds at T=800 K.

The surface diffusion coefficient is given by the equation:

$$D_s = D_0 . \exp\left(-\frac{Q_s}{RT}\right)$$

Where:

R=8.314 J/mol\K,

T is the temperature in Kelvin.

Solution:

Substitute the known values into the equation:

$$D_s = 1x10^{-2} . \exp\left(-\frac{2.5x10^4}{8.314x800}\right)$$

$$D_s = \mathbf{2.35x10^{-4} m^2/s}$$

Thus, the surface diffusion coefficient at T=800 K is approximately Ds=2.35×10^{-4} m^2/s.

Exercise 02:

Given that the surface diffusion coefficient Ds of a material obeys the equation:

$$D_s = D_0 . \exp\left(-\frac{Q_s}{RT}\right)$$

Where:

$D_0=2\times10^{-2}$ m^2/s,

Qs=2.0×10^5 J/mol,

R=8.314 J/mol.

If the surface diffusion coefficient at 800 K is Ds(800 K)=1.2×10^{-4} m^2/s, calculate the surface diffusion coefficient Ds at T=1000 K.

Solution:

Using the equation for Ds, we first find the pre-exponential factor D_0. At T=800K:

$$D_s(800) = D_0.\exp\left(-\frac{Q_s}{Rx800}\right)$$

$$1.2x10^{-4} = D_0.\exp\left(-\frac{2.0x10^5}{8.314x800}\right)$$

$$D_0 = \frac{1.2x10^{-4}}{4.3x10^{-14}} = \mathbf{2.79x10^9 m^2/s}$$

Now, calculate Ds at T=1000 K:

$$D_s(1000) = 2.79x10^9.\exp\left(\frac{2.0x10^5}{8.314x100}\right)$$

$$D_s(1000) = \mathbf{7.53x10^{-2}} \boldsymbol{m^2/s}$$

Thus, the surface diffusion coefficient at T=1000 K is approximately **Ds=7.53×10^{-2} m^2/s**

Exercise 03:

During the deposition of a metal thin film, atoms diffuse across the surface at a rate governed by the surface diffusion coefficient Ds. If the temperature during deposition is T=900 K and the surface diffusion coefficient Ds is 1.5×10^{-4} m^2/s, calculate the mean free path λ for surface diffusion. Assume that the interaction radius r between diffusing atoms is 1.0 A°.

The mean free path for surface diffusion is given by:

$$\lambda = \frac{D_0}{r}$$

Where:

Ds is the surface diffusion coefficient, r is the interaction radius.

Solution:

Substitute the known values into the equation:

$$J_S = (1.2x10^{-4})x(10^{14})x(10^{22})$$

$$J_S = \mathbf{1.2x10^{32}atom/m^2.s}$$

Thus, the surface diffusion flux Js of adatoms **is Js=1.2×10**$^{\mathbf{32}}$ atoms/m^2·s.

Exercise 04:

The rate of surface diffusion of adatoms (atoms adsorbed on a surface) depends on the density of steps (discontinuities) on the surface. In a study of surface diffusion on a crystal, the surface step density is $\rho=10^{14}$ steps/m^2. If the surface diffusion coefficient is $D_s=1.2\times10^{-4}$ m^2/s.

-Calculate the surface diffusion flux Js of adatoms on the surface, assuming that the concentration of adatoms is $C=10^{22}$ atoms/m^2.

The surface diffusion flux is given by:

$$J_s = D_s\rho C$$

Where:

Ds is the surface diffusion coefficient,

ρ is the step density,

C is the adatom concentration.

Solution:

Substitute the known values into the equation:

$$J_s = (1.2x10^{-4})x(10^{14})x(10^{22})$$

$$J_s = \mathbf{1.2x10^{32}atoms/m^2.s}$$

Thus, the surface diffusion flux Js of adatoms is **Js=1.2×10^{32} atoms/m^{2}·s.**

Exercise 05:

For a rough surface with a surface diffusion coefficient Ds=2×10^{-5} m^{2}/s, the surface roughness results in a modified effective diffusion coefficient. If the effective diffusion coefficient is related to the surface roughness factor α by:

$$\mathrm{D_s^{eff}} = \alpha \mathrm{D_s}$$

Where.

α=1.5 is the surface roughness factor.

-Calculate the effective diffusion coefficient D_s^{eff}

Solution:

Substitute the known values into the equation:

$$\mathrm{D_s^{eff}} = 1.5\mathrm{x}2\mathrm{x}10^{-5} = \mathbf{3x10^{-5}m^2/s}$$

Thus, the effective surface diffusion coefficient is
$\mathrm{D_s^{eff}} = \mathbf{3x10^{-5}m^2/s}$

Chapiter VI

Applications of Diffusion

VI.1. Introduction

Diffusion is a key phenomenon in many industrial and technological processes. Its importance extends from material manufacturing processes to cutting-edge applications in the fields of chemistry, metallurgy, electronics, energy, and the environment. In this chapter, we will explore different applications of diffusion, detailing the industrial processes and technologies in which diffusion plays a fundamental role.

VI.2. Diffusion in Metallurgical Processes

1. Carburizing and nitriding of steels:

-*Carburizing:* This process involves increasing the carbon content of the surface of steels to improve their hardness and wear resistance. Carbon diffuses into the surface of the steel at high temperatures (between 850°C and 950°C), generally in a carbon-enriched environment. (gas or liquid). Carbon diffusion modifies the mechanical properties of the material by creating a hardened layer, often called a carburized layer.

-*Nitriding:* Nitriding is similar to carburizing, but it involves introducing nitrogen into the surface of the materials, which creates nitrides on the surface of the metal, thereby improving wear resistance, fatigue, and corrosion. The diffusion of nitrogen is achieved through heat treatment in a gas containing ammonia.

