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A Mathematical Study of A Contact Problem Involving Time-Fractional Derivatives

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List of Symbols

In what follows, we shall use the following notation.

$\Omega \subset \mathbb{R}^d$ Bounded domain occupied by the material body.

Γ Boundary of the domain Ω.

$\Gamma_1, \Gamma_2, \Gamma_3$ Measurable parts of the boundary Γ.

ν Unit outward normal vector on Γ.

$T > 0$ Final time.

u Displacement field.

$\dot{u}$ Velocity field.

$\ddot{u}$ Acceleration field.

σ Cauchy stress tensor.

$\varepsilon(u)$ Linearized strain tensor.

$\varepsilon_{ij}(u)$ Components of the strain tensor.

u_ν Normal displacement component.

u_τ Tangential displacement component.

$\dot{u}_\nu$ Normal velocity.

$\dot{u}_\tau$ Tangential velocity.

σ_ν Normal stress component.

σ_τ Tangential stress component.

f_0 Density of body forces.

f_2 Density of surface tractions.

ρ Mass density.

S^d Space of symmetric second-order tensors.

$v \cdot w$ Euclidean scalar product.

$\sigma \cdot \tau$ Scalar product in S^d.

$\| \cdot \|$ Euclidean norm.

$\mathrm{Div}\, \sigma$ Divergence operator.

$\partial_i u$ Partial derivative of u with respect to x_i.

$u_{i,j}$ Partial derivative of u_i with respect to x_j.

${}_0^C D_t^\alpha$ Caputo fractional derivative.

${}_0 I_t^\alpha$ Riemann–Liouville fractional integral.

$\Gamma(\alpha)$ Gamma function.

α Fractional-order parameter.

A Nonlinear viscosity operator.

F Nonlinear elasticity operator.

λ, μ Positive constitutive constants.

p Instantaneous normal response function.

$j(\cdot, \cdot)$ Contact functional.

$H = L^2(\Omega)^d$ Vector-valued Lebesgue space.

$\mathcal{H} = L^2(\Omega)^{d \times d}$ Tensor-valued Lebesgue space.

$H^1(\Omega)$	Sobolev space.
$H^1(\Omega)^d$	Vector-valued Sobolev space.
V	Space of admissible displacements.
V'	Dual space of V.
$\langle \cdot, \cdot \rangle_{V' \times V}$	Duality pairing between V' and V.
$(\cdot, \cdot)_H$	Inner product in H.
$(\cdot, \cdot)_V$	Inner product in V.
$\| \cdot \|_H$	Norm in H.
$\| \cdot \|_V$	Norm in V.
γ	Trace operator.
H_Γ	Trace space.
$L^p(0, T; X)$	Bochner space.
$W^{k,p}(0, T; X)$	Sobolev space of X-valued functions.
$AC(0, T; X)$	Space of absolutely continuous functions.

Introduction

This work is developed as a detailed continuation and an extended presentation of a previously established article on the mathematical and mechanical modeling of nonlinear fractional viscoelastic contact problems. It aims to provide a rigorous analytical and functional framework for the study of time-fractional partial differential equations arising in continuum mechanics, particularly in the presence of contact phenomena, viscoelastic behavior, and damage effects.

The starting point of this study is a physical setting describing the deformation of a bounded viscoelastic body occupying a domain $\Omega \subset \mathbb{R}^d$ (with $d = 2, 3$), whose boundary is partitioned into disjoint measurable parts associated with different mechanical constraints, including fixed support, applied tractions, and a possible contact zone with a deformable or rigid foundation. The mechanical response of the material is governed by nonlinear constitutive laws incorporating both elastic and viscous effects, as well as memory terms modeled through fractional derivatives in the sense of Caputo.

In this framework, the governing equations combine the fractional form of the balance of momentum with nonlinear stress-strain relations, leading to a coupled system of time-fractional partial differential equations. The presence of contact conditions, possibly with or without friction, introduces additional nonlinear boundary constraints that significantly increase the mathematical complexity of the model.

To properly analyze such a system, this work is also based on a set of essential preliminaries from functional analysis and continuum mechanics. These include Sobolev spaces, vector-valued function spaces, trace theorems, monotone operator theory, convex analysis, and compactness results for time-fractional evolution equations. These tools are fundamental for the derivation of variational and quasi-variational formulations, as well as for establishing existence, uniqueness, and stability results.

The main objective of this study is to reformulate the mechanical problem into an equivalent weak (variational) form, and to develop an analytical framework based on the Faedo-Galerkin approximation method, monotonicity techniques, and fixed point arguments. This allows us to handle the nonlinearities induced by the constitutive laws, the fractional time derivatives, and the contact boundary conditions in a unified functional setting.

Finally, this work is intended as a detailed and structured development of the original article, expanding the theoretical foundations and providing a complete mathematical background necessary for further numerical analysis and future applications in viscoelastic contact mechanics

Chapter 1

Contact Problem Modeling and Functional Analysis Preliminaries

This chapter is devoted to the introduction and the formulation of the mechanical problems which will be dealt with in the sequel, as well as a reminder of the main notions of continuum mechanics and functional analysis necessary for the understanding of this thesis. In this chapter we present the physical framework related to the studied problem, then we consider the laws of behavior of the various materials such as, the nonlinear viscoelastic constitutive law with damage. By then citing the boundary conditions with friction, in order to develop interesting mathematical models. Finally, we introduce a brief useful and essential reminder concerning functional and vector-valued spaces.

1.1 Physical setting-Mathematical models

In this section we will introduce the two physical setting and their mathematical mechanical problems involved in this thesis.

1.1.1 Physical framework of mechanical problem

We consider a viseoplastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$ (with $d = 2, 3$), whose boundary is Lipschitz continuous and decomposed into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\mathrm{meas}(\Gamma_1) > 0$.

Let $T > 0$ and $[0, T]$ be the time interval under consideration. We study, over this time interval, the evolution of the material body under the action of body forces and surface tractions.

The body is fixed on $\Gamma_1 \times (0, T)$; consequently, the displacement field vanishes there. A surface traction of density f_2 acts on $\Gamma_2 \times (0, T)$. In addition, body forces with density f_0 act on $\Omega \times (0, T)$.

The body may be in contact with a foundation on $\Gamma_3 \times (0, T)$. The conditions on this potential contact surface Γ_3 may vary and lead to different contact models, with or without friction (see Figure 1).

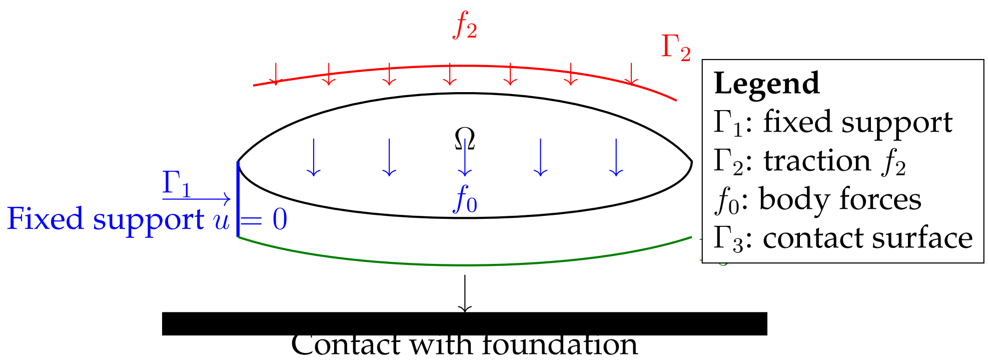

Figure 1.1: Physical framework of the mechanical problem

We denote by $\mathbf{v}_\nu$ and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on the boundary given by

$$\mathbf{v}_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \qquad \mathbf{v}_\tau = \mathbf{v} - \mathbf{v}_\nu \boldsymbol{\nu}. \tag{1.1.1}$$

Noting by $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$, the stress tensor field and we define analogously the normal and tangential components of $\boldsymbol{\sigma}$ on the boundary by the formulas

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \tag{1.1.2}$$

Using (**1**.1.1) and (**1**.1.2), we get the relation

$$(\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \mathbf{v} = \sigma_\nu \mathbf{v}_\nu + \boldsymbol{\sigma}_\tau \cdot \mathbf{v}_\tau, \tag{1.1.3}$$

This formula will be used extensively in the following chapters of this thesis to derive the variational formulations of several contact problems.

The points above a function represent the derivation with respect to time, i.e.

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt}, \qquad \ddot{\mathbf{u}} = \frac{d^2\mathbf{u}}{dt^2}.$$

Where $\dot{\mathbf{u}}$ denotes the velocity field and $\ddot{\mathbf{u}}$ denotes the acceleration field. For the velocity field $\dot{\mathbf{u}}$ the notations $\dot{\mathbf{u}}_\nu$ and $\dot{\mathbf{u}}_\tau$ represent respectively the normal and tangential velocities at the boundary, i.e.

$$\dot{\mathbf{u}}_\nu = \dot{\mathbf{u}} \cdot \boldsymbol{\nu}, \qquad \dot{\mathbf{u}}_\tau = \dot{\mathbf{u}} - \dot{\mathbf{u}}_\nu \boldsymbol{\nu}.$$

1.1.2 Equations of Motion and Equilibrium

The equation of motion governing the evolution of the mechanical state of the body is described by the Cauchy equation of motion

$$\rho\ddot{u} - \operatorname{Div}\sigma = f_0 \quad \text{in } \Omega \times (0, T), \tag{**1**.1.5}$$

where $\rho : \Omega \to \mathbb{R}_+$ denotes the mass density and f_0 is the density of the applied body forces, such as gravitational or electromagnetic forces. Here, Div denotes the divergence operator, namely,

$$\operatorname{Div}\sigma = (\sigma_{ij,j}),$$

and we recall that $\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$. The evolution process defined by (**1**.1.5) is referred to as a *dynamic process*.

In certain situations, this equation can be further simplified. For instance, when the velocity field $\dot{u}$ varies very slowly with respect to time, the acceleration term $\ddot{u}$ may be neglected. In this case, the process is called *quasistatic*, and equation (**1**.1.4) reduces to the equilibrium equation

$$\operatorname{Div}\sigma + f_0 = 0 \quad \text{in } \Omega \times (0, T). \tag{**1**.1.6}$$

Processes governed by the equation of motion (**1**.1.5) are called *dynamic processes*, whereas those governed by the equilibrium equation(**1**.1.6) are referred to as *quasistatic processes*.

1.2 Constitutive Law

As we noted in section 1.1, equations (**1**.1.1) – (**1**.1.6) do not constitute a complete description of the evolution of a continuous body. To obtain a complete model, valid for a given material, it is necessary to add the constitutive law of the material. Although they must satisfy certain basic axioms and principles of invariance, the laws of behavior come mainly from experience. not tonot to mention the subject of damage which is extremely important in design engineering. In this thesis, we develop a constitutive law related to a viscoelastic material with time fractional.

