



PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
UNIVERSITY OF MOUHAMED BOUDIAF / M'SILA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCES
DEPARTMENT OF MATHEMATICS

N° d'ordre:

THESIS

To obtain a doctorate degree in Mathematics

Option:

Pure mathematics and its applications

By:

YAHIAOUI AHLEM

Titled:

Singular Perturbations Method for Partial Differential Equations

Thesis defended publicly on : June 17, 2021, before a jury composed of :

N. BENHAMIDOUCHE	Prof., UNIV. OF M'SILA	Chair
A. SENGOUGA	Dr., UNIV. OF M'SILA	Supervisor
S. GUESMIA	Prof., UNIV. OF THE BAHAMAS	Co-supervisor
B. BENABDERRAHMANE	Prof., UNIV. OF M'SILA	Examiner
M. DILMI	Prof., UNIV. OF SETIF 1	Examiner
B. BOUGHERARA	Dr., UNIV. OF M'SILA	Examiner



To my family, especilly my parents:

Thank you for your encouragement, patience, and assistance over the years. I'm forever indebted to my parents, who have always kept me in their prayers.

ACKNOWLEDGEMENTS

All thanks to God, and praise be to God, who guided the footsteps, opened all doors for us, provided us with the will and illuminated us along the way.

A big thanks to my supervisor: Dr. Abdelmouhcene Sengouga, for his continuous guidance and unconditional support, also for his advice, encouragement, and patience during the realization of this work.

I am also grateful to my co-supervisors: Professor Senoussi Guesmia who has always been in my side for consultation and to guide me during the realization of this work.

I thank the members of the jury for honoring me by participating in the evaluation of this work.

I cannot conclude my thanks without talking about my dearest people, my family, which has an essential and continuous role in my success along with, all my relatives and friends.

A. YAHIAOUI

Author: YAHIAOUI AHLEM
Title : **Singular Perturbations Method for Partial Differential Equations**
Speciality: Mathematics
Option: Pure mathematics and its applications
Supervisor: Dr. ABDELMOUHCENE SENGOUGA
Co-supervisor: Prof. SENOUSSE GUESMIA

Address : Department of Mathematics
Faculty of Mathematics and Computer Sciences
University of Mouhamed Boudiaf / M'sila
BP:166, 28000
M'sila, Algeria

E-mail : ahlem.yahiaoui@univ-msila.dz
ahl.yahyaoui@gmail.com

CONTENTS

INTRODUCTION	1
I ASYMPTOTIC BEHAVIOUR AND EXISTENCE RESULTS FOR SOME NON-LOCAL SEMI-LINEAR ELLIPTIC PROBLEMS	
1 PRELIMINARIES AND FUNCTIONAL SETTING	7
1.1 Lebesgue and Sobolev spaces with weights	7
1.2 Anisotropic weighted Sobolev-type spaces	9
1.3 Problems setting and h -weak formulations	11
1.3.1 The unperturbed problem	11
1.3.2 The perturbed problem	13
1.4 C^1 -functionals involving the non-local term l	14
2 A SEMILINEAR PROBLEM WITH A NON-LOCAL NONLINEARITY	18
2.1 An existence result for the perturbed problem	18
2.2 An existence result for the unperturbed problem	21
3 A NON-LOCAL EIGENVALUE PROBLEM AND AN APPLICATION	26
3.1 A non-local eigenvalue problem	26
3.1.1 l -eigenvalues and l -eigenfunctions	26
3.1.2 Some proprieties of the first l -eigenvalue and l - eigenfunction	29
3.1.3 Convergence of the l -eigenvalues and l -eigenfunctions	30
3.2 Existence and non uniqueness for a semilinear problem with $\lambda < -\lambda_1^\varepsilon$	32
3.2.1 An existence result for the perturbed problem	32
3.2.2 An existence result for the unperturbed problem	34
4 A SEMILINEAR PROBLEM WITH SUBCRITICAL NON-LOCAL NONLINEARITY	38
4.1 A problem with superlinear growth of non-local terms	38
4.1.1 An existence result for the perturbed problem	38
4.1.2 An existence result for the unperturbed problem	42
4.2 Existence and non uniqueness results for a semilinear problem with $\lambda > -\lambda_1^\varepsilon$	44
4.2.1 An existence result for the perturbed problem	45
4.2.2 An existence result for the unperturbed problem	48

II ASYMPTOTIC BEHAVIOUR AND EXISTENCE RESULTS FOR SOME NON-LOCAL QUASI-LINEAR ELLIPTIC PROBLEMS

5	FUNCTIONAL SETTING FOR THE QUASILINEAR PROBLEMS	52
5.1	Anisotropic weighted Sobolev-type spaces	52
5.2	Assumptions and h -weak formulations	56
5.3	C^1 -functionals involving the nonlocal term l	58
6	A QUASILINEAR PROBLEM WITH A NON-LOCAL NONLINEARITY	63
6.1	An existence result for the perturbed problem	63
6.2	An existence result for the unperturbed problem	66
7	A QUASILINEAR NON-LOCAL EIGENVALUE PROBLEM AND AN APPLICATION	71
7.1	A quasilinear non-local eigenvalue problem	71
7.1.1	l -eigenvalues and l -eigenfunctions	71
7.1.2	Convergence of the l -eigenvalues and l -eigenfunctions	74
7.2	A quasilinear problem with subcritical non-local nonlinearity	76
7.2.1	An existence result for the perturbed problem	77
7.2.2	An existence result for the unperturbed problem	79
A	APPENDICES	83
A.1	Notation	83
A.2	Some useful inequalities	84
A.3	On the sum of spaces $\mathcal{V}'_0(\Omega, h)$ and $L^{\frac{r+1}{r}}(w_2)$	85
	BIBLIOGRAPHY	88

INTRODUCTION

The theory of perturbations is an active topic of asymptotic analysis that has an increasing interest in applied mathematics. It deals with mathematical models with some parameters that can be considered as "small" ones when compared to the other parameters involved in the model. If the model is difficult to solve by some analytical or numerical approach, scientists tend to neglect these small parameters to obtain a new model that is easier to solve. This also can go in the other sense. For instance, if the model is some partial differential equation PDE_0 , then adding some small terms (depending on a parameter of perturbation ε) to this equation, we obtain a perturbed equation PDE_ε . If we can solve this later equation for each ε we have a sequence of solution depending on ε . Then, passing to the limit $\varepsilon \rightarrow 0$, we may obtain a solution for the original PDE_0 . This procedure is frequently called the regularization method.

In this thesis, we have more particularly in mind anisotropic cases of singular perturbations. This technique allows us to show the existence of solutions for some non-standard integro-differential problems of non-local type, this method developed in a series of recent works [2, 9, 10, 13–15, 20] for different types of equations (elliptic, parabolic, hyperbolic...).

To simplify the presentation and explain more clearly the main ideas and methods used in this thesis, we consider the following model problem in the unit square $\Omega_1 := (0, 1) \times (0, 1)$,

$$\begin{cases} -\partial_{x_2}(\mathbf{a}(x)\partial_{x_2}u) = g\left(\int_0^1 h(x_1)u dx_1\right) + f(x), & \text{in } \Omega_1, \\ u(x_1, 0) = u(x_1, 1) = 0, & \text{for a.e. } x_1 \in (0, 1) \end{cases} \quad (1)$$

where $x := (x_1, x_2)$, g is a nonlinear function, h and \mathbf{a} are positive functions, with $\mathbf{a} > \alpha$ for some constant $\alpha > 0$. First, we note that due to the presence of the non-local term that involves the values of u on each segment $(0, 1) \times \{x_2\}$, this problem can not be considered as a family of elliptic problems parametrized by x_1 . The above problem, as well as the one arising from the time-dependent diffusion equation

$$\frac{\partial u}{\partial t} - \partial_{x_2}(\mathbf{a}(x)\partial_{x_2}u) = g\left(\int_0^1 h(x_1)u dx_1\right) + f(x), \quad t \geq 0, \quad (2)$$

involve a non-standard type of equations with second-order differentiation only in some of the variables (like $\partial_{x_2}(\mathbf{a}(x)\partial_{x_2}u)$ in (2)) and a non-local term given by a partial integral on the other domain variables. Such problems appear in certain areas of physics and engineering. For instance, an equation similar to (2) appears in the context of time evolution of some populations

of nonlinearly coupled random oscillators, described by a model proposed by Kuramoto and Sakaguchi, see [34, equations (11) and (21)] and [5, 27–29, 35]. In neutron transport theory, the authors of [12] applied the so-called Vladimirov method (see also [30, 37]) to Boltzmann's transport equation. The produced model is an integro-differential problem that is close to (1).

To illustrate the type of problems dealt with in this work, let us take

$$g(s) = -|s|^{r-1}s, \quad r > 1.$$

Then, a weak solution for Problem (1) is a function u satisfying

$$\int_{\Omega_1} \mathbf{a} \partial_{x_2} u \partial_{x_2} v dx + \int_{\Omega_1} \left(\left| \int_0^1 h u dx_1 \right|^{r-1} \int_0^1 h u dx_1 \right) v dx = \int_{\Omega_1} f v dx, \quad (3)$$

for every $v \in \mathcal{D}(\Omega_1)$. To establish some a priori estimates for u , and due to the presence of the non-local term $\int_0^1 h u dx_1$, it will be more convenient to take hv as a test function (rather than v) in the weak formulation (3). By analogy to Kufner and Rákosník [26], we say that u is an " h -weak solution" to (1) whenever u satisfies

$$\int_{\Omega_1} \mathbf{a} h \partial_{x_2} u \partial_{x_2} v dx + \int_0^1 \left(\left| \int_0^1 h u dx_1 \right|^{r-1} \int_0^1 h u dx_1 \int_0^1 h v dx_1 \right) dx_2 = \int_{\Omega_1} f v h dx, \quad (4)$$

for every $v \in \mathcal{D}(\Omega_1)$. The fact that $\int_0^1 h u dx_1$ depends only on x_2 is taken into account.

The lack of differentiation in the direction x_1 in the differential operator made Problem (1) less familiar. To overcome this inconvenience and profit from the theory of elliptic operators, we employ an anisotropic perturbation in the x_1 -direction. Thus, we consider

$$\begin{cases} -\frac{\varepsilon^2}{h} \partial_{x_1}^2 u_\varepsilon - \partial_{x_2} (\mathbf{a} \partial_{x_2} u_\varepsilon) + \left| \int_0^1 h u_\varepsilon dx_1 \right|^{r-1} \int_0^1 h u_\varepsilon dx_1 = f & \text{in } \Omega_1, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_1, \end{cases} \quad (5)$$

where $\varepsilon > 0$ is a small parameter. Of course, the produced problem is elliptic and the function u_ε should have more regularity in the x_1 -direction. The corresponding integral equation is given by

$$\begin{aligned} & \int_{\Omega_1} \varepsilon^2 \partial_{x_1} u_\varepsilon \partial_{x_1} v + \mathbf{a} h \partial_{x_2} u_\varepsilon \partial_{x_2} v dx \\ & + \int_0^1 \left(\left| \int_0^1 h u_\varepsilon dx_1 \right|^{r-1} \int_0^1 h u_\varepsilon dx_1 \int_0^1 h v dx_1 \right) dx_2 = \int_{\Omega_1} h f v dx, \quad \forall v \in \mathcal{D}(\Omega_1). \end{aligned} \quad (6)$$

Dividing by h in the term of perturbation $-\frac{\varepsilon^2}{h} \partial_{x_1}^2 u_\varepsilon$ ensures the symmetry of the bilinear form in the h -weak formulation of (5). This allows us to use a variational approach that is more

adapted to the nonlinearities considered here (see for instance [8, 23, 32]). Therefore, we examine the existence of solutions u_ε to (6) as critical points for the functional

$$J^\varepsilon(v) := \frac{1}{2} \int_{\Omega_1} \varepsilon^2 |\partial_{x_1} v|^2 + \mathbf{a}h |\partial_{x_2} v|^2 dx + \frac{1}{r+1} \int_0^1 \left| \int_0^1 h v dx_1 \right|^{r+1} dx_2 - \int_{\Omega_1} f v h dx.$$

Besides the existence and uniqueness of the solution, our main interest is to study the asymptotic behaviour for anisotropic singular perturbed problems, as $\varepsilon \rightarrow 0$. Roughly speaking, we will show the convergences

$$u_\varepsilon \rightarrow u_0 \text{ and } J^\varepsilon(u_\varepsilon) \rightarrow J^0(u_0), \text{ as } \varepsilon \rightarrow 0$$

where u_0 is an h -weak solution to Problem (1) that is also a critical point to the functional

$$J^0(v) := \frac{1}{2} \int_{\Omega_1} \mathbf{a}h |\partial_{x_2} v|^2 dx + \frac{1}{r+1} \int_0^1 \left| \int_0^1 h v dx_1 \right|^{r+1} dx_2 - \int_{\Omega_1} f v h dx.$$

The problems treated in this work are multidimensional and involve more general nonlinearities, as we will see in the rest. Let $\Omega = \omega_1 \times \omega_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where ω_1 and ω_2 be two open bounded subsets and n_1, n_2 are two positive integers. In the first part of this thesis, we consider the semilinear integro-differential problem

$$\begin{cases} -\nabla_{X_2} \cdot (A(x) \nabla_{X_2} u) = g(l(u)) + f & \text{in } \Omega, \\ u(X_1, \cdot) = 0, \text{ for a.e. } X_1 \in \omega_1 & \text{on } \partial\omega_2, \end{cases} \quad (7)$$

where $x := (X_1, X_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, ∇_{X_i} is the partial gradient in the X_i -direction and A is an $n_2 \times n_2$ matrix that is symmetric, positive definite with bounded coefficients. The non-local term l is defined as

$$l(u) := \int_{\omega_1} h(X_1) u(X_1, X_2) dX_1. \quad (8)$$

Although the function h does not appear in the main operator in (7), it will play a weight role. This forces the study to be established in the framework of weighted Sobolev spaces. Furthermore, the function h may have zeros on the boundary of ω_1 , which makes the functional setting more delicate.

As it is illustrated in the model problem (5), the method of anisotropic singular perturbations will be applied by approximating the first equation in (7) by the following one

$$-\frac{\varepsilon^2}{h} \Delta_{X_1} u_\varepsilon - \nabla_{X_2} \cdot (A(x) \nabla_{X_2} u_\varepsilon) = g(l(u_\varepsilon)) + f \text{ in } \Omega, \quad (9)$$

coupled with some boundary conditions. The results of this part have appeared in [38].

In the second part, we deal with quasilinear problems such as

$$\begin{cases} -\nabla_{X_2} \cdot (b(x)a(|\nabla_{X_2} u|) \nabla_{X_2} u) = g(l(u)) + f & \text{in } \Omega, \\ u(X_1, \cdot) = 0, \text{ for a.e. } X_1 \in \omega_1 & \text{on } \partial\omega_2, \end{cases} \quad (10)$$

where $b(x)$ is a measurable, bounded and positive a.e. in Ω . The real-valued function a is continuous and behaves like t^{p-2} for $p > 1$. This time, the perturbed problem is given by

$$-\frac{\varepsilon^2}{h} \nabla_{X_1} \cdot \left(h(X_1) b(x) a \left(\left| \begin{pmatrix} \varepsilon \nabla_{X_1} u \\ \nabla_{X_2} \end{pmatrix} \right| \right) \nabla_{X_1} u \right) - \nabla_{X_2} \cdot \left(b(x) a \left(\left| \begin{pmatrix} \varepsilon \nabla_{X_1} u \\ \nabla_{X_2} \end{pmatrix} \right| \right) \nabla_{X_2} u \right) = g(l(u_\varepsilon)) + f \text{ in } \Omega, \quad (11)$$

coupled with some boundary conditions. We have to mention that the perturbed and the unperturbed part of the differential operator are not separated as it was the case in (9). By consequence, the adequate assumptions and functional setting for the quasilinear case are not straightforward generalizations of those considered in the semilinear case. The functional setting and the assumptions on the real-valued functions a, b, f, g and h will be further stipulated in both cases.

The authors in [2, 10, 13–15, 20] considered anisotropic singular perturbations for different types of equations and variational inequalities. This technique was reused in [11, 13, 21] as a method to show the existence of solutions for some degenerate non-local semilinear problems. By adding perturbation terms involving a small parameter ε , the problem becomes non-degenerate and the existence of weak solutions, denoted by u_ε , can be obtained by the Schauder fixed point theorem. Then, some a priori estimates independent of ε are established for the solutions u_ε and, letting ε tend to 0, it is shown that the limit is a weak solution for the unperturbed problem. This technique of anisotropic singular perturbation was applied earlier in [1, 28, 29] to show the existence of solutions for certain nonlinear integro-parabolic equations (of the Fokker–Planck type).

In the work at hands, we deal with Problems (9), (11) when g is a nonlinearity that can be handled by a different method as the variational approach. This type of nonlinearity was not considered in the above cited works where the problems are only sublinear. We also appeal to different techniques as convexity and compactness arguments to perform the passage to the limit. Since Problems (9), (11) are strongly related to the eigenvalue problem in the framework of Fredholm theory, we also investigated the associated non-local eigenvalue problem. This was indispensable since the principal eigenvalue (called here l -eigenvalue) is involved in the assumptions that ensure the existence of solutions for the problem considered in the last section.

After the present introduction, we divided this thesis into two parts, the first part includes four chapters and the second one has three chapters. Part 1, titled *Asymptotic behaviour and existence result of some non-local semilinear elliptic problems*. In chapter 1, we present some elementary notions of weighted Sobolev spaces and introduce an anisotropic type of these spaces, as well as some C^1 -functional involving the non-local term $l(u)$. In chapter 2, we show the existence of solutions

(denoted by u_ε) for the perturbed problem (9) under some upper boundedness assumption on the nonlinear term. Then, we investigate the asymptotic behaviour of solutions, as well as of the related functionals, when $\varepsilon \rightarrow 0$. By passage to the limit, we obtain the existence of solutions to the original problem (7) as limits of u_ε . Chapter 3 is devoted to the non-local eigenvalue problem, this was indispensable since the smallest eigenvalue (denoted by λ_ε^1) is involved in the assumptions that ensure the existence of solutions for the semilinear problems considered in this chapter and the two following ones. As a first application of smallest eigenvalue λ_ε^1 , we show the existence of solutions for Problem (9) when

$$g(X_2, s) = -\lambda s - a |s|^{r-1} s, \text{ with } a > 0, \text{ and } \lambda < -\lambda_\varepsilon^1.$$

Then, we investigate the asymptotic behaviour of the solutions as $\varepsilon \rightarrow 0$. In chapter 4, we review the same issues considered in Chapter 2 for Problem (9) under a weaker upper boundedness assumption on the nonlinear term combined with some compactness assumptions. Then, we consider the same issues of chapter 3 for

$$g(X_2, s) = -\lambda s + a |s|^{r-1} s, \text{ with } a > 0, \text{ and } \lambda > -\lambda_\varepsilon^1.$$

The solutions are obtained by constrained minimization this time.

The second part is titled *Asymptotic behaviour and existence result of some non-local quasilinear elliptic problems*. Due to the quasilinear type of the problems considered in this part, the functional setting is updated in Chapter 5. In chapter 6, we deal with a quasilinear problem with some growth condition on the non-local terms. Chapter 7 is devoted to a quasilinear eigenvalue problem, then we revisit the problem of Chapter 6 with a weaker growth condition combined with a compactness assumption. Finally, an appendix and a full list of references are included at the end of the manuscript.

Part I

ASYMPTOTIC BEHAVIOUR AND EXISTENCE RESULTS FOR SOME
NON-LOCAL SEMILINEAR ELLIPTIC PROBLEMS

PRELIMINARIES AND FUNCTIONAL SETTING

In this chapter, we present some results of functional analysis used throughout this part. We recall some known facts on weighted Sobolev space and define some anisotropic type of these spaces. Then, setting the problem, as well as some C^1 –functional involving the non-local term.

1.1 LEBESGUE AND SOBOLEV SPACES WITH WEIGHTS

We recall here some classical definitions and theorems on weighted Sobolev spaces. The proofs can be found in [24, 25]

Definition 1.1 Let Ω be an open set in \mathbb{R}^n . We denote by $\rho(\Omega)$ the set of all measurable, a.e. in Ω positive and finite function $\rho = \rho(x)$, $x \in \Omega$. Elements of $\rho(\Omega)$ will be called weight functions.

Definition 1.2 Let $1 \leq p < \infty$, we denote by $L^p(\Omega, \rho)$ the set of all measurable functions on Ω for which the norm

$$v \mapsto \|v\|_{L^p(\Omega, \rho)} := \left(\int_{\Omega} \rho |v|^p dx \right)^{1/p} \quad (1.1)$$

is finite.

Theorem 1.1 The space $L^p(\Omega, \rho)$ equipped with the norm (1.1) is a Banach space.

Definition 1.3 Let $p > 1$, we shall say that a weight function $\rho \in \rho(\Omega)$ satisfies condition $B_p(\Omega)$ and write: $\rho \in B_p(\Omega)$ if

$$\rho^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega). \quad (1.2)$$

Theorem 1.2 Suppose a weight function $\rho \in B_p(\Omega)$ and Q a compact set in \mathbb{R}^n , where $Q \subset \Omega$. Then

$$L^p(\Omega, \rho) \hookrightarrow L^1(Q).$$

(Here \hookrightarrow stands for the continuous imbedding).

Remark 1.1 From the theorem above, we have $L^p(\Omega, \rho) \hookrightarrow L^1_{loc}(\Omega)$. Using the usual identification of a regular distribution from $\mathcal{D}'(\Omega)$ with a function from $L^1_{loc}(\Omega)$ we conclude that

$$L^p(\Omega, \rho) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega), \quad (1.3)$$

for $\rho \in \mathcal{B}_p(\Omega)$. Therefore, for $u \in L^p(\Omega, \rho)$, the distributional derivatives $D^\alpha u$ of u have a sense.

Definition 1.4 Let α be a multi-indices of length $|\alpha| \leq 1$ and $1 \leq p < \infty$. Suppose Λ_1 be a nonempty set of multi-indices and let $\Lambda = \{\ominus\} \cup \Lambda_1$ with $\ominus = (0, 0, \dots, 0)$. Denote

$$\rho(\Omega) = \{\rho_\alpha(x), x \in \Omega\}$$

with $\rho_\alpha(x) \in \mathcal{B}_p(\Omega)$. We define the Sobolev space with weight $W^{1,p}(\Omega, \rho)$ as the set of all functions $u \in L^p(\Omega, \rho_\ominus) \cap L^1_{loc}(\Omega)$ for which the weak derivatives $D^\alpha u$ with $\alpha \in \Lambda_1$ belong to $L^p(\Omega, \rho_\alpha) \cap L^1_{loc}(\Omega)$. The expression

$$\|v\|_{W^{1,p}(\Omega, \rho)} := \left(\sum_{|\alpha| \in \Lambda} \int_{\Omega} |D^\alpha v|^p \rho_\alpha dx \right)^{\frac{1}{p}} \quad (1.4)$$

is a norm on the linear space $W^{1,p}(\Omega, \rho)$.

Remark 1.2 i) If certain of the weight functions $\rho_\alpha \in \mathcal{B}_p(\Omega)$, then the assumption $D^\alpha u \in L^p(\Omega, \rho_\alpha) \cap L^1_{loc}(\Omega)$ in Definition 1.4 can be replaced in view of (1.3) by the assumption

$$D^\alpha u \in L^p(\Omega, \rho_\alpha).$$

ii) The classical Sobolev space $W^{1,p}(\Omega)$, which represent a special case of weighted spaces $W^{1,p}(\Omega, \rho)$ they result from setting

$$\rho_\alpha \equiv 1 \text{ for } |\alpha| \leq 1.$$

The norm of a function $u \in W^{1,p}(\Omega)$ will be denoted by

$$\|u\|_{W^{1,p}(\Omega)} := \left(\sum_{|\alpha| \leq 1} \int_{\Omega} |D^\alpha u|^p dx \right)^{\frac{1}{p}}.$$

Theorem 1.3 Let $\rho_\alpha \in \mathcal{B}_p(\Omega)$ for all $\alpha \in \Lambda$. Then $W^{1,p}(\Omega, \rho)$ is a Banach space if equipped with the norm (1.4).

The condition $\rho_\alpha \in \mathcal{B}_p(\Omega)$ in Theorem 1.3 is essential. Since when we take u_n be a Cauchy sequence in $W^{1,p}(\Omega, \rho)$, then $D^\alpha u_n$ is a Cauchy sequence in $L^p(\Omega, \rho_\alpha)$ for $\alpha \in \Lambda$ and by Theorem 1.1 we can argue as the proof in [25, Theorem 1.11].

Definition 1.5 Let $\rho_\alpha(x) \in \mathbb{B}_p(\Omega)$ and in addition

$$\rho_\alpha \in L^1_{loc}(\Omega) \text{ for } \alpha \in \Lambda. \quad (1.5)$$

Then we define

$$\overline{C_0^\infty(\Omega)} = W_0^{1,p}(\Omega, \rho). \quad (1.6)$$

the closure being taken with respect to the norm (1.4). It is again a Banach space if additionally (1.2) is satisfied.

Remark 1.3 Condition (1.5) is not only sufficient but also necessary for (1.6) (see [25, Lemma 4.4]).

1.2 ANISOTROPIC WEIGHTED SOBOLEV-TYPE SPACES

Let ω_1 and ω_2 be two open bounded subsets of \mathbb{R}^{n_1} (resp. \mathbb{R}^{n_2}), where n_1 and n_2 are two positive integers. We set

$$\Omega = \omega_1 \times \omega_2,$$

and split the components of a point $x \in \mathbb{R}^{n_1+n_2}$ into the n_1 first components and the n_2 last ones, i.e.

$$x := (X_1, X_2) \text{ where } X_1 := (x_1, \dots, x_{n_1}) \text{ and } X_2 := (x'_1, \dots, x'_{n_2}).$$

With this notation, we set

$$\nabla_{X_1} u := (\partial_{x_1} u, \dots, \partial_{x_{n_1}} u)^T, \quad \nabla_{X_2} u := (\partial_{x'_1} u, \dots, \partial_{x'_{n_2}} u)^T \text{ and } \Delta_{X_1} := \sum_{i=1}^{n_1} \partial_{x_i}^2.$$

Let h be a weight function on ω_1 , satisfying

$$h^{-1} \in L^1_{loc}(\omega_1) \text{ and } h \in L^1(\omega_1), \quad h > 0 \text{ a.e. in } \omega_1. \quad (1.7)$$

Then, with a special choice of the family ρ_α , we can consider the following weighted anisotropic Sobolev space

$$\mathcal{V}(\Omega, h) := \{v \in L^2(\Omega, h) \mid |\nabla_{X_2} v| \in L^2(\Omega, h)\}.$$

The above space is equipped with the norm

$$v \mapsto \left(\|v\|_{L^2(\Omega, h)}^2 + \|\nabla_{X_2} v\|_{L^2(\Omega, h)}^2 \right)^{1/2}. \quad (1.8)$$

For a weight function h satisfying (1.7), $\mathcal{V}(\Omega, h)$ is a Hilbert space. We will denote

$$\mathcal{V}_0(\Omega, h) := \overline{\mathcal{D}(\Omega)}, \text{ the closure with respect to the norm (1.8)}. \quad (1.9)$$

For convenience, this space will be equipped by the norm

$$v \mapsto \|\nabla_{X_2} v\|_{L^2(\Omega, h)},$$

which is equivalent to the norm (1.8), thanks to Poincaré's inequality in the X_2 -direction.

We also consider the weighted Sobolev space

$$\mathcal{W}(\Omega, h) := \{v \in L^2(\Omega, h) \mid |\nabla_{X_1} v| \in L^2(\Omega) \text{ and } |\nabla_{X_2} v| \in L^2(\Omega, h)\},$$

equipped with the norm (without weight for ∇_{X_1})

$$v \mapsto \left(\|v\|_{L^2(\Omega, h)}^2 + \|\varepsilon \nabla_{X_1} v\|_{L^2(\Omega)}^2 + \|\nabla_{X_2} v\|_{L^2(\Omega, h)}^2 \right)^{1/2}. \quad (1.10)$$

The parameter ε is introduced in the norm for convenience. We can easily see that if u is in $\mathcal{W}(\Omega, h)$, it is also in $L^2(\Omega)$. Under the assumption (1.7), $\mathcal{W}(\Omega, h)$ is also a Hilbert space and clearly $\mathcal{W}(\Omega, h) \subset \mathcal{V}(\Omega, h)$. Then, we define

$$\mathcal{W}_0(\Omega, h) := \overline{\mathcal{D}(\Omega)}, \text{ the closure with respect to the norm (1.10)}. \quad (1.11)$$

Using Poincaré's inequality, the application

$$v \mapsto \left(\varepsilon^2 \|\nabla_{X_1} v\|_{L^2(\Omega)}^2 + \|\nabla_{X_2} v\|_{L^2(\Omega, h)}^2 \right)^{1/2}, \quad (1.12)$$

is a norm for $\mathcal{W}_0(\Omega, h)$ equivalent to the norm (1.10).

Due to the presence of the non-local term, we also consider the space

$$\mathcal{L}^p(\Omega) := \{v \text{ measurable in } \Omega \mid l(v) \in L^p(\omega_2)\}, \quad (1.13)$$

equipped with the semi-norm $v \mapsto \|l(v)\|_{L^p(\omega_2)}$.

Remark 1.4 *By Hölder's inequality we have*

$$|l(u)|^p = \left| \int_{\omega_1} h u dX_1 \right|^p \leq \|h\|_{L^1(\omega_1)}^{p-1} \int_{\omega_1} h |u|^p dX_1$$

for $p \geq 1$. Then, integrating on ω_2 , we get

$$\|l(u)\|_{L^p(\omega_2)} \leq \|h\|_{L^1(\omega_1)}^{\frac{p-1}{p}} \|u\|_{L^p(\Omega, h)} \quad (1.14)$$

and thus $L^p(\Omega, h) \subset \mathcal{L}^p(\Omega)$.

Following the usual notation, we set

$$2^* := \begin{cases} 2n_2 / (n_2 - 2) & \text{if } n_2 > 2 \\ +\infty & \text{if } n_2 \leq 2. \end{cases}$$

Note that 2^* is related to n_2 , the dimension of ω_2 , and not to $n_1 + n_2$. The next compactness lemma is very useful to carry out some passage to the limit in the sequel.

Lemma 1.1 *Let w_n be a sequence such that $w_n \rightharpoonup w$ in $\mathcal{V}_0(\Omega, h)$. Then, we have*

$$l(w_n) \rightharpoonup l(w) \text{ in } H_0^1(\omega_2). \quad (1.15)$$

If $1 \leq p < 2^$, we moreover have the strong convergence*

$$l(w_n) \rightarrow l(w) \text{ in } L^p(\omega_2). \quad (1.16)$$

Proof. Using Remark 1.4 and the fact that h is independent of X_2 , we derive for $p = 2$ and $v \in \mathcal{V}_0(\Omega, h)$,

$$\|\nabla_{X_2} l(v)\|_{L^2(\omega_2)}^2 = \sum_{i=1}^{n_2} \int_{\omega_2} |l(\partial_{x_i} v)|^2 dX_2 \leq \|h\|_{L^1(\omega_1)} \int_{\Omega} h |\nabla_{X_2} v(x)|^2 dx.$$

Recalling that $u \mapsto \|\nabla_{X_2} u\|_{L^2(\omega_2)}$ defines a norm on $H_0^1(\omega_2)$, then

$$\|l(v)\|_{H_0^1(\omega_2)} \leq \|h\|_{L^1(\omega_1)}^{1/2} \|\nabla_{X_2} v\|_{L^2(\Omega, h)}. \quad (1.17)$$

Consequently, the linear operator $v \mapsto l(v)$ from $\mathcal{V}_0(\Omega, h)$ into $H_0^1(\omega_2)$ is continuous and the image of the weakly converging sequence $(w_n)_n$ remains weakly converging. This shows (1.15).

If $1 \leq p < 2^*$, the injection $H_0^1(\omega_2) \subset L^p(\omega_2)$ is compact and (1.16) follows. ■

Remark 1.5

i) *We can replace $\mathcal{V}_0(\Omega, h)$ in Lemma 1.1 by a Banach space V such that $V \subset \mathcal{V}_0(\Omega, h)$ with continuous injection. In the sequel, we often take $V = \mathcal{W}_0(\Omega, h)$.*

ii) *Since $h \in L^1(\omega_1)$, if a function $v \in H_0^1(\omega_2)$ then we have also $v \in \mathcal{V}_0(\Omega, h)$ and*

$$\|\nabla_{X_2} v\|_{L^2(\Omega, h)} = \|h\|_{L^1(\omega_1)}^{1/2} \|\nabla_{X_2} v\|_{L^2(\omega_2)}.$$

1.3 PROBLEMS SETTING AND h -WEAK FORMULATIONS

Using the spaces defined in the precedent section, we give a convenient weak formulation for problem (7) and its perturbed version.

1.3.1 The unperturbed problem

Let $r \geq 1$ and consider the space $\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ equipped with the norm

$$v \mapsto \|\nabla_{X_2} v\|_{L^2(\Omega, h)} + \|l(v)\|_{L^{r+1}(\omega_2)}.$$

We can show that $\mathcal{V}'_0(\Omega, h) \subset (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$ and $L^{\frac{r+1}{r}}(\omega_2) \subset (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$ with continuous injections (see Appendix (A.3)), hence

$$\mathcal{V}'_0(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2) \subset \left(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \right)'. \quad (1.18)$$

The space $\mathcal{V}'_0(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2)$ is equipped with the norm

$$v \mapsto \inf_{v=v_1+v_2} \left\{ \|v_1\|_{\mathcal{V}'_0(\Omega, h)} + \|v_2\|_{L^{\frac{r+1}{r}}(\omega_2)}, \text{ for } v_1 \in \mathcal{V}'_0(\Omega, h), v_2 \in L^{\frac{r+1}{r}}(\omega_2) \right\}. \quad (1.19)$$

Arguing as in [15, Corollary 1], we can show that the elements of $\mathcal{V}_0(\Omega, h)$ are the functions with zero trace on $\omega_1 \times \partial\omega_2$. More precisely,

$$\mathcal{V}_0(\Omega, h) = \left\{ v \in \mathcal{V}(\Omega, h) \mid v(X_1, \cdot) \in H^1_0(\omega_2) \text{ a.e. } X_1 \in \omega_1 \right\}. \quad (1.20)$$

Let us suppose that $A = (a_{ij}(x))$ an $n_2 \times n_2$ symmetric matrix satisfying

$$a_{ij} \in L^\infty(\Omega), \forall i, j = 1 \dots n_2, \quad (1.21)$$

$$\alpha |\zeta|^2 \leq A(x) \zeta \cdot \zeta \leq \beta |\zeta|^2, \forall \zeta \in \mathbb{R}^{n_2}, \text{ a.e. } x \in \Omega, \quad (1.22)$$

for some constants $\alpha, \beta > 0$. This allows us to state Problem (7) as follows

$$\begin{cases} -\nabla_{X_2} \cdot (A \nabla_{X_2} u) = g(X_2, l(u)) + f_1(x) + f_2(X_2) & \text{in } \Omega, \\ u \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \end{cases} \quad (1.23)$$

where $g : \omega_2 \times \mathbb{R} \rightarrow \mathbb{R}$ is a (nonlinear) function involving the non-local term l defined as

$$l(u) := \int_{\omega_1} h(X_1) u(X_1, X_2) dX_1. \quad (1.24)$$

As mentioned in the introduction, we shall investigate the existence of solutions in the following sense. We say that a function $u \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ is an h -weak solution to Problem (1.23) if the integral identity

$$\int_{\Omega} h A \nabla_{X_2} u \cdot \nabla_{X_2} v dx = \int_{\omega_2} g(X_2, l(u)) l(v) dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)} + \int_{\omega_2} f_2 l(v) dX_2, \quad (SP_0)$$

holds for every $v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. The two source terms f_1 and f_2 are supposed to satisfy

$$f_1 \in \mathcal{V}'_0(\Omega, h), f_2 \in L^{\frac{p}{p-1}}(\omega_2) \quad (1.25)$$

where $\mathcal{V}'_0(\Omega, h)$ is the dual space of $\mathcal{V}_0(\Omega, h)$.