2. Diffusion in powder metallurgy:

Powder metallurgy involves processes where metal powders are compacted and then heated to high temperatures to promote diffusion and partial melting of the particles. This process is commonly used to manufacture complex metallic components, such as filters or gears, while preserving the physical and mechanical properties of the material.

3. Diffusion of alloying elements:

In the manufacturing of metallic alloys, elements such as nickel, copper, zinc, or manganese diffuse into a metallic matrix to enhance certain properties (for example, corrosion resistance or heat resistance). The diffusion of these alloying elements occurs during melting or annealing.

VI.3. Applications in the field of semiconductors

1. Doping of semiconductors:

In the manufacturing of semiconductor devices, diffusion is used to introduce impurities, called dopants, into the crystalline structure of semiconductor materials. (for example, silicon or germanium). Doping modifies the electrical properties of the materials, thereby creating n-type regions (excess electrons) and p-type regions (excess holes). This process is carried out by thermal diffusion of the dopants into the surface layer of the semiconductor, generally at high temperatures, to promote their incorporation into the crystalline matrix.

2. Fabrication of transistors and electronic chips:

Doping by diffusion is a fundamental element in the fabrication of transistors, ICs (integrated circuits), and electronic chips. Dopants are diffused to form PN (positive-negative) junctions in devices such as diodes and bipolar transistors, which are the basic elements of electronic components.

3. Deposition and diffusion in thin films:

In the fabrication of thin films for electronic devices, such as solar panels, light-emitting diodes (LEDs), and sensors, the diffusion of elements in the surface layers is essential for controlling the electronic properties of the devices.

VI.4. Applications in Chemical Catalysis

1. Solid catalysts and surface diffusion:

Surface diffusion is a key factor in the functioning of solid catalysts. The chemical reactions that occur on the surface of catalysts strongly depend on the ability of reactive molecules to diffuse to the surface and interact with the active sites of the catalyst. This phenomenon is crucial in chemical reactors, catalytic converters in automotive engines, and chemical production processes such as hydrogenation, desulfurization, and ammonia synthesis. (Processus Haber-Bosch). In these applications, the diffusion of reactants and products on the surface of the catalysts directly influences the efficiency of the catalytic process.

2. Surface reactions in high-temperature reactors:

Surface reactions, such as those involving gases in reactors, are also influenced by the diffusion of atoms or molecules on the surface of the catalyst. For example, in catalysts for the hydrogenation of hydrocarbons, hydrogen and carbon molecules diffuse on the surface of the catalyst before reacting.

VI.5. Applications in Energy and Environment

1. Diffusion in batteries and supercapacitors:

The diffusion of ions in the electrode materials of batteries and supercapacitors is a key process that determines the energy storage capacity and lifespan of these devices. For example, in lithium-ion batteries, the diffusion of lithium ions between the anode and the cathode during charge and discharge cycles is fundamental to the system's operation. The optimization of diffusion in these systems allows for improved energy efficiency and lifespan.

2. CO2 Capture and Storage (CCS):

In the context of combating climate change, diffusion is used in CO2 capture systems, where carbon dioxide is captured from industrial exhaust gases and transported to underground storage sites. Diffusion also plays a role in the materials used to trap CO2, such as silica or alumina-based adsorbents.

3. Diffusion in materials for hydrogen:

In hydrogen-related technologies, the diffusion of hydrogen atoms in high-temperature materials is a limiting factor. This phenomenon is important for applications such as hydrogen fuel cells and the storage of hydrogen at high pressure in metal tanks.

VI.6. Applications in materials and material aging

1. Diffusion in polymer materials:

The diffusion of molecules in polymers (for example, in polymer membranes used for ion exchangers) is a critical factor in the performance of materials used for medical applications, sensors, or water purification systems. The diffusion performance of molecules in these materials can be used to control drug release or for filtration applications.

2. Corrosion and aging of materials:

The diffusion of oxygen and ions in metals is involved in the phenomenon of corrosion. The rusting process, for example, results from the diffusion of oxygen and water on the surface of metals, leading to a degradation of mechanical and chemical properties. Understanding diffusion in this context allows for the development of more corrosion-resistant materials and improves the durability of metal structures in aggressive environments.

Appendix

Annexe C.1 : Principales transformations intégrales : Laplace, Fourier, Hankel.

Transformée de Laplace

Définition

$L[T(t)] = \theta(p) = \int_0^{\infty} \exp(-p\,t)\,T(t)\,dt$ et $L^{-1}[\theta(p)] = T(t)$ (Transformée inverse)

Propriétés

Linéarité	$L[a_1\,T(t) + a_2\,T(t)] = a_1\,L[T(t)] + a_2\,L[T(t)]$, idem pour L^{-1}	
Translation	$L[\exp(a\,t)\,T(t)] = \theta(p-a)$	$L^{-1}[\theta(p-a)] = \exp(a\,t)\,T(t)$
		$L^{-1}[\exp(-a\,p)\,\theta(p)] = T(t-a)$ si $t > a$ 0 si $t < a$
Changement d'échelle	$L[T(a\,t)] = \frac{1}{a}\,\theta\left(\frac{p}{a}\right)$	$L^{-1}[\theta(a\,p)] = \frac{1}{a}\,T\left(\frac{t}{a}\right)$
Dérivation	$L[T'(t)] = p\,\theta(p) - T(0)$	$L^{-1}\left[\theta^{(n)}(p)\right] = (-1)^n\,t^n\,T(t)$
	$L[T''(t)] = p^2\,\theta(p) - p\,T(0) - T'(0)$	
Intégration	$L\left[\int_0^t T(u)\,du\right] = \frac{\theta(p)}{p}$	$L^{-1}\left[\int_p^{\infty} \theta(u)\,du\right] = \frac{F(t)}{t}$
Multiplication par t^n	$L[t^n\,T(t)] = (-1)^n\,\theta^{(n)}(p)$	$L^{-1}[p\,\theta(p)] = T'(t) - T(0)\,\delta(t)$
Division par t	$L\left[\frac{T(t)}{t}\right] = \int_p^{\infty} \theta(u)\,du$	$L^{-1}\left[\frac{\theta(p)}{p}\right] = \int_0^t T(u)\,du$
Fonctions périodiques (Période P)	$L[T(t)] = \dfrac{\int_0^P \exp(-p\,t)\,T(t)\,dt}{1-\exp(-p\,P)}$	

