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Theme

Some new critical point theorems and application

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in front of the jury :

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Dedication

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Contents

Notation

We introduce the necessary notations and definitions that are used

$\mathbb{N}$	Set of positive integer numbers.
$\mathbb{N} \cup \{0\}$	Set of non negative integer numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}_+$	Set of positive real numbers.
$\mathbb{R}^N$	Euclidean space of dimension N.
Ω	Open of $\mathbb{R}^N$ equipped with the measure of Lebesgue.
u	Defined measurable function of Ω in $\mathbb{R}$.
∇u	gradient of u, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_N}\right)$.
$C(\Omega)$	Space of continuous functions on Ω.
$C^k(\Omega)$	space of continuous functions on Ω whose partial derivatives of order k are continuous on, Ω, k positive integer.
$C^1(\Omega, \mathbb{R})$	The set of differentiable functions and the derivative is continuous.
$C^\infty(\Omega)$	space $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$.
$L^p(\Omega)$	$L^p(\Omega) = \left\{f : \Omega \to \mathbb{R}^N ; f \text{ is measurable and } \int_\Omega \lvert f(x)\rvert^p \, dx < \infty\right\}$ $(1 < p < \infty, constant)$.
$L^\infty(\Omega)$	$L^\infty(\Omega) = \left\{f : \Omega \to \mathbb{R}^N ; f \text{ is measurable and } \lvert f(x)\rvert \leq c \text{ a.e.on } \Omega\right\}$.
$\mathcal{D}(\Omega)$	Space of indefinitely differentiable functions in Ω, with compact support in Ω.
$W^{1.p}(\Omega)$	Sobolev space of functions of $L^p(\Omega)$ whose derivatives partial in the sense of first-order distributions are also in $L^p(\Omega)$, provided with the norm $\lVert u\rVert_{W^{1.p}} = \sum \lVert\partial_i u\rVert_{L^p}$.
$W_0^{1.p}(\Omega)$	The closing of $\mathcal{D}(\Omega)$ in $W^{1.p}(\Omega)$, $W_0^{1.p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1.p}}$.
$H^1(\Omega)$	$W^{1,2}(\Omega)$.
$H_0^1(\Omega)$	$W_0^{1,2}(\Omega)$.
a.e	Almost everywhere.
$\rightharpoonup$	Weakly converges.
$\rightarrow$	Converges strongly.
$\langle\cdot,\cdot\rangle$	the scalar product.
$E \hookrightarrow F$	E is continuously injected into F.
$E \hookrightarrow_c F$	E is injected in a compact way into F.
p^*	The sobolev exponent, such that $p^* = \frac{Np}{N-p}$.
p'	*Hölder*'s conjugate of p, $p' = \frac{p}{p-1}$, if $p > 1$ and $p' = \infty$ if $p = 1$.

General Introduction

In the past two to three decades, critical point theory and variational methods have witnessed significant advancement in their applications to solving differential equations, as evidenced by the works cited, including [3, 14, 17, 19]. These theories have achieved remarkable success, evolving into highly complex and refined frameworks. Their utility spans across thousands of research papers, underscoring their widespread adoption and impact. Within this vast literature, our work relies heavily on articles, namely [4, 9, 10]. These references serve as cornerstones for our research, offering detailed proofs and consolidating fundamental concepts related to critical point theory and variational methods.

In the first chapter, we give some reminders on functional spaces and notion of differentiability and their main properties, critical point and Palais-Smale condition.

In the second chapter, we establish the existence of critical points for a class of functional according to the Ekland variational principle without satisfying the Palais-Smale condition. We have applied the main result in this section at a simple example and we showed there was a critical point in Riesz-Banach space ordered by a cone k and an application

$$\begin{cases} -(p(t)u^{'}(t))^{'} = f(t,u(t)), a.e.t \in [0,+\infty) \\ u(0) = u(+\infty) = 0, \end{cases} \tag{1}$$

where $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, and may change sign, $p : [0,+\infty) \to (0,+\infty)$ satisfies $\frac{1}{p} \in L^1[0,+\infty)$, and

$$\int_0^{+\infty} \left(\int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

In the last chapter, we establish new fixed point theorems in Hilbert spaces for potential $\alpha-$positively homogeneous operators using the weak Ekeland variational principle. We have considered, we apply our abstract results are motivated by the existence of solution of the following problem:

$$\begin{cases} -u''(t) = q(t)f(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $q \in L^2(0,1)$.

Chapter 1

Preliminary

In this chapter, we introduce fundamental notions to be used later we focus mainly on [3, 5]. Let $\Omega \in \mathbb{R}^N$ be an open set, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

1.1 Functional spaces

1.1.1 L^p Spaces

Definition 1.1 *Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N; f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\},$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, dx \right]^{\frac{1}{p}}.$$

Definition 1.2 *We set*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N; f \text{is measurable and } |f(x)| \leq c \ \text{ a.e.on } \Omega \right\},$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = inf \left\{ c \geq 0 : |f(x)| \leq c \ \text{ a.e.on } \Omega \right\},$$

with c is a constant.

Theorem 1.1 *$L^p(\Omega)$ is a Banach space, for any $1 \leq p \leq \infty$, and reflexive for $1 \leq p < \infty$, and separable for $1 < p < \infty$.*

Notation 1.2 *.Let $1 < p < \infty$, we denote by p' the conjugate exponent*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.3 (Hölder's inequality) *Assume that $f \in L^p$ and $g \in L^p$ with $1 \leq p \leq \infty$, then $fg \in L^1$ and*

$$\int |fg| \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

Theorem 1.4 ***(Cauchy-Schwarz inequality)*** *Let f and $g \in L^2(\Omega)$. Then we have*

$$\int\limits_\Omega |fg| \leq \Big(\int\limits_\Omega |f|^2 \Big)^{\frac{1}{2}} \Big(\int\limits_\Omega |g|^2 \Big)^{\frac{1}{2}}.$$

Theorem 1.5 *[5]**(Lebesgue's dominated convergence theorem)** Let (f_n) be a sequence of functions of L^1. We suppose that*

1. $f_n(x) \longrightarrow f(x)$ *a.e.on* Ω,
2. *there exists a function* $g \in L^1$ *such that for every* n, $|f_n(x)| \leq g(x)$ *a.e.in* Ω.

We have $f \in L^1(\Omega)$ *and* $\|f_n - f\|_{L^1} \longrightarrow 0$.

Theorem 1.6 *[5](Lebesgue's dominated convergence inverse theorem)Let (f_n) be a sequence in L^p and let $f \in L^p$ be such that $\|f_n - f\|_{L^p} \longrightarrow 0$.*
Then, there exist a subsequence (f_{n_k}) and a function $h \in L^p$ such that

1. $f_{n_k}(x) \longrightarrow f(x)$ *a.e.on* Ω,
2. $|f_{n_k}(x)| \leq h(x) \ \ \forall k$, *a.e.in* Ω.

1.1.2 $W^{1.p}(\Omega)$ Spaces

Definition 1.3 *We denote by $\mathcal{D}(\Omega)$ the set of function of class $\mathcal{C}^\infty(\Omega)$ with compact support include in Ω. The Sobolev space $W^{1.p}(\Omega)$ is defined by*

$$W^{1.p}(\Omega) = \left\{ u \in L^p(\Omega), \exists g_i \in L^p(\Omega) \text{such that: } \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega g_i \varphi dx, \forall \varphi \in \mathcal{D}(\Omega), \forall i = 1, 2, ..., N \right\}.$$

We set

$$H^1(\Omega) = W^{1.2}(\Omega).$$

For $u \in W^{1.p}(\Omega)$, we set $\frac{\partial u}{\partial x_i} = g_i$, and we write

$$\nabla u = grad\ u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}).$$

The space $W^{1.p}(\Omega)$ is equipped with the norm $\|u\|_{W^{1.p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}$, or sometimes with the equivalent norm $\|u\|_{W^{1.P}} = (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{\frac{1}{p}}$ if $(1 \leq p < \infty)$.
The space $H^1(\Omega)$ is equipped with the scalar product

$$\int_\Omega uv dx + \int_\Omega \nabla u \nabla v dx.$$

The associated norm $\|u\|_{H^1} = (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{\frac{1}{2}}$ is equivalent to the $W^{1.2}$ norm.

Proposition 1.1 *$W^{1.p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$, reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$.*
In particular $H^1(\Omega)$ is reflexive, separable and Hilbert space.

1.1.3 Sobolev Injections and Inequalities

Let $(E, \|.\|_E), (F, \|.\|_F)$ be Banach spaces

Notation 1.7

1. E is injected continuously into F, means that the canonical injection $j : E \to F$ is continuous i.e, *exists* $c > 0$, *forall* $x \in E : \|x\|_F \leq c\,\|x\|_E$, and we denote by $E \hookrightarrow F$.
2. E is injected in compact into F means that the canonical injection $j : E \to F$ is compact i.e for all sequence bounded (u_n) in E we can extract subsequence (u_{nk}) convergent in F, and we denote by $E \hookrightarrow_c F$.
   If $1 \leq p < \infty$, the sobolev exponent of p defined by $p^* = \frac{Np}{N-p}$ or $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ $\quad (p < N)$.

Theorem 1.8 *Let $1 \leq p \leq \infty$, we suppose that Ω is on open set of class $\mathcal{C}^1$ a bounded bordary, and we take $\Omega = \mathbb{R}^N_+$*

1. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*[\quad$ *if* $p < N$.
2. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, \infty[\quad$ *if* $p = N$.
3. $W^{1.p}(\Omega) \hookrightarrow L^\infty(\Omega)$ *if* $p > N$.

Theorem 1.9 (Rellich-Kondrachon) *[5].Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class c^1*

1. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1.p^*[\quad$ *if* $p < N$.
2. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [p.\infty[\quad$ *if* $p = N$.
3. $W^{1.p}(\Omega) \subset \mathcal{C}\left(\overline{\Omega}\right)$ *if* $p > N$.

1.1.4 $W^{1.p}_0(\Omega)$ Space

Definition 1.4 *.Let $1 \leq p \leq \infty$, $W^{1.p}_0$ means the closing of $\mathcal{D}(\Omega)$ in $W^{1.p}$, we notice*

$$\begin{aligned} W^{1.p}_0(\Omega) &= \overline{\mathcal{D}(\Omega)}^{W^{1.p}} \\ &= \left\{u \in W^{1.p}(\Omega) : u = 0 \ sur \ \partial\Omega\right\}, \end{aligned}$$

and

$$H^1_0(\Omega) = W^{1.2}_0(\Omega).$$

The space $W^{1.p}_0(\Omega)$ provided with norm induced by $W^{1.p}(\Omega)$, $H^1_0(\Omega)$ is a Hilbert space for the scalar product of H^1 we put $\|u\|_{H^1_0} = \|u\|$.

