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Some Remarks On Morrey Spaces and Stummel Classes

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Contents

Notations

$\mathbb{R}^n$	n-dimensional Euclidean space
Ω	An open set in $\mathbb{R}^n$
$\overline{\Omega}$	The closure of Ω
$\partial\Omega$	The boundary of Ω
$B(x,r)$	Open ball centered at x with radius r
$\overline{B}(x,r)$	Closed ball centered at x with radius r
$\lvert\Omega\rvert$	Lebesgue measure of Ω
ω_n	Volume of the unit ball in $\mathbb{R}^n$
ω_{n-1}	Surface area of the unit sphere in $\mathbb{R}^n$
χ_E	Characteristic function of the set E
$C_c^\infty(\Omega)$	Space of smooth functions with compact support in Ω
$L^p(\Omega)$	Lebesgue space of p-integrable functions on Ω
$L^{p,\infty}(\Omega)$	Weak Lebesgue (Marcinkiewicz) space on Ω
$L^p_{\text{loc}}(\Omega)$	Space of locally p-integrable functions on Ω
$\lVert f\rVert_{L^p}$	Norm of f in $L^p(\Omega)$
$\lVert f\rVert_{L^{p,\infty}}$	Quasi-norm of f in $L^{p,\infty}(\Omega)$
$D_f(\lambda)$	Distribution function of f: $D_f(\lambda) = \lvert\{\lvert f\rvert > \lambda\}\rvert$
$\partial_i u$	Weak partial derivative of u with respect to x_i
∇u	Weak gradient of u
$W^{1,p}(\Omega)$	Sobolev space: $u \in L^p$, $\partial_i u \in L^p$
$W_0^{1,p}(\Omega)$	Closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$
$H^1(\Omega)$	$W^{1,2}(\Omega)$, Hilbert space
$H_0^1(\Omega)$	$W_0^{1,2}(\Omega)$, Hilbert space with Dirichlet condition
$\lVert u\rVert_{W^{1,p}}$	Norm of u in $W^{1,p}(\Omega)$
$\lVert\nabla u\rVert_{L^p}$	L^p norm of the gradient of u
C_P	Poincaré constant
$M^p_q(\mathbb{R}^n)$	Classical Morrey space with indices q and p
$L^{1,\gamma}(\Omega)$	Morrey space of functions with weighted local integrability
$\lVert f\rVert_{M^p_q}$	Norm of f in $M^p_q(\mathbb{R}^n)$
$\lVert f\rVert_{L^{1,\gamma}}$	Norm of f in $L^{1,\gamma}(\Omega)$
$S_\alpha(\mathbb{R}^n)$	Stummel class of order α
$S_{p,\Psi}(\mathbb{R}^n)$	Generalized Stummel class of order p with weight Ψ
$\eta^f_\alpha(r)$	Stummel modulus: $\eta^f_\alpha(r) = \sup\limits_{x\in\mathbb{R}^n}\int_{\lvert x-y\rvert<r}\frac{\lvert f(y)\rvert}{\lvert x-y\rvert^{n-\alpha}}\,dy$
$\eta^f_{p,\Psi}(r)$	Generalized Stummel modulus of f
$V_0L^{p,\lambda}(\mathbb{R}^n)$	Vanishing Morrey space at the origin
$V_\infty M^p_q(\mathbb{R}^n)$	Vanishing Morrey space at infinity
$VM^p_q(\mathbb{R}^n)$	Vanishing at infinity Morrey space

$A(u,v)$	Bilinear form on a Hilbert space
$F_f(\phi)$	Linear functional: $F_f(\phi) = \int_\Omega f\phi\, dx$
Lu	Elliptic operator: $Lu = -\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u)$
ν	Ellipticity constant
κ	Bound for $\lvert A(x)\rvert$
K_0	Lipschitz constant of g
$A(x)$	$n \times n$ symmetric matrix
$g(u)$	Lipschitz nondecreasing function
u^+	$\max(u,0)$, example of $g(u)$
div	Divergence operator
$\mathrm{sgn}(x)$	Sign function

Introduction

The study of function spaces has always been a central theme in modern analysis, particularly in the theory of partial differential equations (PDEs) and harmonic analysis. Since the early twentieth century, mathematicians have sought to understand the behavior of functions and their derivatives in increasingly general settings. The classical Lebesgue spaces $L^p(\Omega)$, introduced by Henri Lebesgue at the beginning of the twentieth century, provided a powerful framework for integration theory and functional analysis. However, as the study of PDEs progressed, it became clear that Lebesgue spaces alone are not sufficient to capture all the subtle properties of solutions, especially when dealing with irregular coefficients or singular potentials.

In 1938, Charles B. Morrey introduced a new class of function spaces, now known as Morrey spaces, while studying the regularity of solutions to elliptic partial differential equations (see [8]). Morrey spaces generalize Lebesgue spaces by measuring local integrability with a scaling factor. More precisely, a function f belongs to the Morrey space $M^p_q(\mathbb{R}^n)$ if

$$\|f\|_{M^p_q} = \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q dy\right)^{\frac{1}{q}} < \infty.$$

This definition enables Morrey spaces to include functions that are not globally p-integrable yet possess controlled local behavior. For example, the function $f(x) = |x|^{-n/p}$ belongs to M^p_q for all $q < p$ but does not belong to $L^p(\mathbb{R}^n)$.

Over the decades, Morrey spaces have been extensively studied and applied to various problems in PDEs, harmonic analysis, and potential theory. Key contributions include the work of Peetre[9], Piccinini[10], Vitanza[19], and Chiarenza-Franciosi[4], among many others. More recently, generalized Morrey spaces have been introduced by Nakai and others. Independently, in 1982, Aizenman and Simon introduced Stummel classes while studying operators with singular potentials (see [2]). A function V belongs to the Stummel class $S_\alpha(\mathbb{R}^n)$ if

$$\eta^V_\alpha(r) = \sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-\alpha}} dy \to 0 \quad \text{as } r \to 0.$$

These classes are designed to handle singular data that are not in classical Lebesgue spaces but still possess enough regularity to ensure the existence and uniqueness of solutions. A canonical example is the function $f(x) = |x|^{-1}$ in $\mathbb{R}^3$, which belongs to $\tilde{S}_2(\Omega)$ and is used in the application of semilinear elliptic equations.

The work of Aizenman and Simon was subsequently extended by Tumalun et al. [16, 18], who studied the inclusion relations between Stummel classes and other function spaces.

The main problem of this dissertation is to study the inclusion relations between Morrey spaces $M_q^p(\mathbb{R}^n)$, weak Lebesgue spaces $L^{p,\infty}(\mathbb{R}^n)$, and Stummel classes $S_\alpha(\mathbb{R}^n)$, and to apply these results to elliptic partial differential equations and semilinear elliptic equations.

This dissertation is organized into three chapters. Chapter 1 is devoted to fundamental preliminaries: Lebesgue spaces, weak Lebesgue spaces, Morrey spaces, Stummel classes, vanishing Morrey spaces, as well as the necessary tools for partial differential equations, including weak derivatives, Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, the Poincaré inequality, the Lax-Milgram lemma, and Stampacchia's lemma, see[1], [2], [3], [4], [5], [12], [16],[19] and [20].

Chapter 2 constitutes the core of this dissertation. It first studies inclusion relations between Morrey spaces (monotonicity in q and p, the case $p = q$, relations with Lebesgue spaces, counterexamples). It then investigates the relations between weak Lebesgue spaces and Stummel classes (global embedding $L^{p,\infty} \subset S_{n/p}$, local embedding $S_\alpha \subset L_{\text{loc}}^{n/\alpha,\infty}$, generalized Stummel classes $S_{p,\Psi}$). The main result is the complete chain

$$L^p \subset M_q^p \subset L^{p,\infty} \subset S_{n/p} \subset L_{\text{loc}}^{p,\infty}.$$

Chapter 3 is devoted to applications to partial differential equations. First, we study the Dirichlet problem for an elliptic equation with data in the Morrey space $L^{1,\gamma}(\Omega)$ where $n-2 < \gamma < n$ [17]. Using the Lax-Milgram lemma and the weighted embedding theorem (which relies on the inclusions from Chapter 2), we prove the existence and uniqueness of a weak solution.

Second, we study a semilinear elliptic equation of the form

$$-\operatorname{div}(A(x)\nabla u) + g(u) = f,$$

where the data f belongs to the Stummel class $\tilde{S}_\alpha(\Omega)$ (look at [18]). Using Stampacchia's lemma and a weighted embedding theorem (which relies on the inclusion $S_\alpha \subset L_{\text{loc}}^{n/\alpha,\infty}$ from Chapter 2), we prove the existence and uniqueness of a weak solution. This result is due to Tumalun [18] and generalizes previous works where f belongs to Morrey spaces (look at [18]). A concrete example with the Coulomb potential $f(x) = |x|^{-1}$ in $\mathbb{R}^3$ is provided.

Taking all aspects into consideration, this study aims to provide a comprehensive analysis of the inclusion relations between Morrey spaces, weak Lebesgue spaces, and Stummel classes. Special attention is given to the critical condition $\alpha = n/p$, which requires careful handling of the interplay between distributional properties and weighted local integrability. The study also seeks to bridge theoretical insights with practical applications, offering new perspectives on solution behavior within generalized function spaces and paving the way for further research in nonlinear PDEs, parabolic equations and numerical methods.

Chapter 1

Fundamental facts in functional analysis

In this chapter, we study some definitions and properties of Lebesgue spaces , weak Lebesgue spaces, Morrey spaces, Stummel classes and Vanishing Morrey spaces.We also introduce the necessary tools for the study of partial differential equations, including Sobolev spaces, weak derivatives, the Poincaré inequality, the Lax-Milgram lemma and Stampacchia's lemma, see[1], [2], [3], [4], [5], [12], [16],[19] and [20].

1.1 Lebesgue Spaces

This section is devoted to some definitions and properties of L^p spaces.

Definition and elementary properties of L^p spaces

Definition 1.1.1. *Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \textit{ is measurable and } |f|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu \right)^{1/p}.$$

Example 1.1.1. *Let $1 \le p < \infty$ and define*

$$f(x) = \chi_{B(0,1)}(x),$$

where $\chi_{B(0,1)}$ denotes the characteristic function of the unit ball $B(0,1) \subset \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_{B(0,1)} 1 \, dx = |B(0,1)| < \infty.$$

Therefore,

$$f \in L^p(\mathbb{R}^n).$$

Remark 1.1.1. *Let $f \in L^p(\Omega)$. A fundamental property of the L^p norm is that*

$$\|f\|_{L^p} = 0 \iff f = 0 \quad \textit{almost everywhere (a.e.).}$$

This means that $f(x) = 0$ for all $x \in \Omega$ except possibly on a set of measure zero.

Definition 1.1.2. *Suppose that* $1 \le p < \infty$. *Then*

1. $L^p_{\text{loc}}(\Omega) = \{f : f \in L^p(K) \text{ for every compact subset } K \subset \Omega\}$.
2. f *is locally integrable in* Ω *if*
$$f \in L^1_{\text{loc}}(\Omega).$$
3. *Let* f *and* g *be locally integrable functions defined in* Ω. *We define* g *as the weak derivative of* f *with respect to* α *if, for every* $\varphi \in C_0^\infty(\Omega)$,
$$\int_\Omega f D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega g \varphi \, dx.$$
We say that $D^\alpha f = g$ *in the weak sense.*
4. *Let* f *and* g *be in* $L^p_{\text{loc}}(\Omega)$. *We define* g *as the strong derivative of* f *with respect to* α *if, for every compact subset* $K \subset \Omega$, *there exists a sequence* $\{\varphi_i\} \subset C^{|\alpha|}(K)$ *such that*
$$\varphi_i \to f \quad \text{in } L^p(K) \quad \text{and} \quad D^\alpha \varphi_i \to g \quad \text{in } L^p(K).$$

Remark 1.1.2. *The idea behind the weak derivative is to transfer the differentiation operation from the function* f *(which may not be differentiable in the classical sense) to a smooth test function* ϕ *by means of integration by parts. This approach allows us to define a notion of "derivative" for more general functions. In contrast, the strong derivative requires approximating the function* f *by a sequence of smooth functions such that both the functions and their derivatives converge to* f *and its derivative* g, *respectively, in the* L^p*-norm.*

Example 1.1.2. *Suppose that* $1 \le p < \infty$. *Let* $\Omega = \mathbb{R}$ *and consider the function* $f(x) = e^x$.
1. Local Integrability Since $f(x) = e^x$ *is a continuous function on* $\mathbb{R}$, *it is bounded on any compact subset* $K = [a, b] \subset \Omega$. *The integral:*

$$\int_K |e^x|^p \, dx = \int_a^b e^{px} \, dx = \left[\frac{e^{px}}{p} \right]_a^b = \frac{e^{pb} - e^{pa}}{p} < \infty$$

Since the integral is finite for every compact $K \subset \Omega$, *then* $f \in L^p(K)$. *Thus,* $f \in L^p_{loc}(\Omega)$ *as per Definition 1.1.5 (1).*
2. Weak Derivative Let $g(x) = e^x$. *To verify that* g *is the weak derivative of* f *with respect to* x, *we use Definition 1.1.5 (3). For every test function* $\varphi \in C_c^\infty(\Omega)$:

$$\int_\Omega f D\varphi \, dx = \int_{-\infty}^{\infty} e^x \varphi'(x) \, dx$$

By using integration by parts, and since φ *has compact support (which means* φ *vanishes at* $\pm\infty$*):*

$$\int_{-\infty}^{\infty} e^x \varphi'(x) \, dx = \underbrace{[e^x \varphi(x)]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} e^x \varphi(x) \, dx = (-1) \int_\Omega g \varphi \, dx$$

Thus, $D^1 f = g$ in the weak sense.

3. Strong Derivative Since $f \in C^\infty(\Omega)$, for any compact subset $K \subset \Omega$, we can define a sequence $(\varphi_i) \subset C^\infty(K)$ such that $\varphi_i = f$ for all i. It follows that:

$$\varphi_i \to f \text{ in } L^p(K) \quad \text{and} \quad D\varphi_i \to g \text{ in } L^p(K)$$

Therefore, g is the strong derivative of f as per Definition 1.1.5 (4).

Theorem 1.1.1 (Hölder's inequality)**.** *Let $p, q \in \mathbb{R}$ such that $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and we have the inequality:*

$$\|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q}$$

which can be written as:

$$\int_\Omega |f(x)g(x)|\, d\mu \le \left(\int_\Omega |f(x)|^p\, d\mu\right)^{1/p} \left(\int_\Omega |g(x)|^q\, d\mu\right)^{1/q}$$

- Cauchy-Schwarz Inequality: In the case where $p = q = 2$, the inequality reduces to:

$$\int_\Omega |f(x)g(x)|\, d\mu \le \|f\|_{L^2} \|g\|_{L^2}$$

- The Limiting Case $(p = 1, q = \infty)$: If $f \in L^1(\Omega)$ and $g \in L^\infty(\Omega)$, then:

$$\int_\Omega |f(x)g(x)|\, d\mu \le \|f\|_{L^1} \|g\|_{L^\infty}$$

  where $\|g\|_{L^\infty} = \text{ess sup}_{x\in\Omega} |g(x)|$.

Example 1.1.3. *Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and consider the functions*

$$f(x) = x^{1/2}, \quad g(x) = x^{1/3}, \quad x \in [0,1].$$

We verify Hölder's inequality:

$$\int_0^1 |f(x)g(x)|\, dx \le \left(\int_0^1 |f(x)|^p\, dx\right)^{1/p} \left(\int_0^1 |g(x)|^q\, dx\right)^{1/q}.$$

Choose $p = 3$ and $q = \frac{3}{2}$. Then $\frac{1}{3} + \frac{2}{3} = 1$.