Transformée de Fourier complexe

Définition

$$F[T(x)] = \theta(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{i\omega x}\;T(x)\,dx$$

$$T(x) = F^{-1}[\theta(\omega)] = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-i\omega x}\;\theta(\omega)\,d\omega$$

Propriétés

$$F\left[\frac{\partial T}{\partial x}\right] = -i\,\omega\,\theta(\omega)$$

$$F\left[\frac{\partial^2 T}{\partial x^2}\right] = -\,\omega^2\,\theta(\omega)$$

Annexe C.2 : Transformation de Laplace inverse

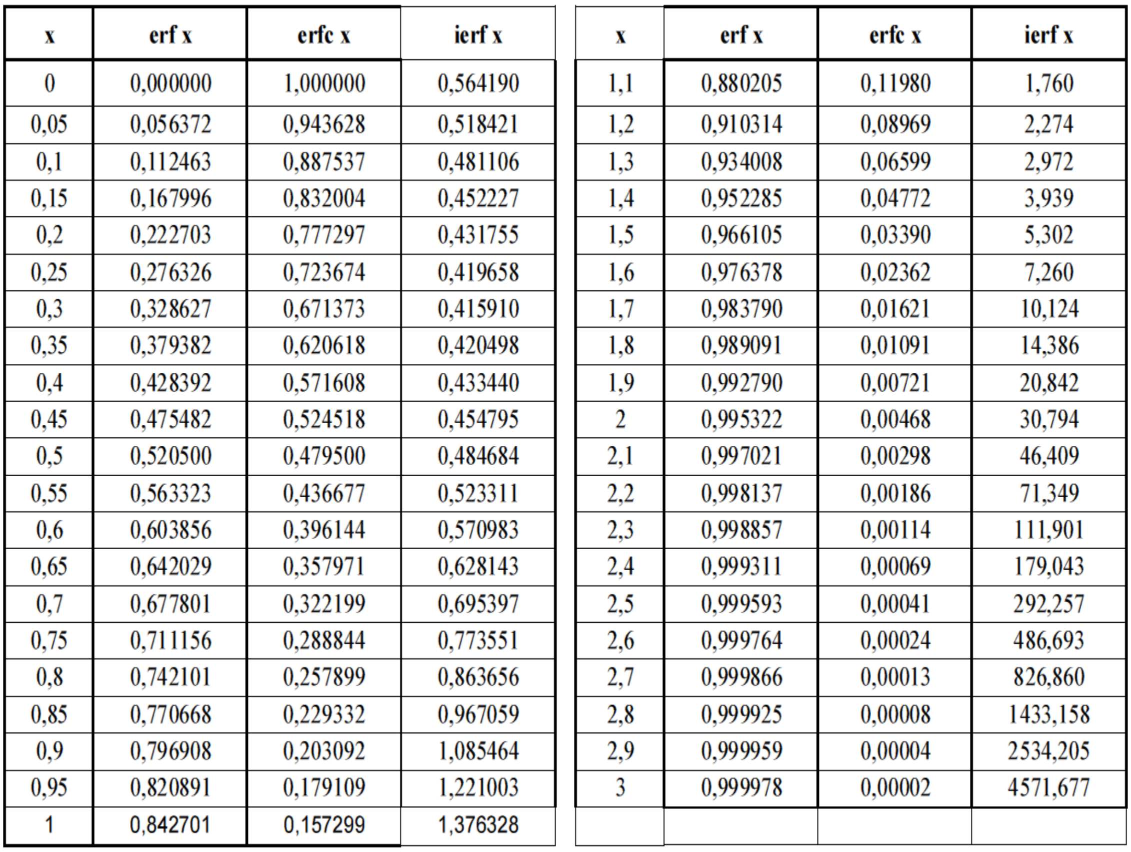
$$q = \sqrt{\frac{p}{a}}$$

$\theta(p) = L\{T(t)\}$	$T(t)$
$\frac{1}{p}$	1
1	$\delta(t)$ Dirac
$\frac{1}{p+\beta}$	$e^{-\beta t}$
$\frac{\omega}{p^2+\omega^2}$	$\sin(\omega t)$
$\frac{p}{p^2+\omega^2}$	$\cos(\omega t)$
$\frac{b}{p\left(b+\sqrt{p}\right)}$	$1-\exp\left(b^2\,t\right)\mathrm{erfc}\left(b\sqrt{t}\right)$

$\theta(p) = L\{T(t)\}$	$T(t)$
$\frac{\ln(p)}{p}$	$-\ln(t)-\gamma \quad ; \quad \gamma = 0{,}57721$
$\frac{1}{\sqrt{p}}$	$\frac{1}{\sqrt{\pi\,t}}$
$\frac{1}{p\sqrt{p}}$	$\frac{2}{\sqrt{\pi}}\sqrt{t}$
$\frac{\omega}{p^2-\omega^2}$	$\mathrm{sh}(\omega t)$
$\frac{p}{p^2-\omega^2}$	$\mathrm{ch}(\omega t)$
$\frac{1}{p^n} \quad n = 1, 2, 3\ldots$	$\frac{t^{n-1}}{(n-1)!}$

