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A Tensor representations of Cohen p-nuclear two- Lipschitz operator ideals

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Dedication

I dedicate this humble work to those whose names I carry with pride and whose sacrifices and unwavering support have accompanied me throughout my journey.

To my beloved mother, whose patience, devotion, and endless sacrifices made this achievement possible. May Allah bless her with health, happiness, and abundant reward.

To the cherished memory of my late father. May Allah envelop him in His mercy and grant him the highest place in Paradise.

To my dear children — **Ilyas, Maryam, Mohamed, and Yahya** — who were my source of joy, strength, and motivation, and who shared with me every moment of this journey, with all its challenges and successes.

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Contents

Introduction

The theory of operator ideals constitutes one of the most important branches of modern Functional Analysis. Since the pioneering works of Grothendieck, Pietsch and Cohen, operator ideals have provided a unified framework for studying linear operators through factorization techniques, tensor products, duality principles and domination theorems. Among the numerous classes of operators that have been investigated, Cohen p-nuclear operators occupy a distinguished place because of their deep connections with absolutely summing operators, tensor norms and Banach space geometry [8, 6, 9].

In recent years, considerable attention has been devoted to the extension of classical linear theories to nonlinear settings. A major breakthrough in this direction was the development of the theory of Lipschitz mappings between metric spaces. The introduction of Lipschitz-free spaces and Lipschitz tensor products has made it possible to transfer many concepts and techniques from linear operator theory to the metric framework. Consequently, nonlinear analogues of several classical operator ideals have been introduced and studied, including Lipschitz summing operators, Lipschitz nuclear operators and Lipschitz Cohen-type operators [11, 15].

The study of Lipschitz operator ideals has become an active area of research, revealing strong interactions between metric geometry, Banach space theory and tensor products. One of the central objectives in this field is the construction of nonlinear counterparts of classical linear ideals while preserving their fundamental structural properties. Particular attention has been devoted to domination theorems of Pietsch type and to tensorial representations, which play a fundamental role in understanding the geometry of the corresponding classes of operators [7, 17, 18].

More recently, Hamidi, et all. introduced the notion of two-Lipschitz operators as a nonlinear extension of bilinear operators acting on metric spaces [13]. These mappings satisfy a Lipschitz condition in two variables simultaneously and admit natural bilinearization and linearization procedures. Their work established the foundations of a new theory of two-Lipschitz operator ideals and provided several constructions based on composition methods and tensor products.

Subsequently, additional classes of two-Lipschitz operators were developed and studied, including interpolative ideals and other nonlinear analogues of classical operator classes [14]. These investigations demonstrated that the theory of two-Lipschitz operators offers a natural framework for extending many fundamental concepts of multilinear analysis to metric spaces.

The main objective of this dissertation is to introduce and investigate a nonlinear version of Cohen p-nuclearity in the setting of two-Lipschitz operators. Inspired by the classical theory of Cohen p-nuclear linear operators developed by Cohen and later generalized by Apiola [8, 6], we define the class of two-Lipschitz Cohen p-nuclear operators and study its fundamental properties. Particular attention is given to domination results, ideal properties and relationships with other classes of two-Lipschitz operators.

Another important goal of this work is to establish tensorial representations for these operator ideals. Tensor products constitute one of the most powerful tools in the study of operator ideals, allowing the translation of operator-theoretic problems into geometric properties of tensor normed spaces [9]. By combining techniques from Lipschitz tensor products and nonlinear operator ideals, we obtain representation theorems that characterize two-Lipschitz Cohen p-nuclear operators in terms of suitable tensor norms and duality relations.

The theory developed in this dissertation is also motivated by recent advances on tensor representations of summing and nuclear classes of polynomials and multilinear mappings [1, 5, 2, 4]. These results illustrate the fundamental role played by tensor methods in the study of nonlinear operator ideals and provide useful tools for extending classical concepts to broader settings.

This dissertation is organized as follows.

Chapter 1 contains the preliminary material required throughout the dissertation. We review the basic notions concerning Banach spaces, operator ideals, Lipschitz mappings, Lipschitz-free spaces and Lipschitz tensor products. We also present the theory of two-Lipschitz operators, including their bilinearization and linearization procedures, together with the construction of two-Lipschitz operator ideals.

Chapter 2 is devoted to the study of two-Lipschitz Cohen p-nuclear operators. After reviewing the classical theory of Cohen p-nuclear operators and its Lipschitz counterparts, we introduce the new class and establish its principal properties. A Pietsch-type domination theorem is obtained and several characterizations are derived.

Finally, Chapter 3 develops the tensor representation theory associated with two-Lipschitz Cohen p-nuclear operators. Suitable tensor norms are introduced and their properties are investigated. The main representation theorems of the dissertation are established, providing a tensorial description of these nonlinear operator ideals and highlighting the connections between metric geometry, Lipschitz tensor products and Cohen-type nuclearity.

List of Abbreviations and Symbols

$\mathbb{K}$ The field of real or complex numbers.

p^* The conjugate of the number p $(1 \leq p < \infty)$, that is

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

B_E The closed unit ball of Banach space E.

E^* The topological dual of Banach space E.

$X^\#$ The Lipschitz dual of the pointed metric space X.

$m_{xx'}$ The molecule defined by

$$m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}},$$

for $x, x' \in X$, where χ_A is the characteristic function of the set A.

$\mathcal{M}(X)$ The vector space of all molecules on the metric space X.

$\mathcal{F}(X)$ The Lipschitz free space of X.

Æ(X) The Arens–Eells space of X.

T^* The adjoint operator of T.

A_L The linearization of the operator T.

A_b The bi-linearization of the operator T.

δ_X The isometric embedding from X to Æ(X).

Lip_0 The class of all Lipschitz operators that vanish at 0.

$\mathcal{I}_{\mathrm{BLip}}$ two-Lipschitz operator ideals.

