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Fourier transform and Sobolev-Hardy spaces

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Dedication

To my father

To my mother

To my dear brother and sisters

To my dear friends

To all family and relatives.

DAIKACHE Oumelkheir

Acknowledgments

First of all, I thank my God who I have given the capacity and the strength, the will, the power, and the health to carry out this modest work.

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Contents

List of Symbols

Notation

Ω	open set in $\mathbb{R}^n$
$\mathbb{R}^n$	Euclidean,n-dimensional space
L^p	Lebesgue spaces
S'	space of tempered distributions
S	space of Schwartz functions
$W^{m,p}$	Sobolev spaces
H^s	non homogeneous Sobolev spaces
$\dot{H}^s$	homogeneous Sobolev spaces
H^p	Hardy spaces
H	Hilbert space

Introduction

The twentieth century is known as the century of functional analysis, where the discussion of spaces in the theory of functions began, such as the space of arbitrary functions $D(\Omega)$, the space of distributions $D'(\Omega)$, Sobolov spaces, and Hardy space. Schwartz space, the space of distributions , and what is called Fourier analysis by Schwartz also appeared, which plays a prominent role in solving partial differential equations.

Fourier transform is the process of converting a mathematical functions from the time domain to the frequency domain. This allows us to understand and analyze complex signals through their basic frequency components. It can be applied in many fields such as image and sound analysis and also in the field of mechanics such as analyzing frequencies resulting from the movement of a car and its various parts.

Where the Fourier transform is used to express many mathematical operations on functions such as derivation and convolution, and through it the scientist Sobolov defined Sobolov space after the emergence of problems that scientists could not find a solution for. It can be used in Hardy space to provide powerful analytical tools to solve problems related to harmonic functions and signal analysis.

This note is organized into four chapters.

The first chapter covers all the essential reminders and supplements for the following chapters.

The second chapter is devoted to the definitions and basic properties of the different Sobolav spaces and the multiplication theorem by taking an example in the Sobolav space, which is Theorem 2.19 below and other results.

The third chapter talks about the Fourier transform and Hardy space with some properties.

The last chapter is devoted to applications.

Finally, we do not claim that this work is original, but we have given some *classical concepts of the Fourier analysis, in order to open a way of research for us.*

Chapter 1

General Notions : Fourier Transform of Tempered Distributions

1.1 Lp Spaces and Fourier Transform

In all this work $\int$ means $\int_{\mathbb{R}^n}$.

We discuss in this chapter general concepts about Fourier transform on the space of tempered distributions to address the other chapters. For writing this, the following references [BCD], [Haim .96], [Zuily.2006], [Les.2012], and [ROD], have been used:

1.1.1 $\mathrm{L^p}$ Spaces

Definition 1.1

For $1 \leq p < \infty$, and $\Omega \subset \mathbb{R}^n$ a measurable set in the Lebesgue sense; we set

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \Big/ \begin{array}{l} \blacktriangleright f \text{ is measurable} \\ \blacktriangleright \int |f|^p \, dx < \infty \end{array} \right\}$$

The norm of f in $L^p(\Omega)$ is defind by:

$$\|f\|_{L^P} = \|f\|_p = \left(\int |f|^p \, dx \right)^{\frac{1}{p}}$$

we shall check later on that $\|.\|_p$ is a norm .

If $p = \infty$, we define

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \Big/ \begin{array}{c} \blacktriangleright f \text{ is measurable} \\ \blacktriangleright \exists c \geq 0, |f(x)| \leq c \; ; \mu - p.p \text{ on } \Omega \end{array} \right\}$$

$$\|f\|_\infty = \inf \{ c \geq 0 : \; |f(x)| \leq c \; \mu - p.p \}$$

where $\mu - p.p$ is the Lebesgue measure.

Example 1.2 *$f(x) = \frac{1}{1+x^2} \in L^1(\mathbb{R})$; as f is measurable and*

$$\begin{aligned} \int_{\mathbb{R}} |f(x)|\, dx &= \int_{\mathbb{R}} \left| \frac{1}{1+x^2} \right| dx \\ &= \int_{\mathbb{R}} \frac{1}{1+x^2} dx = \pi < \infty \end{aligned}$$

Theorem 1.3 (*Hölder´s inequality*)

Let $1 \leq p \leq \infty$; we denote by q its conjugate exponent

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For $f \in L^p$ and $g \in L^q$, then $f \times g \in L^1$, and$\|f \times g\|_1 \leq \|f\|_p \times \|g\|_q$.

Proof. see[Haim .96], p92. □

Theorem 1.4 *L^p is a vector space and $\|.\|_p$ is a norm for any p, $1 \leq p \leq \infty$.*

Proof. see [Haim .96], p93. □

Theorem 1.5 (*Fischer–Riesz*)

L^P is a complet space, then L^P is Banach space for any p such that $1 \leq p \leq \infty$.

Proof. see[Haim .96], p93. □

1.1.2 The Fourier transform of $\mathbf{L}^1$ functions

Definition 1.6 *Let $f \in L^1(\mathbb{R}^n)$. We define the Fourier transform of f by*:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) :\; = \int_{\mathbb{R}^n} f(x) \exp(-2\pi i x \cdot \xi)\, dx,\ \forall \xi \in \mathbb{R}^n.$$

Where $x \cdot \xi = x_1 \cdot \xi_1 + x_2 \cdot \xi_2 + \cdots + x_n \cdot \xi_n$ is a scalar product.

Then the Fourier inversion formula is :

$$f(x) = \mathcal{F}^{-1}(f(x)) :\; = \int_{\mathbb{R}^n} \exp(2\pi i x \cdot \xi)\, f(\xi)\, d\xi.$$

Proposition 1.7 ***Linearity***

Let $f, g \in L^1(\mathbb{R}^n)$*, and* $\alpha, \beta \in \mathbb{C}$

$$\mathcal{F}[\alpha f + \beta g] = \alpha \hat{f} + \beta \hat{g}.$$

Translation

Let $f \in L^1(\mathbb{R}^n)$*; we see the Fourier transform of* $f(x-t)$*. We put* $w = x - t$

$$\begin{aligned}
\mathcal{F}[f(x-t)](\xi) &= \int f(x-t)\exp(-2\pi i \xi x)\,dx \\
&= \int f(w)\exp(-2\pi i \xi (w+t))\,dw \\
&= \exp(-2\pi i \xi t)\int f(w)\exp(-2\pi i \xi w)\,dw \\
\mathcal{F}[f(x-t)](\xi) &= \exp(-2\pi i \xi t)\,\hat{f}(\xi).
\end{aligned}$$

Modulation

Let $f \in L^1(\mathbb{R}^n)$ *and* $\xi_0 \in \mathbb{R}^n$*; the Fourier transform of* $\exp(2\pi i \xi_0 x) f(x)$

$$\mathcal{F}[\exp(2\pi i \xi_0 x) f(x)] = \int f(x)\exp(2\pi i \xi_0 x)\exp(-2\pi i \xi x)\,dx$$

then

$$\mathcal{F}[\exp(2\pi i \xi_0 x) f(x)](\xi) = \hat{f}(\xi - \xi_0)$$

change of scale (***Delation***)

Let $f \in L^1(\mathbb{R}^n)$ *and* $t > 0$ *; we put* $w = t.x$

$$\begin{aligned}
\mathcal{F}[f(t.x)] &= \int f(t.x)\exp(-2\pi i \xi x)\,dx \\
&= \frac{1}{t^n}\int f(t)\exp\left(-2\pi i\left(\frac{\xi}{t}\right)w\right)dw \\
\mathcal{F}[f(t.x)](\xi) &= \frac{1}{t^n}\hat{f}\left(\frac{\xi}{t}\right).
\end{aligned}$$

Transposition

Let $f \in L^1(\mathbb{R}^n)$*; the transpose of* $f(x)$ *being* $f(-x)$

$$\begin{aligned}
\mathcal{F}[f(-x)](\xi) &= \int f(-x)\exp(-2\pi i \xi x)\,dx \\
&= \int f(x)\exp(2\pi i \xi x)\,dx \\
\mathcal{F}[f(-x)](\xi) &= \hat{f}(-\xi).
\end{aligned}$$