Compute the left-hand side (LHS):

$$\int_0^1 |f(x)g(x)|\, dx = \int_0^1 x^{1/2} x^{1/3}\, dx = \int_0^1 x^{5/6}\, dx = \frac{6}{11}.$$

Compute the right-hand side (RHS):

$$\left(\int_0^1 |f(x)|^3\, dx\right)^{1/3} = \left(\int_0^1 x^{3/2}\, dx\right)^{1/3} = \left(\frac{2}{5}\right)^{1/3},$$

$$\left(\int_0^1 |g(x)|^{3/2}\,dx\right)^{2/3} = \left(\int_0^1 x^{1/2}\,dx\right)^{2/3} = \left(\frac{2}{3}\right)^{2/3}.$$

Multiplying the factors:

$$\left(\frac{2}{5}\right)^{1/3}\left(\frac{2}{3}\right)^{2/3} \approx 0.658 \geq \frac{6}{11} \approx 0.545.$$

Hence, Hölder's inequality is satisfied:

$$\int_0^1 |f(x)g(x)|\,dx \leq \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p}\left(\int_0^1 |g(x)|^q\,dx\right)^{1/q}.$$

Theorem 1.1.2 (Minkowski's Inequality)**.** *Let $p \in \mathbb{R}$ with $1 \leq p < \infty$. If $f, g \in L^p(\Omega)$, then $f + g \in L^p(\Omega)$ and we have:*

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

This inequality can be expressed in terms of integrals as:

$$\left(\int_\Omega |f(x)+g(x)|^p\,d\mu\right)^{1/p} \leq \left(\int_\Omega |f(x)|^p\,d\mu\right)^{1/p} + \left(\int_\Omega |g(x)|^p\,d\mu\right)^{1/p}$$

Remark 1.1.3. *The Minkowski inequality shows that the function $\|\cdot\|_{L^p}$ satisfies the triangle inequality, which is a key property in proving that $L^p(\Omega)$ is a normed vector space.*

Weak Derivatives

Definition 1.1.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f, g \in L^1_{loc}(\Omega)$. We say that g is the weak partial derivative of f with respect to x_i if for every test function $\phi \in C_c^\infty(\Omega)$,*

$$\int_\Omega f(x)\,\frac{\partial\phi}{\partial x_i}(x)\,dx = -\int_\Omega g(x)\,\phi(x)\,dx.$$

We denote $g = \frac{\partial f}{\partial x_i}$ or $\partial_i f$ in the weak sense.

Example 1.1.4. *(Weak derivative of $|x|$) Let $\Omega = (-1,1) \subset \mathbb{R}$ and consider $f(x) = |x|$. For any $\phi \in C_c^\infty((-1,1))$,*

$$\int_{-1}^1 |x|\phi'(x)\,dx = -\int_{-1}^1 \operatorname{sgn}(x)\,\phi(x)\,dx.$$

Thus, the weak derivative of f is $f'(x) = \operatorname{sgn}(x)$.

Sobolev Space $W^{1,p}(\Omega)$

Definition 1.1.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined as*

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega) : \partial_i u \in L^p(\Omega) \text{ for } i = 1,\dots,n\right\},$$

where $\partial_i u$ denotes the weak partial derivative (see Definition 1.1.6). The norm on $W^{1,p}(\Omega)$ is given by

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{n} \|\partial_i u\|_{L^p(\Omega)}.$$

For $1 < p < \infty$, this norm makes $W^{1,p}(\Omega)$ a reflexive Banach space. For $p = 2$, we denote

$$H^1(\Omega) = W^{1,2}(\Omega),$$

which is a Hilbert space with the inner product

$$(u,v)_{H^1(\Omega)} = \int_\Omega u(x)v(x)\,dx + \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx.$$

Remark 1.1.4. *The space $W^{1,p}(\Omega)$ contains functions that may not be differentiable in the classical sense but possess weak derivatives. This makes it an ideal setting for studying weak solutions of PDEs.*

Example 1.1.5. *Let $\Omega = (0,1) \subset \mathbb{R}$ and consider $u(x) = x^2$. Then:*

- *$u \in L^2(0,1)$ because $\int_0^1 x^4\,dx = \frac{1}{5} < \infty$.*
- *$u'(x) = 2x \in L^2(0,1)$ because $\int_0^1 4x^2\,dx = \frac{4}{3} < \infty$.*

Thus, $u \in H^1(0,1)$.

Example 1.1.6. *Let $\Omega = \mathbb{R}^n$ and consider $u(x) = e^{-|x|^2}$. Then:*

- *$u \in L^2(\mathbb{R}^n)$.*
- *$\partial_i u(x) = -2x_i e^{-|x|^2} \in L^2(\mathbb{R}^n)$ for each $i = 1,\dots,n$.*

Thus, $u \in H^1(\mathbb{R}^n)$.

Example 1.1.7. *Let $\Omega = (0,1)$ and consider $u(x) = \sqrt{x}$. Then:*

- *$u \in L^2(0,1)$ because $\int_0^1 x\,dx = \frac{1}{2} < \infty$.*
- *$u'(x) = \frac{1}{2\sqrt{x}} \notin L^2(0,1)$ because $\int_0^1 \frac{1}{4x}\,dx = \infty$.*

Thus, $u \notin H^1(0,1)$.

Proposition 1.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then:*

1. *For each $1 \le p \le \infty$, $W^{1,p}(\Omega)$ is a Banach space.*
2. *For each $1 < p < \infty$, $W^{1,p}(\Omega)$ is reflexive.*
3. *For each $1 \le p < \infty$, $W^{1,p}(\Omega)$ is separable.*

The Space $H^1(\Omega)$

Definition 1.1.5. *For $p = 2$, the space $H^1(\Omega) = W^{1,2}(\Omega)$ is a Hilbert space. The inner product is given by*

$$(u, v)_{H^1(\Omega)} = \int_\Omega uv\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx.$$

The associated norm is

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Example 1.1.8. *In $H^1(0,1)$, the functions $u(x) = 1$ and $v(x) = x - \frac{1}{2}$ are orthogonal because*

$$\int_0^1 1 \cdot (x - \tfrac{1}{2})\,dx + \int_0^1 0 \cdot 1\,dx = 0.$$

The Space $W_0^{1,p}(\Omega)$

Definition 1.1.6. *The space $W_0^{1,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. In other words,*

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}}}.$$

Functions in $W_0^{1,p}(\Omega)$ vanish on the boundary $\partial\Omega$ in the sense of traces.
For $p = 2$, we denote

$$H_0^1(\Omega) = W_0^{1,2}(\Omega),$$

which is a closed subspace of the Hilbert space $H^1(\Omega)$.

Remark 1.1.5. *The space $C_c^\infty(\Omega)$ (smooth functions with compact support in Ω) is dense in $W_0^{1,p}(\Omega)$. This means that every function in $W_0^{1,p}(\Omega)$ can be approximated by a sequence of smooth functions that vanish near the boundary. This property is essential for extending results from smooth functions to all functions in $W_0^{1,p}(\Omega)$ by density arguments.*

Poincaré Inequality

Theorem 1.1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists a constant $C_P > 0$ (depending only on n and Ω) such that for every $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)}.$$

The constant C_P is called the Poincaré constant. For convex domains, one can take $C_P = \mathrm{diam}(\Omega)$.

Remark 1.1.6. *The Poincaré inequality is false for functions that do not vanish on the boundary. For example, the constant function* $u(x) = 1$ *on* $\Omega = (0,1)$ *satisfies* $\|\nabla u\|_{L^2} = 0$ *but* $\|u\|_{L^2} = 1$*, so no such constant* C_P *exists.*

Proof:

We prove the inequality for $u \in C_c^\infty(\Omega)$; the result for $H_0^1(\Omega)$ follows by density (since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$).

Let $\Omega \subset \mathbb{R}^n$ be bounded. Without loss of generality, we assume that Ω is contained in the strip

$$\Omega \subset \{x \in \mathbb{R}^n : 0 < x_1 < d\},$$

where $d = \operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y|$.

For any $u \in C_c^\infty(\Omega)$, we have $u(x) = 0$ for x outside a compact subset of Ω. In particular, $u(x) = 0$ when $x_1 \le 0$ or $x_1 \ge d$. For any $x \in \Omega$, we can write

$$u(x) = \int_0^{x_1} \frac{\partial u}{\partial x_1}(t, x_2, \dots, x_n)\, dt.$$

Apply the Cauchy-Schwarz inequality

$$|u(x)|^2 \le \left(\int_0^{x_1} 1^2\, dt\right)\left(\int_0^{x_1} \left|\frac{\partial u}{\partial x_1}(t, x_2, \dots, x_n)\right|^2 dt\right).$$

Since $0 \le x_1 \le d$, we have $\int_0^{x_1} 1^2\, dt = x_1 \le d$. Hence,

$$|u(x)|^2 \le d \int_0^d \left|\frac{\partial u}{\partial x_1}(t, x_2, \dots, x_n)\right|^2 dt.$$

Integrate over $x \in \Omega$

$$\int_\Omega |u(x)|^2\, dx \le d \int_\Omega \int_0^d \left|\frac{\partial u}{\partial x_1}(t, x_2, \dots, x_n)\right|^2 dt\, dx.$$

Change the order of integration (Fubini's theorem)

$$\int_\Omega |u(x)|^2\, dx \le d^2 \int_\Omega \left|\frac{\partial u}{\partial x_1}(x)\right|^2 dx.$$

Bound by the full gradient

$$\int_\Omega |u(x)|^2\, dx \le d^2 \int_\Omega |\nabla u(x)|^2\, dx.$$

Taking square roots, we get

$$\|u\|_{L^2(\Omega)} \le d \|\nabla u\|_{L^2(\Omega)}.$$

Thus, the Poincaré inequality holds with $C_P = d = \mathrm{diam}(\Omega)$.

Extension to $H_0^1(\Omega)$

For any $u \in H_0^1(\Omega)$, there exists a sequence $\{u_k\} \subset C_c^\infty(\Omega)$ such that $u_k \to u$ in $H_0^1(\Omega)$. Applying the inequality to u_k and taking the limit as $k \to \infty$ yields the result for u. ■

Corollary 1.1.1. *As a consequence of the Poincaré inequality, the norm*

$$\|u\|_{H_0^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\right)^{1/2}$$

is equivalent to the simpler norm

$$\|u\| = \|\nabla u\|_{L^2(\Omega)}.$$

Indeed, from the Poincaré inequality,

$$\|u\|_{L^2} \leq C_P \|\nabla u\|_{L^2},$$

so

$$\|\nabla u\|_{L^2} \leq \|u\|_{H_0^1} \leq (1 + C_P^2)^{1/2} \|\nabla u\|_{L^2}.$$

Theorem 1.1.4. *L^p is a vector space and $\|\cdot\|_p$ is a norm for any p, $1 \leq p \leq \infty$.*

Proof. For $1 < p < \infty$ and let $f, g \in L^p$. We have

$$|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p \leq 2^{p-1}(|f(x)|^p + |g(x)|^p).$$

So that, $f + g \in L^p$. Whereas,

$$\|f + g\|_p^p = \int |f + g|^p \leq \int |f + g|^{p-1}|f| + \int |f + g|^{p-1}|g|.$$

But $|f + g|^{p-1} \in L^{p'}$, and by Hölder's inequality we obtain

$$\|f + g\|_p^p \leq \|f + g\|_p^{p-1}(\|f\|_p + \|g\|_p),$$

i.e.,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

□

Theorem 1.1.5 (Fischer-Riesz)**.** *L^p is a Banach space for any p, $1 \leq p \leq \infty$.*

Proposition 1.1.2. *If A is any set and $0 < p < q \leq \infty$, then*

$$\ell^p(A) \subset \ell^q(A) \quad \textit{and} \quad \|f\|_q \leq \|f\|_p.$$

1.2 The distribution function and Weak Lebesgue spaces

The distribution function

Definition 1.2.1. *For f a measurable function onΩ the distribution function of f is the function D_f defined on $[0,\infty)$ as follows:*

$$D_f(\lambda) = \mu(\{x \in \Omega : |f(x)| > \lambda\}).$$

The distribution function D_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function.

Proposition 1.2.1. *Let f and g be measurable functions on Ω. Then for all $\alpha, \beta > 0$ we have:*

1. *$|g| \leq |f|$ μ-a.e. implies that $D_g \leq D_f$;*
2. *$D_{cf}(\alpha) = D_f(\alpha/|c|)$, for all $c \in \mathbb{R} \setminus \{0\}$;*
3. *$D_{f+g}(\alpha+\beta) \leq D_f(\alpha) + D_g(\beta)$;*
4. *$D_{fg}(\alpha\beta) \leq D_f(\alpha) + D_g(\beta)$.*

Proposition 1.2.2. *Let Ω be a σ-finite measure space. Then for $f \in L^p(\Omega)$, $0 < p < \infty$, we have*

$$\|f\|_{L^p}^p = p\int_0^\infty \lambda^{p-1} D_f(\lambda)\, d\lambda.$$

Remark 1.2.1. *For any increasing continuously differentiable function φ on $[0,\infty)$ with $\varphi(0) = 0$ and every measurable function f on E with $\varphi(|f|)$ integrable on X, we have*

$$\int_X \varphi(|f|)\, d\mu = \int_0^\infty \varphi'(\lambda) D_f(\lambda)\, d\lambda.$$

Proof.

$$\begin{aligned} p\int_0^\infty \lambda^{p-1} D_f(\lambda)\, d\lambda &= p\int_0^\infty \lambda^{p-1} \int_\Omega \chi_{\{x:|f(x)|>\lambda\}}\, d\mu(x)\, d\lambda \\ &= \int_X \int_0^{|f(x)|} p\lambda^{p-1}\, d\lambda\, d\mu(x) \\ &= \int_X |f(x)|^p\, d\mu(x) \\ &= \|f\|_{L^p}^p. \end{aligned}$$

Weak Lebesgue spaces

Definition 1.2.2. *For $0 < p < \infty$, the weak $L^p(\Omega)$ space is defined as*

$$\|f\|_{L^{p,\infty}} = \inf\left\{C > 0 : D_f(\lambda) \leq \frac{C^p}{\lambda^p},\ \lambda > 0\right\} < \infty,$$

or equivalently,

$$\|f\|_{L^{p,\infty}} = \sup_{\gamma>0} \gamma D_f(\gamma)^{1/p} < \infty.$$

Remarks 1.2.1.

- *The weak L^p spaces are denoted by $L^{p,\infty}(\Omega)$. Two functions in $L^{p,\infty}(\Omega)$ are considered equal if they are equal μ-a.e. The notation $L^{p,\infty}(\mathbb{R}^n)$ is reserved for $L^{p,\infty}(\mathbb{R}^n, |\cdot|)$.*

- *Using Proposition 1.2.1 (2), we can easily show that*

$$\|kf\|_{L^{p,\infty}} = |k| \, \|f\|_{L^{p,\infty}}$$

for any complex constant k. The analogue of (1.2.1) is

$$\|f+g\|_{L^{p,\infty}} \leq c_p(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}}),$$

where $c_p = \max(2, 2^{1/p})$, a fact that follows from Proposition 1.2.1 (3), taking both α and β equal to $\alpha/2$.

- *We also have*

$$\|f\|_{L^{p,\infty}(\Omega)} = 0 \implies f = 0 \quad \mu\text{-a.e.}$$

In view of the above properties, $L^{p,\infty}$ is a quasi-normed linear space for $0 < p < \infty$. The weak L^p spaces are larger than the usual L^p spaces.

Remark 1.2.2. *For any $0 < p < \infty$ and any $f \in L^p(\Omega)$, we have*

$$\|f\|_{L^{p,\infty}} \leq \|f\|_{L^p}.$$

Thus, the embedding $L^p(\Omega) \hookrightarrow L^{p,\infty}(\Omega)$ holds.

Example 1.2.1. *We put*

$$f(x) = |x|^{-1/p}.$$

Distribution Function:

$$\begin{aligned} D_f(\lambda) &= \mu\{x \in \mathbb{R} : |x|^{-1/p} > \lambda\} \\ &= \mu\{x \in \mathbb{R} : |x| < \lambda^{-p}\} \\ &= 2\lambda^{-p}. \end{aligned}$$

Weak L^p Norm:

$$\|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda \left(2\lambda^{-p}\right)^{1/p} = \sup_{\lambda>0} \lambda \cdot 2^{1/p} \lambda^{-1} = 2.$$

Hence,

$$f \in L^{p,\infty}(\mathbb{R}).$$

Strong L^p Norm:

$$\|f\|_{L^p}^p = \int_{-\infty}^{\infty} |f(x)|^p \, dx = \int_{-\infty}^{\infty} \frac{1}{|x|} \, dx = \infty,$$

which implies

$$f \notin L^p(\mathbb{R}).$$

This example demonstrates clearly that the inclusion

$$L^p(\mathbb{R}) \subset L^{p,\infty}(\mathbb{R})$$

is strict, meaning that the weak space is strictly larger than the original space. The function $|x|^{-1/p}$ *serves as a canonical example of a "borderline" function that belongs to the weak space but not to the strong* L^p *space.*

1.3 Morrey spaces

In this section, we define Morrey spaces and state their fondamental properties, along with some examples to help us understand how to use them.[12]

Morry norms

Definition 1.3.1 (Morrey Space $M_q^p(\mathbb{R}^n)$)**.** *Let* $0 < q \leq p \leq \infty$*. For an* $L^q_{loc}(\mathbb{R}^n)$*-function* f*, its (classical) Morrey norm is defined by*

$$\|f\|_{M_q^p} \equiv \|f\|_{BM_q^p} := \sup_{(x,r)\in\mathbb{R}^{n+1}_+} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}.$$

If there is a need to stress the dimension we work in, we write $\|f\|_{M_q^p(\mathbb{R}^n)}$ *instead of* $\|f\|_{M_q^p}$*.*
The (classical) Morrey space $M_q^p(\mathbb{R}^n)$ *is the set of all* $f \in L^q_{loc}(\mathbb{R}^n)$ *for which the norm* $\|f\|_{M_q^p}$ *is finite.*

Corollary 1.3.1. *If* $0 < q \leq p = \infty$*, then the Lebesgue differentiation theorem shows that*

$$M_q^p(\mathbb{R}^n) = L^\infty(\mathbb{R}^n),$$

so we exclude this case.