Π_p^L The class of all Lipschitz p-summing operators.

$L_{as,(p;p_1,p_2)}$ The class of absolutely $(p;p_1,p_2)$-summing bilinear operators.

BLip The class of all two-Lipschitz operators.

BLip_0 The class of all two-Lipschitz operators that vanish at 0.

Im_{BLip} The two-Lipschitz image.

$\mathrm{BLip}_{0\mathcal{K}}$ The class of two-Lipschitz compact operators.

$\mathrm{BLip}_{0\mathcal{W}}$ The class of two-Lipschitz weakly compact operators.

$\mathrm{BL}_{as,(p;p_1,p_2)}$ The class of two-Lipschitz $(p;p_1,p_2)$-summing operators.

Chapter 1

Preliminaries

1.1 Banach Spaces and Linear Operator Ideals

In this subsection, we briefly recall the basic notions on Banach spaces, bounded linear operators, and linear operator ideals that will serve as a model for the nonlinear two-Lipschitz framework developed later.

Let E and F be Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We denote by $\mathcal{L}(E, F)$ the space of all bounded linear operators $T : E \to F$, endowed with the operator norm

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|.$$

Equivalently,

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

It is well known that $\mathcal{L}(E, F)$ is a Banach space with respect to this norm.

If E^* denotes the topological dual of E, we write $\langle x^*, x\rangle = x^*(x)$ for $x^* \in E^*$ and $x \in E$. We now recall the classical concept of a linear operator ideal.

Definition 1.1.1.

An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}(E, F)$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components

$$\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$$

satisfy:

(i) $\mathcal{I}(E,F)$ *is a vector subspace of* $\mathcal{L}(E,F)$ *which contains the mappings of the form* $x^* \otimes y$ *where* $x^* \in E^*$ *and* $y \in F$.

(ii) The ideal property: if $R \in \mathcal{L}(E,H)$ *and* $T \in \mathcal{I}(H,G)$ *and* $U \in \mathcal{L}(G,F)$, *then the composition* $U \circ T \circ R$ *is in* $\mathcal{I}(E,F)$.

An operator ideal $\mathcal{I}$ *is a normed (Banach) operator ideal if there is* $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \longrightarrow [0,+\infty[$, *satisfies*

(i') $(\mathcal{I}(E,F), \|\cdot\|_{\mathcal{I}})$ *is a normed (Banach) space for all Banach spaces* E *and* F

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$

(iii') If $R \in \mathcal{L}(E,H)$, $T \in \mathcal{I}(H,G)$ *and* $U \in \mathcal{L}(G,F)$, *then*

$$\|U \circ T \circ R\|_{\mathcal{I}} \leq \|U\| \, \|T\|_{\mathcal{I}} \, \|R\| .$$

This abstract framework provides the linear prototype for the two-Lipschitz operator ideals considered later.

1.2 Metric Spaces and Lipschitz Maps

We begin by recalling the standard definition of a metric space, which serves as the basic setting for all the constructions that follow.

Definition 1.2.1.
A metric space *is a pair* (X, d_X) *where* X *is a nonempty set and* $d_X : X \times X \to \mathbb{R}_+$ *is a function satisfying, for all* $x, y, z \in X$:

1. $d_X(x,y) = 0$ *if and only if* $x = y$,

2. $d_X(x,y) = d_X(y,x)$,

3. $d_X(x,z) \leq d_X(x,y) + d_X(y,z)$.

Throughout this manuscript, unless otherwise stated, (X, d_X) and (Y, d_Y) will denote metric spaces. Moreover, whenever Y is a Banach space, it will be considered as a metric space equipped with the metric induced by its norm, namely $d_Y(y_1, y_2) = \|y_1 - y_2\|$.

We now introduce one of the central constructions used in this work, namely the Arens–Eells space, also known as the Lipschitz-free space. This space allows us to linearize metric structures and plays a fundamental role in the study of Lipschitz mappings.

Definition 1.2.2.

Let X be a metric space. A molecule *on X is a scalar-valued function m on X with finite support such that $\sum_{x \in X} m(x) = 0$ We denote by $\mathcal{M}(X)$ the vector space of all molecules on X.*

For any $x, y \in X$, the elementary molecule m_{xy} is defined by $m_{xy} = \chi_{\{x\}} - \chi_{\{y\}}$ where χ_A denotes the characteristic function of a subset $A \subset X$.

Every molecule $m \in \mathcal{M}(X)$ can be written in the form $m = \sum_{j=1}^{m} \lambda_j \, m_{x_j y_j}$ for suitable scalars λ_j. Based on such representations, we define the norm on $\mathcal{M}(X)$ by $\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^{m} |\lambda_j| \, d(x_j, y_j) \ : \ m = \sum_{j=1}^{m} \lambda_j m_{x_j y_j} \right\}$

The infimum is taken over all admissible representations of m.

The *Arens–Eells space* Æ(X) is defined as the completion of the normed space $(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)})$. It is thus a Banach space canonically associated with the metric structure of X.

A fundamental feature of this construction is the existence of a canonical embedding. Indeed, the mapping $\delta_X : X \longrightarrow$ Æ(X) defined by $\delta_X(x) = m_{x0}$ is an isometric injection. Consequently, the metric space X can be identified with a subset of Æ(X).

We now recall the notion of Lipschitz mappings, which will later be linked to linear operators through the Arens–Eells space.

Definition 1.2.3.

Let (X, d_X) and (Y, d_Y) be pointed metric spaces. A mapping $T : X \to Y$ is said to be Lipschitz *if there exists a constant $C \geq 0$ such that $d_Y(Tx, Ty) \leq C \, d_X(x, y)$ for all $x, y \in X$ The smallest such constant is called the* Lipschitz constant *of T and is denoted by* Lip(T).