1.2.1 Viscoelastic constitutive law with time fractional

The most common constitutive law of a viscoelastic material with time fractional is given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}({}_0^C D_t^\alpha \mathbf{u}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}(t)) \tag{1.2.1}$$

Where $\mathbf{u}$ denotes the displacement field and $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}(\mathbf{u})$ represent respectively the stress and the linearized strain tensor. Furthermore $\mathcal{A}$ and $\mathcal{F}$ refer to the viscosity and elasticity tensr respectively. Where ${}_0^C D_t^\alpha \mathbf{u}(t)$ represents Caputo derivative of order $0 < \alpha \leq 1$. Finally, in order to complete the mathematical model which describes the evolution of the body, it is necessary to specify the boundary conditions on Γ_3, it is the object of the conditions of contact and laws with or without friction which we will describe in the following paragraph.

1.3 Contact boundary conditions and friction laws

The modeling of a contact phenomenon is determined by the hypotheses taken into account in its description. These hypotheses can influence either the form and the structure of the system of partial differential equations, or the boundary conditions of the associated mathematical model. The boundary conditions on the contact surface are described both in the direction of the normal and in the tangent plane, being called friction conditions. Recall that the boundary Γ is divided into three disjoint and measurable parts Γ_1, Γ_2 and Γ_3 such that $\text{meas}(\Gamma_1) > 0$, we give in this paragraph the boundary conditions on each of the three parts.

1.3.1 Displacement-traction boundary conditions

In all the problems that will be studied in this thesis, the body is embedded on the part Γ_1, so

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0, T). \tag{1.3.1}$$

This relation represents the boundary condition of displacement. A surface traction of density $\mathbf{f}_2$ acts on Γ_2 is consequently the vector of the stresses of Cauchy $\boldsymbol{\sigma\nu}$ satisfied

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0, T). \tag{1.3.2}$$

This condition is called the condition at the limits of traction.

1.3.2 Contact conditions

1.3.2.1 Bilateral contact condition

The contact is made bilaterally, that is to say the contact is maintained during the movement and there is no separation between the body and the obstacle. The normal component of the field of displacements cancels on the surface of contact and thus

$$\mathbf{u}_\nu = 0. \tag{1.3.3}$$

1.3.2.2 Unilateral contact

This condition models the contact with a rigid foundation. Since the foundation is considered rigid, it will therefore not undergo any deformation. The body will therefore not be able to enter it. This property results in the mathematical relationship

$$\mathbf{u}_\nu \leq 0. \tag{1.3.4}$$

At points of Γ_3 such that $\mathbf{u}_\nu < 0$, the deformable body leaves the rigid base and the normal stresses are then zero there. Therefore we obtain:

$$\mathbf{u}_\nu < 0 \Rightarrow \sigma_\nu = 0. \tag{1.3.5}$$

At points of Γ_3 such that $\mathbf{u}_\nu = 0$, the contact is maintained and the rigid base exerts a normal reaction oriented towards Ω and therefore, we can write

$$\mathbf{u}_\nu = 0 \Rightarrow \sigma_\nu \leq 0. \tag{1.3.6}$$

The contact conditions written by (1.3.4), (1.3.5) and (1.3.6), are called unilateral contact conditions.

1.3.2.3 Contact with Normal Damped Response with or without friction

When dealing with granular materials or lubricated surfaces, the reaction force or the normal stress may depend on the surface velocity. Therefore, we employ the condition known as the *instantaneous normal response*, given by

$$-\sigma_\nu = p(\dot{u}_\nu). \tag{1.3.7}$$

Here, p is a positive function that vanishes whenever its argument is negative, while u_ν and σ_ν denote, respectively, the normal components of the displacement field u and the stress field σ.

Another formulation describing the same condition by using fractional-order derivatives can also be considered, and is written as

$$-\sigma_\nu = p_\nu({}_0^C D_t^\alpha u_\nu(t)), . \tag{1.3.8}$$

1.3.2.4 Laws of contact without friction

We turn now to the conditions in the tangential directions, called also frictional conditions or friction laws. The simplest one is the so-called frictionless condition in which the friction force vanishes during the process, i.e.,

$$\boldsymbol{\sigma}_\tau = 0 \quad \text{on } \Gamma_3 \times (0, T). \tag{1.3.9}$$

1.4 Function Spaces

We introduce in this section spaces of the Sobolev type used in contact mechanics, namely the Hilbert spaces associated with the divergence and deformation operators, as well as the spaces of vector-valued functions. We also present their main properties, in particular the trace theorems. We adopt here the convention of the silent index and we also specify that all the notations as well as the functional spaces used in this thesis are introduced in this section. Also, in writing this section we have used [1]. For more details on Sobolev spaces and distribution spaces, we refer for example to [5].

1.4.1 Sobolev spaces

We begin with a brief reminder of some results on the Sobolev space $H^1(\Omega)$ defined by

$$H^1(\Omega) = \{\mathbf{u} \in L^2(\Omega) \mid \partial_i \mathbf{u} \in L^2(\Omega),\, i = 1, \ldots, d\}.$$

First, we denote by $\nabla \mathbf{u}$ the component vector $\partial_i \mathbf{u}$. We have $\nabla \mathbf{u} \in L^2(\Omega)^d$ for all $\mathbf{u} \in H^1(\Omega)$. We know that $H^1(\Omega)$ is a Hilbert space for the scalar product

$$(\mathbf{u}, \mathbf{v})_{H^1(\Omega)} = (\mathbf{u}, \mathbf{v})_{L^2(\Omega)} + (\partial_i \mathbf{u}, \partial_i \mathbf{v})_{L^2(\Omega)},$$

And the associated norm

$$\|\mathbf{u}\|^2_{H^1(\Omega)} = (\mathbf{u},\mathbf{u})_{H^1(\Omega)} \quad \text{and we write } \|\mathbf{u}\|^2_{H^1(\Omega)} = \|\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla\mathbf{u}\|^2_{L^2(\Omega)},$$

We have the following results:

$$C^1(\overline{\Omega}) \text{ is dense in } H^1(\Omega).$$

Theorem 1.4.1 (Rellich)**.** *$H^1(\Omega) \subset L^2(\Omega)$ with compact injection.*

Theorem 1.4.2 (Sobolev trace)**.** *There exists a linear and continuous map $\gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ such that $\gamma\mathbf{u} = \mathbf{u}|_\Gamma$ for $\mathbf{u} \in C^1(\overline{\Omega})$.*

Remark 1.4.3. The space $L^2(\Gamma)$ above represents the space of real functions on Γ which are L^2 for the surface measure $d\Gamma$. The map γ is called trace map; it is defined as the extension by density of the map $\mathbf{u} \longrightarrow \mathbf{u}|_\Gamma$ defined for $\mathbf{u} \in C^1(\overline{\Omega})$. Note that the trace map $\gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ is a compact operator.

1.4.2 Hilbert spaces associated with the divergence and deformation operators

In this Subsection we present the notation we shall use and some preliminary material. For further details, we refer to [8]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), while $\cdot$ and $|\cdot|$ will represent the inner product and the Euclidean norm on S^d and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces

$$\begin{aligned}
H &= \{\mathbf{u} = (u_i) \mid u_i \in L^2(\Omega),\, i = \overline{1,d}\} = L^2(\Omega)^d,\\
\mathcal{H} &= \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\, i,j = \overline{1,d}\} = L^2(\Omega)^{d\times d},\\
H_1 &= \{\mathbf{u} \in H \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H}\} = \{\mathbf{u} = (u_i) \mid u_i \in H^1(\Omega),\, i = \overline{1,d}\} = H^1(\Omega)^d,\\
\mathcal{H}_1 &= \{\boldsymbol{\sigma} \in \mathcal{H} \mid \sigma_{ij} \in H\}.
\end{aligned}$$

Here $\varepsilon : H_1 \to H$ and $\mathrm{Div} : H_1 \to H$ are the deformation and divergence operators defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}),\ 1 \le i, j \le d.$$

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad |\mathbf{v}| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\sigma}| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S^d.$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products

$$\begin{aligned}
(\mathbf{u}, \mathbf{v})_H &= \int_\Omega u_i v_i \, dx \qquad \forall \mathbf{u}, \mathbf{v} \in H, \\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} \, dx \qquad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}, \\
(\mathbf{u}, \mathbf{v})_{H_1} &= (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in H_1, \\
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau})_H \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.
\end{aligned}$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H, \|\cdot\|_{\mathcal{H}}, \|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$. For every element $\mathbf{v} \in H_1$ we also use the notation $\mathbf{v}$ for the trace $\gamma_{\mathbf{u}}$ of $\mathbf{v}$ on Γ and we denote by $\mathbf{v}_\nu$ and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on Γ given by

$$\mathbf{v}_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - \mathbf{v}_\nu \boldsymbol{\nu}.$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$. If $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

We recall that the trace map $\gamma : H_1 \to L^2(\Gamma)^d$ is continuous linear, but not surjective. The image of H_1 by this map is denoted by H_Γ, this subspace is continuously injected into $L^2(\Gamma)^d$. Let H'_Γ be the dual of H_Γ, and $\langle \cdot, \cdot \rangle_{H'_\Gamma \times H_\Gamma}$ the duality product between H'_Γ and H_Γ.

For every $\boldsymbol{\sigma} \in H_1$, there exists an element denoted $\boldsymbol{\sigma}\boldsymbol{\nu} \in H'_\Gamma$ such that

$$\langle \boldsymbol{\sigma}\boldsymbol{\nu}, \gamma\mathbf{v} \rangle_{H'_\Gamma \times H_\Gamma} = (\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathbf{v})_H \quad \forall \mathbf{v} \in H_1. \tag{1.4.1}$$

Also, if $\boldsymbol{\sigma}$ is regular enough (For example C^1), we have the formula

$$\langle \boldsymbol{\sigma}\boldsymbol{\nu}, \gamma\mathbf{v} \rangle = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \mathbf{v} \, da \qquad \forall \mathbf{v} \in H_1.$$

So, if $\boldsymbol{\sigma}$ is regular enough we have the following formula:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \mathbf{v} \, da \qquad \forall \mathbf{v} \in H_1(\Omega). \tag{1.4.2}$$

Green's formula allows in particular to define the restriction of $\boldsymbol{\sigma\nu}$ to a measurable part Γ_2 of Γ. More precisely, for $g \in L^2(\Gamma_2)$, we have $\boldsymbol{\sigma\nu} = g$ on Γ_2 if and only if $\langle \boldsymbol{\sigma\nu}, \gamma\mathbf{v} \rangle = \int_{\Gamma_2} g \cdot \mathbf{v} \, da$, for all $\mathbf{v} \in H_1$ vanishing on $\Gamma \setminus \Gamma_2$.

Throughout the thesis, in mechanical problems, Γ is partitioned into three measurable parts Γ_1, Γ_2 and Γ_3 such that $\operatorname{meas} \Gamma_1 > 0$.