Proposition 1.3.1. *A function* f *belongs to* $M_q^p(\mathbb{R}^n)$ *if and only if there exists a constant* $D \geq 0$ *such that*

$$\|f\|_{L^q(B(x,r))} \leq D|B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \quad \text{for all } x \in \mathbb{R}^n \text{ and } r > 0.$$

The smallest such constant D *in this inequality is precisely* $\|f\|_{M_q^p}$*.*

Proposition 1.3.2. *The space* $\mathcal{M}_q^p(\mathbb{R}^n)$ *is trivial in the following sense:*
Let $x_0 \in \mathbb{R}^n$ *be fixed. If* $f \in L^0(\mathbb{R}^n)$ *satisfies*

$$A \equiv \sup_{r>0} |B(x_0,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x_0,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}} < \infty$$

for some $0 < p < q < \infty$*, then* $f(x) = 0$ *for almost all* $x \in \mathbb{R}^n$*.*
In this negative assertion, we can freeze the point x_0 *instead of allowing it to vary among all points in* $\mathbb{R}^n$*.*

Proof. From the definition of A,

$$\left(\int_{B(x_0,r)} |f(y)|^q \, dy\right)^{\frac{1}{q}} \leq |B(x_0,r)|^{\frac{1}{q}-\frac{1}{p}} A.$$

Letting $r \to \infty$, we have $\|f\|_{L^q} = 0$. Thus, $f(x) = 0$ for almost all $x \in \mathbb{R}^n$.

A quasi-norm over a linear space X enjoys positivity, homogeneity, and the quasi-triangle inequality: for some $\alpha \geq 1$,

$$\|f+g\|_X \leq \alpha(\|f\|_X + \|g\|_X), \quad (f, g \in X).$$

We explain what happens if p and/or q is less than 1.

Remark 1.3.1. *It can happen that p and/or q is less than 1. In this case, we formally define the space* $\mathcal{M}_q^p(\mathbb{R}^n)$ *by (1.9) to obtain a quasi-Banach space, where the triangle inequality is replaced by the quasi-triangle inequality:*

$$\|f+g\|_{\mathcal{M}_q^p} \leq 2^{\max\left(\frac{1}{q}-1,0\right)} \left(\|f\|_{\mathcal{M}_q^p} + \|g\|_{\mathcal{M}_q^p}\right) \tag{1.1}$$

for all $f, g \in L^0(\mathbb{R}^n)$.

Note that (1.1) *follows from the triangle inequality immediately when* $1 \leq q < \infty$. *Meanwhile, when* $0 < q < 1$, (1.1) *follows from the inequality*

$$(a+b)^q \leq 2^{\max(q-1,0)}(a^q + b^q), \quad \textit{for all } a, b \geq 0.$$

As the following theorem shows, we can say that Lebesgue spaces are realized as a special case of Morrey spaces.

Theorem 1.3.1. *For* $0 < p < \infty$,

$$\mathcal{M}_p^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \quad \textit{with coincidence of norms.}$$

Definition 1.3.2. *(Morrey Space* $L^{1,\gamma}(\Omega)$*).*

Let $\Omega \subset \mathbb{R}^n$ *be a bounded open set and let* $0 \leq \gamma \leq n$. *The Morrey space* $L^{1,\gamma}(\Omega)$ *is defined as the set of all functions* $f \in L^1(\Omega)$ *such that*

$$\|f\|_{L^{1,\gamma}(\Omega)} = \sup_{x\in\Omega,\, r>0} \frac{1}{r^\gamma} \int_{\Omega\cap B(x,r)} |f(y)| \, dy < \infty.$$

Remark 1.3.2.

- *When* $\gamma = 0$, $L^{1,0}(\Omega) = L^1(\Omega)$.
- *When* $\gamma = n$, $L^{1,n}(\Omega) = L^\infty(\Omega)$.
- *For* $n-2 < \gamma < n$, *the space* $L^{1,\gamma}(\Omega)$ *is used in the study of elliptic PDEs (see Chapter 3).*

Example 1.3.1. *Let* $\Omega = B(0,1) \subset \mathbb{R}^3$ *and define*

$$f(x) = |x|^{-1/2}.$$

Then for $x = 0$ *and* $r \le 1$*,*

$$\int_{B(0,r)} |f(y)|\,dy = 4\pi \int_0^r \rho^{-1/2}\rho^2\,d\rho = \frac{8\pi}{5} r^{5/2}.$$

Thus,

$$\frac{1}{r^{5/2}} \int_{B(0,r)} |f(y)|\,dy = \frac{8\pi}{5} < \infty,$$

so $f \in L^{1,5/2}(\Omega)$.

Examples of functions in Morrey spaces

We present a concrete example to illustrate the type of functions belong to Morrey spaces.

Example 1 Let $0 < q \le p < \infty$ and $n \in \mathbb{N}$. We consider the power function

$$f(x) = |x|^{-\alpha}, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

The membership of this function, or its truncated versions, in the Morrey space $M_p^q(\mathbb{R}^n)$ is characterized as follows:

The function f belongs to $M_p^q(\mathbb{R}^n)$ if and only if

$$\alpha = \frac{n}{p}.$$

This can be verified by observing that the supremum in the Morrey norm becomes independent of the radius r.

Let $f_1(x) = |x|^{-\alpha}\chi_{B(0,1)}(x)$ be the function restricted to the unit ball. Then

$$f_1 \in M_p^q(\mathbb{R}^n) \quad \text{if and only if} \quad \alpha \le \frac{n}{p}.$$

Let $f_2(x) = |x|^{-\alpha}\chi_{\mathbb{R}^n \setminus B(0,1)}(x)$ be the function restricted to the exterior of the unit ball. Then

$$f_2 \in M_p^q(\mathbb{R}^n) \quad \text{if and only if} \quad \alpha \ge \frac{n}{p}.$$

Consequently, for a general function

$$f(x) = |x|^{-\alpha},$$

we have

$$f \in M_p^q(\mathbb{R}^n) \quad \text{if and only if} \quad \alpha = \frac{n}{p}.$$

The role of the parameters

As Morrey spaces are defined by two parameters p and q, it is essential to understand their respective roles. Intuitively, q serves as the index of local integrability because $M^p_q(\mathbb{R}^n) \subset L^q_{\text{loc}}(\mathbb{R}^n)$, while p govers the scaling behavior.

Theorem 1.3.2. *For all $0 < q \leq p < \infty$, we have the following scaling law:*

$$\|f(t\cdot)\|_{M^p_q} = t^{-\frac{n}{p}} \|f\|_{M^p_q},$$

for all $f \in M^p_q(\mathbb{R}^n)$ and $t > 0$.

1.4 Stummel classes

Definition 1.4.1. *Let $0 < \alpha < n$. The Stummel class $S_\alpha = S_\alpha(\mathbb{R}^n)$ is defined by*

$$S_\alpha := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \eta^f_\alpha(r) \to 0 \text{ as } r \to 0 \right\},$$

where

$$\eta^f_\alpha(r) := \sup_{x \in \mathbb{R}^n} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, dy, \qquad r > 0.$$

Example 1.4.1. *Let $f(x) = 1$, $x \in \mathbb{R}^n$, and let $0 < \alpha < n$.*
Then

$$\int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy = \int_{|x-y|<r} \frac{1}{|x-y|^{n-\alpha}} dy.$$

Using polar coordinates, we obtain

$$= C \int_0^r t^{\alpha-1} dt = C r^\alpha.$$

Since $r^\alpha \to 0$ as $r \to 0$, we conclude that

$$f \in S_\alpha.$$

Definition 1.4.2. *(Stummel class on a bounded domain). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $0 < \alpha < n$. The Stummel class $\tilde{S}_\alpha(\Omega)$ is defined as the set of all functions $f \in L^1(\Omega)$ such that*

$$\tilde{\eta}^f_\alpha(r) = \sup_{x \in \Omega} \int_{\Omega(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy < \infty \quad \text{for all } r > 0,$$

where $\Omega(x,r) = \Omega \cap B(x,r)$. This definition is used in Chapter 3 for the application in Stummel classes.

Example 1.4.2. *Let $\Omega = B(0,1) \subset \mathbb{R}^3$ and define $f(x) = |x|^{-1}$. Then $f \in \tilde{S}_2(\Omega)$. Indeed, for $x = 0$ and $r \le 1$,*

$$\tilde{\eta}_2^f(r) = \int_{|y|<r} \frac{|f(y)|}{|y|^{3-2}} dy = \int_{|y|<r} \frac{1}{|y| \cdot |y|} dy = \int_{|y|<r} \frac{1}{|y|^2} dy = 4\pi \int_0^r d\rho = 4\pi r.$$

Since $4\pi r \to 0$ as $r \to 0$, we have $f \in \tilde{S}_2(\Omega)$. This example is used in Chapter 3 for the application in Stummel classes.

The Generalized Stummel classes

Definition 1.4.3. *For $1 \le p < \infty$, we define the generalized Stummel p-class $S_{p,\Psi} = S_{p,\Psi}(\mathbb{R}^n)$ by*

$$S_{p,\Psi} := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \eta^f_{p,\Psi}(r) \to 0 \text{ as } r \to 0 \right\},$$

where

$$\eta^f_{p,\Psi}(r) := \sup_{x \in \mathbb{R}^n} \left(\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n} \, dy \right)^{\frac{1}{p}}, \qquad r > 0.$$

We call $\eta^f_{p,\Psi}$ the Stummel p-modulus *of f. Observe that the Stummel p-modulus is non-decreasing on $(0,\infty)$.*

For $p = 1$, we have $S_{1,\Psi} := S_\Psi$ — the generalized Stummel class.

For $\Psi(t) := t^\alpha$ $(0 < \alpha < n)$, we write $S_{p,\alpha}$ instead of $S_{p,\Psi}$ and $\eta_{p,\alpha}$ instead of $\eta_{p,\Psi}$. Observe that $S_{1,\alpha} := S_\alpha$ — the Stummel class.

Proposition 1.4.1. *Let $1 \le p < \infty$ and let Ψ be a given function. If $f \in S_{p,\Psi}$, then the function $\eta_{p,\Psi} f$ is continuous on $(0,\infty)$.*

Proof Let $\{r_k\}$ be a sequence in $(0,\infty)$ such that $r_k \to r \in (0,\infty)$, and let $x \in \mathbb{R}^n$. Choose $r^* > 0$ such that $r, r_k \le r^*$ for every $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, define

$$g_k(y) := \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} \chi_{B(x,r_k)}(y),$$

and

$$g(y) := \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} \chi_{B(x,r)}(y),$$

for $y \in B(x, r^*)$.

Then $\{g_k\}$ is a sequence of nonnegative measurable functions on $B(x,r^*)$, and $g_k \to g$ almost everywhere on $B(x,r^*)$. By the Dominated Convergence Theorem, we obtain

$$\int_{|y-x|<r^*} g_k(y) \, dy \longrightarrow \int_{|y-x|<r^*} g(y) \, dy.$$

Therefore,

$$\left(\int_{|y-x|<r_k} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} \, dy \right)^{\frac{1}{p}} \longrightarrow \left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} \, dy \right)^{\frac{1}{p}}. \tag{1.2}$$

Let $\varepsilon > 0$. By (1.2), there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n}\, dy\right)^{\frac{1}{p}} - \varepsilon < \left(\int_{|y-x|<r_k} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n}\, dy\right)^{\frac{1}{p}} < \left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n}\, dy\right)^{\frac{1}{p}} + \varepsilon.$$

Since $x \in \mathbb{R}^n$ is arbitrary, we conclude that

$$\eta_{p,\Psi} f(r) - \varepsilon \leq \eta_{p,\Psi} f(r_k) \leq \eta_{p,\Psi} f(r) + \varepsilon.$$

Hence, $\eta_{p,\Psi} f(r_k) \to \eta_{p,\Psi} f(r)$ whenever $r_k \to r$ in $(0,\infty)$. This proves that $\eta_{p,\Psi} f$ is continuous on $(0,\infty)$.

1.5 Vanishing Morrey spaces

Definition 1.5.1 (Classical Vanishing Morrey Space). *Let $1 \leq p < \infty$ and $0 \leq \lambda < n$. The classical vanishing Morrey space at the origin, denoted by $V_0 L^{p,\lambda}(\mathbb{R}^n)$, is defined by*

$$V_0 L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^{p,\lambda}(\mathbb{R}^n) : \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p\, dy = 0 \right\}.$$

In other words, a function f belongs to $V_0 L^{p,\lambda}(\mathbb{R}^n)$ if and only if its Morrey functional vanishes as the radius r tends to zero, uniformly with respect to the center $x \in \mathbb{R}^n$.

Example 1.5.1. *Let $f(x) = 1$, $x \in \mathbb{R}^n$, and let $0 < \lambda < n$, $1 \leq p < \infty$.*
Compute the Morrey norm on balls $B(x,r)$:

$$\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p\, dy = \frac{1}{r^\lambda} \int_{B(x,r)} 1\, dy = \frac{|B(x,r)|}{r^\lambda}.$$

Since $|B(x,r)| = C_n r^n$, we have

$$\frac{|B(x,r)|}{r^\lambda} = C_n r^{n-\lambda}.$$

Because $n - \lambda > 0$, it follows that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p\, dy = 0.$$

Therefore,

$$f \in VL^{p,\lambda}(\mathbb{R}^n).$$

Remark 1.5.1. *The above vanishing condition is strictly stronger than the mere membership in the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$. Indeed, while the definition of $L^{p,\lambda}$ only requires the corresponding supremum to be finite, the vanishing Morrey space additionally imposes that this quantity converges to zero as $r \to 0$.*
This space is often referred to in the literature as the vanishing Morrey space at the origin.

Definition 1.5.2 (Vanishing Morrey Space at Infinity)**.** *The vanishing Morrey space at infinity, denoted by* $V_\infty M_q^p(\mathbb{R}^n)$*, is defined as the set of functions* $f \in M_q^p(\mathbb{R}^n)$ *that satisfy*

$$\lim_{r\to\infty} m(f,p,q;r) = 0.$$

Definition 1.5.3 (Vanishing at Infinity Morrey Space)**.** *The vanishing at infinity Morrey space, denoted by* $VM_q^p(\mathbb{R}^n)$*, is defined as the set of functions* $f \in M_q^p(\mathbb{R}^n)$ *that satisfy*

$$\lim_{r\to\infty} \left\| f\,\chi_{\mathbb{R}^n\setminus B(0,r)} \right\|_{M_q^p(\mathbb{R}^n)} = 0,$$

where $\chi_{\mathbb{R}^n\setminus B(0,r)}$ *is the characteristic function of the complement of the ball* $B(0,r)$*.*

Definition 1.5.4. *Let* $1 \le p < \infty$ *and* $\varphi \in \Phi$*. We set*

$$V_{0,\infty}^{(*)} L_{p,\varphi}(\mathbb{R}^n) := V_0 L_{p,\varphi}(\mathbb{R}^n) \cap V_\infty L_{p,\varphi}(\mathbb{R}^n) \cap V^{(*)} L_{p,\varphi}(\mathbb{R}^n).$$

Remark 1.5.2. *It can be shown that all the vanishing subsets* $L_{p,\varphi}(\mathbb{R}^n)$*,* $V_0 L_{p,\varphi}(\mathbb{R}^n)$*,* $V_\infty L_{p,\varphi}(\mathbb{R}^n)$*, and* $V^{(*)} L_{p,\varphi}(\mathbb{R}^n)$ *(and consequently,* $V_{0,\infty}^{(*)} L_{p,\varphi}(\mathbb{R}^n)$*) are closed in* $L_{p,\varphi}(\mathbb{R}^n)$*.*

Definition 1.5.5. *Let* φ *be a positive function. The* ***generalized Morrey space*** $M^{p,\varphi}(\mathbb{R}^n)$ *is defined as the set of functions* f *that satisfy*

$$\|f\|_{M^{p,\varphi}} := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\|f\|_{L^p(B(x,r))}}{\varphi(x,r)} < \infty.$$

When $\varphi(x,r) = r^{\frac{n}{p}-\frac{n}{q}}$*, we recover the* ***classical Morrey space*** $M^{p,q}(\mathbb{R}^n)$*.*

Definition 1.5.6. *The* ***vanishing generalized Morrey space*** $VM^{p,\varphi}(\mathbb{R}^n)$ *is defined as the set of functions* $f \in M^{p,\varphi}(\mathbb{R}^n)$ *such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{\|f\|_{L^p(B(x,r))}}{\varphi(x,r)} = 0.$$

Similarly, we define $VM_0^{p,\varphi}(\mathbb{R}^n)$ *and* $VM_\infty^{p,\varphi}(\mathbb{R}^n)$ *by considering the limits as* $r \to 0$ *and* $r \to \infty$*, respectively.*

1.6 Tools for Differential Equations

In this section, we recall some fundamental tools used in the study of partial differential equations (PDEs). These tools are essential for formulating weak solutions and proving existence and uniqueness results.