Equivalently,

$$\mathrm{Lip}(T) = \sup_{x \neq y} \frac{d_Y(Tx, Ty)}{d_X(x, y)}$$

We denote by $\mathrm{Lip}(X, Y)$ the space of all Lipschitz mappings from X into Y.

Proposition 1.2.4.

If Y is a Banach space, then $(\mathrm{Lip}_0(X, Y), \mathrm{Lip}\,(\cdot))$ is a Banach space.

Remark 1.2.5.

If X is a Banach space and $T : X \to Y$ is linear, then T is Lipschitz if and only if it is bounded, and in this case $\mathrm{Lip}(T) = \|T\|$ Thus, Lipschitz mappings extend the classical theory of bounded linear operators to a nonlinear setting.

These notions form the basic framework for the study of Lipschitz-type operator ideals, including in particular two-Lipschitz mappings, which will be introduced later.

A key feature of this theory is the linearization principle, which connects Lipschitz mappings with bounded linear operators on the Lipschitz-free space.

Theorem 1.2.6.

Let $T \in \mathrm{Lip}_0(X, Y)$. Then there exists a unique linear operator $\widehat{T} \in \mathcal{L}\big(\text{Æ}(X), \text{Æ}(Y)\big)$ such that $\widehat{T} \circ \delta_X = \delta_Y \circ T$ that is, the diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\delta_X"'] & Y \arrow[d, "\delta_Y"] \\ \text{Æ}(X) \arrow[r, "\widehat{T}"'] & \text{Æ}(Y) \end{tikzcd} \tag{1.1}$$

commutes. Moreover, $\|\widehat{T}\| = \mathrm{Lip}(T)$.

This result shows that every Lipschitz map admits a canonical isometric linear extension between the corresponding Lipschitz-free spaces.

When the target space is a Banach space, the linearization simplifies and leads to a direct representation by bounded linear operators.

Theorem 1.2.7.

Let $T \in \mathrm{Lip}_0(X, E)$. Then there exists a unique bounded linear operator $T_L : \text{Æ}(X) \longrightarrow E$

such that $T = T_L \circ \delta_X$ that is, the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "\delta_X"'] & & E \\ & \text{Æ}(X) \arrow[ur, "T_L"'] & \end{tikzcd} \tag{1.2}$$

commutes. Furthermore, $\|T_L\| = \mathrm{Lip}(T)$

The operator T_L is called the *linearization* of T. This correspondence establishes an isometric isomorphism

$$\mathrm{Lip}_0(X, E) \cong \mathcal{L}\big(\text{Æ}(X), E\big).$$

In particular, the Lipschitz dual space $X^{\#} = Lip_0(X, \mathbb{K})$ is isometrically isomorphic to $\text{Æ}(X)^*$ via the linearization map $R(f) = f_L$, where for every molecule $m \in \text{Æ}(X)$,

$$f_L(m) = \sum_{x \in X} f(x)m(x).$$

1.3 Lipschitz Tensor Products

Let X be a pointed metric space and E a Banach space. The *Lipschitz tensor product* $X \boxtimes E$ is defined as the linear span of elementary tensors of the form $\delta_{x,y} \boxtimes e$ where $x, y \in X$, $e \in E$ and $\delta_{(x,y)}$ represents the molecule $\delta_x - \delta_y$.

Following the construction of the Lipschitz tensor product, the *algebraic Lipschitz tensor product* $X \boxtimes E$ is defined as the linear span of the elementary tensors $\delta_{xy} \boxtimes e$, where $x, y \in X$ and $e \in E$. An arbitrary element $u \in X \boxtimes E$ can therefore be written in the form $u = \sum_{j=1}^{m} \delta_{x_j y_j} \boxtimes e_j$

The *Lipschitz projective norm* on $X \boxtimes E$ is given by

$$\|u\|_\pi = \left\| \sum_{j=1}^{m} \delta_{x_j y_j} \boxtimes e_j \right\|_\pi := \inf \left\{ \sum_{j=1}^{m} d(x_j, y_j)\|e_j\| : u = \sum_{j=1}^{m} \delta_{x_j y_j} \boxtimes e_j \right\}$$

The completion of $X \boxtimes E$ with respect to $\| \cdot \|_\pi$ is denoted by $X \widehat{\boxtimes}_\pi E$

A fundamental structural fact is the existence of a canonical isometric isomorphism

$$X \widehat{\boxtimes}_\pi E \cong \text{Æ}(X) \widehat{\otimes}_\pi E, \tag{1.3}$$

where $\widehat{\otimes}_\pi$ denotes the classical projective tensor product of Banach spaces defined by The projective norm on $E \otimes F$ the tensor product of E and F, is defined by

$$\pi(u) = \inf \sum_{j=1}^{m} \|x_j\| \, \|y_j\|, \tag{1.4}$$

where the infimum is taken over all finite representations of $u \in E \otimes F$ of the form $u = \sum_{j=1}^{m} x_j \otimes y_j$

The algebraic tensor product $E \otimes F$, equipped with the projective norm π, is denoted by $E \otimes_\pi F$. Its completion with respect to this norm is written $E \widehat{\otimes}_\pi F$ and is called the *projective tensor product* of E and F.

The canonical isomorphism $I : X \widehat{\boxtimes}_\pi E \longrightarrow Æ(X) \widehat{\otimes}_\pi E$ is defined on elementary tensors by $I\left(\sum_{j=1}^{m} \delta_{x_j y_j} \boxtimes e_j\right) = \sum_{j=1}^{m} (\delta_{x_j 0} - \delta_{y_j 0}) \otimes e_j$

1.4 Two-Lipschitz operator ideals

1.4.1 Two-Lipschitz Operators

In order to develop a nonlinear tensor theory adapted to metric spaces, we first introduce the class of two-Lipschitz operators. This notion extends bilinear mappings between Banach spaces to the metric setting and will serve as the basic object throughout this work. The results presented in this subsection form part of the contributions of Hamidi et all.