Remark 1.1 *.when $\Omega = \mathbb{R}^N$, we know that $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1.p}(\mathbb{R}^N)$, and there for*

$$W^{1.p}_0(\mathbb{R}^N) = W^{1.p}(\mathbb{R}^N).$$

Proposition 1.2 (Poincaré's inequality) *[5].Let $\Omega \subset \mathbb{R}^N$ be on open set, Then there exists a constant $C > 0$ such that*

$$\|u\|_{W^{1.p}(\Omega)} \leq c\,\|\nabla u\|_{L^p(\Omega)}\,, \quad \forall u \in W_0^{1.p}(\Omega).$$

In other words, on $W_0^{1.p}(\Omega)$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ is a norm equivalent to the $W^{1.p}(\Omega)$ norm.

Theorem 1.10 (Young's inequality) . *For $a, b \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

1.2 Some notions of convergence

Definition 1.5 ***(Strong convergence)*** *Convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ in a normed vector space $(E, \|.\|_E)$ to an element $x \in E$, defined in the following way:*

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow \|x_n - x\|_E \leq \epsilon.$$

Or,

$$\lim_{n\to+\infty} \|x_n - x\|_E = 0.$$

We denote by $x_n \to x$ as $n \to +\infty$

Definition 1.6 *We say that the sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $x \in E$ if and only if*

$$\langle f, x_n\rangle \to \langle f, x\rangle\,, \quad \forall f \in E^*$$

for $n \to +\infty$. This weak convergence is written as $x_n \rightharpoonup x$.

1.3 Operators on Banach spaces

For further details on these concepts, refer to [3, 5, 17].

Definition 1.7 ***(Dual space)*** *We denote by E^* the dual space of E, that is, the space of all continuous linear functionals on E; the (dual) norm on $E^*$is defined by*

$$\|f\|_{E^*} = \sup_{x\in E, \|x\|\leq 1} |f(x)| = \sup_{x\in E, \|x\|=1} |f(x)|\,.$$

Definition 1.8 ***(Reflexive space)*** *The space E is said to be reflexive if the canonical injection $i : E \to E^{**}$ is surjective, i.e. $i(E) = E^{**}$.*
When E is reflexive, E^{**} is usually identified with E; $(E^{**} = E)$.

Definition 1.9 *Let $A : E \to F$ be a linear operator. One says that A is bounded if there is a constant $c \geq 0$ such that:*

$$\|Au\|_F \leq c\,\|u\|_E\,, \quad \forall u \in E.$$

Definition 1.10 ***(Continuous Operator)*** *Let A and B be two Banach spaces. An operator $T : A \to B$ is said to be continuous if for any sequence $(u_n) \subset A$ that converges to a point u_0 in A, the sequence $(T(u_n))$ converges to $T(u_0)$ in B.*

Definition 1.11 ***(Compact Operator)*** *Let A and B be two Banach spaces. An operator $T : A \to B$ is said to be compact if for every bounded sequence (u_n) in A, the sequence $(T(u_n))$ in B has a convergent subsequence.*

Definition 1.12 ***(convex Operator)*** *1) We say that a part K of E is convex if It is said that part K of X is convex if:*

$$\forall x, y \in K, \ \forall \theta \in [0,1], \quad \theta x + (1-\theta)y \in K.$$

2) When K is convex and $J : K \to \mathbb{R}$ is a functional. We say that J is convex if:

$$\forall x, y \in K, \ \forall \in [0,1], \quad J(\theta x + (1-\theta)y) \leq \theta J(x) + (1-\theta)J(y).$$

Definition 1.13 ***(Minimizing sequence)****. A minimizing sequence of a functional $J : E \longrightarrow]-\infty, +\infty]$ is a sequence (u_n), such that*

$$\lim_{n \longrightarrow +\infty} J(u_n) = \inf_{u \in E} J(u).$$

Definition 1.14 *Let $J : E \longrightarrow \mathbb{R}$ be a functional, we say that J is weakly lower semi-continuous if for all $u_n \subset E : u_n \rightharpoonup u_0$ in E, we have*

$$J(u_0) \leq \lim_{n \longrightarrow +\infty} \inf J(u_n).$$

Definition 1.15 ***(coercive)*** *Let $J : E \longrightarrow \mathbb{R} \cup +\infty$ a functional on E . We say that J is coercive if and only if*

$$\lim_{\|u\|_E \longrightarrow +\infty} \frac{J(u)}{\|u_E\|} = +\infty.$$

Definition 1.16 ***(cone)*** *Let X be a real Banach space. A nonempty set C is said to be a cone if C is a closed convex set and $C \cap (-C) = 0$.*

Definition 1.17 ***(weakly closed)*** *We say that A is weakly closed if For every sequence (x_n) in A converges weakly to x then x in A.*

Definition 1.18 ***(strongly continuous)*** *we say that f is strongly continuous if for all sequence (x_n) converges weakly to x then $f(x_n)$ converges to $f(x)$*

Definition 1.19 ***(bounded from below)*** *The oprator $TE \longrightarrow \mathbb{R}$ is bounded below if there exists m in $\mathbb{R}$ such that*

$$Tx \geq m. \ \ \textit{For all } x \textit{ in } \mathbb{R}$$

Definition 1.20 ***(Carathéodory conditions)*** *function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is said to satisfy the Carathéodory conditions if*

- *$f(x,u)$ is a continuous function of u for almost all $x \in \Omega$;*

- *$f(x,u)$ is a measurable function of x for all $u \in \mathbb{R}^m$.*

Definition 1.21 ***(Nemytskii operator)*** *Given a function f satisfying the Carathéodory conditions and function $u : \Omega \to \mathbb{R}^m$, define a new function $F(u) : \Omega \to \mathbb{R}$ by*

$$F(u)(x) = f(x, u(x)).$$

The function F is called a Nemytskii operator

1.4 Differentiability of Functionals and Critical Points

In what follows, we introduce some notion of derivatives for functions defined on Banach spaces, we start with the directional derivative.

1.2.1 Derivative in the sense of Gateau

Definition 1.22 *.Let E be a Banach space, $\Omega \subseteq E$ an open set, and let $I : o \longrightarrow \mathbb{R}$ a functional, we say that I is ad differentiable in the sense of Gateau (G-differentiable) at $u \in \Omega$, if there exists $A \in E'$ (A linear and continuous), denoted by $I'_G(u)$ such that, for all $v \in E$, where $I(u+tv)$ exists for $t > 0$ small enough, the directional derivative $DI(u)$ exists i.e:*

$$\lim_{t \longrightarrow 0} \frac{I(u+tv) - I(u)}{t} = \langle A, v \rangle .$$

If I is differentiable in the sense of Gateau in u, there exists only one verified linear functional.

Example 1.1 *.Let $I : L^p(\Omega) \to \mathbb{R}$, $I(u) = \int_\Omega |u|^p dx$. I is G-differentiable and we have $\langle I'_G(u), v\rangle = p\int_\Omega |u|^{p-2} uvdx$, indeed, let $x \in \Omega$, t sufficient small fixed and we define $g_{u.v}(s) = |u+tv|^p$. $s \in [0,t]$ let $g_{u.v} : [0,t] \to \mathbb{R}$, we have $g_{u.v}$ is continuous on the closed interval $[0,t]$ and differentiable on the interval open $]0,t[$, then according to the mean value theorem, there exists a real $c_t \in]0,t[$ and when $t \to 0$, we have $c_t \to 0$*

$$\begin{aligned} \lim_{t \to 0} \frac{|u+tv|^p - |u|^p}{t} &= \lim_{c_t \longrightarrow 0} p|u+c_t v|^{p-2}(u+c_t v)v \\ &= p|u|^{p-2}uv. \end{aligned}$$

According to Lebesgue's convergence theorem, we have

$$\langle I'(u), v\rangle = p\int_\Omega |u|^{p-2} uvdx.$$

We define $A : L^p \to \mathbb{R}$

$$v \to A(v) = p\int_\Omega |u|^{p-2} uvdx$$

We prove that $A \in (L^p(\Omega))' = L^{p'}(\Omega)$, *i.e* A *is linear and continuous*
A *is linear, in effect, let* $v_1, v_2 \in L^p(\Omega)$ *and let* $\alpha.\beta \in \mathbb{R}$

$$\begin{aligned} A(\alpha v_1 + \beta v_2) &= p\int_\Omega |u|^{p-2}u(\alpha v_1 + \beta v_2)dx \\ &= p[\int_\Omega |u|^{p-2}u\alpha v_1 dx + \int_\Omega |u|^{p-2}u\beta v_2 dx] \\ &= \alpha p\int_\Omega |u|^{p-2}uv_1 dx + \beta p\int_\Omega |u|^{p-2}uv_2 dx \\ &= \alpha A(v_1) + \beta A(v_2). \end{aligned}$$

A *is linear.* A *is continuous, in effect, let* $u, v \in L^p(\Omega)$

$$\begin{aligned} |p\int_\Omega |u|^{p-2}uvdx| &\leq p\int_\Omega |u|^{p-1}|v|dx \\ &\leq p(\int_\Omega |u|^{(p-1)p'}dx)^{\frac{1}{p'}}(\int_\Omega |v|^p dx)^{\frac{1}{p}} \\ &\leq p(\int_\Omega |u|^{(p-1)(\frac{p}{p-1})}dx)^{\frac{p-1}{p}}(\int_\Omega |v|^p dx)^{\frac{1}{p}} \\ &= \|u\|_{L^p}^{p-1}\|v\|_{L^p}. \end{aligned}$$

Then A *is a continuous, so the function is G-differentiable.*

1.2.2 Derivative in the sense of Frechet

Definition 1.23 *.Let* E *be a Banach space* ,$\Omega \subseteq E$ *an open set and let* $I : \Omega \longrightarrow \mathbb{R}$ *a functional, we say that* I *is ad differentiable in the sense of Frechet, at* $u \in \Omega$, *if there exists* $A \in E'$ *such as*

$$\lim_{\|v\|\to 0} \frac{I(u+tv) - I(v) - Av}{\|v\|} = 0$$

or

$$I(u+tv) - I(u) = Av + o(\|v\|).$$

If I *is differentiable, then* A *is unique and we denote* $I'(u) = A$, *the set of differentiable function, we will be denote* $\mathcal{C}^1(\Omega.\mathbb{R})$.