We will constantly need the space of admissible displacements V defined as being a closed subspace of the space H_1

$$V = \left\{ \mathbf{v} \in H_1(\Omega)^d \mid \mathbf{v} = 0 \quad \text{on } \Gamma_1 \right\}. \tag{1.4.3}$$

Since $\operatorname{meas} \Gamma_1 > 0$, Korn's inequality applies to V: there exists a constant $C_k > 0$ depending only on Ω and Γ_1 such that

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \geq C_k \|\mathbf{v}\|_{H_1(\Omega)^d} \qquad \forall \mathbf{v} \in V. \tag{1.4.4}$$

On V we consider the scalar product $(\cdot, \cdot)_V$ and the norm $\|\cdot\|_V$ associated with this product, given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \qquad \|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{1.4.5}$$

By the Korn inequality, it comes that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V and so $(V, \|\cdot\|_V)$ is a Hilbert space.

Also, by Sobolev's trace theorem, there exists constant $C_0 > 0$ depending only on Ω, Γ_1 and Γ_2 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq C_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \tag{1.4.6}$$

1.4.3 Spaces of Vector-valued Functions

Let H be a Hilbert space. Let $k \in \mathbb{N}$, $1 \leq p \leq +\infty$ and $T > 0$. We recall that $W^{k,p}(0, T; H)$ is the space of vector distributions $u \in \mathcal{D}'(0, T; H)$ such that $D^j u \in L^p(0, T; H)$ for $j = \overline{0, k}$, D^j denoting the derivative of order j in the sense of distributions.

For $1 \leq p < +\infty$, $W^{k,p}(0,T;H)$ is a real Banach space for the norm defined by

$$\|u\|_{W^{k,p}(0,T;H)} = \Big(\sum_{j=0}^{k} \int_0^T \|D^j u(x)\|_H^p \, dx\Big)^{1/p} \quad \forall u \in W^{k,p}(0,T;H). \tag{\textbf{1}.4.7}$$

In particular, $W^{k,2}(0,T;H)$ is a real Hilbert space for the inner product defined by

$$(u,v)_{W^{k,2}(0,T;H)} = \sum_{j=0}^{k} \int_0^T (D^j u(t), D^j v(t))_H \, dt \quad \forall u,v \in W^{k,2}(0,T;H). \tag{\textbf{1}.4.8}$$

On the other hand, $W^{k,\infty}(0,T;H)$ is a Banach space for the norm defined by

$$\|u\|_{W^{k,\infty}(0,T;H)} = \sum_{j=0}^{k} \operatorname{ess\,sup}_{[0,T]} \|D^j u(t)\|_H \quad \forall u \in W^{k,\infty}(0,T;H). \tag{\textbf{1}.4.9}$$

For the particular case $k = 0$ it is observed that

$$W^{0,p}(0,T;H) = L^p(0,T;H),$$

And we then denote the norm $L^p(0,T;H)$ by $\|\cdot\|_{L^p(0,T;H)}$ for all $1 \leq p \leq +\infty$. We also define, for all $k \in \mathbb{N}$, the space $C^k([0,T];H)$ of functions $\mathbf{u} : [0,T] \to H$ such that for all $j = \overline{0,k}$ the derivatives $\frac{d^j \mathbf{u}}{dt^j}$ exist and are continuous on $[0,T]$. We denote, in particular, $C^0([0,T];H)$ by $C([0,T];H)$. The space $C^k([0,T];H)$ is a Banach space for the norm defined by

$$\|\mathbf{u}\|_{C^k([0,T];H)} = \sum_{j=0}^{k} \max_{t\in[0,T]} \Big\| \frac{d^j \mathbf{u}}{dt^j}(t) \Big\|_H \quad \forall u \in C^k([0,T],H). \tag{\textbf{1}.4.10}$$

In particular, the norms on the spaces $C([0,T];H)$ and $C^1([0,T];H)$ are given by

$$\|\mathbf{u}\|_{C([0,T];H)} = \max_{t\in[0,T]} \|\mathbf{u}(t)\|_H \quad \forall u \in C([0,T],H), \tag{1.1}$$

$$\|\mathbf{u}\|_{C^1([0,T];H)} = \|\mathbf{u}\|_{C([0,T];H)} + \|\dot{\mathbf{u}}\|_{C([0,T];H)} \quad \forall u \in C^1([0,T],H). \tag{\textbf{1}.4.11}$$

We specify that the point above an expression designates the derivative of this expression with respect to time, represented by the variable $t \in [0,T]$.

CHAPTER 2

FRACTIONAL CALCULUS AND NONLINEAR ANALYSIS

In this chapter, we recall some fundamental results from fractional calculus and nonlinear functional analysis which are essential for the study of time-fractional partial differential equations. These results can be found in standard references such as 5,12,15, Elements of convex analysis; Gronwall's lemmas, and the fixed-point theorem.

2.1 Fractional Calculus

The Gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad \Re(z) > 0,$$

where $t^{z-1} = e^{(z-1)\log(t)}$. This integral is convergent for all complex numbers $z \in \mathbb{C}$ such that $\Re(z) > 0$. For this function, the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \qquad \Re(z) > 0,$$

holds. In particular, if $z = n \in \mathbb{N}_0$, then

$$\Gamma(n+1) = n!, \qquad n \in \mathbb{N}_0,$$

with, as usual,

$$0! = 1.$$

Let us consider some of the starting points for a discussion of fractional calculus.

One development begins with a generalization of repeated integration. Thus, if f is locally integrable on (c, ∞), then the n-fold iterated integral is given by

$${}_cD_t^{-n} f(t) = \int_c^t ds_1 \int_c^{s_1} ds_2 \cdots \int_c^{s_{n-1}} f(s_n)\, ds_n = \frac{1}{(n-1)!} \int_c^t (t-s)^{n-1} f(s)\, ds,$$

for almost all t with $-\infty \leq c < t < \infty$ and $n \in \mathbb{N}$. Writing $(n-1)! = \Gamma(n)$, an immediate generalization is the integral of f of fractional order $\alpha > 0$,

$$ {}_cD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} f(s)\, ds, $$

(left-hand integral), and similarly for $-\infty < t < d \leq \infty$,

$$ {}_tD_d^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^d (s-t)^{\alpha-1} f(s)\, ds, $$

(right-hand integral), both being defined for suitable functions f.
In this memory, we restrict our attention to the use of the Riemann–Liouville and Caputo fractional derivatives.

2.1.1 Riemann–Liouville Fractional Integral

Definition 2.1.1 (Riemann–Liouville Fractional Integral)**.** Let X be a Banach space and $(0, T)$ a finite time interval. For $\alpha > 0$ and $f \in L^1(0, T; X)$, the Riemann–Liouville fractional integral is defined by

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad \forall t \in (0, T), \tag{2.1.1} $$

where $\Gamma(\cdot)$ is the Gamma function defined by

$$ \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt. $$

Remark 2.1.2. For $\alpha = 0$, we define ${}_0I_t^0 = I$, the identity operator, i.e.,

$$ {}_0I_t^0 f(t) = f(t), \quad \text{a.e. } t \in (0, T). $$

Caputo Fractional Derivative

Definition 2.1.3 (Caputo Derivative)**.** Let X be a Banach space, $0 < \alpha \leq 1$, and $(0, T)$ a finite interval. For $f \in AC(0, T; X)$, the Caputo fractional derivative is defined by

$$ {}_0^CD_t^\alpha f(t) = {}_0I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds. \tag{2.1.2} $$

Remark 2.1.4. If $\alpha = 1$, the Caputo derivative reduces to the classical derivative:

$$ {}_0^CD_t^1 f(t) = f'(t). $$

Proposition 2.1.5. *Let X be a Banach space and $\alpha, \beta > 0$. Then:*

(i) For $y \in L^1(0,T;X)$,

$$ {}_0I_t^{\alpha}\left({}_0I_t^{\beta}y(t)\right) = {}_0I_t^{\alpha+\beta}y(t). \tag{2.1.3}$$

(ii) For $y \in AC(0,T;X)$ and $\alpha \in (0,1]$,

$$ {}_0I_t^{\alpha}\left({}_0^CD_t^{\alpha}y(t)\right) = y(t) - y(0). \tag{2.1.4}$$

(iii) For $y \in L^1(0,T;X)$,

$$ {}_0^CD_t^{\alpha}\left({}_0I_t^{\alpha}y(t)\right) = y(t). \tag{2.1.5}$$

Now, we establish the following global behavior of the fractional ODE.

Proposition 2.1.6 (13). *Let $-\infty \le a < b \le \infty$. Assume*

$$f : [0,\infty) \times (a,b) \to \mathbb{R}$$

is continuous and locally Lipschitz in the second variable. In other words, for every $\beta > 0$ and every compact set $K \subset (a,b)$, there exists $L_{\beta,K} > 0$ such that

$$\sup_{0\le t\le \beta} |f(t,v_1) - f(t,v_2)| \le L_{\beta,K}|v_1 - v_2|, \qquad \forall v_1, v_2 \in K. \tag{2.1.6}$$

Let $0 < \alpha < 1$ and $v_0 \in (a,b)$. Then, the IVP

$$ {}^cD_t^{\alpha}v(t) = f(t,v(t)), \qquad v(0) = v_0, \tag{2.1.7}$$

has a unique continuous solution $v(\cdot)$ on $[0,T_\gamma)$, where

$$T_\gamma = \sup\left\{h > 0 : \text{the solution } v \in C[0,h),\ v(t) \in (a,b),\ \forall t \in [0,h)\right\}.$$

The quantity $T_\gamma \in (0,\infty]$ is the largest time of existence. Moreover, if $T_\gamma < \infty$, then

$$\liminf_{t\to T_\gamma^-} v(t) = a, \qquad \text{or} \qquad \limsup_{t\to T_\gamma^-} v(t) = b.$$

2.1.2 Compactness Criterion for Time-Fractional PDEs

Additionally, we recall a compactness criterion analogous to the Aubin–Lions lemma, used in the study of weak solutions to time-fractional PDEs (see 15).

Theorem 2.1.7. *Let $T > 0$, $\alpha \in (0,1)$, and $p \in [1,\infty)$. Let B_0, B, B_1 be Banach spaces such that*

$$B_0 \hookrightarrow B \quad \textit{compactly}, \qquad B \hookrightarrow B_1 \quad \textit{continuously}.$$

Let $W \subset L^1_{\mathrm{loc}}(0,T;B_0)$ satisfy:

(i) There exist $r_1 \in [1,\infty)$ and $c_1 > 0$ such that for all $u \in W$,

$$\sup_{t\in(0,T)} \|J^\alpha u(t)\|_{B_0}^{r_1} = \sup_{t\in(0,T)} \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds \right\|_{B_0}^{r_1} \leq c_1;$$

(ii) There exists $p_1 \in (p,\infty]$ such that W is bounded in $L^{p_1}(0,T;B)$;

(iii) There exist $r_2 \in [1,\infty)$ and $c_2 > 0$ such that for all $u \in W$, there exists an initial value u_0 such that the weak Caputo derivative satisfies

$$\|{}_0^C D_t^\alpha u\|_{L^{r_2}(0,T;B_1)} \leq c_2. \tag{2.1.8}$$

Then W is relatively compact in $L^p(0,T;B)$.