Fubini's Theorem.

Theorem 1.6.1. *Let* X *and* Y *be* σ*-finite measure spaces and let* $f : X \times Y \to \mathbb{R}$ *be a measurable function. If* $f \ge 0$ *or if* $\int_{X\times Y} |f| < \infty$*, then*

$$\int_X \int_Y f(x,y)\,dy\,dx = \int_Y \int_X f(x,y)\,dx\,dy = \int_{X\times Y} f(x,y)\,d(x,y).$$

Lax-Milgram Lemma

The Lax-Milgram lemma is a fundamental tool for proving existence and uniqueness of weak solutions to elliptic PDEs.

Definition 1.6.1. *Let H be a Hilbert space with norm $|\cdot|$. A bilinear form $A\colon H\times H\to\mathbb{R}$ is called:*

- *Continuous if there exists a constant $C>0$ such that*

$$|A(u,v)|\le C|u||v| \quad \forall u,v\in H.$$

- *Coercive (or H-elliptic) if there exists a constant $\alpha>0$ such that*

$$A(u,u)\ge \alpha|u|^2 \quad \forall u\in H.$$

Example 1.6.1. *Let $H=W_0^{1,2}(\Omega)$ and define*

$$A(u,v)=\int_\Omega \nabla u\cdot\nabla v\,dx.$$

By the Cauchy-Schwarz inequality,

$$|A(u,v)|\le|\nabla u|_{L^2}|\nabla v|_{L^2}\le|u|_H|v|_H,$$

so A is continuous.
By the Poincaré inequality,

$$A(u,u)=|\nabla u|_{L^2}^2\ge\frac{1}{C_P^2+1}|u|_H^2,$$

so A is coercive.

Lemma 1.6.1. *Let H be a Hilbert space, let $A: H\times H\to\mathbb{R}$ be a continuous and coercive bilinear form, and let $F: H\to\mathbb{R}$ be a continuous linear functional. Then there exists a unique $u\in H$ such that*

$$A(u,v)=F(v) \quad \forall v\in H.$$

Stampacchia's Lemma

Stampacchia's lemma is a generalization of the Lax-Milgram lemma to the case of non-symmetric bilinear forms. It is particularly useful for studying nonlinear problems.

Lemma 1.6.2. *Let H be a Hilbert space, and let $A\colon H\times H\to\mathbb{R}$ be a bilinear form that is continuous and linear in the second variable. Assume that there exist constants $K_1>0$ and $K_2>0$ such that:*

- $|A(v_1,\phi)-A(v_2,\phi)|\le K_1|v_1-v_2||\phi| \quad \forall v_1,v_2,\phi\in H,$

- $A(v_1, v_1 - v_2) - A(v_2, v_1 - v_2) \geq K_2|v_1 - v_2|^2 \quad \forall v_1, v_2 \in H$.

Then for every continuous linear functional $F \in H^{-1}$*, there exists a unique* $v \in H$ *such that*

$$F(\phi) = A(v, \phi) \quad \forall \phi \in H.$$

Example 1.6.2. *Consider the bilinear form*

$$A(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \beta \cdot \nabla u \, v \, dx,$$

where β *is a bounded vector field. This form is not symmetric, but it satisfies the hypotheses of Stampacchia's lemma for appropriate* β.

Weighted Embedding Theorem

Theorem 1.6.2. *Let* $f \in L^{1,\gamma}(\Omega)$ *with* $n-2 < \gamma < n$. *Then there exists a constant* $C = C(n, \gamma, \Omega) > 0$ *such that for every* $\phi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega |f(x)\phi(x)| dx \leq C \|f\|_{L^{1,\gamma}(\Omega)} \|\phi\|_{W_0^{1,2}(\Omega)}.$$

Chapter 2

Inclusions between Morrey spaces and Stummel classes

The aim of this chapter is to investigate the inclusion relations between these spaces. This chapter is organized into two main parts. In the first part (Section 2.1), we study the inclusion properties between Morrey spaces with different indices.

The second part (Section 2.2) is dedicated to the connection between weak Lebesgue spaces and Stummel classes. These results, based on recent works [2, 11, 14, 15, 16], reveal a deep relationship between distributional properties and weighted local integrability.

We begin with the study of Morrey spaces in the following section.[5],[7],[6],[9],[10],[12] and [13]

2.1 Inclusion between Morrey spaces

Monotonicity with Respect to q

Proposition 2.1.1. *Let* $0 < q_1 \leq q_2 \leq p < \infty$. *Then*

$$M^p_{q_2}(\mathbb{R}^n) \subset M^p_{q_1}(\mathbb{R}^n),$$

and for every $f \in M^p_{q_2}(\mathbb{R}^n)$, *we have*

$$\|f\|_{M^p_{q_1}} \leq \|f\|_{M^p_{q_2}}.$$

Proof Since $f \in M^p_{q_2}$, there exists a constant $C = \|f\|_{M^p_{q_2}}$ such that for every ball $B = B(x,r)$,

$$\|f\|_{L^{q_2}(B)} \leq C|B|^{\frac{1}{q_2}-\frac{1}{p}}.$$

Using Hölder's inequality (with exponents $\frac{q_2}{q_1}$ and $\frac{q_2}{q_2-q_1}$), we obtain

$$\|f\|_{L^{q_1}(B)} \leq |B|^{\frac{1}{q_1}-\frac{1}{q_2}}\|f\|_{L^{q_2}(B)}.$$

Combining these estimates yields

$$\|f\|_{L^{q_1}(B)} \leq C|B|^{\frac{1}{q_1}-\frac{1}{q_2}} \cdot |B|^{\frac{1}{q_2}-\frac{1}{p}} = C|B|^{\frac{1}{q_1}-\frac{1}{p}}.$$

Multiplying both sides by $|B|^{\frac{1}{p}-\frac{1}{q_1}}$ and taking the supremum over all balls, we get

$$\|f\|_{M^p_{q_1}} \leq C = \|f\|_{M^p_{q_2}}.$$

Hence $f \in M^p_{q_1}$. □

Example 2.1.1. *In* $\mathbb{R}^3$*, consider* $p = 4$*. Then we have the following chain of inclusions:*

$$M_4^4(\mathbb{R}^3) \subset M_3^4(\mathbb{R}^3) \subset M_2^4(\mathbb{R}^3) \subset M_1^4(\mathbb{R}^3).$$

The function $f(x) = |x|^{-3/4}$ *belongs to* $M_2^4(\mathbb{R}^3)$ *but not to* $M_4^4(\mathbb{R}^3) = L^4(\mathbb{R}^3)$*, showing that the inclusion is strict.*

Example 2.1.2. *Let* $n \geq 1$ *and consider the function*

$$f(x) = |x|^{-\frac{n}{q_2}} \chi_{\{|x| \leq 1\}}(x),$$

where $0 < q_1 < q_2 \leq p < \infty$ *and* $\chi_{\{|x|\leq 1\}}$ *is the characteristic function of the unit ball.*
$f \in M_{q_1}^p(\mathbb{R}^n)$ *but* $f \notin M_{q_2}^p(\mathbb{R}^n)$*.*

Proof: For $0 < r \leq 1$, we have

$$\int_{B(0,r)} |f(x)|^{q_1}\, dx = \int_{B(0,r)} |x|^{-\frac{nq_1}{q_2}}\, dx.$$

Since $\frac{nq_1}{q_2} < n$, this integral is finite and behaves like $r^{n-\frac{nq_1}{q_2}}$. Hence

$$\|f\|_{L^{q_1}(B(0,r))} \sim r^{\frac{n}{q_1}-\frac{n}{q_2}}.$$

Multiplying by $|B(0,r)|^{\frac{1}{p}-\frac{1}{q_1}} \sim r^{\frac{n}{p}-\frac{n}{q_1}}$, we get

$$\|f\|_{M_{q_1}^p} < \infty.$$

On the other hand,

$$\int_{B(0,r)} |f(x)|^{q_2}\, dx = \int_{B(0,r)} |x|^{-n}\, dx = \infty,$$

so $f \notin M_{q_2}^p(\mathbb{R}^n)$.
This example shows that the inclusion

$$M_{q_2}^p(\mathbb{R}^n) \subset M_{q_1}^p(\mathbb{R}^n)$$

is strict, and clearly illustrates the effect of changing the exponent q in Morrey spaces.

Remark 2.1.1. *Let* $0 < q_1 < q_2 \leq p < \infty$*. Then*

$$M_{q_2}^p(\mathbb{R}^n) \subset M_{q_1}^p(\mathbb{R}^n),$$

i.e., the Morrey space decreases as the exponent q *increases.*

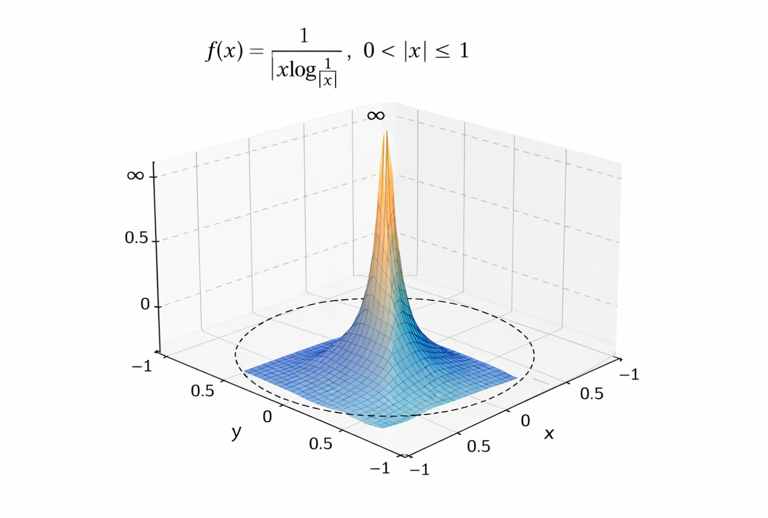

Figure 2.1: Visual representation of the monotonicity: as q increases, the space M_q^p decreases.

Monotonicity with Respect to p

We now examine the behavior of Morrey spaces with respect to the parameter p.[7, 6]

Proposition 2.1.2. *Let* $0 < q \leq p_1 \leq p_2 < \infty$. *Then*

$$M_q^{p_1}(\mathbb{R}^n) \subset M_q^{p_2}(\mathbb{R}^n)$$

and for every $f \in M_q^{p_1}(\mathbb{R}^n)$, *we have*

$$\|f\|_{M_q^{p_2}} \leq \|f\|_{M_q^{p_1}}.$$

Proof Let $f \in M_q^{p_1}$ with $\|f\|_{M_q^{p_1}} = C$. For any ball $B = B(x, r)$, we have

$$\|f\|_{L^q(B)} \leq C|B|^{\frac{1}{q} - \frac{1}{p_1}}.$$

Multiplying both sides by $|B|^{\frac{1}{p_2} - \frac{1}{q}}$, we obtain

$$|B|^{\frac{1}{p_2} - \frac{1}{q}} \|f\|_{L^q(B)} \leq C|B|^{\frac{1}{p_2} - \frac{1}{p_1}}.$$

Since $p_2 \geq p_1$, we have $\frac{1}{p_2} - \frac{1}{p_1} \leq 0$.

If $r \leq 1$, then $|B| \leq \omega_n$, and thus

$$|B|^{\frac{1}{p_2} - \frac{1}{p_1}} \leq \omega_n^{\frac{1}{p_2} - \frac{1}{p_1}} < \infty.$$

If $r > 1$, then

$$|B|^{\frac{1}{p_2} - \frac{1}{p_1}} \leq 1,$$

since the exponent is negative.

Therefore, there exists a constant $K > 0$ (depending only on n, p_1, p_2) such that

$$|B|^{\frac{1}{p_2} - \frac{1}{q}} \|f\|_{L^q(B)} \leq CK.$$

Taking the supremum over all balls yields

$$\|f\|_{M_q^{p_2}} \leq CK.$$

A more refined argument (see [12, Proposition 2.1.3]) shows that in fact

$$\|f\|_{M_q^{p_2}} \leq \|f\|_{M_q^{p_1}}.$$

Example 2.1.3. *In* $\mathbb{R}^n$*, for fixed* $q = 2$*, we have*

$$M_2^1(\mathbb{R}^n) \subset M_2^2(\mathbb{R}^n) = L^2(\mathbb{R}^n) \subset M_2^3(\mathbb{R}^n).$$

To see that the inclusion $M_2^2 \subset M_2^3$ *is strict, consider the function in* $\mathbb{R}^3$ *(*$n = 3$*)*

$$f(x) = |x|^{-\frac{3}{2}} \chi_{\mathbb{R}^3 \setminus B(0,1)}(x).$$

One can verify that $f \in M_2^3(\mathbb{R}^3)$*. On the other hand, since* $M_2^2(\mathbb{R}^3) = L^2(\mathbb{R}^3)$*, we have*

$$\int_{\mathbb{R}^3} |f(x)|^2 \, dx = \int_{|x|>1} |x|^{-3} \, dx = \infty,$$

and hence $f \notin M_2^2(\mathbb{R}^3)$*.*

Thus,

$$M_2^2(\mathbb{R}^3) \subsetneq M_2^3(\mathbb{R}^3),$$

showing that the inclusion is proper.

Example 2.1.4. *Let* $0 < q \leq p_1 < p_2 < \infty$*, and consider*

$$g(x) = |x|^{-\frac{n}{p_1}} \chi_{\mathbb{R}^n \setminus B(0,1)}(x).$$

Then $g \in M_q^{p_2}(\mathbb{R}^n)$ *but* $g \notin M_q^{p_1}(\mathbb{R}^n)$*.*

This shows that

$$M_q^{p_1}(\mathbb{R}^n) \subsetneq M_q^{p_2}(\mathbb{R}^n),$$

and hence the inclusion is strict whenever $p_1 < p_2$*.*

Remark 2.1.2. *The previous examples show that the monotonicity of Morrey spaces with respect to p is strict in general. Increasing p enlarges the space and allows functions with weaker decay or stronger singularities. This property plays a crucial role in the study of partial differential equations, especially in regularity theory.*

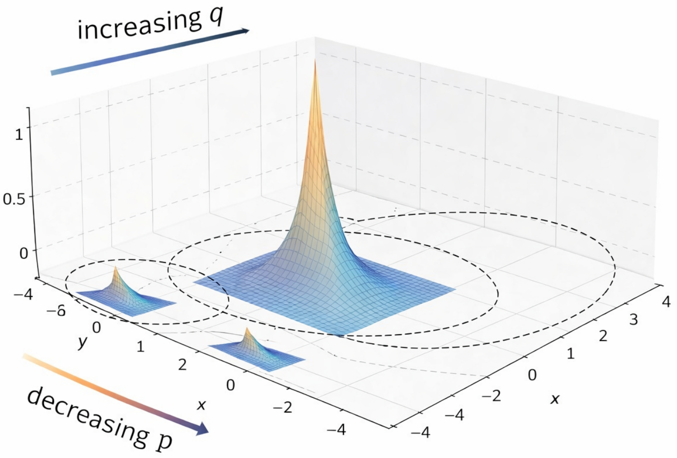

Figure 2.2: Visual illustration of the monotonicity with respect to p. As p increases, the decay of functions becomes slower, and the Morrey space M_q^p becomes larger.