We begin with the fundamental definition, which captures the mixed Lipschitz behavior with respect to both variables simultaneously.

Definition 1.4.1.
Let (X, d_X) and (Y, d_Y) be pointed metric spaces and let E be a Banach space. A mapping $A : X \times Y \to E$ is called a two-Lipschitz operator *if there exists $C > 0$ such that for all $x, x' \in X$ and $y, y' \in Y$,*

$$\|A(x,y) - A(x,y') - A(x',y) + A(x',y')\| \le C \, d_X(x,x') \, d_Y(y,y'). \tag{1.5}$$

We denote by $\mathrm{BLip}_0(X, Y; E)$ the set of all two-Lipschitz operators satisfying $A(x,0) = A(0,y) = 0$ for every $x \in X$ and $y \in Y$.

For $A \in \mathrm{BLip}_0(X, Y; E)$ we define

$$\mathrm{BLip}(A) = \sup_{x \neq x', \, y \neq y'} \frac{\|A(x, y) - A(x, y') - A(x', y) + A(x', y')\|}{d_X(x, x') \, d_Y(y, y')}. \tag{1.6}$$

This definition can be interpreted in terms of partial mappings. For a fixed $y \in Y$ define $A_y : X \to E$ by $A_y(x) = A(x, y)$, and for a fixed $x \in X$ define $A_x : Y \to E$ by $A_x(y) = A(x, y)$. The two-Lipschitz condition ensures that these slices are Lipschitz and vary in a Lipschitz way with respect to the remaining variable. The next result makes this relationship precise.

Proposition 1.4.2.

For a mapping $A : X \times Y \to E$, the following are equivalent:

(i) $A \in \mathrm{BLip}_0(X, Y; E)$.

(ii) For every fixed $x \in X$, $A_x \in \mathrm{Lip}_0(Y, E)$ and the map $G : X \to \mathrm{Lip}_0(Y, E)$ given by $G(x) = A_x$ belongs to $\mathrm{Lip}_0(X, \mathrm{Lip}_0(Y, E))$.

(iii) For every fixed $y \in Y$, $A_y \in \mathrm{Lip}_0(X, E)$ and the map $H : Y \to \mathrm{Lip}_0(X, E)$ given by $H(y) = A_y$ belongs to $\mathrm{Lip}_0(Y, \mathrm{Lip}_0(X, E))$.

An important structural feature of two-Lipschitz operators is their stability under composition. This property will later allow us to define two-Lipschitz operator ideals.

Proposition 1.4.3.

Let X, Y, Z, W be pointed metric spaces and let E, F be Banach spaces. If $f \in \mathrm{Lip}_0(Z, X)$, $g \in \mathrm{Lip}_0(W, Y)$, $A \in \mathrm{BLip}_0(X, Y; E)$ and $u \in \mathcal{L}(E, F)$, then $u \circ A \circ (f, g) \in \mathrm{BLip}_0(Z, W; F)$ where $(f, g)(z, w) = (f(z), g(w))$. Moreover,

$$\mathrm{BLip}(u \circ A \circ (f, g)) \leq \|u\| \, \mathrm{BLip}(A) \, \mathrm{Lip}(f) \, \mathrm{Lip}(g). \tag{1.7}$$

The concept of two-Lipschitz operator extends the classical notion of bilinear mappings. Indeed, when the metric spaces are Banach spaces, the two notions coincide.

Remark 1.4.4.

If X and Y are Banach spaces, then every bilinear operator $A : X \times Y \to E$ is two-Lipschitz and $\mathrm{BLip}(A) = \|A\|$

Indeed, $\|A(x,y)-A(x,y')-A(x',y)+A(x',y')\| = \|A(x-x',y-y')\| \le \|A\|\,\|x-x'\|\,\|y-y'\|$ *The reverse inequality follows by evaluating* (1.6) *at* $x'=0$ *and* $y'=0$.

We now turn to the completeness of this space, which guarantees a solid functional-analytic framework.

Theorem 1.4.5.

The space $\mathrm{BLip}_0(X,Y;E)$ *is a Banach space with respect to the norm* $\mathrm{BLip}(\cdot)$.

To connect the nonlinear setting with tensor products, we consider the canonical embeddings into the Arens–Eells spaces. Equip $X\times Y$ with the metric $d\big((x,y),(x',y')\big) = d_X(x,x') + d_Y(y,y')$ and for Banach spaces E and F equip $E\times F$ with $\|(x,y)\|_{E\times F} = \|x\|_E + \|y\|_F$

Proposition 1.4.6.

The mapping $(\delta_X,\delta_Y) : X\times Y \longrightarrow \text{Æ}(X)\times \text{Æ}(Y), \qquad (\delta_X,\delta_Y)(x,y) = (m_{x0}, m_{y0})$ *is an isometric embedding.*

The key step in the tensor approach is the bilinearization of a two-Lipschitz operator. This construction transforms a nonlinear object into a continuous bilinear mapping.

Theorem 1.4.7.

For every $A\in \mathrm{BLip}_0(X,Y;E)$ *there exists a unique continuous bilinear mapping* $A_b : \text{Æ}(X)\times \text{Æ}(Y)\to E$ *such that* $A = A_b\circ(\delta_X,\delta_Y)$ *and* $\mathrm{BLip}(A) = \|A_b\|$

The mapping A_b *is called the bilinearization of* A.

Since bilinear operators admit linearizations on the projective tensor product, we obtain a purely linear representation of A.

Remark 1.4.8.

The bilinear operator A_b *induces a bounded linear operator* $A_L : \text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y)\to E$ *such that* $A = A_L\circ\sigma_2\circ(\delta_X,\delta_Y)$ *and* $\mathrm{BLip}(A) = \|A_L\|$ *where* $\sigma_2(m_{x0},m_{y0}) = m_{x0}\otimes m_{y0}$. *The operator* A_L *is called the linearization of* A.