2.2 Nonlinear analysis elements

In this section we will recall some notions of nonlinear analysis which will be of great use for the realization of this work. In particular results on weak convergence and weak star convergence, bounded operators, monotone, lower convex and lower semi-continuous functions, differentiability and subdifferentiability.

2.2.1 Weak convergence and Weak star convergence

We now turn our attention to the notion of convergence. Let X is a Banach space and X' the dual space of X and denote $\langle \cdot,\cdot \rangle$ the duality product between X and its topological dual space X'.

Definition 2.2.1 (Strong convergence)**.** A sequence u_n is said to strongly converge to u if $u_n, u \in X$ and if

$$\lim_{n\to\infty} \|u_n - u\|_X = 0,$$

we will denote this convergence by $u_n \to u$ in X.

Now, let's recall some definitions and results on weak topology.

Definition 2.2.2 (Weak convergence)**.** **[?]** Let X be a normed space with X' its dual space. A sequence $\{u_n\} \subset X$ weakly converges to $u \in X$, if

$$\ell(u_n) \to \ell(u) \quad \forall \ell \in X'.$$

This convergence will be denoted by $u_n \rightharpoonup u$ in X.

Proposition 2.2.3. *Let u_n be a sequence in X. Then*

If $u_n \to u$ strongly in X, then $u_n \rightharpoonup u$ weakly in X. If $u_n \rightharpoonup u$ weakly in X, then $\|u_n\|_X$ is bounded and $\|u\|_X \leq \liminf_{n\to\infty} \|u_n\|_X$.If $u_n \rightharpoonup u$ weakly in X and if $\ell_n \to \ell$ strongly in X', then $\langle \ell_n, u_n \rangle_{X'\times X} \to \langle \ell, u \rangle_{X'\times X}$.

Definition 2.2.4. Let X be a Hilbert space endowed with the scalar product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$ and let $A : X \to X$ be an operator.

The operator A is said to be *monotone* if

$$(Au - (Au - Av, u - v)_X \geq 0 \quad \forall u, v \in X.$$

The operator A is *strictly monotone* if

$$(Au - Av, u - v)_X > 0 \quad \forall u, v \in X,\ u \neq v.$$

The operator A is said to be *strongly monotone* if there exists $m > 0$ such that

$$(Au - Av, u - v)_X \geq m\|u - v\|_X^2 \quad \forall u, v \in X.$$

Operator A is *Lipschitz* if there exists $M > 0$ such that

$$\|Au - Av\|_X \leq M\|u - v\|_X \quad \forall u, v \in X.$$

2.3 Elements of convex analysis

2.3.1 Convex and lower semicontinuous functions

Definition 2.3.1. Let X be a real vector space and let the function $\phi : X \to]-\infty, +\infty]$. We say that the function ϕ is *proper* if $\phi(u) > -\infty$ for all $u \in X$ and $\phi(u) < +\infty$ for at least one $u \in X$. The function ϕ is *convex* if

$$\phi(tu + (1-t)v) \leq t\phi(u) + (1-t)\phi(v) \qquad \forall u, v \in X,\ t \in]0, 1[.$$

If the strict inequality holds for any $u \neq v$ then ϕ is said to be *strictly convex* in X. The set

$$D(\phi) = \{u \in X \mid \phi(u) < +\infty\}$$

is called the effective domain of ϕ. For a real number α, the set $\{(u, \alpha) \in X \times \mathbb{R} \mid \phi(u) \leq \alpha\}$ is called epigraph of ϕ ($\operatorname{epi} \phi$).

Note that if $\phi, \psi : X \to]-\infty, +\infty]$ are convex functions and $\lambda \geq 0$, then the functions $\phi + \psi$ and $\lambda\phi$ are also convex.

Definition 2.3.2. A function $\phi : X \to]-\infty, +\infty]$ is said to be *lower semi-continuous (l.s.c.)* at $u \in X$ if

$$\liminf_{n \to +\infty} \phi(u_n) \geq \phi(u) \tag{2.3.1}$$

for each sequence $\{u_n\} \subset X$ convergent to u in X. The function ϕ is l.s.c. if it is l.s.c. at each point $u \in X$. When inequality (1.5.1) holds for any sequence $\{u_n\} \subset X$ which converges weakly to u, we say that the function ϕ is weakly lower semicontinuous at u. The function ϕ is weakly l.s.c. if it is weakly l.s.c. at each point $u \in X$.

Note that if $\phi, \psi : X \to]-\infty, +\infty]$ are l.s.c. functions and $\lambda \geq 0$, then the functions $\phi + \psi$ and $\lambda\phi$ are also l.s.c. Furthermore, if ϕ is a continuous function, then it is also l.s.c. However, the reverse is not true since lower semi-continuity does not imply continuity.

As strong convergence in X implies weak convergence, it follows that a weakly lower semi-continuous function is lower semi-continuous. Moreover, one can show that a proper convex function $\phi : X \to]-\infty, +\infty]$ is lower semicontinuous if and only if it is weakly lower semicontinuous.

Example 2.3.3 (Indicator function)**.** Let X be a normed space and $K \subset X$. We call the indicator function of K the function $\Psi_K : X \to]-\infty, +\infty]$ defined by:

$$\Psi_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3.2}$$

We can prove that K is a non-empty, closed and convex set of X if and only if the indicator function Ψ_K is proper, convex and l.s.c.

2.3.1.1 Differentiability and sub-differentiability

We now recall the definition of *Gateaux differentiable* functions.

Definition 2.3.4. Let $\phi : X \to \mathbb{R}$ and let $u \in X$. Then ϕ is *Gateaux differentiable* at u if there exists an element $\nabla\phi(u) \in X$ such that

$$\lim_{t \to 0} \frac{\phi(u + tv) - \phi(u)}{t} = (\nabla\phi(u), v)_X \quad \forall v \in X. \tag{2.3.3}$$

The element $\nabla\phi(u)$ that satisfies (1.5.3) is unique and is called the gradient of ϕ at u. The function $\phi : X \to \mathbb{R}$ is said to be *Gateaux differentiable* if it is *Gateaux differentiable* at every point of X. In thIn this case, the operator $\nabla\phi : X \to X$ that maps every element $u \in X$ into the element $\nabla\phi(u)$ is called the gradient operator of ϕ.

The convexity of *Gateaux differentiable* functions can be characterized as follows.

Proposition 2.3.5. *Let $(X, (\cdot,\cdot)_X)$ be a pre-Hilbertian space and let $\phi : X \to \mathbb{R}$ be a* Gateaux differentiable *function. The following statements are equivalent:*

1. *ϕ is a convex function;*

2. *ϕ satisfies the inequality*

$$\phi(v) - \phi(u) \geq (\nabla\phi(u), v - u)_X \qquad \forall u, v \in X. \tag{2.3.4}$$

3. *the gradient of ϕ is a monotone operator*

$$(\nabla\phi(u) - \nabla\phi(v), u - v)_X \geq 0 \qquad \forall u, v \in X. \tag{2.3.5}$$

Corollary 2.3.6. *Let $\phi : X \to \mathbb{R}$ be a convex* Gateaux differentiable *function. Then ϕ is lower semicontinuous.*

Definition 2.3.7. Let $\phi : X \rightarrow]-\infty, +\infty]$ and let $u \in X$. Then the *subdifferential* of ϕ at u is the set

$$\partial\phi(u) = \{f \in X \mid \phi(v) - \phi(u) \geq (f, v-u)_X \quad \forall v \in X\}. \tag{2.3.6}$$

Denote

$$D(\partial\phi) = \{u \in X \mid \partial\phi(u) \neq \emptyset\}. \tag{2.3.7}$$

A function ϕ is said to be subdifferentiable at $u \in X$ if $u \in D(\partial\phi)$, and each element $f \in \partial\phi(u)$ is called a subgradient of ϕ at u. A function ϕ is said to be subdifferentiable if it is subdifferentiable at each point $u \in X$, i.e., if $D(\partial\phi) = X$.

2.4 The Galerkin Method

The Galerkin method is a fundamental and widely used technique for the approximation of solutions to boundary value problems, particularly in the context of partial differential equations.

The main idea of the Galerkin method consists in reducing a problem posed in an infinite-dimensional functional space to a sequence of problems defined in finite-dimensional subspaces.

Let V be a Hilbert space and let us consider the following variational problem: find $u \in V$ such that

$$a(u, v) = L(v), \quad \forall v \in V, \tag{2.4.1}$$

where $a(\cdot, \cdot)$ is a bilinear form and L is a linear functional.

The Galerkin approximation consists in choosing a finite-dimensional subspace $V_h \subset V$ and seeking an approximate solution $u_h \in V_h$ such that

$$a(u_h, v_h) = L(v_h), \quad \forall v_h \in V_h. \tag{2.4.2}$$

This approach transforms the original infinite-dimensional problem into a finite system of algebraic equations. The Galerkin method is characterized by its stability, consistency, and convergence properties, and it constitutes the theoretical foundation of the finite element method.

2.5 Miscellaneous Complements

In this section, we recall the classical Gronwall-type lemmas which arise in many estimation problems, particularly in establishing the uniqueness of solutions. We also present several theorems used throughout this thesis, including a version of the Riesz–Fréchet representation theorem and Banach's fixed point theorem. For further details concerning the material reviewed in this section, the reader may consult, for example, references [**?**, **?**].

Gronwall Lemmas

Lemma 2.5.1 ([**?**, **?**]). *Let* $m, n \in C(0,T;\mathbb{R})$ *such that*

$$m(t) \geq 0, \qquad n(t) \geq 0,$$

for all $t \in [0,T]$*, and let* $a \geq 0$*. If* $\varphi \in C(0,T;\mathbb{R})$ *satisfies*

$$\varphi(t) \leq a + \int_0^t m(s)\,ds + \int_0^t n(s)\varphi(s)\,ds, \quad \forall t \in [0,T],$$

then

$$\varphi(t) \leq \left(a + \int_0^t m(s)\,ds\right) \exp\left(\int_0^t n(s)\,ds\right), \quad \forall t \in [0,T].$$

Corollary 2.5.2. *Let* $n \in C(0,T;\mathbb{R})$ *with* $n(t) \geq 0$ *for all* $t \in [0,T]$*, and let* $a \geq 0$*. If* $\varphi \in C(0,T;\mathbb{R})$ *satisfies*

$$\varphi(t) \leq a + \int_0^t n(s)\varphi(s)\,ds,$$

then

$$\varphi(t) \leq a \exp\left(\int_0^t n(s)\,ds\right).$$

Lemma 2.5.3. *Let* $m, n \in C(0,T;\mathbb{R})$ *be such that*

$$m(t) \geq 0, \qquad n(t) \geq 0,$$

for all $t \in [0,T]$*, and let* $a \geq 0$*. If* $\varphi \in C(0,T;\mathbb{R})$ *satisfies*

$$\frac{1}{2}\varphi^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)\varphi(s)\,ds + \int_0^t n(s)\varphi^2(s)\,ds, \qquad \forall t \in [0,T],$$

then

$$|\varphi(t)| \leq \left(a + \int_0^t m(s)\,ds\right) \exp\left(\int_0^t n(s)\,ds\right), \qquad \forall t \in [0,T].$$

2.5.1 Statements of Some Theorems

We now present several important theorems that will be used throughout this thesis.