Derivation with respect to the variable x

Let $f \in L^1(\mathbb{R})$*; suppose that* f *is differentiable and that* $f' \in L^1(\mathbb{R})$*, then*

$$\begin{aligned} \mathcal{F}[f'(x)](\xi) &= \int f'(x)\exp(-2\pi i\xi x)\,dx \\ \mathcal{F}[f'(x)](\xi) &= 2\pi i\xi\mathcal{F}[f(x)] \\ &= 2\pi i\xi\hat{f}(\xi) \end{aligned}$$

more generally , for the derivative of order k

$$\mathcal{F}\left[f^{(k)}(x)\right] = (2\pi i\xi)^k \hat{f}(\xi) \ \ in\ \mathbb{R}.$$

Derivation with respect to the frequency ξ

Let $f \in L^1(\mathbb{R})$*; if* $xf(x) \in L^1(\mathbb{R})$*, then* $\hat{f}$ *is differentiable and we have*

$$\begin{aligned} \frac{d\hat{f}(\xi)}{d\xi} &= \frac{d}{d\xi}\int f(x)\exp(-2\pi i\xi x)\,dx \\ &= \int -2\pi i x f(x)\exp(-2\pi i\xi x)\,dx \\ \frac{d\hat{f}(\xi)}{d\xi} &= \mathcal{F}[-2\pi i x f(x)] \end{aligned}$$

more generally, for the derivative of order k

$$\begin{aligned} \frac{d^K\hat{f}(\xi)}{d\xi^K} &= \mathcal{F}\left[(-2\pi i x)^K f(x)\right] \\ &= \hat{f}^{(K)}(\xi). \end{aligned}$$

Convolution

Let $f, g \in L^1(\mathbb{R}^n)$*; then* $f * g \in L^1(\mathbb{R}^n)$ *and*

$$\mathcal{F}[f * g] = \int \exp(-2\pi i\xi x)\int f(w)\int g(x-w)\,dw dx$$

we used Fubini's theorem, then we obtain the follewing assertion

$$\mathcal{F}[(f * g)](\xi) = \mathcal{F}[f](\xi)\cdot\mathcal{F}[g](\xi).$$

Theorem 1.8 (*Parseval – Plancherel formula*)

For all integrable functions f *and* g*, one has*:

$$\int f(x)\overline{g(x)}dx = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Example 1.9 *Calculate the Fourier transform for*

$$\begin{aligned}
f(x) &= \exp(-\alpha|x|);\alpha>0 \\
\hat{f}(\xi) &= \mathcal{F}[f(x)](\xi) \\
&= \int_{-\infty}^{+\infty} f(x)\exp(-2\pi i\xi x)\,dx \\
&= \int_{-\infty}^{+\infty} \exp(-\alpha|x|)\exp(-2\pi i\xi x)\,dx \\
&= \int_{-\infty}^{+\infty} \exp((-\alpha)|x|-(2\pi i\xi)x) \\
&= \int_{-\infty}^{0} \exp(\alpha x)\exp(-2\pi i\xi x)\,dx+\int_{0}^{+\infty} \exp(-\alpha x)\exp(-2\pi i\xi x)\,dx \\
&= \int_{-\infty}^{0} \exp((\alpha-2\pi i\xi)x)\,dx+\int_{0}^{+\infty} \exp((-\alpha-2\pi i\xi)x)\,dx \\
&= \left[\frac{1}{\alpha-2\pi i\xi}\exp((\alpha-2\pi i\xi)x)\right]_{-\infty}^{0}+\left[\frac{1}{-(\alpha+2\pi i\xi)}\exp(-(\alpha+2\pi i\xi)x)\right]_{0}^{+\infty} \\
\frac{1}{\alpha-2\pi i\xi}+\frac{1}{\alpha+2\pi i\xi} &= \frac{2\alpha}{\alpha^2+4\pi^2\xi^2},
\end{aligned}$$

since we have :

$$|\exp((\alpha-2\pi i\xi)x)| = \exp(\alpha x)\to 0$$

then

$$\lim_{x\to-\infty}\exp((\alpha-2\pi i\xi)x)=0$$

Recall that

if $z\in\mathbb{C}$ *and* $|z|=0$, *then* $z=0$. *We have the some for :*

$$\begin{aligned}
\lim_{x\to+\infty}\exp(-(\alpha+2\pi i\xi)x) &= 0 \\
\mathcal{F}(\exp(-\alpha|x|))(\xi) &= \frac{2\alpha}{\alpha^2+4\pi^2\xi^2}
\end{aligned}$$

we can take $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}\alpha>0$ *in the preceding example, i.e.*

$$f(x)=\exp(-\alpha|x|),\operatorname{Re}\alpha>0.$$

1.2 Tempered distributions and the Fourier transform

1.2.1 Schwartz Space

Definition 1.10 *The space of decreasing functions refers to functions that are differentiable, and their derivatives decrease rapidly then*

$$S = \left\{ \psi \in C^{\infty} : \forall k, m \in \mathbb{N}; \sup_{x \in \mathbb{R}^n} \left| x^k \psi^m(x) \right| < \infty \right\}.$$

Remark 1.11 ►*we have D is a subspace and dense in S.*

►*If $f, g \in S$ then $f * g \in S$.*

Example 1.12 *Let $f(x) = \exp(-x^2), x \in \mathbb{R}$, is indefinitely defferentiable since*

$$\lim_{|x| \to \infty} x^m \exp\left(-x^2\right) = 0, \text{ for all } m \in \mathbb{N}$$

it decreases rapidly, and therefor $\exp(-x^2) \in S$.

Proposition 1.13 *If $\psi \in S$, then $\hat{\psi} \in S$, hence the space S is stable under the Fourier transform .*

1.2.2 The Fourier transform on Schwartz Space

Definition 1.14 *If $\psi \in S$, then Fourier transform exists and is given by:*

$$\hat{\psi}(\xi) = \int \psi(x) \exp(-2i\pi\xi \cdot x)\, dx.$$

1.2.3 Tempered distributions space S'

Definition 1.15 *A tempered distributions is essentially any linear and continuous functional defined on the Schwartz space, denoted by S'.*

Example 1.16 *δ and $pv\frac{1}{x}$ are tempered distributions.*

1.2.4 The Fourier transform

The Fourier transform of a termpered distribution T is the distribution of Fourier transform, and is defined by:

$$\langle \mathcal{F}(T), \psi \rangle = \langle T, \mathcal{F}(\psi) \rangle, \ \psi \in S.$$

Proposition 1.17 *If $T \in S'$, then $\mathcal{F}T \in S'$.*

Remark 1.18 *We define the Fourier transform (inversion) $\overline{\mathcal{F}}$ de $\mathcal{F}$ by setting*

$$\begin{aligned} \left\langle \overline{\mathcal{F}}\mathcal{F}(T), \psi \right\rangle &= \left\langle \mathcal{F}T, \overline{\mathcal{F}}(\psi) \right\rangle \\ &= \left\langle T, \mathcal{F}\overline{\mathcal{F}}(\psi) \right\rangle \\ &= \langle T, \psi \rangle. \end{aligned}$$

Example 1.19 *We have*

$$\begin{aligned} \langle \mathcal{F}(1), \psi \rangle &= \langle 1, \mathcal{F}(\psi) \rangle \\ &= \int_{-\infty}^{+\infty} (\mathcal{F}(\psi))(x)\, d\xi \\ &= \int_{-\infty}^{+\infty} (\mathcal{F}(\psi))(x) \exp(2\pi i \xi 0)\, d\xi \\ &= \mathcal{F}^{-1}(\mathcal{F}(\psi))(0) \\ &= \psi(0) \\ \langle \mathcal{F}(1), \psi \rangle &= \langle \delta, \psi \rangle, \ \textit{then } \mathcal{F}(1) = \delta. \end{aligned}$$

Proposition 1.20 ***Translation***

Let $\hat{T}(\xi)$ the Fourier transform of $T(x)$ for all $\psi \in S$, we have

$$\begin{aligned} \left\langle T(x-\lambda), \hat{\psi}(x) \right\rangle &= \langle T(x), \psi(x+\lambda) \rangle \\ &= \left\langle \hat{T}(\xi), \exp(-2i\pi\lambda\xi)\, \psi(\xi) \right\rangle \\ &= \left\langle \exp(-2i\pi\lambda\xi)\, \hat{T}(\xi), \psi(\xi) \right\rangle \\ \mathcal{F}[T(x-\lambda)](\xi) &= \exp(-2i\pi\lambda\xi)\, \hat{T}(\xi). \end{aligned}$$