Proposition 1.3 *.Suppose that* $\Omega \subseteq E$ *is on open set, such that* I *G-differentiable in* Ω *and that* I'_G *is continuous at* $\Omega \in E$, *then* I *is also differentiable at* u, *and of cours* $I'_G = I'(u)$.

Remark 1.2 *.The importance of Proposition 1.3 reside in the fact that it is often technically easier to calculate the derivative in the sense of Gateau an then to prove that it is continuous, rather than to directly prove the differentiability in the sense of Frechet.*

Example 1.2 *.We prove that the function* $J : L^p(]0,1[) \to \mathbb{R}$ $(p > 2)$ *and by using Holder inequality we get*

$$u \to J(u) = \int_0^1 |u|^p dx.$$

is a Frechet differentiable on $L^p(]0,1[)$.
we already prove that J *is G-differentiable, with*

$$J'_G : L^p(]0,1[) \to (L^p(]0,1[))' = L^q(]0,1[)$$

$$u \to J'_G(u) : L^p(]0,1[) \to \mathbb{R}$$

$$v \to \langle J'(u), v\rangle = p\int_0^1 |u|^{p-2}uvdx$$

it remains to prove that $J'_G : L^p(]0,1[) \to (L^p(]0,1[))' = L^q(]0,1[)$

$$u \to J'_G(u)$$

is continuous.
Let $(u_n) \subset L^p(]0,1[)$ *such as* $u_n \to u$ *in* $L^p(]0,1[)$ *we prove that* $J'_G(u_n) \to J'_G(u)$ *in* $(L^p(]0,1[))'$

$$\|J'_G(u_n) - J'_G(u)\|_{(L^p(]0,1[))'} = \sup_{v\in L^p(]0,1[)} |\langle J'_G(u_n) - J'_G(u), v\rangle|$$

let $v \in L^p(]0,1[)$, *such as* $\|v\|_{L^p} = 1$, *we have*

$$\begin{aligned} |\,\langle J'_G(u_n) - J'_G(u), v\rangle\,| &= |\,\langle J'_G(u_n), v\rangle - \langle J'_G(u), v\rangle\,| \\ &= p|\int_0^1 |u_n|^{p-2}u_n vdx - \int_0^1 |u|^{p-2}uvdx| \\ &\le p[\int_0^1 ||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}}dx](\int_0^1 |v|^p dx)^{\frac{1}{p}}. \end{aligned}$$

We pose $u_n = |u_n|^{p-2}\,u_n, u = |u|^{p-2}\,u$, *we know that* $u_n \to u$ *in* $L^p(0,1)$, *there is a subsequence* $u_{nk} = u_n$, *such that*

$$u_n(x) \to u(x) \ a.e \ in \ ]0,1[$$

.

moreover there exists $g \in L^p$ *such that*

$$|u_n(x)| \le g(x) \quad a.e \ in \ ]0,1[,$$

so

$$|u_n(x)|^{p-1} \le |g(x)|^{p-1} \quad \in L^{\frac{p}{p-1}}.$$

According to $u_n \to u$ *in* $L^{\frac{p}{p-1}}$ *we recall that* $\|J'_G(u_n) - J'(u)\| \le \int_\Omega |u_n|^{p-2}u_n - |u|^{p-2}udx \to 0$ *and then* $\lim_{n\to+\infty} \|J'_G(u_n) - J'(u)\|_{(L^p(\Omega))'} = 0$
as J *is G-differentiable and* J'_G *is continuous, so* J *is Frechet differentiable and* $J'_G = J'$.

1.2.3 Critical points

Definition 1.24 *[17].Let* Ω *an open set of Banach space* E, *suppose that* $I \in \mathcal{C}^1(\Omega.\mathbb{R})$, *we say that* $u \in \Omega$ *is critical point of* I, *if*

$$I'(u) = 0$$

if u *is not critical point, then we say that* u *is regular point of* I.
if $c \in \mathbb{R}$, *then we say that* c *is a critical value of* I, *if there exists* $u \in \Omega$ *such as*

$$I(u) = c \ \ and \ \ I'(u) = 0.$$

if c *is not a critical value, then* c *is said to be a regular value of* I.

Palais-Smale's sequence and condition

Definition 1.25 [15] *Let X be a Banach space and $J : X \to \mathbb{R}$ a C^1-functional. We say that J satisfies the Palais-Smale condition, denoted $(P.S)$, if any sequence (u_n) in X such that*

$$(J(u_n)) \text{ is bounded, and } J'(u_n) \to 0, \text{ as } n \to +\infty \tag{1.1}$$

admits a convergent subsequence.
Any sequence satisfying (1.1) is called a Palais-Smale sequence.

Definition 1.26 [15] *Let X be a Banach space and $J : X \to \mathbb{R}$ a C^1-functional, and $c \in \mathbb{R}$. The functional J is said to satisfy the (local) Palais-Smale condition at the level c, denoted by $(P.S)_c$, if any sequence (u_n) in X such that*

$$J(u_n) \to c, \text{ and } J'(u_n) \to 0, \text{ as } n \to +\infty \tag{1.2}$$

admits a convergent sub-sequence.

Remark 1.3 *The condition $(P.S)_c$ is a compactness condition on the functional J, in the sense that the set $\mathbb{K}_c$ of critical points of J at the level c,*

$$K_c = \{u \in X : J(u) = c \text{ and } J'(u) = 0\}$$

is compact.

Theorem 1.11 ***(Mountain Pass Theorem)***([3, 15]) *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ with $J(0) = 0$. Suppose $J(u)$ satisfies $(P.S)$ condition and*

- *(a) there exist $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ when $\|u\|_E = \rho$,*
- *(b) there exists a $e \in E$, $\|e\|_E > \rho$ such that $J(e) < 0$.*

Define

$$\Gamma = \{\gamma \in C([0,1], E) \mid \quad \gamma(0) = 0, \gamma(1) = e\}. \tag{1.3}$$

Then

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)) \geq \alpha \tag{1.4}$$

is a critical value of $J(u)$.

CHAPTER 2

A few critical point theorems using Ekeland variational principle

The weak Ekeland variational principle is an important tool in critical point theory and nonlinear analysis, see [14] in this work we use this principle to establish some new results in critical point theory.

2.1 Critical point theorems without the Palais–Smale condition

We need the following weak Ekeland variational principle which can be found for example in [11, 14]

Lemma 2.1 *(Weak Ekeland variational principle). Let* (E, d)*be a complete metric space and let* $\varphi : E \to \mathbb{R}$ *be a lower semicontinuous functional, bounded from below. Then for every* $\epsilon > 0$*, there exists a point* $u_\epsilon \in E$ *with*

$$\varphi(u_\epsilon) \leq \inf_E \varphi + \epsilon$$

Such that

$$\varphi(u_\epsilon) < \varphi(v) + \epsilon d(u_\epsilon, v), \quad \forall v \in E \quad \textit{such that } v \neq u_\epsilon$$

Definition 2.1 (Sequence of almost critical points) *We say that a functional* $\varphi \in C^1(E, \mathbb{R})$ *has a sequence of almost critical points if there exists a sequence* $(v_n)_n$ *in* E *such that* $\varphi'(v_n) \to 0$ *in* E^* *as* $n \to \infty$

Lemma 2.2 (Minimization principle) *[15] Let* E *be a Banach space and* $\varphi : E \to \mathbb{R}$ *a functional, bounded from below and Gâteaux differentiable. Then, there exists a minimizing sequence* $(v_n)_n$ *of almost critical points of* φ *in the sense that*

$$\lim_{n\to+\infty} \varphi(v_n) = \inf_{v\in E} \varphi(v) \textit{ and } \lim_{n\to+\infty} \varphi'(v_n) = 0$$

A slight modification of Theorem 1.26 in [20] (see also Corollary 2.4 in this work) gives the following lemma with a dilatation type condition

Lemma 2.3 *Let* $(E, \|.\|_1)$*be a Banach space and* $(F, \|.\|_2)$ *be a normed space. If* A *is a closed set in* $E, f : A \to F$ *is continuous,and*

$$\exists \kappa > 0 : \exists \theta > 0, \|f(x) - f(y)\|_2 \geq \kappa \|x - y\|_1^{\theta} \qquad \forall x, y \in A$$

then $f(A)$ *is closed.*

proof. We show $f(A)$ is closed.Let $g \in \overline{f(A)}$ there exists point $x_n \in f(A)$ so that $g = \lim f(x_n)$ thus $f(x_n)$ is Cauchy in F. Our hypothesis implies therefore that x_n is Cauchy in E. Being a closed subset of a complete metric space, A is complete; hence there exists $x = \lim x_n \; in \; A$ Since f is continuous,

$$f(x) = \lim f(x_n) = g$$

Thus $g \in f(A)$. ■

2.1.1 Main results

The first main result of section is the following :

Theorem 2.1 *Let* E *be a reflexive Banach space,* F *be a bounded and weakly closed set of* E *with* $J \in C^1(E, \mathbb{R})$*, and let* J' *be strongly continuous on* F *. Suppose also that* J *satisfies* $\|J'(\varphi(u))\|_{E^*} \leq \kappa \|J'(u)\|_{E^*}$ *for all* $u \in F$*, where* $\varphi : F \to F$ *is a function such that* $\varphi(u) \neq u$ *for all* $u \in F$*, and* $0 < \kappa < 1$ *is a constant . Then,* J *has at least one critical point in* F*.*

proof. We first show that $J'(F)$is closed recall that $J'(F) = \{J'(u); u \in F\}$ we need to verify $\overline{J'(F)} = J'(F)$. Let $g \in \overline{J'(F)}$, there exist a sequence $g_n \in J'(F)$, such that $\lim_{n\to+\infty} g_n = g$, and so there exist $(u_n) \subset F$ with $\lim_{n\to+\infty} J'(u_n) = g$. Since F is bounded and E is reflexive, there exist $(u_{n_k}) \subset (u_n)$such that $(u_{n_k}) \rightharpoonup u \in E$. Since J' is strongly continuous then

$$g = \lim_{n\to+\infty} J'(u_{n_k}) = J'(u) \in J'(F).$$

We consider the complete metric space $J'(F)$, and define the functional ψ on $J'(F)$ by

$$\psi : J'(F) \to \mathbb{R}$$

$$J'(u) \to \psi(J'(u)) = \|J'(u)\|_{E^*} .$$

ψ is lower semicontinuous; Indeed, let $(v_n) \subset J'(F)$. Such that

$$v_n \to v \; in \; E^*.$$

Since J' is strongly continuous, then we have

$$J'(u_n) \to J'(u) \; in \; E^*,$$

which means that :

$$\|J'(u_n) - J'(u)\|_{E^*} \to 0 \; as \; n \to +\infty.$$

This implies that
$$\|J'(u)\|_{E^*} = \lim_{n\to+\infty} \|J'(u_n)\|_{E^*}.$$
Coniquently, ψ is lower semicontinuous.
Obviousely, ψ is bounded from below because

$$\|J'(u)\|_{E^*} \geq 0,\ \forall v \in J'(F)$$

Then, ψ is lower semicontinuous and bounded from below on $J'(F)$. Let $\epsilon = \frac{1-\kappa}{1+\kappa} \in (0,1)$. From Lemma 2.1 , there exists u^* in F such that

$$\begin{aligned} \psi(J'(u^*)) &< \psi J'(v) + \epsilon d(J'(u^*), J'(v)) \\ &\textit{With compensation} \\ \|J'(u^*)\|_{E^*} &< \|J'(v)\|_{E^*} + \epsilon \|J'(u^*) - J'(v)\|,\quad \forall v \in F \textit{ such that } v \neq u^* \end{aligned}$$