To characterize the elements of $\mathcal{V}'_0(\Omega, h)$, consider $f \in \mathcal{V}'_0(\Omega, h)$. Then, by Riesz's representation theorem there exists $\phi \in \mathcal{V}_0(\Omega, h)$, such that

$$\langle f, v \rangle_{\mathcal{V}'_0(\Omega, h)} = \int_{\Omega} h \nabla_{X_2} \phi \cdot \nabla_{X_2} v dx, \quad \forall v \in \mathcal{V}_0(\Omega, h).$$

Here and in the sequel, the brackets $\langle \cdot, \cdot \rangle_X$ denotes the dual product between a Banach space X and its dual. Setting $\phi_i = -\partial_{x'_i} \phi \in L^2(\Omega, h)$, $i = 1, \dots, n_2$, we can write

$$f = h \sum_{i=1}^{n_2} \partial_{x'_i} \phi_i.$$

This defines a one-to-one correspondence between $\mathcal{V}'_0(\Omega, h)$ and the set

$$\left\{ \sum_{i=1}^{n_2} \partial_{x'_i} \phi_i, \phi_i \in L^2(\Omega, h) \right\}.$$

From now on, we shall mean by $f \in \mathcal{V}'_0(\Omega, h)$ its corresponding element $\sum_{i=1}^{n_2} \partial_{x'_i} \phi_i$ in the above set. May be we have to point out that such element should be tested by hv to recover an element of $\mathcal{V}'_0(\Omega, h)$, i.e.

$$\left\langle -\sum_{i=1}^{n_2} \partial_{x'_i} \phi_i, hv \right\rangle := \left\langle -h \sum_{i=1}^{n_2} \partial_{x'_i} \phi_i, v \right\rangle_{D(\Omega)} = \int_{\Omega} h \sum_{i=1}^{n_2} \phi_i \partial_{x'_i} v dx = \langle f, v \rangle_{\mathcal{V}'_0(\Omega, h)}, \quad \forall v \in D(\Omega).$$

1.3.2 The perturbed problem

We deal with the following anisotropic perturbation of Problem (1.23)

$$\begin{cases} -\frac{\varepsilon^2}{h} \Delta_{X_1} u_\varepsilon - \nabla_{X_2} \cdot (A \nabla_{X_2} u_\varepsilon) = g(X_2, l(u_\varepsilon)) + f_1(x) + f_2(X_2) & \text{in } \Omega, \\ u_\varepsilon \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \end{cases} \quad (1.26)$$

where the space $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ is equipped with the norm

$$v \mapsto \left(\varepsilon^2 \|\nabla_{X_1} v\|_{L^2(\Omega)}^2 + \|\nabla_{X_2} v\|_{L^2(\Omega, h)}^2 \right)^{1/2} + \|l(v)\|_{L^{r+1}(\omega_2)}. \quad (1.27)$$

We can show as for (1.18) that

$$\mathcal{W}'_0(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2) \subset \left(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \right)',$$

where the norm of $\mathcal{W}'_0(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2)$ is defined as in (1.19), with $\mathcal{V}'_0(\Omega, h)$ is replaced by $\mathcal{W}'_0(\Omega, h)$.

We say that a function $u_\varepsilon \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ is an h -weak solution to Problem (1.26) if the integral identity

$$\begin{aligned} \int_{\Omega} \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + h A \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v dx &= \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(v) dX_2 \\ &+ \langle f_1, v \rangle_{\mathcal{V}'_0(\Omega, h)} + \int_{\omega_2} f_2 l(v) dX_2 \quad (SP_\varepsilon) \end{aligned}$$

holds for every $v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Remark 1.6 Let $v \in \mathcal{W}_0(\Omega, h)$. On the boundary $\omega_1 \times \partial\omega_2$, we have $v(X_1, \cdot) \in H_0^1(\omega_2)$ for a.e. $X_1 \in \omega_1$ since $\mathcal{W}_0(\Omega, h) \subset \mathcal{V}_0(\Omega, h)$. There is no issue to define the trace of v on the boundary $\partial\omega_1 \times \omega_2$ since the L^2 -norm of the gradient ∇_{X_1} is independent of h . In general, a definition of the boundary values depends on the speed at which the weight degenerates on the boundary, see for instance [19, 33].

1.4 C^1 -FUNCTIONALS INVOLVING THE NON-LOCAL TERM l

Concerning the nonlinearity, we assume that $g : \omega_2 \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, i.e.

$$\begin{cases} i) s \mapsto g(X_2, s) \text{ is continuous for a.e. } X_2 \in \omega_2, \\ ii) X_2 \mapsto g(X_2, s) \text{ is measurable for every } s \in \mathbb{R}, \end{cases} \quad (1.28)$$

and satisfies

$$|g(X_2, s)| \leq \gamma_1 (|s| + |s|^r), \text{ for a.e. } X_2 \in \omega_2 \text{ and } \forall s \in \mathbb{R}, \quad (1.29)$$

where $1 \leq r < +\infty$ and $\gamma_1 > 0$ are constants. Then, define

$$G(X_2, t) := \int_0^t g(X_2, s) ds, \text{ for } X_2 \in \omega_2 \text{ and } t \in \mathbb{R}. \quad (1.30)$$

As a consequence of (1.29), it follows that

$$|G(X_2, t)| \leq \gamma_1 \left(\frac{|t|^2}{2} + \frac{|t|^{r+1}}{r+1} \right), \text{ for a.e. } X_2 \in \omega_2 \text{ and } t \in \mathbb{R}. \quad (1.31)$$

This allows us to define the functional

$$J_g(\bar{v}) := \int_{\omega_2} G(X_2, \bar{v}) dX_2, \text{ for every } \bar{v} \in L^{r+1}(\omega_2). \quad (1.32)$$

To determine the derivative of J_g , we first need the following lemma.

Lemma 1.2 Assume that h satisfies (1.7) and let $g_0 : \omega_2 \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying (1.28) and

$$|g_0(X_2, s)| \leq \gamma_0 |s|^r, \text{ for } s \in \mathbb{R} \text{ and a.e. } X_2 \in \omega_2, \quad (1.33)$$

for $1 \leq r < +\infty$ and some constant $\gamma_0 \geq 0$. Set

$$J_{g_0}(u) := \int_{\omega_2} G_0(X_2, l(u)) dX_2, \text{ for } u \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega),$$

where G_0 is defined by g_0 as (1.30). Then, the mapping

$$u \mapsto g_0(\cdot, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \text{ into } L^{\frac{r+1}{r}}(\omega_2),$$

$J_{g_0} \in C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and its derivative is given by

$$\left\langle J'_{g_0}(u), v \right\rangle_{\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} = \int_{\omega_2} g_0(X_2, l(u)) l(v) dX_2, \quad (1.34)$$

for every $u, v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Proof. Let us consider the functional

$$j_{g_0}(\bar{v}) = \int_{\omega_2} G_0(X_2, \bar{v}) dX_2, \quad \text{for } \bar{v} \in L^{r+1}(\omega_2). \quad (1.35)$$

Thanks to (1.33), we can argue as in [8, page 37], to show that

$$\bar{u} \mapsto g_0(\cdot, \bar{u}) \text{ is continuous from } L^{r+1}(\omega_2) \text{ into } L^{\frac{r+1}{r}}(\omega_2),$$

$j_{g_0} \in C^1(L^{r+1}(\omega_2), \mathbb{R})$ and its derivative is given by

$$\langle j'_{g_0}(\bar{u}), \bar{v} \rangle_{L^{r+1}(\omega_2)} = \int_{\omega_2} g_0(X_2, \bar{u}) \bar{v} dX_2, \quad \forall \bar{u}, \bar{v} \in L^{r+1}(\omega_2).$$

From (1.17), we infer in particular that

$$u \mapsto l(u), \text{ from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \text{ into } L^{r+1}(\omega_2)$$

is a continuous linear mapping. Consequently, the composite mapping

$$u \mapsto (g_0 \circ l)(u) = g_0(\cdot, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \text{ into } L^{\frac{r+1}{r}}(\omega_2),$$

$J_{g_0} = j_{g_0} \circ l(\cdot) \in C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and its derivative is given by

$$\langle J'_{g_0}(u), v \rangle_{\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} = \langle j'_{g_0}(l(u)), l(v) \rangle_{L^{r+1}(\omega_2)}, \text{ for } u, v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega).$$

This shows (1.34) and the lemma follows. ■

In the next section, we investigate the existence of solutions as critical points of the functional

$$\begin{aligned} \mathbf{J}^\varepsilon(u) := & \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u|^2 + hA \nabla_{X_2} u \cdot \nabla_{X_2} u dx \\ & - \int_{\omega_2} G(X_2, l(u)) dX_2 - \langle f_1, u \rangle_{\mathcal{V}_0(\Omega, h)} - \int_{\omega_2} f_2 l(u) dX_2. \end{aligned} \quad (1.36)$$

Thus, we need to compute the derivative of \mathbf{J}^ε .

Theorem 1.4 *Let $1 \leq r < +\infty$. Assume that g satisfies (1.28) and (1.29). Then, the mapping*

$$u \mapsto g(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \text{ into } \mathcal{W}'_0(\Omega, h) + L^{\frac{r+1}{r}}(\Omega),$$

$\mathbf{J}^\varepsilon \in C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and its derivative is given by

$$\begin{aligned} \langle \mathbf{J}'^\varepsilon(u), v \rangle_{\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} = & \int_{\Omega} \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + hA \nabla_{X_2} u \cdot \nabla_{X_2} v dx \\ & - \int_{\omega_2} g(X_2, l(u)) l(v) dX_2 - \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)} - \int_{\omega_2} f_2 l(v) dX_2, \end{aligned} \quad (1.37)$$

for all $u, v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Proof. First, to show the continuity of $u \mapsto g(X_2, l(u))$ we split g into two parts g_1 and g_2

$$g(X_2, s) = g_1(X_2, s) + g_2(X_2, s),$$

where $g_1(X_2, s) = g(X_2, s)$ for $|s| \leq 1$ and $g_1(X_2, s) = 0$ for $|s| \geq 2$, such that

$$|g_1(X_2, s)| \leq C|s|, \quad |g_2(X_2, s)| \leq C|s|^r,$$

for a.e. $X_2 \in \omega_2$, and some positive constant C .

Thanks to Lemma 1.2 and Inequality (1.14), for $p = 2$, we see that the mapping

$$u \mapsto g_1(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^2(\Omega) \text{ into } L^2(\omega_2).$$

Since $\mathcal{W}_0(\Omega, h) \subset L^2(\Omega, h) \subset \mathcal{L}^2(\Omega)$ and $L^2(\omega_2) \subset L^2(\Omega, h) \subset \mathcal{W}'_0(\Omega, h)$, we infer that

$$u \mapsto g_1(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \text{ into } \mathcal{W}'_0(\Omega, h).$$

As well, due to Lemma 1.2, the mapping

$$u \mapsto g_2(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega) \text{ into } L^{\frac{r+1}{r}}(\omega_2).$$

Recalling that $\mathcal{W}'_0(\Omega, h) + L^{\frac{r+1}{r}}(\Omega)$ is equipped with the norm defined in (1.19), then the mapping

$$u \mapsto g(X_2, l(u)) = (g_1 + g_2)(X_2, l(u))$$

is continuous from $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ into $\mathcal{W}'_0(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2)$ and

$$\langle J'_g(u), v \rangle_{\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} = \int_{\omega_2} g(X_2, l(u)) l(v) dX_2, \quad \text{for } u, v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega). \quad (1.38)$$

To give the derivatives for the remaining terms in the definition of J^ε , we set

$$J^\varepsilon_{\nabla} : \mathcal{W}_0(\Omega, h) \mapsto \mathbb{R}, J^\varepsilon_{\nabla}(u) := \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u|^2 + hA \nabla_{X_2} u \cdot \nabla_{X_2} u dx,$$

$$J_{f_1} : \mathcal{V}_0(\Omega, h) \mapsto \mathbb{R}, J_{f_1}(u) := \langle f_1, u \rangle_{\mathcal{V}_0(\Omega, h)},$$

$$J_{f_2} : \mathcal{L}^{r+1}(\Omega) \cap \mathcal{V}_0(\Omega, h) \mapsto \mathbb{R}, J_{f_2}(u) := \int_{\omega_2} f_2 l(u) dX_2.$$

First, $J^\varepsilon_{\nabla} \in C^1(\mathcal{W}_0(\Omega, h), \mathbb{R}) \subset C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and J^ε_{∇} is given by

$$\begin{aligned} \langle J^{\varepsilon'}_{\nabla}(u), v \rangle_{\mathcal{W}_0(\Omega, h)} &:= \lim_{t \rightarrow 0} \frac{J^\varepsilon_{\nabla}(u + tv) - J^\varepsilon_{\nabla}(u)}{t} \\ &= \lim_{t \rightarrow 0} \frac{1}{2t} \int_{\Omega} \varepsilon^2 |\nabla_{X_1}(u + tv)|^2 + hA_{22} \nabla_{X_2}(u + tv) \cdot \nabla_{X_2}(u + tv) dx \\ &\quad - \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u|^2 + hA_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u dx \\ &= \lim_{t \rightarrow 0} \int_{\Omega} \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + t \varepsilon^2 |\nabla_{X_1} v|^2 + hA_{22} \nabla_{X_2} u \cdot \nabla_{X_2} v \\ &\quad + thA_{22} \nabla_{X_2} v \cdot \nabla_{X_2} v dx, \end{aligned}$$

hence

$$\langle J_{\nabla}^{\varepsilon l}(u), v \rangle_{\mathcal{W}_0(\Omega, h)} = \int_{\Omega} \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + h A \nabla_{X_2} u \cdot \nabla_{X_2} v dx. \quad (1.39)$$

Next, as the functionals J_{f_1} and J_{f_2} are linear, then $J_{f_1}, J_{f_2} \in C^1(\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and their derivatives are given by

$$\langle J'_{f_1}(u), v \rangle_{\mathcal{V}_0(\Omega, h)} = \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)} \quad \text{and} \quad \langle J'_{f_2}(u), v \rangle_{\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} = \int_{\omega_2} f_2 l(v) dX_2. \quad (1.40)$$

Taking into account (1.38), (1.39) and (1.40), we obtain (1.37) since $\mathbf{J}^{\varepsilon} = J_{\nabla}^{\varepsilon} - J_g - J_{f_1} - J_{f_2}$. ■

Corollary 1.1 *Let $1 \leq r < 2^* - 1$. Assume that g satisfies (1.28) and (1.29). Then,*

$$u \rightarrow g(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0(\Omega, h) \rightarrow \mathcal{W}'_0(\Omega, h),$$

$\mathbf{J}^{\varepsilon} \in C^1(\mathcal{W}_0(\Omega, h), \mathbb{R})$ and (1.37) holds for all $u, v \in \mathcal{W}_0(\Omega, h)$.

Proof. By Sobolev's embedding $H_0^1(\omega_2) \subset L^{r+1}(\omega_2)$ and (1.17), we get

$$\begin{aligned} \|l(v)\|_{L^{r+1}(\omega_2)} &\leq C_S \|h\|_{L^1(\omega_1)}^{\frac{1}{2}} \|\nabla_{X_2} v\|_{L^2(\Omega, h)} \\ &\leq C_S \|h\|_{L^1(\omega_1)}^{\frac{1}{2}} \left(\varepsilon^2 \|\nabla_{X_1} v\|_{L^2(\Omega)}^2 + \|\nabla_{X_2} v\|_{L^2(\Omega, h)}^2 \right)^{\frac{1}{2}} \end{aligned} \quad (1.41)$$

for all $v \in \mathcal{W}_0(\Omega, h)$. Here, C_S denotes the Sobolev constant. Thus, $\mathcal{W}_0(\Omega, h) \subset \mathcal{L}^{r+1}(\Omega)$ and the corollary results follow from Theorem 1.4. ■

Remark 1.7 *In a similar manner, we can also show that Theorem 1.4 and Corollary 1.1 still hold if we replace $\mathcal{W}_0(\Omega, h)$ by $\mathcal{V}_0(\Omega, h)$ and \mathbf{J}^{ε} by the functional*

$$\mathbf{J}^0(v) := \frac{1}{2} \int_{\Omega} h A \nabla_{X_2} v \cdot \nabla_{X_2} v dx - \int_{\omega_2} G(X_2, l(v)) dX_2 - \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)} - \int_{\omega_2} f_2 l(v) dX_2. \quad (1.42)$$

A SEMILINEAR PROBLEM WITH A NON-LOCAL NONLINEARITY

In this chapter, we aim to establish the existence of solutions u_ε for Problem (SP_ε) as global minimizers for \mathbf{J}^ε , defined by (1.36), provided that G satisfies some upper boundedness assumption. Then we study the asymptotic behavior of u_ε and $\mathbf{J}^\varepsilon(u_\varepsilon)$, as $\varepsilon \rightarrow 0$.

2.1 AN EXISTENCE RESULT FOR THE PERTURBED PROBLEM

We begin with the following existence result.

Theorem 2.1 *Let $\varepsilon > 0$ and assume that g satisfies (1.28) and (1.29) for $r \geq 1$. Suppose further that G , defined by (1.30) satisfies,*

$$G(X_2, t) \leq C_1 - a|t|^{r+1}, \quad \forall t \in \mathbb{R}, \quad (2.1)$$

for a.e. $X_2 \in \omega_2$ and some constants $C_1 \geq 0, a > 0$. Then, there exists $u \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ such that

$$\mathbf{J}^\varepsilon(u) = \inf_{v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} \mathbf{J}^\varepsilon(v). \quad (2.2)$$

In particular, u is a solution of (SP_ε) .

Proof. *A priori estimates.* Thanks to (2.1), we deduce that

$$\begin{aligned} \mathbf{J}^\varepsilon(v) \geq & \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h A \nabla_{X_2} v \cdot \nabla_{X_2} v dx + a \int_{\omega_2} |l(v)|^{r+1} dX_2 \\ & - \langle f_1, v \rangle_{\mathcal{V}'_0(\Omega, h)} - \int_{\omega_2} f_2 l(v) dX_2 - C_1 |\omega_2|, \end{aligned} \quad (2.3)$$

where $|\omega_2|$ denotes the Lebesgue measure of the set ω_2 . Since $\mathcal{V}_0(\Omega, h)$ is equipped with the norm $v \mapsto \|\nabla_{X_2} v\|_{L^2(\Omega, h)}$, we infer that

$$\begin{aligned} \langle f_1, v \rangle_{\mathcal{V}'_0(\Omega, h)} & \leq \|f_1\|_{\mathcal{V}'_0(\Omega, h)} \left(\int_{\Omega} h |\nabla_{X_2} v|^2 dx \right)^{1/2} \\ & \leq \|f_1\|_{\mathcal{V}'_0(\Omega, h)} \left(\int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx \right)^{1/2}. \end{aligned} \quad (2.4)$$

Then, by (1.22) and Hölder's inequality it comes that

$$\begin{aligned} \mathbf{J}^\varepsilon(v) &\geq \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h \alpha |\nabla_{X_2} v|^2 dx + a \int_{\omega_2} |l(v)|^{r+1} dX_2 \\ &\quad - \|f_1\|_{\mathcal{W}'_0(\Omega, h)} \left(\int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx \right)^{1/2} - \|f_2\|_{L^{\frac{r+1}{r}}(\omega_2)} \|l(v)\|_{L^{r+1}(\omega_2)} - C_1 |\omega_2|. \end{aligned}$$

Applying Young's inequality

$$ts \leq \frac{\min\{1, \alpha\}}{4} t^2 + \frac{s^2}{\min\{1, \alpha\}} \text{ and } \left(\frac{(r+1)a}{2} \right)^{\frac{1}{r+1}} t \cdot \left(\frac{2}{a(r+1)} \right)^{\frac{1}{r+1}} s \leq \frac{a}{2} t^{r+1} + \frac{2^{\frac{1}{r}} r}{a^{\frac{1}{r}} (r+1)^{\frac{r+1}{r}}} s^{\frac{r+1}{r}},$$

we get

$$\begin{aligned} \mathbf{J}^\varepsilon(v) &\geq \frac{\min\{1, \alpha\}}{4} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx + \frac{a}{2} \int_{\omega_2} |l(v)|^{r+1} dX_2 \\ &\quad - \frac{1}{\min\{1, \alpha\}} \|f_1\|_{\mathcal{W}'_0(\Omega, h)}^2 - \frac{2^{\frac{1}{r}} r}{a^{\frac{1}{r}} (r+1)^{\frac{r+1}{r}}} \|f_2\|_{L^{\frac{r+1}{r}}(\omega_2)}^{\frac{r+1}{r}} - C_1 |\omega_2|, \quad (2.5) \end{aligned}$$

for all $v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. Thus \mathbf{J}^ε is bounded below in $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Passage to the limit. We set

$$d^\varepsilon := \inf_{v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} \mathbf{J}^\varepsilon(v),$$

and let $(u_n)_{n \geq 0}$ be a minimizing sequence. By (2.5), such a sequence is necessarily bounded in $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$, i.e.

$$\begin{aligned} u_n, |\nabla_{X_2} u_n| &\text{ are bounded in } L^2(\Omega, h), \\ |\nabla_{X_1} u_n| &\text{ is bounded in } L^2(\Omega), \\ l(u_n) &\text{ is bounded in } L^{r+1}(\omega_2). \end{aligned}$$

The boundedness of u_n is a consequence of Poincaré's inequality in the X_2 -direction. Then, there exist $u \in \mathcal{W}_0(\Omega, h)$, $U_1 \in (L^2(\Omega))^{n_1}$ and $U_2 \in (L^2(\Omega, h))^{n_2}$ such that –up to a subsequence–

$$\begin{aligned} u_n &\rightharpoonup u, \nabla_{X_2} u_n \rightharpoonup U_2 \text{ in } L^2(\Omega, h) \text{ and } \nabla_{X_1} u_n \rightharpoonup U_1, \text{ in } L^2(\Omega), \\ l(u_n) &\rightharpoonup l(u) \text{ in } L^{r+1}(\omega_2), \end{aligned}$$

as $n \rightarrow +\infty$. (These convergences meant component by component convergences). Since the above convergences imply the convergence in the distributional sense, we can easily see that

$$u_n \rightharpoonup u, \nabla_{X_2} u_n \rightharpoonup \nabla_{X_2} u \text{ in } L^2(\Omega, h) \quad (2.6)$$

$$\nabla_{X_1} u_n \rightharpoonup \nabla_{X_1} u \text{ in } L^2(\Omega). \quad (2.7)$$

Using Lemma 1.1, with $p = 2$, we obtain the strong convergence $l(u_n) \rightarrow l(u)$ in $L^2(\omega_2)$. Then, up to a subsequence

$$l(u_n) \rightarrow l(u) \text{ a.e. in } \omega_2 \quad (2.8)$$

and the continuity of G implies that

$$G(X_2, l(u_n)) \rightarrow G(X_2, l(u)) \text{ a.e. in } \omega_2, \text{ as } n \rightarrow +\infty. \quad (2.9)$$

To perform the passage to the limit in \mathbf{J}^ε , we write

$$\begin{aligned} \mathbf{J}^\varepsilon(v) &= \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_n|^2 + hA \nabla_{X_2} u_n \cdot \nabla_{X_2} u_n dx \\ &\quad + \int_{\omega_2} (C_1 - G(X_2, l(u_n))) dX_2 - C_1 |\omega_2| - \langle f_1, u_n \rangle_{\mathcal{V}_0(\Omega, h)} - \int_{\omega_2} f_2 l(u_n) dX_2, \end{aligned} \quad (2.10)$$

where $C_1 - G(X_2, v) \geq 0$, for a.e. $X_2 \in \omega_2$. Then, Fatou's lemma and the convergences (2.8) and (2.9) imply

$$\int_{\omega_2} C_1 - G(X_2, l(u)) dX_2 \leq \liminf_{n \rightarrow \infty} \int_{\omega_2} C_1 - G(X_2, l(u_n)) dX_2.$$

Besides, since A is symmetric and satisfies (1.22), then the mapping

$$v \mapsto \left(\int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx \right)^{1/2}$$

defines a norm on $\mathcal{W}_0(\Omega, h)$. Thanks to (2.6), (2.7) and the lower semi-continuity of norms, we deduce that

$$\int_{\Omega} \varepsilon^2 |\nabla_{X_1} u|^2 + hA \nabla_{X_2} u \cdot \nabla_{X_2} u dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_n|^2 + hA \nabla_{X_2} u_n \cdot \nabla_{X_2} u_n dx.$$

Passing to the limit in (2.10), we get

$$\mathbf{J}^\varepsilon(u) \leq \liminf_{n \rightarrow \infty} \mathbf{J}^\varepsilon(u_n) = d^\varepsilon.$$

Therefore $\mathbf{J}^\varepsilon(u) = d^\varepsilon$.

Finally, by Theorem 1.4 we have $(\mathbf{J}^\varepsilon)'(u) = 0$, i.e. u is a solution to (SP_ε) . This ends the proof of the theorem. ■

Remark 2.1 Problems (SP_ε) may have one or several solutions, depending on the assumptions on g, f_1 and f_2 . If the mapping $t \mapsto g(\cdot, t)$ is non increasing, then the operator

$$u \mapsto -\frac{\varepsilon^2}{h} \Delta_{X_1} u - \nabla_{X_2} \cdot (A \nabla_{X_2} u) - g(\cdot, u) \quad (2.11)$$

is strictly monotone. In this case the solution of Problem (SP_ε) given by Theorem 2.1 is unique.

Example 2.1 *Theorem 2.1 applies to the problem*

$$\begin{cases} -\frac{\varepsilon^2}{h} \Delta_{X_1} u - \nabla_{X_2} \cdot (A \nabla_{X_2} u) + a_1 |l(u)|^{r-1} l(u) \\ \quad - a_2 |l(u)|^{q-1} l(u) = f_1 + f_2 \quad \text{in } \mathcal{W}'_0(\Omega, h) + L^{\frac{r+1}{r}}(\omega_2), \\ u \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega). \end{cases}$$

Here we have taken

$$g(X_2, t) = -a_1 |t|^{r-1} t + a_2 |t|^{q-1} t,$$

where $a_1, a_2 \in \mathbb{R}, a_1 > 0$ and $1 \leq q < r$. The assumptions (1.29) and (2.1) hold, since

$$|t|^q < |t| + |t|^r \quad \text{and} \quad a_2 |t|^{q+1} \leq C + \frac{a_1(q+1)}{2(r+1)} |t|^{r+1},$$

for some constant $C > 0$. Indeed

$$\begin{aligned} \left| -a_1 |t|^{r-1} t + a_2 |t|^{q-1} t \right| &\leq a_1 |t|^r + a_2 |t|^q \\ &\leq a_1 |t|^r + a_2 (|t| + |t|^r) \\ &\leq \max(a_1 + a_2, a_2) (|t| + |t|^r) \end{aligned}$$

and

$$\begin{aligned} G(X_2, t) &= \frac{-a_1}{r+1} |t|^{r+1} + \frac{a_2}{q+1} |t|^{q+1} \\ &\leq \frac{-a_1}{r+1} |t|^{r+1} + \frac{1}{q+1} \left(C + \frac{a_1(q+1)}{2(r+1)} |t|^{r+1} \right) \\ &\leq \frac{-a_1}{r+1} |t|^{r+1} + C + \frac{a_1}{2(r+1)} |t|^{r+1} \\ &\leq C - \frac{a_1}{r+1} |t|^{r+1}. \end{aligned}$$

2.2 AN EXISTENCE RESULT FOR THE UNPERTURBED PROBLEM

Letting now $\varepsilon \rightarrow 0$, we have the following convergence results that can be improved in the next theorem.

Lemma 2.1 *Let u_ε be a solution of Problem (SP $_\varepsilon$). Under the assumptions of Theorem 2.1, we have –up to a subsequence–*

$$u_\varepsilon \rightharpoonup u_0, \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h), \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \text{ in } L^2(\Omega), \quad (2.12)$$

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^2(\omega_2), \quad (2.13)$$

$$l(u_\varepsilon) \rightharpoonup l(u_0) \text{ in } L^{r+1}(\omega_2), \quad (2.14)$$

as $\varepsilon \rightarrow 0$, where u_0 is a solution of Problem (SP $_0$).

Moreover, the above convergences hold for the whole sequence if the solution of Problem (SP $_0$) is unique.

Proof. Without loss of generality, we assume that $0 < \varepsilon < 1$. Then, we know that

$$\mathbf{J}^\varepsilon(u_\varepsilon) \leq \mathbf{J}^\varepsilon(v) \leq \mathbf{J}^1(v), \forall v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega). \quad (2.15)$$

Thanks to (2.5), we deduce that

$$\begin{aligned} & \frac{\min\{1, \alpha\}}{4} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 dx + h |\nabla_{X_2}(u_\varepsilon)|^2 dx + \frac{a}{2} \|l(u_\varepsilon)\|_{L^{r+1}(\omega_2)}^{r+1} \\ & \leq \mathbf{J}^1(v) + \|f_1\|_{\mathcal{V}'_0(\Omega, h)}^2 + C_1 |\omega_2| + \frac{2^{\frac{1}{r}} r}{a^{\frac{1}{r}} (r+1)^{\frac{r+1}{r}}} \|f_2\|_{L^{\frac{r+1}{r}}(\omega_2)}^{\frac{r+1}{r}}. \end{aligned} \quad (2.16)$$

and it follows that

$$\begin{aligned} & u_\varepsilon, |\nabla_{X_2} u_\varepsilon| \text{ are bounded in } L^2(\Omega, h), \\ & |\varepsilon \nabla_{X_1} u_\varepsilon| \text{ is bounded in } L^2(\Omega), \end{aligned}$$

and

$$l(u_\varepsilon) \text{ is bounded in } L^{r+1}(\omega_2). \quad (2.17)$$

Then, there exist $u_0 \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$, $\tilde{U}_1 \in (L^2(\Omega))^{n_1}$, $\tilde{U}_2 \in (L^2(\Omega, h))^{n_2}$ such that –up to a subsequence –

$$u_\varepsilon \rightharpoonup u_0, \nabla_{X_2} u_\varepsilon \rightharpoonup \tilde{U}_2 \text{ in } L^2(\Omega, h) \text{ and } \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup \tilde{U}_1 \text{ in } L^2(\Omega),$$

as $\varepsilon \rightarrow 0$. Since the above convergences imply the convergence in the distributional sense, we can check that

$$u_\varepsilon \rightharpoonup u_0 \text{ and } \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h). \quad (2.18)$$

As well, since $|\varepsilon \nabla_{X_1} u_\varepsilon|$ is bounded in $L^2(\Omega)$ and $\frac{\operatorname{div}_{X_1} \Phi}{\sqrt{h}} \in L^2(\Omega, h)$, $\forall \Phi \in (D(\Omega))^{n_1}$, then

$$\int_{\Omega} \varepsilon \nabla_{X_1} u_\varepsilon \cdot \Phi dx = \int_{\Omega} \varepsilon \sqrt{h} u_\varepsilon \left(-\frac{\operatorname{div}_{X_1} \Phi}{\sqrt{h}} \right) dx \rightarrow 0, \quad \forall \Phi \in (D(\Omega))^{n_1}.$$

It follows that

$$\varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0, \text{ in } L^2(\Omega). \quad (2.19)$$

Besides, we have

$$l(u_\varepsilon) \rightharpoonup l(u_0) \text{ in } L^{r+1}(\omega_2), \quad (2.20)$$

and also the strong convergence $l(u_\varepsilon) \rightarrow l(u_0)$ in $L^2(\omega_2)$ from Lemma 1.1 with $p = 2$. Then, we have –up to a new subsequence–

$$l(u_\varepsilon) \rightarrow l(u_0), \text{ a.e. in } \omega_2. \quad (2.21)$$

Assumption (1.29) and Young's inequality $t \leq (r + |t|^{r+1})/r + 1$ imply that

$$|g(X_2, l(u_\varepsilon))|^{r+1} \leq C \left(1 + |l(u_\varepsilon)|^{r+1}\right). \quad (2.22)$$

Thus $g(X_2, l(u_\varepsilon))$ is bounded in $L^{\frac{r+1}{r}}(\omega_2)$ since $l(u_\varepsilon)$ is bounded in $L^{r+1}(\omega_2)$. Moreover, (2.21) and the continuity of g ensures that

$$g(X_2, l(u_\varepsilon)) \rightarrow g(X_2, l(u_0)), \text{ a.e. in } \omega_2.$$

Then, due to Lemma 1.3 in [31, page 12], we infer that

$$g(X_2, l(u_\varepsilon)) \rightharpoonup g(X_2, l(u_0)) \text{ in } L^{\frac{r+1}{r}}(\omega_2). \quad (2.23)$$

The convergences (2.18) – (2.20) and (2.23) allow us to pass to the limit in (SP_ε) . It comes that u_0 satisfies

$$\int_{\Omega} h A \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dx = \int_{\omega_2} g(X_2, l(u_0)) l(v) dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)} + \int_{\omega_2} f_2 l(v) dX_2, \quad (2.24)$$

for all $v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. By density of $\mathcal{W}_0(\Omega, h)$ in $\mathcal{V}_0(\Omega, h)$, this means that the limit u_0 is a solution to Problem (SP_0) . ■

Remark 2.2 Under the assumption (1.29) and (2.1), the function $g(X_2, t)$ behaves like $-|t|^{r-1}t$ for large t , but it is not necessarily monotone. This prevents the use of the monotonicity argument to get the strong convergence as in [14, 15]. An example illustrating this situation can be given by $g(t) = -t - (2 + \cos(e^t))|t|^{r-1}t$.

We have the following result about the convergence for the minimum of J^ε as $\varepsilon \rightarrow 0$.