Change of variables

Let $\hat{T}(\xi)$ the Fourier transform of $T(x)$, we have

$$\begin{aligned}
\left\langle \hat{T}(\xi), \psi(\xi) \right\rangle &= \left\langle T(x), \hat{\psi}(x) \right\rangle \\
\left\langle T(\lambda x), \hat{\psi}(x) \right\rangle &= \left\langle T(x), \frac{1}{|\lambda|}\hat{\psi}\left(\frac{x}{\lambda}\right) \right\rangle \\
\left\langle T(x), \mathcal{F}[\psi(\lambda\xi)] \right\rangle &= \left\langle \frac{1}{|\lambda|}\hat{T}\left(\frac{\xi}{\lambda}\right), \hat{\psi}(\xi) \right\rangle \\
\mathcal{F}[T(\lambda x)] &= \frac{1}{|\lambda|^n}\hat{T}\left(\frac{\xi}{\lambda}\right), \text{ for all } \lambda \in \mathbb{R}.
\end{aligned}$$

Derivation

$$\mathcal{F}\left[\frac{d}{dx}T(x)\right] = 2i\pi\xi\hat{T}(\xi).$$

1.3 Taylor series

The Taylor series of function f, that is differentiable at α is

$$f(x) = f(\alpha) + f'(\alpha)\frac{(x-\alpha)}{1!} + f^{(2)}(\alpha)\frac{(x-\alpha)^2}{2!} + f^{(3)}(\alpha)\frac{(x-\alpha)^3}{3!} + \ldots = \sum_{n=0}^{\infty} f^{(n)}(\alpha)\frac{(x-\alpha)^n}{n!}$$

here, $n!$ denotes the factorial of n.The function $f^{(n)}(\alpha)$ denotes the n-th derivative of f.

Example 1.21 ► $\ln x = 0 + (x-1)^1 - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \ldots$

► $\exp(2x) = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots$

Chapter 2

Fourier transform and Sobolev spaces

All facts given in this chapter can be found in [BCD], [Bou.95], [Haim .96], and [Zuily.2006].

2.1 The spaces $W^{m,p}$

Definition 2.1 *For any positive integer $m \in \mathbb{N}$ and $p \in [1, +\infty]$. The Sobolev space $W^{m,p}(\Omega)$ is defined by :*

$$W^{m,p}(\Omega) = \{ f \in L^p(\Omega) \, | D^\alpha f \in L^p(\Omega) \text{ for } 0 \le |\alpha| \le m \}$$

with

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$$

and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \; ; \; D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}}$$

is derivatives in the sense of distributions, that

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle, \forall \varphi \in D(\Omega)$$

the space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|f\|_{W^{m,p}} = \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha f\|_p^p \right)^{\frac{1}{p}} \text{ if } 1 \le p \le \infty.$$

Theorem 2.2 *The space $W^{m,p}(\Omega)$ is a Banach space .*

Proof. see[Haim .96]. □

2.2 Sobolev spaces $W^{1,p}(\Omega)$

Definition 2.3 *Let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}(\Omega)$ is difined by*

$$W^{1,p} = \{f \in L^p(\Omega); \nabla f \in (L^p(\Omega))^n\}$$

the space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|f\|_{W^{1,P}} = \|f\|_p + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_p$$

if $p = \infty$, we provide the space $W^{1,\infty}(\Omega)$ with the norm

$$\|f\|_{W^{1,\infty}} = \max(\|f\|_\infty, \|\nabla f\|_\infty)$$

Proposition 2.4 ▶ *$W^{1,p}(\Omega)$ is a Banach space for every $1 \leq p \leq \infty$.*

▶ *$W^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$, and it is separable for $1 \leq p \leq \infty$.*

Proof. see[Haim .96], P 264. □

Proposition 2.5 (*differentiation of a product*)

Let $f, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \leq p \leq \infty$, then $f, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i} g + \frac{\partial g}{\partial x_i} f; i = 1, 2, \ldots, n.$$

Proof. see[Haim .96], P 269-270. □

Example 2.6 *The function defined on $\Omega =]-1, 1[$ by*

$$f(x) = \frac{1}{2}(|x| + x) \in W^{1,p}(\Omega) \text{ for any } 1 \leq p \leq \infty$$

f is continuous and bounded on Ω. So $f \in L^p(\Omega)$ for all $1 \leq p \leq \infty$.

Using the differentiation in terms of distributions is the Heaviside function on $]-1, +1[$

$$H(x) = \begin{cases} 1 \text{ if } 0 < x < 1 \\ 0 \text{ if } -1 < x \leq 0 \end{cases}$$

for any $1 \leq p \leq \infty$. We say that H is bounded, then $H \in L^P(\Omega)$. So $f \in W^{1,p}(\Omega)$.

We note that $H \notin W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$, because $H' = \delta_0$ $(\delta_0 \notin L^p(\Omega))$.

Remark 2.7 *For $p = 2$; we have the Sobolev spaces which is: $H^m(\Omega) = W^{m,2}(\Omega)$, for any $m \in \mathbb{N}$.*

2.3 Non homogeneous Sobolev spaces on $\mathbb{R}^n$

Definition 2.8 *Let s be in $\mathbb{R}$, $H^s(\mathbb{R}^n)$ is the vector space of elements $f \in S'(\mathbb{R}^n)$ and S*

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|)^2 \left|\hat{f}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}$$

we equip $H^s(\mathbb{R}^n)$ with the scalar product

$$(f, g)_{H^s} = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \hat{f}(\xi)\, \overline{\hat{g}(\xi)} d\xi$$

for any $f, g \in H^s$.

Example 2.9 *Let $\delta_0 \in H^s(\mathbb{R}^n)$ if $s < \frac{-n}{2}$.*

Indeed

$$\left(1 + |\xi|^2\right)^{\frac{s}{2}} \in L^2(\mathbb{R}^n) \text{ if } s < \frac{-n}{2}.$$

Example 2.10 *We have*

$$h(x) = \exp\left(-|x|^2\right)$$

then $h \in L^2$ because

$$\int_{\mathbb{R}^n} \left|\exp\left(-|x|^2\right)\right|^2 dx = \int_{\mathbb{R}^n} \exp\left(-2|x|^2\right)$$

this integral converges. Compute the derivatives of h

$$D^\alpha h(x) = D^\alpha\left(\exp\left(-|x|^2\right)\right) = P_\alpha(x) \exp\left(-|x|^2\right)$$

where $P_\alpha(x)$ is a polynomial derived from the differentiation process for example

$$\frac{\partial h}{\partial x_i} = -2x_i \exp\left(-|x|^2\right)$$

we need to show:

$$\int \left|P_\alpha(x) \exp\left(-|x|^2\right)\right|^2 dx < \infty.$$

Given the rapid decay of $\exp(-|x|^2)$, this integral converges for all polynomials $P_\alpha(x)$. Since both $h(x)$ and all its partial derivatives up. To order s are square integrable. So, $h \in H^s(\mathbb{R}^n)$ for any s.

Proposition 2.11 *If $s_0 \leq s \leq s_1$, then we have*

$$\|f\|_{H^s} \leq \|f\|_{H^{s_0}}^{1-\theta} \|f\|_{H^{s_1}}^{\theta} ; \text{with } s = (1-\theta) s_0 + \theta s_1$$

when s is a non negative integer, the Fourier Plancherel formula ensures that the space H^s coincides with the set of L^2 functions f such that $\partial^\alpha f$ belongs to L^2 for any α in $\mathbb{N}^m$ with $|\alpha| \leq s$.
In the case where s is a negative integer the space H^s is described by the following proposition.