We claim that u^* is a critical point of J. If this is not true then $J'(u^*) \neq 0$.
According to the triangular inequality we have

$$\begin{aligned} \|J'(u^*)\| < \|J'(v)\| + \epsilon \|J'(u^*)\| + \epsilon \|J'(v)\|_{E^*} &\Leftrightarrow (1-\epsilon)\|J'(u^*)\|_{E^*} < (1+\epsilon)\|J'(v)\|_{E^*} \\ &\Leftrightarrow \|J'(u^*)\|_{E^*} < \frac{(1+\epsilon)}{(1-\epsilon)} \|J'(v)\|_{E^*} \end{aligned}$$

for any $v \in F$ such that $v \neq u^*$ and in particular for $v = \varphi(u^*)$, we have

$$\|J'(u^*)\|_{E^*} < \frac{(1+\epsilon)}{(1-\epsilon)} \|J'(\varphi(u^*))\|_{E^*}$$

let $\epsilon = \frac{1-\kappa}{1+\kappa}$ then $\kappa = \frac{1-\epsilon}{1+\epsilon}$, so we have

$$\|J'(u^*)\|_{E^*} < \frac{1}{\kappa} \|J'(\varphi(u^*))\|_{E^*}.$$

This contradicts the hypothesis $\|J'(\varphi(u))\|_{E^*} \leq \kappa \|J'(u)\|_{E^*}$. ■
Now, we introduce the second main result :

Theorem 2.2 *Let E be a Banach space and let the functional $J \in C^1(E,\mathbb{R})$ with $J'(E)$a closed set in E^*. Suppose also that J admits a sequence of almost critical points. Then J has at least one critical point in E.*

proof. We consider the complete metric space $J'(E)$ of E^* and define the functional φ on $J'(E)$ by

$$\varphi : J'(E) \to \mathbb{R}$$
$$J'(u) \to \varphi(J'(u)) = \|J'(u)\|_{E^*}.$$

Then φ is lower semicontinuous and bounded from below on $J'(E)$. Let $\epsilon \in (0,1)$.From the weak Ekeland variational principle, there exists u^* in E such that

$$\|J'(u^*)\|_{E^*} \leq \|J'(v)\|_{E^*} + \epsilon \|J'(u^*) - J'(v)\|_{E^*} \qquad \forall v \in E.$$

We deduce that u^* is a critical point of J. If this is not true then $J'(u^*) \neq 0$. Let (v_n) the sequence of almost critical point of J. Then we obtain that

$$\|J'(u^*)\|_{E^*} \leq \|J'(v_n)\|_{E^*} + \epsilon \|J'(u^*) - J'(v_n)\|_{E^*}, \qquad \forall n \in \mathbb{N}.$$

Because $J'(v_n) \to 0$ as $n \to \infty$, by passing to the limit,we obtain that

$$\|J'(u^*)\|_{E^*} \leq \epsilon \|J'(u^*)\|_{E^*},$$

which implies that $1 \leq \epsilon$ and this is contradiction with hypothesis on $\epsilon \in (0,1)$. ■
As a consequence of theorem 2.2, we obtain the following corollaries.

Remark 2.1 *The two geometric conditions in the Mountain pass theorem suffice to get a sequence of almost critical points [15]*

Corollary 2.1 *Let E be a Banach space, and let $J \in C^1(E,\mathbb{R})$ satisfy $J(0) = 0$. Assume that $J'(E)$ is a closed set in E^* and there exist positive numbers ρ and α such that*

1 $j(u) \geq \alpha$ if $\|u\| = \rho$,

2 there exists $e \in E$ such that $\|e\| > \rho$ and $j(e) < \alpha$.

Then, J admits at least one critical point u. It is characterized by

$$J'(u) = 0, j(u) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) | \gamma(0) = 0, \gamma(1) = e\}.$$

Corollary 2.2 *Let E be a Banach space and let the functional $J \in C^1(E,\mathbb{R})$satisfy*

$$\exists\kappa > 0 : \exists\theta > 0, \|J'(u) - J'(v)\|_{E^*} \geq \kappa \|u - v\|_E^\theta \tag{2.1}$$

Suppose also that J admits a sequence of almost critical points. Then J has at least one critical point in E.

proof. This is a direct consequence of Theorem 2.2 using Lemma 2.3. ■

Corollary 2.3 *Let E be a Banach space, and let $J \in C^1(E, \mathbb{R})$ with $J'(E)$ a closed set in E^*. Suppose that J is bounded from below. Then J has at least one critical point.*

proof. The minimization principle ensures the existence of almost critical points. The conclusion follows from Theorem 2.2. ■

Corollary 2.4 *Let E be a reflexive Banach space, let F a bounded and weakly closed subset of E with $J \in C^1(E, \mathbb{R})$, and let J' be strongly continuous on F. Suppose also that J admits a sequence of almost critical point in F. Then J has at least one critical point in F.*

proof. We show $J'(F)$ is closed recall that $J'(F) = \{J'(u); u \in F\}$ we need to verify Indeed let $g \in J'(F)$. There exists $(g_n) \in J'(F)$

$$\exists (g_n) \subset J'(F) : g_n \to g \text{ } in \text{ } E^*.$$

Hense

$$(u_n) \in F : g_n = J'(u_n) \to g \text{ } in \text{ } E^*.$$

Since F is bounded in E then the sequence (u_n) is bounded in E which is reflexive space, then we can extract a subsequence the $(u_{n_k}) \to (u_n) \in F \subset E$. So, Since J' is strongly continuous, then

$$J'(u_{n_k}) \to J'(u) \in J'(F)$$

But, we know that

$$J'(u_{n_k}) \to g$$

We deduce that $J'(u) = g$, thus $g \in J'(F)$. We conclude that $J'(F)$ is closed set in E^*, consenquently by theorem 2.2 J has at least one critical point in E^*

■

2.1.2 Application

We consider the functional J defined on $E = H_0^1(0,1)$ by

$$J(u) = \int_0^1 \left(\int_0^{u(t)} f(t, \xi) d\xi \right) dt,$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ is a continuous function . Suppose that there exist a function $\phi : \mathbb{R} \to \mathbb{R}$ and $\kappa \in]0,1[$ such that

$$\left| \int_0^1 f(t, \phi(u(t))) h(t) dt \right| \leq \kappa \left| \int_0^1 f(t, u(t)) h(t) dt \right|, forall \text{ } u, h \in H_0^1(0,1). \tag{2.2}$$

Example 2.1 *One may take as examples of f and ϕ,*

$$f(t,u)=q(t)(u+\kappa)^2,\ \phi(s)=\kappa s+\kappa^2-\kappa,\ t\in[0,1],\ \kappa\in]0,1[$$

where q is a positive function defined on $[0,1]$.
By substituting , we find

$$\begin{aligned}\left|\int_0^1 f(t,\phi(u(t)))h(t)dt\right| &= \left|\int_0^1 q(t)(\kappa u(t)+\kappa^2-\kappa+\kappa)^2h(t)dt\right|\\ &= \kappa^2\left|\int_0^1 q(t)(u(t)+\kappa)^2h(t)dt\right|\\ &= \kappa^2\left|\int_0^1 f(t,u(t))h(t)dt\right|\\ &\leq \kappa\left|\int_0^1 f(t,u(t))h(t)dt\right| \quad Since\ \kappa\in[0,1]\end{aligned}$$

Theorem 2.3 *Suppose that f satisfies condition 2.2. Then J has at least one critical point.*

proof. Note that J is well defined and $J\in C^1(H_0^1(0,1),\mathbb{R})$ with

$$J'(u).h=\int_0^1 f(t,u(t))h(t)dt,\ for\ all\ u,h\in H_0^1(0,1).$$

We first show J is well defined, let f is continuous function is on $[0,1]\times\mathbb{R}$ so f is bounded on $[0,1]\times\kappa,\quad \kappa\ is\ compact$

$$\begin{aligned}|J(u)| &= \left|\int_0^1(\int_0^{u(t)} f(t,\xi)d\xi)dt\right|\\ &\leq \int_0^1(\int_0^{u(t)} Cd\xi)dt,\quad (|f|\leq C\ on\ [0,1]\times[0,u])\\ &= C\int_0^1 u(t)dt\end{aligned}$$

$Using\ Cauchy-Schwarz\ inequality$

$$\begin{aligned}|J(u)| &\leq C(\int_0^1 u(t)^2)^{\frac{1}{2}}(\int_0^1 1^2dt)^{\frac{1}{2}}\\ &=C\ \ \|u\|_{L^2}<\infty\end{aligned}$$

Coniquently, J is well defined.
Now, we show $J\in C^1(H_0^1(0,1),\mathbb{R})$
We put

$$F(t,u)=\int_0^{u(t)} f(t,\xi)d\xi$$

and

$$J(u) = \int_0^1 F(t, u(t))dt$$

Using mean value theorem

$$\lim_{s \to 0} \frac{J(u(t) + sh(t)) - J(u(t))}{s} = \int_0^1 f(t, u(t))h(t)dt$$