Let X be a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. We denote by X' the dual space of X and by $\langle\cdot,\cdot\rangle_{X'\times X}$ the duality pairing between X' and X.

Theorem 2.5.4 (Riesz–Fréchet Representation Theorem)**.** *Given any $\ell \in X'$, there exists a unique element $f \in X$ such that*

$$\langle \ell, v\rangle_{X'\times X} = (f, v)_X, \qquad \forall v \in X, \tag{2.5.1}$$

and moreover

$$\|\ell\|_{X'} = \|f\|_X.$$

This theorem shows that every continuous linear functional on X can be represented uniquely through the inner product. The mapping

$$\ell \mapsto f$$

is an isometric isomorphism, which allows one to identify X with its dual space X'.

Theorem 2.5.5 (Schauder Fixed Point Theorem)**.** *Let X be a Banach space, let K be a nonempty, convex, and compact subset of X, and let*

$$F : K \to K$$

be a continuous mapping. Then F admits at least one fixed point.

In Schauder's theorem, the mapping F is not necessarily Lipschitz continuous. Consequently, the theorem guarantees the existence of a solution but not necessarily its uniqueness, which will be addressed in the following theorems.

Banach Fixed Point Theorem

Theorem 2.5.6 (Banach Fixed Point Theorem [**?**])**.** *Let K be a nonempty closed subset of a Banach space $(X, \|\cdot\|_X)$. Assume that*

$$\Phi : K \to K$$

is a contraction, that is, there exists a constant $\lambda \in [0,1)$ such that

$$\|\Phi(u) - \Phi(v)\|_X \leq \lambda \|u - v\|_X, \qquad \forall u, v \in K.$$

Then there exists a unique element $u \in K$ such that

$$\Phi(u) = u.$$

We shall also need the following version of Banach's fixed point theorem. For an operator Φ, its powers are defined inductively by

$$\Phi^m = \Phi \circ \Phi^{m-1}, \qquad m \geq 2.$$

Theorem 2.5.7 **([?]).** *Let K be a nonempty closed subset of a Banach space X, and let*

$$\Phi : K \to K.$$

Assume that Φ^m is a contraction for some positive integer m. Then Φ possesses a unique fixed point.

CHAPTER 3

MATHEMATICAL ANALYSIS OF A NONLINEAR FRACTIONAL VISCOELASTIC PROBLEM WITH CONTACT CONDITIONS

3.1 Mechanical Problem Statement

The nonlinear problem considered in this chapter consists in determining the pair (u, σ) satisfying the following problem:

Problem P^1**:** Find a displacement field $u : \Omega\times]0, T[\longrightarrow \mathbb{R}^d$ and a stress tensor $\sigma : \Omega\times]0, T[\longrightarrow \mathbb{S}^d$ such that

$$\sigma(t) = \mathcal{A}\,\varepsilon\big({}_0^C D_t^\alpha u(t)\big) + \mathcal{F}\,\varepsilon(u(t)) \qquad \text{in } \Omega \times (0, T), \tag{3.1.1}$$

$$\rho\,{}_0^C D_t^{2\alpha} u(t) - \operatorname{Div}\sigma(t) = f_0(t) \qquad \text{in } \Omega \times (0, T), \tag{3.1.2}$$

$$u = 0 \qquad \text{on } \Gamma_D \times (0, T), \tag{3.1.3}$$

$$\sigma(t)\nu = f_1(t) \qquad \text{on } \Gamma_N \times (0, T), \tag{3.1.4}$$

$$u(0, x) = u_0, \qquad {}_0^C D_t^\alpha u(0, x) = w_0 \qquad \text{in } \Omega, \tag{3.1.5}$$

$$-\sigma_\nu = p_\nu({}_0^C D_t^\alpha u_\nu(t)), \qquad \sigma_\tau = 0 \qquad \text{on } \Gamma_C \times (0, T). \tag{3.1.6}$$

Equation (3.2.1) represents the time-fractional Kelvin–Voigt viscoelastic constitutive law of Caputo type, as described in the literature. In this equation, $\mathcal{A} = (a_{ijkl})$ and $\mathcal{F} = (f_{ijkl})$ denote the fourth-order viscosity tensor and elasticity tensor, respectively. Relation (3.2.2) corresponds to the fractional equation of motion, where ρ represents the mass density of the material. Here and throughout the paper, Div denotes the divergence operator acting on tensor fields, namely, $\operatorname{Div}\sigma = (\sigma_{ij,j})$. Moreover, relations (3.2.2)-(3.2.4) describe the mechanical boundary conditions imposed on the body. The initial conditions are given by equation ((3.2.5). Finally, equation (3.2.6) represents the frictionless contact condition with a Caputo-type time-fractional Normal Damped Response, in which p_ν is a prescribed function. For positive values of p_ν, the quantity

u_ν describes the penetration of the surface asperities into the foundation.

Assumptions

For the analysis of this problem, we consider the following assumptions. We assume that the linear operator $(\mathcal{G} = \mathcal{A}, \mathcal{F}) : \Omega \times S_d \longrightarrow S_d$ satisfies :

$$\begin{cases} \text{(a) The usual property of symmetry } \mathcal{G}_{ijlk} = \mathcal{G}_{jilk} = \mathcal{G}_{lkji} \in L^\infty(\Omega). \\ \text{(b) There exists a constant } L_\mathcal{G} > 0 \text{ such that} \\ \qquad |\mathcal{G}(x,\varepsilon_1) - \mathcal{G}(x,\varepsilon_2)| \le L_\mathcal{G}|\varepsilon_1 - \varepsilon_2|, \quad \forall\, \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Gamma_3. \\ \text{(c) There exists a constant } m_\mathcal{G} > 0 \text{ such that} \\ \qquad \big(\mathcal{G}(x,\varepsilon_1) - \mathcal{G}(x,\varepsilon_2)\big)(\varepsilon_1 - \varepsilon_2) \ge m_\mathcal{G}|\varepsilon_1 - \varepsilon_2|^2, \quad \forall\, \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Gamma_3. \\ \text{(d)} \qquad \mathcal{G}(x,0) = 0, \quad \text{a.e. } x \in \Gamma_3. \\ \text{(e) The mapping } x \mapsto \mathcal{G}(x,0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.1.7}$$

The Normal Damped Response function $p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies the following assumptions:

$$\begin{cases} \text{(a) There exists a constant } L > 0 \text{ such that} \\ \qquad |p(x,u_1) - p(x,u_2)| \le L|u_1 - u_2|, \quad \forall\, u_1, u_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3. \\ \text{(b) For every } u \in \mathbb{R}, \text{ the mapping } x \mapsto p(x,u) \text{ is measurable on } \Gamma_3. \\ \text{(c)} \qquad \big(p(x,u_1) - p(x,u_2)\big)(u_1 - u_2) \ge 0, \\ \qquad \forall\, u_1, u_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3. \\ \text{(d)} \qquad p(x,u) = 0, \quad \forall\, u \le 0, \text{ a.e. } x \in \Gamma_3. \\ \text{(e) The mapping } x \mapsto p(x,0) \text{ belongs to } L^2(\Gamma_3). \end{cases} \tag{3.1.8}$$

The mass density satisfies

$$\rho \in L^\infty(\Omega), \qquad \text{and there exists } \rho^* > 0 \text{ such that } \rho(x) \ge \rho^* \quad \text{a.e. } x \in \Omega. \tag{3.1.9}$$

We also assume that the body and surface forces satisfy

$$f_0 \in L^2(0,T;H), \qquad f_2 \in L^2(0,T;L^2(\Gamma_2)^d). \tag{3.1.10}$$

The boundary and initial conditions satisfy

$$u_0 \in V, \qquad v_0 \in V, \tag{3.1.11}$$

We define the vector-valued function

$$f : [0,T] \to V'$$

by

$$\langle f(t), v\rangle_{V'\times V} = \int_\Omega f_0(t)\cdot v\,dx + \int_{\Gamma_2} f_2(t)\cdot v\,da, \qquad \forall v\in V, \tag{3.1.12}$$

and we note that

$$f \in L^2(0,T;V'). \tag{3.1.13}$$

Next, we define the functional

$$j : V\times V \to \mathbb{R}$$

by

$$j(u,v) = \int_{\Gamma_3} p_\nu(u_\nu)\, v_\nu\, da, \qquad \forall u, v\in V. \tag{3.1.14}$$

Let $V \subset H \subset V'$ be an evolution triple of spaces, where V is a reflexive and separable Banach space, H is a separable Hilbert space, and the embedding $V \hookrightarrow H$ is dense and continuous. The embedding operator between V and H is denoted by ι, and it is assumed to be compact. The dual space of V is denoted by V', and the adjoint operator $\iota^* : H \to V^*$ of ι is also linear, continuous, and compact.

3.2 Weak formulation

In this section, we present the Weak formulation of Problem P^1.

Using the following Green formula

$$(\mathrm{Div}\,\sigma, v)_H + (\sigma(t), \varepsilon(v))_H = \int_\Gamma \sigma\nu\cdot v\,da, \qquad \forall v\in H^1(\Omega)^d,\ \text{a.e. } t\in(0,T),$$

and since

$$\int_\Gamma \sigma\nu\cdot v\,da = \int_{\Gamma_1}\sigma\nu\cdot v\,da + \int_{\Gamma_2}\sigma\nu\cdot v\,da + \int_{\Gamma_3}\sigma\nu\cdot v\,da, \qquad \forall v\in V, \tag{3.2.1}$$

then, using the definition of the space V together with (3.2.1), and (3.1.12), we obtain

$$(\rho\, {}_0^C D_t^{2\alpha} u(t), v)_H + (\sigma(t), \varepsilon(v))_H = \int_\Omega f_0\cdot v\,dx + \int_{\Gamma_2} f_2\cdot v\,da + \int_{\Gamma_3}\sigma\nu\cdot v\,da,$$

for all $v\in V$, and for a.e. $t\in(0,T)$.