We finish with a fundamental example that generates the finite-type two-Lipschitz operators and plays a central role in the tensor representation theory.

Example 1.4.9.

Let $f \in X^{\#}$, $g \in Y^{\#}$ *and* $e \in E$. *Define* $(f \cdot g \cdot e)(x, y) = f(x)g(y)e$

Then $f \cdot g \cdot e \in \mathrm{BLip}_0(X, Y; E)$ *and* $\mathrm{BLip}(f \cdot g \cdot e) = \mathrm{Lip}(f)\,\mathrm{Lip}(g)\,\|e\|$ *Finite linear combinations of mappings of this form constitute the basic building blocks of the tensorial structure developed in the subsequent sections.*

Definition 1.4.10.

We denote by $BLip_{0F}(X, Y; E)$ *the vector subspace of all two-Lipschitz operators generated by the mappings of the special form* $(f \cdot g \cdot e)(x, y) = f(x)g(y)e, \qquad x \in X,\ y \in Y$. *All elements* A *of this space are called* finite type *operators.*

Thus, any $A \in BLip_{0F}(X, Y; E)$ *admits a finite representation of the form* $A = \sum_{j=1}^{M} f_j \cdot g_j \cdot e_j$ *where* $(f_j)_{j=1}^{n} \subset X^{\#}, \qquad (g_j)_{j=1}^{n} \subset Y^{\#}, \qquad (e_j)_{j=1}^{n} \subset E$

1.4.2 Two-Lipschitz Operators and the Composition Method

In order to develop a coherent theory parallel to the classical theory of linear and multilinear operator ideals, we introduce the notion of two-Lipschitz operator ideals. This framework allows us to organize two-Lipschitz mappings into stable classes that behave well under composition and linearization. The results of this section follow the approach developed by Hamidi and et all [13].

Definition 1.4.11.

A two-Lipschitz operator ideal between pointed metric spaces and Banach spaces, denoted by $\mathcal{I}_{\mathrm{BLip}}$, *is a subclass of* BLip_0 *such that for every pointed metric spaces* X, Y *and every Banach space* E, *the component* $\mathcal{I}_{\mathrm{BLip}}(X, Y; E) := \mathrm{BLip}_0(X, Y; E) \cap \mathcal{I}_{\mathrm{BLip}}$ *satisfies:*

(i) $\mathcal{I}_{\mathrm{BLip}}(X, Y; E)$ *is a vector subspace of* $\mathrm{BLip}_0(X, Y; E)$.

(ii) For every $f \in X^{\#}$, $g \in Y^{\#}$ *and* $e \in E$, *the map* $f \cdot g \cdot e$ *belongs to* $\mathcal{I}_{\mathrm{BLip}}(X, Y; E)$.

(iii) (Ideal property) If $f \in \mathrm{Lip}_0(Z, X)$, $g \in \mathrm{Lip}_0(W, Y)$, $A \in \mathcal{I}_{\mathrm{BLip}}(X, Y; E)$ *and* $u \in \mathcal{L}(E, F)$, *then the composition* $u \circ A \circ (f, g)$ *belongs to* $\mathcal{I}_{\mathrm{BLip}}(Z, W; F)$.

A two-Lipschitz operator ideal $\mathcal{I}_{\mathrm{BLip}}$ *is called a normed (Banach) two-Lipschitz operator ideal if there exists a function* $\|\cdot\|_{\mathcal{I}_{\mathrm{BLip}}} : \mathcal{I}_{\mathrm{BLip}} \longrightarrow [0, +\infty[$ *such that:*

(i') *For every pointed metric spaces* X, Y *and Banach space* E*, the pair* $\big(\mathcal{I}_{\mathrm{BLip}}(X, Y; E), \|\cdot\|_{\mathcal{I}_{\mathrm{BLip}}}\big)$ *is a normed (Banach) space and* $\mathrm{BLip}(A) \leq \|A\|_{\mathcal{I}_{\mathrm{BLip}}}$ *for all* $A \in \mathcal{I}_{\mathrm{BLip}}(X, Y; E)$

(ii') $\|Id_{\mathbb{K}^2} : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K}, \quad Id_{\mathbb{K}^2}(\alpha, \beta) = \alpha\beta\|_{\mathcal{I}_{\mathrm{BLip}}} = 1$

(iii') *If* $f \in \mathrm{Lip}_0(Z, X)$*,* $g \in \mathrm{Lip}_0(W, Y)$*,* $A \in \mathcal{I}_{\mathrm{BLip}}(X, Y; E)$ *and* $u \in \mathcal{L}(E, F)$*, then*

$$\|u \circ A \circ (f, g)\|_{\mathcal{I}_{\mathrm{BLip}}} \leq \|u\| \, \|A\|_{\mathcal{I}_{\mathrm{BLip}}} \, \mathrm{Lip}(f) \, \mathrm{Lip}(g).$$

All Banach spaces considered in this definition are taken over the same fixed scalar field.

The two-Lipschitz operator ideal $\mathcal{I}_{\mathrm{BLip}}$ is said to be *closed* if, for every X, Y, E, the space $\mathcal{I}_{\mathrm{BLip}}(X, Y; E)$ is closed in $\mathrm{BLip}(X, Y; E)$ with respect to the norm $\mathrm{BLip}(\cdot)$.

The following proposition determines the norm of elementary tensors inside a normed two-Lipschitz operator ideal.

Proposition 1.4.12.

Let $\mathcal{I}_{\mathrm{BLip}}$ *be a normed two-Lipschitz operator ideal,* X, Y *pointed metric spaces and* E *a Banach space. Then* $\|f \cdot g \cdot e\|_{\mathcal{I}_{\mathrm{BLip}}} = \|e\| \, Lip(f) \, Lip(g)$ *for every* $f \in X^{\#}$*,* $g \in Y^{\#}$ *and* $e \in E$*.*

Remark 1.4.13.