Now, we show J' is strongly continuous on $F = B(0, \rho) \subset H_0^1(0, 1)$.
Let (u_n) a sequence with $(u_n) \subset F$ and $u_n \rightharpoonup u$ (F *is bounded in* $H_0^1(0, 1)$) and note it converges uniformly to u on $[0, 1]$. Since F is weakly closed, $u \in F$. Using cauchy Schwardz inequality and Poincaré's inequality

$$\begin{aligned}
\|J'(u_n) - J'(u)\|_{E^*} &= \sup_{\|h\|_{H_0^1} \leq 1} |J'(u_n)h - J'(u)h| \\
&= \sup_{\|h\|_{H_0^1} \leq 1} \left| \int_0^1 f(t, u_n(t)) - f(t, u(t)))h(t)dt \right| \\
&\leq \sup_{\|h\|_{H_0^1} \leq 1} (\int_0^1 f(t, u_n(t)) - f(t, u(t))dt)^2)^{\frac{1}{2}} (\int_0^1 h^2(t)dt)^{\frac{1}{2}} \\
&= \sup_{\|h\|_{H_0^1} \leq 1} (\int_0^1 f(t, u_n(t)) - f(t, u(t))dt)^2)^{\frac{1}{2}} \|h\|_{L^2(0,1)} \\
&\leq C \sup_{\|h\|_{H_0^1} \leq 1} (\int_0^1 f(t, u_n(t)) - f(t, u(t))dt)^2)^{\frac{1}{2}} \|h\|_{H_0^1} \\
&\leq C(\int_0^1 f(t, u_n(t)) - f(t, u(t))dt)^2)^{\frac{1}{2}}
\end{aligned}$$

Let K be the constant of the continuous embedding of $H_0^1(0, 1)$
1) in $C[0, 1]$,and note that

$$|f(t, u_n(t)) - f(t, u(t))| \leq 2 \sup_{(t,y) \in [0,1] \times [-k\rho, k\rho]} |f(t, y)|$$

$$\lim_{n \to +\infty} f(t, u_n(t)) = f(t, u(t))$$

From the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n \to +\infty} \int_0^1 f(t, u_n(t)) - f(t, u(t))dt)^2)^{\frac{1}{2}} = 0,$$

and so

$$\lim_{n \to +\infty} \|J'(u_n) - J'(u)\|_{E^*} = 0.$$

Finally we show J' satisfies $\|J'(\varphi(u))\|_{E^*} \leq \kappa \|J'(u)\|_{E^*}$for all $u \in F$, where φ is the Nemytskii's operator associated with φ. Now from 2.2 we have

$$\begin{aligned}\|J'(\varphi(u))\|_{E^*} &= \sup_{\|h\|_{H_0^1}\leq 1}\left|\int_0^1 f(t,\phi(u(t)))h(t)dt\right| \\ &\leq \kappa \sup_{\|h\|_{H_0^1}\leq 1}\left|\int_0^1 f(t,u(t))h(t)dt\right| \\ &= \kappa \left\|J'(u)\right\|_{E^*}.\end{aligned}$$

From Theorem 2.1, J has at least one critical point in F.

■

2.2 Minimization Principle in Ordered Banach Spaces and Application via Ekeland's variational principle

2.2.1 Ordered Banach Spaces

Definition 2.2 (Ordered Banach Spaces) *Let $(E, \|.\|)$ be a real Banach space. Now E is called an ordered Banach space if the following conditions hold:*

(1) $(E, \leq)$ is an ordered set.

(2) Given $u, v, w \in E$, if $u \leq v$, then $u + w \leq v + w$. If $u \leq v$, then $\lambda u \leq \lambda v$ for any $\lambda \in [0, +\infty)$.

(3) $E^+ = \{u \in E : 0 \leq u\}$ is a closed subset of E.

Definition 2.3 *[18]*

1) We say that a Banach space E is ordered by a cone K, that is $u \leq v$ if and only if $v - u \in K$.

2) An ordered Banach space E is called a Riesz-Banach space if $u \vee v = \sup\{u, v\}$, $u \wedge v = \inf\{u, v\}$ exist for any $u, v \in E$.

For a Riesz-Banach space E, we define $|u| = u \vee (^{-}u)$, $u^+ = u \vee 0$, $u^- = (^{-}u) \vee 0$.

Remark 2.2 *[16]*

(1) The Lebesgue space $L^p(\Omega)$ and the Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are ordered Banach spaces, where we define the order $u \leq v$ if $u(x) \leq v(x)$ a.e. $x \in \Omega$. Note $L^p(\Omega)$ and the first order Sobolev spaces $W^{1,p}(\Omega), W_0^{1,p}(\Omega)$ are Riesz-Banach spaces.

(2) If $u \in W^{m,p}(\Omega)$, then $|u| \in W^{m,p}(\Omega)$. Moreover we have

$$\nabla |u(x)| = \begin{cases} \nabla u(x), & if \quad u(x) > 0 \\ 0, & if \quad u(x) = 0 \\ -\nabla u(x) & if \quad u(x) < 0 \end{cases}$$

$$|\nabla |u(x)||_p = \|u\|_p \ \ for \ u \in W^{1,p}(\Omega),$$

where $|u|$ denotes the $L^p(\Omega)$-norm.

According to Lemma 2.1, we have the following result.

Proposition 2.1 *[17] Let E be a Banach space and $J : E \to \mathbb{R}$, a C^1- functional that is bounded from below and satisfies the (PS) condition. Then there exists a critical point $u \in E$ of J.*

proof. Let E be a Banach space and $J \in C^1$, by definition we have

$$J(u+v) - J(v) = \langle J'(u), v\rangle + o(\|v\|).$$

Using Ekeland lemma, there exist $(u_n) \in E$ such that:

$$\begin{cases} m \leq J(u_n) < m + \frac{1}{n} \\ J(v) + \frac{1}{n}\|v - u_n\| \geq J(u_n), \quad \forall v \in E \end{cases} \tag{2.3}$$

We show that : $\|J'(u_n)\|_{E^*} \leq \frac{1}{n}$. Let $w \in E$ such that $\|w\|_E = 1$ and we set $v = u_n + tw \in E$ in (2.3)

$$J(u_n + tw) + \frac{t}{n}\|w\| \geq J(u_n).$$

Then

$$\lim_{t\to 0} \frac{J(u_n + tw) - J(u_n)}{t} \geq \lim_{t\to 0} -\frac{1}{n}$$

Which means that

$$\langle J'(u_n), w\rangle \geq -\frac{1}{n}$$

We can replace w with $-w$, we get $\langle J'(u_n), -w\rangle \geq -\frac{1}{n}$

So

$$|\langle J'(u_n), w\rangle| \leq \frac{1}{n}$$

Consequently

$$\|J'(u_n)\|_{E^*} = \sup_{\|w\|=1} |\langle J'(u_n), w\rangle| \leq \frac{1}{n}.$$

We deduce that $\lim_{n\to+\infty} \|J'(u_n)\|_{E^*} = 0$, so $J'(u_n) \to 0$ in E^*.

We can deduce that there exist $u_0 \in E$ such that

$$\begin{cases} J'(u_0) = 0 \\ J(u_0) = m \end{cases}$$

Indeed, knowing that

$$\begin{cases} J(u_n) \to m \\ J'(u_n) \to 0 \ in \ E^* \end{cases}$$

So (u_n) is a sequence of $(PS)_m$. Since J satisfies the $(PS)_m$ condition, We deduce there exist a subsequence $(u_{n_k}) \subset E$ and $u_0 \in E$ such that:

$$u_{n_k} \to u_0 \ \ in \ E.$$

Since J is continuous, then

$$J(u_{n_k}) \to J(u_0),$$

and since J' is continuous, then

$$J'(u_{n_k}) \to J'(u_0).$$

By the uniqueness of the limit, we have

$$J'(u_0) = 0, \quad J(u_0) = m \tag{2.4}$$

So, J admit a critical point. ■

2.2.2 Main result and application

Our goal in this section is to prove a version of proposition 2.1 in Riesz-Banach spaces.

Theorem 2.4 *Let E be a Riesz-Banach space ordered by a cone K and let the functional $J \in C^1(E, \mathbb{R})$ be bounded from below, and satisfy the (PS) condition. Suppose that*

$$J(|u|) \leq J(u), \forall u \in E.$$

Then, J admits a critical point u in K.

proof. For $\epsilon = \frac{1}{n}$, let $u_0 \in E$ be such that

$$J(u_0) \leq \inf_E J(u) + \frac{1}{n}.$$

From Ekeland's variational principle, there exists $(u_n) \subset E$, such that

$$J(u_n) < J(v) + \tfrac{1}{n} \|u_n - v\| \ \ for \ all \ v \in E \ such \ that \ v \neq u_n.$$

Let $v = u_n + th, t > 0, h \neq 0$. Then by a standard technique, one has

$$\lim_{n \to +\infty} \|J'(u_n)\| = 0.$$

Now

$$\inf J(u)_{u \in E} \leq J(u_n) \leq J(u_0) \leq \inf J(u)_E + \frac{1}{n},$$

So (u_n) is a Palais-Smale sequence, and since J satisfies the (PS) condition, then there exists a subsequence $(u_{n_k}) \subset (u_n)$ such that $u_{n_k} \to w$ and $J(w) = \inf J(u)_{u \in E}, J'(w) = 0$. Since $J(|w|) \leq J(w)$, we have $J(|w|) = \inf J(u)_{u \in E}$ and because $J \in C^1(E, \mathbb{R})$, then $|w| \in K$ is a critical point of J. ■

As an application of the above result, we consider the problem

$$\begin{cases} -(p(t)u^{'}(t))^{'} = f(t, u(t)), a.e. t \in [0, +\infty) \\ u(0) = u(+\infty) = 0, \end{cases} \tag{2.5}$$

where $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, and may change sign,
$p : [0, +\infty) \to (0, +\infty)$ satisfies $\frac{1}{p} \in L^1[0, +\infty)$, and

$$\int_0^{+\infty} \left(\int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

Examples of p are the exponential function or

$$p(t) = \begin{cases} \sqrt{t} & if \quad t \in [0, 1], \\ \frac{1}{2} t(1 + t^2) & if \quad t \geq 1. \end{cases} \tag{2.6}$$

2.2.3 The functional framework

Define the space
$H^1_{0,p}(0, +\infty) = \left\{ u \in AC([0, +\infty), \mathbb{R}) | u(0) = u(+\infty) = 0, \sqrt{p} u' \in L^2[0, +\infty] \right\}$ and the cone

$$K = \left\{ u \in H^1_{0,p}(0, +\infty), 0 \leq u \right\}.$$

Lemma 2.4 *$H^1_{0,p}(0, +\infty)$ is embedded in $L^2[0, +\infty]$.*

$$\|u\|_p^2 = \int_0^{+\infty} p(t) u'^2(t) dt + \int_0^{+\infty} u^2(t) dt,$$

proof. For $u \in H^1_{0,p}(0, +\infty)$, we have

$$|u(t)| = \left| \int_t^{+\infty} u'(s) ds \right| = \left| \int_t^{+\infty} \sqrt{p(s)} u'(s) \frac{1}{\sqrt{p(s)}} ds \right|.$$