Since

$$\sigma\nu\cdot v = \sigma_\nu v_\nu + \sigma_\tau\cdot v_\tau = -p_\nu({}_0^C D_t^\alpha u_\nu(t))v_\nu,$$

it follows that

$$(\ddot{u}, v)_H + (\sigma(t), \varepsilon(v))_H = \int_\Omega f_0 \cdot v\, dx + \int_{\Gamma_2} f_2 \cdot v\, da - \int_{\Gamma_3} p_\nu({}_0^C D_t^\alpha u_\nu(t)) v_\nu da, \tag{\textbf{3}.2.2}$$

for all $v \in V$, and for a.e. $t \in (0, T)$.

From (2.1.20), (2.1.21), and (2.1.23), we obtain

$$\left({}_0^C D_t^{2\alpha} u(t), v\right)_V (\sigma(t), (v))_{\mathcal{H}} + j({}_0^C D_t^\alpha u(t), v) = (f(t), v)_V,$$

for all $v \in V$, and for a.e. $t \in (0, T)$.

From (2.1.1), (2.1.8), (2.2.1), and (2.2.2), we obtain the following frac tional formulation of of the mechanical problem $P1$.

Problem PV^1**.** Find the displacement field $u : [0, T] \to V$, and the stress tensor field $\sigma : [0, T] \to H_1$, for $v \in V$ and $\alpha \in (0, 1]$, such that

$$\sigma_\eta(t) = \mathcal{A}\varepsilon\left({}_0^C D_t^\alpha u_\eta(t)\right) \mathcal{F}\varepsilon(u_\eta(t)), , \tag{3.7}$$

$$\left({}_0^C D_t^{2\alpha} u_\eta(t), v\right)_V + \left(\mathcal{A}\varepsilon\left({}_0^C D_t^\alpha u_\eta(t)\right), \varepsilon(v)\right) + \left(\mathcal{F}\varepsilon(u(t)), \varepsilon(v)\right) + j({}_0^C D_t^\alpha u(t), v) = (f(t), v)_V, \tag{3.8}$$

$$u(0, x) = u_0, \qquad {}_0^C D_t^\alpha u(0, x) = w_0. \tag{3.9}$$

3.3 Main Result

Our principal existence theorem, proved in the next section, is formulated as follows:

Theorem 3.1. Assuming that assumptions (**3**.1.7) to (**3**.1.14) hold, Then there exists at least one solution u, σ to the problem P_V^1 with the regularity

$$u \in W^{2,2}(0, T; V), \tag{\textbf{3}.3.1}$$

$$\sigma \in L^2(0,T;L^2(\Omega,S^d)) \tag{3.3.2}$$

A pair of functions (u,σ) satisfying (**3**.1.1), and (**3**.3.1)-(**3**.3.2) is called a weak solution of the viscoelastic contact problem P^1.

The proof of **Theorem 3.1** is carried out in several steps. It is based on the abstract of monotone operator, Caputo derivative, Galerkin method and Banach fixed point theorem.

In the first step, we consider the following auxiliary problem for the displacement field, in which $\eta \in L^2(0,T;V)$ is given.

Problem $P^1_{u\eta}$**.** Find the displacement field $u_\eta : [0,T] \to V$ and the stress tensor field $\sigma_\eta : [0,T] \to$

H_1, for $v \in V$ and $\alpha \in (0,1]$, such that

$$\sigma_\eta(t) = \mathcal{A}\varepsilon\big({}_0^C D_t^\alpha u_\eta(t)\big) + \mathcal{F}\varepsilon(u_\eta(t)), \tag{3.3.3}$$

$$\left({}_0^C D_t^{2\alpha} u_\eta(t), v\right)_V + \lambda\left(\mathcal{A}\varepsilon\big({}_0^C D_t^\alpha u_\eta(t)\big), \varepsilon(v)\right) + (\eta(t), v) = (f(t), v)_V, \tag{3.3.4}$$

$$u_\eta(0) = u_0, \qquad {}_0^C D_t^\alpha u_\eta(0) = v_0. \tag{3.3.5}$$

We define $(f_\eta(t) \in V$ for a.e. $t \in (0,T)$ by

$$\langle f_\eta(t), v\rangle_V = \langle f(t) - \eta(t), v\rangle_V, \qquad \forall v \in V. \tag{3.3.6}$$

Using the Riesz-Frechet representation theorem, we define the operator $(G = A, F) : V \to V'$ by

$$\langle Gu, v\rangle_V = (\mathcal{G}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \qquad \forall u, v \in V. \tag{3.3.7}$$

Introducing the variable ${}_0^C D_t^\alpha u_\eta(t) = w_\eta(t).$, we obtain $u_\eta(t) ={}_0 I_t^\alpha w_\eta(t) + u_0$, the problem $P^1_{u_\eta}$ can be rewritten, for a.e. $t \in (0,T)$, as follows:

Problem $P^1_{w_\eta}$**.** Find $w_\eta : [0,T] \to V$ such that

$$\sigma_{w\eta}(t) = \mathcal{A}\varepsilon(w_\eta(t)) + \mathcal{F}\varepsilon({}_0I_t^\alpha w_\eta(t) + u_0), \tag{3.3.8}$$

$$\left({}_0^C D_t^\alpha w_\eta(t), v\right)_V + (Aw_\eta(t), v) = (f_\eta(t), v)_V, \tag{3.3.9}$$

$$w_\eta(0) = v_0. \tag{3.3.10}$$

To solve Problem $P^1_{w_\eta}$, we prove the following theorem.

Lemma 3.3.1. *For all $v \in V$ and $t \in (0,T)$, Problem $P^1_{w_\eta}$ admits at least one solution*

$$w_\eta \in W^{1,2}(0,T;V) \textit{ and } \sigma_{w\eta} \in L^2\big(0,T;L^2(\Omega;S^d)\big).$$

Proof. The proof is carried out in four steps:

1. **Approximate solution.** We construct approximate solutions using the Faedo–Galerkin method.

2. **A priori estimates.** We establish a priori estimates (upper bounds) for the approximate solutions.

3. **Passage to the limit.** We pass to the limit by exploiting compactness properties in the nonlinear terms and the lower semicontinuity properties of an appropriate operator.

4. **Verification of the initial conditions.** We verify that the obtained solution satisfies the prescribed initial conditions.

a) Step 1: Approximate Solution

We introduce a sequence (φ_n) of functions satisfying the following properties:

- $\varphi_j \in H^1(\Omega)$, for all j;
- The family $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ is a Hilbertian basis of $H^1(\Omega)$.
- The space $V_m = \mathrm{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ generated by the family $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ is dense in $H^1(\Omega)$.

For each $m \in \mathbb{N}^*$, we consider the following approximate problem, denoted by $P^1_{w_{\eta m}}$: find $w_m = w_{\eta m}(t) \in V_m$ of the form

$$w_m(t) = \sum_{j=1}^{m} K_{jm}(t)\varphi_j, \tag{3.3.11}$$

Approximate Problem

Problem $P^1_{w_{\eta m}}$. Find $w_m \in L^2(0,T;V_m)$ such that ${}^C_0D^\alpha_t w_m \in L^2(0,T;V_m)$, and

$$\big({}^C_0D^\alpha_t w_m(t), \varphi_j\big)_V + \lambda\big(Aw_m(t), \varphi_j\big) = \big(f_\eta(t), \varphi_j\big)_V, \qquad j = 1, \dots, m. \tag{3.3.12}$$

$$w_{0m} = \sum_{j=1}^{m} \varphi_{jm} K_j m \longrightarrow v_0 \quad \text{in } V, \qquad \text{as } m \to \infty. \tag{3.3.13}$$

Inserting the expansion (**3**.3.11) into (**3**.3.9), we arrive at the following system of ordinary differential equations: Find $K_m(t) = (K_{1m}(t), \ldots, K_{mm}(t)) \in H^1(\Omega)^d$ such that

$$\sum_{j=1}^{n} {}_0^C D_t^\alpha K_{jm}(t)\,(\varphi_j, \varphi_i) + \left(A \sum_{j=1}^{n} K_{jm}(t)\,(\varphi_j, \varphi_i) \right) = (f_\eta(t), \varphi_i), \tag{3.3.14}$$

Then, we obtain

$${}_0^C D_t^\alpha K_{im}(t) + \mathcal{A} K_{im}(t) = f_{i\eta}(t), \tag{3.3.15}$$

Then, (**3**.3.15) can be written as follows

$${}_0^C D_t^\alpha K_{im}(t) = g(t, K_{im}(t)), \tag{3.3.16}$$

where $g(t, K_{im}(t)) = f_{i\eta}(t) - \mathcal{A} K_{im}(t)$.

Therefore, assumption (**3**.1.7) ensures that the function h satisfies a Lipschitz condition, namely

$$\Big| g(t, K_{im_1}(t))\big) - g(t, K_{im_2}(t))\big) \Big| \leq L_{\mathcal{A}} \left| K_{im_1}(t) - K_{im_2}(t) \right|.$$

An application of **Proposition 2.1.6** ensures there exists a unique continuous solution $K_m(t) = (K_{1m}(t), \ldots, K_{mm}(t))$ on $[0, T)$ satisfying equation (**3**.3.16). Throughout this work, c_1 and c_2 denote positive generic constants whose values may change from one line to another.

Step 2: Estimates on $(u_m)_m$

We multiply equation (**3**.3.12) by $K_{jm}(t)$ and sum over $j = 1, \ldots, m$. We obtain:

$$\big({}_0^C D_t^\alpha w_m(t), w_m(t)\big)_V + \lambda\big(A w_m(t), w_m(t)\big) = \big(f_\eta(t), w_m(t)\big)_V. \tag{3.3.17}$$

We can verify that for all $a \geq 0$, $b \geq 0$ such that $a + b = 1$, and for all $x, y \in \mathbb{R}$, one has

$$(ax + by)^2 \leq ax^2 + by^2.$$

Hence, since the functional

$$w_m \mapsto \frac{1}{2} \|w_m\|_V^2$$

is convex, the convexity inequality for the Caputo fractional derivative (Proposition 2.1.5(i)) implies that

$${}_0^C D_t^\alpha \left(\frac{1}{2} \|w_m(t)\|_V^2 \right) \leq \big({}_0^C D_t^\alpha w_m(t), w_m(t)\big)_V.$$

Therefore,

$${}_0^C D_t^\alpha \left(\frac{1}{2} \|w_m(t)\|_V^2 \right) + (A w_m(t), w_m(t)) \leq (f_\eta(t), w_m(t))_V. \tag{3.3.18}$$

By assumption (**3**.1.7),Taking $v = w_m(t)$, we obtain

$$m_{\mathcal{A}}\|w_m(t)\|_V^2 \leq (Aw_m(t), w_m(t)). \tag{\textbf{3}.3.19}$$

Using the Cauchy–Schwarz inequality,

$$|(f_\eta(t), w_m(t))_V| \leq \|f_\eta(t)\|_V \|w_m(t)\|_V.$$

Applying Young's inequality,

$$ab \leq \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2,$$

we obtain

$$|(f_\eta(t), w_m(t))_V| \leq \frac{1}{2\varepsilon}\|f_\eta(t)\|_V^2 + \frac{\varepsilon}{2}\|w_m(t)\|_V^2.$$