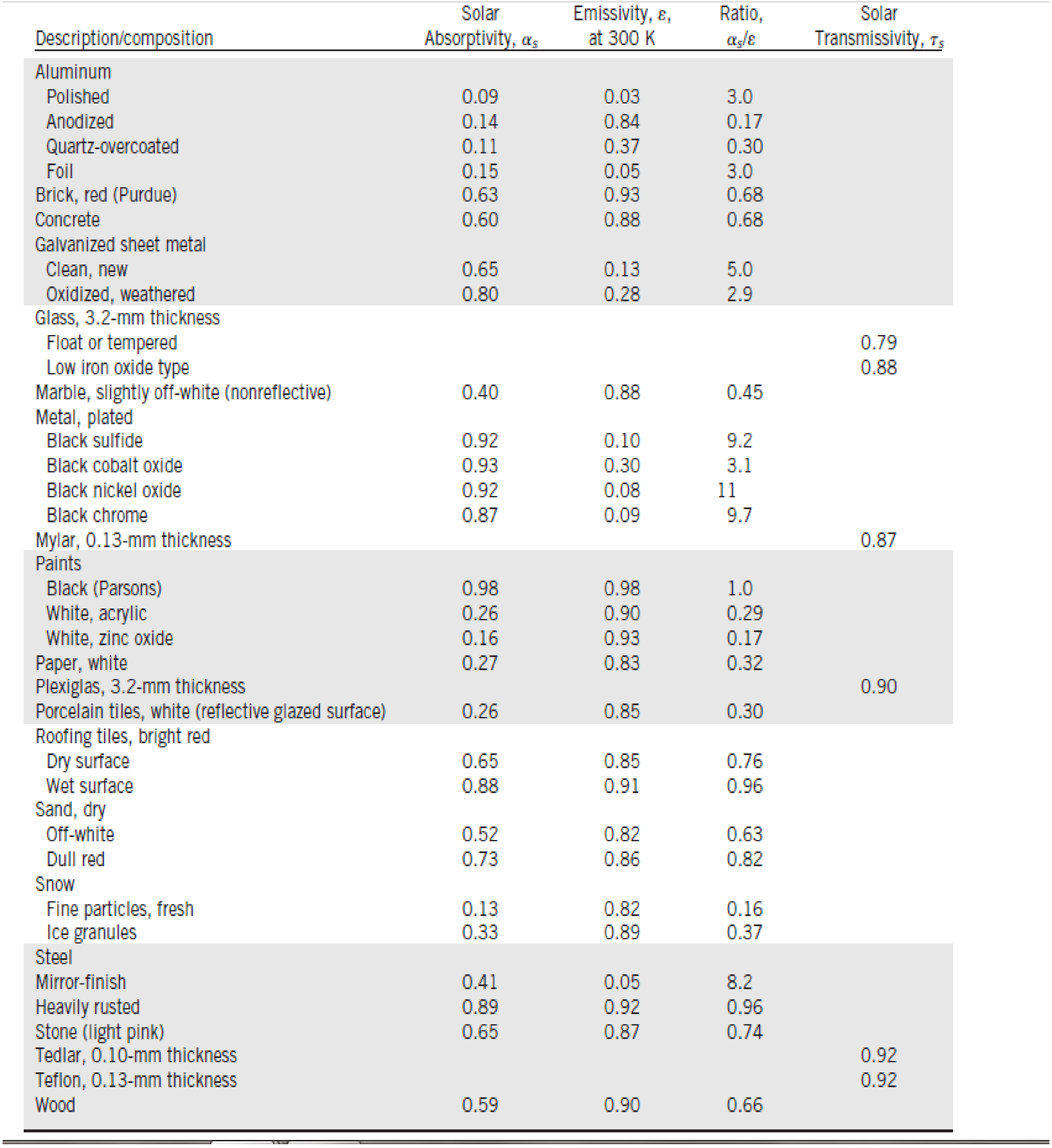
$\theta(p) = L\{T(t)\}$	$T(t)$
$e^{-q\,x}$	$\frac{x}{2\sqrt{\pi\,a\,t^3}}\exp\left(-\frac{x^2}{4\,a\,t}\right)$
$\frac{e^{-q\,x}}{q}$	$\left(\frac{a}{\pi\,t}\right)^{1/2}\exp\left(-\frac{x^2}{4\,a\,t}\right)$
$\frac{e^{-q\,x}}{p}$	$\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right)$
$\frac{e^{-q\,x}}{pq}$	$2\left(\frac{\alpha t}{\pi}\right)^{1/2}\exp\left(-\frac{x^2}{4at}\right) - x\,\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right)$
$\frac{e^{-q\,x}}{p^2}$	$\left(t+\frac{x^2}{2a}\right)\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right) - x\left(\frac{t}{\pi a}\right)^{1/2}\exp\left(-\frac{x^2}{4at}\right)$
$\frac{e^{-q\,x}}{q+h}$	$\left(\frac{a}{\pi t}\right)^{1/2}\exp\left(-\frac{x^2}{4at}\right) - h\,a\exp\left(hx+a\,t\,h^2\right)\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}+h\sqrt{at}\right)$
$\frac{e^{-q\,x}}{q\,(q+h)}$	$a\exp\left(hx+a\,t\,h^2\right)\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}+h\sqrt{at}\right)$
$\frac{e^{-q\,x}}{p\,(q+h)}$	$\frac{1}{h}\mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right)-\frac{1}{h}\exp\left(hx+a\,t\,h^2\right)\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}+h\sqrt{at}\right)$
$\frac{e^{-q\,x}}{p\,q\,(q+h)}$	$\frac{2}{h}\left(\frac{a}{\pi t}\right)^{1/2}\exp\left(-\frac{x^2}{4at}\right)-\frac{1+h\,x}{h^2}\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right)+\frac{1}{h^2}\exp\left(hx+a\,t\,h^2\right)\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}+h\sqrt{at}\right)$
$\frac{e^{-q\,x}}{(q+h)^2}$	$-2\,h\left(\frac{a^3\,t}{\pi}\right)\exp\left(\frac{-x^2}{2\,a\,t}\right)+a\left(1+h\,x+2\,h^2\,a\,t\right)\exp\left(h\,x+a\,t\,h^2\right)\mathrm{erfc}\left(\frac{x}{2\sqrt{at}}+h\sqrt{at}\right)$