The class BLip_{0F} *is the smallest two-Lipschitz operator ideal, whereas the class of all two-Lipschitz operators between arbitrary pointed metric spaces and Banach spaces is the largest two-Lipschitz operator ideal.*

We now describe a systematic way to construct two-Lipschitz operator ideals starting from classical linear operator ideals. This procedure, known as the *composition method*, extends structural properties of linear ideals to the two-Lipschitz setting.

Definition 1.4.14.

Let $\mathcal{I}$ *be an operator ideal. A two-Lipschitz operator* $A \in \mathrm{BLip}_0(X, Y; E)$ *belongs to the composition two-Lipschitz operator ideal* $\mathcal{I} \circ \mathrm{BLip}_0(X, Y; E)$ *if there exist a Banach space* F*,* *a two-Lipschitz operator* $S \in \mathrm{BLip}_0(X, Y; F)$ *and a linear operator* $u \in \mathcal{I}(F, E)$ *such that* $A = u \circ S$ *If* $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ *is a normed operator ideal, we define* $\|A\|_{\mathcal{I}\circ\mathrm{BLip}_0} = \inf \|u\|_{\mathcal{I}} \, \mathrm{BLip}(S)$ *where the infimum is taken over all factorizations* $A = u \circ S$.

The next theorem provides the fundamental link between the composition method and the linearization via the Lipschitz-free spaces.

Theorem 1.4.15. *[12]*

Let $\mathcal{I}$ *be an operator ideal. A two-Lipschitz operator* $A \in \mathrm{BLip}_0(X, Y; E)$ *belongs to* $\mathcal{I} \circ \mathrm{BLip}_0(X, Y; E)$ *if and only if its linearization* $A_L \in \mathcal{I}\big(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E\big)$

Furthermore, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ *is a normed operator ideal, then* $\|A\|_{\mathcal{I}\circ\mathrm{BLip}_0} = \|A_L\|_{\mathcal{I}}$ *and we have the isometric identification*

$$\big(\mathcal{I} \circ \mathrm{BLip}_0(X, Y; E), \|\cdot\|_{\mathcal{I}\circ\mathrm{BLip}_0}\big) = \big(\mathcal{I}(\text{Æ}(X)\widehat{\otimes}_\pi \text{Æ}(Y), E), \|\cdot\|_{\mathcal{I}}\big). \tag{1.8}$$

This result shows that the structure of two-Lipschitz operator ideals of composition type is completely determined by the corresponding linear operator ideals acting on projective tensor products of Lipschitz-free spaces. It constitutes a key step toward the tensor representation of Cohen p-nuclear two-Lipschitz operator ideals developed in the sequel.

1.5 Two-Lipschitz Operator Ideals via Trace Duality

1.5.1 Definition of Two-Lipschitz Operators

In this section, we recall and reformulate the notion of two-Lipschitz operators, which extends the classical Lipschitz framework to mappings defined on product spaces involving a Banach space structure. This notion plays a central role in the tensorial representation developed in the sequel.

Starting from the general definition of a two-Lipschitz mapping (1.5) and using the linearity of A with respect to the variable E, we introduce a useful simplification by fixing $y = 0$. This leads naturally to the following definition.

Definition 1.5.1.

Let X be a pointed metric space, and let E and F be Banach spaces. We say that a mapping $A : X \times E \longrightarrow F$ is a two-Lipschitz operator if A is linear in the variable E and there exists a constant $C > 0$ such that, for all $x, y \in X$ and $z \in E$, $\|A(x,z) - A(y,z)\| \leq C\, d(x,y)\, \|z\|$

We denote by $\mathrm{BLip}_0(X, E; F)$ the collection of all two-Lipschitz operators from $X \times E$ into F satisfying the normalization condition $A(0,e) = 0$ for all $e \in E$. For any $A \in \mathrm{BLip}_0(X, E; F)$, we define its two-Lipschitz seminorm by

$$\mathrm{BLip}(A) = \inf C = \sup_{x \neq y,\, e \neq 0} \frac{\|A(x,e) - A(y,e)\|}{d(x,y)\, \|e\|}$$

To better understand the structure of such mappings, we introduce partial evaluations. For a given mapping $A : X \times E \longrightarrow F$, we define $A_e : X \longrightarrow F, \quad A_e(x) = A(x,e)$ for each fixed $e \in E$, and $A_x : E \longrightarrow F, \quad A_x(e) = A(x,e)$ for each fixed $x \in X$.

According to Hamidi et al. in [13], a mapping A is two-Lipschitz if and only if A_x is linear for every $x \in X$, A_e is Lipschitz for every $e \in E$, and the mapping $G : X \longrightarrow L(E,F), \quad G(x) = A_x$ belongs to $\mathrm{Lip}_0(X, \mathcal{L}(E,F))$ (equivalently, the mapping $H : E \longrightarrow \mathrm{Lip}_0(X,F)$ given by $H(e) = A_e$ belongs to $\mathcal{L}(E, \mathrm{Lip}_0(X,F))$).

Remark 1.5.2.