Proof. see[BCD], p39. □

Proposition 2.12 *Let k be a positive integer. the space $H^{-K}(\mathbb{R}^n)$ consists of distributions which are the sums of derivatives of order k of $L^2(\mathbb{R}^n)$ functions.*

Proof. see[BCD], p39. □

Proposition 2.13 *Let s be a nonnegative real number and K a compact subset of $\mathbb{R}^n$. Let $H_K^s(\mathbb{R}^n)$ be the space of those distributions of $H_K^s(\mathbb{R}^n)$ which are supported in K. There; then; exists a positive constant c such that*

$$\forall f \in H_K^s(\mathbb{R}^n), \frac{1}{c}\|f\|_{H^s} \leq \|f\|_{\dot{H}^s} \leq \|f\|_{H^s}.$$

Proof. see[BCD]. □

Corollary 2.14 *Let $0 \leq t \leq s$ .A constant c exists such that for any positive δ and any function $f \in \dot{H}^s(\mathbb{R}^n)$ supported in a ball of radius δ, we have*

$$\|f\|_{\dot{H}^t} \leq c\delta^{s-t}\|f\|_{\dot{H}^s}$$

and

$$\|f\|_{H^t} \leq c\delta^{s-t}\|f\|_{H^s}.$$

Proof. see[BCD], p 40. □

We have the density property of Sobolev spaces:

Proposition 2.15 *The space $S(\mathbb{R}^n)$ is dense in H^s, for $s \in \mathbb{R}$.*

Proof. see[BCD], p 41. □

We can extend our study to K–diffeomorphism.

Definition 2.16 *Let $f\colon X \to Y$ is a C^K diffeomorphism if :*

► *f is bijective.*

► *f and its inverse f^{-1} are K– times continuously differentiable.$\left(i.e.,\ f^{-1} \in C^K(Y, X)\right)$.*

Example 2.17 *The exponential function* exp *on $\mathbb{R}$, and its inverse, the natural logarithm* log*, are classic examples of C^∞ (smooth) diffeomorphisms.*

Corollary 2.18 *Let φ be a global K–diffeomorphism on $\mathbb{R}^n$, $0 \le s \le k$ and $f \in H^s(\mathbb{R}^n)$. then, $f \circ \varphi \in H^s(\mathbb{R}^n)$.*

Proof. see[BCD], p 41. □

The multipllius theory is one of application fields in sobolev spaces.

Then we can state :

Theorem 2.19 ► *if $\varphi \in S(\mathbb{R}^n)$ and $f \in H^s$. Then $\varphi f \in H^s(\mathbb{R}^n)$ with*

$$\|\varphi f\|_s \le (2\pi)^{-n} 2^{\frac{|s|}{2}} \left(\int \left(1 + |\xi|^2\right)^{\frac{|s|}{2}} |\hat{\varphi}(\xi)|\, d\xi \right) \|f\|_s$$

► *if $\alpha \in \mathbb{N}^n$ and $f \in H^s$ then $\partial^\alpha f \in H^{s-|\alpha|}$ and $\|\partial^\alpha f\|_{s-|\alpha|(\mathbb{R}^n)} \le \|f\|_s$.*

Proof. see[Zuily.2006], p135-136. □

Theorem 2.20 *Multipliction by a function of $S(\mathbb{R}^n)$ is a continuous map form $H^s(\mathbb{R}^n)$ into itself.*

Proof. see[BCD], p 42. □

Example 2.21 *For*

$$f(x) = \exp\left(-x^2\right) \in S$$

and we have

$$g(x) = \cos(x)$$

is bounded. The product

$$\varphi = g \times f$$

then

$$\varphi(x) = \cos(x) \cdot \exp\left(-x^2\right)$$

also belongs to $S(\mathbb{R}^n)$. *We have*

$$h(x) = \exp\left(-|x|^2\right) \in H^s(\mathbb{R}^n)$$

using the multiplication theorem, we have the product

$$(\varphi \cdot h)(x) = \left[\cos(x) \cdot \exp\left(-x^2\right)\right] \cdot \exp\left(-|x|^2\right)$$

so

$$\varphi \cdot h \in H^s(\mathbb{R}^n).$$

Theorem 2.22 *Let s be in $\mathbb{R}$ and $s > \frac{1}{2}$. The restriction map γ is defined by*

$$\gamma\colon \begin{cases} S(\mathbb{R}^n) \to S(\mathbb{R}^{n-1}) \\ \varphi \mapsto \gamma(\varphi) = (x_2, \ldots, x_n) \mapsto \varphi(0, x_2, \ldots, x_n) \end{cases}$$

can be continuously extended from $H^s(\mathbb{R}^n)$ on to $H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$.

Proof. see[BCD], p 43. □

2.3.1 Duality

Definition 2.23 *Let $s \in \mathbb{R}$ and $g \in H^{-s}(\mathbb{R}^n)$. If $f \in H^s(\mathbb{R}^n)$, the function $\hat{f}(\xi)\,\hat{g}(-\xi)$ belongs to $L^1(\mathbb{R}^n)$. Indeed,*

$$\hat{f}(\xi)\,\hat{g}(-\xi) = \left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{f}(\xi) \left(1 + |\xi|^2\right)^{\frac{-s}{2}} \hat{g}(-\xi)$$

and in the second term, on the other hand, we have according to Holder´s inquality

$$\left| \int \hat{f}(\xi)\,\hat{g}(-\xi)\,d\xi \right| \leq \|f\|_s \|g\|_{-s}$$

we have the application L is linear, bijective and bicontinuous, then

$$f \mapsto L_g(f) = (2\pi)^{-n} \int \hat{f}(\xi)\, \hat{g}(-\xi)\, d\xi = \int \hat{f}(\xi)\, \overline{\mathcal{F}}g(\xi)\, d\xi$$

is a continuous linear form on H^s and

$$\|L_g\|_{(H^s)'} \leq (2\pi)^{-n} \|g\|_{-s}$$

then

$$H^{-s} \to (H^s)', g \mapsto L_g.$$

Remark 2.24 *if $g \in L^2$ and $f \in L^2$, we have*

$$L_g(f) = \int \hat{f}(\xi)\, \overline{\mathcal{F}}g(\xi)\, d\xi = \int f(x)\, g(x)\, dx.$$

Lemma 2.25 *Let $\varphi \in D$ such that $0 \notin \sup \varphi$, there exists $C(r) > 0$ such that, for all $f \in C^r$ $(r \in \mathbb{R}^+ \diagdown \mathbb{N})$, the function g is defined by :*

$$\hat{g}(\xi) = \varphi\left(\frac{\xi}{T}\right) \hat{f}(\xi)\, ; (T > 0)$$

check if

$$\|g\|_\infty \leq CT^{-r} \|f\|_{C^r}.$$

Proof. we have

$$g(x) = T^n \int \check{\varphi}(Ty)\, f(x-y)\, dy$$

$f(x-y)$ is expressed using the Taylor formula (where $m = [r]$)

$$\begin{aligned} f(x-y) &= \sum_{|\alpha|<m} \frac{(-y)^\alpha f^{(\alpha)}(x)}{\alpha!} + \sum_{|\alpha|=m} \frac{(-y)^\alpha}{\alpha!} m \int_0^1 (1-t)^{m-1} f^{(\alpha)}(x-ty)\, dt \\ &= \sum_{|\alpha|<m} \frac{(-y)^\alpha}{\alpha!} f^{(\alpha)}(x) + \sum_{|\alpha|=m} \frac{(-y)^\alpha}{\alpha!} m \int_0^1 (1-t)^{m-1} \left[f^{(\alpha)}(x-ty) - f^{(\alpha)}(x)\right] dt \\ &= \sum_{|\alpha|\leq m} \frac{(-y)^\alpha}{\alpha!} f^{(\alpha)}(x) + R(x,y) \end{aligned}$$

then

$$|R(x,y)| \leq C(m,n)\, |y|^r \|f\|_{C^r}$$

by substituting into, we find

$$\begin{aligned} g(x) &= \sum_{|\alpha|\leq m} \frac{f^{(\alpha)}(x)}{\alpha!}\left[T^n \int \check{\varphi}(Ty)(-y)^\alpha\, dy\right] + \int T^n \check{\varphi}(Ty)\, R(x,y)\, dy; \\ & T^n \int \check{\varphi}(Ty)(-y)^\alpha\, dy \\ & (iT)^{|\alpha|} \int \check{\varphi}(y)(-iy)^\alpha\, dy \end{aligned}$$