Annexe C.3 : Valeur de la fonction erf



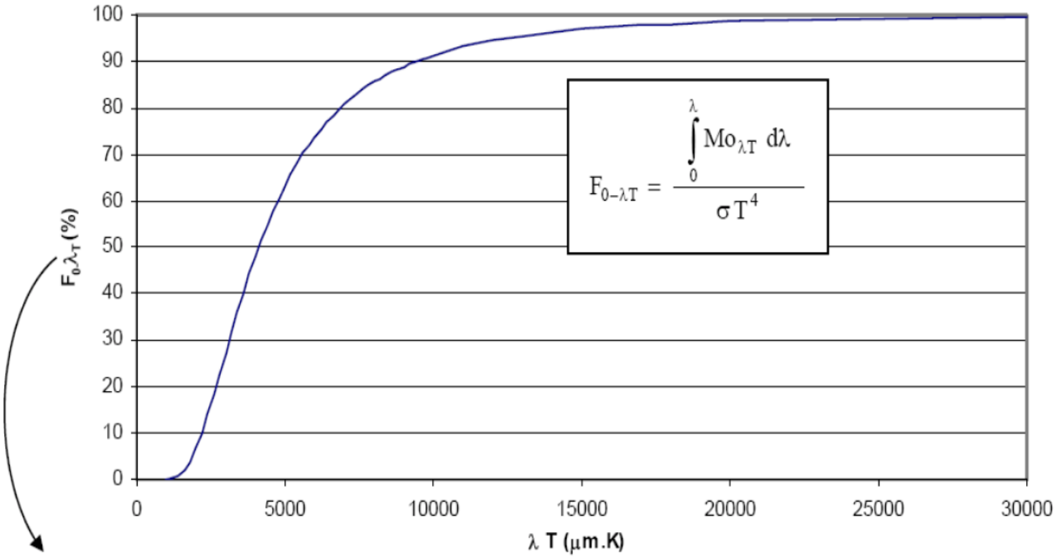
x	erf x	erfc x	ierf x
0	0,000000	1,000000	0,564190
0,05	0,056372	0,943628	0,518421
0,1	0,112463	0,887537	0,481106
0,15	0,167996	0,832004	0,452227
0,2	0,222703	0,777297	0,431755
0,25	0,276326	0,723674	0,419658
0,3	0,328627	0,671373	0,415910
0,35	0,379382	0,620618	0,420498
0,4	0,428392	0,571608	0,433440
0,45	0,475482	0,524518	0,454795
0,5	0,520500	0,479500	0,484684
0,55	0,563323	0,436677	0,523311
0,6	0,603856	0,396144	0,570983
0,65	0,642029	0,357971	0,628143
0,7	0,677801	0,322199	0,695397
0,75	0,711156	0,288844	0,773551
0,8	0,742101	0,257899	0,863656
0,85	0,770668	0,229332	0,967059
0,9	0,796908	0,203092	1,085464
0,95	0,820891	0,179109	1,221003
1	0,842701	0,157299	1,376328
1,1	0,880205	0,11980	1,760
1,2	0,910314	0,08969	2,274
1,3	0,934008	0,06599	2,972
1,4	0,952285	0,04772	3,939
1,5	0,966105	0,03390	5,302
1,6	0,976378	0,02362	7,260
1,7	0,983790	0,01621	10,124
1,8	0,989091	0,01091	14,386
1,9	0,992790	0,00721	20,842
2	0,995322	0,00468	30,794
2,1	0,997021	0,00298	46,409
2,2	0,998137	0,00186	71,349
2,3	0,998857	0,00114	111,901
2,4	0,999311	0,00069	179,043
2,5	0,999593	0,00041	292,257
2,6	0,999764	0,00024	486,693
2,7	0,999866	0,00013	826,860
2,8	0,999925	0,00008	1433,158
2,9	0,999959	0,00004	2534,205
3	0,999978	0,00002	4571,677