Theorem 2.2 Under the assumptions of Lemma 2.1, the converging subsequences give a smooth minimizing sequences for the functional J^0 . That is

$$J^\varepsilon(u_\varepsilon) \rightarrow J^0(u_0) = \inf_{v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} J^0(v), \quad (2.25)$$

as $\varepsilon \rightarrow 0$, where the functional J^0 is defined by (1.42). Moreover, we have the strong convergences

$$u_\varepsilon \rightarrow u_0, \nabla_{X_2} u_\varepsilon \rightarrow \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h), \varepsilon \nabla_{X_1} u_\varepsilon \rightarrow 0 \text{ in } L^2(\Omega), \quad (2.26)$$

$$G(\cdot, l(u_\varepsilon)) \rightarrow G(\cdot, l(u_0)) \text{ in } L^1(\omega_2), \quad (2.27)$$

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^{r+1}(\omega_2). \quad (2.28)$$

Proof. We begin by establishing (2.25). Since u_ε realizes the minimum of J^ε , then

$$J^\varepsilon(u_\varepsilon) \leq J^\varepsilon(v), \quad \forall v \in \mathcal{D}(\Omega).$$

Passing to the lim sup, we get

$$\limsup_{\varepsilon \rightarrow 0} \mathbf{J}^\varepsilon(u_\varepsilon) \leq \mathbf{J}^0(v), \quad \forall v \in \mathcal{D}(\Omega).$$

By density and the fact that $\mathbf{J}^0 \in C^1(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ (see Remark 1.7), the precedent inequality also holds for every $v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. Thus,

$$\limsup_{\varepsilon \rightarrow 0} \mathbf{J}^\varepsilon(u_\varepsilon) \leq \inf_{v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} \mathbf{J}^0(v). \quad (2.29)$$

Since G satisfies (1.31), then we have

$$|G(X_2, l(u_\varepsilon))| \leq C \left(1 + |l(u_\varepsilon)|^{r+1}\right), \quad (2.30)$$

for some constant $C > 0$. Thanks to (2.17), this means that $G(\cdot, l(u_\varepsilon))$ is a bounded sequence in $L^1(\omega_2)$. Due to (2.21) and the continuity of $t \mapsto G(X_2, t)$, for a.e. $X_2 \in \omega_2$, we get –up to a subsequence–

$$G(X_2, l(u_\varepsilon)) \rightarrow G(X_2, l(u_0)), \quad \text{a.e. in } \omega_2, \text{ as } \varepsilon \rightarrow 0. \quad (2.31)$$

Writing \mathbf{J}^ε as in (2.10), then the convergences (2.12), (2.14) and (2.31) allow us to pass to the infimum limit in $\mathbf{J}^\varepsilon(u_\varepsilon)$. We deduce that

$$\mathbf{J}^0(u_0) \leq \liminf_{\varepsilon \rightarrow 0} \mathbf{J}^\varepsilon(u_\varepsilon).$$

Reporting this to (2.29), we obtain $\mathbf{J}^0(u_0) \leq \inf_{v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} \mathbf{J}^0(v)$ and thus (2.25) is proved.

To show the strong convergences, we consider

$$\begin{aligned} I_\varepsilon := & \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + hA \nabla_{X_2} (u_\varepsilon - u_0) \cdot \nabla_{X_2} (u_\varepsilon - u_0) dx \\ & + \int_{\omega_2} |G(X_2, l(u_\varepsilon)) - G(X_2, l(u_0))| dX_2. \end{aligned} \quad (2.32)$$

Then, expending I_ε , we get

$$\begin{aligned} I_\varepsilon = & \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + hA \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_\varepsilon dx - \int_{\Omega} hA \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_0 dx \\ & + \frac{1}{2} \int_{\Omega} hA \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx + \int_{\omega_2} |G(X_2, l(u_\varepsilon)) - G(X_2, l(u_0))| dX_2. \end{aligned}$$

Going back to the definition of \mathbf{J}^ε , given in (1.36), we can write

$$\begin{aligned} I_\varepsilon = & \mathbf{J}^\varepsilon(u_\varepsilon) + \langle f_1, u_\varepsilon \rangle_{\mathcal{V}_0(\Omega, h)} + \int_{\omega_2} f_2 l(u_\varepsilon) dX_2 \\ & - \int_{\Omega} hA \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_0 dx + \frac{1}{2} \int_{\Omega} hA \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx \\ & - \int_{\omega_2} C_1 - G(X_2, l(u_\varepsilon)) - |G(X_2, l(u_\varepsilon)) - G(X_2, l(u_0))| dX_2 + C_1 |\omega_2|. \end{aligned} \quad (2.33)$$

(Recall that $C_1 - G(X_2, t) \geq 0$, for $t \in \mathbb{R}$). Thanks to the pointwise convergence (2.31), we can apply Brezis–Lieb’s Lemma [7] to the sequence $C_1 - G(X_2, l(u_\varepsilon))$, we get

$$\int_{\omega_2} C_1 - G(X_2, l(u_\varepsilon)) - |G(X_2, l(u_\varepsilon)) - G(X_2, l(u_0))| dX_2 \rightarrow \int_{\omega_2} C_1 - G(X_2, l(u_0)) dX_2, \quad (2.34)$$

as $\varepsilon \rightarrow 0$. Using (2.25), (2.34) and the convergences (2.12), (2.14) from Lemma 2.1, we can pass to the limit in (2.33). It comes that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} I_\varepsilon &= \mathbf{J}^0(u_0) + \langle f_1, u_0 \rangle_{\mathcal{V}_0(\Omega, h)} + \int_{\omega_2} f_2 l(u_0) dX_2 \\ &\quad - \frac{1}{2} \int_{\Omega} h A \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx + \int_{\omega_2} G(X_2, l(u_0)) dX_2 = 0, \end{aligned}$$

since \mathbf{J}^0 is defined by (1.42). The convergences (2.26) and (2.27) are consequences of the inequality

$$I_\varepsilon \geq \min\{1, \alpha\} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + h |\nabla_{X_2} (u_\varepsilon - u_0)|^2 dx + \int_{\omega_2} |G(X_2, l(u_\varepsilon)) - G(X_2, l(u_0))| dX_2.$$

Finally, due to the convergences (2.21), (2.27) and the fact that $a |l(u_\varepsilon)|^{r+1} \leq C_1 - G(X_2, l(u_\varepsilon))$, we deduce (2.28) by applying Lebesgue’s theorem. This ends the proof. ■

Remark 2.3 Taking $\varepsilon = 0$, replacing $\mathcal{W}_0(\Omega, h)$ by $\mathcal{V}_0(\Omega, h)$ and arguing as in Theorem 2.1, we can show directly that under the assumptions of Theorem 2.1 there exists $u \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ such that

$$\mathbf{J}^0(u) = \inf_{v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} \mathbf{J}^0(v), \quad (2.35)$$

where u is a solution of (SP_0) . Moreover, this solution is unique when $t \mapsto g(\cdot, t)$ is non increasing.

A NON-LOCAL EIGENVALUE PROBLEM AND AN APPLICATION

In this chapter, we aim to establish the existence of solutions for some non-local eigenvalue problems. This allow us to go further and study problems (SP_0) and (SP_ε) under some assumptions weaker than (2.1). As a first application, we show the existence of solutions for Problem (SP_ε) when $g(s) = -\lambda s - a|s|^{r-1}s$ and $f_1 = f_2 = 0$. Passing to the limit $\varepsilon \rightarrow 0$, we obtain an existence result for problem (SP_0) under the same assumptions.

3.1 A NON-LOCAL EIGENVALUE PROBLEM

We need to consider the following non-local eigenvalue problem.

3.1.1 l -eigenvalues and l -eigenfunctions

We say that λ^ε is an l -eigenvalue of the linear problem

$$\begin{cases} -\frac{\varepsilon^2}{h}\Delta_{X_1}\varphi - \nabla_{X_2} \cdot (A\nabla_{X_2}\varphi) = \lambda^\varepsilon l(\varphi) & \text{in } \mathcal{W}'_0(\Omega, h) \\ \varphi \in \mathcal{W}_0(\Omega, h), \end{cases} \quad (3.1)$$

provided that there exists a solution φ such that $l(\varphi)$ is not identically null. A solution φ to (3.1), called an l -eigenfunction associated with λ^ε , is taken in the h -weak sense, i.e. φ satisfies

$$\int_{\Omega} \varepsilon^2 \nabla_{X_1} \varphi \cdot \nabla_{X_1} v + h A \nabla_{X_2} \varphi \cdot \nabla_{X_2} v \, dx = \lambda^\varepsilon \int_{\omega_2} l(\varphi) l(v) \, dX_2, \quad \forall v \in \mathcal{W}_0(\Omega, h). \quad (EVP_\varepsilon)$$

Theorem 3.1 *The smallest l -eigenvalue of Problem (EVP_ε) exists, is positive and defined by*

$$\lambda_1^\varepsilon := \inf_{v \in \mathcal{W}_0(\Omega, h), l(v) \neq 0} \frac{\int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h A \nabla_{X_2} v \cdot \nabla_{X_2} v \, dx}{\|l(v)\|_{L^2(\omega_2)}^2}. \quad (3.2)$$

The associated l -eigenfunctions are the minimizers of the above functional.

Proof. Let λ^ε be an l -eigenvalue of Problem (EVP_ε) . Taking $v = \varphi \in \mathcal{W}_0(\Omega, h)$ in (EVP_ε) , we get

$$\int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi|^2 + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} \varphi dx = \lambda^\varepsilon \int_{\omega_2} |l(\varphi)|^2 dX_2$$

and we have clearly $\lambda_1^\varepsilon \leq \lambda^\varepsilon$ and $\lambda^\varepsilon > 0$ since $l(\varphi) \not\equiv 0$.

Let us show that the infimum in (3.2) is indeed achieved for a function in $\mathcal{W}_0(\Omega, h)$. Thanks to the density of $\mathcal{D}(\Omega)$ in $\mathcal{W}_0(\Omega, h)$, we can take a minimizing sequence $\varphi_n \in \mathcal{D}(\Omega)$, i.e.,

$$\frac{\int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi_n|^2 + hA \nabla_{X_2} \varphi_n \cdot \nabla_{X_2} \varphi_n dx}{\|l(\varphi_n)\|_{L^2(\omega_2)}^2} \longrightarrow \inf_{v \in \mathcal{W}_0(\Omega, h), l(v) \neq 0} \frac{\int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx}{\|l(v)\|_{L^2(\omega_2)}^2} := \text{Inf}, \quad (3.3)$$

as $n \rightarrow +\infty$. Replacing $h\varphi_n$ by $h\varphi_n / \|l(\varphi_n)\|_{L^2(\omega_2)}$, we can assume that $\|l(\varphi_n)\|_{L^2(\omega_2)} = 1, \forall n \in \mathbb{N}$. By (1.22), (3.3) and for n large enough, we derive

$$\alpha \int_{\Omega} h |\nabla_{X_2} \varphi_n|^2 dx \leq \int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi_n|^2 + hA \nabla_{X_2} \varphi_n \cdot \nabla_{X_2} \varphi_n dx \leq \text{Inf} + 1, \quad (3.4)$$

which guarantees that

$$|\nabla_{X_1} \varphi_n| \text{ is bounded in } L^2(\Omega) \text{ and } |\nabla_{X_2} \varphi_n| \text{ is bounded in } L^2(\Omega, h).$$

Then, we have -up to a subsequence-

$$\begin{aligned} \varphi_n &\rightharpoonup \varphi \text{ and } \nabla_{X_2} \varphi_n \rightharpoonup \nabla_{X_2} \varphi \text{ in } L^2(\Omega, h), \\ \nabla_{X_1} \varphi_n &\rightharpoonup \nabla_{X_1} \varphi \text{ in } L^2(\Omega), \end{aligned}$$

for some $\varphi \in \mathcal{W}_0(\Omega, h)$. The first convergence $\varphi_n \rightharpoonup \varphi$ is a consequence of Poincaré's inequality in the X_2 -direction. Moreover, using Lemma 1.1, with $p = 2$, we obtain

$$l(\varphi_n) \rightarrow l(\varphi) \text{ in } L^2(\omega_2) \text{ and } \|l(\varphi_n)\|_{L^2(\omega_2)} \rightarrow \|l(\varphi)\|_{L^2(\omega_2)} = 1. \quad (3.5)$$

This means that $l(\varphi) \not\equiv 0$ and a fortiori $\varphi \not\equiv 0$. By lower semi-continuity of norms, we deduce that

$$\text{Inf} \leq \int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi|^2 + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} \varphi dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi_n|^2 + hA \nabla_{X_2} \varphi_n \cdot \nabla_{X_2} \varphi_n dx := \text{Inf}.$$

To summarize, $\varphi \in \mathcal{W}_0(\Omega, h)$ is a function for which the infimum (3.2) is achieved, i.e.,

$$\int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi|^2 + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} \varphi dx = \lambda_1^\varepsilon, \quad \|l(\varphi)\|_{L^2(\omega_2)} = 1. \quad (3.6)$$

Let us show that this φ is an l -eigenfunction. Let $v \in \mathcal{D}(\Omega)$ and consider

$$T(t) := \int_{\Omega} \varepsilon^2 |\nabla_{X_1}(\varphi + tv)|^2 + hA \nabla_{X_2}(\varphi + tv) \cdot \nabla_{X_2}(\varphi + tv) dx \Big/ \int_{\omega_2} |l(\varphi + tv)|^2 dX_2.$$

Then, we should have $T'(0) = 0$. That is

$$\left. \frac{d}{dt} \frac{\int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi|^2 + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} \varphi + 2t (\varepsilon^2 \nabla_{X_1} \varphi \cdot \nabla_{X_1} v + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} v) dx + \int_{\Omega} t^2 (\varepsilon^2 |\nabla_{X_1} v|^2 + hA \nabla_{X_2} v \cdot \nabla_{X_2} v) dx}{\int_{\omega_2} |l(\varphi)|^2 + 2tl(\varphi)l(v) + t^2 |l(v)|^2 dX_2} \right|_{t=0} = 0,$$

hence

$$\int_{\Omega} \varepsilon^2 \nabla_{X_1} \varphi \cdot \nabla_{X_1} v + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} v dx \cdot \int_{\omega_2} |l(\varphi)|^2 dX_2 - \int_{\omega_2} l(\varphi)l(v) dX_2 \cdot \int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi|^2 + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} \varphi dx = 0.$$

Using (3.6), we obtain

$$\int_{\Omega} \varepsilon^2 \nabla_{X_1} \varphi \cdot \nabla_{X_1} v + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} v dx = \lambda_1^\varepsilon \int_{\omega_2} l(\varphi)l(v) dX_2, \quad \forall v \in \mathcal{D}(\Omega).$$

This shows that φ is an l -eigenfunction associated with the smallest l -eigenvalue λ_1^ε . ■

Concerning the (expected) limit problem, we say that λ^0 is an l -eigenvalue of the linear problem

$$\begin{cases} -\nabla_{X_2} \cdot (A \nabla_{X_2} \varphi) = \lambda^0 l(\varphi) & \text{in } \mathcal{V}'_0(\Omega, h) \\ \varphi \in \mathcal{V}_0(\Omega, h), \end{cases} \quad (3.7)$$

provided that there exists a solution φ such that $l(\varphi) \not\equiv 0$. This solution φ is also taken in the h -weak sense, i.e. φ satisfies

$$\int_{\Omega} hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} v dx = \lambda^0 \int_{\omega_2} l(\varphi)l(v) dX_2, \quad \forall v \in \mathcal{V}_0(\Omega, h). \quad (EVP_0)$$

The proof of Theorem 3.1 can be easily reproduced, by taking $\varepsilon = 0$ and replacing $\mathcal{W}_0(\Omega, h)$ by $\mathcal{V}_0(\Omega, h)$, to prove that:

Theorem 3.2 *The smallest l -eigenvalue of Problem (EVP₀) exists, is positive and defined by*

$$\lambda_1^0 := \inf_{v \in \mathcal{V}_0(\Omega, h), l(v) \neq 0} \frac{\int_{\Omega} hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx}{\|l(v)\|_{L^2(\omega_2)}^2}. \quad (3.8)$$

Remark 3.1 *If the matrix A is independent of X_1 , it is clear from (EVP₀) that an l -eigenfunction $\tilde{\varphi}_0$, associated with an l -eigenvalue λ^0 , is independent of X_1 and*

$$-\nabla_{X_2} \cdot (A(X_2) \nabla_{X_2} \tilde{\varphi}_0) = \lambda^0 \|h\|_{L^1(\omega_1)} \tilde{\varphi}_0, \quad \text{with } \tilde{\varphi}_0 \in H_0^1(\omega_2).$$

This means that $\lambda^0 \|h\|_{L^1(\omega_1)}$ and $\tilde{\varphi}_0$ are respectively an eigenvalue and eigenfunction (in the classical sense) of the operator $-\nabla_{X_2} \cdot (A(X_2) \nabla_{X_2} \cdot)$ on ω_2 , with Dirichlet boundary conditions. Denoting by

λ_{1,ω_2} the usual first eigenvalue $-\nabla_{X_2} \cdot (A(X_2) \nabla_{X_2} \cdot)$ on ω_2 , with Dirichlet boundary conditions, then by definition $\lambda_{1,\omega_2} \leq \lambda_1^0 \|h\|_{L^1(\omega_1)}$. Besides, if $\tilde{\varphi}_0 \in \mathcal{V}_0(\Omega, h)$ is the eigenvalue corresponding to λ_{1,ω_2} , then (3.8) yields

$$\lambda_1^0 \leq \frac{\|h\|_{L^1(\omega_1)} \int_{\omega_2} A \nabla_{X_2} \tilde{\varphi}_0 \cdot \nabla_{X_2} \tilde{\varphi}_0 dX_2}{\|l(\tilde{\varphi}_0)\|_{L^2(\omega_2)}^2} = \frac{\lambda_{1,\omega_2}}{\|h\|_{L^1(\omega_1)}}. \quad (3.9)$$

Thus $\lambda_1^0 = \lambda_{1,\omega_2} / \|h\|_{L^1(\omega_1)}$.

Even if A depends on x , we still have:

Corollary 3.1 For every $\varepsilon > 0$, the l -eigenvalues λ_1^ε and λ_1^0 satisfy

$$\lambda_1^\varepsilon \geq \lambda_1^0 \geq \alpha \lambda_{1,\omega_2}^\Delta / \|h\|_{L^1(\omega_1)}, \quad (3.10)$$

where α is the ellipticity constant given in (1.22) and $\lambda_{1,\omega_2}^\Delta$ denotes the first eigenvalue of $-\Delta_{X_2}$ on ω_2 with Dirichlet boundary conditions.

Proof. By the density of $\mathcal{W}_0(\Omega, h)$ in $\mathcal{V}_0(\Omega, h)$ and from the definitions (3.2) and (3.8), it is clear that $\lambda_1^\varepsilon \geq \lambda_1^0$. Let φ_0 be a normalized l -eigenfunction of Problem (EVP₀) associated with λ_1^0 , then

$$\lambda_1^0 = \int_{\Omega} h A \nabla_{X_2} \varphi_0 \cdot \nabla_{X_2} \varphi_0 dx \geq \alpha \int_{\Omega} h |\nabla_{X_2} \varphi_0|^2 dx. \quad (3.11)$$

On the other hand, taking $p = 2$ in (1.14) then using Poincaré's inequality in the X_2 -direction, we get

$$1 = \|l(\varphi_0)\|_{L^2(\omega_2)}^2 \leq \|h\|_{L^1(\omega_1)} \int_{\Omega} h |\varphi_0|^2 dx \leq \frac{\|h\|_{L^1(\omega_1)}}{\lambda_{1,\omega_2}^\Delta} \int_{\Omega} h |\nabla_{X_2} \varphi_0|^2 dx. \quad (3.12)$$

Combining this with (3.11), we deduce (3.10). ■

3.1.2 Some proprieties of the first l -eigenvalue and l -eigenfunction

As for the classical eigenvalue problems, we can obtain some proprieties like simplicity of λ_ε^1 , the positivity of φ_ε and its uniqueness up to a multiplicative constant.

Theorem 3.3 Let $\varepsilon > 0$ and φ_ε be an l -eigenfunction for (EVP _{ε}) associated with λ_ε^1 . Then, λ_ε^1 is simple and φ_ε does not change sign in Ω .

Proof. Let $\varphi_\varepsilon \in \mathcal{W}_0(\Omega, h)$, $l(\varphi_\varepsilon) \neq 0$, be an h -weak solution of (EVP _{ε}) and thus φ_ε realizes the infimum (3.2). Set $\varphi_\varepsilon^+ := \max\{0, \varphi_\varepsilon\}$. Arguing by density as in [22, Lemma 1.19], we can show that

$$\varphi_\varepsilon^+ \in \mathcal{W}_0(\Omega, h) \quad \text{and} \quad \partial_{x_i} \varphi_\varepsilon^+ = (\partial_{x_i} \varphi_\varepsilon) \chi_{\{\varphi_\varepsilon > 0\}}, \quad \text{for } i = 1, \dots, n_1 + n_2.$$

In particular, this means that $|\varphi_\varepsilon| = (2\varphi_\varepsilon^+ - \varphi_\varepsilon) \in \mathcal{W}_0(\Omega, h)$. Since $\|l(\varphi_\varepsilon)\|_{L^2(\omega_2)} \leq \|l(|\varphi_\varepsilon|)\|_{L^2(\omega_2)}$, we infer that

$$\frac{\int_{\Omega} \varepsilon^2 |\nabla_{X_1} |\varphi_\varepsilon||^2 + hA \nabla_{X_2} |\varphi_\varepsilon| \cdot \nabla_{X_2} |\varphi_\varepsilon| dx}{\|l(|\varphi_\varepsilon|)\|_{L^2(\omega_2)}^2} \leq \frac{\int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi_\varepsilon|^2 + hA \nabla_{X_2} \varphi_\varepsilon \cdot \nabla_{X_2} \varphi_\varepsilon dx}{\|l(\varphi_\varepsilon)\|_{L^2(\omega_2)}} = \lambda_1^\varepsilon, \quad (3.13)$$

hence the function $|\varphi_\varepsilon|$ also realizes the infimum (3.2). Thanks to Theorem 3.1, $|\varphi_\varepsilon|$ is also an l -eigenfunction for (3.1).

Next, we show that φ_ε does not change sign in Ω . Going back to (3.1), the non-negative function $|\varphi_\varepsilon|$ satisfies

$$-\frac{\varepsilon^2}{h} \Delta_{X_1} |\varphi_\varepsilon| - \nabla_{X_2} \cdot (A \nabla_{X_2} |\varphi_\varepsilon|) \geq 0,$$

in the weak sense. According to [36], under the assumption (1.7) we must have $|\varphi_\varepsilon| > 0$, a.e. in Ω and thus λ_1^ε has a positive l -eigenfunction. Therefore we must have $\varphi_\varepsilon = \pm |\varphi_\varepsilon|$. Otherwise $\varphi_\varepsilon^+ = (\varphi_\varepsilon + |\varphi_\varepsilon|)/2$ would be a non-negative l -eigenfunction for (3.1) vanishing on a set of positive measure in contradiction with the precedent assertion $|\varphi_\varepsilon^+| = \varphi_\varepsilon^+ > 0$, a.e. in Ω .

Finally, if λ_1^ε is not a simple l -eigenvalue, then there is a second l -eigenfunction ψ_ε corresponding to λ_1^ε and orthogonal to φ_ε . This ψ_ε would not be of one sign (a.e.) which leads to a contradiction. This shows the simplicity of λ_1^ε . ■

Remark 3.2 Let $\varphi_0 \in \mathcal{V}_0(\Omega, h)$ be an l -eigenfunction associated with λ_1^0 . Taking $\varepsilon = 0$ and replacing φ by φ_0 in (3.13), we deduce that the function $|\varphi_0| \in \mathcal{V}_0(\Omega, h)$ is also an l -eigenvalue. The function φ_0^+ (or $\varphi_0^- := \max\{0, -\varphi_0\}$) is another non-negative l -eigenvalues whenever it is not identically null.

Remark 3.3 Arguing as in [10, Page 185] and if $\partial_{x_k} a_{ij} \in L^\infty(\Omega)$ for $k = 1, \dots, n_1$, we can show that a solutions of (EVP₀) satisfies $\nabla_{X_1} \varphi_0 \in [L^2(\Omega, h)]^{n_1}$.

3.1.3 Convergence of the l -eigenvalues and l -eigenfunctions

We are now interested in the behaviour of the l -eigenvalue λ_1^ε and its associated l -eigenfunction φ_ε , as $\varepsilon \rightarrow 0$.

Theorem 3.4 Let λ_1^ε be defined by (3.2) and φ_ε an associated l -eigenfunction such that $\|l(\varphi_\varepsilon)\|_{L^2(\omega_2)} = 1$. Then, we have

$$\lim_{\varepsilon \rightarrow 0} \lambda_1^\varepsilon = \lambda_1^0, \quad (3.14)$$

where λ_1^0 is defined by (3.8). Moreover, -up to a subsequence-

$$\varphi_\varepsilon \rightarrow \varphi_0, \nabla_{X_2} \varphi_\varepsilon \rightarrow \nabla_{X_2} \varphi_0 \text{ in } L^2(\Omega, h), \varepsilon \nabla_{X_1} \varphi_\varepsilon \rightarrow 0 \text{ in } L^2(\Omega) \quad (3.15)$$

$$l(\varphi_\varepsilon) \rightarrow l(\varphi_0) \text{ in } L^2(\omega_2), \quad (3.16)$$

where φ_0 is a normalized l -eigenfunction associated with λ_1^0 , i.e. a solution to (EVP_0) , satisfying $\|l(\varphi_0)\|_{L^2(\omega_2)} = 1$.

Proof. The sequence λ_1^ε is decreasing as $\varepsilon \rightarrow 0$ and clearly $\lambda_1^\varepsilon \geq \lambda_1^0 > 0, \forall \varepsilon > 0$. Thus λ_1^ε converges to some limit

$$\lambda_1^\varepsilon \rightarrow \bar{\lambda}_1 := \inf_{\varepsilon > 0} \lambda_1^\varepsilon \geq \lambda_1^0. \quad (3.17)$$

Let φ_ε be an l -eigenfunction associated with λ_1^ε such that $\|l(\varphi_\varepsilon)\|_{L^2(\omega_2)} = 1$, then

$$\int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi_\varepsilon|^2 + \alpha |\nabla_{X_2} \varphi_\varepsilon|^2 h dx \leq \int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi_\varepsilon|^2 + h A \nabla_{X_2} \varphi_\varepsilon \cdot \nabla_{X_2} \varphi_\varepsilon dx = \lambda_1^\varepsilon \leq \lambda_1^1, \quad (3.18)$$

for $0 < \varepsilon \leq 1$. It follows that

$$\begin{aligned} \varphi_\varepsilon, |\nabla_{X_2} \varphi_\varepsilon| &\text{ are bounded in } L^2(\Omega, h), \\ |\varepsilon \nabla_{X_1} \varphi_\varepsilon| &\text{ is bounded in } L^2(\Omega). \end{aligned}$$

Of course, the first estimate follows from Poincaré's inequality in the X_2 -direction. Then, there exists $\varphi_0 \in \mathcal{V}_0(\Omega, h)$ such that -up to a subsequence -

$$\varphi_\varepsilon \rightarrow \varphi_0, \nabla_{X_2} \varphi_\varepsilon \rightarrow \nabla_{X_2} \varphi_0 \text{ in } L^2(\Omega, h) \text{ and } \varepsilon \nabla_{X_1} \varphi_\varepsilon \rightarrow 0, \text{ in } L^2(\Omega). \quad (3.19)$$

In addition, Lemma 1.1 with $p = 2$ ensures that

$$l(\varphi_\varepsilon) \rightarrow l(\varphi_0) \text{ in } L^2(\omega_2). \quad (3.20)$$

Let us now prove that $\bar{\lambda}_1 = \lambda_1^0$ and φ_0 is its associated l -eigenvalue. On one hand, taking $\varphi = \varphi_\varepsilon$ in (EVP_ε) , then using the convergences (3.19) and (3.20) to pass to the limit yields

$$\int_{\Omega} h A \nabla_{X_2} \varphi_0 \cdot \nabla_{X_2} v dx = \bar{\lambda}_1 \int_{\omega_2} l(\varphi_0) l(v) dX_2, \forall v \in \mathcal{D}(\Omega). \quad (3.21)$$

Besides, the strong convergence (3.20) means that $\|l(\varphi_0)\|_{L^2(\omega_2)} = 1$, hence $l(\varphi_0) \not\equiv 0$ and the identity (3.21) shows that φ_0 is an l - eigenvalue associated with $\bar{\lambda}_1$.

On the other hand, by definition we have

$$\lambda_1^\varepsilon \leq \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h A \nabla_{X_2} v \cdot \nabla_{X_2} v dx, \quad \forall v \in \mathcal{D}(\Omega), \quad \|l(v)\|_{L^2(\omega_2)} = 1,$$

and passing to the limit, when $\varepsilon \rightarrow 0$, we get

$$\bar{\lambda}_1 \leq \int_{\Omega} h A \nabla_{X_2} v \cdot \nabla_{X_2} v dx, \quad \forall v \in \mathcal{D}(\Omega), \quad \|l(v)\|_{L^2(\omega_2)} = 1.$$

By density of $\mathcal{D}(\Omega)$ in $\mathcal{V}_0(\Omega, h)$ we infer that $\bar{\lambda}_1 \leq \lambda_1^0$. Taking (3.17) into account, we obtain $\bar{\lambda}_1 = \lambda_1^0$.

It remains to show the strong convergences (3.15). Set

$$\Lambda_\varepsilon := \int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi_\varepsilon|^2 + hA \nabla_{X_2} (\varphi_\varepsilon - \varphi_0) \cdot \nabla_{X_2} (\varphi_\varepsilon - \varphi_0) dx.$$

Developing Λ_ε and using the convergences (3.14) and (3.19), we get

$$\begin{aligned} \Lambda_\varepsilon &= \lambda_1^\varepsilon - 2 \int_{\Omega} hA \nabla_{X_2} \varphi_0 \cdot \nabla_{X_2} \varphi_\varepsilon dx + \int_{\Omega} hA \nabla_{X_2} \varphi_0 \cdot \nabla_{X_2} \varphi_0 dx \\ &\rightarrow \lambda_1^0 - \int_{\Omega} hA \nabla_{X_2} \varphi_0 \cdot \nabla_{X_2} \varphi_0 dx = 0. \end{aligned}$$

The strong convergences (3.15) follow from the fact that

$$\Lambda_\varepsilon \geq \int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi_\varepsilon|^2 + \alpha h |\nabla_{X_2} (\varphi_\varepsilon - \varphi_0)|^2 dx, \quad \forall \varepsilon > 0.$$

This ends the proof of the theorem. ■

3.2 EXISTENCE AND NON UNIQUENESS FOR A SEMILINEAR PROBLEM WITH $\lambda < -\lambda_1^\varepsilon$

In this section we take

$$g(s) = -\lambda s - a |s|^{r-1} s,$$

and consider the h -weak problem to find u_ε satisfying

$$\int_{\Omega} \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + hA \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v dx = - \int_{\omega_2} \lambda l(u_\varepsilon) l(v) dX_2 - a |l(u_\varepsilon)|^{r-1} l(u_\varepsilon) l(v) dX_2, \quad (3.22)$$

for every $v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

Clearly $u = 0$ is a trivial solution for (3.22). We will show the existence of other nontrivial solutions as global minimizers of the functional

$$\begin{aligned} J^\varepsilon(v) &:= \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx \\ &\quad + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2 + \frac{a}{r+1} \int_{\omega_2} |l(v)|^{r+1} dX_2, \quad (3.23) \end{aligned}$$

on $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$.

3.2.1 An existence result for the perturbed problem

Theorem 3.5 *Let $\varepsilon > 0$ and assume that*

$$a > 0, r > 1 \text{ and } \lambda < -\lambda_1^\varepsilon.$$

Then, Problem (3.22) has a solution u that minimizes J^ε on $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$, where $l(u) \neq 0$. In addition, there are at least two other different solutions 0 and $-u$.

Proof. Applying Young's inequality $t^2 \leq C + \frac{a}{2(r+1)}t^{r+1}$, for some constant $C > 0$ depending on r and a , we get

$$J^\varepsilon(v) \geq \frac{\min\{1, \alpha\}}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx + \frac{a}{2(r+1)} \int_{\omega_2} |l(v)|^{r+1} dX_2 - C.$$

This means that a minimizing sequence $(u_n)_{n \geq 0}$ is necessarily bounded in $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$, i.e.

$$\begin{aligned} u_n, |\nabla_{X_2} u_n| &\text{ are bounded in } L^2(\Omega, h), \\ |\nabla_{X_1} u_n| &\text{ is bounded in } L^2(\Omega), \\ l(u_n) &\text{ is bounded in } L^{r+1}(\omega_2). \end{aligned}$$

The boundedness of u_n is a consequence of Poincaré's inequality in the X_2 -direction. Then, there exist $u \in \mathcal{W}_0(\Omega, h)$, $U_1 \in (L^2(\Omega))^{n_1}$ and $U_2 \in (L^2(\Omega, h))^{n_2}$ and a subsequence $(u_{n_k})_{k \geq 0}$ such that

$$\begin{aligned} u_{n_k} &\rightharpoonup u, \nabla_{X_2} u_{n_k} \rightharpoonup U_2 \text{ in } L^2(\Omega, h) \text{ and } \nabla_{X_1} u_{n_k} \rightharpoonup U_1 \text{ in } L^2(\Omega), \\ l(u_{n_k}) &\rightharpoonup l(u) \text{ in } L^{r+1}(\omega_2), \end{aligned}$$

as $k \rightarrow +\infty$. (These convergences meant component by component convergences). Since the above convergences imply the convergence in the distributional sense, we can easily see that

$$u_{n_k} \rightharpoonup u, \nabla_{X_2} u_{n_k} \rightharpoonup \nabla_{X_2} u \text{ in } L^2(\Omega, h), \quad (3.24)$$

$$\nabla_{X_1} u_{n_k} \rightharpoonup \nabla_{X_1} u \text{ in } L^2(\Omega). \quad (3.25)$$

Using Lemma 1.1, with $p = 2$, we obtain the strong convergence

$$l(u_{n_k}) \rightarrow l(u) \text{ in } L^2(\omega_2).$$

These convergences suffices to pass to the \liminf in (3.23), it comes that

$$J^\varepsilon(u) \leq \liminf_{n_k \rightarrow \infty} J^\varepsilon(u_{n_k}) = \inf_{v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} J^\varepsilon(v). \quad (3.26)$$

Therefore u_ε realizes the infimum of J^ε on $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. By Theorem 1.4 we have $(J^\varepsilon)'(u) = 0$, i.e. u is a solution to Problem (3.22).

Since $-u$ is a solution whenever u is a solution, it remains to show that the infimum of J^ε is negative, so that this solution is not identically 0. Indeed, since $\lambda < -\lambda_1^\varepsilon$, there exists $\varphi \in$

$\mathcal{W}_0(\Omega, h)$, sufficiently close to the l -eigenfunction associated to λ_1^ε , such that $\|l(\varphi)\|_{L^2(\omega_2)} = 1$ and

$$\int_{\Omega} \varepsilon^2 |\nabla_{X_1} \varphi|^2 + hA \nabla_{X_2} \varphi \cdot \nabla_{X_2} \varphi dx = \mu \in (\lambda_1^\varepsilon, -\lambda).$$

Given $t > 0$, we have

$$\begin{aligned} J^\varepsilon(t\varphi) &= \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1}(t\varphi)|^2 + hA \nabla_{X_2}(t\varphi) \cdot \nabla_{X_2}(t\varphi) dx \\ &\quad + \frac{\lambda}{2} \int_{\omega_2} |l(t\varphi)|^2 dX_2 + \frac{a}{r+1} \int_{\omega_2} |l(t\varphi)|^{r+1} dX_2 \\ &= \frac{t^2}{2} (\mu + \lambda) + \frac{at^{r+1}}{r+1} \int_{\omega_2} |l(\varphi)|^{r+1} dX_2. \end{aligned}$$

Since $\mu + \lambda < 0$, and $t^2 > t^{r+1}$ for $t \in (0, 1)$, we have $J^\varepsilon(t\varphi) < 0$ for t small enough. Thus $\inf J^\varepsilon(u) < 0$ and necessarily $\|l(u)\|_{L^2(\omega_2)} > 0$. The proof is complete. ■

3.2.2 An existence result for the unperturbed problem

Now, we let ε goes to 0 in Problem (3.22).

Theorem 3.6 *Let $\varepsilon > 0$ and assume that*

$$a > 0, r > 1 \text{ and } \lambda < -\lambda_1^0.$$

If u_ε is a nontrivial solution of Problem (3.22), then we have -up to a subsequence-

$$u_\varepsilon \rightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \rightarrow \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h), \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightarrow 0 \text{ in } L^2(\Omega), \quad (3.27)$$

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^2(\omega_2), \quad (3.28)$$

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^{r+1}(\omega_2), \quad (3.29)$$

where u_0 is a solution to the problem

$$\int_{\Omega} hA \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dx = -\lambda \int_{\omega_2} l(u_0) l(v) dX_2 - a \int_{\omega_2} |l(u_0)|^{r-1} l(u_0) l(v) dX_2, \quad (3.30)$$

for every $v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. Moreover, $l(u_0) \not\equiv 0$ and Problem (3.30) have two other different solutions 0 and $-u_0$.