Since every two-Lipschitz operator $A : X \times E \longrightarrow F$ is linear in the second variable E, we impose the following stability condition in order to guarantee the ideal property in Definition 1.4.11 of a two-Lipschitz ideal. Let $f \in \mathrm{Lip}_0(Z,X)$, $v \in \mathcal{L}(W,E)$, $A \in \mathcal{I}_{\mathrm{BLip}}(X,E;F)$, and $u \in \mathcal{L}(F,G)$. Then the composition $u \circ A \circ (f,v)$ belongs to $\mathcal{I}_{\mathrm{BLip}}(Z,W;G)$ and satisfies $\|u \circ A \circ (f,v)\|_{\mathcal{I}_{\mathrm{BLip}}} \leq \|u\|\, \|A\|_{\mathcal{I}_{\mathrm{BLip}}} \mathrm{Lip}(f)\, \|v\|$

More generally, if $f \in \mathrm{Lip}_0(Z,X)$, $v \in L(W,E)$, $A \in \mathrm{BLip}_0(X,E;F)$, and $u \in \mathcal{L}(F,G)$, then $u \circ A \circ (f,v)$ belongs to $\mathrm{BLip}_0(Z,W;G)$ and we have

$$\mathrm{BLip}(u \circ A \circ (f,v)) \leq \|u\|\, \mathrm{BLip}(A)\, \mathrm{Lip}(f)\, \|v\|$$

In order to pass to a tensorial framework, we consider the canonical embedding $(\delta_X, id_E) : X \times E \longrightarrow Æ(X) \times E$ defined by $(\delta_X, id_E)(x,e) = (\delta_X(x), e) = (m_{x0}, e)$ which is an isometric embedding of $X \times E$ into $Æ(X) \times E$, where id_E denotes the identity operator on E.

This embedding allows us to associate with each two-Lipschitz operator a bilinear representation on the Arens–Eells space. More precisely, for every $A : X \times E \longrightarrow F$ we define a bilinear mapping $A_b : \text{Æ}(X) \times E \longrightarrow F$ by $A_b(u, e) \sum_{j=1}^{m} \lambda_j \big(A(x_j, e) - A(y_j, e)\big)$, where

$$u = \sum_{j=1}^{m} \lambda_j, m_{x_j y_j} \in \mathcal{M}(X),$$

And every $e \in E$. Consequently, A_b serves as the bilinear representation associated with the original two-Lipschitz operator A.

Theorem 1.5.3. *[3]*

For every two-Lipschitz operator $A \in \mathrm{BLip}_0(X, E; F)$, there exists a unique continuous bilinear operator $A_b : \text{Æ}(X) \times E \longrightarrow F$ satisfying(1.9)and such that

$$A = A_b \circ (\delta_X, id_E) : X \times E \xrightarrow{(\delta_X, id_E)} \text{Æ}(X) \times E \xrightarrow{A_b} F \tag{1.9}$$

Moreover, $\mathrm{BLip}(A) = \|A_b\|$

The bilinear mapping A_b is called the bi-linearization of the two-Lipschitz operator A.

Remark 1.5.4.

The bilinear operator A_b admits a canonical linearization $(A_b)_L : \text{Æ}(X)\widehat{\otimes}_\pi E \longrightarrow F$ satisfying

$$A = A_b \circ (\delta_X, id_E) = (A_b)_L \circ \sigma_2 \circ (\delta_X, id_E),$$

where $\sigma_2 : \text{Æ}(X) \times E \longrightarrow \text{Æ}(X) \otimes_\pi E$ is the canonical bilinear mapping defined in (2). Moreover,

$$\mathrm{BLip}(A) = \|A_b\| = \|(A_b)_L\|.$$

The operator $(A_b)_L$ is called the linearization of A, and for simplicity we write A_L instead of $(A_b)_L$.

This yields the canonical identification

$$\mathrm{BLip}_0(X, E; F) \cong \mathcal{L}(\text{Æ}(X)\widehat{\otimes}_\pi E, F).$$

In the particular case $F = \mathbb{K}$, we obtain the dual representation

$$\mathrm{BLip}_0(X, E) \cong (\text{Æ}(X)\widehat{\otimes}_\pi E)^*$$

Under this identification, every continuous form $A \in \mathrm{BLip}_0(X,E)$ acts on elementary tensors by

$$\sum_{i=1}^{n} m_i \otimes e_i \longmapsto sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ji}\ in\big(A(x_{ij},e_i) - A(y_{ij},e_i)\big),$$

whenever each m_j admits a representation of the form $m_j = \sum_{j=1}^{m} \lambda_{ji} \in m_{x_{ji}y_{ji}}$

1.5.2 Universal Property

We now turn to the universal property of the Lipschitz tensor product, which provides the structural link between two-Lipschitz mappings and their linear representations.

To begin with, we introduce the canonical mapping

$$\widetilde{\sigma}_2 : X \times E \longrightarrow X\widehat{\boxtimes}_\pi E, \qquad \widetilde{\sigma}_2(x,e) = \delta_{x0} \boxtimes e.$$

It is straightforward to verify that $\widetilde{\sigma}_2$ is a two-Lipschitz operator and that $\mathrm{BLip}(\widetilde{\sigma}_2) = 1$

Having fixed this canonical embedding, we now consider an arbitrary target Banach space F and a two-Lipschitz operator $A : X \times E \longrightarrow F$ which vanishes at $(x,0)$ and $(0,e)$ In order to associate to A a linear object, we define a mapping $A^{\boxtimes} : X\widehat{\boxtimes}_\pi E \longrightarrow F$ on elementary tensors by the rule

$$A^{\boxtimes}(\delta_{xy} \boxtimes e) = A(x,e) - A(y,e), \qquad x,y \in X,\ e \in E. \tag{1.10}$$

The operator $A^{\boxtimes}$ is called the *operator associate* (or linearization) of A.

The following result shows that this construction is not only well-defined but also universal.