$y \mapsto (-iy)^\alpha \check{\varphi}(y)$ is the Fourier cosine transform of $\varphi^{(\alpha)}$, so

$$\int \check{\varphi}(y)(-iy)^\alpha\, dy = \varphi^{(\alpha)}(0) = 0 \ \text{(since the support of } \varphi \text{ does not contain } 0).$$

There remains

$$\begin{aligned} |g(x)| &\leq T^n \int |\check{\varphi}(Ty)|\, |R(x,y)|\, dy \\ &\leq C(r,n)\, \|f\|_{C^r}\, T^n \int |\check{\varphi}(Ty)|\, |y|^r\, dy \\ &= C(r,n)\, \|f\|_{C^r} \left(\int |\check{\varphi}(y)|\, |y|^r\, dy\right) T^{-r} : \end{aligned}$$

□

Proposition 2.26 *Let $r \in \mathbb{R}^+ \diagdown \mathbb{N}$; there exists $C > 0$ such that every function $f \in C^r$, then*

$$f = \sum_{j\geq 0} f_j$$

where

- ► $\|f_j\|_\infty \leq C 2^{-rj} \|f\|_{C^r}$.
- ► $\hat{f}_j$ *is supported by the ring* $2^{j-1} \leq |\xi| \leq 2^{j+1}$ *for* $j \geq 1$ *and* $\hat{f}_0$ *by the ball* $|\xi| \leq 2$.

2.4 Homogeneous Sobolev spaces on $\mathbb{R}^n$

Definition 2.27 *For $s \in \mathbb{R}$. we say that $f \in S'(\mathbb{R}^n)$ belongs to the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ if*

$$\|f\|^2_{H^s} = \int_{\mathbb{R}^n} |\xi|^{2s} \left|\hat{f}(\xi)\right|^2 d\xi < \infty.$$

Proposition 2.28 *Let $s_0 \leq s \leq s_1$. Then, $\dot{H}^{s_0} \cap \dot{H}^{s_1}$ is included in $\dot{H}^s$, and we have*

$$\|f\|_{\dot{H}^s} \leq \|f\|^{1-\theta}_{\dot{H}^{s_0}} \|f\|^{\theta}_{\dot{H}^{s_1}}$$

with

$$s = (1-\theta)\, s_0 + \theta s_1.$$

Proof. see[BCD], p 25. $\square$

Proposition 2.29 *Let k be a positive integer. The space $\dot{H}^{-k}(\mathbb{R}^n)$ consists of distributions which are the sums of derivatives of order k of $L^2(\mathbb{R}^n)$ functions .*

Proof. see[BCD], p 25-26. $\square$

Proposition 2.30 *$\dot{H}^s(\mathbb{R}^n)$ is a Hilbert space if and only if $s < \frac{n}{2}$.*

Proof. see[BCD], p 26. $\square$

Proposition 2.31 *If $s < \frac{n}{2}$, then the space $S_0(\mathbb{R}^n)$ of functions of $S(\mathbb{R}^n)$, the Fourier transform of which vanishes near the origin, is dense in $\dot{H}^s$.*

Proof. *see[BCD], p 27.* $\square$

Proposition 2.32 *If $|s| < \frac{n}{2}$, then the bilinear functional*

$$B\colon \begin{cases} s_0 \times s_0 \to \mathbb{C} \\ (\varphi, \psi) \mapsto \int_{\mathbb{R}^n} \varphi(x)\, \psi(x)\, dx \end{cases}$$

can be extended to a continuous bilinear functional on $\dot{H}^{-s} \times \dot{H}^s$. Moreover, if L is a continuous linear functional on $\dot{H}^s$, then a unique tempered distributions f exists in $\dot{H}^s$ such that

$$\forall \psi \in \dot{H}^s, \langle L, \psi \rangle = B(f, \psi) \text{ and } \|L\|_{(\dot{H}^s)'} = \|f\|_{\dot{H}^{-s}}.$$

Proof. see[BCD], P 27-28. $\square$

Proposition 2.33 *Let s be a real number in the interval $]0,1[$ and f be in $\dot{H}^s(\mathbb{R}^n)$. Then,*

$$f \in L^2_{Loc}(\mathbb{R}^n) \text{ and} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x+y) - f(x)|^2}{|y|^{n+2s}} dxdy < \infty$$

Moreover, a constant C_s exists such that for any function f in $\dot{H}^s(\mathbb{R}^n)$, we have

$$\|f\|^2_{\dot{H}^s} = C_s \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x+y) - f(x)|^2}{|y|^{n+2s}} dxdy.$$

Proof. see[BCD], P 28. $\square$

2.4.1 Sobolev Embedding in Lebesgue spaces

Theorem 2.34 *If s is in $\left[0, \frac{n}{2}\right[$, then the space $\dot{H}^s(\mathbb{R}^n)$ is continuously embedded in $L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$.*

Proof. see[BCD], P 29. □

Corollary 2.35 *If p belongs to $]1,2]$, then $L^p(\mathbb{R}^n)$ embeds continuously in $\dot{H}^s(\mathbb{R}^n)$ with $s = \frac{n}{2} - \frac{2}{p}$.*

Proof. see[BCD], P 29-30. □

2.4.2 Hardy Inequality

Theorem 2.36 *If $n \geq 3$, then*

$$\left(\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} dx\right)^{\frac{1}{2}} \leq \frac{2}{n-2} \|\nabla f\|_{L^2} \text{ for any } f \text{ in } \dot{H}^1(\mathbb{R}^n).$$

Proof. see[BCD]. □

2.4.3 The Embedding Theorem in Hölder Spaces

Definition 2.37 *Let(k,p)be in $\mathbb{N}\times]0,1]$. The Hölder space $C^{k,p}(\mathbb{R}^n)$ $\big($or $C^{k,p}$, if no confusion is possibl is the space of C^k functions f on $\mathbb{R}^n$ such that*

$$\|f\|_{C^{k,p}} = \sup_{|\alpha|\leq k}\left(\|\partial^\alpha f\|_{L^\infty} + \sup_{x\neq y}\left(\frac{\partial^\alpha f(x) + \partial^\alpha f(y)}{|x-y|^P}\right)\right) < \infty.$$

Proving that the sets $C^{k,p}$ are Banach spaces is left as an exercise. We point out that $C^{0,1}$ is the space of bounded Lipschitz functions.

Chapter 3

Fourier transform and Hardy spaces

Hardy space theory began in the last century and has undergone many uses extensions. In this section, we want to focus on using the Fourier transform in these spaces. For this chapter we refer to [Gra.2014], [Stein.1993].

3.1 Hardy spaces

Definition 3.1 *The Hardy spaces* $H^p(\mathbb{R}^n)$, $0 < p < \infty$, *are spaces of distributions which become more singular as p decreases. these function spaces have remarkable similarities to* L^p *and, in many ways, serve as a substitute for* L^p *when* $p < 1$ *and it can be defined as : the spaces of all tempered distributions* f *modulo polynomials for which*

$$\|f\|_{H^p} = \left\| \left(\sum_{j\in\mathbb{Z}} |\triangle_j (f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} < \infty.$$

Definition 3.2 *Let* f *be a bounded tempered distribution on* $\mathbb{R}^n$ *and let* $0 < p < \infty$. *We say that* f *lies in the Hardy space* $H^p(\mathbb{R}^n)$ *if the Poisson maximal function*

$$M(f;P)(x) = \sup_{t>0} |(P_t * f)(x)|$$

lies in $L^P(\mathbb{R}^n)$. *Then the Poisson Kernel* P *is the function*

$$P(x) = \frac{T\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x^2|)^{\frac{n+1}{2}}}$$

for $t > 0$, *let*

$$P_t(x) = t^{-n} P\left(t^{-1}x\right)$$

Since $P_t \in L^1$ *and an important property of bounded tempered distributions* f *is that*

$$P_t * f \to f \, in \; S(\mathbb{R}^n) \; as \; t \to 0$$

then the convolution $P_t * f$ *can be defined as a distribution via the convergent integral*

$$\langle P_t * f, \varphi \rangle = \left\langle \tilde{\varphi} * f, \tilde{P}_t \right\rangle = \int_{\mathbb{R}^n} (\tilde{\varphi} * f)(x) \, \tilde{p}_t(x) \, dx$$

The space $H^p(\mathbb{R}^n)$ *equipped with the norm*

$$\|f\|_{H^p} = \|M(f; P)\|_{L^p}.$$

Theorem 3.3 ▶*Let* $1 < p < \infty$. *Then every bounded tempered distribution* f *in* H^p *is an element of* L^p.