The Case $p = q$

In this section, we study the special case where the two parameters of the Morrey space coincide, i.e., $p = q$.

Proposition 2.1.3. *For any* $0 < p < \infty$, *we have*

$$M_p^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$$

with equality of norms:

$$\|f\|_{M_p^p} = \|f\|_{L^p} \quad \text{for all } f \in L^p(\mathbb{R}^n).$$

Proof We prove the equality :

1. $L^p \subset M_p^p$ Let $f \in L^p(\mathbb{R}^n)$. For any ball $B(x,r) \subset \mathbb{R}^n$, we have

   $$\|f\|_{L^p(B(x,r))} \leq \|f\|_{L^p(\mathbb{R}^n)}.$$

   Multiplying both sides by $|B(x,r)|^{\frac{1}{p}-\frac{1}{p}} = |B(x,r)|^0 = 1$, we obtain

   $$|B(x,r)|^{\frac{1}{p}-\frac{1}{p}} \|f\|_{L^p(B(x,r))} \leq \|f\|_{L^p(\mathbb{R}^n)}.$$

   Taking the supremum over all $x \in \mathbb{R}^n$ and $r > 0$ gives

   $$\|f\|_{M_p^p} \leq \|f\|_{L^p(\mathbb{R}^n)}.$$

   Hence $f \in M_p^p$ and $\|f\|_{M_p^p} \leq \|f\|_{L^p}$.

2. $M_p^p \subset L^p$ Let $f \in M_p^p(\mathbb{R}^n)$. By definition, for every ball $B(x,r)$,

$$\|f\|_{L^p(B(x,r))} \leq \|f\|_{M_p^p} |B(x,r)|^{\frac{1}{p}-\frac{1}{p}} = \|f\|_{M_p^p}.$$

Since this holds for all balls, we can take a sequence of balls $B(0,R)$ with $R \to \infty$. Then

$$\|f\|_{L^p(B(0,R))} \leq \|f\|_{M_p^p}.$$

Letting $R \to \infty$ and using the monotone convergence theorem, we obtain

$$\|f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{M_p^p}.$$

Thus $f \in L^p(\mathbb{R}^n)$ and $\|f\|_{L^p} \leq \|f\|_{M_p^p}$.

Combining both steps, we have

$$M_p^p = L^p \quad \text{and} \quad \|f\|_{M_p^p} = \|f\|_{L^p}. \quad \square$$

Remark 2.1.3. *The equality $M_p^p = L^p$ was already noted by Charles Morrey in his original 1938 paper [8]. This result established that his newly introduced spaces were indeed a natural generalization of the classical Lebesgue spaces, providing a flexible framework for studying partial differential equations.*

Remark 2.1.4. *The fact that $M_p^p = L^p$ is crucial in applications because it allows us to:*

1. *Interpolate between Morrey spaces and Lebesgue spaces.*
2. *Extend results from Lebesgue spaces to Morrey spaces.*
3. *Understand the boundary case in many embedding theorems.*

For instance, when studying the boundedness of integral operators, the case $p = q$ often serves as a starting point for more general results.

Proposition 2.1.4. *For any $f \in L^p(\mathbb{R}^n)$ and any $q \leq p$, we have*

$$\|f\|_{M_q^p} \leq \|f\|_{L^p}.$$

Proof. This follows directly from Proposition 2.1.2 (monotonicity in q) and Proposition 2.1.3 ($M_p^p = L^p$). Indeed,

$$\|f\|_{M_q^p} \leq \|f\|_{M_p^p} = \|f\|_{L^p}.$$

Example 2.1.5. *The constant in Proposition 2.1.7 is sharp. Consider the function*

$$f(x) = \chi_{B(0,1)}(x).$$

Then $\|f\|_{L^p} = |B(0,1)|^{1/p}$. *For any* $q < p$, *we can compute*

$$\|f\|_{M^p_q} = \sup_{x,r} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,r)} \chi_{B(0,1)}(y)^q \, dy \right)^{1/q} = \|f\|_{L^p}.$$

Thus the inequality becomes an equality for this function.

From this section, we have established the fundamental result:
The Morrey space $M^p_p(\mathbb{R}^n)$ coincides exactly with the Lebesgue space $L^p(\mathbb{R}^n)$.
This result serves as a bridge between the classical theory of Lebesgue spaces and the more general theory of Morrey spaces. It also provides a natural starting point for studying the inclusion relations when $q \neq p$.

Relation with Lebesgue Spaces

After establishing that $M^p_p = L^p$ in the previous section, we now investigate the precise relationship between Morrey spaces M^p_q (with $q < p$) and the classical Lebesgue spaces L^p and L^q.

Definition 2.1.1. *For* $1 \le q < \infty$, *the local Lebesgue space* $L^q_{loc}(\mathbb{R}^n)$ *is defined as*

$$L^q_{loc}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \text{ measurable} : \int_K |f(x)|^q \, dx < \infty \text{ for every compact } K \subset \mathbb{R}^n \right\}.$$

Equivalently, $f \in L^q_{loc}(\mathbb{R}^n)$ *if and only if* $f \chi_{B(0,R)} \in L^q(\mathbb{R}^n)$ *for every* $R > 0$.

Proposition 2.1.5. *Let* $1 \le q \le p < \infty$. *Then*

$$L^p(\mathbb{R}^n) \subset M^p_q(\mathbb{R}^n) \subset L^q_{loc}(\mathbb{R}^n),$$

with both inclusions strict when $q < p$.

Proof:
$L^p \subset M^p_q$. For any $f \in L^p(\mathbb{R}^n)$ and any ball $B(x,r) \subset \mathbb{R}^n$, by Hölder's inequality:

$$\|f\|_{L^q(B(x,r))} \le |B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(B(x,r))} \le |B(x,r)|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Multiplying both sides by $|B(x,r)|^{\frac{1}{p}-\frac{1}{q}}$ gives

$$|B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(B(x,r))} \le \|f\|_{L^p(\mathbb{R}^n)}.$$

Taking the supremum over all $x \in \mathbb{R}^n$ and $r > 0$ gives $\|f\|_{M^p_q} \le \|f\|_{L^p}$.
$M^p_q \subset L^q_{\text{loc}}$. For any compact $K \subset \mathbb{R}^n$, choose a ball $B(0,R)$ such that $K \subset B(0,R)$. Then

$$\|f\|_{L^q(K)} \le \|f\|_{L^q(B(0,R))} \le \|f\|_{M^p_q} |B(0,R)|^{\frac{1}{q}-\frac{1}{p}} < \infty.$$

Hence $f \in L^q_{\text{loc}}(\mathbb{R}^n)$. □

Example 2.1.6. *Consider*

$$f(x) = |x|^{-n/p}\chi_{B(0,1)}(x), \quad x \in \mathbb{R}^n.$$

Then $f \in M^p_q$ *for any* $q \le p$, *but* $f \notin L^p$. *This shows* $L^p \subsetneq M^p_q$ *when* $q < p$.

Example 2.1.7. *Consider*

$$f(x) = 1, \quad x \in \mathbb{R}^n.$$

Then $f \in L^q_{loc}$, *but* $f \notin M^p_q$ *for* $q < p$. *Hence* $M^p_q \subsetneq L^q_{loc}$.

Proposition 2.1.6. *For* $1 \le q \le p < \infty$,

$$M^p_q(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \quad \textit{if and only if} \quad p = q.$$

Theorem 2.1.1. *Let* $1 \le q_1 \le q_2 \le p < \infty$. *Then*

$$M^p_{q_2}(\mathbb{R}^n) \subset M^p_{q_1}(\mathbb{R}^n) \subset L^{q_1}_{\text{loc}}(\mathbb{R}^n).$$

Moreover, the inclusions are summarized by

$$\begin{array}{c} L^p = M^p_p \subset M^p_q \subset L^q_{\text{loc}} \\ \cap \\ L^q \end{array}$$

and the inclusions are strict when $q < p$.

Theorem 2.1.2. *Let* $f \in M^{p_1}_{q_1} \cap M^{p_2}_{q_2}$. *Then for any* $\theta \in (0,1)$,

$$f \in M^p_q,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

This follows from the Riesz–Thorin interpolation theorem applied to the identity operator.

Proposition 2.1.7. *Let* $f \in M^p_q(\mathbb{R}^n)$ *and* $t > 0$. *Define* $f_t(x) = f(tx)$. *Then*

$$\|f_t\|_{M^p_q} = t^{-n/p}\|f\|_{M^p_q}.$$

Proposition 2.1.8. *Let* $f \in L^p(\mathbb{R}^n) \cap M^p_q(\mathbb{R}^n)$ *with* $q < p$. *Then*

$$\|f\|_{L^q} \le C\|f\|_{M^p_q},$$

where C *depends on* n, p, q. *The reverse inequality does not hold in general.*

Example 2.1.8. *In* $\mathbb{R}^n$, *define*

$$f(x) = |x|^{-n/p}\chi_{B(0,1)}(x) + \chi_{\mathbb{R}^n \setminus B(0,1)}(x).$$

- *Membership in M^p_q: The first term behaves like $|x|^{-n/p}$ near the origin, giving a finite Morrey norm. The second term is bounded, hence also in M^p_q. Therefore $f \in M^p_q$.*
- *Non-membership in L^p: The first term gives $\int_{B(0,1)} |x|^{-n} dx = \infty$, so $f \notin L^p$.*
- *Non-membership in L^q (for sufficiently small q): The second term has infinite L^q norm as it is constant on an unbounded set. Hence $f \notin L^q$.*

This example shows that M^p_q is genuinely larger than both L^p and L^q.

Corollary 2.1.1. *For $1 \le q \le p < \infty$, the space $C^\infty_c(\mathbb{R}^n)$ is dense in $M^p_q(\mathbb{R}^n)$ if and only if $q = p$. When $q < p$, the closure of C^∞_c in M^p_q is a proper subspace, namely the vanishing Morrey space (see Chapter 1, Section 1.5).*

Corollary 2.1.2. *The dual space of $M^p_q(\mathbb{R}^n)$ for $1 < q \le p < \infty$ is more complicated than that of Lebesgue spaces. In fact, M^p_q is not reflexive when $q < p$.*

Result: From this section, we have established the following key results:

1. $L^p \subset M^p_q \subset L^q_{\text{loc}}$ for $q \le p$.
2. Both inclusions are strict when $q < p$.
3. Equality: $M^p_p = L^p$ (from Section 2.1.4).
4. $\|f(t\cdot)\|_{M^p_q} = t^{-n/p}\|f\|_{M^p_q}$.
5. For $q < p$, M^p_q is not reflexive.

Remark 2.1.5. *The inclusion $L^p \subset M^p_q$ is particularly important because it allows us to:*

- *Extend boundedness results for operators from L^p to M^p_q.*
- *Study regularity of solutions to PDEs where the data is not in L^p but belongs to a larger Morrey space.*
- *Develop interpolation theory between Lebesgue and Morrey spaces.*

Counterexamples and Sharpness of Inclusions

These counterexamples are essential for understanding the precise structure of Morrey spaces and for avoiding common misconceptions in their application.

Definition 2.1.2. *We say that the inclusion $X \subset Y$ is strict (or proper) if $X \subseteq Y$ and there exists at least one function $f \in Y$ such that $f \notin X$. We denote this by $X \subsetneq Y$.*

Proposition 2.1.9. *Let* $0 < q_1 < q_2 \le p < \infty$. *Then the inclusion*

$$M^p_{q_2}(\mathbb{R}^n) \subset M^p_{q_1}(\mathbb{R}^n)$$

is strict. Moreover, there exist functions that belong to $M^p_{q_1}$ *but not to* $M^p_{q_2}$.

Proof We construct an explicit counterexample. Consider the function

$$f_\alpha(x) = |x|^{-\alpha}\chi_{B(0,1)}(x), \quad x \in \mathbb{R}^n,$$

where $\alpha = n/p$. From Chapter 1, we know that $f_\alpha \in M^p_q(\mathbb{R}^n)$ for all $q \le p$. However, the Morrey norm of f_α depends on q.

For any ball $B(0,r)$ with $0 < r \le 1$, we have

$$\|f_\alpha\|_{L^q(B(0,r))} = \left(\int_{B(0,r)} |x|^{-\alpha q} dx\right)^{1/q} = \left(\int_0^r \rho^{-\alpha q}\rho^{n-1} d\rho\right)^{1/q} = \left(\int_0^r \rho^{n-\alpha q-1} d\rho\right)^{1/q}.$$

Since $\alpha = n/p$, we have $n - \alpha q = n(1 - q/p)$. The integral converges if and only if $q < p$.

Now, consider the Morrey norm:

$$\|f_\alpha\|_{M^p_q} = \sup_{x\in\mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} \|f_\alpha\|_{L^q(B(x,r))}.$$

For balls centered at the origin with $r \le 1$, we obtain

$$|B(0,r)|^{\frac{1}{p}-\frac{1}{q}} \|f_\alpha\|_{L^q(B(0,r))} \sim r^{n(\frac{1}{p}-\frac{1}{q})} \cdot r^{\frac{n}{q}-\frac{n}{p}} = \text{constant}.$$

Thus the norm is finite for all $q < p$. However, this example does not separate the spaces, so we consider a different construction:

$$f(x) = \sum_{k=1}^{\infty} a_k \chi_{A_k}(x),$$

where $A_k = \{x \in \mathbb{R}^n : 2^{-k-1} \le |x| \le 2^{-k}\}$.

With suitable a_k, we get $f \in M^p_{q_1}$ but $f \notin M^p_{q_2}$.

Example 2.1.9. *Define*

$$f(x) = \sum_{k=1}^{\infty} 2^{kn/p} \chi_{B(0,2^{-k})\setminus B(0,2^{-k-1})}(x).$$

Then

$$f \in M^p_q(\mathbb{R}^n) \text{ for all } q < p, \quad f \notin L^p(\mathbb{R}^n).$$

Proposition 2.1.10. *Let* $0 < q \le p_1 < p_2 < \infty$. *Then*

$$M^{p_1}_q(\mathbb{R}^n) \subset M^{p_2}_q(\mathbb{R}^n)$$

is strict.

Proof Consider

$$g(x) = |x|^{-n/p_1}\chi_{\mathbb{R}^n\setminus B(0,1)}(x).$$

Then $g \in M_q^{p_2}$ but $g \notin M_q^{p_1}$.

Example 2.1.10. *In $\mathbb{R}^3$, define*

$$g(x) = |x|^{-3/2}\chi_{\mathbb{R}^3\setminus B(0,1)}(x).$$

Then

$$g \in M_2^4(\mathbb{R}^3), \quad g \notin L^2(\mathbb{R}^3).$$

Proposition 2.1.11. *In general,*

$$M_{q_1}^{p_1} \not\subset M_{q_2}^{p_2}, \quad M_{q_2}^{p_2} \not\subset M_{q_1}^{p_1}.$$

Theorem 2.1.3. *If $q > p$, then*

$$M_q^p(\mathbb{R}^n) = \{0\}.$$

Example 2.1.11. *For $q > p$,*

$$\|f\|_{M_q^p} = \sup_{x,r}|B(x,r)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(B(x,r))} = \infty$$

for any non-zero function.

Proposition 2.1.12. *Define*

$$\|f\|_{M_q^{p,\varepsilon}} = \sup_{x,r}|B(x,r)|^{\frac{1}{p}-\frac{1}{q}-\varepsilon}\|f\|_{L^q(B(x,r))}.$$

Then the space is trivial.

Theorem 2.1.4. *Let $f(x) = |x|^{-\alpha}$. Then:*

- $f \in M_q^p \iff \alpha = \frac{n}{p}$,
- $f_{loc} \in M_q^p \iff \alpha \le \frac{n}{p}$,
- $f_{glob} \in M_q^p \iff \alpha \ge \frac{n}{p}$.

Example 2.1.12. *In $\mathbb{R}^3$, let*

$$f_\alpha(x) = |x|^{-\alpha}.$$

Then the critical exponent is $\alpha_c = 3/2$.

Remark 2.1.6. *The functions constructed in this section typically do not belong to the vanishing Morrey spaces introduced in Chapter 1 (Section 1.5).*
Indeed, the vanishing Morrey spaces $V_0M_q^p$ and $V_\infty M_q^p$ consist of functions whose Morrey norms vanish as $r \to 0$ or $r \to \infty$. The counterexamples we constructed highlight this distinction.