Choosing $\varepsilon > 0$ appropriately, there exists a positive constant c such that

$$|(f_\eta(t), w_m(t))_V| \leq c\|f_\eta(t)\|_V^2 + c\|w_m(t)\|_V^2. \tag{\textbf{3}.3.20}$$

Substituting (**3**.3.19) and (**3**.3.20) into (**3**.3.18), we obtain

$${}_0^C D_t^\alpha \left(\frac{1}{2}\|w_m(t)\|_V^2\right) + m_{\mathcal{A}}\|w_m(t)\|_V^2 \leq \frac{1}{2\varepsilon}\|f_\eta(t)\|_V^2 + \frac{\varepsilon}{2}\|w_m(t)\|_V^2. \tag{\textbf{3}.3.21}$$

Hence,

$${}_0^C D_t^\alpha \left(\frac{1}{2}\|w_m(t)\|_V^2\right) + \left(m_{\mathcal{A}} - \frac{\varepsilon}{2}\right)\|w_m(t)\|_V^2 \leq \frac{1}{2\varepsilon}\|f_z(t)\|_V^2.$$

Let

$$c_1 = m_{\mathcal{A}} - \frac{\varepsilon}{2} > 0, \qquad c_2 = \frac{1}{2\varepsilon}.$$

Then

$${}_0^C D_t^\alpha \left(\frac{1}{2}\|w_m(t)\|_V^2\right) + c_1\|w_m(t)\|_V^2 \leq c_2\|f_\eta(t)\|_V^2. \tag{\textbf{3}.3.22}$$

Applying Proposition **2.1.5** (ii) to $y(t) = \frac{1}{2}\|w_m(t)\|_V^2$, we deduce

$$\|w_m(t)\|_V^2 + \frac{2c_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|w_m(s)\|_V^2\,ds \leq c_2\|f_\eta(t)\|_V^2 + \|w_0\|_V^2 . tag\mathbf{3}.3.23$$

This estimate is independent of m, which implies that $\|w_m(t)\|_V$ remains bounded on every finite time interval. Consequently, no blow-up can occur and the maximal existence time satisfies $T^* = +\infty$.

Let $v \in V$ with $\|v\|_V \leq 1$. Decompose $v = v_1 + v_2$, where

$$v_1 \in \operatorname{span}\{\beta_1, \ldots, \beta_m\},$$

and $(v_2, \beta_k)_V = 0, \qquad k = 1, \dots, m$. Since $\{\beta_k\}_{k=1}^m$ is an orthogonal family in V,

$$\|v_1\|_V \leq \|v\|_V \leq 1. \tag{3.3.24}$$

Using (**3**.3.12), we obtain

$$\left({}_0^C D_t^\alpha w_(t), v_1\right)_V + (Aw_m(t), v_1) = (f_\eta(t), v_1)_V. \tag{3.3.25}$$

The continuity of A implies

$$|(Aw_m(t), v_1)| \leq M_A \|w_m(t)\|_V \|v_1\|_V \leq M_A \|w_m(t)\|_V,$$

while

$$|(f_\eta(t), v_1)_V| \leq \|f_\eta(t)\|_V \|v_1\|_V \leq \|f_\eta(t)\|_V.$$

Therefore,

$$\left|\left({}_0^C D_t^\alpha w_m(t), v_1\right)_V\right| \leq c_1 + c_2 \left(\|w_m(t)\|_V + \|f_\eta(t)\|_V\right).$$

Taking the supremum over all $v \in V$ such that $\|v\|_V \leq 1$, we obtain

$$\|{}_0^C D_t^\alpha w_m(t)\|_{V^*} \leq c_1 + c_2 \left(\|w_m(t)\|_V + \|f_\eta(t)\|_V\right). \tag{3.3.26}$$

Finally, estimate (**3**.3.23) shows that the sequence $\{w_m\}$ is bounded in $L^2(0,T;V)$. Combining this fact with (**??**), we conclude that there exists a positive constant c, independent of m, such that

$$\left\|{}_0^C D_t^\alpha w_m\right\|_{L^2(0,T;V^*)} \leq c. \tag{3.3.27}$$

Hence,

$$\{w_m\} \quad \text{is bounded in} \quad L^2(0,T;V),$$

and

$$\left\{{}_0^C D_t^\alpha w_m\right\} \quad \text{is bounded in} \quad L^2(0,T;V^*).L^2(0,T;V),$$

Passage to the Limit

By virtue of the a priori estimates established above and the compactness result stated in Theorem **5.1.18.** for the Caputo fractional derivative, there exist a subsequence, still denoted by $\{w_{m_\ell \eta}\}$, and a function $w_\eta \in L^2(0,T;V)$ such that

$$w_{m_\ell \eta} \longrightarrow w_\eta \qquad \text{strongly in } L^2(0,T;V), \tag{3.3.28}$$

and

$$ {}_0^C D_t^\alpha w_{m_\ell \eta} \rightharpoonup {}_0^C D_t^\alpha w_\eta \qquad \text{weakly in } L^2(0,T;V^*). \tag{3.3.29}$$

The strong convergence (**3**.3.27) and the continuity of the bilinear form b imply that

$$ \big(A w_{m_\ell \eta}(t), v\big) \longrightarrow \big(A w_\eta(t), v\big) \qquad \text{in } \mathbb{R}, \tag{3.3.30}$$

for every $v \in V$.

Furthermore, the weak convergence (**3**.3.28) yields

$$ \big\langle {}_0^C D_t^\alpha w_{m_\ell \eta}, v \big\rangle_{V^* \times V} \longrightarrow \big\langle {}_0^C D_t^\alpha w_\eta, v \big\rangle_{V' \times V} \qquad \text{in } \mathbb{R}, \tag{3.3.31}$$

for every $v \in V$.

Moreover, since the fractional integral operator

$$ {}_0I_t^\alpha : L^2(0,T;V) \longrightarrow L^2(0,T;V) $$

is linear and continuous, it follows from (**3**.3.27) that

$$ {}_0I_t^\alpha w_{m_\ell \eta} \rightharpoonup {}_0I_t^\alpha w_\eta \eta \qquad \text{weakly in } L^2(0,T;V). $$

Passing to the limit in the Galerkin formulation, we conclude that w_η satisfies the variational problem associated with equation (**3**.3.4). Therefore, there exists a weak solution $w_\eta \in W^{1,2}(0,T;V)$ of equation (**3**.3.4). Next, using relation (**3**.3.8) together with assumptions (**3**.1.7), we deduce that

$$ \sigma_{w_\eta} \in L^2\big(0,T;L^2(\Omega;S^d)\big). $$

This completes the proof of Lemma **3.3.1.** $\square$

Final step

, for a given $\eta \in L^2(0,T;V)$, we consider the solution w_η obtained in Lemma **3.3.1** We then introduce the operator $\Lambda : L^2(0,T;V) \longrightarrow L^2(0,T;V)$, defined by

$$ (\Lambda\eta(t), v)_V := F({}_0I_t^\alpha w_\eta(t) + u_0, v) + j(w_\eta(t), v)\,, \qquad \forall v \in V. \tag{3.3.32}$$

We obtain the following result.

Lemma3.3.2. For all $\eta \in L^2(0,T;V)$ and $\alpha \in (0,1]$, the operator $\Lambda z : (0,T) \to V$ is continuous. Moreover, there exists a unique element $\eta^* \in L^2(0,T;V)$ such that

$$\Lambda\eta^* = \eta^*.$$

Let $\eta \in L^2(0,T;V)$ and $t_1, t_2 \in (0,T)$. From (**3**.3.32), we deduce

$$(\Lambda\eta(t_1) - \Lambda\eta(t_2), v) = F\big({}_0I_t^\alpha w_\eta(t_1) - {}_0I_t^\alpha w_\eta(t_2), v\big) + j\big(w_\eta(t_1), v\big) - j\big(w_\eta(t_2), v\big). \tag{3.3.33}$$

By applying Definition **2.1.3,** and assumption (**3**.1.7), we obtain

$$\big|F\big({}_0I_t^\alpha w_\eta(t_1) - {}_0I_t^\alpha w_\eta(t_2), v\big)\big| \leq \frac{L_{\mathcal{F}}T^\alpha}{\Gamma(\alpha+1)}\|w_\eta(t_1) - w_\eta(t_2)\|_V\|v\|_V. \tag{3.3.34}$$

Based on (**1**.4.6), (**3**.1.14), and hypothesis (**3**.1.8), we derive

$$\big|j\big(w_\eta(t_1), v\big) - j\big(w_\eta(t_2), v\big)\big| \leq \frac{L_\nu C_0 T^\alpha}{\Gamma(\alpha+1)}\|w_\eta(t_1) - w_\eta(t_2)\|_V\|v\|_V. \tag{3.3.35}$$

By combining the previous inequalities, we conclude that there exists a positive constant c, depending on M_F, L_ν, C_0, α, and T, such that

$$\|\Lambda\eta(t_1) - \Lambda\eta(t_2)\|_V \leq c\|w_\eta(t_1) - w_\eta(t_2)\|_V. \tag{3.3.36}$$

Using the regularity of w_η, we deduce that $\Lambda\eta \in L^2(0,T;V)$.
Now, let $\eta_1, \eta_2 \in L^2(0,T;V)$ and $t \in [0,T]$. Similarly, we obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq c\|w_{\eta_1}(t) - w_{\eta_2}(t)\|_V. \tag{3.3.37}$$

Starting from the weak formulation associated with (**3**.3.9), we have for $i = 1, 2$,

$$\big({}_0^CD_t^\alpha w_{\eta_i}(t), v\big)_V + (Aw_{\eta_i}(t), v) + (\eta_i(t), v)_V = 0, \qquad \forall v \in V.$$

Subtracting the two relations corresponding to η_1 and η_2 yields

$$\big({}_0^CD_t^\alpha w_{\eta_1}(t) - {}_0^CD_t^\alpha w_{\eta_2}(t), v\big)_V + (Aw_{\eta_1}(t) - Aw_{\eta_2}(t), v) + (\eta_1(t) - \eta_2(t), v)_V = 0.$$

Choosing the test function $v = w_{\eta_1}(t) - w_{\eta_2}(t)$, we obtain

$$\left({}_0^C D_t^\alpha w_{\eta_1}(t) - {}_0^C D_t^\alpha w_{\eta_2}(t), w_{\eta_1}(t) - w_{\eta_2}(t)\right)_V$$

$$(Aw_{\eta_1}(t) - Aw_{\eta_2}(t), w_{\eta_1}(t) - w_{\eta_2}(t))$$

$$+(\eta_1(t) - \eta_2(t), w_{\eta_1}(t) - w_{\eta_2}(t))_V = 0. \qquad (\mathbf{3}.3.38)$$