Moreover, there is a constant $C_{n,p}$ *such that*

for all such f *we have*

$$\|f\|_{L^p} \leq \|f\|_{H^p} \leq C_{n,p} \|f\|_{L^p}$$

and therefore $H^p(\mathbb{R}^n)$ *coincides with* $L^p(\mathbb{R}^n)$.

▶*When* $P = 1$, *every element of* H^1 *is an integrable function. In other words,* $H^1(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ *and for all* $f \in H^1$ *we have*

$$\|f\|_{L^1} \leq \|f\|_{H^1}$$

We now obtain some characterisations of these spaces.

Definition 3.4 *Let* $a, b > 0$. *Let* φ *be a Schwartz function and let* f *be the tempered distributions on* $\mathbb{R}^n$. *We define the smooth maximal function of* f *with respect to* φ *as*

$$M_\varphi f(x) = M(f; \varphi)(x) = \sup |(\varphi_t * f)(x)|.$$

Proposition 3.5 *Let* $0 < p \leq 1$. *Then the following statements are valid :*

▶*Convergence in* H^P *implies convergence in* S'

▶*If* $f_k \in H^P$ *satifies*

$$\sup_{k \in z^+} \|f_k\|_{H^P} \leq C < \infty$$

and $f_k \to f$ *in* S' *as* $k \to \infty$ *then* $f \in H^p$.

Proof. see[Gra.2014]. □

3.1.1 The maximal characterization

1. We come now to the formulation of the maximal characterisation of H^p.

For any $\varphi \in S$ and any distribution f, we define $M_\varphi f(x)$ by

$$M_\varphi f(x) = \sup_{t>0} |(f * \varphi_t)(x)|$$

To extend beyond single approximations of the identity, we shall also consider "grand maximal functions" using collections of such φ. For this purpose, let $\mathcal{F} = \left\{\|.\|_{\alpha_i,\beta_i}\right\}$be any finite collection of seminorms on S. We denote by $S_\mathcal{F}$ the subset of S controlled by this collection of seminorms; more precisely, we set

$$S_\mathcal{F} = \left\{\varphi \in S : \|\varphi\|_{\alpha_i,\beta_i} \leq 1 \text{ for all } \|.\|_{\alpha_i,\beta_i} \in \mathcal{F}\right\}$$

we then write

$$\mathcal{M}_\mathcal{F} f(x) = \sup_{\varphi \in S_\mathcal{F}} M_\varphi f(x)$$

Finally whenever f is a bounded distribution, let

$$u(x,t) = (f * P_t)(x)$$

be the Pisson integral of f and let

$$u^*(x) = \sup_{|x-y|\leq t} |u(y,t)|$$

denote the nontangential maximal function of u.

Theorem 3.6 *Let f be a distribution and let $0 < p \leq \infty$. Then the following conditions are equivalent :*

► *There is a $\varphi \in S$ with $\int \varphi dx \neq 0$ so that $M_\varphi f \in L^p(\mathbb{R}^n)$.*

► *There is a collection $\mathcal{F}$ so that $\mathcal{M}_\mathcal{F} f \in L^p(\mathbb{R}^n)$.*

► *The distribution f is bounded and $u^* \in L^p(\mathbb{R}^n)$.*

When any of these three equivalent conditions are met, we say that f belongs to $H^p(\mathbb{R}^n)$. Before delving into the theorem´s proof, let´s clarify some aspects of the spaces $H^p(\mathbb{R}^n)$.

1. (a) For $1 < p < \infty$, these conditions are equivalent to f being a function in $L^p(\mathbb{R}^n)$ .Thus, for this range $H^p(\mathbb{R}^n)$ can be identified with $L^p(\mathbb{R}^n)$. Specifically, if $f \in L^p(\mathbb{R}^n)$, then $M_\varphi f(x) \leq cMf(x)$, implying $M_\varphi f \in L^p$ by the maximal theorem. Conversely, suppose $\int \varphi = 1$ and $M_\varphi f \in L^p$. Then f is a bounded sequence in L^p, and by to the weak compactness of L^p $\left(\text{as the dual of } L^{p'}\right)$, there exists $f_0 \in L^p$ and a subsequence $f * \varphi_{\frac{1}{n_j}} \to f_0$ weakly. However, $f * \varphi_{\frac{1}{n_j}} \to f$ in the sense of distributions; thus $f = f_0 \in L^p$.

   (b) Let us consider the case $p = 1$.By a suitable modification of what was said above it follows that $H^1 \subset L^1$. However, the converse is false. In the first instance, for a function f to belong to H^1, it must satisfy the moment condition $\int f dx = 0$. Besides, cancellation conditions of this kind (which implicitly occur at every level), there must also be restrictions on the size of f. Thus if f is positive on a ball B, it can belong to H^1 only if $f\log(1+f)$ is integrable on every ball that is strictly contained in B.

   (c) The size and cancellation conditions required of H^1 functions are most neatly expressed in spacial examples (called atoms) that serve also as basic building blocks for H^1 functions. A function a is an H^1 (atoassociated to a ball B)if:

      i. α is supported in B

      ii. $|\alpha| \leq |B|^{-1}$ almost everywhere, and

      iii. $\int \alpha dx = 0$.

   For fixed $\varphi \in S$, it is easy to show that

$$\|M_\varphi \alpha\|_{L^1} \leq c$$

   with c independent of α and B.

Proof.

we begin the proof of our theorem by considering some further maximal operators (associated with given that provide the crucial tools in what follows. First we define the "non tangential" version of M_φ, it is given by

$$M_\varphi^* f(x) = \sup_{|x-y|<t} |(f * \varphi_t)(y)| = \sup_{|y|<t} |(f * \varphi_t)(x-y)|$$

and an even larger "tangential"variant M_N^{**} (depending on a parameter N), given by

$$M_\varphi^{**} f(x) = \sup_{y\in\mathbb{R}^n, t>0} |(f * \varphi_t)(x-y)| \left(1 + \frac{|y|}{t}\right)^{-N}$$

we note the obvious pointwise inequalities

$$M_\varphi f \leq M_\varphi^* \leq 2^N M_N^{**} f.$$

□

Lemma 3.7 *If $M_\varphi^* f \in L^p(\mathbb{R}^n)$ and $N > \frac{n}{p}$, then $M_N^{**} f \in L^p(\mathbb{R}^n)$ with*

$$\|M_N^{**} f\|_{L^p} \leq c_{N,p} \left\|M_\varphi^* f\right\|_{L^p}$$

Lemma 3.8 *suppose we are given φ and ψ in S, with $\int_{\mathbb{R}^n} \varphi dx = 1$, then there is a sequence$\{\eta^{(k)}\}$, $\eta^{(k)} \in S$, so that*

$$\psi = \sum \eta^{(k)} * \varphi_{2^{-k}}$$

with $\eta^{(k)} \to 0$ rapidly, in the sense that whenever $\|.\|_{\alpha,\beta}$ is a seminorm, and $M \geq 0$ is fixed, then

$$\left\|\eta^{(k)}\right\|_{\alpha,\beta} = O\left(2^{-kM}\right) \text{ as } k \to \infty$$

this idea comes from either using the Fourier transform or dealing directly with the relevant convolutions. The former approach, which is also useful in extensions to different settings. From the Fourier transform point of view, can be thought of as an expression of Wiener's principle: The linear span of the set of all translates of a given family of functions is dense in L^1, provided the family satisfies the obvious necessary condition that there are no points at which their Fourier transforms simultaneously vanish. In our case, the family is $\{\varphi_t\}_t > 0$.