Then, by the Cauchy-Schwartz inequality

$$|u(t)|^2 \quad \leq \quad (\int_t^{+\infty} p(s) u'^2(s) ds) (\int_t^{+\infty} \frac{1}{p(s)} ds)$$

Hense

$$\int_0^{+\infty} |u(t)|^2 \, dt \leq \left(\int_0^{+\infty} (\int_t^{+\infty} \frac{1}{p(s)} ds) dt \right) \left(\int_0^{+\infty} p(s) \left| u' \right|^2 (s) ds \right),$$

that is

$$\|u\|_{L^2} \leq \sqrt{M} \, \|\sqrt{p} u'\|_{L^2} .$$

■

Now $H^1_{0,p}$ is a Hilbert space equipped with the norm.

$$\|u\|_p^2 = \int_0^{+\infty} p(t) u'^2(t) dt + \int_0^{+\infty} u^2(t) dt$$

associated with the scalar product

$$(u, v) = \int_0^{+\infty} p(t) u'(t) v'(t) dt + \int_0^{+\infty} u(t) v(t) dt.$$

Lemma 2.5 *On $H^1_{0,p}(0, +\infty)$, the quantity $\|u\|^2 = \int_0^{+\infty} p(t) u'^2 dt$ is a norm which is equivalent to the $H^1_{0,p}(0, +\infty)-$ norm.*

proof. Given $u \in H^1_{0,p}(0; +\infty)$, in view of Lemma 2.4, we have

$$\int_0^{+\infty} |u(t)|^2 \, dt \leq M \, \|u\|^2 .$$

Then

$$\int_0^{+\infty} p(t) u'^2(t) dt \leq \int_0^{+\infty} (u^2(t) + p(t) u'^2(t)) \leq (1 + M) \int_0^{+\infty} p(t) u'^2(t) dt$$

that is

$$\|u\| \leq \|u\|_{0,p} \leq \sqrt{1 + M} \, \|u\|$$

■

Lemma 2.6 *$H^1_{0,p}(0, +\infty), \|.\|)$ is embedded in $(C_l[0, +\infty), \|u\|_\infty)$, where $C_l[0, +\infty] = u \in C$ $([0, +\infty], \mathbb{R})$: $\lim_{t \to +\infty} u(t)$ exists and $\|u\|_\infty = \sup_{t \in [0, +\infty)} |u(t)|$ with $d = \sqrt{\|1/p\|_{L^1}}$ the constant of the embedding.*

proof. Let $t \in [0, +\infty[$:

$$\begin{aligned} u(t) &= \left| \int_t^{+\infty} u'(s) ds \right| \\ u(t) &= \left| \int_t^{+\infty} \frac{1}{\sqrt{p(s)}} u'(s) \sqrt{p(s)} ds \right| \\ \textit{Then, by the Cauchy} - \textit{Schwartz inequality} & \\ &\leq \left(\int_0^{+\infty} \frac{1}{p(s)} ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} p(s) u'^2 ds \right)^{\frac{1}{2}} \\ \sup |u(t)| &\leq \sqrt{\left\| \frac{1}{p} \right\|_{L^1}} \, \|\sqrt{p} u'\|_{L^2} \\ \|u\|_\infty &\leq d \, \|\sqrt{p} u'\|_{L^2} \end{aligned}$$

■

In order to prove the compactly embedding , we need the following important lemma:

Lemma 2.7 *[13] Let $H \subset (C_l[0, +\infty); \mathbb{R})$ be a bounded set. Then H is relatively compact if the following conditions hold:*

(a) H is equicontinuous on any compact sub-interval of $[0; +\infty)$; i.e.,
$\forall J \subset [0; +\infty)$ compact subinterval , $\forall \epsilon > 0$, $\exists \delta > 0$, $\forall t_1, t_2 \in J$:

$$|t_1 - t_2| < \delta \Rightarrow |u(t_1) - u(t_2)| \leq \epsilon, \quad \forall u \in H.$$

(b) H is equiconvergent at $+\infty$, i.e.,
$\forall \epsilon > 0, \exists T = T(\epsilon) > 0$ such that

$$\forall t_1, t_2 \in J : t_1, t_2 \geq T(\epsilon) = |u(t_1) - u(t_2)| \leq \epsilon, \forall u \in H.$$

Lemma 2.8 *The embedding*

$$H^1_{0,p}(0, +\infty) \hookrightarrow C_l[0, +\infty)$$

is compact.

proof. Let $D \subset H^1_{0,p}(0; +\infty)$ be a bounded set; then it is bounded in $C_l[0, +\infty)$ by Lemma 2.3. Let $R > 0$ be such that for all $u \in D$, $\|u\| \leq R$, we have

(a) D is equicontinuous on every compact interval of $[0; +\infty)$. For $u \in D$ and $t1, t_2 \in [0; +\infty)$; we have

$$\begin{aligned} |u(t_1) - u(t_2)| &= \left| \int_{t_2}^{t_1} u'(\tau) d\tau \right| = \left| \int_{t_2}^{t_1} \sqrt{p(\tau)} u'(\tau) \frac{1}{\sqrt{p(\tau)} d\tau} \right| \\ &\leq \left(\int_{t_2}^{t_1} \frac{1}{p(\tau) d\tau} \right)^{\frac{1}{2}} \|u\| \\ &\leq R \left(\int_{t_2}^{t_1} \frac{1}{p(\tau) d\tau} \right)^{\frac{1}{2}} \end{aligned}$$

and the right-hand side tends to 0, as $|t_1 - t_2| \to 0$ for $\frac{1}{p} \in L^1(0; +\infty)$.

(b) is equiconvergent. For $t \in [0; +\infty)$ and $u \in D$, using the fact that $u(+\infty) = 0$, we have

$$\begin{aligned} |u(t) - u(+\infty)| &= |u(t)| \\ &= \left| \int_t^{+\infty} u'(\tau) d\tau \right| \\ &\leq \left(\int_t^{+\infty} \frac{1}{p(\tau)} \right)^{\frac{1}{2}} \|u\| \\ &\leq R \left(\int_t^{+\infty} \frac{1}{p(\tau)} \right)^{\frac{1}{2}} \to 0, as\ t \to +\infty. \end{aligned}$$

The result then follows from Lemma 2.7.

■

2.2.4 Existence of weak solutions

Take $v \in H^1_{0,p}(0,+\infty)$, and multiply the equation in 2.5 by v and integrate between 0 and $+\infty$, to obtain

$$-\int_0^{+\infty}(p(t)u'(t))'v(t)dt = \int_0^{+\infty} f(t,u(t))v(t)dt.$$

Hence

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt = \int_0^{+\infty} f(t,u(t))v(t)dt.$$

This leads to the natural concept of a weak solution for (2.5).

Definition 2.4 *We say that a function $u \in H^1_{0,p}(0,+\infty)$ weak solution of (2.5) if*

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt - \int_0^{+\infty} f(t,u(t))v(t)dt = 0.$$

for all $v \in H^1_{0,p}(0,+\infty)$ To study (2.5), consider the functional $J : H^1_{0,p}(0,+\infty) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t))dt,$$

Where

$$F(t,u) = \int_0^u f(t,s)ds.$$

Let the operator $A : H^1_{0,p}(0,+\infty) \to H^1_{0,p}(0,+\infty)$ be defined by

$$Au(t) = \int_0^{+\infty} G(t,s)f(s,u(s))ds$$

with the Green's function

$$G(t,s) = \frac{1}{\|\frac{1}{p}\|_{L^1}}\begin{cases}\varphi_1(t)\varphi_2(s) & t \leq s \quad ,\\ \varphi_1(s)\varphi_2(t) & s \leq t \quad .\end{cases}$$

and the fundamental system of solutions $\varphi_1(t) = \int_0^t \frac{ds}{p(s)}$ and $\varphi_2(t) = \int_t^{+\infty} \frac{ds}{p(s)}$

Theorem 2.5 *Suppose the following condition holds:*
f is an odd function in u and there exist a constant $\mu \in [0,1)$, and positive functions $a_1, b_1 \in L^1[0,+\infty)$ such that

$$|f(t,u)| \leq a_1(t)\,|u|^{\mu} + b_1(t), fora.e.t \in [0,+\infty) \text{ and all } u \in \mathbb{R}. \tag{2.7}$$

Then (2.5) has at least one weak solution in K.
We start by the axillary result

Lemma 2.9 *Under assumption (2.7), we have*

(1) A is well defined,

(2) A is compact.

proof. We confine to prove the second assersion; Let (u_n) be a bounded sequence in the reflexive separable space $H^1_{0,p}$ Then there exists a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup u$ in $H^1_{0,p}$, as $k \to +\infty$. We will prove that the sequence (Au_{n_k}) is convergent. We have the estimates:

$$\begin{aligned}
\|Au_{n_k} - Au\| &= \sup_{\|v\|\leq 1} |(Au_{n_k} - Au, v)| \\
&= \sup_{\|v\|\leq 1} \left| \int_0^{+\infty} p(t)\left(Au_{n_k} - Au\right)'(t) v'(t)dt \right| \\
&= \sup_{\|v\|\leq 1} \left| \int_0^{+\infty} -\left(p(t)(Au_{n_k} - Au)'(t)\right)' v(t)dt \right| \\
&= \sup_{\|v\|\leq 1} \left| \int_0^{+\infty} \left[-(p(t)(Au_{n_k})'(t))' - (-p(t)(Au)'(t))'\right] v(t)dt \right| \\
&= \sup_{\|v\|\leq 1} \left| \int_0^{+\infty} \left[f(t, u_{n_k}(t)) - f(t, u(t))\right] v(t)dt \right| \\
&\leq \sup_{\|v\|\leq 1} \left[\int_0^{+\infty} \|v\|_\infty \left|f(t, u_{n_k}(t)) - f(t, u(t))\right| dt \right] \\
&\leq d \sup_{\|v\|\leq 1} \left[\int_0^{+\infty} \|v\| \left|f(t, u_{n_k}(t)) - f(t, u(t))\right| dt \right] \\
&\leq d \int_0^{+\infty} \left|f(t, u_{n_k}(t)) - f(t, u(t))\right| dt
\end{aligned}$$