Proof. Thanks to Theorem 3.4, we have $\lambda_1^\varepsilon \rightarrow \lambda_1^0$. So we can always assume that $\lambda < -\lambda_1^\varepsilon$, for $0 < \varepsilon \leq \varepsilon_0$ and ε_0 small enough. We set

$$J^0(v) := \frac{1}{2} \int_{\Omega} hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2 + \frac{a}{r+1} \int_{\omega_2} |l(v)|^{r+1} dX_2. \quad (3.31)$$

and

$$d_0 := \inf_{v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} J^0(v), \quad d_\varepsilon := \inf_{v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} J^\varepsilon(v)$$

For any $v \in \mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ we have $J^0(v) \leq J^\varepsilon(v) < J^{\varepsilon_0}(v)$. It follows that the sequence d_ε is decreasing as $\varepsilon \rightarrow 0$ and

$$d_0 < d_\varepsilon < d_{\varepsilon_0} < 0 \text{ for } 0 < \varepsilon \leq \varepsilon_0.$$

By Theorem 2.1 we know that there exists $u_\varepsilon \neq 0$ realizing the negative minimum of the functional J^ε on $\mathcal{W}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. It comes that

$$d_0 \leq J^\varepsilon(u_\varepsilon) < d_{\varepsilon_0} < 0.$$

Clearly $d_0 > -\infty$ since t^2 is dominated by t^{r+1} for large t . Taking $v = u_\varepsilon$ in (3.22), we can rewrite J^ε to deduce that

$$d_0 \leq J^\varepsilon(u_\varepsilon) = a \left(\frac{1}{r+1} - \frac{1}{2} \right) \int_{\omega_2} |l(u_\varepsilon)|^{r+1} dX_2 < d_{\varepsilon_0} < 0. \quad (3.32)$$

As $a \left(\frac{1}{r+1} - \frac{1}{2} \right) < 0$, it follows that

$$\int_{\omega_2} |l(u_\varepsilon)|^{r+1} dX_2 \leq \frac{2|d_0|}{a} \left(\frac{r+1}{r-1} \right). \quad (3.33)$$

Since $r+1 > 2$, we have

$$\int_{\omega_2} |l(u_\varepsilon)|^2 dX_2 \leq |\omega_2|^{\frac{2(r-1)}{r+1}} \left(\frac{2|d_0|}{a} \left(\frac{r+1}{r-1} \right) \right)^{\frac{2}{r+1}}. \quad (3.34)$$

Using (3.22) with $v = u_\varepsilon$, we can also derive that

$$J^\varepsilon(u_\varepsilon) = \left(\frac{1}{2} - \frac{1}{r+1} \right) \left(\int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + hA \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_\varepsilon dx + \lambda \int_{\omega_2} |l(u_\varepsilon)|^2 dX_2 \right) < 0.$$

Combined with (3.34), we get (recall that $\lambda < 0$)

$$\begin{aligned} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + hA \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_\varepsilon dx &< -\lambda \int_{\omega_2} |l(u_\varepsilon)|^2 dX_2 \\ &\leq |\omega_2|^{\frac{2(r-1)}{r+1}} |\lambda| \left(\frac{2|d_0|}{a} \left(\frac{r+1}{r-1} \right) \right)^{\frac{2}{r+1}}, \end{aligned} \quad (3.35)$$

The estimations (3.33), (3.34), (3.35) can be summarized as

$$u_\varepsilon, |\nabla_{X_2} u_\varepsilon| \text{ are bounded in } L^2(\Omega, h), \quad (3.36)$$

$$|\varepsilon \nabla_{X_1} u_\varepsilon| \text{ is bounded in } L^2(\Omega), \quad (3.37)$$

$$l(u_\varepsilon) \text{ is bounded in } L^2(\omega_2) \text{ and in } L^{r+1}(\omega_2). \quad (3.38)$$

Then, up to a subsequence, the convergences (3.27) and (3.29) hold in a weak sense. Thanks to Lemma 1.1 with $p = 2$, we have $l(u_\varepsilon) \rightarrow l(u_0)$ in $L^2(\omega_2)$ and –up to a new subsequence–

$$l(u_\varepsilon) \rightarrow l(u_0), \text{ a.e. in } \omega_2. \quad (3.39)$$

In addition $|l(u_\varepsilon)|^{r-1} l(u_\varepsilon)$ is bounded in $L^{\frac{r+1}{r}}(\omega_2)$ since

$$\left\| |l(u_\varepsilon)|^{r-1} l(u_\varepsilon) \right\|_{L^{\frac{r+1}{r}}(\omega_2)} = \|l(u_\varepsilon)\|_{L^{r+1}(\omega_2)}^r.$$

Then, due to Lemma 1.3 in [31, page 12], we infer that

$$|l(u_\varepsilon)|^{r-1} l(u_\varepsilon) \rightharpoonup |l(u_0)|^{r-1} l(u_0) \text{ in } L^{\frac{r+1}{r}}(\omega_2). \quad (3.40)$$

Passing to the limit $\varepsilon \rightarrow 0$ in the h -weak formulation (3.22), we deduce that the limit u_0 is a solution to Problem (3.30) since $\mathcal{W}_0(\Omega, h)$ is dense in $\mathcal{V}_0(\Omega, h)$.

To show that (3.27) and (3.29) hold in the strong sense, we consider

$$\begin{aligned} M^\varepsilon := & \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + h A \nabla_{X_2} (u_\varepsilon - u_0) \cdot \nabla_{X_2} (u_\varepsilon - u_0) dx \\ & + a \int_{\omega_2} \left(|l(u_\varepsilon)|^{r-1} l(u_\varepsilon) - |l(u_0)|^{r-1} l(u_0) \right) (l(u_\varepsilon) - l(u_0)) dX_2. \end{aligned}$$

Thanks to the inequality

$$|s|^{r-1} s - |t|^{r-1} t \geq c_r |s - t|^r, \text{ for } t, s \in \mathbb{R}, \text{ and some constant } c_r > 0,$$

we see that

$$M^\varepsilon \geq \min\{1, \alpha\} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + h |\nabla_{X_2} (u_\varepsilon - u_0)|^2 dx + a c_r \int_{\omega_2} |l(u_\varepsilon) - l(u_0)|^{r+1} dX_2. \quad (3.41)$$

Expanding M^ε and using the h -weak formulation (3.22) with $v = u_\varepsilon$, we get

$$\begin{aligned} M^\varepsilon = & -2 \int_{\Omega} h A \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_0 dx + \int_{\Omega} h A \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx - \lambda \int_{\omega_2} |l(u_\varepsilon)|^2 dX_2 \\ & - a \int_{\omega_2} |l(u_\varepsilon)|^{r-1} l(u_\varepsilon) l(u_0) dX_2 - a \int_{\omega_2} |l(u_0)|^{r-1} l(u_0) (l(u_\varepsilon) - l(u_0)) dX_2. \end{aligned}$$

Passing to the limit in M^ε , it comes that

$$\lim_{\varepsilon \rightarrow 0} M^\varepsilon = - \int_{\Omega} h A \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx - \lambda \int_{\omega_2} |l(u_0)|^2 dX_2 - a \int_{\omega_2} |l(u_0)|^{r+1} dX_2.$$

Going back to (3.30) and taking $v = u_0$, we infer that $\lim_{\varepsilon \rightarrow 0} M^\varepsilon = 0$. Thanks to (3.41), this shows that the convergence (3.27)–(3.29) hold in a strong sense.

Finally, going back to (3.32), we also infer that

$$\int_{\omega_2} |l(u_\varepsilon)|^{r+1} dX_2 \geq \frac{2|d_{\varepsilon_0}|}{a} \left(\frac{r+1}{r-1} \right),$$

where the right hand side is positive and independent from ε . Passing to limit, we see that $|l(u_0)|_{L^{r+1}(\omega_2)} > 0$, i.e. $l(u_0) \not\equiv 0$. This ends the proof of the theorem. ■

In addition, we have the following result on the convergence of $J^\varepsilon(u_\varepsilon)$ as $\varepsilon \rightarrow 0$.

Corollary 3.2 *Each limit u_0 of a converging subsequence from Theorem 3.6 realizes the minimum of J^0 defined by (3.31) and*

$$J^\varepsilon(u_\varepsilon) \rightarrow J^0(u_0) = \inf_{v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} J^0(v). \quad (3.42)$$

Proof. The strong convergences (3.27)–(3.29) in Theorem 3.6 ensures that $J^\varepsilon(u_\varepsilon) \rightarrow J^0(u_0)$. Since u_ε realize the minimum of J^ε , then

$$J^\varepsilon(u_\varepsilon) \leq J^\varepsilon(v), \quad \forall v \in \mathcal{D}(\Omega).$$

Passing to the limit, we get

$$J^0(u_0) \leq J^0(v), \quad \forall v \in \mathcal{D}(\Omega).$$

By density and the fact that $J^0 \in C^1(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ (see Remark 1.7), the precedent inequality also holds for every $v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$. Thus u_0 realizes the minimum of $J^0(v)$ on $\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$ and the corollary follows. ■

A SEMILINEAR PROBLEM WITH SUBCRITICAL NON-LOCAL NONLINEARITY

In this chapter, we aim to establish the existence of solution for Problem (SP_ε) where the assumption (2.1) is relaxed by considering an upper quadratic bound of $G(\cdot, t)$ for t large. Besides, we revisit the same issues considered in Section 3.2 for $g(s) = -\lambda s + a|s|^{r-1}s$ and investigate the existence of solutions under the assumption

$$1 \leq r < 2^* - 1.$$

We recall that in this case $\mathcal{W}_0(\Omega, h) \subset \mathcal{L}^{r+1}(\Omega)$ and $L^{\frac{r+1}{r}}(\omega_2) \subset \mathcal{W}'_0(\Omega, h)$. Without loss of generality, we assume that $f_2 = 0$

4.1 A PROBLEM WITH SUPERLINEAR GROWTH OF NON-LOCAL TERMS

The h -weak formulation of Problem (1.26) reads

$$\int_{\Omega} \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + hA \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v dx = \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(v) dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)}, \quad (\tilde{P}_\varepsilon)$$

for every $v \in \mathcal{W}_0(\Omega, h)$. The functional associated to (\tilde{P}_ε) is given here by

$$\tilde{\mathbf{J}}^\varepsilon(u) := \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u|^2 + hA \nabla_{X_2} u \cdot \nabla_{X_2} u dx - \int_{\omega_2} G(X_2, l(u)) dX_2 - \langle f_1, u \rangle_{\mathcal{V}_0(\Omega, h)}. \quad (4.1)$$

We aim to establish the existence of solutions u_ε for (\tilde{P}_ε) as global minimizers for $\tilde{\mathbf{J}}^\varepsilon$.

4.1.1 An existence result for the perturbed problem

First, we establish the following existence result.

Theorem 4.1 *Let $\varepsilon > 0$. Assume that g satisfies (1.28) and (1.29) with $1 \leq r < 2^* - 1$. In addition, assume that G defined by (1.30), satisfies*

$$G(X_2, t) \leq \frac{-\lambda}{2} t^2 \quad \text{a.e. } X_2 \in \omega_2, \text{ for } |t| \text{ large enough} \quad (4.2)$$

and some $\lambda > -\lambda_1^\varepsilon$. Then, there exists $u \in \mathcal{W}_0(\Omega, h)$ such that

$$\tilde{\mathbf{j}}^\varepsilon(u) = \inf_{v \in \mathcal{W}_0(\Omega, h)} \tilde{\mathbf{j}}^\varepsilon(v), \quad (4.3)$$

where $\tilde{\mathbf{j}}^\varepsilon$ is defined by (4.1). In particular, u is a solution of (\tilde{P}_ε) .

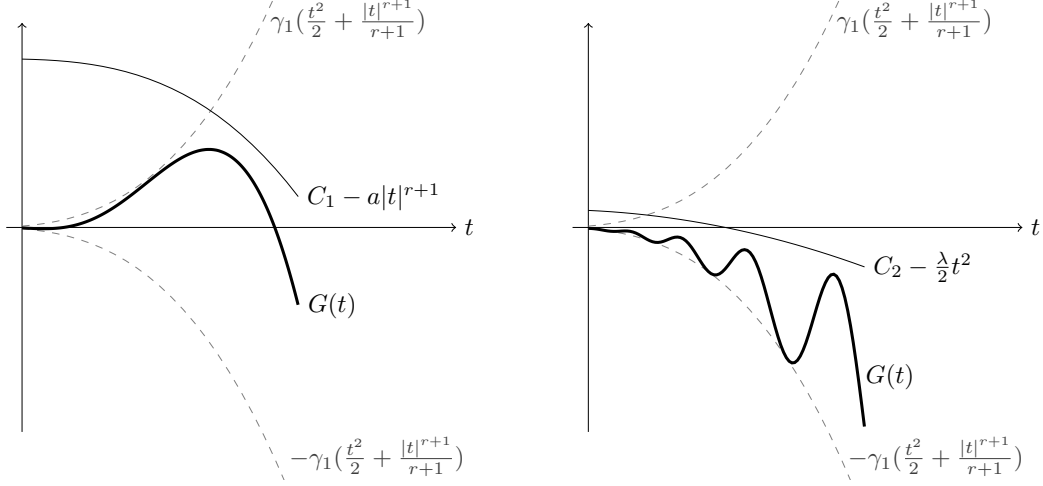


Figure 4.1: An illustration of the assumptions (2.1) and (4.2), as well as their compatibility with (1.31).

Proof. *A priori estimates.* Due to (4.2) we have

$$G(X_2, t) \leq C_2 - \frac{\lambda}{2} t^2, \quad \forall t \in \mathbb{R}, \quad (4.4)$$

for some constant $C_2 \geq 0$. Reporting this to (4.1), we have the lower boundedness

$$\tilde{\mathbf{j}}^\varepsilon(v) \geq \Phi_\lambda^\varepsilon(v) - \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)} - C_2 |\omega_2|, \quad (4.5)$$

where

$$\Phi_\lambda^\varepsilon(v) := \frac{1}{2} \int_\Omega \varepsilon^2 |\nabla_{X_1} v|^2 + h A \nabla_{X_2} v \cdot \nabla_{X_2} v dx + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2.$$

Thanks to (3.2) we have, for any given $0 < \eta < 1$,

$$\begin{aligned} \Phi_\lambda^\varepsilon(v) &= \frac{\eta + (1-\eta)}{2} \int_\Omega \varepsilon^2 |\nabla_{X_1} v|^2 + h A \nabla_{X_2} v \cdot \nabla_{X_2} v dx + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2 \\ &\geq \frac{\eta}{2} \min\{1, \alpha\} \int_\Omega \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx + \frac{(1-\eta)\lambda_1^\varepsilon + \lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2. \end{aligned}$$

Since $\lambda + \lambda_1^\varepsilon > 0$, then for $0 < \eta \leq \eta_\varepsilon := \min\{1, (\lambda + \lambda_1^\varepsilon) / \lambda_1^\varepsilon\}$, the last term above is nonnegative. Denoting $\delta_\varepsilon := \eta_\varepsilon \min\{1, \alpha\}$, we have

$$\Phi_\lambda^\varepsilon(v) \geq \frac{\delta_\varepsilon}{2} \int_\Omega \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx, \quad \forall \varepsilon > 0. \quad (4.6)$$

Going back to (4.5) and using (2.4), we obtain

$$\begin{aligned} \tilde{\mathbf{J}}^\varepsilon(v) \geq & \frac{\delta_\varepsilon}{2} \int_\Omega \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx \\ & - \|f_1\|_{\mathcal{V}'_0(\Omega, h)} \left(\int_\Omega \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx \right)^{1/2} - C_2 |\omega_2|. \end{aligned}$$

Applying Young's inequality $ab \leq \delta_\varepsilon^{-1} a^2 + \delta_\varepsilon b^2/4$, we get

$$\tilde{\mathbf{J}}^\varepsilon(v) \geq \frac{\delta_\varepsilon}{4} \int_\Omega \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx - \frac{1}{\delta_\varepsilon} \|f_1\|_{\mathcal{V}'_0(\Omega, h)}^2 - C_2 |\omega_2|. \quad (4.7)$$

Thus $\tilde{\mathbf{J}}^\varepsilon$ is bounded below, i.e.

$$d_\varepsilon := \inf_{v \in \mathcal{W}_0(\Omega, h)} \tilde{\mathbf{J}}^\varepsilon(v) > -\infty.$$

Passage to the limit. The estimation (1.31) implies that

$$G(X_2, t) \leq \gamma_2 t^2, \text{ for small } t \text{ and some constant } \gamma_2 \geq 0.$$

Combining this with (4.2), which holds for t large, we deduce that for some constant $\mu \geq 0$ it holds that

$$G(X_2, t) \leq \mu t^2, \text{ for all } t \in \mathbb{R}. \quad (4.8)$$

Then, we rewrite $\tilde{\mathbf{J}}^\varepsilon$ as

$$\tilde{\mathbf{J}}^\varepsilon(v) = \Phi_\lambda^\varepsilon(v) + J_1(v) - \left(\frac{\lambda}{2} + \mu \right) \int_{\omega_2} |l(v)|^2 dX_2 - \langle f_1, v \rangle_{\mathcal{V}'_0(\Omega, h)}, \quad (4.9)$$

where

$$J_1(v) := \int_{\omega_2} -G(X_2, l(v)) + \mu |l(v)|^2 dX_2. \quad (4.10)$$

Let $(u_n)_{n \geq 0}$ be a minimizing sequence. By (4.7), we deduce that $(u_n)_{n \geq 0}$ is necessarily bounded in $\mathcal{W}_0(\Omega, h)$. Then, there exist $u \in \mathcal{W}_0(\Omega, h)$ and a weakly converging subsequence $(u_{n_k})_{n_k \geq 0}$ in $\mathcal{W}_0(\Omega, h)$ such that

$$\begin{aligned} \nabla_{X_1} u_{n_k} &\rightharpoonup \nabla_{X_1} u \text{ in } L^2(\Omega), \quad \nabla_{X_2} u_{n_k} \rightharpoonup \nabla_{X_2} u, \text{ in } L^2(\Omega, h), \\ l(u_{n_k}) &\rightarrow l(u) \text{ in } L^2(\omega_2), \end{aligned}$$

as $n_k \rightarrow +\infty$. The last strong convergence is a consequence of Lemma 1.1 with $p = 2$. Up to a new subsequence, we have $l(u_{n_k}) \rightarrow l(u)$ a.e. in ω_2 and since the function $-G(X_2, t) + \mu t^2$ is nonnegative, Fatou's lemma implies that

$$J_1(u) \leq \liminf_{n_k \rightarrow \infty} J_1(u_{n_k}).$$

Using the weak lower semi-continuity of norms, we infer that

$$\Phi_\lambda^\varepsilon(u) \leq \liminf_{n_k \rightarrow \infty} \Phi_\lambda^\varepsilon(u_{n_k}).$$

Passing to the limit in (4.9) yields

$$\tilde{\mathbf{J}}^\varepsilon(u) \leq \liminf_{n_k \rightarrow \infty} \tilde{\mathbf{J}}^\varepsilon(u_{n_k}) = d_\varepsilon.$$

Therefore, the limit u realizes the minimum of $\tilde{\mathbf{J}}^\varepsilon$ on $\mathcal{W}_0(\Omega, h)$.

As $1 \leq r < 2^* - 1$, then by Corollary 1.1 we have $\tilde{\mathbf{J}}^\varepsilon \in C^1(\mathcal{W}_0(\Omega, h), \mathbb{R})$. Thus $(\tilde{\mathbf{J}}^\varepsilon)'(u) = 0$ and the limit u is a solution to (\tilde{P}_ε) . This ends the proof. ■

Example 4.1 *Theorem 4.1 applies to the problem*

$$\begin{cases} -\frac{\varepsilon^2}{h} \Delta_{X_1} u - \nabla_{X_2} \cdot (A \nabla_{X_2} u) + \lambda l(u) + a(l(u)) |l(u)|^{r-1} l(u) \\ \qquad \qquad \qquad - b(l(u)) |l(u)|^{q-1} l(u) = f_1 \quad \text{in } \mathcal{W}'_0(\Omega, h) \\ u \in \mathcal{W}_0(\Omega, h), \end{cases}$$

where $\lambda > -\lambda_1^\varepsilon, 1 < q < r < 2^* - 1$ and $a, b : \mathbb{R} \rightarrow \mathbb{R}$ are bounded continuous functions satisfying

$$a \geq 0 \text{ and } b(t) = O(a(t)), \text{ as } |t| \rightarrow +\infty.$$

Here we have taken

$$g(\cdot, t) = -\lambda t - a(t) |t|^{r-1} t + b(t) |t|^{q-1} t, \text{ for } t \in \mathbb{R}.$$

Corollary 4.1 *Let u be a solution of Problem (\tilde{P}_ε) given by Theorem 4.1. If the mapping*

$$s \mapsto g(\cdot, s) + \lambda s \text{ is non increasing,} \tag{4.11}$$

then u is the unique solution of (\tilde{P}_ε) .

Proof. We keep the notation of the previous proof. If (4.11) holds, then the functional

$$v \mapsto - \int_{\omega_2} G(X_2, l(v)) + \frac{\lambda}{2} |l(v)|^2 dX_2$$

is convex. In addition, Φ_λ^ε is strictly convex since it satisfies (4.29). Rewriting $\tilde{\mathbf{J}}^\varepsilon$ as

$$\tilde{\mathbf{J}}^\varepsilon(v) = \Phi_\lambda^\varepsilon(v) - \int_{\omega_2} G(X_2, l(v)) + \frac{\lambda}{2} |l(v)|^2 dX_2 - \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)},$$

we can see that $\tilde{\mathbf{J}}^\varepsilon$ is strictly convex and the uniqueness of the solution realizing the minimum follows. ■

4.1.2 An existence result for the unperturbed problem

The next theorem shows the existence of a solution u_0 to the following unperturbed problem

$$\int_{\Omega} h A \nabla_{X_2} u \cdot \nabla_{X_2} v dx = \int_{\omega_2} g(X_2, l(u)) l(v) dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)}, \quad \forall v \in \mathcal{V}_0(\Omega, h). \quad (\tilde{P}_0)$$

Theorem 4.2 *Under the assumptions of Theorem 4.1 with $\lambda > -\lambda_1^0$, and if u_ε is a solution of Problem (\tilde{P}_ε) , then we have –up to a subsequence–*

$$u_\varepsilon \rightharpoonup u_0, \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h), \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightarrow 0 \text{ in } L^2(\Omega), \quad (4.12)$$

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^{r+1}(\omega_2), \quad (4.13)$$

$$g(\cdot, l(u_\varepsilon)) \rightarrow g(\cdot, l(u_0)) \text{ in } L^{\frac{r+1}{r}}(\omega_2). \quad (4.14)$$

Moreover, the above convergences hold for the whole sequence if the solution of (\tilde{P}_0) is unique.

Proof. First, the assumptions of Theorem 4.1 hold since $\lambda > -\lambda_1^0 \geq -\lambda_1^\varepsilon$. Thus, for any $\varepsilon > 0$, a solution u_ε exists and realizes the minimum of \tilde{J}^ε on $\mathcal{W}_0(\Omega, h)$. For $0 < \varepsilon < 1$, we have

$$\tilde{J}^\varepsilon(u_\varepsilon) \leq \tilde{J}^\varepsilon(v) \leq \tilde{J}^1(v), \quad \forall v \in \mathcal{W}_0(\Omega, h)$$

and $\lambda_1^1 > \lambda_1^\varepsilon > \lambda_1^0$. In particular,

$$\eta_\varepsilon > \eta_0 := \min \left\{ 1, \frac{(\lambda + \lambda_1^0)}{\lambda_1^1} \right\} > 0, \quad \text{for } \lambda > -\lambda_1^0,$$

where η_ε is the constant considered in (4.29). Denoting $\delta_0 := \eta_0 \min\{1, \alpha\}$, we have

$$\Phi_\lambda^\varepsilon(v) \geq \frac{\delta_0}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx, \quad \text{for } 0 < \varepsilon < 1,$$

where this time δ_0 is independent of ε .

The same argument used to derive (4.7) yields

$$\frac{\delta_0}{4} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + h |\nabla_{X_2} u_\varepsilon|^2 dx \leq \tilde{J}^1(v) + \frac{1}{\delta_0} \|f_1\|_{\mathcal{V}'_0(\Omega, h)}^2 + C |\omega_2|, \quad \forall v \in \mathcal{W}_0(\Omega, h). \quad (4.15)$$

This means that the sequences

$$u_\varepsilon, |\nabla_{X_2} u_\varepsilon| \text{ are bounded in } L^2(\Omega, h),$$

$$|\varepsilon \nabla_{X_1} u_\varepsilon| \text{ is bounded in } L^2(\Omega).$$

Then, there exists $u_0 \in \mathcal{V}_0(\Omega, h)$ such that –up to a subsequence–

$$u_\varepsilon \rightharpoonup u_0, \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h), \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \text{ in } L^2(\Omega), \quad (4.16)$$

as $\varepsilon \rightarrow 0$. Thanks to Lemma 1.1, we have also

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^{r+1}(\omega_2) \quad (4.17)$$

since $1 \leq r < 2^* - 1$. Moreover, up to a new subsequence, $l(u_\varepsilon) \rightarrow l(u_0)$ a.e. in ω_2 and there exists $K \in L^{r+1}(\omega_2)$, such that $|l(u_\varepsilon)| \leq K$ a.e. in ω_2 . The continuity of g implies that

$$g(X_2, l(u_\varepsilon)) \rightarrow g(X_2, l(u_0)) \text{ a.e. in } \omega_2. \quad (4.18)$$

In addition, we have

$$|g(X_2, l(u_\varepsilon))|^{\frac{r+1}{r}} \leq C \left(1 + K^{r+1}(X_2)\right) \text{ a.e. in } \omega_2,$$

see (2.22). Applying Lebesgue's theorem, we derive the strong convergence (4.14).

Going back to (\tilde{P}_ε) and passing to the limit, we deduce that the limit u_0 satisfies

$$\int_{\Omega} hA \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dx = \int_{\omega_2} g(X_2, l(u_0)) l(v) dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, h)}, \quad (4.19)$$

for every $v \in \mathcal{W}_0(\Omega, h)$. This means that u_0 is a solution to Problem (\tilde{P}_0) since $\mathcal{W}_0(\Omega, h)$ is dense in $\mathcal{V}_0(\Omega, h)$.

It remains to show the strong convergences (4.12). To this end we consider

$$\tilde{I}_\varepsilon := \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + hA \nabla_{X_2} (u_\varepsilon - u_0) \cdot \nabla_{X_2} (u_\varepsilon - u_0) dx. \quad (4.20)$$

Expanding \tilde{I}_ε and taking $v = u_\varepsilon$ in the h -weak formulation (\tilde{P}_0) , we get

$$\begin{aligned} \tilde{I}_\varepsilon = & \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(u_\varepsilon) dX_2 + \langle f_1, u_\varepsilon \rangle_{\mathcal{V}_0(\Omega, h)} \\ & - 2 \int_{\Omega} hA \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_0 dx + \int_{\Omega} hA \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx. \end{aligned}$$

Thanks to (4.14), (4.16) and (4.17), the passage to the limit in \tilde{I}_ε yields

$$\lim_{\varepsilon \rightarrow 0} \tilde{I}_\varepsilon = \int_{\omega_2} g(X_2, l(u_0)) l(u_0) dX_2 + \langle f_1, u_0 \rangle_{\mathcal{V}_0(\Omega, h)} - \int_{\Omega} hA \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx.$$

Going back to (\tilde{P}_0) and taking $v = u_0$, we infer that $\lim_{\varepsilon \rightarrow 0} \tilde{I}_\varepsilon = 0$. The convergences (4.12) follow from the inequality

$$\tilde{I}_\varepsilon \geq \min\{1, \alpha\} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + h |\nabla_{X_2} (u_\varepsilon - u_0)|^2 dx.$$

This ends the proof of the theorem. ■

Concerning the convergence of $\tilde{J}^\varepsilon(u_\varepsilon)$ as $\varepsilon \rightarrow 0$, we set

$$\tilde{J}^0(v) := \frac{1}{2} \int_{\Omega} hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx - \int_{\omega_2} G(X_2, l(v)) dX_2 - \langle f_1, v \rangle. \quad (4.21)$$

Then, we have:

Corollary 4.2 *Under the assumptions of Theorem 4.2, the converging subsequences satisfy*

$$G(\cdot, l(u_\varepsilon)) \rightarrow G(\cdot, l(u_0)) \text{ in } L^1(\omega_2), \tag{4.22}$$

and give smooth minimizing sequences for the functional \tilde{J}^0 . That is,

$$\tilde{J}^\varepsilon(u_\varepsilon) \rightarrow \tilde{J}^0(u_0) = \inf_{v \in \mathcal{V}_0(\Omega, h)} \tilde{J}^0(v). \tag{4.23}$$

Proof. Since (4.13) holds and G still satisfies (1.31), the convergence (7.43) follows as in the proof of Theorem 2.2. Combining this with (4.12), we can pass to the limit in $\tilde{J}^\varepsilon(u_\varepsilon)$. It comes that

$$\tilde{J}^\varepsilon(u_\varepsilon) \rightarrow \tilde{J}^0(u_0) \geq \inf_{v \in \mathcal{V}_0(\Omega, h)} \tilde{J}^0(v).$$

In addition, $\tilde{J}^\varepsilon(u_\varepsilon) \leq \tilde{J}^\varepsilon(v), \forall v \in \mathcal{D}(\Omega)$. Passing to the limit in both sides, we get

$$\tilde{J}^0(u_0) \leq \tilde{J}^0(v), \forall v \in \mathcal{D}(\Omega). \tag{4.24}$$

Since $\mathcal{D}(\Omega)$ is dense in $\mathcal{V}_0(\Omega, h)$ and $\tilde{J}^0 \in C^1(\mathcal{V}_0(\Omega, h), \mathbb{R})$ (see Remark 1.7), then (4.24) still holds for every $v \in \mathcal{V}_0(\Omega, h)$. So u_0 realizes the minimum of \tilde{J}^0 on $\mathcal{V}_0(\Omega, h)$ and the corollary follows. ■

Remark 4.1 *The arguments used in the proof of Theorem 4.1 can be adapted, by setting $\varepsilon = 0$ and replacing $\mathcal{W}_0(\Omega, h)$ by $\mathcal{V}_0(\Omega, h)$, to obtain the existence of a solution for Problem (\tilde{P}_0) that realizes the infimum of \tilde{J}^0 on $\mathcal{V}_0(\Omega, h)$. Moreover, this solution is unique if (4.11) holds.*

This chapter is devoted to some applications, as we will deal with different types of nonlinearity g in (SP_0) . we study the existence of solutions (denoted by u_ε) for the perturbed problems as well as the asymptotic behavior of u_ε and the related functional J^ε . Then, passage to the limit as $\varepsilon \rightarrow 0$.

4.2 EXISTENCE AND NON UNIQUENESS RESULTS FOR A SEMILINEAR PROBLEM WITH $\lambda > -\lambda_1^\varepsilon$

In this section, we take

$$g(s) = -\lambda s + a |s|^{r-1} s$$

Then, we consider the h -weak problem to find u_ε satisfying

$$\begin{aligned} \int_{\Omega} \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + h A \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v dx \\ = -\lambda \int_{\omega_2} l(u_\varepsilon) l(v) dX_2 + a \int_{\omega_2} |l(u_\varepsilon)|^{r-1} l(u_\varepsilon) l(v) dX_2, \end{aligned} \tag{4.25}$$

for every $v \in \mathcal{W}_0(\Omega, h)$. The associated functional

$$\begin{aligned} \mathbf{J}^\varepsilon(v) := & \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx \\ & + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2 - \frac{a}{r+1} \int_{\omega_2} |l(v)|^{r+1} dX_2 \end{aligned} \quad (4.26)$$

is not bounded from below (it suffices to let $t \rightarrow +\infty$ in $\mathbf{J}^\varepsilon(tv)$). This led us to consider a constrained minimization approach to investigate the existence of non trivial solutions for this problem.

4.2.1 An existence result for the perturbed problem

Theorem 4.3 *Let $\varepsilon > 0$ and assume that*

$$a > 0, 1 < r < 2^* - 1 \text{ and } \lambda > -\lambda_1^\varepsilon.$$

Then Problem (4.25) has a solutions u that minimizes \mathbf{J}^ε on $\mathcal{W}_0(\Omega, h)$, where $l(u) \not\equiv 0$. In addition, there are at least two other different solutions 0 and $-u$.

Proof. We split the proof to several steps.

Step 1. We consider¹ the set (of constraints)

$$\mathcal{S} := \left\{ v \in \mathcal{W}_0(\Omega, h), l(v) \not\equiv 0; \|l(v)\|_{L^{r+1}(\omega_2)} = 1 \right\}.$$

Then \mathbf{J}^ε restricted to \mathcal{S} reads

$$\mathbf{J}^\varepsilon(v) = \mathbf{I}^\varepsilon(v) - \frac{a}{r+1}, \quad (4.27)$$

where

$$\mathbf{I}^\varepsilon(v) := \frac{1}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2. \quad (4.28)$$

Now minimizing \mathbf{J}^ε on \mathcal{S} is equivalent to minimizing just \mathbf{I}^ε on \mathcal{S} .

Thanks to (3.2) we have, for any given $0 < \eta < 1$,

$$\begin{aligned} \mathbf{I}^\varepsilon(v) &= \frac{\eta + (1-\eta)}{2} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2 \\ &\geq \frac{\eta}{2} \min\{1, \alpha\} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx + \frac{(1-\eta)\lambda_1^\varepsilon + \lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2. \end{aligned}$$

Since $\lambda + \lambda_1^\varepsilon > 0$, then for $0 < \eta < \min\left\{1, \left(\lambda + \lambda_1^0\right) / \lambda_1^1\right\}$ the last term above is nonnegative.

Denoting $\alpha_\lambda := \eta \min\{1, \alpha\} / 2$, which is independent of ε , we have

$$\mathbf{I}^\varepsilon(v) \geq \alpha_\lambda \int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + h |\nabla_{X_2} v|^2 dx, \quad \forall \varepsilon > 0. \quad (4.29)$$

¹ We argue as in [3, page 56].