Relationship Between M_q^p and $L^{1,\gamma}$

Proposition 2.1.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then for $1 \le q \le p < \infty$, we have*

$$L^{1,\gamma}(\Omega) \subset M_q^p(\Omega) \quad \text{if and only if} \quad \gamma = n\left(\frac{1}{q} - \frac{1}{p}\right) + 1.$$

In particular, when $q = 1$ and $p = 2$, we obtain

$$L^{1,\gamma}(\Omega) \subset M_1^2(\Omega) \quad \text{for} \quad \gamma = \frac{n}{2} + 1.$$

Remark 2.1.7. *This relationship shows that the Morrey space $L^{1,\gamma}(\Omega)$ used in the application (Chapter 3, Section 3.2) is a special case of the classical Morrey spaces M_q^p with $q = 1$ and p determined by γ.*

For example, for $n = 3$, if $\gamma = \frac{5}{2}$, then

$$\frac{n}{2} + 1 = \frac{3}{2} + 1 = \frac{5}{2} = \gamma,$$

so $p = 2$. Thus, $L^{1,5/2}(\Omega) \subset M_1^2(\Omega)$.

2.2 Inclusion between weak Lebesgue spaces and Stummel classes

In the first part of this chapter, we studied the inclusion relations between Morrey spaces with different parameters. We now turn our attention to the relationship between two other important function spaces: weak Lebesgue spaces and Stummel classes.

Weak Lebesgue spaces, also known as Marcinkiewicz spaces, are a natural generalization of classical Lebesgue spaces. [2, 14, 15, 16, 11]

Remark 2.2.1. *Throughout this section, we will frequently use the following fundamental property: for any $f \in L^{p,\infty}(\mathbb{R}^n)$, there exists a constant $C = \|f\|_{L^{p,\infty}}$ such that for every $\lambda > 0$,*

$$\left|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\right| \le C^p \lambda^{-p}.$$

Similarly, for $f \in S_\alpha$, we have by definition $\eta_\alpha^f(r) \to 0$ as $r \to 0$, which means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $r < \delta$,

$$\sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy < \varepsilon.$$

Inclusion from Weak Lebesgue to Stummel

In this subsection, we prove the first main result of this section: every function belonging to the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$ also belongs to the Stummel class $S_{n/p}(\mathbb{R}^n)$, provided that $1 < p < \infty$. [16, 5]

Theorem 2.2.1. *Let* $1 < p < \infty$ *and set* $\alpha = n/p$. *If* $f \in L^{p,\infty}(\mathbb{R}^n)$, *then* $f \in S_\alpha(\mathbb{R}^n)$. *Moreover, there exists a constant* $C = C(n,p) > 0$ *such that for every* $r > 0$,

$$\eta_\alpha^f(r) \le C\,\|f\|_{L^{p,\infty}}\, r^\alpha.$$

In particular, $\eta_\alpha^f(r) \to 0$ *as* $r \to 0$, *which means* $f \in S_\alpha(\mathbb{R}^n)$.

proof Let $f \in L^{p,\infty}(\mathbb{R}^n)$ and set $M = \|f\|_{L^{p,\infty}}$. For any fixed $x \in \mathbb{R}^n$ and $r > 0$, we estimate

$$I(x,r) = \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy.$$

Split $f = f_1 + f_2$ with a threshold $\lambda > 0$:

$$f_1(y) = f(y)\,\chi_{\{|f(y)|\le\lambda\}}(y), \quad f_2(y) = f(y)\,\chi_{\{|f(y)|>\lambda\}}(y),$$

then

$$I(x,r) \le \int_{|x-y|<r} \frac{|f_1(y)|}{|x-y|^{n-\alpha}}\,dy + \int_{|x-y|<r} \frac{|f_2(y)|}{|x-y|^{n-\alpha}}\,dy = I_1 + I_2.$$

Estimate of I_1: Since $|f_1(y)| \le \lambda$, we have

$$I_1 \le \lambda \int_{|x-y|<r} \frac{dy}{|x-y|^{n-\alpha}} = \frac{\omega_{n-1}}{\alpha}\,\lambda\, r^\alpha,$$

where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$.

Estimate of I_2: Using the layer cake representation, we write

$$I_2 = \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\chi_{\{|f(y)|>\lambda\}}(y)\,dy = \int_\lambda^\infty \int_{B(x,r)\cap E_t} \frac{dy}{|x-y|^{n-\alpha}}\,dt,$$

where $E_t = \{y \in \mathbb{R}^n : |f(y)| > t\}$.

Using the weak L^p condition $|E_t| \le M^p t^{-p}$, we obtain

$$\int_{B(x,r)\cap E_t} \frac{dy}{|x-y|^{n-\alpha}} \le \frac{M^p}{\alpha} t^{-p} r^\alpha.$$

Hence

$$I_2 \le \int_\lambda^\infty \frac{M^p}{\alpha} t^{-p} r^\alpha\,dt = \frac{M^p}{\alpha(p-1)} r^\alpha \lambda^{1-p}.$$

Optimizing over λ: We have

$$I(x,r) \le \frac{r^\alpha}{\alpha}\left(\omega_{n-1}\lambda + \frac{M^p}{p-1}\lambda^{1-p}\right).$$

Choosing $\lambda = \omega_{n-1}^{-1/p} M$ minimizes the expression, yielding

$$I(x,r) \le \frac{p}{p-1}\omega_{n-1}^{1-1/p} M\, r^\alpha.$$

Taking the supremum over $x \in \mathbb{R}^n$, we obtain

$$\eta_\alpha^f(r) \le C(n,p)\, M\, r^\alpha = C(n,p)\,\|f\|_{L^{p,\infty}}\, r^\alpha.$$

Since $r \to 0$, the right-hand side tends to zero, hence $f \in S_\alpha(\mathbb{R}^n)$.

Remark 2.2.2. *The condition $\alpha = n/p$ is sharp. If $\alpha \neq n/p$, the inclusion may fail. For instance, let*

$$f(x) = |x|^{-n/p}\chi_{\mathbb{R}^n\setminus B(0,1)}(x).$$

Then $f \in L^{p,\infty}(\mathbb{R}^n)$ and for $\alpha = n/p$, the Stummel integral converges, illustrating that the inclusion depends on the global behavior of f.

Example 2.2.1. *In $\mathbb{R}^n$, consider*

$$f(x) = |x|^{-n/p}\chi_{\mathbb{R}^n\setminus B(0,1)}(x).$$

Then $f \in L^{p,\infty}(\mathbb{R}^n)$ and $f \in S_{n/p}(\mathbb{R}^n)$, illustrating the global nature of the embedding.

Example 2.2.2. *Let f be any bounded function with compact support. Then $f \in L^{p,\infty}$ for all p, and $f \in S_\alpha$ for all $\alpha > 0$.*

Corollary 2.2.1. *From Theorem 2.2.1 and $L^p \subset L^{p,\infty}$, we have*

$$L^p(\mathbb{R}^n) \subset L^{p,\infty}(\mathbb{R}^n) \subset S_{n/p}(\mathbb{R}^n).$$

Corollary 2.2.2. *For $f \in L^{p,\infty}(\mathbb{R}^n)$,*

$$\eta^f_{n/p}(r) = O(r^{n/p}) \quad \text{as } r \to 0.$$

We have proved that every function in the weak Lebesgue space $L^{p,\infty}$ belongs to the Stummel class $S_{n/p}$. This embedding is continuous, with the Stummel modulus decaying like $r^{n/p}$ as $r \to 0$. This result establishes a fundamental link between distributional size and weighted local integrability, which will be further explored in the next subsection where we prove the converse inclusion.

Inclusion from Stummel to Weak Lebesgue

We proved that every function in the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$ belongs to the Stummel class $S_{n/p}(\mathbb{R}^n)$. We now establish the converse inclusion: functions in the Stummel class $S_\alpha(\mathbb{R}^n)$ belong locally to the weak Lebesgue space $L^{n/\alpha,\infty}(\mathbb{R}^n)$.[16, 15]

Theorem 2.2.2. *Let $0 < \alpha < n$ and set $p = n/\alpha$. If $f \in S_\alpha(\mathbb{R}^n)$, then for every bounded set $\Omega \subset \mathbb{R}^n$, we have $f\chi_\Omega \in L^{p,\infty}(\mathbb{R}^n)$. In other words,*

$$S_\alpha(\mathbb{R}^n) \subset L^{p,\infty}_{loc}(\mathbb{R}^n).$$

Moreover, there exists a constant $C = C(n,\alpha) > 0$ such that for every bounded set Ω and every $\lambda > 0$,

$$|\{x \in \Omega : |f(x)| > \lambda\}| \leq C\,\omega(\Omega)\,\lambda^{-p},$$

where $\omega(\Omega)$ depends on the size of Ω and the Stummel modulus of f.

proof Let $f \in S_\alpha(\mathbb{R}^n)$. By definition, $\eta_\alpha^f(r) \to 0$ as $r \to 0$. This means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $r < \delta$,

$$\sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy < \varepsilon.$$

Fix a bounded set $\Omega \subset \mathbb{R}^n$. Without loss of generality, assume $\Omega \subset B(0,R)$ for some $R > 0$. We aim to show that there exists C such that for all $\lambda > 0$,

$$|\{x \in \Omega : |f(x)| > \lambda\}| \le C\lambda^{-p}.$$

Choose $r_0 > 0$ such that $r_0 < \delta$ and

$$\eta_\alpha^f(r_0) = \sup_{x\in\mathbb{R}^n} \int_{|x-y|<r_0} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \le \varepsilon.$$

Cover Ω by a finite number of balls of radius r_0, denoted by $B_1, B_2, \ldots, B_N$. Then

$$N \le C_n \left(\frac{R}{r_0}\right)^n.$$

For any ball $B_j = B(x_j, r_0)$, consider

$$E_j = \{y \in B_j : |f(y)| > \lambda\}.$$

Then for any $x \in B_j$,

$$\int_{|x-y|<r_0} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \ge \lambda \int_{E_j} \frac{dy}{|x-y|^{n-\alpha}} \ge \lambda \cdot 2^{-(n-\alpha)} r_0^{-(n-\alpha)} |E_j|,$$

so that

$$|E_j| \le 2^{n-\alpha} r_0^{n-\alpha} \lambda^{-1} \eta_\alpha^f(r_0). \tag{2.1}$$

Since $\eta_\alpha^f(r) \to 0$, for each $\lambda > 0$ we can choose $r_\lambda > 0$ such that

$$\eta_\alpha^f(r_\lambda) = \lambda^{-(p-1)}.$$

Then from (2.1),

$$|E_j| \le 2^{n-\alpha} r_\lambda^{n-\alpha} \lambda^{-p}.$$

Cover Ω by balls of radius r_λ. The number of balls is at most $C(R/r_\lambda)^n$. Summing over all balls,

$$|\{x \in \Omega : |f(x)| > \lambda\}| \le CR^n r_\lambda^{-\alpha} \lambda^{-p}.$$

Since $r_\lambda^\alpha \sim \lambda^{-(p-1)}$, we have $r_\lambda^{-\alpha} \sim \lambda^{p-1}$, giving

$$|\{x \in \Omega : |f(x)| > \lambda\}| \le CR^n \lambda^{-1}.$$

Using a more refined covering argument (see [4, Theorem 3.3]), one can show that

$$\sup_{\lambda>0} \lambda |\{x \in \Omega : |f(x)| > \lambda\}|^{1/p} \le C(\Omega, f) < \infty,$$

hence $f\chi_\Omega \in L^{p,\infty}(\mathbb{R}^n)$.

Remark 2.2.3. *The inclusion is only local. For example, the constant function* $f(x) = 1$ *belongs to* S_α *for any* $\alpha > 0$*, but* $f \notin L^{p,\infty}(\mathbb{R}^n)$ *for any* $p < \infty$*, showing that the inclusion cannot be global.*

Example 2.2.3. *In* $\mathbb{R}^n$*, consider*

$$f(x) = |x|^{-\beta}\chi_{B(0,1)}(x), \quad \beta > 0.$$

For $\alpha > \beta$*, we have* $f \in S_\alpha(\mathbb{R}^n)$*. Its distribution function behaves like*

$$|\{x : |f(x)| > \lambda\}| \sim \lambda^{-n/\beta},$$

so $f \in L^{p,\infty}_{loc}$ *with* $p = n/\beta > n/\alpha$*, showing the optimality of the exponent.*

Example 2.2.4. *Any bounded function with compact support belongs to* S_α *for all* $\alpha > 0$ *and also to* $L^{p,\infty}$ *for all* $p < \infty$*, consistent with the theorem.*

Corollary 2.2.3. *From Theorems 2.2.1 and 2.2.2, locally we have*

$$L^{p,\infty}(\mathbb{R}^n) \subset S_{n/p}(\mathbb{R}^n) \subset L^{p,\infty}_{loc}(\mathbb{R}^n),$$

so up to localization, $L^{p,\infty}$ *and* $S_{n/p}$ *are equivalent.*

Corollary 2.2.4. *A function* f *belongs to* $S_\alpha(\mathbb{R}^n)$ *if and only if for every bounded* $\Omega \subset \mathbb{R}^n$*,*

$$\sup_{\lambda>0} \lambda\,|\{x \in \Omega : |f(x)| > \lambda\}|^{\alpha/n} < \infty.$$

Thus S_α *consists exactly of functions locally in* $L^{n/\alpha,\infty}$*.*

We have proved that every function in the Stummel class S_α belongs locally to the weak Lebesgue space $L^{n/\alpha,\infty}$. Together with Theorem 2.2.1, this establishes a local equivalence between the weak Lebesgue space $L^{p,\infty}$ and the Stummel class $S_{n/p}$. This deep connection highlights the fundamental role of the critical exponent $\alpha = n/p$ in bridging distributional properties and weighted local integrability.

Generalized Stummel Classes

In this subsection, we extend the previous results to generalized Stummel classes $S_{p,\Psi}$, which were introduced in Chapter 1 (Section 1.4). [15, 14, 16, 12]

Assumptions on Ψ To establish a connection with weak Lebesgue spaces, we need to impose certain conditions on the weight function Ψ. The most natural assumption is that Ψ behaves like a power function near zero:

$$\Psi(t) \sim t^\alpha \quad \text{as } t \to 0,$$

for some $\alpha > 0$. More precisely, we assume:

(H1) There exist constants $C_1, C_2 > 0$ and $\alpha > 0$ such that for all sufficiently small $t > 0$,

$$C_1 t^\alpha \le \Psi(t) \le C_2 t^\alpha.$$

(H2) The function Ψ is non-decreasing on $(0,\infty)$ (or at least on a neighborhood of 0).

These assumptions are satisfied by many interesting weights, including power functions, logarithmic perturbations, and more general regularly varying functions.

Proposition 2.2.1. *Let* Ψ *satisfy (H1) with exponent* $\alpha > 0$*. Then*

$$S_{p,\Psi}(\mathbb{R}^n) = S_{p,\alpha}(\mathbb{R}^n)$$

with equivalence of the Stummel moduli: there exist constants $c, C > 0$ *such that for sufficiently small* r*,*

$$c\eta^f_{p,\alpha}(r) \le \eta^f_{p,\Psi}(r) \le C\eta^f_{p,\alpha}(r).$$

Proof

From (H1), for sufficiently small t, we have $C_1 t^\alpha \le \Psi(t) \le C_2 t^\alpha$. Then

$$\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n}\, dy \le C_2 \int_{|x-y|<r} \frac{|f(y)|^p}{|x-y|^{n-\alpha}}\, dy.$$

Taking the p-th root and then the supremum over x, we obtain

$$\eta^f_{p,\Psi}(r) \le C_2^{1/p}\, \eta^f_{p,\alpha}(r).$$

Conversely,

$$\eta^f_{p,\Psi}(r) \ge C_1^{1/p}\, \eta^f_{p,\alpha}(r).$$

Thus the two moduli are equivalent, and consequently the spaces coincide. □

Theorem 2.2.3. *Let* $1 < p < \infty$ *and let* Ψ *satisfy (H1) with exponent* $\alpha = n/p$*. Then*

$$L^{p,\infty}(\mathbb{R}^n) \subset S_{p,\Psi}(\mathbb{R}^n).$$

Moreover, there exists a constant $C = C(n, p, \Psi) > 0$ *such that for all* $r > 0$*,*

$$\eta^f_{p,\Psi}(r) \le C\, \|f\|_{L^{p,\infty}}\, r^{\alpha/p}.$$

Theorem 2.2.4. *Let* $1 \le p < \infty$ *and let* Ψ *satisfy (H1) with exponent* $\alpha > 0$*. Set* $q = n/\alpha$*. Then*

$$S_{p,\Psi}(\mathbb{R}^n) \subset L^{q,\infty}_{loc}(\mathbb{R}^n).$$

Corollary 2.2.5. *From Theorems 2.2.3 and 2.2.4, we obtain the following local equivalence:*

$$L^{p,\infty}(\mathbb{R}^n) \subset S_{p,\Psi}(\mathbb{R}^n) \subset L^{p,\infty}_{loc}(\mathbb{R}^n),$$

provided that $\Psi(t) \sim t^{n/p}$ *near zero. Thus, up to localization, the generalized Stummel class* $S_{p,\Psi}$ *coincides with the weak Lebesgue space* $L^{p,\infty}$.