Appendix R.1 : Solar radiative properties of materials



Description/composition	Solar Absorptivity, α_s	Emissivity, ε, at 300 K	Ratio, α_s/ε	Solar Transmissivity, τ_s
Aluminum				
Polished	0.09	0.03	3.0	
Anodized	0.14	0.84	0.17	
Quartz-overcoated	0.11	0.37	0.30	
Foil	0.15	0.05	3.0	
Brick, red (Purdue)	0.63	0.93	0.68	
Concrete	0.60	0.88	0.68	
Galvanized sheet metal				
Clean, new	0.65	0.13	5.0	
Oxidized, weathered	0.80	0.28	2.9	
Glass, 3.2-mm thickness				
Float or tempered				0.79
Low iron oxide type				0.88
Marble, slightly off-white (nonreflective)	0.40	0.88	0.45	
Metal, plated				
Black sulfide	0.92	0.10	9.2	
Black cobalt oxide	0.93	0.30	3.1	
Black nickel oxide	0.92	0.08	11	
Black chrome	0.87	0.09	9.7	
Mylar, 0.13-mm thickness				0.87
Paints				
Black (Parsons)	0.98	0.98	1.0	
White, acrylic	0.26	0.90	0.29	
White, zinc oxide	0.16	0.93	0.17	
Paper, white	0.27	0.83	0.32	
Plexiglas, 3.2-mm thickness				0.90
Porcelain tiles, white (reflective glazed surface)	0.26	0.85	0.30	
Roofing tiles, bright red				
Dry surface	0.65	0.85	0.76	
Wet surface	0.88	0.91	0.96	
Sand, dry				
Off-white	0.52	0.82	0.63	
Dull red	0.73	0.86	0.82	
Snow				
Fine particles, fresh	0.13	0.82	0.16	
Ice granules	0.33	0.89	0.37	
Steel				
Mirror-finish	0.41	0.05	8.2	
Heavily rusted	0.89	0.92	0.96	
Stone (light pink)	0.65	0.87	0.74	
Tedlar, 0.10-mm thickness				0.92
Teflon, 0.13-mm thickness				0.92
Wood	0.59	0.90	0.66	

Appendix R.2: Fraction of energy F0-λT radiated by a black body between 0 and λ



$$F_{0-\lambda T} = \frac{\int_0^\lambda Mo_{\lambda T}\, d\lambda}{\sigma T^4}$$

a \ b	0	40	80	120	160
1 000	0,03	0,05	0,08	0,11	0,16
1 200	0,21	0,29	0,38	0,49	0,62
1 400	0,78	0,96	1,17	1,41	1,68
1 600	1,97	2,30	2,66	3,06	3,48
1 800	3,94	4,42	4,94	5,49	6,07
2 000	6,68	7,31	7,97	8,65	9,36
2 200	10,09	10,84	11,61	12,40	13,21
2 400	14,03	14,86	15,71	16,57	17,44
2 600	18,32	19,20	20,09	20,99	21,89
2 800	22,79	23,70	24,61	25,51	26,42
3 000	27,33	28,23	29,13	30,03	30,92
3 200	31,81	32,70	33,58	34,45	35,32
3 400	36,18	37,03	37,88	38,71	39,54
3 600	40,36	41,18	41,98	42,78	43,56
3 800	44,34	45,11	45,87	46,62	47,36
4 000	48,09	48,81	49,53	50,23	50,92
4 200	51,60	52,28	52,94	53,60	54,25
4 400	54,88	55,51	56,13	56,74	57,34
4 600	57,93	58,51	59,09	59,65	60,21
4 800	60,66	61,30	61,83	62,35	62,87
5 000	63,38	63,88	64,37	64,85	65,33
5 200	65,80	66,26	66,72	67,16	67,60
5 400	68,04	68,46	68,88	69,30	69,70
5 600	70,11	70,50	70,89	71,27	71,65
5 800	72,02	72,38	72,74	73,09	73,44
6 000	73,78	74,12	74,45	74,78	75,10
6 200	75,41	75,72	76,03	76,33	76,63
6 400	76,92	77,21	77,49	77,77	78,05
6 600	78,32	78,59	78,85	79,11	79,36
6 800	79,61	79,86	80,10	80,34	80,58
7 000	80,90	81,04	81,26	81,47	81,70
7 200	81,92	82,13	82,34	82,55	82,75
7 400	82,95	83,15	83,34	83,53	83,72
7 600	83,91	84,09	84,27	84,45	84,62

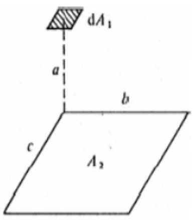
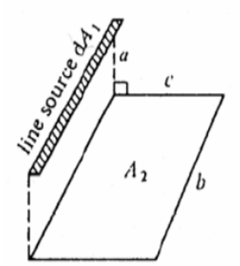
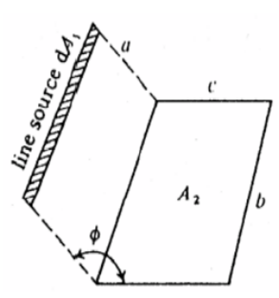
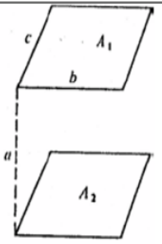
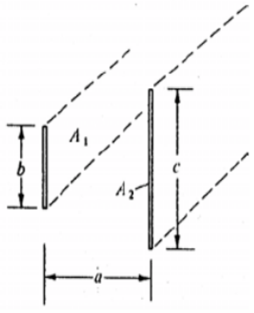
a \ b	0	40	80	120	160
7 800	84,80	84,97	85,14	85,30	85,47
8 000	85,63	85,78	85,94	86,10	86,25
8 200	86,40	86,55	86,69	86,83	86,98
8 400	87,12	87,25	87,39	87,52	87,66
8 600	87,80	87,92	88,04	88,17	88,29
8 800	88,41	88,53	88,65	88,77	88,88
9 000	88,89	89,11	89,22	89,33	89,44
9 200	89,55	89,65	89,76	89,86	89,96
9 400	90,06	90,16	90,26	90,35	90,45
9 600	90,54	90,63	90,72	90,81	90,90
9 800	90,99	91,08	91,16	91,25	91,33
10 000	91,42				

a \ b	0	200	400	600	800
10 000	91,42	91,81	92,19	92,54	92,87
11 000	93,18	93,48	93,76	94,02	94,27
12 000	94,50	94,73	94,94	95,14	95,33
13 000	95,51	95,68	95,84	96,00	96,14
14 000	96,29	96,42	96,54	96,67	96,78
15 000	96,89	97,00	97,10	97,19	97,29
16 000	97,37	97,46	97,54	97,62	97,69
17 000	97,77	97,83	97,90	97,96	98,02
18 000	98,08	98,14	98,19	98,24	98,29
19 000	98,34	98,38	98,43	98,47	98,51
20 000	98,55				
30 000	99,53				
40 000	99,78				
50 000	99,89				
60 000	99,93				
70 000	99,96				
80 000	99,97				
90 000	99,98				
100 000	99,98				