Using Lemma 2.6, 2.8, and condition (2.7), and the Lebesgue dominated convergence theorem, we obtain that

$$\|Au_{n_k} - Au\| \to 0, \quad as\ k \to +\infty.$$

The same estimates show that $\|Au_{n_k} - Au\| \to 0$ whenever $\|u_n - u\| \to 0$, as $n \to +\infty$. The compactness of A is then proved. In the same way, one can show that A is continuous. ■

proof. We will apply Theorem 2.4.First we note that J is well defined. In fact, given $u \in H^1_{0,p}(0, +\infty)$, then condition (2.2) guarantees that

$$|F(t, u(t))| \leq \frac{a_1}{\mu + 1} |u(t)|^{\mu+1} + b_1(t) |u(t)|.$$

Hence using Lemma 2.6 we have

$$\begin{aligned}
\left| \int_0^{+\infty} F(t, u(t))dt \right| &\leq \frac{d^{\mu+1}}{\mu + 1} \|u\|^{\mu+1} \int_0^{+\infty} a_1(t)dt + d\|u\| \int_0^{+\infty} b_1(t)dt \\
&\leq \frac{d^{\mu+1}}{\mu + 1} \|u\|^{\mu+1} |a_1|_1 + d\|u\| |b_1|_1,
\end{aligned}$$

So

$$|J(u)| \leq \frac{1}{2}\|u\|^2 + \frac{d^{\mu+1}}{\mu + 1} \|u\|^{\mu+1} |a_1|_1 + d\|u\| |b_1|_1$$

.

Now we show that J is bounded from below. To see this note (f1) and Lemma2.3 guarantee that

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{d^{\mu+1}}{\mu+1}\|u\|^{\mu+1}|a_1|_1 - d\|u\||b_1|_1 \tag{2.8}$$

Since $\mu < 1$, (2.8) implies

$$\lim_{\|u\|\to+\infty} J(u) = +\infty.$$

Next from (f_1), J is continuously differentiable and satisfies

$$(J'(u),v) = \int_0^{+\infty} p(t)u'(t)v'(t)dt - \int_0^{+\infty} f(t,u(t))v(t)dt$$

for all $u, v \in H^1_{0,p}(0,+\infty)$ and

$$J' = I - A.$$

Finally J satisfies the (PS) condition. Indeed, suppose that $(u_n) \in H^1_{0,p}(0,+\infty)$ and there exists $M > 0$ such that $|J(u_n)| \leq M$ and $J'(u_n) = u_n - Au_n \to 0$ on $H^1_{0,p}(0,+\infty)$ when $n \to +\infty$ From the above (J is bounded from below) we see that (u_n) is bounded in $H^1_{0,p}(0,+\infty)$. From the compactness of A there is a subsequence (Au_{n_k}) such that $Au_{n_k} \to w$. Then

$$\|u_{n_k} - w\| \leq \|u_{n_k} - Au_{n_k}\| + \|Au_{n_k} - w\|,$$

and since $u_{n_k} - Au_{n_k} \to 0$ in $H^1_{0,p}(0,+\infty)$, when $n \to +\infty$, we have that (u_n) has a convergent subsequence (u_{n_k}) with $u_{n_k} \to w$. Now $J(|u|) = J(u), \forall u \in H^1_{0,p}(0,+\infty)$ since f is odd and now apply Theorem 2.4. ■

Example 2.2 *An example of f is the odd function*

$$f(t,u) = a(t)u^{\frac{1}{3}} - b(t)u^{\frac{1}{5}},$$

with $a, b \in L^1(0,+\infty)$.

CHAPTER 3

Fixed point theorem via critical point theory

The aim of this chapter is to prove new fixed point theorems on Hilbert spaces for potential $\alpha-$positively homogeneous operators by using weak Ekeland variational principle. To do this, we begin with the following.

3.1 A fixed point theorem on Hilbert spaces for potential $\alpha-$positively homogeneous operators

Let H be a real Hilbert space endowed with scalar product denoted (. , .).

Definition 3.1 ($\alpha-$positively homogeneous operator) *A mapping $T : H \to H$ is said to be $\alpha-$positively homogeneous operator if*

$$T(tx) = t^{\alpha}T(x), \forall t > 0, \forall x \in H, \quad for\ some\ \alpha > 0.$$

Let ψ be a Frechet differentiable functional defined on H and let $\psi'(x)$ denote the differential of ψ at $x \in H$. We denote by $\nabla\psi(x)$ the unique vector of H such that

$$\psi'(x).y = (\nabla\psi(x), y) \ \ \forall x, y \in H.$$

Definition 3.2 (Potential operator) *A mapping $T : H \to H$ is said to be a potential (or gradient) operator if there exists a differentiable functional ψ on H (the potential of T) such that $T = \nabla\psi$. Let $T : H \to H$ be a potential operator and ψ be its potential, so that $\nabla\psi = T$. If T is assumed to be continuous, then ψ is of class C^1 and ψ and T are related by the formula*

$$\psi(x) = \int_0^1 (T(tx), x)dt.$$

For more details on potential operators and their properties, see the pioneering book of Chabrowski [12]. In particular, the page 02 for the above property.

If T is $\alpha-$positively homogeneous, then

$$\psi(x) = \frac{1}{\alpha + 1}(T(x), x).$$

If we set

$$G(x) = (T(x), x),$$

we have that G is of class C^1 and

$$\nabla G = (\alpha + 1)T.$$

We need also the following theorem

3.1.1 Main results

We are now in a position to give the main results.

Theorem 3.1 *Let $T : \overline{U} \to H$ be a compact potential operator and α-positively homogeneous, where U is an open and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that*

$$(T(u), u) \leq \frac{\alpha+1}{2} \|u\|^2 - C \|u\| \;\; for \; all \; u \in \partial U, \tag{3.1}$$

then T has a fixed point in $\overline{U}$

proof. We consider the complete metric space $\overline{U}$ endowed with the distance induced by the norm of H and the functional φ defined on $\overline{U}$ by

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 (T(su), u)ds \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{1+\alpha}(T(u), u). \end{aligned}$$

It is clear that ϕ is differentiable with $\varphi' = I - T \left(\varphi'(u) = u - T(u), \; for \; any \; u \; \in H' = H\right)$. So, $u \in H$ is a fixed point of T if and only if u is a critical point of φ.
Hence, in order to prove the existence of critical point of φ, we need to apply Weak Ekeland Variational Principle (see Lemma [14]).

Claim 1. The functional φ is bounded from below. Indeed, because $\overline{U}$ is a bounded set with $0 \in U$ and the operator T is compact, there exists $M > 0$ such that $\|T(u)\| \leq M$ for all $u \in \overline{U}$. By using CauchySchwarz inequality, we obtain

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{1+\alpha}(T(u), u). \\ &\geq \frac{1}{2}\|u\|^2 - \frac{M}{1+\alpha}\|u\| \end{aligned}$$

Then, φ is bounded from below. Let $0 < \epsilon \leq \frac{C}{\alpha+1}$, then, by the weak Ekeland variational principle, there exists $u_\epsilon \in \overline{U}$ with

$$\varphi(u_\epsilon) \leq \inf_{\overline{U}} \varphi + \epsilon$$

and whenever $v \in \overline{U}$ with $v \neq u_\epsilon$, then

$$\varphi(u_\epsilon) < \varphi(v) + \epsilon \|u_\epsilon - v\|.$$

Claim 2 We have $u_\epsilon \notin \partial U$. Indeed, if $u_\epsilon \in \partial U$, then, for $v = 0$, we have $\varphi(u_\epsilon) < \varphi(0) + \epsilon \|u_\epsilon - 0\|$. Because $\varphi(0) = 0$, we obtain that

$$\varphi(u_\epsilon) < \epsilon \|u_\epsilon\| \leq \frac{C}{\alpha+1} \|u_\epsilon\|$$

i.e.

$$\frac{1}{2}\|u_\epsilon\|^2 - \frac{1}{\alpha+1}(T(u_\epsilon), u_\epsilon) < \frac{C}{\alpha+1}\|u_\epsilon\|$$

and this is a contradiction with the hypothesis (3.1).

Claim 3. u_ϵ is an approximate fixed point of T. Indeed, let $t > 0$ and $h \in H$. We put $v_\epsilon = u + th$. We remark that because $u_\epsilon \in U$ and U is open then $u_\epsilon + th \in U$ for t small enough. We have then

$$\frac{\varphi(u_\epsilon) - \varphi(u_\epsilon + th)}{t} \leq \epsilon \|h\|.$$

By passing to the limit as $t \to 0^+$, we obtain that

$$-\langle \varphi'(u_\epsilon), h\rangle \leq \epsilon \|h\|$$

. As $h \in H$ is arbitrary, we obtain

$$|\langle \varphi'(u_\epsilon), h\rangle| \leq \epsilon \|h\|$$

which means that

$$\|\varphi'(u_\epsilon)\| \leq \epsilon$$

. This means that u_ϵ is an approximate critical point of φ and then it is an approximate fixed point of T ($T(u_\epsilon) - u_\epsilon \to 0$ in H).

Claim 4. Existence of a fixed point. Indeed, for $\epsilon = \frac{1}{n}$, we remark that $\frac{1}{n} \leq C$ when $n \to +\infty$. We obtain that

$$\|\varphi'(u_n)\| \leq \frac{1}{n}$$

which means that

$$\|\varphi'(u_n)\| \to 0 \ as \ n \to +\infty$$

and thus we have

$$\|u_n - T(u_n)\| \to 0 \ as \ n \to +\infty$$

Since the operator T is compact, there exists a subsequence

$$(u_{n_k}) \subset (u_n) \ such \ that \ T(u_{n_k}) \to \ w \ with \ w \in \overline{U}.$$

Then $u_{n_k} \to w$. Indeed, we have

$$\|u_{n_k} - w\| \leq \|u_{n_k} - T(u_{n_k})\| + \|T(u_{n_k}) - w\| \to 0$$

which means that $u_{n_k} \to w$ and then $T(u_{n_k}) \to T(w)$. Thus, we have
$T(w) = w$ and w is a fixed point of T. ■

As a direct consequence of the above theorem and by using Cauchy-Schwarz inequality, we obtain:

Corollary 3.1 *Let $T : \overline{U} \to H$ be a compact potential operator and $\alpha-$positively homogeneous, where U is an open and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that*

$$\|T(u)\| \leq \frac{\alpha+1}{2}\|u\| - C \quad for \ all \ u \in \partial U,$$

then T has a fixed point in $\overline{U}$.