Step 2. Let us show that the infimum

$$\mathbf{d}_\varepsilon := \inf_{v \in \mathcal{S}} \mathbf{I}^\varepsilon(v)$$

is attained for some $\tilde{u} \in \mathcal{S}$. Let $u_n \in \mathcal{S}$ be a minimizing sequence. By (4.29), such sequence is bounded in $\mathcal{W}_0(\Omega, h)$. Then we have –up to a subsequence–

$$u_n \rightharpoonup \tilde{u} \text{ in } \mathcal{W}_0(\Omega, h), \quad (4.30)$$

for some $\tilde{u} \in \mathcal{W}_0(\Omega, h)$. Since $r < 2^* - 1$, Lemma 1.1 yields the strong convergences

$$l(u_n) \rightarrow l(\tilde{u}) \text{ in } L^{r+1}(\omega_2) \text{ and also in } L^2(\omega_2). \quad (4.31)$$

Besides, since A is symmetric and satisfy (1.22), then the mapping

$$v \mapsto \left(\int_{\Omega} \varepsilon^2 |\nabla_{X_1} v|^2 + hA \nabla_{X_2} v \cdot \nabla_{X_2} v dx \right)^{1/2},$$

defines a norm on $\mathcal{W}_0(\Omega, h)$. Thanks to (4.30) and the lower semi-continuity of norms, we deduce that

$$\int_{\Omega} \varepsilon^2 |\nabla_{X_1} \tilde{u}|^2 + hA \nabla_{X_2} \tilde{u} \cdot \nabla_{X_2} \tilde{u} dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \varepsilon^2 |\nabla_{X_1} u_n|^2 + hA \nabla_{X_2} u_n \cdot \nabla_{X_2} u_n dx.$$

Combining this with the strong convergence in (4.31), we infer that

$$\mathbf{I}^\varepsilon(\tilde{u}) \leq \liminf_{n \rightarrow +\infty} \mathbf{I}^\varepsilon(u_n) = \mathbf{d}_\varepsilon \text{ and } \|l(\tilde{u})\|_{L^{r+1}(\omega_2)} = 1.$$

Thus $\tilde{u} \in \mathcal{S}$, and this shows that $\mathbf{I}^\varepsilon(\tilde{u}) = \mathbf{d}_\varepsilon$. Taking into account that λ_1^ε is given by (3.2), we infer that

$$\mathbf{d}_\varepsilon = \mathbf{I}^\varepsilon(\tilde{u}) \geq \frac{\lambda + \lambda_1^\varepsilon}{2} \int_{\omega_2} |l(\tilde{u})|^2 dX_2 > 0,$$

since $l(\tilde{u}) \not\equiv 0$ and $\lambda + \lambda_1^\varepsilon > 0$.

Step 3. We check that \tilde{u} , the minimizer found with the previous step, satisfies

$$\int_{\Omega} \varepsilon^2 \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1} v + hA \nabla_{X_2} \tilde{u} \cdot \nabla_{X_2} v dx + \lambda \int_{\omega_2} l(\tilde{u}) l(v) dX_2 = \mathbf{d}_\varepsilon \int_{\omega_2} |l(\tilde{u})|^{r-1} l(\tilde{u}) l(v) dX_2, \quad (4.32)$$

for all $v \in \mathcal{W}_0(\Omega, h)$. To this aim, fix $v \in \mathcal{W}_0(\Omega, h)$. For $t \in \mathbb{R}$ small enough, say $t \in (-\delta, \delta)$, the function $l(\tilde{u} + tv)$, is not identically zero. Therefore there exists $\eta : (-\delta, \delta) \rightarrow (0, +\infty)$ such that

$$\int_{\omega_2} |l(\eta(t)(\tilde{u} + tv))|^{r+1} dX_2 = 1,$$

i.e. $\eta(t) = \left(\int_{\omega_2} |l(\tilde{u} + tv)|^{r+1} dX_2 \right)^{-1/(r+1)}$. The function η is differentiable on $(-\delta, \delta)$, and

$$\eta'(t) = - \left(\int_{\omega_2} |l(\tilde{u} + tv)|^{r+1} dX_2 \right)^{-\frac{r+2}{r+1}} \int_{\omega_2} |l(\tilde{u} + tv)|^{r-1} l(\tilde{u} + tv) l(v) dX_2.$$

Then, we have

$$\eta(0) = 1, \eta'(0) = - \int_{\omega_2} |l(\tilde{u})|^{r-1} l(\tilde{u}) l(v) dX_2.$$

We define $\gamma : (-\delta, \delta) \rightarrow \mathbb{R}$ as

$$\gamma(t) := \mathbf{I}^\varepsilon(\eta(t)(\tilde{u} + tv)).$$

The function γ is differentiable and

$$\begin{aligned} \gamma'(t) &= 2\eta'(t)\eta(t)\mathbf{I}^\varepsilon(\tilde{u} + tv) + 2\eta^2(t) \int_{\Omega} \varepsilon^2 \nabla_{X_1}(\tilde{u} + tv) \cdot \nabla_{X_1} v + hA \nabla_{X_2}(\tilde{u} + tv) \cdot \nabla_{X_2} v dx \\ &\quad + 2\lambda \eta^2(t) \int_{\omega_2} l(\tilde{u} + tv) l(v) dX_2. \end{aligned}$$

Since $\eta(t)(\tilde{u} + tv) \in \mathcal{S}$ for every $t \in (-\delta, \delta)$, the point $t = 0$ is a local minimum for γ . So we should have

$$0 = \gamma'(0) = 2\eta'(0)\mathbf{I}^\varepsilon(\tilde{u}) + 2 \int_{\Omega} \varepsilon^2 \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1} v + hA \nabla_{X_2} \tilde{u} \cdot \nabla_{X_2} v dx + 2\lambda \int_{\omega_2} l(\tilde{u}) l(v) dX_2,$$

i.e. \tilde{u} satisfies (4.32).

Finally, the function² $u = \pm (a/\mathbf{d}_\varepsilon)^{1/(r-1)} \tilde{u}$ is a solution to Problem (4.25) with

$$\int_{\omega_2} |l(u)|^{r+1} dX_2 = (a/\mathbf{d}_\varepsilon)^{(r+1)/(r-1)} > 0.$$

This ends the proof. ■

Corollary 4.3 *Under the assumptions of Theorem 4.3, with $-\lambda_1^\varepsilon < \lambda \leq 0$, Problem (4.25) has one positive solution and one negative solution.*

Proof. Let $v \in \mathcal{W}_0(\Omega, h)$, $l(v) \not\equiv 0$. Set $v^+ := \max\{0, v\}$. Arguing by density as in [22, Lemma 1.19], we can show that

$$v^+ \in \mathcal{W}_0(\Omega, h) \quad \text{and} \quad \partial_{x_i} v^+ = (\partial_{x_i} \varphi_\varepsilon) \chi_{\{\varphi_\varepsilon > 0\}}, \quad \text{for } i = 1, \dots, n_1 + n_2.$$

In particular, this means that $|v| = (2v^+ - v) \in \mathcal{W}_0(\Omega, h)$ and one can check that

$$0 \leq \mathbf{I}^\varepsilon(|v|) \leq \mathbf{I}^\varepsilon(v) \quad \text{when} \quad -\lambda_1^\varepsilon < \lambda \leq 0. \quad (4.33)$$

Let now $\tilde{u} \in \mathcal{S}$ be a minimizer to \mathbf{J}^ε and set $b_\varepsilon := \|l(|\tilde{u}|)\|_{L^{r+1}(\omega_2)}$. Clearly

$$b_\varepsilon \geq \|l(\tilde{u})\|_{L^{r+1}(\omega_2)} = 1 \quad \text{and} \quad \frac{|u|}{b_\varepsilon} \in \mathcal{S}.$$

² \tilde{u} minimises \mathbf{J}^ε on \mathcal{S} but u is the one that solve Problem (4.25).

Thanks to (4.27) and (4.33), we have

$$\mathbf{J}^\varepsilon \left(\frac{|\tilde{u}|}{b_\varepsilon} \right) = \frac{1}{b_\varepsilon^2} \mathbf{I}^\varepsilon (|\tilde{u}|) - \frac{a}{r+1} \leq \frac{1}{b_\varepsilon^2} \mathbf{I}^\varepsilon (\tilde{u}) - \frac{a}{r+1}.$$

Taking into account that $b_\varepsilon \geq 1$ and $\mathbf{I}^\varepsilon (\tilde{u}) \geq 0$, we infer that

$$\mathbf{J}^\varepsilon \left(\frac{|\tilde{u}|}{b_\varepsilon} \right) \leq \mathbf{J}^\varepsilon (\tilde{u}).$$

This means that $|\tilde{u}|/b_\varepsilon$ is also a minimizer to \mathbf{J}^ε on \mathcal{S} . Arguing as above, $u = + (a/\mathbf{d}_\varepsilon)^{1/(r-1)} |\tilde{u}|/b_\varepsilon$ is a non-negative solution to Problem (4.25). It also satisfies

$$-\frac{\varepsilon^2}{h} \Delta_{X_1} u - \nabla_{X_2} (A \nabla_{X_2} u) \geq 0,$$

in the (classical) weak sense. Under the assumption (1.21), (1.22), (1.7) and according to [36], we must have $u > 0$, a.e. in Ω . Of course $-u$ is a negative solution to (4.25). This ends the proof. ■

4.2.2 An existence result for the unperturbed problem

Theorem 4.4 *Under the assumptions of Theorem 4.3 with $\lambda > -\lambda_1^0$, and if u_ε is a solution for Problem (4.25) such that $l(u_\varepsilon) \not\equiv 0$, then we have –up to a subsequence–*

$$u_\varepsilon \rightarrow u_0, \nabla_{X_2} u_\varepsilon \rightarrow \nabla_{X_2} u_0 \text{ in } L^2(\Omega, h), \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightarrow 0 \text{ in } L^2(\Omega), \quad (4.34)$$

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^{r+1}(\omega_2). \quad (4.35)$$

where $u_0 \in \mathcal{V}_0(\Omega, h)$ is solution to problem

$$\int_{\Omega} h A \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dx + \lambda \int_{\omega_2} l(u_0) l(v) dX_2 = a \int_{\omega_2} |l(u_0)|^{r-1} l(u_0) l(v) dX_2, \quad (4.36)$$

for every $v \in \mathcal{V}_0(\Omega, h)$. Moreover, $l(u_0) \not\equiv 0$ and Problem (4.36) have two other different solutions 0 and $-u_0$.

Proof. First, since $\lambda > -\lambda_1^0 > -\lambda_1^\varepsilon$, all the assumptions of Theorem 4.3 hold and there exists $\tilde{u}_\varepsilon \in \mathcal{S}$ such that

$$\mathbf{I}^\varepsilon (\tilde{u}_\varepsilon) = \inf_{v \in \mathcal{S}} \mathbf{I}^\varepsilon (v) = \mathbf{d}_\varepsilon > 0, \quad \text{for } 0 < \varepsilon < 1.$$

Recall that \tilde{u}_ε solves (4.32). Thanks to (4.29) and the fact that $\mathbf{d}_\varepsilon < d_1$ for $0 < \varepsilon < 1$, we get

$$\alpha_\lambda \int_{\Omega} \varepsilon^2 |\nabla_{X_1} \tilde{u}_\varepsilon|^2 + h |\nabla_{X_2} \tilde{u}_\varepsilon|^2 dx \leq \mathbf{I}^\varepsilon (\tilde{u}_\varepsilon) = \mathbf{d}_\varepsilon \leq d_1, \quad (4.37)$$

hence

$$\tilde{u}_\varepsilon, |\nabla_{X_2} \tilde{u}_\varepsilon| \text{ are bounded in } L^2(\Omega, h),$$

$$|\varepsilon \nabla_{X_1} \tilde{u}_\varepsilon| \text{ is bounded in } L^2(\Omega).$$

We can argue as in the proof of Theorem 4.2 to show that –up to a subsequence–

$$\tilde{u}_\varepsilon \rightarrow u_0, \nabla_{X_2} \tilde{u}_\varepsilon \rightarrow \nabla_{X_2} \tilde{u}_0 \text{ in } L^2(\Omega, h), \quad \varepsilon \nabla_{X_1} \tilde{u}_\varepsilon \rightarrow 0 \text{ in } L^2(\Omega), \quad (4.38)$$

$$l(\tilde{u}_\varepsilon) \rightarrow l(\tilde{u}_0) \text{ in } L^2(\omega_2) \text{ and in } L^{r+1}(\omega_2), \quad (4.39)$$

The last two convergences follows from Lemma 1.1 since $2 < r < 2^* - 1$.

Passing to the limit in (4.32), we deduce that limit \tilde{u}_0 is a solution to the problem

$$\int_{\Omega} h A \nabla_{X_2} \tilde{u}_0 \cdot \nabla_{X_2} v dx + \lambda \int_{\omega_2} l(\tilde{u}_0) l(v) dX_2 = \mathbf{d}_0 \int_{\omega_2} |l(\tilde{u}_0)|^{r-1} l(\tilde{u}_0) l(v) dX_2, \quad (4.40)$$

for all $v \in \mathcal{V}_0(\Omega, h)$. Here $\mathbf{d}_0 = \lim_{\varepsilon \rightarrow 0} \mathbf{d}_\varepsilon$. Such limit exists since $(\mathbf{d}_\varepsilon)_\varepsilon$ decreases as $\varepsilon \rightarrow 0$ and $\mathbf{d}_\varepsilon > 0$. Since $\|l(\tilde{u}_\varepsilon)\|_{L^{r+1}(\omega_2)} = 1$ the strong convergence $l(\tilde{u}_\varepsilon) \rightarrow l(\tilde{u}_0)$ in $L^{r+1}(\omega_2)$ implies that

$$\|l(\tilde{u}_0)\|_{L^{r+1}(\omega_2)} = 1, \quad (4.41)$$

and necessarily $l(u_0) \not\equiv 0$. Letting $\varepsilon \rightarrow 0$ in (4.37), we deduce in particular that

$$\alpha_\lambda \int_{\Omega} h |\nabla_{X_2} \tilde{u}_0|^2 dx \leq \mathbf{d}_0,$$

(Recall that α_λ is independent from ε). This means that $\mathbf{d}_0 > 0$ since $\tilde{u}_0 \not\equiv 0$.

The function $u_0 = \pm (a/\mathbf{d}_0)^{1/(r-1)} \tilde{u}_0$ is a solution to Problem (4.36) and $u_\varepsilon = \pm (a/\mathbf{d}_\varepsilon)^{1/(r-1)} \tilde{u}_\varepsilon$ satisfies the convergences (4.34) and (4.35). This ends the proof. ■

Remark 4.2 *If $\lambda > -\lambda_1^0$, then Problem (4.36) has one non-positive solution and one non-negative solution. It suffices to take a sequence of positive (negative resp.) solutions, which exists by Theorem 4.3, then pass to the limit $\varepsilon \rightarrow 0$ as in Theorem 4.3.*

Finally, give an additional information on u_0 by characterizing .

Corollary 4.4 *Let u_0 be a limit of a converging subsequence from Theorem 4.4. Then u_0 realizes the minimum of*

$$\mathbf{J}^0(u) := \frac{1}{2} \int_{\Omega} h A \nabla_{X_2} u \cdot \nabla_{X_2} u dx + \frac{\lambda}{2} \int_{\omega_2} |l(u)|^2 dX_2 - \frac{a}{r+1} \int_{\omega_2} |l(u)|^{r+1} dX_2,$$

on the set

$$\mathcal{S}_0 := \left\{ u \in \mathcal{V}_0(\Omega, h), l(v) \not\equiv 0; \|l(v)\|_{L^{r+1}(\omega_2)} = 1 \right\},$$

and we have

$$\mathbf{J}^\varepsilon(u_\varepsilon) \rightarrow \mathbf{J}^0(u_0) \text{ and } \mathbf{J}^\varepsilon(\tilde{u}_\varepsilon) \rightarrow \mathbf{J}^0(\tilde{u}_0) = \inf_{v \in \mathcal{S}_0} \mathbf{J}^0(v), \quad (4.42)$$

where $\tilde{u}_0 = \pm (a/\mathbf{d}_0)^{-1/(r-1)} u_0 \in \mathcal{S}_0$.

Proof. We use the notation of the proof of Theorem 4.4 and we set

$$\mathbf{I}^0(v) := \frac{1}{2} \int_{\Omega} h A \nabla_{X_2} v \cdot \nabla_{X_2} v dx + \frac{\lambda}{2} \int_{\omega_2} |l(v)|^2 dX_2. \quad (4.43)$$

The function $\tilde{u}_0 \in \mathcal{V}_0(\Omega, h)$ and by (4.41) we know that $\tilde{u}_0 \in \mathcal{S}_0$. In addition, we have

$$\mathbf{J}^\varepsilon(\tilde{u}_\varepsilon) \leq \mathbf{J}^\varepsilon(v), \forall v \in \mathcal{S}.$$

The strong convergences (4.38) and (4.39) implies that

$$\mathbf{J}^\varepsilon(\tilde{u}_\varepsilon) \rightarrow \mathbf{J}^0(\tilde{u}_0) \leq \mathbf{J}^0(v), \forall v \in \mathcal{S}.$$

Let $\tilde{v} \in \mathcal{S}_0$. By the density of $\mathcal{W}_0(\Omega, h)$ in $\mathcal{V}_0(\Omega, h)$, there exists a sequence $(v_n)_{n \geq 0}$ in $\mathcal{W}_0(\Omega, h)$, such that

$$v_n \rightarrow \tilde{v} \text{ in } \mathcal{V}_0(\Omega, h), \text{ as } n \rightarrow +\infty.$$

Since $r < 2^* - 1$, Lemma 1.1 ensures the strong convergence $v_n \rightarrow \tilde{v}$ in $L^{r+1}(\omega_2)$. Having $\|l(\tilde{v})\|_{L^{r+1}(\omega_2)} = 1$, we can always assume that $\|l(v_n)\|_{L^{r+1}(\omega_2)} \neq 0$. The sequence $w_n := v_n / \|l(v_n)\|_{L^{r+1}(\omega_2)}^{r+1}$ satisfies

$$w_n \in \mathcal{S}, w_n \rightarrow \tilde{v} \text{ in } \mathcal{V}_0(\Omega, h), \text{ as } n \rightarrow +\infty.$$

As $\mathbf{J}^0 \in C^1(\mathcal{V}_0(\Omega, h), \mathbb{R})$, see Remark 1.7-ii), this implies in particular

$$\mathbf{J}^0(\tilde{u}_0) \leq \mathbf{J}^0(w_n) \rightarrow \mathbf{J}^0(\tilde{v}).$$

Since \tilde{v} is arbitrary chosen in \mathcal{S}_0 , this means that \tilde{u}_0 realizes the minimum of \mathbf{J}^0 on \mathcal{S}_0 and the corollary follows. ■

Part II

ASYMPTOTIC BEHAVIOUR AND EXISTENCE RESULTS FOR SOME
NON-LOCAL QUASILINEAR ELLIPTIC PROBLEMS

 FUNCTIONAL SETTING FOR THE QUASILINEAR PROBLEMS

In this chapter, we define the functional spaces related to the quasilinear problems considered in this second part. Then, we state the assumptions used in the remainnig of this thesis. Finally, we consider some C^1 –functionals involving non-local terms.

5.1 ANISOTROPIC WEIGHTED SOBOLEV-TYPE SPACES

We consider a measurable function b positive a.e. on $\Omega := \omega_1 \times \omega_2$ and a second function h , depending only on X_1 , positive a.e. on ω_1 . By consequence, the function η given by

$$\eta(x) = h(X_1) b(x)$$

is also measurable and positive a.e. on Ω . For $1 < p < +\infty$, we consider the following weighted anisotropic Sobolev-type space

$$\mathcal{V}^p(\Omega, \eta) := \left\{ v \in L^p(\Omega) \mid \partial_{x_i'} v \in L^p(\Omega, \eta), i = 1, \dots, n_2 \right\},$$

equipped with the norm (the weight is only in the gradiant)

$$v \mapsto \left(\|\nabla_{X_2} v\|_{L^p(\Omega, \eta)}^p + \|v\|_{L^p(\Omega)}^p \right)^{1/p}. \quad (5.1)$$

As we will assume Dirichlet boundary conditions on $\omega_1 \times \partial\omega_2$, we also consider the subspace $\mathcal{V}_0^p(\Omega, \eta) := \overline{\mathcal{D}(\Omega)}$, the closure for the norm (5.1).

We also consider the weighted Sobolev space

$$\mathcal{W}^p(\Omega, \eta) := \left\{ v \in L^p(\Omega) \mid |\nabla^\varepsilon v| \in L^p(\Omega, \eta) \right\},$$

equipped with the norm

$$v \mapsto \left(\|\nabla^\varepsilon v\|_{L^p(\Omega, \eta)}^p + \|v\|_{L^p(\Omega)}^p \right)^{1/p}. \quad (5.2)$$

The parameter ε is introduced in the norm for convenience. Clearly $\mathcal{W}^p(\Omega, \eta) \subset \mathcal{V}^p(\Omega, \eta)$. Then, we define $\mathcal{W}_0^p(\Omega, \eta) := \overline{\mathcal{D}(\Omega)}$, the closure with respect to the norm (5.2).

Arguing as in [25], we can show that for a weight function η satisfying

$$\eta \in L^1_{loc}(\Omega) \text{ and } \eta^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \tag{5.3}$$

both spaces $\mathcal{V}^p(\Omega, \eta), \mathcal{V}_0^p(\Omega, \eta), \mathcal{W}^p(\Omega, \eta)$ and $\mathcal{W}_0^p(\Omega, \eta)$ are reflexive Banach spaces for $1 < p < +\infty$.

Let us suppose that the weight function b satisfies¹

$$b^{-s}(X_1, \cdot) \in L^1(\omega_2), \text{ for a.e. } X_1 \in \omega_1, \tag{5.4}$$

with a certain $s > 0$ which will be specified later. Since h is almost positive a.e. on ω_1 , this means that η satisfies, besides (5.3), the condition

$$\eta^{-s}(X_1, \cdot) = h^{-s}(X_1) b^{-s}(X_1, \cdot) \in L^1(\omega_2), \text{ for a.e. } X_1 \in \omega_1.$$

Introducing the parameter p_s by

$$p_s := \frac{ps}{s+1} < p$$

and using the Hölder inequality with the parameters $\frac{p_s}{p} + \frac{1}{s+1} = 1$, we obtain

$$\begin{aligned} \int_{\omega_2} |v|^{p_s} dX_2 &= \int_{\omega_2} |v|^{p_s} \eta^{p_s/p}(x) \eta^{-p_s/p}(x) dX_2 \\ &\leq \left(\int_{\omega_2} \eta^{-s}(x) dX_2 \right)^{\frac{1}{s+1}} \left(\int_{\omega_2} |v|^p \eta(x) dX_2 \right)^{p_s/p}, \text{ for a.e. } X_1 \in \omega_1. \end{aligned}$$

Replacing v by $\partial_{x'_i} v$, for $i = 1, 2, \dots, n_2$, we get

$$\left(\int_{\omega_2} |\partial_{x'_i} v|^{p_s} dX_2 \right)^{1/p_s} \leq \left(\int_{\omega_2} \eta^{-s}(x) dX_2 \right)^{1/p_s} \left(\int_{\omega_2} |\partial_{x'_i} v|^p \eta(x) dX_2 \right)^{1/p}.$$

Thus ²

$$\left(\int_{\omega_2} |\nabla_{X_2} v|^{p_s} dX_2 \right)^{1/p_s} \leq \left(\int_{\omega_2} \eta^{-s}(x) dX_2 \right)^{1/p_s} \left(\int_{\omega_2} |\nabla_{X_2} v|^p \eta(x) dX_2 \right)^{1/p}, \tag{5.5}$$

for a.e. $X_1 \in \omega_1$. It follows that a function $v \in W^{1,p}(\omega_2, \eta(X_1, \cdot))$ belongs to the (non weighted) space $W^{1,p_s}(\omega_2)$, i.e.

$$W^{1,p}(\omega_2, \eta(X_1, \cdot)) \subset W^{1,p_s}(\omega_2), \text{ for a.e. } X_1 \in \omega_1.$$

¹ See [17, Section 1.5]

² Clearly if $X = (x_1, \dots, x_n) \in \mathbb{R}^n, x_i \geq 0, |x_i| \leq \sqrt{x_1^2 + \dots + x_n^2} = |X|$ So $\sum_{i=1}^n |x_i|^p \leq n |X|^p$ and by Hölder inequality for sums, we get

$$|X|^p \leq n^{\frac{p}{2}} \left(\sum_{i=1}^n |x_i| \right)^p \Rightarrow |X|^p \leq n^{\frac{p}{2}} n^{\frac{p}{2}} \sum_{i=1}^n |x_i|^p.$$

Of course, the parameter p_s is less than p . This means that

$$\|v\|_{W^{1,p_s}(\omega_2)} \leq B^{\frac{1}{p_s}}(X_1) \|v\|_{W^{1,p}(\omega_2, \eta(X_1, \cdot))},$$

where

$$B(X_1) := \int_{\omega_2} \eta^{-s}(x) dX_2.$$

To be more precise, we are considering the norms

$$\begin{aligned} \|v\|_{W^{1,p}(\omega_2, \eta(X_1, \cdot))} &:= \left(\|\nabla_{X_2} v\|_{L^p(\omega_2, \eta(X_1, \cdot))}^p + \|v\|_{L^p(\omega_2)}^p \right)^{1/p}, \text{ for a.e. } X_1 \in \omega_1, \\ \|v\|_{W^{1,p_s}(\omega_2)} &:= \left(\|\nabla_{X_2} v\|_{L^p(\omega_2)}^{p_s} + \|v\|_{L^p(\omega_2)}^{p_s} \right)^{1/p_s}. \end{aligned}$$

Now, we can use the Sobolev injection $W^{1,p_s}(\omega_2) \subset L^q(\omega_2)$ and obtain that

$$W^{1,p}(\omega_2, \eta(X_1, \cdot)) \subset L^q(\omega_2), \text{ for a.e. } X_1 \in \omega_1, \quad (5.6)$$

where

$$1 \leq q \leq p_s^* := \begin{cases} \frac{n_2 p_s}{n_2 - p_s} = \frac{n_2 p s}{n_2(s+1) - p s} & \text{if } p s < n_2(s+1), \\ +\infty & \text{if } p s \geq n_2(s+1). \end{cases} \quad (5.7)$$

Moreover, the injection (5.6) is compact provided $1 \leq q < p_s^*$.

In particular, we have $p_s^* > p$ if $s > \frac{n_2}{p}$, and consequently, the injection

$$W^{1,p}(\omega_2, \eta(X_1, \cdot)) \subset L^p(\omega_2), \text{ for a.e. } X_1 \in \omega_1. \quad (5.8)$$

is compact. To summarize, the inclusion (5.8) holds if

$$\eta^{-s}(X_1, \cdot) \in L^1(\omega_2) \text{ and } s \in \left(\frac{n_2}{p}, +\infty \right) \cap \left[\frac{1}{p-1}, +\infty \right) \quad (5.9)$$

since, to be in accordance with (5.3), we have to suppose also $s \geq \frac{1}{p-1}$.

The embeddings, derived for $W^{1,p}(\omega_2, \eta(X_1, \cdot))$ hold also for $W_0^{1,p}(\omega_2, \eta(X_1, \cdot))$. For $p < p_s^*$ we have, in virtue of the imbedding $W_0^{1,p_s}(\omega_2) \subset L^{p_s^*}(\omega_2)$, that

$$\left(\int_{\omega_2} |v|^p dX_2 \right)^{1/p} \leq C_{1,\omega_2} \left(\int_{\omega_2} |v|^{p_s^*} dX_2 \right)^{1/p_s^*} \leq C_{2,\omega_2} \left(\int_{\omega_2} |\nabla_{X_2} v|^{p_s} dX_2 \right)^{1/p_s},$$

$C_{i,\omega_2}, i = 1, 2, \dots$, denotes here and in the sequel, a constant depending at most on p, s and ω_2 .

Thanks to (5.5), we immediately have the estimate

$$\int_{\omega_2} |v|^p dX_2 \leq C_{3,\omega_2} B^{\frac{1}{s}}(X_1) \int_{\omega_2} |\nabla_{X_2} v|^p \eta dX_2, \text{ for a.e. } X_1 \in \omega_1.$$

Assuming that B is bounded and integrating on ω_1 , we have shown the following lemma.

Lemma 5.1 Under the assumptions (5.3), (5.9), and

$$h^{-s} \int_{\omega_2} b^{-s}(\cdot, X_2) dX_2 \in L^\infty(\omega_1), \tag{5.10}$$

then we have the following weighted Poincaré inequality

$$\int_{\Omega} |v|^p dx \leq C_1 \int_{\Omega} |\nabla_{X_2} v|^p \eta(x) dx, \quad \forall v \in \mathcal{V}_0^p(\Omega, \eta) \tag{5.11}$$

with a constant C_1 depending at most on b, h, p, s and Ω .

In particular, Inequality (5.11) implies that the mapping

$$v \mapsto \|\nabla_{X_2} v\|_{L^p(\Omega, \eta)}$$

is a norm on the space $\mathcal{V}_0^p(\Omega, \eta)$ equivalent to norm (5.1).

Similarly, under the assumptions (5.3), (5.9) and (5.10), we still have the weighted Poincaré inequality (5.11) for the space $\mathcal{W}_0^p(\Omega, \eta)$. By consequence the application

$$v \mapsto \|\nabla^\varepsilon v\|_{L^p(\Omega, \eta)}, \tag{5.12}$$

is a norm for $\mathcal{W}_0^p(\Omega, \eta)$ equivalent to the norm (5.2).

The next compactness lemma is very important for the sequel.

Lemma 5.2 Let $(w_n)_n$ be a sequence in $\mathcal{V}_0^p(\Omega, \eta)$ such that $w_n \rightharpoonup w$ in $\mathcal{V}_0^p(\Omega, \eta)$. Under the assumptions (5.3), (5.9), (5.10) and

$$\int_{\omega_1} h(X_1) b^{-\frac{1}{p-1}}(x) dX_1 \in L^\infty(\omega_2), \tag{5.13}$$

we have

$$l(w_n) \rightharpoonup l(w) \text{ in } W_0^{1,p}(\omega_2). \tag{5.14}$$

Moreover, for every $q \geq 1$ satisfying

$$1 \leq q < p^* := \begin{cases} \frac{n_2 p}{n_2 - p} & \text{if } n_2 > p \\ +\infty & \text{if } n_2 \leq p \end{cases} \tag{5.15}$$

we have the strong convergence

$$l(w_n) \rightarrow l(w) \text{ in } L^q(\omega_2). \tag{5.16}$$

Proof. By Hölder's inequality it holds that

$$\left| \partial_{x'_i} l(w_n) \right|^p = \left| \int_{\omega_1} \left(h^{\frac{p-1}{p}} b^{-\frac{1}{p}} \right) \left(\eta^{\frac{1}{p}} \partial_{x'_i} w_n \right) dX_1 \right|^p \leq \left(\int_{\omega_1} h b^{-\frac{1}{p-1}} dX_1 \right)^{p-1} \int_{\omega_1} \left| \partial_{x'_i} w_n \right|^p \eta dX_1,$$

for $i = 1, \dots, n_2$, and a.e. $X_2 \in \omega_2$, hence

$$|\nabla_{X_2} l(w_n)|^p \leq \left(\int_{\omega_1} h b^{-\frac{1}{p-1}} dX_1 \right)^{p-1} \int_{\omega_1} |\nabla_{X_2} w_n|^p \eta dX_1.$$

Taking (5.13) into account and integrating on ω_2 , we deduce that

$$\int_{\omega_2} |\nabla_{X_2} l(w_n)|^p dX_2 \leq C_2 \int_{\Omega} |\nabla_{X_2} w_n|^p \eta dx. \quad (5.17)$$

Recalling that the mapping $v \mapsto \|\nabla_{X_2} v\|_{L^p(\omega_2)}$ defines a norm on $W_0^{1,p}(\omega_2)$, then (5.17) means that the linear operator $v \mapsto l(v)$, from $\mathcal{V}_0^p(\Omega, \eta)$ into $W_0^{1,p}(\omega_2)$, is continuous and the image of the weakly converging sequence $(w_n)_n$ remains weakly converging. This shows (5.14).

Finally, for $1 \leq q < p^*$ the injection $W_0^{1,p}(\omega_2) \subset L^q(\omega_2)$ is compact and (5.16) follows. ■

5.2 ASSUMPTIONS AND h -WEAK FORMULATIONS

Let $r \geq 1$ and $1 < p < \infty$. We assume that the function a satisfies the following conditions:

$$a \in C(\mathbb{R}^+, \mathbb{R}). \quad (5.18)$$

- There exist constants $\alpha, \beta > 0$ satisfying

$$\alpha t^{p-1} \leq a(t)t \leq \beta t^{p-1}, \quad \text{for } t \geq 0. \quad (5.19)$$

- We denote $A(t) = \int_0^t a(s) s dx$, for $t \geq 0$, and assume that the mapping

$$v \rightarrow \int_{\Omega} \eta A(|\nabla v|^p) dx \quad (5.20)$$

is weakly lower semi-continuous on $\mathcal{W}_0^p(\Omega, \eta)$.

- For every $\forall \zeta, \xi \in \mathbb{R}^{n_1+n_2}$, we have

$$\{a(|\zeta|)\zeta - a(|\xi|)\xi\} \cdot (\zeta - \xi) \geq 0. \quad (5.21)$$

Remark 5.1 *i) Note that $A(t)$ is well defined since $a(t)t$ satisfies (5.19) for $p > 1$ and $t > 0$.*

ii) A sufficient condition for (5.20) to hold is to have the mapping $\xi \mapsto A(|\xi|^p)$ convex for $\xi \in \mathbb{R}^{n_1+n_2}$, (See [16, Theorem 1.3]). This holds for instance if the function $t \rightarrow a(t)t^{2-p}$ is nondecreasing from $\mathbb{R}^+ \rightarrow \mathbb{R}^+$.

Concerning the non-local terms, we assume that $g : \omega_2 \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function and satisfies the growth condition

$$|g(X_2, s)| \leq \gamma_1 \left(|s|^{p-1} + |s|^r \right), \quad \text{for a.e. } X_2 \in \omega_2, \quad (5.22)$$

for some constant $\gamma_1 \geq 0$.

The two source terms f_1 and f_2 are supposed to satisfy

$$f_1 \in \mathcal{V}'_0(\Omega, \eta) \text{ and } f_2 \in L^{\frac{p}{p-1}}(\omega_2), \quad (5.23)$$

where $\mathcal{V}'_0(\Omega, \eta)$ is the dual space of $\mathcal{V}_0^p(\Omega, \eta)$.

We consider the space $\mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$ equipped with the norm

$$v \mapsto \|\nabla_{X_2} v\|_{L^p(\Omega, \eta)} + \|l(v)\|_{L^{r+1}(\omega_2)}.$$

As $\mathcal{V}'_0(\Omega, \eta) \subset (\mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega))'$ and $L^{\frac{r+1}{r}}(\omega_2) \subset (\mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega))'$ with continuous injections, we also have

$$\mathcal{V}'_0(\Omega, \eta) + L^{\frac{r+1}{r}}(\omega_2) \subset (\mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega))'. \quad (5.24)$$

The space $\mathcal{V}'_0(\Omega, \eta) + L^{\frac{r+1}{r}}(\omega_2)$ is equipped with the norm

$$v \mapsto \inf_{v=v_1+v_2} \left\{ \|v_1\|_{\mathcal{V}'_0(\Omega, \eta)} + \|v_2\|_{L^{\frac{r+1}{r}}(\omega_2)} \right\}, \text{ for } v_1 \in \mathcal{V}'_0(\Omega, \eta), v_2 \in L^{\frac{r+1}{r}}(\omega_2). \quad (5.25)$$

The unperturbed problem

Arguing as in [15, Corollary 1], we can show that the elements of $\mathcal{V}_0^p(\Omega, \eta)$ are the functions with zero trace on $\omega_1 \times \partial\omega_2$. More precisely,

$$\mathcal{V}_0^p(\Omega, \eta) = \left\{ v \in \mathcal{V}^p(\Omega, \eta) \mid v(X_1, \cdot) \in W_0^{1,p}(\omega_2) \text{ a.e. } X_1 \in \omega_1 \right\}. \quad (5.26)$$

Thus, Problem (10) from the introduction, see page 4, can be stated as follows

$$\begin{cases} -\nabla_{X_2} \cdot (b(x)a(|\nabla_{X_2} u|) \nabla_{X_2} u) = g(X_2, l(u)) + f_1(x) + f_2(X_2) & \text{in } \Omega, \\ u \in \mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega). \end{cases} \quad (5.27)$$

As mentioned in the introduction, we shall investigate the existence of solutions in the following sense. We say that a function $u \in \mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$ is an h -weak solution to Problem (5.27) if the integral identity

$$\begin{aligned} \int_{\Omega} \eta a(|\nabla_{X_2} u_0|) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dx &= \int_{\omega_2} g(X_2, l(u_0)) l(v) \, dX_2 \\ &+ \langle f_1, v \rangle_{\mathcal{V}'_0(\Omega, \eta)} + \int_{\omega_2} f_2 l(v) \, dX_2, \quad (QP_0) \end{aligned}$$

holds for every $v \in \mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$. This formulation follows from testing (5.27) by hv (instead of v only), then performing some integrations by parts.