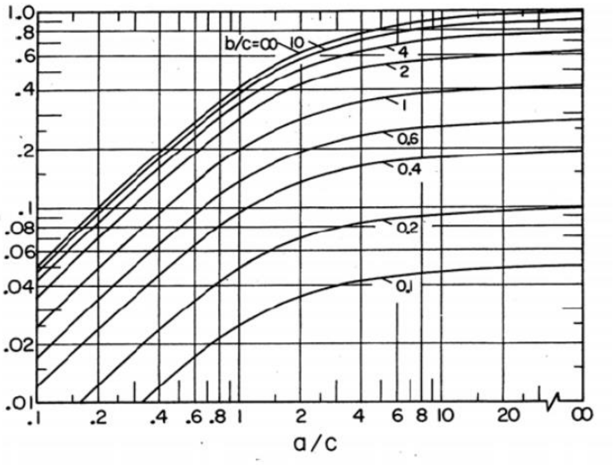
Utilisation :
λ T = a + b

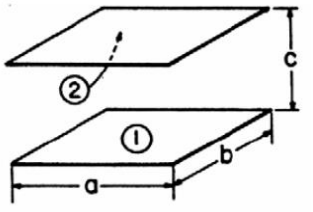
Exemple : λ T = 2720 μm.K
se lit à 2600 + 120
d'où : $F_{0-\lambda T}$ = 20,99 %

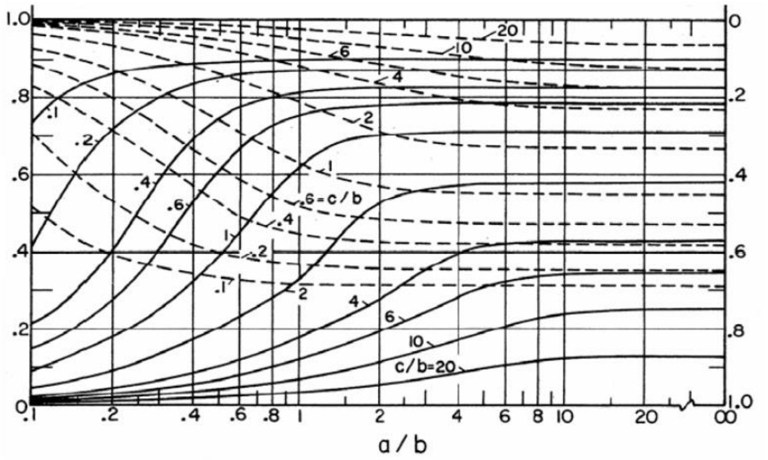
Appendix R.3: Geometric Radiation Shape Factors

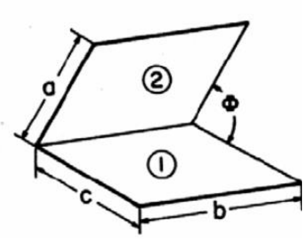
Configuration	Schema	Form factor value
Surface dS parallel to a rectangular plane		
Linear source parallel to a rectangular plane		$F_{dA_1-A_2} = \frac{1}{\pi B}\left[\sqrt{1+B^2}\tan^{-1}\left(\frac{C}{\sqrt{1+B^2}}\right) - \tan^{-1} C\right.$ $\left. + \frac{BC}{\sqrt{1+C^2}}\tan^{-1}\left(\frac{B}{\sqrt{1+C^2}}\right)\right]$ $B = \frac{b}{a}, \quad C = \frac{c}{a}$
linear source parallel and rectangular plane intersecting at an angle ϕ		$F_{dA_1-A_2} = \frac{1}{\pi}\left\{\tan^{-1} B + \frac{\sin^2\phi}{2B}\ln\left[\frac{B^2+X^2}{(1+B^2)X^2}\right]\right.$ $- \frac{\sin 2\phi}{2B}\left[\frac{\pi}{2} - \phi + \tan^{-1}\left(\frac{C-\cos\phi}{\sin\phi}\right)\right]$ $+ \frac{Y}{B}\left[\tan^{-1}\left(\frac{C-\cos\phi}{Y}\right) + \tan^{-1}\left(\frac{\cos\phi}{Y}\right)\right]$ $\left.\times \cos\phi + \frac{C\cos\phi - 1}{X}\tan^{-1}\left(\frac{B}{X}\right)\right\}$ $B = \frac{b}{a}, \quad C = \frac{c}{a}, \quad X = \sqrt{C^2 - 2C\cos\phi + 1},$ $Y = \sqrt{B^2 + \sin^2\phi}$
Two parallel rectangular planes of the same area		$F_{A_1-A_2} = \frac{1}{\pi}\left[\frac{1}{BC}\ln\left(\frac{XY}{X+Y-1}\right) + \frac{2\sqrt{X}}{B}\tan^{-1}\frac{C}{\sqrt{X}}\right.$ $\left. + \frac{2\sqrt{Y}}{C}\tan^{-1}\frac{B}{\sqrt{Y}} - \frac{2}{C}\tan^{-1} B - \frac{2}{B}\tan^{-1} C\right]$ $B = \frac{b}{a}, \quad C = \frac{c}{a}, \quad X = 1 + B^2, \quad Y = 1 + C^2$
Two infinite parallel bands of different widths		$F_{A_1-A_2} = \frac{1}{2B}\left[\sqrt{(B+C)^2+4} - \sqrt{(C-B)^2+4}\right]$ $F_{A_2-A_1} = \frac{1}{2C}\left[\sqrt{(B+C)^2+4} - \sqrt{(B-C)^2+4}\right]$ $B = \frac{b}{a}, \quad C = \frac{c}{a}$ $F_{A_1-A_2} = F_{A_2-A_1} = \frac{1}{B}\left[\sqrt{B^2+1} - 1\right]$ for $b = c$