When we put $U = B(0, R)$ for some $R > 0$, we obtain the following

Corollary 3.2 *Let $T : H \to H$ be a compact potential operator and $\alpha-$positively homogeneous satisfying*

$$\exists R > 0, \quad \exists C < \frac{1}{2}(\alpha+1)R,$$

such that $T(\partial B(0,R)) \subset B(0, \frac{1}{2}(\alpha+1)R - C)$,
then T has a fixed point in $\overline{B}(0,R)$.

3.1.2 Application

We consider the Dirichlet boundary value problem

$$\begin{cases} -u''(t) = q(t)f(u(t))), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{3.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $q \in L^2(0,1)$. The following lemma is standard in the literature.

Lemma 3.1 *If u is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,s)q(s)f(u(s))ds,$$

Where

$$G(t,s) = \begin{cases} t(1-s), & t \leq s \quad , \\ s(1-t), & s \leq t \quad . \end{cases} \tag{3.3}$$

then u is a solution of problem (3.2) and conversely.

Let T be the operator defined on the standard Sobolev space $H_0^1(0,1)$ endowed with the norm $\|u\| = \int_0^1 u'^2(t)dt$ by

$$Tu(t) = \int_0^1 G(t,s)q(s)f(u(s))ds.$$

Then, T satisfies the problem

$$\begin{cases} -(Tu)''(t) = q(t)f(u(t)), t \in [0,1], \\ (Tu)(0) = (Tu)(1) = 0. \end{cases} \tag{3.4}$$

We remark that the operator T satisfies also the following property

$$(Tu, v) = \int_0^1 q(s)f(u(s))v(s)ds, \quad \forall u, v \in H_0^1(0,1).$$

Let φ be the functional defined on $H_0^1(0,1)$ by

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_0^1 \left(\int_0^{u(t)} q(s)f(s)ds \right) dt.$$

Definition 3.3 *We say that* $u \in H_0^1(0,1)$ *is a weak solution of (3.2) if*

$$\int_0^1 [u'(t)v'(t) - q(t)f(u(t))v(t)]dt = 0, \quad for\ all\ \ v \in H_0^1(0,1).$$

Lemma 3.2 *[7] The operator* $T : H_0^1(0,1) \to H_0^1(0,1)$ *is compact.*

Theorem 3.2 *Assume that the following conditions hold:*

(H1) $\exists \alpha > 0,$ *such that* $f(su) = s^{\alpha} f(u), \forall s > 0, \forall u \in \mathbb{R},$

(H2) $\exists R > 0, \max_{|\xi| \le R} |f(\xi)| < \frac{\pi}{2\|q\|_{L^2}}(\alpha+1)R.$

Then problem (3.2) has a solution $u \in C^2[0,1]$.

proof. Integrating by parts, we obtain for all $u, v \in H_0^1(0,1)$

$$\begin{aligned} \langle \varphi'(u),(v)\rangle &= \int_0^1 u'(t)v'(t)dt - \int_0^1 q(t)f(u(t))v(t)dt \\ &= \int_0^1 u'(t)v'(t)dt + \int_0^1 (Tu)''(t)v(t)dt \\ &= \int_0^1 (u'(t)v'(t) - (Tu)'(t)v'(t))dt \\ &= (u,v) - (Tu,v) = (u - Tu, v) = ((I-T)u, v). \end{aligned}$$

Thus

$$\varphi' = I - T.$$

We prove that the operator T verifies the conditions of Corollary 3.2 . Take any $C > 0$ such that

$$C \le \frac{1}{2}(\alpha+1)R - \frac{\|q\|_{L^2}}{\pi} \sup_{|\xi| \le R} |f(\xi)|$$

Such a number exists in view of (H2). Then by using the Cauchy-Schwarz inequality and Poincare's inequality we obtain

$$\begin{aligned}
\|Tu\| &= \sup_{\|v\|\leq 1} |(Tu,v)| \\
&= \sup_{\|v\|\leq 1} |\int_0^1 q(s)f(u(s))v(s)ds| \\
&\leq \sup_{\|v\|\leq 1} (\int_0^1 |q(s)f(u(s))|^2 ds)^{\frac{1}{2}} (\int_0^1 |v(s)|^2 ds)^{\frac{1}{2}} \\
&\leq \sup_{\|v\|\leq 1} (\int_0^1 |q(s)f(u(s))|^2 ds)^{\frac{1}{2}} \, \|v\|^2 \\
&\leq \frac{\|q\|_{L^2}}{\sqrt{\lambda_1}} \sup_{|\xi|\leq R} |f(\xi)| \\
&\leq \frac{1}{2}(\alpha+1)R - C
\end{aligned}$$

Here $\lambda_1 = \pi^2$ is the first eigenvalue of the linear Dirichlet problem

$$\begin{cases} -u''(t) = \lambda u(t), \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

We have also used the Poincare's inequality

$$\|u\|_{L^2} \leq \|u'\| - C \ for \ all \ u \in \partial U,$$

then T has a fixed point in $\overline{U}$. ■

Conclusion

In this memory, we established a new critical point theorem according to the Ekeland variational principle without satisfying the Palais-Smale condition, we showed there was a critical point in Riesz-Banach space ordered by a cone k and prove new fixed point theorems on Hilbert spaces for potential $\alpha-$positively homogeneous operators.

Ordered Banach spaces play a crucial role in the study of differential equations due to their structured framework. They facilitate the analysis of positive operators and the development of spectral theory, which are essential for understanding the behavior of solutions, such as stability and periodicity. These spaces are also vital for handling boundary conditions and constraints in partial differential equations (PDEs), particularly those with positivity constraints like heat distribution and wave propagation. This makes ordered Banach spaces invaluable in solving practical problems across physics, engineering, biology, and economics.

Overall, this work underscores the profound impact of critical point theory in advancing mathematical knowledge and solving practical problems. Through continued exploration and innovation, we can expect to uncover even deeper insights and broader applications in the years to come.

In my work I found difficulty the lack of references on my subjects.

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ملخص

في هذه المذكرة قمنا بدراسة نظريات جديدة :
أولا قمنا باثبات نظرية النقطة الحرجة التي لا يتوفر فيها شرط بالي سميل والتي تضمن وجود نقطة حرجة ثم برهنا ان هناك نقطة حرجة في فضاء ريز باناخ مرتب بمخروط k ثم تطبيق النتيجة المجردة على المسألة التالية:

$$\begin{cases}-(p(t)u^{'}(t))^{'}=f(t,u(t)), a.e.t\in[0,+\infty)\\ u(0)=u(+\infty)=0,\end{cases} \tag{١}$$

حيث $f:[0,+\infty)\mathbb{R}\to\mathbb{R}$ دالة كاراثيودورية $[0,+\infty)\to(0,+\infty)$ محققة $\frac{1}{p}\in L^1[0,+\infty),$ و

$$\int_0^{+\infty}\left(\int_t^{+\infty}\frac{1}{p(s)}ds\right)dt<+\infty.$$

أما في الحالة الثانية برهنا نظريات جديدة للنقاط الثابتة في مساحات هيلبرت للمشغلين المحتملين ذري التجانس الايجابي بدرجة α باستخدام مبدأ ايكلاند الضعيف ثم تطبيق نتيجتنا المجردة على المسألة التالية :

$$\begin{cases}-u''(t)=q(t)f(u(t))), \quad t\in(0,1),\\ u(0)=u(1)=0,\end{cases} \tag{٢}$$

حيث $f:\mathbb{R}\to\mathbb{R}$ هي دالة مستمرة ، $q\in L^2(0,1).$

كلمات مفتاحية

نقطة حرجة، نقطة ثابتة، مبدأ إيكلاند، حل ضعيف، ، مسالة حدية .

Abstract

In this memory, we have studied new theorems : First, we proved the critical point theorem without satisfying the Palais-Smale condition, ensuring the existence of a critical point. Then, we demonstrated the existence of a critical point in Riesz-Banach space ordered by a cone k, followed by applying the abstract result to the following problem :

$$\begin{cases} -(p(t)u^{'}(t))^{'} = f(t,u(t)), a.e.t \in [0,+\infty) \\ u(0) = u(+\infty) = 0, \end{cases} \tag{3}$$

Where $f : [0,+\infty)\mathbb{R} \to \mathbb{R}$ is a Caratheodory function, and may change sign, $p : [0,+\infty) \to (0,+\infty)$ satisfies $\frac{1}{p} \in L^1[0,+\infty)$, and

$$\int_0^{+\infty} \left(\int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

In the second case, we established new theorems of fixed points in Hilbert spaces for potential $\alpha-$positively homogeneous operators using the weak Ekeland principle, then applied our abstract result to the following problem :

$$\begin{cases} -u''(t) = q(t)f(u(t))), \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4}$$

Where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $q \in L^2(0,1)$.

Key words

Critical point,Fixed point, Ekeland principle, Weak solution, Boundary problem.

Résumé

Dans ce mémoire, nous avons étudié de nouvelles théories : Tout d'abord, nous avons démontré la théorie du point critiquesans satisfaire la condition de palais-smale et qui garantit l'existence d'un point critique. Ensuite, nous avons prouvé l'existence d'un point critique dans l'espace de Banach ordonné avec un cône k, puis appliqué le résultat abstrait au problème suivant :

$$\begin{cases} -(p(t)u^{'}(t))^{'} = f(t, u(t)), a.e.t \in [0, +\infty) \\ u(0) = u(+\infty) = 0, \end{cases} \tag{5}$$

où $f : [0, +\infty)\mathbb{R} \to \mathbb{R}$ est une fonction Caratheodory, et peut changer de signe, $p : [0, +\infty) \to (0, +\infty)$ satisfies $\frac{1}{p} \in L^1[0, +\infty)$, and

$$\int_0^{+\infty} \left(\int_t^{+\infty} \frac{1}{p(s)} ds \right) dt < +\infty.$$

Dans le deuxième cas, nous avons démontré de nouvelles théories des points fixes dans les espaces de Hilbert pour les opérateurs α potentiels homogènes en utilisant le principe d'Ekeland faible, puis appliqué notre résultat abstrait au problème suivant :

$$\begin{cases} -u''(t) = q(t)f(u(t))), \quad t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{6}$$

Où $f : \mathbb{R} \to \mathbb{R}$ est une fonction continue, $q \in L^2(0, 1)$.

Mot-clés

Point critique,Point fixe, principe d'Ekeland, Solution faible, Probléme aux limites.