Theorem 3.9 *Let $p \leq 1$, then every $f \in H^p$ can be written as a sum of H^p atoms, as in $f = \sum_k \lambda_k \alpha_k$ above, that converges in H^p norm; moreover*

$$\sum_k |\lambda_k|^p \leq c \, \|f\|_{H^p}^p .$$

Theorem 3.10 *If satisfies the assumptions*

$$\hat{T} f(\xi) = m(\xi) \hat{f}(\xi), |m(\xi)| \leq A$$

and

$$\int_{|x| \geq 2|y|} |k(x-y) - k(x)| \, dy \leq A, \text{ whenever } y \neq 0$$

above, then it is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$; that is

$$\|Tf\|_{L^1(\mathbb{R}^n)} \leq A' \|f\|_{H^1(\mathbb{R}^n)}$$

where the bound A' depends only on the constant A appearing in

$$\|Tf\|_{L^2} \leq A \, \|f\|_{L^2}$$

and

$$\int_{|x| \geq 2|y|} |k(x-y) - k(x)| \, dy \leq A$$

whene ver $y \neq 0$.

Theorem 3.11 *Let $0 < p < \infty$. Then the following statements are valid : there exists a Schwartz function φ_0 with $\int_{\mathbb{R}^n} \varphi^0(x) \, dx = 1$ such that*

$$\left\| M\left(f; \varphi^0\right) \right\|_{L^p} \leq 500 \, \|f\|_{H^p}$$

for all bounded distributions $f \in S'(\mathbb{R}^n)$.

Proof. see[Gra.2014], p 62-63. □

Example 3.12 *we have*

$$\psi(t) = \frac{e}{\pi} \frac{1}{t} e^{\frac{-\sqrt{2}}{2}(t-1)^{\frac{1}{4}}} \sin\left(\frac{\sqrt{2}}{2} (t-1)^{\frac{1}{4}} \right)$$

then

$$|\psi(t)| \leq c_N t^{-N} \text{ for all } N > 0$$

and such that

$$\int_1^{\infty} t^k \psi(x)\, dx = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ if } k = 1, 2, 3, \ldots \end{cases}$$

we now define the function

$$\varphi^0(x) = \int_1^{\infty} \psi(t)\, P_t(x)\, dt$$

where P_t is the Poisson kernel. Note that the double integral

$$\int_{\mathbb{R}^n} \int_1^{\infty} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}} t^{-N} dt dx$$

converges and from the above, with the use of the Fubini´s theorem that

$$\int_{\mathbb{R}^n} \varphi^0(x)\, dx = 1.$$

Proposition 3.13 *Let $0 < p < 1$ and let r satisfy $p \leq r \leq \infty$. Then $L^r \cap H^p$ is dense in H^p. Then, $H^p \cap L^2$ and $H^p \cap L^1$ are dense in H^p.*

Proof. see[Gra.2014], p 73. □

Chapter 4

Application : Fourier transform and Hilbert transform

For this paragraph, we refer to the book of Bourdaud [Bou.95], and [Les.2012], [CMQ], [Haim .96].

4.1 Hilbert space

Definition 4.1 *Let $x, y \in H$, and H is complex vector space provided with the scalar product $\langle , \rangle$, and application $\{H \times H \to \mathbb{C}, (x, y) \longmapsto \langle x, y \rangle\}$ verifying*

1. $\forall x \in H, \langle x, x \rangle \geq 0.$
2. $\forall x, y \in H, \forall \lambda \in \mathbb{C}, \langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle .$
3. $\forall x, y \in H, \langle y, x \rangle = \overline{\langle x, y \rangle}$ (Conjugate complex of (x, y)).
4. $[\langle x, x \rangle = 0] \Leftrightarrow [x = 0]$.

   The norm of x in H is defined by:

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}} .$$

4.1.1 Hilbert transform

Definition 4.2 *Let $f \in \mathbb{R}$, we define the Hilbert transform of f by:*

$$H(f)(y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(x)}{x - y} dx$$

the integral considered here is in the sense of the main value, that is to say that

$$H(f)(y) = \lim_{\substack{\xi \to 0 \\ m \to +\infty}} \left(\frac{1}{\pi} \int_{-m}^{-\xi} \frac{f(x)}{x-y} dx + \frac{1}{\pi} \int_{\xi}^{m} \frac{f(x)}{x-y} dx \right)$$

for $p > 1$*, we have if* $f \in L^p(\mathbb{R})$*, then* $Hf \in L^p(\mathbb{R})$.

Example 4.3 *Let* $f \in \mathbb{R}$ *we have*

$$f(x) = \frac{1}{x^2+1}$$

the Hilbert transform of f *is*

$$H(f) = \frac{-x}{x^2+1}.$$

Example 4.4 *Let* $H(\varphi)$ *be Hilbert transformation*

where $H(\varphi) = f * \varphi$ *and*

$$\langle f, \varphi \rangle = \lim_{\xi \to 0} \int_{|x| \geq \xi} \frac{\varphi(x)}{x} dx, \varphi \in S(\mathbb{R})$$

we put $\xi = \frac{1}{j}$

$$\langle f, \varphi \rangle = \lim_{j \to \infty} \int_{|x| \geq \frac{1}{j}} \frac{\varphi(x)}{x} dx$$

we want to show that H *take* L^2 *into itself i.e*

$$H \colon L^2 \to L^2$$

by applying Per areal equality we obtain

$$\|H(\varphi)\|_2 = \left\| \widehat{H(\varphi)} \right\|_2$$

now we go to calculate $\widehat{H(\varphi)}$ *as* $\varphi \in S(\mathbb{R})$

we have

$$\begin{aligned}
\widehat{H(\varphi)} &= \left\langle \hat{f}, \varphi \right\rangle = \langle f, \hat{\varphi} \rangle \\
&= \lim_{j \to \infty} \int_{|x| \geq \frac{1}{j}} \frac{\hat{\varphi}(x)}{x} dx \\
&= \lim_{j \to \infty} \int_{|x| \geq \frac{1}{j}} \frac{1}{x} \int_{\mathbb{R}} \exp(-ix\xi) \varphi(\xi) d\xi dx \\
\widehat{H(\varphi)} &= \langle f, \hat{\varphi} \rangle, \forall \varphi \in S(\mathbb{R}) \\
&= \lim_{j \to \infty} \int_{|x| \geq \frac{1}{j}} \frac{\hat{\varphi}(\xi)}{\xi} d\xi
\end{aligned}$$

we first calculate

$$\begin{aligned} & \int_{\frac{1}{j}<|x|<j} \frac{\exp(-ix\xi)}{x} dx \\ = & \cdots = -2i \int_{\frac{\xi}{j}}^{j\xi} \frac{\sin f}{f} df \end{aligned}$$

next : $\forall \psi \in S(\mathbb{R})$ *we have*

$$\begin{aligned} & -i\pi \int_{-\infty}^{+\infty} \psi(\xi)\, sgn(\xi)\, d\xi \\ = & \langle -i\pi sgn(\xi), \psi \rangle \\ = & \lim_{j\to\infty} \langle m_j(\xi), \psi \rangle \\ = & \cdots \end{aligned}$$

we set

$$\begin{aligned} m_j(\xi) &= \int_{1\le|x|\le j} \frac{\exp(-ix\xi)}{x} dx \\ &= \int_{-j}^{-\frac{1}{j}} \frac{\exp(-ix\xi)}{x} dx + \int_{\frac{1}{j}}^{j} \frac{\exp(-ix\xi)}{x} dx \\ &= -\int_{\frac{1}{j}}^{j} \frac{\exp(ix\xi)}{x} dx + \int_{\frac{1}{j}}^{j} \frac{\exp(-ix\xi)}{x} dx \\ &= -\int_{\frac{1}{j}}^{j} \frac{1}{x} \frac{(-\exp(-ix\xi) + \exp(-ix\xi))}{2i} 2idx \\ &= -2i \int_{\frac{1}{j}}^{j} \frac{\sin(x\xi)}{x} dx \end{aligned}$$

by taking $y = x\xi$

$$\begin{aligned} m_j(\xi) &= -2i \int_{\frac{1}{j}\xi}^{j\xi} \frac{\sin(y)}{y} dy \\ \lim_{j\to\infty} m_j(\xi) &= -2i \int_0^{\infty} \frac{\sin(y)}{y} dy = -2i\frac{\pi}{2} = -i\pi, \textit{if } \xi \ge 0 \\ \lim_{j\to\infty} m_j(\xi) &= -2i \int_0^{-\infty} \frac{\sin(y)}{y} dy = 2i \int_0^{\infty} \frac{\sin(y)}{y} dy = 2i\frac{\pi}{2} = i\pi, \textit{if } \xi < 0 \\ \lim_{j\to 0} m_j(\xi) &= i\pi sgn(\xi) \end{aligned}$$

now we obtain that $|m_j(\xi)| \leq c$ *for all* j *and* ξ *then* $m_j \in L^\infty$

$$\begin{aligned}\widehat{H(\varphi)} &= i\pi \int_{\mathbb{R}} \varphi(\xi)\, sgn(\xi)\, d\xi \\ \|H(\varphi)\|_2 &= \left\|\widehat{H(\varphi)}\right\|_2 \\ &\leq c\,\|\varphi\|_2 .\end{aligned}$$

4.2 Laplace transform of a distribution

Definition 4.5 *Let* $T \in \mathbb{R}$ *whose support is contained in* $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$.