The perturbed problem

Let $\varepsilon > 0$. We introduce the following problem with an anisotropic singular perturbation

$$\begin{cases} -\frac{\varepsilon^2}{h} \nabla_{X_1} \cdot (\eta a (|\nabla^\varepsilon u_\varepsilon|) \nabla_{X_1} u_\varepsilon) \\ \quad - \nabla_{X_2} \cdot (b a (|\nabla^\varepsilon u_\varepsilon|) \nabla_{X_2} u_\varepsilon) = g(X_2, l(u_\varepsilon)) + f_1 + f_2 & \text{in } \Omega, \\ u_\varepsilon \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega), \end{cases} \quad (5.28)$$

where the space $\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$ is equipped with the norm

$$v \mapsto \|\nabla^\varepsilon v\|_{L^p(\Omega, \eta)} + \|l(v)\|_{L^{r+1}(\omega_2)}. \quad (5.29)$$

It is clear, as for (5.24), that

$$\mathcal{W}_0^{p'}(\Omega, \eta) + L^{\frac{r+1}{r}}(\omega_2) \subset \left(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega) \right)',$$

where the norm of $\mathcal{W}_0^{p'}(\Omega, \eta) + L^{\frac{r+1}{r}}(\omega_2)$ is defined as in (5.25), with $\mathcal{V}_0^p(\Omega, \eta)$ is replaced by $\mathcal{W}_0^{p'}(\Omega, \eta)$. Problem (5.28) is a nonlocal quasilinear elliptic problem with homogeneous Dirichlet boundary condition. Using a variational method, we aim to show that it has a solution in the sense that

$$\int_{\Omega} \eta a (|\nabla^\varepsilon u|) \nabla^\varepsilon u \cdot \nabla^\varepsilon v \, dx = \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(v) \, dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, \eta)} + \int_{\omega_2} f_2 l(v) \, dX_2, \quad (QP_\varepsilon)$$

for every $v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$.

5.3 C^1 -FUNCTIONALS INVOLVING THE NONLOCAL TERM l

Taking the assumption (5.22) into account, we consider the primitive of g given by

$$G(X_2, t) := \int_0^t g(X_2, s) \, ds, \text{ for } X_2 \in \omega_2 \text{ and } t \in \mathbb{R} \quad (5.30)$$

and as a consequence of (5.22), it satisfies

$$|G(X_2, t)| \leq \gamma_1 \left(\frac{|t|^p}{p} + \frac{|t|^{r+1}}{r+1} \right), \text{ for a.e. } X_2 \in \omega_2 \text{ and } t \in \mathbb{R}. \quad (5.31)$$

This allows us to define the functional

$$J_g(v) := \int_{\omega_2} G(X_2, l(v)) \, dX_2, \text{ for } v \in \mathcal{W}_0^p(\Omega, \eta) \cap L^{r+1}(\omega_2). \quad (5.32)$$

We investigate the existence of solutions for (QP_ε) as critical points of the functional $J^\varepsilon(v)$ such that

$$J^\varepsilon(v) := \int_{\Omega} \eta A(|\nabla^\varepsilon v|^p) \, dx - \int_{\omega_2} G(X_2, l(v)) \, dX_2 - \langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, \eta)} - \int_{\omega_2} f_2 l(v) \, dX_2, \quad (5.33)$$

where A is the function considered in (5.20). In order to determine the derivative of J^ε , we need the following lemma

Lemma 5.3 *Assume that $g_0 : \omega_2 \times \mathbb{R} \mapsto \mathbb{R}$ a Carathéodory function satisfying*

$$|g_0(X_2, s)| \leq \gamma_0 |s|^q, \quad s \in \mathbb{R} \text{ and a.e. } X_2 \in \omega_2, \quad (5.34)$$

for $1 \leq q < +\infty$ and some constant $\gamma_0 \geq 0$. Let

$$J_{g_0}(v) := \int_{\omega_2} G_0(X_2, l(v)) dX_2, \quad \text{for } v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega)$$

where G_0 is defined by g_0 as in (5.30). Then, the mapping

$$u \mapsto g_0(\cdot, l(u)) \text{ is continuous from } \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega) \text{ into } L^{\frac{q+1}{q}}(\omega_2).$$

Moreover, $J_{g_0} \in C^1(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega), \mathbb{R})$ and its derivative is given by

$$\langle J'_{g_0}(u), v \rangle_{\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega)} := \int_{\omega_2} g(X_2, l(u)) l(v) dX_2,$$

for every $u, v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega)$.

Proof. Let us consider the functional

$$j_{g_0}(\bar{v}) := \int_{\omega_2} G_0(X_2, \bar{v}) dX_2, \quad \text{for all } \bar{v} \in L^{q+1}(\omega_2).$$

We can argue as in [38, page 10], to show that the composite mapping

$$u \mapsto g_0(\cdot, l(u)) \text{ is continuous from } \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega) \text{ into } L^{\frac{q+1}{q}}(\omega_2).$$

$J_{g_0} = j_{g_0} \circ l(\cdot) \in C^1(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega), \mathbb{R})$ and its derivative is given by

$$\langle J'_{g_0}(u), v \rangle_{\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega)} = \int_{\omega_2} g(X_2, l(u)) l(v) dX_2,$$

for every $u, v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{q+1}(\Omega)$. ■

Let us now give the derivative of J^ε .

Theorem 5.1 *Let $1 < p \leq r+1 < +\infty$. Assume (5.3), (5.9), (5.10) and that g satisfies (5.22). Let J^ε be defined by (5.33), then the mapping*

$$u \mapsto g(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega) \text{ into } \left(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega) \right)',$$

$J^\varepsilon \in C^1(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and its derivative is given by

$$\begin{aligned} \langle J^{\varepsilon'}(u), v \rangle_{\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} &= \int_{\Omega} \eta a(|\nabla^\varepsilon u|) \nabla^\varepsilon u \cdot \nabla^\varepsilon v dx - \int_{\omega_2} g(X_2, l(u)) l(v) dX_2 \\ &\quad - \langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, \eta)} - \int_{\omega_2} f_2 l(v) dX_2, \end{aligned} \quad (5.35)$$

for all $u, v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$.

Proof. First, to show the continuity of $U \mapsto g(X_2, U)$ we split g into Carathéodory functions g_1 and g_2

$$g(X_2, s) = g_1(X_2, s) + g_2(X_2, s),$$

where $g(X_2, s) = g_1(X_2, s)$ for $|s| \leq 1$ and $g_1(X_2, s) = 0$ for $|s| \geq 2$, for a.e. $X_2 \in \omega_2$. So

$$|g_1(X_2, s)| \leq C |s|^{p-1}, \quad |g_2(X_2, s)| \leq C |s|^r,$$

by possibly modifying the value of C .

Thanks to Lemma 5.3 for $q = p - 1$, we see that the mapping

$$u \mapsto g_1(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^p(\Omega) \text{ into } L^{\frac{p}{p-1}}(\omega_2).$$

By the Poincaré inequality, it follows from (5.17) that

$$\int_{\omega_2} |l(w_n)|^p dX_2 \leq C_0 \int_{\omega_2} |\nabla_{X_2} l(w_n)|^p dx \leq C_3 \int_{\Omega} |\nabla_{X_2} w_n|^p \eta dx, \quad (5.36)$$

this means $\mathcal{W}_0^p(\Omega, \eta) \subset \mathcal{L}^p(\Omega)$ and $L^{\frac{p}{p-1}}(\omega_2) \subset \mathcal{W}_0^p(\Omega, \eta)$.

As well, by Lemma 5.3 with $q = r$, the mapping

$$u \mapsto g_2(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega) \rightarrow L^{\frac{r+1}{r}}(\omega_2).$$

Then the mapping

$$u \mapsto g(X_2, l(u)) = (g_1 + g_2)(X_2, l(u)),$$

is continuous from $\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$ to $\mathcal{W}_0^p(\Omega, \eta) + L^{\frac{r+1}{r}}(\omega_2)$ and

$$\left\langle J'_g(u), v \right\rangle_{\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} = \int_{\omega_2} g(X_2, l(u)) l(v) dX_2, \text{ for } u, v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega).$$

To give the derivatives for the remaining terms in the definition of J^ε , we set

$$\begin{aligned} J_A & : \mathcal{W}_0^p(\Omega, \eta) \mapsto \mathbb{R}, J_A(u) = \int_{\Omega} \eta A(|\nabla^\varepsilon u|^p) dx, \\ J_{f_1} & : \mathcal{V}_0^p(\Omega, \eta) \mapsto \mathbb{R}, J_{f_1}(u) = \langle f_1, u \rangle_{\mathcal{V}_0^p(\Omega, \eta)} \\ J_{f_2} & : \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega) \mapsto \mathbb{R}, J_{f_2}(u) = \int_{\omega_2} f_2 l(u) dX_2. \end{aligned}$$

First, note that if conditions (5.18) and (5.19) are satisfied, then J_A is a C^1 -function i.e. Indeed, since $A(t) = \int_0^{t^{1/p}} a(s) ds$, we see that

$$A'(t) = \frac{1}{p} a\left(t^{1/p}\right) t^{\frac{2}{p}-1}. \quad (5.37)$$

This allows us to show that

$$\lim_{\delta \rightarrow 0} \frac{A(|\nabla^\varepsilon u + \delta \nabla^\varepsilon v|^p) - A(|\nabla^\varepsilon u|^p)}{\delta} = a(|\nabla^\varepsilon u|) \nabla^\varepsilon u \cdot \nabla^\varepsilon v, \text{ a.e. in } \Omega.$$

On the other hand, by (5.19) for $0 < \delta < 1$

$$\left| \frac{A(|\nabla^\varepsilon u + \delta \nabla^\varepsilon v|^p) - A(|\nabla^\varepsilon u|^p)}{\delta} \right| = \frac{1}{\delta} \left| \int_{|\nabla^\varepsilon u|^{1/p}}^{|\nabla^\varepsilon u + \delta \nabla^\varepsilon v|^{1/p}} a(s) s ds \right| \leq \frac{\beta}{\delta} \left| \int_{|\nabla^\varepsilon u|^{1/p}}^{|\nabla^\varepsilon u + \delta \nabla^\varepsilon v|^{1/p}} s^{p-1} ds \right|,$$

hence

$$\left| \frac{A(|\nabla^\varepsilon u + \delta \nabla^\varepsilon v|^p) - A(|\nabla^\varepsilon u|^p)}{\delta} \right| \leq \frac{\beta}{p\delta} (|\nabla^\varepsilon u + \delta \nabla^\varepsilon v| - |\nabla^\varepsilon u|) \leq \frac{\beta}{p} (|\nabla^\varepsilon v|) \in L^p(\Omega, \eta).$$

Then, by the dominated convergence theorem we deduce that

$$\lim_{\delta \rightarrow 0} \int_{\Omega} \eta \frac{A(|\nabla^\varepsilon u + \delta \nabla^\varepsilon v|^p) - A(|\nabla^\varepsilon u|^p)}{\delta} dx = \int_{\Omega} \eta a(|\nabla^\varepsilon u|) \nabla^\varepsilon u \cdot \nabla^\varepsilon v dx,$$

i.e.,

$$\langle J'_A(u), v \rangle_{W_0^{1,p}(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} = \int_{\Omega} \eta a(|\nabla^\varepsilon u|) \nabla^\varepsilon u \cdot \nabla^\varepsilon v dx. \quad (5.39)$$

Finally, since $u \mapsto g(X_2, l(u))$ is continuous from $\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$ into $\mathcal{W}_0^p(\Omega, \eta) \cap L^{\frac{r+1}{r}}(\Omega)$, then $J_g \in C^1(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$.

Next, we easily check that $J_{f_1} \in C^1(\mathcal{V}_0^p(\Omega, \eta), \mathbb{R}) \subset C^1(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and that

$$\langle J'_{f_1}(u), v \rangle_{\mathcal{V}_0^p(\Omega, \eta)} = \langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, \eta)}. \quad (5.40)$$

Also, as $J_{f_2} \in C^1(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ and that

$$\langle J'_{f_2}(u), v \rangle_{\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} = \int_{\omega_2} f_2 l(v) dX_2. \quad (5.41)$$

Summing up, we get

$$J_A + J_{f_1} + J_{f_2} \in C^1(\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R}).$$

Taking into account (5.40), (5.41), we obtain (5.35) since $J^\varepsilon = J_A - J_g - J_{f_1} - J_{f_2}$. ■

Corollary 5.1 *Under the assumption of Theorem 5.1, assume in addition that $1 < p \leq r+1 < p^*$, and (5.13) holds, then*

$$u \rightarrow g(X_2, l(u)) \text{ is continuous from } \mathcal{W}_0^p(\Omega, \eta) \rightarrow \mathcal{W}_0^p(\Omega, \eta),$$

$J^\varepsilon \in C^1(\mathcal{W}_0^p(\Omega, \eta), \mathbb{R})$ and (5.35) holds for all $u \in \mathcal{W}_0^p(\Omega, \eta)$.

Proof. By Sobolev's embedding theorem $W_0^{1,p}(\omega_2) \subset L^{r+1}(\omega_2)$ and taking into account (5.17)

$$\|l(v)\|_{L^{r+1}(\omega_2)} \leq C_S \|l(v)\|_{W_0^{1,p}(\omega_2)} \leq C_S C_3 \|\nabla_{X_2} v\|_{L^p(\Omega,\eta)} \leq C_S C_3 \|v\|_{\mathcal{W}_0^p(\Omega,\eta)}, \quad (5.42)$$

for all $v \in \mathcal{W}_0^p(\Omega,\eta)$. Here, C_S denotes the Sobolev constant and C_3 is defined in (5.17). This means that $\mathcal{W}_0^p(\Omega,\eta) \subset \mathcal{L}^{r+1}(\Omega)$, the result follows from Theorem 5.1. ■

Remark 5.2 *In a similar manner, we can also show the following results:*

i) Let $1 < p \leq r+1 < +\infty$. Theorem 5.1 holds if we replace $\mathcal{W}_0^p(\Omega,\eta)$ by $\mathcal{V}_0^p(\Omega,\eta)$ and J^ε by the functional J^0 defined by

$$J^0(v) := \int_{\Omega} \eta A(|\nabla_{X_2} u|^p) dx - \int_{\omega_2} G(X_2, l(v)) dX_2 - \langle f_1, v \rangle_{\mathcal{V}_0(\Omega,b)} - \int_{\omega_2} f_2 l(u) dX_2. \quad (5.43)$$

ii) Let $1 < p \leq r+1 < p^*$. Then, Corollary 5.1 holds if we replace $\mathcal{W}_0^p(\Omega,\eta)$ by $\mathcal{V}_0^p(\Omega,\eta)$, and J^ε by J^0 .

A QUASILINEAR PROBLEM WITH A NON-LOCAL NONLINEARITY

This chapter is devoted to study the existence of solutions u_ε for (QP_ε) as global minimizers for J^ε , defined by (5.33), provided that G satisfies some upper growth condition. Then, we study the asymptotic behaviour of u_ε and J^ε , as $\varepsilon \rightarrow 0$.

6.1 AN EXISTENCE RESULT FOR THE PERTURBED PROBLEM

We have the following existence result.

Theorem 6.1 *Let $\varepsilon > 0$ and assume that g satisfies (5.22) for $1 < p \leq r + 1 < \infty$. Suppose further that*

$$G(X_2, t) \leq C_1 - \gamma_2 |t|^{r+1}, \quad \forall t \in \mathbb{R}, \quad (6.1)$$

for a.e. $X_2 \in \omega_2$ and some constants $C_1 \geq 0$, $\gamma_2 > 0$. Then, there exists $u \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$ such that

$$J^\varepsilon(u) = \inf_{v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} J^\varepsilon(v). \quad (6.2)$$

In particular, u is a solution of (QP_ε) .

Proof. *A priori estimates.* First, observe that for some constant $C_1 \geq 0$ we have

$$J^\varepsilon(v) \geq \int_{\Omega} \eta A(|\nabla^\varepsilon v|^p) dx + \gamma_2 \int_{\omega_2} |l(v)|^{r+1} dX_2 - \langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, \eta)} - \int_{\omega_2} f_2 l(v) dX_2 - C_1 |\omega_2|, \quad (6.3)$$

where $|\omega_2|$ denotes the Lebesgue measure of the set ω_2 . Since $\mathcal{V}_0^p(\Omega, \eta)$ is equipped with the norm $v \mapsto \|\nabla_{X_2} v\|_{L^p(\Omega, \eta)}$, we infer that

$$\langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, \eta)} \leq \|f_1\|_{\mathcal{V}_0^p(\Omega, \eta)} \left(\int_{\Omega} \eta |\nabla_{X_2} v|^p dx \right)^{1/p} \leq \|f_1\|_{\mathcal{V}_0^p(\Omega, \eta)} \left(\int_{\Omega} \eta |\nabla^\varepsilon v|^p dx \right)^{1/p}. \quad (6.4)$$

Then, by (5.19) and Hölder's inequality it comes that

$$J^\varepsilon(v) \geq \frac{\alpha}{p} \int_{\Omega} \eta |\nabla^\varepsilon v|^p dx + \gamma_2 \int_{\omega_2} |l(v)|^{r+1} dX_2 \\ - \|f_1\|_{\mathcal{V}_0^p(\Omega, \eta)} \left(\int_{\Omega} \eta |\nabla^\varepsilon v|^p dx \right)^{1/p} + \|f_2\|_{L^{\frac{r+1}{r}}(\omega_2)} \|l(v)\|_{L^{r+1}(\omega_2)} - C_1 |\omega_2|.$$

Applying Young's inequality, we deduce

$$J^\varepsilon(v) \geq \frac{\alpha}{2p} \int_{\Omega} \eta |\nabla^\varepsilon v|^p dx + \frac{\gamma_2}{2} \int_{\omega_2} |l(v)|^{r+1} dX_2 \\ - \frac{2(p-1)}{p\alpha} \|f_1\|_{\mathcal{V}_0^p(\Omega, \eta)}^{\frac{p-1}{p}} - \frac{2^{\frac{1}{r}} r}{\gamma_2^{\frac{1}{r}} (r+1)^{\frac{r+1}{r}}} \|f_2\|_{L^{\frac{r+1}{r}}(\omega_2)}^{\frac{r+1}{r}} - C_1 |\omega_2|, \quad (6.5)$$

for all $u \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$. Thus J^ε is bounded from below on $\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$.

Passage to the limit. We set

$$d_\varepsilon = \inf_{v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} J^\varepsilon(v),$$

and let $(u_n)_{n \geq 0}$ be a minimizing sequence. By (6.5), we deduce that $(u_n)_{n \geq 0}$ is bounded in $\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$, i.e.

$$u_n \text{ is bounded in } L^p(\Omega) \text{ and } |\nabla^\varepsilon u_n| \text{ is bounded in } L^p(\Omega, \eta), \\ l(u_n) \text{ is bounded in } L^{r+1}(\omega_2)$$

independently of n . The boundedness of u_n is a consequence of the weighted Poincaré inequality in the X_2 -direction (5.11). Then, there exist $u \in \mathcal{W}_0^p(\Omega, \eta)$, $U \in (L^p(\Omega, \eta))^{n_1+n_2}$ such that -up to a subsequence -

$$u_n \rightharpoonup u \text{ in } L^p(\Omega) \text{ and } \nabla^\varepsilon u_n \rightharpoonup U \text{ in } L^p(\Omega, \eta), \\ l(u_n) \rightharpoonup l(u) \text{ in } L^{r+1}(\omega_2),$$

as $n \rightarrow +\infty$. Since the above convergences also hold in the distributional sense, we can easily see that

$$u_n \rightharpoonup u \text{ in } L^p(\Omega) \text{ and } \nabla^\varepsilon u_n \rightharpoonup \nabla^\varepsilon u \text{ in } L^p(\Omega, \eta). \quad (6.6)$$

Using Lemma 5.2, with $q = p$, we obtain the strong convergence $l(u_n) \rightarrow l(u)$ in $L^p(\omega_2)$ and up to a subsequence

$$l(u_n) \rightarrow l(u) \text{ a.e. in } \omega_2, \quad (6.7)$$

and the continuity of G implies that

$$G(X_2, l(u_n)) \rightarrow G(X_2, l(u)) \text{ a.e. in } \omega_2. \quad (6.8)$$

To perform the passage to the limit in J^ε , we write

$$J^\varepsilon(v) = \int_{\Omega} \eta A(|\nabla^\varepsilon v|^p) dx + \int_{\omega_2} (C_1 - G(X_2, l(u_n))) dX_2 - C_1 |\omega_2| - \langle f_1, u_n \rangle_{\mathcal{V}_0^p(\Omega, \eta)} - \int_{\omega_2} f_2 l(u_n) dX_2, \quad (6.9)$$

where $C_1 - G(X_2, l(u_n)) \geq 0$, for a.e. $X_2 \in \omega_2$. Then, Fatou's lemma and the convergence (6.7), (6.8) imply

$$\int_{\omega_2} (C_1 - G(X_2, l(u))) dX_2 \leq \liminf_{n_k \rightarrow \infty} \int_{\omega_2} (C_1 - G(X_2, l(u_n))) dX_2. \quad (6.10)$$

Due to lower semi-continuity assumption (5.20), we deduce from (6.6) that

$$\int_{\Omega} \eta A(|\nabla^\varepsilon u|^p) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \eta A(|\nabla^\varepsilon u_n|^p) dx. \quad (6.11)$$

Passing to the limit in (6.9), we get

$$J^\varepsilon(u) \leq \liminf_{n \rightarrow \infty} J^\varepsilon(u_n) = d_\varepsilon.$$

Therefore $J^\varepsilon(u) = d_\varepsilon$.

Finally, by Theorem 5.1, we have $J^{\varepsilon'}(u) = 0$, i.e. u is a solution to (QP_ε) . This ends the proof of the theorem. ■

Note that Problem (QP_ε) may have one or several solution, depending on the assumptions on g, f_1, f_2 . We can ensure the uniqueness under a monotonicity on g .

Corollary 6.1 *Under the assumptions of Theorem 6.1, and if the mapping*

$$s \mapsto g(X_2, s) \text{ is non increasing,}$$

for a.e. $X_2 \in \omega_2$, then the solution of (QP_ε) is unique.

Example 6.1 *Theorem 6.1 applies to*

$$\begin{cases} -\frac{\varepsilon^2}{h} \nabla_{X_1} \cdot (\eta a(|\nabla^\varepsilon u|) \nabla_{X_1} u) - \nabla_{X_2} \cdot (b(x) a(|\nabla^\varepsilon u|) \nabla_{X_2}) \\ \quad + a_1 |l(u)|^{r-1} l(u) - a_2 |l(u)|^{q-1} l(u) = f_1 + f_2 \\ u_\varepsilon \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega). \end{cases} \quad \text{in } \Omega,$$

Here we have taken

$$g(X_2, t) = -a_1 |t|^{r-1} t + a_2 |t|^{q-1} t,$$

where $a_1, a_2 \in \mathbb{R}, a_1 > 0$ and $1 \leq q < r$. We can check that (5.22) and (6.1) holds.¹

¹ Since

$$|t|^q < |t|^r + |t|^{p-1} \quad \text{and} \quad a_2 |t|^{q+1} \leq C + \frac{a_1(q+1)}{2(r+1)} |t|^{r+1},$$

for some constant $C > 0$.

6.2 AN EXISTENCE RESULT FOR THE UNPERTURBED PROBLEM

Letting $\varepsilon \rightarrow 0$, we have the following convergence result.

Lemma 6.1 *Under the assumptions of Theorem 6.1. Let u_ε be a solution of Problem (QP_ε) and for all $\varepsilon > 0$, we have - up to a subsequence -*

$$u_\varepsilon \text{ is bounded in } L^p(\Omega), \quad (6.12)$$

$$\nabla_{X_2} u_\varepsilon, \varepsilon \nabla_{X_1} u_\varepsilon \text{ are bounded in } L^p(\Omega, \eta), \quad (6.13)$$

$$l(u_\varepsilon) \text{ is bounded in } L^{r+1}(\omega_2), \quad (6.14)$$

$$a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \text{ is bounded in } L^{\frac{p}{p-1}}(\Omega, \eta) \quad (6.15)$$

independently of ε .

Proof. First, we know

$$J^\varepsilon(u_\varepsilon) \leq J^\varepsilon(v) \leq J^1(v), \forall v \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega). \quad (6.16)$$

Thanks to (6.5), we deduce that

$$\begin{aligned} & \frac{\alpha}{2p} \int_{\Omega} \eta |\nabla^\varepsilon v|^p dx + \frac{\gamma_2}{2} \|l(u_\varepsilon)\|_{L^{r+1}(\omega_2)}^{r+1} \\ & \leq J^1(v) + \frac{2(p-1)}{p\alpha} \|f_1\|_{\mathcal{V}_0^p(\Omega, \eta)}^{\frac{p-1}{p}} + \frac{2^{\frac{1}{r}} r}{\gamma_2^{\frac{1}{r}} (r+1)^{\frac{r+1}{r}}} \|f_2\|_{L^{\frac{r+1}{r}}(\omega_2)}^{\frac{r+1}{r}} + C_1 |\omega_2|, \end{aligned} \quad (6.17)$$

and it follows that

$$\begin{aligned} & u_\varepsilon \text{ is bounded in } L^p(\Omega), \\ & |\nabla_{X_2} u_\varepsilon|, |\varepsilon \nabla_{X_1} u_\varepsilon| \text{ are bounded in } L^p(\Omega, \eta), \\ & l(u_\varepsilon) \text{ is bounded in } L^{r+1}(\omega_2). \end{aligned}$$

For the last estimate (6.15), we use (5.19) to derive

$$\|a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon\|_{L^{\frac{p}{p-1}}(\Omega, \eta)} \leq \beta \int_{\Omega} \eta |\nabla^\varepsilon u_\varepsilon|^{(p-1)\frac{p}{p-1}} dx = \beta \|\nabla^\varepsilon u_\varepsilon\|_{L^p(\Omega, \eta)}^p. \quad (6.18)$$

■

Theorem 6.2 *Under the assumptions of Theorem 6.1, (5.21) and*

$$-(g(X_2, s) - g(X_2, t)) \cdot (s - t) \geq 0, \quad (6.19)$$

we have

$$u_\varepsilon \rightharpoonup u_0 \text{ in } L^p(\Omega), \quad (6.20)$$

$$\nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0, \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \text{ in } L^p(\Omega, \eta), \quad (6.21)$$

$$l(u_\varepsilon) \rightharpoonup l(u_0) \text{ in } L^{r+1}(\omega_2), \quad (6.22)$$

$$a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \rightharpoonup a\left(|\nabla^0 u_0|\right) \nabla^0 u_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta) \quad (6.23)$$

as $\varepsilon \rightarrow 0$, where u_0 is a solution to Problem (QP₀). Moreover, if the function a satisfies,

$$(a(\xi) \xi - a(\xi') \xi') (\xi - \xi') \geq \gamma |\xi - \xi'|^p, \quad (6.24)$$

for $\xi, \xi' \in \mathbb{R}^n$ and some $\gamma > 0$, then we have the strong convergences

$$u_\varepsilon \rightarrow u_0 \text{ in } L^p(\Omega), \quad (6.25)$$

$$\nabla_{X_2} u_\varepsilon \rightarrow \nabla_{X_2} u_0, \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightarrow 0 \text{ in } L^p(\Omega, \eta), \quad (6.26)$$

$$a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \rightarrow a\left(|\nabla^0 u_0|\right) \nabla^0 u_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta). \quad (6.27)$$

Proof. First, thanks to Lemma 6.1, there exist $u_0 \in \mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$, $\tilde{U}_1 \in (L^p(\Omega, \eta))^{n_1}$ and $\tilde{U}_2 \in (L^p(\Omega, \eta))^{n_2}$ such that –up to a subsequence–

$$u_\varepsilon \rightharpoonup u_0 \text{ in } L^p(\Omega), \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \tilde{U}_2 \text{ and } \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup \tilde{U}_1 \text{ in } L^p(\Omega, \eta),$$

as $\varepsilon \rightarrow 0$. Since the above convergences imply the convergence in the distributional sense, we can check that

$$u_\varepsilon \rightharpoonup u_0 \text{ in } L^p(\Omega), \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \text{ and } \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \text{ in } L^p(\Omega, \eta). \quad (6.28)$$

Besides, we have

$$l(u_\varepsilon) \rightharpoonup l(u_0) \text{ in } L^{r+1}(\omega_2)$$

and (6.15) give us

$$a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \rightharpoonup \chi_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta) \quad (6.29)$$

for some $\chi_0 \in \text{in } \left[L^{\frac{p}{p-1}}(\Omega, \eta) \right]^{n_1+n_2}$.

Going back to Lemma 5.2, with $q = p$ we still have the strong convergence $l(u_\varepsilon) \rightarrow l(u_0)$ in $L^p(\omega_2)$ and –up to a new subsequence–

$$l(u_\varepsilon) \rightarrow l(u_0), \text{ a.e. in } \omega_2. \quad (6.30)$$

Assumption (5.22) and the boundedness of $l(u_\varepsilon)$ in $L^{r+1}(\omega_2)$ implies that

$$|g(X_2, l(u_\varepsilon))|^{\frac{r+1}{r}} \leq C \left(1 + |l(u_\varepsilon)|^{r+1} \right). \quad (6.31)$$

Thus $g(X_2, l(u_\varepsilon))$ is bounded in $L^{\frac{r+1}{r}}(\omega_2)$. Moreover, (6.30) and the continuity of g ensure that

$$g(X_2, l(u_\varepsilon)) \rightarrow g(X_2, l(u_0)), \quad \text{a.e. in } \omega_2.$$

Then, due to Lemma 1.3 in [31, page 12], we infer that

$$g(X_2, l(u_\varepsilon)) \rightharpoonup g(X_2, l(u_0)) \quad \text{in } L^{\frac{r+1}{r}}(\omega_2). \quad (6.32)$$

The convergences (6.20), (6.21), (6.29) and (6.32) allow us to pass to the limit in (QP_ε) . It comes that

$$\int_{\Omega} \eta \chi_0 \cdot \nabla_{X_2} v \, dx = \int_{\omega_2} g(X_2, l(u_0)) l(v) \, dX_2 + \langle f_1, v \rangle_{V_0^p(\Omega, \eta)} + \int_{\omega_2} h_2 l(v) \, dX_2. \quad (6.33)$$

Taking $v = u_\varepsilon$ and pass to the limit again, we have

$$\int_{\Omega} \eta \chi_0 \cdot \nabla_{X_2} u_0 \, dx = \int_{\omega_2} g(X_2, l(u_0)) l(u_0) \, dX_2 + \langle f_1, u_0 \rangle_{V_0^p(\Omega, \eta)} + \int_{\omega_2} h_2 l(u_0) \, dX_2. \quad (6.34)$$

To prove that u_0 satisfies (QP_0) we must show that $\chi_0 = a(|\nabla^0 u_0|) \nabla^0 u_0$. To this end, consider

$$\begin{aligned} M_\varepsilon := & \int_{\Omega} \eta \left(a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon - a(|\nabla^0 v|) \nabla^0 v \right) \cdot \nabla^\varepsilon (u_\varepsilon - v) \, dx \\ & - \int_{\omega_2} (g(X_2, l(u_\varepsilon)) - g(X_2, l(v))) \cdot (l(u_\varepsilon) - l(v)) \, dX_2. \end{aligned}$$

Due to (5.21) and (6.19), we have $M_\varepsilon \geq 0$. Developing the product and using (5.35) with $v = u_\varepsilon$, we get

$$\begin{aligned} M_\varepsilon = & \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(v) \, dX_2 + \int_{\omega_2} g(X_2, l(v)) (l(u_\varepsilon) - l(v)) \, dX_2 + \langle f_1, u_\varepsilon \rangle_{V_0^p(\Omega, \eta)} \\ & + \int_{\omega_2} f_2 l(u_\varepsilon) \, dX_2 - \int_{\Omega} \eta a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon v \, dx - \int_{\Omega} \eta a(|\nabla^0 v|) \nabla^0 v \cdot \nabla^\varepsilon (u_\varepsilon - v) \, dx. \end{aligned}$$

Using again the convergences (6.20), (6.21), we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} M_\varepsilon = & \int_{\omega_2} g(X_2, l(u_0)) l(v) \, dX_2 + \int_{\omega_2} g(X_2, l(v)) (l(u_0) - l(v)) \, dX_2 + \langle f_1, u_0 \rangle_{V_0^p(\Omega, \eta)} \\ & + \int_{\omega_2} f_2 l(u_0) \, dX_2 - \int_{\Omega} \eta \chi \cdot \nabla^0 v \, dx - \int_{\Omega} \eta a(|\nabla^0 v|) \nabla^0 v \cdot \nabla^0 (u_0 - v) \, dx. \end{aligned}$$

Thanks to (6.34), then we have

$$\begin{aligned} & \int_{\Omega} \eta \left(\chi_0 - a(|\nabla^0 v|) \nabla^0 v \right) \cdot \nabla^0 (u_0 - v) \, dx \\ & - \int_{\omega_2} (g(X_2, l(u_0)) - g(X_2, l(v))) (l(u_0) - l(v)) \, dX_2 \geq 0. \quad (6.35) \end{aligned}$$

Let $v = u_0 - tw$, $t > 0$, $w \in \mathcal{W}_0^p(\Omega, \eta)$, then (6.35) give us

$$\begin{aligned} & t \int_{\Omega} \eta \left(\chi_0 - a(|\nabla^0 (u_0 - tw)|) \nabla^0 (u_0 - tw) \right) \cdot \nabla^0 w \, dx \\ & - t \int_{\omega_2} (g(X_2, l(u_0)) - g(X_2, l(u_0 - tw))) l(w) \, dX_2 \geq 0. \end{aligned}$$

So, we have

$$\begin{aligned} \int_{\Omega} \eta \left(\chi_0 - a \left(\left| \nabla^0 (u_0 - tw) \right| \right) \nabla^0 (u_0 - tw) \right) \cdot \nabla^0 w dx \\ - \int_{\omega_2} (g(X_2, l(u_0)) - g(X_2, l(u_0 - tw))) l(w) dX_2 \geq 0. \end{aligned}$$

Thanks to (5.18), letting $t \rightarrow 0$, we deduce that

$$\int_{\Omega} \eta \left(\chi_0 - a \left(\left| \nabla^0 u_0 \right| \right) \nabla^0 u_0 \right) \cdot \nabla^0 w dx \geq 0, \quad \forall w \in \mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega),$$

hence $\chi_0 = a \left(\left| \nabla^0 u_0 \right| \right) \nabla^0 u_0$.