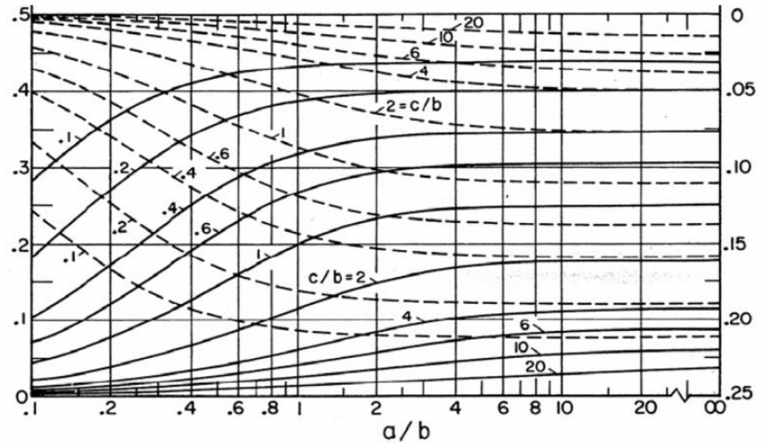
Appendix R.4: Geometric Radiation Shape Factors



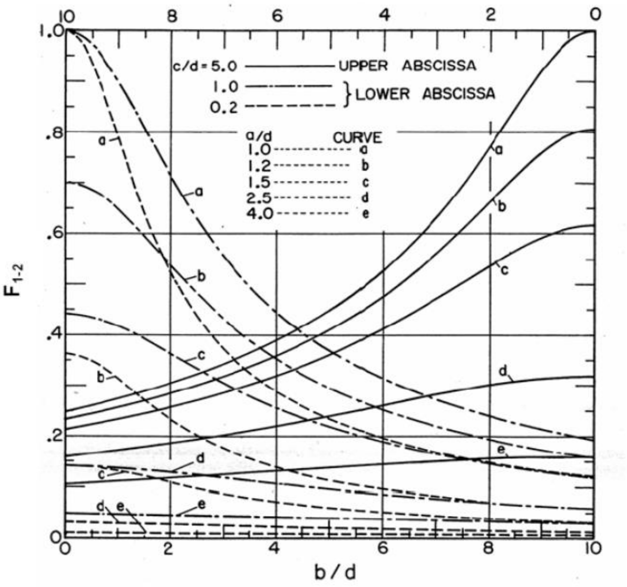


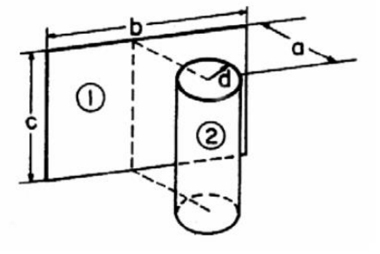


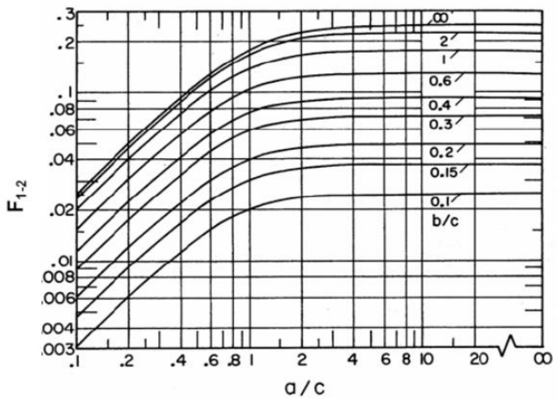


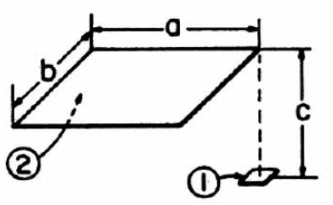


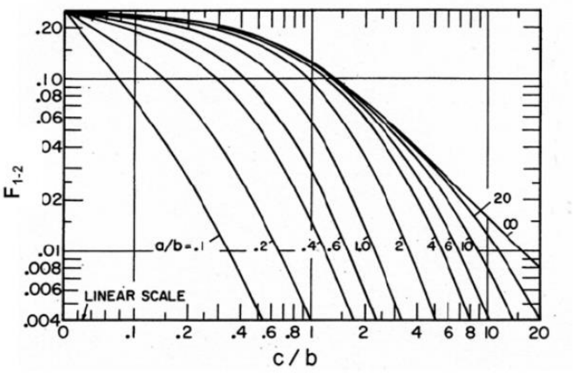
Appendix R.5: Geometric Radiation Shape Factors

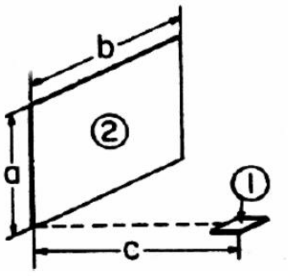












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