We have

$$\exp(-\xi x)\, T \in S'(\mathbb{R}^n)$$

with $\xi \in \mathbb{R}$*; then we define the Laplace transform* $\mathcal{L}T$ *of the distribution* T :

$$(\mathcal{L}T)(p) = \langle T, \exp(-px)\rangle, \text{ with } p \in \mathbb{C}\,(\operatorname{Re} p > \xi)$$

Example 4.6 *The Laplace transforrm of the Dirac distribution* δ *is*

$$\mathcal{L}\delta = 1$$

Indeed, we have

$$\mathcal{L}\delta = \langle \delta, \exp(-px)\rangle = 1.$$

4.2.1 Laplace transform of a function

Definition 4.7 *Let* $f\colon \mathbb{R}_+ \to \mathbb{R}\,(or\ \mathbb{C})$*, we define the Laplace transform* $\mathcal{L}\{f(x)\}$ *of the function* f:

$$\mathcal{L}\{f(x)\} = F(p) = \int_0^\infty f(x)\exp(-px)\,dx$$

with $F(p)$ *of the complex variable* $p = \xi + iw$.

Example 4.8 *The Laplace transform of the Heaviside function* $H(x)$.

We have

$$\begin{aligned}H(x) &= \begin{cases}1 \ \textit{if}\ x \geq 0\\ 0 \ \textit{if}\ x \leq 0\end{cases} \\ \mathcal{L}\{H(x)\} &= \int_0^\infty \exp(-px)\,dx = \lim_{\alpha\to\infty} \frac{1-\exp(\alpha p)}{p} = \frac{1}{p} \ \textit{with}\ \operatorname{Re} p > 0\end{aligned}$$

In this example, we are talking about the effect of the Fourier transform in differential equations.

Example 4.9 *We have the heat equation*

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}, \ \text{for } \alpha > 0$$

with the initial condition : $f(x,0) = \varphi(x)$, *and* $\varphi(x)$ *is the temperature at time* $t = 0$. *Using the Fourier transform, we have*

$$\int_{-\infty}^{+\infty} \frac{\partial f}{\partial t}(x,t) \exp(-2\pi i w x)\, dx = \alpha \int_{-\infty}^{+\infty} \frac{\partial^2 f}{\partial t^2}(x,t) \exp(-2\pi i w x)\, dx$$

and

$$\begin{aligned}
\mathcal{F}\{f(x,t)\}\hat{f}(w,t) &= \int_{-\infty}^{+\infty} f(x,t) \exp(-2\pi i w x)\, dx \\
\mathcal{F}\left\{\frac{\partial f}{\partial x}(x,t)\right\} &= \widehat{\frac{\partial f}{\partial w}}(w,t) = \int_{-\infty}^{+\infty} \frac{\partial f}{\partial x}(x,t) \exp(-2\pi i w x)\, dx \\
&= 2\pi i w \widehat{f}(x,t) \\
\mathcal{F}\left\{\frac{\partial^2 f}{\partial x^2}(x,t)\right\} &= \widehat{\frac{\partial^2 f}{\partial w^2}}(w,t) = \int_{-\infty}^{+\infty} \frac{\partial^2 f}{\partial x^2}(x,t) \exp(-2\pi i w x)\, dx \\
&= -4\pi^2 w^2 \hat{f}(w,t)
\end{aligned}$$

therefore, the previous equation is written in the form

$$\frac{\partial \hat{f}}{\partial t}(w,t) = -4\alpha\pi^2 w^2 \hat{f}(w,t)$$

by integration this equation into the unknown $\hat{f}(w,t)$ *of the variable* t, *we have*

$$\hat{f}(w,t) = c \exp\left(-4\alpha\pi^2 w^2 t\right)$$

and

$$\begin{aligned}
\hat{f}(w,0) &= c \\
&= \int_{-\infty}^{+\infty} f(x,0) \exp(-2\pi i w x)\, dx \\
&= \int_{-\infty}^{+\infty} \varphi(x) \exp(-2\pi i w x)\, dx
\end{aligned}$$

so

$$\begin{aligned}\hat{f}(w,t) &= \exp\left(-4\alpha\pi^2 w^2 t\right)\int_{-\infty}^{+\infty}\varphi(x)\exp(-2\pi iwx)\,dx \\ &= \exp\left(-4\alpha\pi^2 w^2 t\right)\hat{\varphi}(w).\end{aligned}$$

Then

$$\mathcal{F}\{f(x,t)\} = \exp\left(-4\alpha\pi^2 w^2 t\right)\mathcal{F}\{\varphi(x)\}.$$

As

$$\mathcal{F}\{h * g\} = \mathcal{F}\{h\}\mathcal{F}\{g\}$$

we have

$$f(x,t) = h * g$$

with

$$h = \varphi(x)$$

and

$$g = \frac{1}{4\pi\sqrt{\alpha\pi t}}\exp\left(\frac{-x^2}{16\alpha\pi^2 t}\right)$$

because

$$\mathcal{F}\{g\} = \hat{g}(w) = \exp\left(-4\alpha\pi^2 w^2 t\right).$$

Finally, we have

$$f(x,t) = \varphi * \frac{1}{4\pi\sqrt{\alpha\pi t}}\exp\left(\frac{-x^2}{16\alpha\pi^2 t}\right) = \frac{1}{4\pi\sqrt{\alpha\pi t}}\int_{-\infty}^{+\infty}\varphi(y)\exp\left(\frac{-(x-y)^2}{16\alpha\pi^2 t}\right)dy.$$

4.3 Conclusion

Functional spaces theory is a classical branch of mathematical analysis, and since these spaces are widely used in mathematical modeling in many fields such as physics and mechanics....

It is important to explore and study this type of spaces such as Sobolev spaces, Hardy spaces, by studying the Fourier transform, and defining their on different scales.

We sought in the applications of the Fourier transform to solve differential equations, and we took a good example of the Hilbert transform.

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الملخص

الهدف من هذا العمل هو دراسة استعمالات تحويل فورييه في مختلف الفضاءات مثل الفضاءات سوبولاف و فضاء هاردي.

حيث توفر كل من هذه الأدوات والفضاءات مفاهيم وتقنيات مكملة لفهم ودراسة الدوال والتحولات الرياضية في سياقات متعددة.

مما يسهم هذا الترابط في تشكيل إطار متكامل للتحليل الرياضي.

كلمات مفتاحية: التوزيعات المعتدلة، تحويل فورييه، الفضاءات الدالية.

Abstract

The aim of this work is to study the uses of the Fourier transform in different spaces such as Sobolav spaces and Hardy spaces.

Each of these tools and spaces provides complementary. concepts and techniques for understanding and analyzing functions and mathematical transformations in multiple contexts.

This interrelation contributes to forming a comprehensive framework for mathematical analysis.

Key words: Tempered Distributions, Fourier transform, Functional spaces

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Résumé

L'objectif de ce travail est d'étudier les utilisations des transformations de Fourier dans différents espaces tels que les espaces Sobolev et le espace de Hardy.

Ces outils et espaces fournissent des concepts et des techniques complémentaires pour comprendre et étudier les fonctions et les transformations mathématiques dans des contextes variés.

Cette interconnexion contribue à former un cadre intégré pour l'analyse mathématique.

Mots Clés : Distributions tempérées, Transformation de Fourier, Espaces Fonctionnels