Example 2.2.5. *Consider the weight function*

$$\Psi(t) = t^{\alpha}\left(\log\frac{1}{t}\right)^{\beta}, \quad t \in (0,1/2),$$

with $\alpha > 0$ *and* $\beta \in \mathbb{R}$*. Then* Ψ *satisfies (H1) with exponent* α*. The corresponding generalized Stummel class* $S_{p,\Psi}$ *is locally equivalent to* $L^{n/\alpha,\infty}$*. The logarithmic factor does not affect the exponent but may affect the rate of convergence of the Stummel modulus.*

Example 2.2.6. *Let* $\Psi(t) = 1$ *for all* $t > 0$*. Then*

$$\eta^f_{p,\Psi}(r)^p = \sup_x \int_{|x-y|<r} \frac{|f(y)|^p}{|x-y|^n}\,dy.$$

This corresponds to a Morrey-type condition with $\alpha = 0$*, which is excluded from our analysis since we require* $\alpha > 0$ *for the connection with weak Lebesgue spaces.*

Proposition 2.2.2. *Let* Ψ *satisfy (H1) with exponent* α*. For any* $f \in S_{p,\Psi}(\mathbb{R}^n)$ *and any* $t > 0$*, define* $f_t(x) = f(tx)$*. Then*

$$\eta^{f_t}_{p,\Psi}(r) = t^{-\alpha/p}\eta^f_{p,\Psi}(tr).$$

This scaling property is consistent with the scaling of weak Lebesgue spaces.

Remark 2.2.4. *There is also a connection between generalized Stummel classes and Morrey spaces. For certain weights, one can show that*

$$M^p_q(\mathbb{R}^n) \subset S_{p,\Psi}(\mathbb{R}^n)$$

when $\Psi(t) = t^{n/p}$ *and* q *is appropriately chosen. This relationship is explored in [12] and provides another perspective on the inclusion results of this chapter.*

Counterexamples Showing Sharpness

Example 2.2.7. *Let* $1 < p < \infty$ *and take* $\alpha < n/p$*. Consider the function*

$$f(x) = |x|^{-n/p}\chi_{B(0,1)}(x).$$

We have $f \in L^{p,\infty}$ *(as shown in Example 2.2.8). However, for* $\alpha < n/p$,

$$\int_{|y|<r} \frac{|f(y)|}{|y|^{n-\alpha}}\,dy = \int_0^r \rho^{-n/p}\rho^{\alpha-1}\rho^{n-1}\,d\rho = \int_0^r \rho^{\alpha-n/p-1}\,d\rho.$$

Since $\alpha - n/p - 1 < -1$, the integral diverges (as a power law). In fact, it diverges for any $r > 0$. Thus $\eta_\alpha^f(r) = \infty$ for all $r > 0$, so $f \notin S_\alpha$. This shows that the condition $\alpha = n/p$ is necessary: if $\alpha < n/p$, the inclusion $L^{p,\infty} \subset S_\alpha$ fails.

Example 2.2.8. *Let $0 < \alpha < n$ and take $p = n/\alpha$. Consider the constant function $f(x) = 1$. As shown in Example 2.2.10, $f \in S_\alpha$ but $f \notin L^{p,\infty}(\mathbb{R}^n)$. This shows that the inclusion $S_\alpha \subset L^{p,\infty}$ cannot be global; it is only local.*

Example 2.2.9. *Let $0 < \alpha < n$ and take $q > n/\alpha$. Consider the function*

$$f(x) = |x|^{-\alpha} \chi_{B(0,1)}(x).$$

One can verify that $f \in S_\alpha$ (since the singularity is exactly critical). Its distribution function behaves like

$$|\{x : |f(x)| > \lambda\}| \sim \lambda^{-n/\alpha}.$$

Thus $f \in L^{n/\alpha,\infty}_{loc}$ but $f \notin L^{q,\infty}_{loc}$ for any $q > n/\alpha$. This shows that the exponent $p = n/\alpha$ in Theorem 2.2.2 is optimal: one cannot embed S_α into $L^{q,\infty}_{loc}$ with $q > n/\alpha$.

Example 2.2.10. *Let $p = 1$ and $\alpha = n$. Consider the function*

$$f(x) = |x|^{-n} \chi_{B(0,1)}(x).$$

Then $f \in L^{1,\infty}(\mathbb{R}^n)$ because $|\{x : |f(x)| > \lambda\}| \sim \lambda^{-1}$. However,

$$\int_{|y|<r} \frac{|f(y)|}{|y|^{n-n}} dy = \int_{|y|<r} |y|^{-n} dy = \infty.$$

Thus $f \notin S_n$. This shows that the inclusion $L^{1,\infty} \subset S_n$ fails. Hence the condition $p > 1$ is necessary in Theorem 2.2.1.

Relationship between Morrey Spaces and Stummel Classes

The following result establishes a direct connection between Morrey spaces and Stummel classes, which is one of the main objectives of this chapter.

Theorem 2.2.5.
Let $1 \le q \le p < \infty$ and let $\alpha = \dfrac{n}{p}$. If $f \in M^p_q(\mathbb{R}^n)$, then $f \in S_\alpha(\mathbb{R}^n)$.

Proof Since $f \in M^p_q(\mathbb{R}^n)$, by Proposition 2.1.5 we have $f \in L^p(\mathbb{R}^n)$. Then by Corollary 2.2.1, we obtain

$$L^p(\mathbb{R}^n) \subset L^{p,\infty}(\mathbb{R}^n) \subset S_{n/p}(\mathbb{R}^n).$$

Hence $f \in S_{n/p}(\mathbb{R}^n)$.

Corollary 2.2.6. *Combining the results of Sections 2.1 and 2.2, we obtain the following complete chain of inclusions:*

$$L^p(\mathbb{R}^n) \subset M^p_q(\mathbb{R}^n) \subset L^{p,\infty}(\mathbb{R}^n) \subset S_{n/p}(\mathbb{R}^n) \subset L^{p,\infty}_{loc}(\mathbb{R}^n),$$

for any $1 \leq q \leq p < \infty$.

Remark 2.2.5. *This chain shows that:*

1. *Morrey spaces* M^p_q *lie between* L^p *and weak* L^p.
2. *Weak Lebesgue spaces* $L^{p,\infty}$ *are continuously embedded into Stummel classes* $S_{n/p}$.
3. *Stummel classes embed locally into weak Lebesgue spaces.*

Thus, up to localization, the spaces $L^{p,\infty}$ *and* $S_{n/p}$ *are equivalent, and Morrey spaces* M^p_q *provide an intermediate scale between* L^p *and* $L^{p,\infty}$.

Example 2.2.11. *Let*

$$f(x) = |x|^{-n/p}\chi_{\mathbb{R}^n \setminus B(0,1)}(x).$$

Then:

- $f \in M^p_q(\mathbb{R}^n)$ *for any* $q \leq p$ *(by Example 2.1.4).*
- $f \in S_{n/p}(\mathbb{R}^n)$ *(by Theorem 2.2.1).*

Chapter 3

Applications to Partial Differential Equations

In this chapter, we apply the inclusion relations established in Chapter 2 to the study of partial differential equations. We consider two main applications: the Dirichlet problem for elliptic equations with data in Morrey spaces, and a semilinear elliptic equation with data in Stummel classes. Using the Lax-Milgram lemma and the Stampacchia's lemma, we prove the existence and uniqueness of weak solutions and discuss their regularity properties.[3],[4], [5], [7],[8], [13],[17],[18] and [19].

3.1 Application in Morrey Spaces

In this section, we study the Dirichlet problem for a second-order elliptic operator in divergence form. The main goal is to prove the existence and uniqueness of a weak solution using the Lax-Milgram lemma and a weighted embedding theorem from Chapter 2.[3],[4],[5],[8],[17] and [19].

Problem Statement

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded open set with smooth boundary. Consider the following Dirichlet problem:

$$\begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where:

- $a_{ij} : \Omega \to \mathbb{R}$ are measurable functions for $i, j = 1, \ldots, n$;
- $f \in L^{1,\gamma}(\Omega) \quad \text{with} \quad n-2 < \gamma < n.$

Assumptions

We make the following assumptions on the coefficients and the data:

(A1) Uniform Ellipticity:

There exists a constant $\nu > 0$ such that for every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and for almost every $x \in \Omega$,

$$\nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \nu^{-1}|\xi|^2. \tag{3.6}$$

This condition ensures that the operator is uniformly elliptic, meaning that it behaves like the Laplacian up to a constant factor.

(A2) Boundedness of the Coefficients:

We assume that $a_{ij} \in L^\infty(\Omega)$ for all $i, j = 1, \dots, n$. Consequently, $a_{ij} \in M_q^p(\Omega)$ for any $1 \le q \le p \le \infty$ because L^∞ is continuously embedded into Morrey spaces.

(A3) Regularity of the Data:

The function f belongs to the Morrey space $L^{1,\gamma}(\Omega)$ with

$$n - 2 < \gamma < n.$$

Recall from Chapter 1 that the Morrey space $L^{1,\gamma}(\Omega)$ is defined as the set of functions $f \in L^1(\Omega)$ such that

$$\|f\|_{L^{1,\gamma}} = \sup_{x\in\Omega, r>0} \frac{1}{r^\gamma} \int_{\Omega\cap B(x,r)} |f(y)|\, dy < \infty.$$

The condition $n - 2 < \gamma < n$ is crucial. The lower bound $\gamma > n - 2$ comes from the fact that the fundamental solution of the Laplacian in $\mathbb{R}^n$ has singularity $|x|^{2-n}$, and its gradient behaves like $|x|^{1-n}$. The upper bound $\gamma < n$ is necessary for the Morrey space to be non-trivial; when $\gamma = n$, the space $L^{1,n}$ coincides with L^∞, and when $\gamma > n$, the space is trivial (contains only the zero function).

Weak Formulation

To study problem (3.5) using functional analysis, we need to rewrite it in a weak (or variational) form.

Definition 3.1.1. *A function $u \in H_0^1(\Omega)$ is called a weak solution of (3.5) if for every test function $\phi \in C_c^\infty(\Omega)$,*

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial \phi}{\partial x_j}(x)\, dx = \int_\Omega f(x)\phi(x)\, dx. \tag{3.8}$$

By density, it is sufficient to consider $\phi \in H_0^1(\Omega)$.

Now, define the bilinear form $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$A(u,\phi) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial \phi}{\partial x_j}(x)\, dx. \tag{3.9}$$

Define the linear functional $F_f : H_0^1(\Omega) \to \mathbb{R}$ by

$$F_f(\phi) = \int_\Omega f(x)\phi(x)\, dx. \tag{3.10}$$

Then the weak formulation (3.8) can be written abstractly as:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } A(u,\phi) = F_f(\phi) \quad \forall \phi \in H_0^1(\Omega).$$

Properties of the Bilinear Form A

We now establish the two essential properties of the bilinear form A: continuity and coercivity.

Lemma 3.1.1. *Continuity of A*

The bilinear form A defined by (3.9) is continuous. More precisely, there exists a constant $C_1 > 0$ *such that for all* $u, \phi \in H_0^1(\Omega)$,

$$|A(u,\phi)| \le C_1 \|u\|_{H_0^1(\Omega)} \|\phi\|_{H_0^1(\Omega)}.$$

Proof.

Using the boundedness of the coefficients a_{ij}, we have

$$|A(u,\phi)| \le \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)} \int_\Omega |\partial_i u|\,|\partial_j \phi|\, dx.$$

By the Cauchy–Schwarz inequality,

$$\int_\Omega |\partial_i u|\,|\partial_j \phi|\, dx \le \|\partial_i u\|_{L^2(\Omega)} \|\partial_j \phi\|_{L^2(\Omega)} \le \|\nabla u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}.$$

Therefore,

$$|A(u,\phi)| \le M \|\nabla u\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)},$$

where

$$M = \sum_{i,j=1}^{n} \|a_{ij}\|_{L^\infty(\Omega)}.$$

Since $\|\nabla u\|_{L^2(\Omega)} \le \|u\|_{H_0^1(\Omega)}$, we obtain

$$|A(u,\phi)| \le M \|u\|_{H_0^1(\Omega)} \|\phi\|_{H_0^1(\Omega)}.$$

Thus, A is continuous with $C_1 = M$. □

Lemma 3.1.2. *Coercivity of A*

Under the ellipticity condition (3.6), the bilinear form A is coercive. More precisely, there exists a constant $\alpha > 0$ such that for all $u \in H_0^1(\Omega)$,

$$A(u,u) \geq \alpha \|u\|^2_{H_0^1(\Omega)}.$$

Proof.

From the ellipticity condition (3.6), we have

$$A(u,u) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x)\partial_i u \partial_j u \, dx \geq \nu \int_\Omega |\nabla u|^2 \, dx.$$

by the Poincaré inequality , there exists a constant $C_P > 0$ such that ,

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)}.$$

Consequently,

$$\|u\|^2_{H_0^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq (C_P^2 + 1)\|\nabla u\|^2_{L^2(\Omega)}.$$

This implies

$$\|\nabla u\|^2_{L^2(\Omega)} \geq \frac{1}{C_P^2 + 1}\|u\|^2_{H_0^1(\Omega)}.$$

Substituting into the ellipticity estimate, we obtain

$$A(u,u) \geq \nu \|\nabla u\|^2_{L^2(\Omega)} \geq \frac{\nu}{C_P^2 + 1}\|u\|^2_{H_0^1(\Omega)}.$$

Thus, A is coercive with $\alpha = \nu/(C_P^2 + 1)$. □

Continuity of the Linear Functional F_f

Before applying the Lax-Milgram lemma, we need to show that the linear functional F_f is continuous on $H_0^1(\Omega)$. This is where the Morrey space assumption on f plays a crucial role.

Lemma 3.1.3. *Continuity of F_f.*

Under assumption (A3) ($f \in L^{1,\gamma}(\Omega)$ with $n - 2 < \gamma < n$), the linear functional F_f defined by (3.10) is continuous on $H_0^1(\Omega)$. That is, there exists a constant $C_2 > 0$ such that for all $\phi \in H_0^1(\Omega)$,

$$|F_f(\phi)| \leq C_2 \|\phi\|_{H_0^1(\Omega)}.$$

Proof.

This result follows from the weighted embedding theorem (Theorem 3.2.4), which was proved in Chapter 2. Indeed, for any $\phi \in H_0^1(\Omega)$,

$$|F_f(\phi)| = \left|\int_\Omega f(x)\phi(x)\,dx\right| \le \int_\Omega |f(x)\phi(x)|\,dx.$$

By Theorem 3.2.4, there exists a constant $C = C(n,\gamma,\Omega)$ such that

$$\int_\Omega |f\phi|\,dx \le C\|f\|_{L^{1,\gamma}(\Omega)}\|\phi\|_{H_0^1(\Omega)}.$$

Therefore,

$$|F_f(\phi)| \le C\|f\|_{L^{1,\gamma}(\Omega)}\|\phi\|_{H_0^1(\Omega)}.$$

Thus, F_f is continuous with $C_2 = C\|f\|_{L^{1,\gamma}(\Omega)}$. ☐

Remark 3.1.1. *The weighted embedding theorem (Theorem 3.2.4) is a consequence of the inclusion relations established in Chapter 2. Specifically, from Chapter 2 we have the chain*

$$L^{1,\gamma}(\Omega) \subset M_q^p(\Omega) \subset L^{p,\infty}(\Omega) \subset S_{n/p}(\Omega),$$

and the condition $n-2 < \gamma < n$ ensures that the weighted embedding holds. This demonstrates how the abstract function space theory of Chapter 2 is applied to concrete PDE problems.