For the strong convergences, if (6.24) holds, then we have

$$\begin{aligned} \gamma \int_{\Omega} \eta |\nabla^\varepsilon (u_0 - u_\varepsilon)|^p dx \\ \leq \int_{\Omega} \eta (a(|\nabla_{X_2} u_0|) \nabla_{X_2} u_0 - a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon (u_0 - u_\varepsilon) dx \\ \leq \int_{\Omega} \eta a(|\nabla_{X_2} u_0|) \nabla_{X_2} u_0 \cdot \nabla^\varepsilon (u_0 - u_\varepsilon) dx - \int_{\Omega} \eta a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon (u_0 - u_\varepsilon) dx. \end{aligned}$$

Thanks to (6.12) and since u_ε satisfies (QP_ε) we get

$$\begin{aligned} \gamma \int_{\Omega} \eta |\nabla^\varepsilon (u_0 - u_\varepsilon)|^p dx \\ \leq \int_{\Omega} \eta a(|\nabla_{X_2} u_0|) \nabla_{X_2} u_0 \cdot \nabla^\varepsilon (u_0 - u_\varepsilon) dx - \left\langle J^{\varepsilon'}(u_\varepsilon), u_0 - u_\varepsilon \right\rangle_{\mathcal{W}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} \\ - \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(u_0 - u_\varepsilon) dx - \langle f_1, u_0 - u_\varepsilon \rangle_{\mathcal{V}_0^p(\Omega, \eta)} - \int_{\omega_2} f_2 l(u_0 - u_\varepsilon) dx \rightarrow 0. \end{aligned}$$

as $\varepsilon \rightarrow 0$. Hence, we have (6.26). Besides, due to (5.11), we have (6.25) and -up to a new subsequence- $u_\varepsilon \rightarrow u_0$ a.e. in Ω and the continuity of a implies

$$a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \rightarrow a\left(\left|\nabla^0 u_0\right|\right) \nabla^0 u_0, \quad \text{a.e. in } \Omega.$$

Thanks to (5.19), we have

$$|a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon|^{\frac{p}{p-1}} \leq \beta^{\frac{p}{p-1}} |\nabla^\varepsilon u_\varepsilon|^p \in L^p(\Omega, \eta).$$

Then, by dominated convergence theorem, we deduce that

$$a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \rightarrow a\left(\left|\nabla^0 u_0\right|\right) \nabla^0 u_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta),$$

as $\varepsilon \rightarrow 0$. This ends the proof. ■

Corollary 6.2 *Under the assumptions of Theorem 6.2, the converging subsequences give a smooth minimizing sequences for the functional \mathbf{J}^0 . That is*

$$\mathbf{J}^\varepsilon(u_\varepsilon) \rightarrow \mathbf{J}^0(u_0) = \inf_{v \in \mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} \mathbf{J}^0(v), \quad (6.36)$$

as $\varepsilon \rightarrow 0$, where the functional \mathbf{J}^0 is defined by (5.43).

Proof. Since u_ε realizes the minimum of J^ε , then

$$J^\varepsilon(u_\varepsilon) \leq J^\varepsilon(v), \quad \forall v \in \mathcal{D}(\Omega).$$

Passing to the lim sup, we get

$$\limsup_{\varepsilon \rightarrow 0} J^\varepsilon(u_\varepsilon) \leq J^0(v), \quad \forall v \in \mathcal{D}(\Omega).$$

By density and the fact that $J^0 \in C^1(\mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega), \mathbb{R})$ (see Remark 5.2), the precedent inequality also holds for every $v \in \mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)$. Thus,

$$\limsup_{\varepsilon \rightarrow 0} J^\varepsilon(u_\varepsilon) \leq \inf_{v \in \mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} J^0(v). \quad (6.37)$$

Since G satisfies (5.31), then we have

$$|G(X_2, l(u_\varepsilon))| \leq C \left(1 + |l(u_\varepsilon)|^{r+1}\right), \quad (6.38)$$

for some constant $C > 0$. Thanks to (6.14), this means that $G(\cdot, l(u_\varepsilon))$ is a bounded sequence in $L^1(\omega_2)$. Due to (6.30) and the continuity of $t \mapsto G(X_2, t)$, for a.e. $X_2 \in \omega_2$, we get –up to a subsequence–

$$G(X_2, l(u_\varepsilon)) \rightarrow G(X_2, l(u_0)), \quad \text{a.e. in } \omega_2, \text{ as } \varepsilon \rightarrow 0. \quad (6.39)$$

Writing $J^\varepsilon(u_\varepsilon)$ as

$$\begin{aligned} J^\varepsilon(u_\varepsilon) = & \int_{\Omega} \eta A(|\nabla^\varepsilon u_\varepsilon|^p) dx + \int_{\omega_2} (C_1 - G(X_2, l(u_\varepsilon))) dX_2 \\ & - C_1 |\omega_2| - \langle f_1, u_\varepsilon \rangle_{\mathcal{V}_0^p(\Omega, \eta)} - \int_{\omega_2} f_2 l(u_\varepsilon) dX_2. \end{aligned}$$

Then, the convergences (6.11), (6.12), (6.14) and (6.39) allow us to pass to the infimum limit in $J^\varepsilon(u_\varepsilon)$. We deduce that

$$J^0(u_0) \leq \liminf_{\varepsilon \rightarrow 0} J^\varepsilon(u_\varepsilon).$$

Reporting this to (6.37), we obtain $J^0(u_0) \leq \inf_{v \in \mathcal{V}_0^p(\Omega, \eta) \cap \mathcal{L}^{r+1}(\Omega)} J^0(v)$ and thus (6.36) is proved. ■

Remark 6.1 *We still do not have the strong convergence $l(u_\varepsilon) \rightarrow l(u_0)$ in $L^{r+1}(\omega_2)$. Such convergence holds if some strong monotonicity is assumed on $-g$, i.e.*

$$-(g(X_2, s) - g(X_2, t))(s - t) \geq c |s - t|^{r+1}, \quad \text{a.e. in } \omega_2$$

for some constant $c > 0$.

A QUASILINEAR NON-LOCAL EIGENVALUE PROBLEM AND AN APPLICATION

To go further and study Problems (QP_0) and (QP_ε) under some assumptions weaker than (6.1), we need first to consider a quasilinear eigenvalue problem of non-local type.

7.1 A QUASILINEAR NON-LOCAL EIGENVALUE PROBLEM

7.1.1 l -eigenvalues and l -eigenfunctions

The eigenvalue problem we shall consider is the following:

$$\begin{cases} -\frac{\varepsilon^2}{h} \nabla_{X_1} \cdot (\eta a (|\nabla^\varepsilon \varphi|) \nabla_{X_1} \varphi) \\ \quad - \nabla_{X_2} \cdot (b a (|\nabla^\varepsilon \varphi|) \nabla_{X_2} \varphi) = \lambda^\varepsilon |l(\varphi)|^{p-2} l(\varphi) & \text{in } \Omega, \\ \varphi \in \mathcal{W}_0^p(\Omega, \eta). \end{cases} \quad (7.1)$$

provided that there exists a weak solution φ such that $l(\varphi)$ is not identically null. This solution is called an l -eigenfunction associated to λ^ε and taken in the h -weak sense, i.e. φ satisfies

$$\int_{\Omega} \eta a (|\nabla^\varepsilon \varphi|) \nabla^\varepsilon \varphi \cdot \nabla^\varepsilon v dx = \lambda^\varepsilon \int_{\omega_2} |l(\varphi)|^{p-2} l(\varphi) l(v) dX_2, \quad \forall \varphi \in \mathcal{W}_0^p(\Omega, \eta). \quad (QEV P_\varepsilon)$$

Theorem 7.1 *The smallest l -eigenvalue of Problem $(QEV P_\varepsilon)$ exists, is positive and defined by*

$$\lambda_1^\varepsilon := \inf_{v \in \mathcal{W}_0^p(\Omega, \eta), l(v) \neq 0} \frac{\int_{\Omega} \eta A (|\nabla^\varepsilon v|^p) dx}{\|l(v)\|_{L^p(\omega_2)}^p}. \quad (7.2)$$

The associated l -eigenfunctions are the minimizers of the above functional.

Proof. Set

$$\mu_\varepsilon = \inf_{v \in \mathcal{W}_0^p(\Omega, \eta), \|l(v)\|_{L^p(\omega_2)} = 1} \int_{\Omega} \eta A (|\nabla^\varepsilon v|^p) dx.$$

We shall prove that μ_ε is the first eigenvalue of $(QEV P_\varepsilon)$. Let $(\varphi_n)_n$ be minimizing sequence for μ_ε i.e.

$$\|l(\varphi_n)\|_{L^p(\omega_2)} = 1, \int_{\Omega} \eta A (|\nabla^\varepsilon \varphi_n|^p) dx = \mu_\varepsilon + \gamma_n, \tag{7.3}$$

with $\gamma_n \rightarrow 0$ for $n \rightarrow \infty$. It follows from (5.19) and (7.3) that

$$\frac{\alpha}{p} \|\varphi_n\|_{\mathcal{W}_0^p(\Omega, \eta)}^p \leq \int_{\Omega} \eta A (|\nabla^\varepsilon \varphi_n|^p) dx = \mu_\varepsilon + \gamma_n,$$

which guarantees that

$$\begin{aligned} \varphi_n &\text{ is bounded in } L^p(\Omega), \\ |\nabla^\varepsilon \varphi_n| &\text{ is bounded in } L^p(\Omega, \eta). \end{aligned}$$

The boundedness of φ_n is a consequence of the weighted Poincaré inequality in the X_2 -direction (5.11). Then, there exist $\varphi \in \mathcal{W}_0^p(\Omega, \eta)$, $U \in (L^p(\Omega, \eta))^{n_1+n_2}$ such that –up to a subsequence –

$$\varphi_n \rightharpoonup \varphi, \text{ in } L^p(\Omega) \text{ and } \nabla^\varepsilon \varphi_n \rightharpoonup U, \text{ in } L^p(\Omega, \eta),$$

as $n \rightarrow +\infty$. Since the above convergences imply the convergence in the distributional sense, we can easily see that

$$\varphi_n \rightharpoonup \varphi \text{ in } L^p(\Omega) \text{ and } \nabla^\varepsilon \varphi_n \rightharpoonup \nabla^\varepsilon \varphi \text{ in } L^p(\Omega, \eta). \tag{7.4}$$

Moreover, using Lemma 5.2, with $q = p$, we obtain

$$l(\varphi_n) \rightarrow l(\varphi) \text{ in } L^p(\omega_2) \text{ and } \|l(\varphi_n)\|_{L^p(\omega_2)} \rightarrow \|l(\varphi)\|_{L^p(\omega_2)} = 1.$$

By lower semi-continuity of the map $v \rightarrow \int_{\Omega} \eta A (|\nabla^\varepsilon v|^p) dx$, we deduce that

$$\mu_\varepsilon \leq \int_{\Omega} \eta A (|\nabla^\varepsilon \varphi|^p) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \eta A (|\nabla^\varepsilon \varphi_n|^p) dx = \liminf_{n \rightarrow \infty} (\mu_\varepsilon + \gamma_n) = \mu_\varepsilon,$$

i.e.

$$\int_{\Omega} \eta A (|\nabla^\varepsilon \varphi|^p) dx = \mu_\varepsilon.$$

To summarize, $\varphi \in \mathcal{W}_0^p(\Omega, \eta)$ is a function for which the infimum (7.2) is achieved, i.e.

$$\lambda_1^\varepsilon = \mu_\varepsilon = \int_{\Omega} \eta A (|\nabla^\varepsilon \varphi|^p) dx, \|l(\varphi)\|_{L^p(\omega_2)} = 1. \tag{7.5}$$

Let us show that this φ is an l -eigenfunction. Let $v \in \mathcal{W}_0^p(\Omega, \eta)$ and consider

$$T(t) = \frac{\int_{\Omega} \eta A (|\nabla^\varepsilon(\varphi + tv)|^p) dx}{\int_{\omega_2} |l(\varphi + tv)|^p dX_2}.$$

Then, we should have $T'(0) = 0$ i.e.

$$\int_{\Omega} \eta \frac{d}{dt} A(|\nabla^\varepsilon(\varphi + tv)|^p) dx \cdot \int_{\omega_2} |l(\varphi + tv)|^p dX_2 - \int_{\Omega} \eta A(|\nabla^\varepsilon(\varphi + tv)|^p) dx \cdot \int_{\omega_2} \frac{d}{dt} |l(\varphi + tv)|^p dX_2 = 0, \text{ as } t \rightarrow 0.$$

This implies that, see (5.37)

$$\int_{\Omega} \eta a(|\nabla^\varepsilon(\varphi + tv)|) \left(\nabla^\varepsilon(\varphi) \nabla^\varepsilon(v) + t |\nabla^\varepsilon(v)|^2 \right) dx \cdot \int_{\omega_2} |l(\varphi + tv)|^p dX_2 - \int_{\Omega} \eta A(|\nabla^\varepsilon(\varphi + tv)|^p) dx \cdot \int_{\omega_2} |l(\varphi + tv)|^{p-2} \left(l(\varphi)l(v) + t |l(v)|^2 \right) dX_2 = 0.$$

Letting $t \rightarrow 0$, we obtain

$$\int_{\Omega} \eta a(|\nabla^\varepsilon(\varphi)|) \nabla^\varepsilon(\varphi) \cdot \nabla^\varepsilon(v) dx \cdot \int_{\omega_2} |l(\varphi)|^p dX_2 - \int_{\Omega} \eta A(|\nabla^\varepsilon \varphi|^p) dx \cdot \int_{\omega_2} |l(\varphi)|^{p-2} l(\varphi)l(v) dX_2 = 0.$$

Using (7.5), we have

$$\int_{\Omega} \eta a(|\nabla^\varepsilon(\varphi)|) \nabla^\varepsilon(\varphi) \cdot \nabla^\varepsilon(v) dx = \lambda_1^\varepsilon \int_{\omega_2} |l(\varphi)|^{p-2} l(\varphi)l(v) dX_2, \quad v \in \mathcal{W}_0^p(\Omega, \eta). \quad (7.6)$$

This shows that φ is a l -eigenfunction for λ_1^ε . ■

Concerning the (expected) limit problem, we say that λ^0 is an l -eigenvalue of the problem

$$\begin{cases} -\nabla_{X_2} \cdot (ba(|\nabla_{X_2} \varphi|) \nabla_{X_2} \varphi) = \lambda^0 |l(\varphi)|^{p-2} l(\varphi) & \text{in } \Omega, \\ \varphi \in \mathcal{V}_0^p(\Omega, \eta). \end{cases} \quad (7.7)$$

provided that there exists a solution φ such that $l(\varphi) \neq 0$. This solution is also taken in the h -weak sense, i.e. φ satisfies

$$\int_{\Omega} \eta a(|\nabla_{X_2} \varphi|) \nabla_{X_2} \varphi \cdot \nabla_{X_2} v dx = \lambda^0 \int_{\omega_2} |l(\varphi)|^{p-2} l(\varphi)l(v) dX_2, \quad \forall \varphi \in \mathcal{W}_0^p(\Omega, \eta). \quad (QEV P_0)$$

The proof of Theorem 7.1 can be easily reproduced, by taking $\varepsilon = 0$ and replacing $\mathcal{W}_0^p(\Omega, \eta)$ by $\mathcal{V}_0^p(\Omega, \eta)$, to prove that:

Theorem 7.2 *The smallest l -eigenvalue of Problem (QEV P₀) exists, is positive and defined by*

$$\lambda_1^0 := \inf_{\substack{v \in \mathcal{V}_0^p(\Omega, \eta) \\ l(u(\cdot, X_2)) \neq 0 \text{ a.e. in } \omega_2}} \frac{\int_{\Omega} \eta A(|\nabla_{X_2} v|^p) dx}{\|l(v)\|_{L^p(\omega_2)}^p}. \quad (7.8)$$

7.1.2 Convergence of the l -eigenvalues and l -eigenfunctions

We are now interested in the limit of the l -eigenvalues λ_1^ε and its associated l -eigenfunction φ_ε , as $\varepsilon \rightarrow 0$.

Theorem 7.3 *Let λ_1^ε be an l -eigenvalue defined by (7.2) and φ_ε an associated l - eigenfunction such that $\|l(\varphi_\varepsilon)\|_{L^p(\omega_2)} = 1$. Then, we have*

$$\lambda_1^\varepsilon \rightarrow \lambda_1^0, \text{ as } \varepsilon \rightarrow 0, \tag{7.9}$$

where λ_1^0 is defined by (7.8) and -up to a subsequence-

$$\varphi_\varepsilon \rightharpoonup \varphi_0 \text{ in } L^p(\Omega), \tag{7.10}$$

$$\nabla_{X_2} \varphi_\varepsilon \rightharpoonup \nabla_{X_2} \varphi_0, \quad \varepsilon \nabla_{X_1} \varphi_\varepsilon \rightharpoonup 0 \text{ in } L^p(\Omega, \eta), \tag{7.11}$$

$$l(\varphi_\varepsilon) \rightarrow l(\varphi_0) \text{ in } L^p(\omega_2), \tag{7.12}$$

$$a(|\nabla^\varepsilon \varphi_\varepsilon|) \nabla^\varepsilon \varphi_\varepsilon \rightharpoonup a(|\nabla^0 \varphi_0|) \nabla^0 \varphi_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta) \tag{7.13}$$

as $\varepsilon \rightarrow 0$, where φ_0 is an l -eigenfunction associated to λ_1^0 , i.e. a solution to (7.7), satisfying $\|l(\varphi_0)\|_{L^p(\omega_2)} = 1$.

Moreover, if the function a satisfies (6.24), then the convergence (7.10), (7.11) and (7.13) hold in the strong sense.

Proof. The sequence λ_1^ε is decreasing as $\varepsilon \rightarrow 0$ and clearly $\lambda_1^\varepsilon \geq \lambda_1^0, \forall \varepsilon > 0$. Thus λ_1^ε converges to some limit

$$\lambda_1^\varepsilon \rightarrow \bar{\lambda}_1 = \inf_{\varepsilon > 0} \lambda_1^\varepsilon \geq \lambda_1^0. \tag{7.14}$$

Let φ_ε an l -eigenvalues associated to λ_1^ε such that $\|l(\varphi_\varepsilon)\|_{L^p(\omega_2)} = 1$, then by (5.19), we have

$$\frac{\alpha}{p} \int_{\Omega} \eta |\nabla^\varepsilon \varphi_\varepsilon|^p dx \leq \int_{\Omega} \eta A(|\nabla^\varepsilon \varphi_\varepsilon|^p) dx = \lambda_1^\varepsilon \leq \lambda_1^1, \tag{7.15}$$

for $0 < \varepsilon \leq 1$. It follows that

$$\begin{aligned} \varphi_\varepsilon &\text{ is bounded in } L^p(\Omega), \\ |\nabla_{X_2} \varphi_\varepsilon|, |\varepsilon \nabla_{X_1} \varphi_\varepsilon| &\text{ are bounded in } L^p(\Omega, \eta). \end{aligned}$$

The boundedness of φ_ε follows from Poincaré's inequality (5.11). Then, there exists $\varphi_0 \in \mathcal{V}_0^p(\Omega, \eta)$ such that -up to a subsequence-

$$\begin{aligned} \varphi_\varepsilon &\rightharpoonup \varphi_0 \text{ in } L^p(\Omega), \\ \nabla_{X_2} \varphi_\varepsilon &\rightharpoonup \nabla_{X_2} \varphi_0, \quad \varepsilon \nabla_{X_1} \varphi_\varepsilon \rightharpoonup 0 \text{ in } L^p(\Omega, \eta), \end{aligned}$$

as $\varepsilon \rightarrow 0$. Using, Lemma 5.2, with $q = p$, we deduce that

$$l(\varphi_\varepsilon) \rightarrow l(\varphi_0) \text{ in } L^p(\omega_2), \tag{7.16}$$

and, up to a new subsequence, $l(\varphi_\varepsilon) \rightarrow l(\varphi_0)$ a.e. in ω_2 and $|l(\varphi_\varepsilon)| \leq K \in L^p(\omega_2), \forall \varepsilon > 0$. Thus

$$|l(\varphi_\varepsilon)|^{p-2} l(\varphi_\varepsilon) \rightarrow |l(\varphi_0)|^{p-2} l(\varphi_0) \text{ a.e. in } \omega_2.$$

and

$$\left| |l(\varphi_\varepsilon)|^{p-2} |l(\varphi_\varepsilon)| \right| = |l(\varphi_\varepsilon)|^{p-1} \leq K^{p-1} \in L^{\frac{p}{p-1}}(\omega_2), \forall \varepsilon > 0.$$

Then, applying Lebesgue's theorem, we infer that

$$|l(\varphi_\varepsilon)|^{p-2} l(\varphi_\varepsilon) \rightarrow |l(\varphi_0)|^{p-2} l(\varphi_0) \text{ in } L^{\frac{p}{p-1}}(\omega_2). \tag{7.17}$$

Let us now prove that $\bar{\lambda}_1 = \lambda_1^0$ and φ_0 is an associated l -eigenvalue. On one hand, thanks to (6.15), we can deduce that

$$a(|\nabla^\varepsilon \varphi_\varepsilon|) \nabla^\varepsilon \varphi_\varepsilon \text{ is bounded in } L^{\frac{p}{p-1}}(\Omega, \eta), \tag{7.18}$$

and for some $\chi \in L^{\frac{p}{p-1}}(\Omega, \eta)$, we have -up to a new subsequence-

$$a(|\nabla^\varepsilon \varphi_\varepsilon|) \nabla^\varepsilon \varphi_\varepsilon \rightharpoonup \chi_\varepsilon \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta).$$

Next, we use (5.21), (6.19) and argue by monotonicity as the proof of Theorem 6.2 to show that

$$a(|\nabla^\varepsilon \varphi_\varepsilon|) \nabla^\varepsilon \varphi_\varepsilon \rightharpoonup a(|\nabla^0 \varphi_0|) \nabla^0 \varphi_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta), \tag{7.19}$$

as $\varepsilon \rightarrow 0$. Taking $\varphi = \varphi_\varepsilon$ in (QEV P_ε) and using the convergences (7.10), (7.11), (7.17) and (7.19) to pass to the limit yields

$$\int_\Omega \eta a(|\nabla_{X_2} \varphi_0|) \nabla_{X_2} \varphi_0 \cdot \nabla_{X_2} v dx = \bar{\lambda}_1 \int_{\omega_2} |l(\varphi_0)|^{p-2} l(\varphi_0) l(v) dX_2, \forall \varphi \in \mathcal{W}_0^p(\Omega, \eta). \tag{7.20}$$

Besides, the strong convergence (7.16) means that $\|l(\varphi_0)\|_{L^p(\omega_2)} = 1$, hence $l(\varphi_0) \not\equiv 0$ and the identity (7.20) shows that φ_0 is an l -eigenvalue associated with $\bar{\lambda}_1$.

On the other hand, by definition we have

$$\lambda_1^\varepsilon \leq \int_\Omega \eta A(|\nabla^\varepsilon v|^p) dx, \quad \forall v \in \mathcal{D}(\Omega), \quad \|l(v)\|_{L^p(\omega_2)} = 1$$

and passing to the infimum limit in both sides, we get

$$\bar{\lambda}_1 \leq \int_\Omega \eta A(|\nabla_{X_2} v|^p) dx, \quad \forall v \in \mathcal{D}(\Omega), \quad \|l(v)\|_{L^2(\omega_2)} = 1,$$

as $\varepsilon \rightarrow 0$. By density of $\mathcal{D}(\Omega)$ in $\mathcal{V}_0^p(\Omega, \eta)$ we infer that $\bar{\lambda}_1 \leq \lambda_1^0$ since λ_1^0 is defined by (7.8).

Taking (7.14) into account, we obtain $\bar{\lambda}_1 = \lambda_1^0$.

It remains to show the last assertion of the theorem. If (6.24) holds, then we have

$$\gamma \int_{\Omega} \eta |\nabla^{\varepsilon}(\varphi_0 - \varphi_{\varepsilon})|^p dx \leq \int_{\Omega} \eta \{a(|\nabla_{X_2} \varphi_0|) \nabla_{X_2} \varphi_0 - a(|\nabla^{\varepsilon} \varphi_{\varepsilon}|) \nabla^{\varepsilon} \varphi_{\varepsilon}\} \cdot \nabla^{\varepsilon}(\varphi_0 - \varphi_{\varepsilon}) dx,$$

hence

$$\begin{aligned} & \gamma \int_{\Omega} \eta |\nabla^{\varepsilon}(\varphi_0 - \varphi_{\varepsilon})|^p dx \\ & \leq \int_{\Omega} \eta a(|\nabla_{X_2} \varphi_0|) \nabla_{X_2} \varphi_0 \cdot \nabla^{\varepsilon}(\varphi_0 - \varphi_{\varepsilon}) dx - \lambda_1^{\varepsilon} \int_{\omega_2} |l(\varphi_{\varepsilon})|^{p-2} l(\varphi_{\varepsilon}) l(\varphi_0 - \varphi_{\varepsilon}) dX_2. \end{aligned}$$

Thanks to (7.9), (7.10) and since φ_{ε} satisfies (7.6), we get

$$\begin{aligned} & \gamma \int_{\Omega} \eta |\nabla^{\varepsilon}(\varphi_0 - \varphi_{\varepsilon})|^p dx \\ & \leq \int_{\Omega} \eta a(|\nabla_{X_2} \varphi_0|) \nabla_{X_2} \varphi_0 \cdot \nabla^{\varepsilon}(\varphi_0 - \varphi_{\varepsilon}) dx - \lambda_1^{\varepsilon} \int_{\omega_2} |l(\varphi_{\varepsilon})|^{p-2} l(\varphi_{\varepsilon}) l(\varphi_0 - \varphi_{\varepsilon}) dX_2 \rightarrow 0, \end{aligned}$$

as $\varepsilon \rightarrow 0$. This shows that the convergences (7.11) hold in a strong sense. Besides, due to (5.11) we have $\varphi_{\varepsilon} \rightarrow \varphi_0$ in $L^p(\Omega)$ and, up to a new subsequence, $\varphi_{\varepsilon} \rightarrow \varphi_0$ a.e. in Ω . The continuity of a ensures that

$$a(|\nabla^{\varepsilon} \varphi_{\varepsilon}|) \nabla^{\varepsilon} \varphi_{\varepsilon} \rightarrow a(|\nabla^0 \varphi_0|) \nabla^0 \varphi_0, \text{ a.e. in } \Omega.$$

Thanks to (5.19), we have

$$|a(|\nabla^{\varepsilon} \varphi_{\varepsilon}|) \nabla^{\varepsilon} \varphi_{\varepsilon}|^{\frac{p}{p-1}} \leq \beta^{\frac{p}{p-1}} |\nabla^{\varepsilon} \varphi_{\varepsilon}|^p \in L^p(\Omega, \eta).$$

Then, by Lebesgue's theorem, we deduce that

$$a(|\nabla^{\varepsilon} \varphi_{\varepsilon}|) \nabla^{\varepsilon} \varphi_{\varepsilon} \rightarrow a(|\nabla^0 \varphi_0|) \nabla^0 \varphi_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta),$$

as $\varepsilon \rightarrow 0$. This ends the proof. ■

7.2 A QUASILINEAR PROBLEM WITH SUBCRITICAL NON-LOCAL NONLINEARITY

In this section, we relax the assumption (6.1) by considering a weaker upper bound of $G(\cdot, t)$ for t large, and assume that

$$1 < p \leq r + 1 < p^*.$$

We recall that in this case $\mathcal{W}_0^p(\Omega, \eta) \subset \mathcal{L}^{r+1}(\Omega)$ and $L^{\frac{r+1}{r}}(\omega_2) \subset \mathcal{W}_0'(\Omega, \eta)$. So, without loss of generality, we assume that $f_2 = 0$ and the solution of Problem (QP_{ε}) reads

$$\int_{\Omega} \eta a(|\nabla^{\varepsilon} u_{\varepsilon}|) \nabla^{\varepsilon} u_{\varepsilon} \cdot \nabla^{\varepsilon} v dx = \int_{\omega_2} g(X_2, l(u_{\varepsilon})) l(v) dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0(\Omega, \eta)}, \quad (Q\tilde{P}_{\varepsilon})$$

for every $v \in \mathcal{W}_0^p(\Omega, \eta)$. The functional associated to $(Q\tilde{P}_\varepsilon)$ is given here by

$$\tilde{\mathbf{J}}^\varepsilon(u) := \int_\Omega \eta A(|\nabla^\varepsilon u_\varepsilon|^p) dx - \int_{\omega_2} G(X_2, l(u)) dX_2 - \langle f_1, u \rangle_{\mathcal{V}_0^p(\Omega, \eta)}. \quad (7.21)$$

We aim to establish the existence of solutions u_ε for $(Q\tilde{P}_\varepsilon)$ as global minimizers for $\tilde{\mathbf{J}}^\varepsilon$, then we study the asymptotic behaviour of u_ε and $\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon)$, as $\varepsilon \rightarrow 0$.

7.2.1 An existence result for the perturbed problem

First, we establish the following existence result.

Theorem 7.4 *Let $\varepsilon > 0$ and $\lambda > -\lambda_1^\varepsilon$. Assume that g satisfies (5.22). In addition, assume that G , defined by (5.30), satisfies*

$$G(X_2, t) \leq \frac{-\lambda}{p} |t|^p, \text{ for } |t| \text{ large enough}, \quad (7.22)$$

and a.e. $X_2 \in \omega_2$. Then, there exists $u \in \mathcal{W}_0^p(\Omega, \eta)$ such that

$$\tilde{\mathbf{J}}^\varepsilon(u) = \inf_{v \in \mathcal{W}_0^p(\Omega, \eta)} \tilde{\mathbf{J}}^\varepsilon(v), \quad (7.23)$$

where $\tilde{\mathbf{J}}^\varepsilon$ is defined by (7.21). In particular, u is a solution of $(Q\tilde{P}_\varepsilon)$.

Proof. *A priori estimates.* Due to (7.22) we have

$$G(X_2, t) \leq C_4 - \frac{\lambda}{p} |t|^p, \quad \forall t \in \mathbb{R} \quad (7.24)$$

for some constant $C_4 \geq 0$. Reporting this to (7.21), we have the lower boundedness

$$\tilde{\mathbf{J}}^\varepsilon(v) \geq \Phi_\lambda^\varepsilon(v) - \langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, \eta)} - C_4 |\omega_2|, \quad (7.25)$$

where

$$\Phi_\lambda^\varepsilon(v) = \int_\Omega \eta A(|\nabla^\varepsilon v|^p) dx + \frac{\lambda}{p} \int_{\omega_2} |l(v)|^p dX_2.$$

Thanks to (7.2) and (5.19) we have, for any given $0 < \theta < 1$,

$$\begin{aligned} \Phi_\lambda^\varepsilon(v) &= (\theta + (1 - \theta)) \int_\Omega \eta A(|\nabla^\varepsilon v|^p) dx + \frac{\lambda}{p} \int_{\omega_2} |l(v)|^p dX_2 \\ &\geq \frac{\theta\alpha}{p} \int_\Omega \eta |\nabla^\varepsilon v|^p dx + \frac{(1 - \theta)\lambda_1^\varepsilon + \lambda}{p} \int_{\omega_2} |l(v)|^p dX_2. \end{aligned}$$

Since $\lambda + \lambda_1^\varepsilon > 0$, then for

$$0 < \theta < \theta_\varepsilon := \min \left\{ 1, \frac{\lambda + \lambda_1^\varepsilon}{\lambda_1^\varepsilon} \right\} \quad (7.26)$$

the last term above is nonnegative. Denoting $\delta_\varepsilon := \theta_\varepsilon \alpha$, we have

$$\Phi_\lambda^\varepsilon(v) \geq \frac{\delta_\varepsilon}{p} \int_\Omega \eta |\nabla^\varepsilon v|^p dx, \quad \forall \varepsilon > 0. \quad (7.27)$$

Going back to (7.25) and using (6.4), we obtain

$$\tilde{\mathbf{J}}^\varepsilon(v) \geq \frac{\delta_\varepsilon}{p} \int_\Omega \eta |\nabla^\varepsilon v|^p dx - \|f_1\|_{\mathcal{V}'_0(\Omega, \eta)} \left(\int_\Omega \eta |\nabla^\varepsilon v|^p dx \right)^{1/p} - C_4 |\omega_2|.$$

Applying Young's inequality $ab \leq \delta_\varepsilon b^p / 2p + 2a^{\frac{p}{p-1}} (p-1) / p\delta_\varepsilon$, we get

$$\tilde{\mathbf{J}}^\varepsilon(v) \geq \frac{\delta_\varepsilon}{2p} \int_\Omega \eta |\nabla^\varepsilon v|^p dx - \frac{2(p-1)}{p\delta_\varepsilon} \|f_1\|_{\mathcal{V}'_0(\Omega, \eta)}^{\frac{p-1}{p}} - C_4 |\omega_2|. \quad (7.28)$$

Thus $\tilde{\mathbf{J}}^\varepsilon(v)$ is bounded below, i.e.

$$d_\varepsilon := \inf_{v \in \mathcal{W}_0^p(\Omega, \eta)} \tilde{\mathbf{J}}^\varepsilon(v) > -\infty.$$

Passage to the limit. Thanks to (5.31) and since $p \leq r+1$, we can take

$$G(X_2, t) \leq \gamma_2 |t|^p, \quad \text{for small } t \text{ and some constant } \gamma_2 \geq 0.$$

Combining this with (7.22), which holds for t large, we deduce that for some constant $\mu \geq 0$ it holds that

$$G(X_2, t) \leq \mu |t|^p \quad \text{for all } t \in \mathbb{R}. \quad (7.29)$$

Then, we rewrite $\tilde{\mathbf{J}}^\varepsilon$ as

$$\tilde{\mathbf{J}}^\varepsilon(v) = \int_\Omega \eta A(|\nabla^\varepsilon v|^p) dx + J_1(v) - \mu \int_{\omega_2} |l(v)|^p dX_2 - \langle f_1, v \rangle_{\mathcal{V}'_0(\Omega, \eta)}, \quad (7.30)$$

where

$$J_1(v) = \int_{\omega_2} -G(X_2, l(v)) + \mu |l(v)|^p dX_2. \quad (7.31)$$

Let $(u_n)_{n \geq 0}$ be a minimizing sequence. By (7.28), we deduce that $(u_n)_{n \geq 0}$ is necessarily bounded in $\mathcal{W}_0^p(\Omega, \eta)$. Then, there exist $u \in \mathcal{W}_0^p(\Omega, \eta)$ and a weakly converging subsequence $(u_{n_k}) \in \mathcal{W}_0^p(\Omega, \eta)$ such that

$$u_{n_k} \rightharpoonup u \text{ in } L^p(\Omega) \quad \text{and} \quad \nabla^\varepsilon u_{n_k} \rightharpoonup \nabla^\varepsilon u \text{ in } L^p(\Omega, \eta), \quad (7.32)$$

$$l(u_{n_k}) \rightarrow l(u) \text{ in } L^p(\omega_2), \quad (7.33)$$

as $k \rightarrow +\infty$ and up to a new subsequence, we have

$$l(u_{n_k}) \rightarrow l(u) \text{ a.e. in } \omega_2.$$

The first weak convergence is a consequence of Poincaré's inequality in the X_2 -direction (5.11) while the last strong convergence is a consequence of Lemma 5.2 with $q = p$.

Since the function $-G(X_2, t) + \mu |t|^p$ is nonnegative, Fatou's lemma implies that

$$J_1(u) \leq \liminf_{n_k \rightarrow \infty} J_1(u_{n_k}). \quad (7.34)$$

Using (7.32) – (7.34) and the weak lower semi-continuity of the map $v \mapsto \int_{\Omega} \eta A(|\nabla^\varepsilon v|^p) dx$, to pass to the limit in (7.30) with $v = u_{n_k}$, we infer that

$$\tilde{J}^\varepsilon(u) \leq \liminf_{n_k \rightarrow \infty} \tilde{J}^\varepsilon(u_{n_k}) = d_\varepsilon.$$

Therefore, the limit u realizes the minimum of \tilde{J}^ε on $\mathcal{W}_0^p(\Omega, \eta)$.