On the other hand, by Definition (**2**.1.3) of the Caputo fractional derivative, we have the estimate

$$\left\|{}_0^C D_t^\alpha w_{\eta_1}(t) - {}_0^C D_t^\alpha w_{\eta_2}(t)\right\|_V \le \frac{T^{1-\alpha}}{\Gamma(\alpha)} \left\|\dot{w}_{\eta_1}(t) - \dot{w}_{\eta_2}(t)\right\|_V. \qquad (\mathbf{3}.3.39)$$

Moreover, assumption (**3**.1.7)(ii) ensures the coercivity property

$$F(v, v) \ge m_{\mathcal{F}}\|v\|_V^2, \qquad \forall v \in V,$$

Combining this property with relations (**3**.3.38)and (**3**.3.39), we obtain

$$m_F\|w_{\eta_1}(t) - w_{\eta_2}(t)\|_V^2 \le \frac{T^{1-\alpha}}{\Gamma(\alpha)}\|\dot{w}_{\eta_1}(t) - \dot{w}_{\eta_2}(t)\|_V\|w_{\eta_1}(t) - w_{\eta_2}(t)\|_V$$

$$+\|\eta_1(t) - \eta_2(t)\|_V\|w_{\eta_1}(t) - w_{\eta_2}(t)\|_V. \qquad (\mathbf{3}.3.40)$$

Since

$$w_{\eta_i}(t) = \int_0^t \dot{w}_{\eta_i}(s)\, ds + w_{\eta_i}(0), \qquad i = 1, 2,$$

and because $w_{\eta_1}(0) = w_{\eta_2}(0)$, it follows that

$$w_{\eta_1}(t) - w_{\eta_2}(t) = \int_0^t \left(\dot{w}_{\eta_1}(s) - \dot{w}_{\eta_2}(s)\right) ds.$$

Taking the norm in V, we deduce

$$\|w_{\eta_1}(t) - w_{\eta_2}(t)\|_V \le \int_0^t \|\dot{w}_{\eta_1}(s) - \dot{w}_{\eta_2}(s)\|_V\, ds. \qquad (\mathbf{3}.3.41)$$

Integrating inequality (**3**.3.40) over the interval $[0, t]$, and using relation (**3**.3.41), we obtain

$$m_{\mathcal{F}}\int_0^t \|w_{\eta_1}(s) - w_{\eta_2}(s)\|_V^2\, ds \le \frac{T^{1-\alpha}}{\Gamma(\alpha)}\|w_{\eta_1}(t) - w_{\eta_2}(t)\|_V^2$$

$$+\int_0^t \|\eta_1(s) - \eta_2(s)\|_V\|w_{\eta_1}(s) - w_{\eta_2}(s)\|_V\, ds. \qquad (\mathbf{3}.3.42)$$

Next, we use Young's inequality $xy \le \gamma x^2 + \frac{1}{4\gamma}y^2$, with $x = \|w_{\eta_1}(s) - w_{\eta_2}(s)\|_V, y = \|\eta_1(s) - \eta_2(s)\|_V$, in (4.41) and then Gronwall's lemma, we conclude that

$$\|w_{\eta_1}(t) - w_{\eta_2}(t)\|_V^2 \le c\int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2\, ds, \qquad (\mathbf{3}.3.43)$$

for some positive constant c. Combining estimate (**3**.3.43) with relation (**3**.3.37) gives

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V^2 \le c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 \, ds. \tag{3.3.44}$$

Iterating the previous inequality n times, we obtain

$$\|\Lambda^n\eta_1 - \Lambda^n\eta_2\|^2_{L^2(0,T;V)} \le \frac{(cT)^n}{n!} \|\eta_1 - \eta_2\|^2_{L^2(0,T;V)}. \tag{3.3.45}$$

Since $\frac{(cT)^n}{n!} \longrightarrow 0$ as $n \to \infty$, there exists an integer n sufficiently large such that $\frac{(cT)^n}{n!} < 1$. Therefore, the operator Λ^n is a contraction on the Banach space $L^2(0,T;V)$. Consequently, Banach's fixed point theorem guarantees the existence of a unique fixed point $\eta^* \in L^2(0,T;V)$ such that $\Lambda\eta^* = \eta^*$. We now possess all the necessary ingredients to establish Theorem 3.1.

Proof of Theorem 2.3.1

We are now in a position to complete the proof of the main existence result. Let $\eta^* \in L^2(0,T;V)$ be the unique fixed point of the operator $\Lambda : L^2(0,T;V) \longrightarrow L^2(0,T;V)$, that is, $\Lambda(\eta^*) = \eta^*$. According to Lemma 3.3.1, for every $\eta \in L^2(0,T;V)$, there exists a unique function $w_\eta \in W^{1,2}(0,T;V)$ satisfying the auxiliary variational problem

$${}_0^C D_t^\alpha w_\eta(t) + \mathcal{F} w_\eta(t) + \eta(t) = f(t).$$

Therefore, corresponding to the fixed point η^*, there exists a function $w^* := w_{\eta^*}$ such that $w^* \in W^{1,2}(0,T;V)$. Next, we define the stress tensor by

$$\sigma^*(t) = \mathcal{F}\varepsilon(w^*(t)) + \mathcal{A}\varepsilon({}_0I_t^\alpha w^*(t) + u_0)\,.$$

Since the operators $\mathcal{A}$ and $\mathcal{F}$ are bounded linear operators, and since $w^* \in W^{1,2}(0,T;V)$, it follows from the regularity assumptions and the definition of the strain tensor that $\sigma^* \in L^2\big(0,T;L^2(\Omega;S^d)\big)$. On the other hand, using the definition of the fractional integral, we introduce the displacement field

$$u^*(t) = {}_0I_t^\alpha w^*(t) + u_0, \qquad \text{for a.e. } t \in (0,T).$$

By Proposition 2.1.5(iii), the Caputo fractional derivative satisfies

$${}_0^C D_t^\alpha u^*(t) = w^*(t).$$

Since $w^* \in W^{1,2}(0,T;V)$; we deduce that $u^* \in W^{2,2}(0,T;V)$. Moreover, because η^* is a fixed point of the operator Λ, relation (**3**.3.32) gives

$$(\eta^*(t), v)_V = a(u^*(t), v) + j({}_0^C D_t^\alpha u^*(t), v), \qquad \forall v \in V.$$

Substituting this identity into the auxiliary variational equation satisfied by w^*, we obtain

$$\left({}_0^C D_t^\alpha w^*(t), v\right)_V + b(w^*(t), v) + a(u^*(t), v) + j({}_0^C D_t^\alpha u^*(t), v) = (f(t), v)_V,$$

for all $v \in V$. Recalling that

$$w^*(t) = {}_0^C D_t^\alpha u^*(t),$$

we finally derive

$$\left({}_0^C D_t^{2\alpha} u^*(t), v\right)_V + b\left({}_0^C D_t^\alpha u^*(t), v\right) + a(u^*(t), v) + j({}_0^C D_t^\alpha u^*(t), v) = (f(t), v)_V,$$

which is precisely the weak formulation of Problem PV^1.

Furthermore, the initial conditions are satisfied by construction:

$$u^*(0) = u_0, \qquad {}_0^C D_t^\alpha u^*(0) = w_0.$$

Consequently, the pair (u^*, σ^*) constitutes a weak solution of Problem PV^1.

Hence, Problem $PV^1()$ admits at least one solution satisfying

$$u^* \in W^{2,2}(0,T;V), \qquad \sigma^* \in L^2\left(0,T; L^2(\Omega; S^d)\right).$$

This completes the proof of Theorem 3.1. $\square$

Conclusion

This work establishes the existence of a weak solution to a fractional viscoelastic contact problem and lays the foundation for extending the analysis to more advanced physical and numerical models.

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ملخص : تناول هذا العمل دراسة مسألة تماس لزج مرن كسري غير خطية ناشئة في ميكانيكا الأوساط المتصلة. يعتمد النموذج على مشتقات كابوتو الكسرية لوصف تأثيرات الذاكرة والسلوك الوراثي للمواد. تجمع المعادلات الحاكمة بين قانون سلوك كسري زمني وشروط تماس من نوع الاستجابة العادية المخمدة. وقد تم بناء صياغة تغايرية دقيقة ضمن إطار دالي مناسب يعتمد على فضاءات سوبوليف والحساب الكسري. كما تم إثبات وجود حل ضعيف باستعمال طريقة فايدو–غالركين وتقنيات التراص ونظريات النقطة الثابتة. وتوفر النتائج المتحصل عليها أساسًا نظريًا لتطوير المحاكاة العددية ودراسة نماذج تماس كسرية أكثر تقدمًا.

الكلمات المفتاحية:
اللزوجة المرنة الكسرية، ميكانيكا التماس، مشتقة كابوتو الكسرية، الحل الضعيف، الصياغة التغايرية، طريقة فايدو–غالركين، نظرية النقطة الثابتة، المعادلات التفاضلية الجزئية الكسرية

Abstract : This work studies a nonlinear fractional viscoelastic contact problem arising in continuum mechanics. The model incorporates Caputo fractional derivatives to describe memory effects and hereditary material behavior. The governing equations combine a time-fractional constitutive law with contact boundary conditions of normal damped response type. A rigorous variational formulation is established within an appropriate functional framework based on Sobolev spaces and fractional calculus. The existence of a weak solution is proved using the Faedo–Galerkin approximation method, compactness techniques, and fixed-point arguments. The obtained results provide a theoretical basis for future numerical simulations and more advanced fractional contact models.

Keywords

Fractional Viscoelasticity; Contact Mechanics; Caputo Fractional Derivative; Weak Solution; Variational Formulation; Faedo–Galerkin Method; Fixed Point Theory; Fractional Partial Differential Equations..

Résumé : Ce travail porte sur l'étude d'un problème de contact viscoélastique fractionnaire non linéaire issu de la mécanique des milieux continus. Le modèle utilise les dérivées fractionnaires de Caputo afin de décrire les effets de mémoire et le comportement héréditaire du matériau. Les équations gouvernantes combinent une loi de comportement fractionnaire en temps avec des conditions de contact de type réponse normale amortie. Une formulation variationnelle rigoureuse est établie dans un cadre fonctionnel approprié fondé sur les espaces de Sobolev et le calcul fractionnaire. L'existence d'une solution faible est démontrée à l'aide de la méthode de Faedo–Galerkin, des arguments de compacité et des théorèmes du point fixe. Les résultats obtenus fournissent une base théorique solide pour le développement de simulations numériques et l'étude de modèles fractionnaires de contact plus avancés.

Mots-clés :
Viscoélasticité fractionnaire, mécanique du contact, dérivée fractionnaire de Caputo, solution

faible, formulation variationnelle, méthode de Faedo–Galerkin, théorème du point fixe, équations aux dérivées partielles fractionnaires.