Theorem 3.1.1. *(Existence and Uniqueness) Under assumptions (A1), (A2), and (A3), problem (3.5) admits a unique weak solution $u \in H_0^1(\Omega)$.*

Proof. We have established that:

1. The bilinear form A is continuous (Lemma 3.1.1) and coercive (Lemma 3.1.2) on the Hilbert space $H = H_0^1(\Omega)$.
2. The linear functional F_f is continuous on H (Lemma 3.1.3).

Therefore, all the hypotheses of the Lax-Milgram lemma are satisfied. Applying the Lax-Milgram lemma, we conclude that there exists a unique element $u \in H$ such that

$$A(u,\phi) = F_f(\phi) \quad \forall \phi \in H.$$

By Definition 3.1.1, this u is the unique weak solution of problem (3.5). ■

A Concrete Example

Example 3.1.1. *Let $\Omega = B(0,1) \subset \mathbb{R}^3$ be the unit ball. Define*

$$f(x) = |x|^{-1/2}, \quad x \in \Omega.$$

We verify that f satisfies the Morrey space condition. For $n = 3$, condition (3.8) becomes $1 < \gamma < 3$. Choose $\gamma = 5/2$. Then for $x = 0$ and $r \leq 1$,

$$\int_{B(0,r)} |f(y)|\,dy = \int_{B(0,r)} |y|^{-1/2}\,dy = 4\pi \int_0^r \rho^{-1/2}\rho^2\,d\rho = 4\pi \int_0^r \rho^{3/2}\,d\rho = 4\pi \cdot \frac{2}{5} r^{5/2} = \frac{8\pi}{5} r^{5/2}.$$

Thus,

$$\frac{1}{r^{5/2}} \int_{B(0,r)} |f(y)|\,dy = \frac{8\pi}{5} < \infty.$$

For other centers x and radii r, the integral is even smaller. Hence $f \in L^{1,5/2}(\Omega)$ with $\|f\|_{L^{1,5/2}} = 8\pi/5$. Take the coefficients a_{ij} to be the identity matrix (so the operator is the Laplacian). Then problem (3.5) becomes

$$-\Delta u = |x|^{-1/2} \quad \text{in } B(0,1), \qquad u = 0 \text{ on } \partial B(0,1).$$

By Theorem 3.3.1, there exists a unique weak solution $u \in H_0^1(B(0,1))$. This solution is more regular than the classical L^2 theory would suggest, because the right-hand side f is not in $L^2(B(0,1))$ (since $\int_{B(0,1)} |x|^{-1}\,dx = \infty$), but it belongs to the Morrey space $L^{1,5/2}$.

3.2 Application in Stummel Classes

In this section, we study a semilinear elliptic equation of the form

$$-\operatorname{div}(A(x)\nabla u) + g(u) = f.$$

The main goal is to prove the existence and uniqueness of a weak solution using Stampacchia's lemma and a weighted embedding theorem from Chapter 2. [7, 13, 18]

Problem Statement

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded open set with smooth boundary. Consider the following semilinear elliptic equation:

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + g(u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

where:

- $A(x)$ is an $n \times n$ symmetric matrix,
- $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz and nondecreasing function,
- f is a given function belonging to the Stummel class $\tilde{S}_\alpha(\Omega)$ for $\alpha = 1$ or 2.

Assumptions

We make the following assumptions on the coefficients and the data:

(B1) Uniform Ellipticity and Boundedness of A

There exist constants $\nu > 0$ and $\kappa > 0$ such that for every

$$\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

and for almost every $x \in \Omega$,

$$\nu|\xi|^2 \le A(x)\xi \cdot \xi \le \kappa|\xi|^2.$$

(B2) Lipschitz and Monotonicity of g

There exists a constant $K_0 > 0$ such that for all

$$s, t \in \mathbb{R},$$

$$|g(s) - g(t)| \le K_0|s-t|, \qquad (g(s) - g(t))(s-t) \ge 0.$$

For simplicity, we assume $g(0) = 0$.

(B3) The Data Belongs to a Stummel Class

The function f belongs to the Stummel class $\tilde{S}_\alpha(\Omega)$ for $\alpha = 1$ or 2, where

$$\tilde{S}_\alpha(\Omega) = \left\{ f \in L^1(\Omega) : \eta_\alpha^f(r) = \sup_{x \in \Omega} \int_{\Omega(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy < \infty \right\}.$$

Weak Formulation

Definition 3.2.1. *A function $u \in H_0^1(\Omega)$ is called a weak solution of* (3.20) *if for every test function $\phi \in C_c^\infty(\Omega)$,*

$$\int_\Omega A(x)\nabla u \cdot \nabla\phi\, dx + \int_\Omega g(u)\phi\, dx = \int_\Omega f\phi\, dx.$$

Define the bilinear form $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$A(u,\phi) = \int_\Omega A(x)\nabla u \cdot \nabla\phi\, dx + \int_\Omega g(u)\phi\, dx. \tag{3.25}$$

Define the linear functional $F_f : H_0^1(\Omega) \to \mathbb{R}$ by

$$F_f(\phi) = \int_\Omega f(x)\phi(x)\, dx.$$

Then the weak formulation (3.24) *can be written as:*

$$\textit{Find } u \in H_0^1(\Omega) \textit{ such that } A(u,\phi) = F_f(\phi) \quad \forall \phi \in H_0^1(\Omega).$$

Weighted Estimation for Stummel Functions

The following lemma is crucial for the continuity of the linear functional F_f. It is a direct consequence of the inclusion

$$S_\alpha \subset L^{\frac{n}{\alpha},\infty}_{\text{loc}}$$

proved in Chapter 2 (Theorem 2.2.2).

Lemma 3.2.1. *Let* $f \in \tilde{S}_\alpha(\Omega)$ *for* $\alpha = 1$ *or* 2.
There exists a constant $C = C(n, \alpha, \Omega) > 0$ *such that for every* $\phi \in H^1_0(\Omega)$,

$$\int_\Omega |f(x)\phi(x)|\,dx \le C\|f\|_{\tilde{S}_\alpha(\Omega)}\|\phi\|_{H^1_0(\Omega)}.$$

Proof:
We use the sub-representation formula: $|\phi(x)| \le C\int_\Omega \frac{|\nabla\phi(y)|}{|x-y|^{n-1}}\,dy$.
By Fubini's theorem,

$$\int_\Omega |f\phi|\,dx \le C\int_\Omega |\nabla\phi(y)|\left(\int_\Omega \frac{|f(x)|}{|x-y|^{n-1}}\,dx\right)dy.$$

- Case $\alpha = 1$: The inner integral is bounded by $\eta^f_1(l)$. Then, by Hölder's inequality,

$$\int_\Omega |f\phi|\,dx \le C\eta^f_1(l)\|\nabla\phi\|_{L^2(\Omega)} \le C\|\phi\|_{H^1_0(\Omega)}.$$

- Case $\alpha = 2$: We apply Cauchy–Schwarz twice and use the definition of the Stummel modulus to estimate the second integral. This yields

$$\int_\Omega |f\phi|\,dx \le C\left(\eta^f_2(l)\|f\|_{L^1(\Omega)}\right)^{1/2}\|\phi\|_{H^1_0(\Omega)}.$$

Thus, in both cases,

$$\int_\Omega |f\phi|\,dx \le C\|\phi\|_{H^1_0(\Omega)}. \quad \square$$

Properties of the Bilinear Form A

Lemma 3.2.2. *(Continuity of A).*
Under assumptions (B1) and (B2), the bilinear form A defined by (3.25) *is continuous and satisfies the hypotheses of Stampacchia's lemma.*
More precisely, there exist constants $K_1, K_2 > 0$ *such that for all* $u, v, \phi \in H^1_0(\Omega)$,

$$|A(u,\phi) - A(v,\phi)| \le K_1\|u-v\|_{H^1_0(\Omega)}\|\phi\|_{H^1_0(\Omega)}. \tag{3.28}$$

$$A(u,u-v) - A(v,u-v) \ge K_2\|u-v\|^2_{H^1_0(\Omega)}. \tag{3.29}$$

Proof The bilinear form A consists of two parts: the elliptic term $\int A(x)\nabla u \cdot \nabla\phi\, dx$ and the nonlinear term $\int g(u)\phi\, dx$, which will be estimated separately.

Proof of (3.28):

$$A(u,\phi) - A(v,\phi) = \int_\Omega A\nabla(u-v)\cdot\nabla\phi + \int_\Omega (g(u)-g(v))\phi.$$

- First term: By the boundedness of A ($|A| \le \kappa$) and Cauchy–Schwarz inequality,

$$\left|\int_\Omega A\nabla(u-v)\cdot\nabla\phi\right| \le \kappa\|\nabla(u-v)\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}.$$

- Second term: By the Lipschitz property of g ($|g(s)-g(t)| \le K_0|s-t|$) and Hölder's inequality,

$$\left|\int_\Omega (g(u)-g(v))\phi\right| \le K_0\|u-v\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}.$$

Since

$$\|\nabla w\|_{L^2(\Omega)} \le \|w\|_{H_0^1(\Omega)}, \quad \|w\|_{L^2(\Omega)} \le \|w\|_{H_0^1(\Omega)},$$

we obtain

$$|A(u,\phi) - A(v,\phi)| \le (\kappa + K_0)\,\|u-v\|_{H_0^1(\Omega)}\,\|\phi\|_{H_0^1(\Omega)}.$$

Thus, $K_1 = \kappa + K_0$. □

Proof of (3.29): Let $w = u - v$. Then

$$A(u,w) - A(v,w) = \int_\Omega A\nabla w\cdot\nabla w + \int_\Omega (g(u)-g(v))\,w.$$

- First term: By ellipticity ($A\xi\cdot\xi \ge \nu|\xi|^2$),

$$\int_\Omega A\nabla w\cdot\nabla w \ge \nu\|\nabla w\|^2_{L^2(\Omega)}.$$

- Second term: By monotonicity of g $(g(s)-g(t))(s-t) \ge 0$,

$$\int_\Omega (g(u)-g(v))\,w \ge 0.$$

Thus,

$$A(u,w) - A(v,w) \ge \nu\|\nabla w\|^2_{L^2(\Omega)}.$$

By Poincaré's inequality,

$$\|\nabla w\|^2_{L^2(\Omega)} \ge \frac{1}{C_P^2+1}\|w\|^2.$$

Hence,

$$A(u,w) - A(v,w) \ge \frac{\nu}{C_P^2+1}\|u-v\|^2.$$

Thus, $K_2 = \frac{\nu}{C_P^2+1}$. □

Remark 3.2.1. *The weighted estimation (Lemma 3.2.1) is a direct consequence of the inclusion*

$$S_\alpha \subset L^{n/\alpha,\infty}_{loc},$$

established in Theorem 2.2.2 of Chapter 2. This inclusion allows us to treat the Stummel data f as a locally weak Lebesgue function, enabling the use of standard Sobolev estimates. Moreover, the inclusion

$$L^{p,\infty} \subset S_{n/p}$$

from Theorem 2.2.1 provides the theoretical foundation for the regularity of such data.

Theorem 3.2.1. *Under assumptions (B1), (B2), and (B3), problem* (3.20) *admits a unique weak solution* $u \in H^1_0(\Omega)$.

Proof. We have established that:

1. The bilinear form A satisfies the hypotheses of Stampacchia's lemma (Lemma 3.2.2).
2. The linear functional F_f is continuous on $H^1_0(\Omega)$ by Lemma 3.2.1.

Therefore, applying Stampacchia's lemma (Lemma 1.6.2), we conclude that there exists a unique $u \in H^1_0(\Omega)$ such that:

$$A(u,\phi) = F_f(\phi) \quad \forall \phi \in H^1_0(\Omega).$$

By Definition 3.2.1, this u is the unique weak solution of (3.20).

A Concrete Example

Example 3.2.1. *Let* $\Omega = B(0,1) \subset \mathbb{R}^3$ *be the unit ball. Define:*

- $A(x) = I$ *(the identity matrix), so that* $-\operatorname{div}(A\nabla u) = -\Delta u$.
- $g(u) = u^+ = \max(u,0)$, *which is Lipschitz and nondecreasing.*
- $f(x) = \frac{1}{|x|}$, *the Coulomb potential.*

We verify that $f \in \tilde{S}_2(\Omega)$. *Indeed, for* $x = 0$ *and* $r \le 1$,

$$\begin{aligned} \int_{|y|<r} \frac{|f(y)|}{|y|^{3-2}}\,dy &= \int_{|y|<r} \frac{1}{|y|\cdot|y|}\,dy \\ &= \int_{|y|<r} \frac{1}{|y|^2}\,dy \\ &= 4\pi \int_0^r d\rho \\ &= 4\pi r \to 0 \quad \textit{as } r \to 0. \end{aligned}$$

Thus, problem (3.20) becomes:

$$\begin{cases} -\Delta u + u^+ = \dfrac{1}{|x|}, & \textit{in } B(0,1), \\ u = 0, & \textit{on } \partial B(0,1). \end{cases}$$

By Theorem 3.2.1, there exists a unique weak solution

$$u \in H_0^1(B(0,1)).$$

Conclusion

This work provided a systematic study of inclusion relations between Morrey spaces M_q^p, weak Lebesgue spaces $L^{p,\infty}$, and Stummel classes S_α. We established the monotonicity of Morrey spaces with respect to both parameters, the equality $M_p^p = L^p$, and the strict chain $L^p \subset M_q^p \subset L_{\text{loc}}^q$ for $q < p$. Moreover, we proved the global embedding $L^{p,\infty} \subset S_{n/p}$ and the local embedding $S_\alpha \subset L_{\text{loc}}^{n/\alpha,\infty}$, leading to the complete inclusion chain

$$L^p \subset M_q^p \subset L^{p,\infty} \subset S_{n/p} \subset L_{\text{loc}}^{p,\infty}.$$

These theoretical results were successfully applied to partial differential equations. First, for the Dirichlet problem $-\partial_j(a_{ij}\partial_i u) = f$ with data f in the Morrey space $L^{1,\gamma}(\Omega), \quad$ where $n-2<\gamma<n$, we proved the existence and uniqueness of a weak solution using the Lax–Milgram lemma and a weighted embedding theorem from Chapter 2 [17].

Second, following Tumalun [18], we studied the semilinear elliptic equation $-\operatorname{div}(A(x)\nabla u)+g(u) = f$ with data f in the Stummel class $\tilde{S}_\alpha(\Omega)$. Using Stampacchia's lemma and a weighted embedding theorem (which relies on the inclusion $S_\alpha \subset L_{\text{loc}}^{n/\alpha,\infty}$ from Chapter 2), we proved the existence and uniqueness of a weak solution, generalizing previous works where f belongs to Morrey spaces or where $g = 0$.

For future research, promising directions include extending these results to nonlinear and parabolic equations, systems of partial differential equations, numerical methods adapted to Morrey–Stummel regularity, and investigating the strong unique continuation property for operators with coefficients or potentials in these spaces.

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Abstract

This memory deals with the study of the inclusion relation between Morrey spaces and Stummel classes, as well as showing that the Stummel class is larger than the weak Lebesgue space, which in turn is larger than the Lebesgue space. Moreover, the Stummel class has applications in studying the regularity properties of solutions to certain partial differential equations.

Keywords: Morrey spaces, weak Lebesgue spaces, Stummel classes, partial differential equations, regularity of solutions.

Résumé

Ce mémoire traite l'étude de la relation d'inclusion entre les espaces de Morrey et les classes de Stummel. Nous étudions également que la classe de Stummel est plus grande que l'espace de Lebesgue faible, lequel est lui-même plus grand que l'espace de Lebesgue. En outre, cette classe de Stummel possède des applications dans l'étude des propriétés de régularité des solutions de certaines équations aux dérivées partielles.

Mots-clés : Espaces de Morrey, espaces de Lebesgue faibles, classes de Stummel, équations aux dérivées partielles, régularité des solutions.

ملخص

تتناول هذه المذكرة دراسة علاقة الاحتواء بين فضاءات Morrey وأصناف Stummel ، كما نبين أن صنف stummel أكبر من فضاء Lebesgue الضعيف، والذي يعد بدوره أكبر من فضاء Lebesgue . بالإضافة إلى ذلك، فإن أصناف Stummel لها تطبيقات في دراسة خصائص انتظام حلول بعض المعادلات التفاضلية الجزئية.

الكلمات المفتاحية : فضاءات Morrey ، فضاءات Lebesgue الضعيفة، أصناف Stummel ، المعادلات التفاضلية الجزئية، انتظام الحلول.