As $p \leq r+1 < p^*$, then by Corollary 5.1 we have $\tilde{J}^\varepsilon \in C^1(\mathcal{W}_0^p(\Omega, \eta), \mathbb{R})$. Thus $(\tilde{J}^\varepsilon)'(u) = 0$ and the limit u is a solution to $(Q\tilde{P}_\varepsilon)$. This ends the proof. ■

Example 7.1 *Theorem 7.4 applies to the problem*

$$\begin{cases} -\frac{\varepsilon^2}{h} \nabla_{X_1} \cdot (\eta a(|\nabla^\varepsilon u|) \nabla_{X_1} u) - \nabla_{X_2} \cdot (b(x) a(|\nabla^\varepsilon u|) \nabla_{X_2} u) \\ \quad + \lambda |l(u)|^{p-1} l(u) + a_1 |l(u)|^{r-1} l(u) - a_2 |l(u)|^{q-1} l(u) = f_1 & \text{in } \Omega, \\ u_\varepsilon \in \mathcal{W}_0^p(\Omega, \eta) \end{cases}$$

where $\lambda > -\lambda_1^\varepsilon$, $a_1, a_2 > 0$, $1 < p \leq r+1 < p^*$, $1 \leq q < r+1$. Here we have taken

$$g(X_2, l(u_\varepsilon)) = -\lambda |l(u)|^{p-1} l(u) - a_1 |l(u)|^{r-1} l(u) + a_2 |l(u)|^{q-1} l(u).$$

7.2.2 An existence result for the unperturbed problem

The next theorem shows the existence of a solution u_0 to following unperturbed problem

$$\int_{\Omega} \eta a(|\nabla_{X_2} u_0|) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v dx = \int_{\omega_2} g(X_2, l(u_0)) l(v) dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, \eta)}, \quad (Q\tilde{P}_0)$$

for every $v \in \mathcal{V}_0^p(\Omega, \eta)$.

Theorem 7.5 *Under the assumptions of Theorem 7.4, (5.21), (6.19) with $\lambda > -\lambda_1^0$, we have -up to a subsequence-*

$$u_\varepsilon \rightharpoonup u_0 \text{ in } L^p(\Omega), \quad (7.35)$$

$$\nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0, \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \text{ in } L^p(\Omega, \eta), \quad (7.36)$$

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^{r+1}(\omega_2), \quad (7.37)$$

$$a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \rightharpoonup a(|\nabla^0 u_0|) \nabla^0 u_0 \text{ in } L^{p'}(\Omega, \eta), \quad (7.38)$$

where u_0 is a solution to Problem $(Q\tilde{P}_0)$. Moreover, if the function $a(t)$ satisfies (6.24), then the convergence (7.35), (7.36) and (7.38) hold in the strong sense.

Proof. First, since $\lambda > -\lambda_1^0 > -\lambda_1^\varepsilon$, the assumptions of Theorem 7.4 hold and thus u_ε exists $\forall \varepsilon > 0$. Since u_ε realize the minimum of $\tilde{\mathbf{J}}^\varepsilon$ on $\mathcal{W}_0^p(\Omega, \eta)$, then, we have

$$\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon) \leq \tilde{\mathbf{J}}^\varepsilon(v) \leq \tilde{\mathbf{J}}^1(v), \quad \forall v \in \mathcal{W}_0^p(\Omega, \eta),$$

for $0 < \varepsilon < 1$. recall that we have also $\lambda_1^1 > \lambda_1^\varepsilon > \lambda_1^0$. In particular,

$$\theta_\varepsilon > \theta_0 := \min \left\{ 1, \frac{\lambda + \lambda_1^0}{\lambda_1^1} \right\}, \quad \text{for } \lambda > -\lambda_1^0,$$

where θ_ε is the constant considered in (7.26). Denoting $\delta_0 = \theta_0 \alpha$, we have

$$\Phi_\lambda^\varepsilon(v) \geq \frac{\delta_0}{p} \int_\Omega \eta |\nabla^\varepsilon v|^p dx, \quad \forall \varepsilon > 0,$$

where this time δ_0 is independent of ε . Then, the same argument used to derive (7.28) yields

$$\frac{\delta_0}{2p} \int_\Omega \eta |\nabla^\varepsilon v|^p dx \leq \tilde{\mathbf{J}}^\varepsilon(v) + \frac{2(p-1)}{p\delta_0} \|f_1\|_{\mathcal{V}_0^p(\Omega, \eta)}^{\frac{p-1}{p}} + C_4 |\omega_2|, \quad \forall v \in \mathcal{W}_0^p(\Omega, \eta), \quad (7.39)$$

and it follows that

$$u_\varepsilon \text{ is bounded in } L^p(\Omega),$$

$$|\nabla_{X_2} u_\varepsilon|, |\varepsilon \nabla_{X_1} u_\varepsilon| \text{ are bounded in } L^p(\Omega, \eta)$$

independently of ε . Then, there exists $u_0 \in \mathcal{W}_0^p(\Omega, \eta)$ such that –up to a subsequence–

$$u_\varepsilon \rightharpoonup u_0 \text{ in } L^p(\Omega), \quad (7.40)$$

$$\nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0, \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \text{ in } L^p(\Omega, \eta), \quad (7.41)$$

as $\varepsilon \rightarrow 0$. Arguing by monotonicity as in Theorem 6.2 we have

$$a(|\nabla^\varepsilon u_\varepsilon|) \nabla^\varepsilon u_\varepsilon \rightharpoonup \chi_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta),$$

for some $\chi_0 \in \left[L^{\frac{p}{p-1}}(\Omega, \eta) \right]^{n_1+n_2}$. As $1 < p \leq r+1 < p^*$, by Lemma 5.2 we have

$$l(u_\varepsilon) \rightarrow l(u_0) \text{ in } L^{r+1}(\omega_2), \quad (7.42)$$

and –up to a new subsequence– $l(u_\varepsilon) \rightarrow l(u_0)$, a.e. in ω_2 and the continuity of g ensure that

$$g(X_2, l(u_\varepsilon)) \rightarrow g(X_2, l(u_0)), \quad \text{a.e. in } \omega_2.$$

Thanks to (6.31), we infer that

$$|g(X_2, l(u_\varepsilon))|^{\frac{r+1}{r}} \leq C \left(1 + |l(u_\varepsilon)|^{r+1} \right).$$

Applying Lebesgue's theorem, we derive the strong convergence

$$g(X_2, l(u_\varepsilon)) \rightarrow g(X_2, l(u_0)) \text{ in } L^{r+1}(\omega_2).$$

Going back to $(Q\tilde{P}_\varepsilon)$ and passing to the limit, we deduce that the limit u_0 satisfies

$$\int_{\Omega} \eta \chi_0 \cdot \nabla_{X_2} v dx = \int_{\omega_2} g(X_2, l(u_0)) l(v) dX_2 + \langle f_1, v \rangle_{\mathcal{V}_0^p(\Omega, b)}, \forall v \in \mathcal{W}_0^p(\Omega, b).$$

By density of $\mathcal{W}_0^p(\Omega, \eta)$ in $\mathcal{V}_0^p(\Omega, \eta)$, this also holds for every $v \in \mathcal{V}_0^p(\Omega, \eta)$.

Taking $v = u_\varepsilon$ and pass again to the limit, we get

$$\int_{\Omega} \eta \chi_0 \cdot \nabla^0 u_0 dx = \int_{\omega_2} g(X_2, l(u_0)) l(u_0) dX_2 + \langle f_1, u_0 \rangle_{\mathcal{V}_0^p(\Omega, \eta)}.$$

Using this identity, (5.18), (5.21) and argue by monotonicity as in the proof of Theorem 6.2, we show that $\chi_0 = a\left(\left|\nabla^0 u_0\right|\right) \nabla^0 u_0$. This means that u_0 solves Problem $(Q\tilde{P}_0)$.

For the strong convergences, if (6.24) holds, then we have

$$\gamma \int_{\Omega} \eta |\nabla^\varepsilon(u_0 - u_\varepsilon)|^p dx \leq \int_{\Omega} \eta \left(a\left(\left|\nabla^0 u_0\right|\right) \nabla^0 u_0 - a\left(|\nabla^\varepsilon u_\varepsilon|\right) \nabla^\varepsilon u_\varepsilon \right) \cdot \nabla^\varepsilon(u_0 - u_\varepsilon) dx,$$

hence

$$\begin{aligned} \gamma \int_{\Omega} \eta |\nabla^\varepsilon(u_0 - u_\varepsilon)|^p dx &\leq \int_{\Omega} \eta a\left(\left|\nabla^0 u_0\right|\right) \nabla^0 u_0 \cdot \nabla^\varepsilon(u_0 - u_\varepsilon) dx \\ &\quad - \int_{\Omega} \eta a\left(|\nabla^\varepsilon u_\varepsilon|\right) \nabla^\varepsilon u_\varepsilon \cdot \nabla^\varepsilon(u_0 - u_\varepsilon) dx. \end{aligned}$$

Thanks to (7.40) and since u_ε satisfies $(Q\tilde{P}_\varepsilon)$, we get

$$\begin{aligned} \gamma \int_{\Omega} \eta |\nabla^\varepsilon(u_0 - u_\varepsilon)|^p dx &\leq \int_{\Omega} \eta a\left(\left|\nabla^0 u_0\right|\right) \nabla^0 u_0 \cdot \nabla^\varepsilon(u_0 - u_\varepsilon) dx \\ &\quad - \int_{\omega_2} g(X_2, l(u_\varepsilon)) l(u_0 - u_\varepsilon) dx - \langle f_1, u_0 - u_\varepsilon \rangle_{\mathcal{V}_0^p(\Omega, \eta)} \rightarrow 0. \end{aligned}$$

Then, (7.36) holds in a strong sense. Finally, due to (5.11) we have the strong convergence of (7.35) and –up to a new subsequence– $u_\varepsilon \rightarrow u_0$ a.e. in Ω and thus

$$a\left(|\nabla^\varepsilon u_\varepsilon|\right) \nabla^\varepsilon u_\varepsilon \rightarrow a\left(\left|\nabla^0 u_0\right|\right) \nabla^0 u_0, \text{ a.e. in } \Omega.$$

Thanks to (5.19), we have

$$|a\left(|\nabla^\varepsilon u_\varepsilon|\right) \nabla^\varepsilon u_\varepsilon|^{\frac{p}{p-1}} \leq \beta^{\frac{p}{p-1}} |\nabla^\varepsilon u_\varepsilon|^p \in L^p(\Omega, \eta).$$

Then, by the dominated convergence theorem, we deduce that

$$a\left(|\nabla^\varepsilon u_\varepsilon|\right) \nabla^\varepsilon u_\varepsilon \rightarrow a\left(\left|\nabla^0 u_0\right|\right) \nabla^0 u_0 \text{ in } L^{\frac{p}{p-1}}(\Omega, \eta),$$

as $\varepsilon \rightarrow 0$. This ends the proof. ■

Corollary 7.1 *Under the assumptions of Theorem 7.5, without (6.24) the converging subsequences satisfy*

$$G(\cdot, l(u_\varepsilon)) \rightarrow G(\cdot, l(u_0)) \text{ in } L^1(\omega_2) \tag{7.43}$$

and give smooth minimizing sequences for the functional $\tilde{\mathbf{J}}^0$. That is,

$$\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon) \rightarrow \tilde{\mathbf{J}}^0(u_0) = \inf_{v \in \mathcal{V}_0^p(\Omega, \eta)} \tilde{\mathbf{J}}^0(v). \quad (7.44)$$

Proof. Since the convergence $l(u_\varepsilon) \rightarrow l(u_0)$, a.e. in ω_2 still holds, then by the continuity of G with respect to the second variable implies

$$G(X_2, l(u_\varepsilon)) \rightarrow G(X_2, l(u_0)) \text{ a.e. in } \omega_2. \quad (7.45)$$

Besides, G satisfies $|G(X_2, l(u_\varepsilon))| \leq C(1 + |l(u_\varepsilon)|^{r+1})$. Since the sequence in the right hand side is strongly converging in $L^1(\omega_2)$, applying Lebesgue's theorem we obtain (7.43).

Combining (7.43) and the weak convergence $u_\varepsilon \rightarrow u_0$ in $\mathcal{V}_0^p(\Omega, \eta)$ with the lower semi continuity of $v \mapsto \int_\Omega \eta A(|\nabla^\varepsilon v|^p) dx$, we can pass to the infimum limit in $\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon)$. It comes that

$$\liminf_{\varepsilon \rightarrow 0} \tilde{\mathbf{J}}^\varepsilon(u_\varepsilon) \rightarrow \tilde{\mathbf{J}}^0(u_0) \geq \inf_{v \in \mathcal{V}_0^p(\Omega, \eta)} \tilde{\mathbf{J}}^0(v).$$

In addition, $\tilde{\mathbf{J}}^\varepsilon(u_\varepsilon) \leq \tilde{\mathbf{J}}^\varepsilon(v)$, $\forall v \in \mathcal{D}(\Omega)$. Passing to the limit in both sides, we get

$$\limsup_{\varepsilon \rightarrow 0} \tilde{\mathbf{J}}^\varepsilon(u_\varepsilon) \leq \tilde{\mathbf{J}}^0(v), \quad \forall v \in \mathcal{D}(\Omega). \quad (7.46)$$

Since $\mathcal{D}(\Omega)$ is dense in $\mathcal{V}_0^p(\Omega, \eta)$ and $\tilde{\mathbf{J}}^0 \in C^1(\mathcal{V}_0^p(\Omega, \eta), \mathbb{R})$ (see Remark 5.2), then (7.46) still holds for every $v \in \mathcal{V}_0^p(\Omega, h)$. So u_0 realize the minimum of $\tilde{\mathbf{J}}^0$ on $\mathcal{V}_0^p(\Omega, \eta)$ and the corollary follows. ■

Remark 7.1 *The arguments used in the proof of Theorem 7.4 can be adapted, by setting $\varepsilon = 0$ and replacing $\mathcal{W}_0^p(\Omega, \eta)$ by $\mathcal{V}_0^p(\Omega, \eta)$, to obtain the existence of a solution for Problem (QP₀) that realizes the infimum of $\tilde{\mathbf{J}}^0$ on $\mathcal{V}_0^p(\Omega, \eta)$.*

A

APPENDICES

This appendix comprehends notation, some inequalities and definitions of spaces used in the thesis. Proofs and more advanced results can be found in the standard books of functional analysis and partial differential equations, see for instance [4, 6, 18].

A.1 NOTATION

General Notations

$$\begin{aligned}
 & := && \text{equal by definition} \\
 E' & = && \text{dual of space } E \\
 p' & = && \text{conjugate exponent of } p, \text{ i.e. } p' = \frac{p}{p-1}, \text{ for } 1 \leq p \leq \infty \\
 \alpha & = && (\alpha_1, \alpha_2, \dots, \alpha_N), \alpha_i \in \mathbb{N} - \text{N-dimensional multi-index} \\
 |\alpha| & = && \alpha_1 + \alpha_2 + \dots + \alpha_N - \text{length of multi-index } \alpha \\
 \rho(\Omega) & = && \{\rho_\alpha = \rho_\alpha(x), x \in \Omega, |\alpha| \leq 1\} \\
 \rho_\alpha & = && \text{weight functions measurable, positive almost everywhere} \\
 D^\alpha & = && \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_N}^{\alpha_N}}, \text{ derivative of order } |\alpha| \\
 \|\cdot\|_E & = && \text{norm of space } E \\
 \langle \cdot, \cdot \rangle_E & = && \text{duality product of a space } E', E \\
 x & = && (X_1, X_2) \in \mathbb{R}^n \text{ where } X_1 = (x_1, \dots, x_{n_1}) \in \mathbb{R}^{n_1} \text{ and } X_2 = (x'_1, \dots, x'_{n_2}) \in \mathbb{R}^{n_2} \\
 \nabla_{X_1} & = && \left(\partial_{x_1}, \dots, \partial_{x_{n_1}} \right)^T \\
 \nabla_{X_2} & = && \left(\partial_{x'_1}, \dots, \partial_{x'_{n_2}} \right)^T \\
 \nabla^\varepsilon & = && \begin{pmatrix} \varepsilon \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix} \\
 \nabla^0 & = && \begin{pmatrix} 0 \\ \nabla_{X_2} u \end{pmatrix}
 \end{aligned}$$

- $\Delta_{x_1} = \sum_{i=1}^{n_1} \partial_{x_i}^2$
- $x \cdot y = \sum_{i=1}^n x_i y_i$, for $x, y \in \mathbb{R}^n$ (Euclidean scalar product)
- $|x| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$, for $x \in \mathbb{R}^n$
- \hookrightarrow symbol denoting continuous imbedding
- $\omega_1 =$ open bounded set of \mathbb{R}^{n_1} , n_1 positive integer
- $\omega_2 =$ open bounded set of \mathbb{R}^{n_2} , n_2 positive integer
- $\Omega =$ open bounded set of $\mathbb{R}^{n_1+n_2}$
- $\bar{\Omega} =$ closure of Ω
- $\partial\Omega =$ Γ boundary of Ω

Function Spaces

- $C^k(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \mid u \text{ is } k \text{ times continuously differentiable}\}$, $k \in \mathbb{N}$
- $\mathcal{D}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \mid u \text{ is infinitely differentiable with compact support in } \Omega\}$
- $\mathcal{D}'(\Omega)$ space of distributions on Ω
- $L^p(\Omega, \rho) = \left\{ u : \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable and } \int_{\Omega} |u|^p \rho dx < \infty \right\}$, $1 \leq p < \infty$
- $W^{k,p}(\Omega, \rho) = \left\{ u = u(x) \mid \int_{\Omega} |D^\alpha u|^p \rho_\alpha < \infty, \text{ for orders } |\alpha| \leq k \right\}$
- $W_0^{1,p}(\Omega, \rho)$ the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega, \rho)$
- \rightharpoonup denotes the weak convergences
- \rightarrow denotes the strong convergences

A.2 SOME USEFUL INEQUALITIES

The following inequalities are often used to derive estimates in Analysis.

A polynomial inequality

Let $1 < p < +\infty$ and $a, b > 0$, then

$$a^p + b^p \leq (a + b)^p \leq 2^{p-1}(a^p + b^p)$$

Young's inequality

Assume $1 < p, p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then for any $a, b > 0$, it holds

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

It is sometimes convenient to use the form

$$ab \leq \alpha a^p + C_\alpha b^{p'}, \quad C_\alpha = \alpha^{-1/(p-1)}, \quad \alpha \in \mathbb{R}^+.$$

Hölder's inequality

Let $p, q \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Cauchy-Schwartz inequality

$$|\langle x, y \rangle| \leq |x| |y|, \quad \forall x, y \in \mathbb{R}^n.$$

A.3 ON THE SUM OF SPACES $\mathcal{V}'_0(\Omega, h)$ AND $L^{\frac{r+1}{r}}(w_2)$

The goal of this subsection is to establish the inclusion $\mathcal{V}'_0(\Omega, h) + L^{\frac{r+1}{r}}(w_2) \subset (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$.

We split the proof to several steps

1. $\mathcal{V}'_0(\Omega, h) \hookrightarrow (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$:

- This inclusion is algebraic: i.e. $f \in \mathcal{V}'_0(\Omega, h)$ implies that $f \in (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$. Given $f \in \mathcal{V}'_0(\Omega, h)$, then the map $u \mapsto (f, v)_{\mathcal{V}_0(\Omega, h)}$ defines an element from $(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$. Indeed, this map is linear and its continuity can be shown as follows:

$$\begin{aligned} |(f, v)_{\mathcal{V}_0(\Omega, h)}| &\leq \|f\|_{\mathcal{V}'_0(\Omega, h)} \|v\|_{\mathcal{V}_0(\Omega, h)} \\ &\leq \|f\|_{\mathcal{V}'_0(\Omega, h)} \left(\|v\|_{\mathcal{V}_0(\Omega, h)} + \|l(v)\|_{L^{r+1}(w_2)} \right) \\ &\leq \|f\|_{\mathcal{V}'_0(\Omega, h)} \|v\|_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)}. \end{aligned} \quad (*)$$

- This inclusion is an injection: so that any element in $\mathcal{V}'_0(\Omega, h)$ can be thought as an element of $(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$. This means that there exist a unique $\tilde{f} \in (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$ such that

$$\langle \tilde{f}, v \rangle_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} = (f, v)_{\mathcal{V}_0(\Omega, h)} \quad \text{for all } v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega).$$

i) First, such \tilde{f} is unique. By linearity it is sufficient to show that $\tilde{f} = 0 \implies f = 0$. Assume that $\tilde{f} = 0$, then

$$(f, v)_{\mathcal{V}_0(\Omega, h)} = 0, \quad \forall v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$$

since $\mathcal{D}(\Omega) \subset \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$, then

$$(f, v)_{\mathcal{V}_0(\Omega, h)} = 0, \quad \forall v \in \mathcal{D}(\Omega).$$

By density of $\mathcal{D}(\Omega)$ in $\mathcal{V}_0(\Omega, h)$, we deduce that $f = 0$.

ii) from (A.1) above, we have

$$\langle \tilde{f}, v \rangle_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} = (f, v)_{\mathcal{V}_0(\Omega, h)} \leq \|f\|_{\mathcal{V}'_0(\Omega, h)} \|v\|_{\mathcal{V}_0(\Omega, h)}.$$

Then,

$$\frac{\langle \tilde{f}, v \rangle_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)}}{\|v\|_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)}} \leq \|f\|_{\mathcal{V}'_0(\Omega, h)}.$$

Taking the sup on the left hand, we get

$$\|\tilde{f}\|_{(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'} \leq \|f\|_{\mathcal{V}'_0(\Omega, h)}.$$

This shows the continuity of the injection $\mathcal{V}'_0(\Omega, h) \subset (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$.

2. $L^{\frac{r+1}{r}}(w_2) \hookrightarrow (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$:

- This inclusion is algebraic i.e. given $f \in L^{\frac{r+1}{r}}(w_2)$, then the map $v \mapsto \int_{w_2} fl(v) dX_2$ define an element on $(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$. Clearly, it is a linear map and it continuity follows from the fact that

$$\begin{aligned} \left| \int_{w_2} fl(v) dX_2 \right| &\leq \|f\|_{L^{\frac{r+1}{r}}(w_2)} \|l(v)\|_{L^{r+1}(w_2)} \\ &\leq \|f\|_{L^{\frac{r+1}{r}}(w_2)} \left(\|v\|_{\mathcal{V}_0(\Omega, h)} + \|l(v)\|_{L^{r+1}(w_2)} \right) \\ &\leq \|f\|_{L^{\frac{r+1}{r}}(w_2)} \|v\|_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)}. \end{aligned} \quad (**)$$

- This inclusion is an injection: there exist a unique $\tilde{f} \in (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$ such that

$$\langle \tilde{f}, v \rangle_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} = \int_{w_2} fl(v) dX_2 \quad \text{for all } v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega).$$

i) if $\tilde{f} = 0$, then

$$\int_{w_2} fl(v) dX_2 = 0 \quad \forall v \in \mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)$$

Then, we have

$$\int_{w_2} fl(v) dX_2 = 0 \quad \forall v \in \mathcal{D}(\Omega)$$

this means

$$\int_{\Omega} f(X_2) h(X_1) v(X_1, X_2) dx = 0, \quad \forall v \in \mathcal{D}(\Omega)$$

since $f(X_2) h(X_1) \in L^1_{loc}(\Omega)$, this means that

$$fh = 0 \quad \text{a.e in } \Omega$$

since $h > 0$ a.e in Ω , we deduce that (from [6, Corollary 4.24 page 110])

$$f = 0 \quad \text{a.e in } \Omega$$

ii) From (A.1) above, we have

$$|\langle \tilde{f}, v \rangle| = \left| \int_{w_2} fl(v) dX_2 \right| \leq \|f\|_{L^{\frac{r+1}{r}}(w_2)} \|v\|_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)}.$$

Then, similarly to the step 1 we get

$$\|\tilde{f}\|_{(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'} \leq \|f\|_{L^{\frac{r+1}{r}}(w_2)}.$$

In the finally we have should the continuity

3. It remain to show the continuity of the injection $\mathcal{V}'_0(\Omega, h) + L^{\frac{r+1}{r}}(w_2) \subset (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$

Given $f \in \mathcal{V}'_0(\Omega, h) + L^{\frac{r+1}{r}}(w_2)$, then by definition there exist f_1 and f_2 such that

$$f = f_1 + f_2, \quad f_1 \in \mathcal{V}'_0(\Omega, h) \quad \text{and} \quad f_2 \in L^{\frac{r+1}{r}}(w_2).$$

So we get

$$\begin{aligned} |\langle f, v \rangle| &\leq \left| \langle f_1, v \rangle_{\mathcal{V}'_0(\Omega, h)} \right| + \left| \langle f_2, v \rangle_{L^{\frac{r+1}{r}}(w_2)} \right| \\ &\leq \left| \langle f_1, v \rangle_{\mathcal{V}'_0(\Omega, h)} \right| + \left| \int_{w_2} f_2 l(v) dX_2 \right| \\ &\leq \|f_1\|_{\mathcal{V}'_0(\Omega, h)} \|v\|_{\mathcal{V}_0(\Omega, h)} + \|f_2\|_{L^{\frac{r+1}{r}}(w_2)} \|l(v)\|_{L^{r+1}(w_2)} \\ &\leq \|f_1\|_{\mathcal{V}'_0(\Omega, h)} \|v\|_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} + \|f_2\|_{L^{\frac{r+1}{r}}(w_2)} \|v\|_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} \\ &\leq \|v\|_{\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega)} \left(\|f_1\|_{\mathcal{V}'_0(\Omega, h)} + \|f_2\|_{L^{\frac{r+1}{r}}(w_2)} \right) \end{aligned}$$

Thus $f \in (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$ and

$$\|f\|_{(\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'} \leq \|f\|_{\mathcal{V}'_0(\Omega, h) + L^{\frac{r+1}{r}}(w_2)}.$$

This shows that $\mathcal{V}'_0(\Omega, h) + L^{\frac{r+1}{r}}(w_2) \subset (\mathcal{V}_0(\Omega, h) \cap \mathcal{L}^{r+1}(\Omega))'$.

BIBLIOGRAPHY

- [1] D. R. Akhmetov and R. Spigler. Existence and uniqueness of classical solutions to certain nonlinear integro-differential fokker-planck type equations. *Electron. J. Differential Equations*, 2002(24):1–17, 2002.
- [2] S. Azouz and S. Guesmia. Asymptotic development of anisotropic singular perturbation problems. *Asymptot. Anal*, 100(3-4):131–152, 2016. doi: 10.3233/asy-161389.
- [3] M. Badiale and E. Serra. *Semilinear Elliptic Equations for Beginners: Existence Results via the Variational Approach*. Universitext. Springer, 2011. doi: 10.1007/978-0-85729-227-8.
- [4] J. Bergh and J. Löfström. *Interpolation spaces: an introduction*, volume 223. Springer Science & Business Media, 2012.
- [5] L. L. Bonilla, J. C. Neu, and R. Spigler. Nonlinear stability of incoherence and collective synchronization in a population of coupled oscillators. *J. Statistical Physics*, 67(1-2):313–330, 1992. doi: 10.1007/BF01049037.
- [6] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.
- [7] H. Brezis and E. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88(3):486–490, 1983. doi: 10.2307/2044999.
- [8] T. Cazenave. An introduction to semilinear elliptic equations. *Editora do Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro*, 164, 2006.
- [9] M. Chipot. *Asymptotic issues for some partial differential equations*. World Scientific., 2016.
- [10] M. Chipot and S. Guesmia. On the asymptotic behaviour of elliptic, anisotropic singular perturbations problems. *Com. Pure Appl. Ana.*, 8(1):179–193, 2009. doi: 10.3934/cpaa.2009.8.179.
- [11] M. Chipot and S. Guesmia. Correctors for some asymptotic problems. *Proc. Steklov Inst. Math.*, 270:263–277, 2010. doi: 10.1134/s0081543810030211.
- [12] M. Chipot and S. Guesmia. On a class of integro-differential problems. *Commun. Pure Appl. Anal.*, 9(5):1249–1262, 2010. doi: 10.3934/cpaa.2010.9.1249.

- [13] M. Chipot and S. Guesmia. On some anisotropic, nonlocal, parabolic singular perturbations problems. *Appl. Anal.*, 90(11-12):1775–1789, 2011. doi: 10.1080/00036811003627542.
- [14] M. Chipot, S. Guesmia, and A. Sengouga. Singular perturbations of some nonlinear problems. *J. Math. Sci. (N. Y.)*, 176:828–843, 2011. doi: 10.1007/s10958-011-0439-y.
- [15] M. Chipot, S. Guesmia, and A. Sengouga. Anisotropic singular perturbations of variational inequalities. *Calc. Var. Partial Differential Equations*, 57(1):7, 2018. doi: 10.1007/s00526-017-1286-6.
- [16] B. Dacorogna. *Direct methods in the calculus of variations*, volume 78. Springer Science & Business Media, 2008.
- [17] P. Drábek, A. Kufner, and F. Nicolosi. *Quasilinear elliptic equations with degenerations and singularities*. Walter de Gruyter, 2011.
- [18] L. C. Evans. *Partial differential equations: Second edition*, volume 19 of *Grad. Stud. Math.* Am. Math. Soc., Providence, RI, second edition, 2010.
- [19] P. Grisvard. Espaces intermédiaires entre espaces de sobolev avec poids. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 17(3):255–296, 1963.
- [20] S. Guesmia and A. Sengouga. Anisotropic singular perturbations of hyperbolic problems. *Appl. Math. Comput.*, 217(22):8983–8996, 2011. doi: 10.1016/j.amc.2011.03.104.
- [21] S. Guesmia, R. Kechkar, and M.-S. Moulay. Existence results for some partial integro-differential equations. *Mediterr. J. of Math.*, 13(6):4063–4079, 2016. doi: 10.1007/s00009-016-0732-6.
- [22] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, 2006.
- [23] O. Kavian. *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques*, volume 13 of *Mathématiques et applications*. Springer-Verlag, 1993.
- [24] A. Kufner. *Weighted sobolev spaces*. John Wiley & Sons Incorporated., 1985.
- [25] A. Kufner and B. Opic. How to define reasonably weighted sobolev spaces. *Comment. Math. Univ. Carolin.*, 25(3):537–554, 1984.
- [26] A. Kufner and J. Rákosník. Linear elliptic boundary value problems and weighted sobolev spaces: a modified approach. *Math. Slovaca*, 34(2):185–197, 1984.

- [27] M. M. Lavrentiev and R. Spigler. Existence and uniqueness of solutions to the kuramoto-sakaguchi nonlinear parabolic integro-differential equation. *Differential and Integral Equations*, 13(4-6):649–667, 2000.
- [28] M. M. Lavrentiev, R. Spigler, and D. R. Akhmetov. Nonlinear integro-parabolic equations on unbounded domains: Existence of classical solutions with special properties. *Sib. Math. J.*, 42(3):495–516, 2001.
- [29] M. M. Lavrentiev, R. Spigler, and D. R. Akhmetov. Regularizing a nonlinear integro-parabolic fokker-planck equation with space-periodic solutions: existence of strong solutions. *Sib. Math. J.*, 42(4):693–714, 2001.
- [30] E. E. Lewis and W. F. Miller. *Computational methods of neutron transport*. John Wiley and Sons, 1984.
- [31] J. L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod-Gautier Villars, 1969.
- [32] J. L. Lions. *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*, volume 323 of *Lecture Notes in Math*. Springer-Verlag, 1973. doi: 10.1007/bfbo060528.
- [33] R. D. Meyer. Some embedding theorems for generalized sobolev spaces and applications to degenerate elliptic differential operators. *J. Math. Mechanics*, 16(7):739–760, 1967.
- [34] H. Sakaguchi. Cooperative phenomena in coupled oscillator systems under external fields. *PTEP. Prog. Theor. Exp. Phys*, 79(1):39–46, 1988. doi: 10.1143/PTP.79.39.
- [35] S. H. Strogatz and R. E. Mirollo. Stability of incoherence in a population of coupled oscillators. *J. Statistical Physics*, 63(3-4):613–635, 1991. doi: 10.1007/BF01029202.
- [36] N. S. Trudinger. On the positivity of weak supersolutions of non-uniformly elliptic equations. *Bull. Aust. Math. Soc.*, 19(3):321–324, 1978. doi: 10.1017/s0004972700008868.
- [37] V. S. Vladimirov. Mathematical problems of one-speed particle transport theory. *Trudy Mat. Inst. Steklov. Akad. Nauk SSSR*, 61:3–158, 1961.
- [38] A. Yahiaoui, S. Guesmia, and A. Sengouga. Anisotropic non-local problems: Asymptotic behaviour and existence results. *Complex Var. Elliptic Equ.*, to appear. doi: 10.1080/17476933.2020.1860952.

ملخص:

في هذه الأطروحة نتعامل مع اضطرابات شاذة متباينة الخواص لبعض المسائل التكاملية-التفاضلية التي تتعلق بمعامل صغير للإضطراب $\varepsilon > 0$. تتضمن هذه المسائل معادلات إهليجية ذات نوع نصف خطي و شبه خطي مع بعض الأطراف غير المحلية وغير الخطية. باستخدام مقاربة تغايرية نثبت وجود حلول لهذه المسائل كنقاط حرجة لبعض الدوال من صنف C^1 في فضاءات ذات أوزان من نوع سوبولوف متباينة الخواص. الهدف الرئيسي من هذه الأطروحة هو دراسة السلوك المقارب لهذه الحلول بالنسبة لمعامل الاضطراب، ولما $\varepsilon \rightarrow 0$ نستخلص وجود حلول لبعض المسائل التكاملية-التفاضلية الغير قياسية.

الكلمات المفتاحية:

الاضطرابات المتباينة الشاذة، السلوك المقارب، المعادلات من النوع الإهليجي، الأطراف الغير محلية، النقاط الحرجة، المعادلات النصف خطية، المعادلات الشبه خطية.

ABSTRACT

In this thesis we deal with anisotropic singular perturbations of some integro-differential problems, depending on a small parameter of perturbation $\varepsilon > 0$. These problems involve elliptic equations of semilinear and quasilinear types with some non-local nonlinearities. Using a variational approach, we establish the existence of their solutions as critical points of some C^1 -functionals in anisotropic weighted Sobolev type spaces. The main goal of this thesis is to study the asymptotic behaviour of these solutions with respect to the parameter of perturbation ε . As $\varepsilon \rightarrow 0$, some existence results for non-standard integro-differential problems are established.

Keywords and phrases:

Anisotropic singular perturbations, asymptotic behaviour, elliptic problems, non-local terms, critical points, semilinear problems, quasilinear problems.

RÉSUMÉ:

Dans cette thèse on traite des perturbations singulières anisotropes de certains problèmes intégro-différentiels, dépendant d'un petit paramètre de perturbation $\varepsilon > 0$. Ces problèmes impliquent des équations elliptiques de types semi-linéaires et quasi-linéaires avec quelques non-linéarités non locales. En utilisant une approche variationnelle, nous établissons l'existence de leurs solutions comme points critiques de certaines C^1 -fonctionnelles dans des espaces avec poids de type Sobolev anisotropes. L'objectif principal de cette thèse est d'étudier le comportement asymptotique de ces solutions par rapport au paramètre de perturbation ε . Lorsque $\varepsilon \rightarrow 0$, on établie des résultats d'existence pour certains problèmes intégro-différentiels non standards.

Mots Clés:

Perturbations singulières anisotropes, comportement asymptotique, problèmes elliptic, termes non locaux, points critiques, problèmes semi linéaires, problèmes quasi linéaires.