Theorem 1.5.5. *[3]*
For every $A \in \mathrm{BLip}_0(X,E;F)$ there exists a unique continuous linear operator $A^{\boxtimes} : X\widehat{\boxtimes}_\pi E \longrightarrow F$ satisfying (1.10) *and such that $A = A^{\boxtimes} \circ \widetilde{\sigma}_2$*

Moreover, $\mathrm{BLip}(A) = \|A^{\boxtimes}\|$

Proof. We start by defining a linear map $A_0^{\boxtimes}$ on the algebraic tensor product $X \boxtimes E$ by setting

$$A_0^{\boxtimes}\left(\sum_{j=1}^{m} \delta_{x_j y_j} \boxtimes e_j\right) = \sum_{j=1}^{m} \big(A(x_j,e_j) - A(y_j,e_j)\big).$$

Next, we use the two-Lipschitz property of A to obtain a uniform control. Indeed, there exists $C = \mathrm{BLip}(A)$ such that

$$\|A(x_j, e_j) - A(y_j, e_j)\| \leq C\, d(x_j, y_j)\|e_j\|.$$

Consequently,

$$\left\|A_0^{\boxtimes}(u)\right\| \leq C \sum_{j=1}^{n} d(x_j, y_j)\|e_j\|.$$

By taking the infimum over all possible representations of u, we obtain $\|A_0^{\boxtimes}(u)\| \leq C\|u\|_\pi$.

Thus $A_0^{\boxtimes}$ is bounded and therefore extends uniquely to a continuous linear operator $A^{\boxtimes} : X\widehat{\boxtimes}_\pi E \longrightarrow F$ satisfying (**??**).

Conversely, suppose that $A^{\boxtimes}$ is a bounded linear operator on $X\widehat{\boxtimes}_\pi E$. Then, for every $(x, e) \in X \times E$, we have

$$A^{\boxtimes} \circ \widetilde{\sigma}_2(x, e) = A^{\boxtimes}(\delta_{x0} \boxtimes e) = A(x, e) - A(0, e) = A(x, e),$$

since $A(0, e) = A(x, 0) = 0$.

From this representation, a standard norm comparison argument yields $\|A^{\boxtimes}\| \leq \mathrm{BLip}(A)$ and $\mathrm{BLip}(A) \leq \|A^{\boxtimes}\|$, which implies the equality of the norms.

Finally, uniqueness follows from the fact that the tensors $\delta_{xy} \boxtimes e$ generate a dense subspace of $X\widehat{\boxtimes}_\pi E$. □

As a consequence of this universal construction, we obtain a dual representation result.

Theorem 1.5.6. *[3]*
The space $\mathrm{BLip}_0(X, E)$ *is isometrically isomorphic to* $\left(X\widehat{\boxtimes}_\pi E\right)^*$.

More precisely, the mapping

$$\Psi : \mathrm{BLip}_0(X, E) \longrightarrow \left(X\widehat{\boxtimes}_\pi E\right)^*$$

given by

$$\Psi(\varphi)\left(\sum_{j=1}^{m} \delta_{x_j y_j} \boxtimes e_j\right) = \sum_{j=1}^{m} \big(\varphi(x_j, e_j) - \varphi(y_j, e_j)\big)$$

is a surjective linear isometry.

Finally, we relate this construction to the classical tensor framework. Using the canonical identification (1.3), we obtain compatibility with the standard linearization.

Theorem 1.5.7. *[3]*
For every $A \in \mathrm{BLip}_0(X, E; F)$, we have $A^{\boxtimes} = A_L \circ I^{-1}$, where $I : \text{Æ}(X)\widehat{\otimes}_\pi E \to X\widehat{\boxtimes}_\pi E$ is the canonical isomorphism. Moreover,

$$\mathrm{BLip}(A) = \|A^{\boxtimes}\| = \|A_L\|.$$

Chapter 2

Two-Lipschitz Cohen p-Nuclear Operators

2.1 Background on Cohen p-Nuclear Operators and Their Extensions

The class of p-nuclear operators was originally introduced by Cohen [8] and later extended to the framework of Cohen (p, q)-nuclear operators by Apiola [6]. We begin by recalling the linear setting.

Definition 2.1.1.

Let E and F be Banach spaces. A linear operator $T : E \to F$ is said to be Cohen p-nuclear $(1 < p < \infty)$ *if there exists a constant $C > 0$ such that, for every $m \in \mathbb{N}$, every finite family $(x_j)_{j=1}^m \subset E$ and every $(a_j^*)_{j=1}^m \subset F^*$, one has*

$$\sum_{j=1}^{m} \langle T(x_j), a_j^* \rangle \leq C \sup_{x^* \in B_{E^*}} \left\| (x^*(x_j))_{j=1}^m \right\|_{\ell_p^m} \sup_{a \in B_F} \left\| (a_j^*(a))_{j=1}^m \right\|_{\ell_{p^*}^m} .$$

The least constant C for which the above inequality holds is denoted by $\eta_p(T)$ and is called the *Cohen p-nuclear norm.* The collection of all Cohen p-nuclear operators from E into F is denoted by $\mathcal{N}_p(E, F)$; endowed with the norm $\eta_p(\cdot)$, it becomes a Banach space.

The notion of Cohen p-nuclear m-linear operators was introduced by . Achour and Alouani(see [2]) as a multilinear extension of the concept of p-nuclear linear operators.

Definition 2.1.2.

A bilinear operator $T : E_1 \times E_2 \to F$ is called Cohen p-nuclear $(1 < p < \infty)$ *if there exists a constant $C > 0$ such that, for every $m \in \mathbb{N}$, every $x_1, \dots, x_m \in E_1$, every $y_1, \dots, y_m \in E_2$, and every $a_1^*, \dots, a_m^* \in F^*$ the following inequality holds:*

$$\sum_{j=1}^{m} \langle T(x_j, y_j), a_j^* \rangle \leq C \left(\sup_{\varphi_1 \in B_{E_1^*} \ \varphi_2 \in B_{E_2^*}} \sum_{j=1}^{m} |\varphi_1(x_j)|^p |\varphi_2(y_j)|^p \right)^{1/p} \sup_{a \in B_F}, \left\| (a_j^*(a))_{j=1}^m \right\|_{\ell_{p^*}^m}. \quad (2.1)$$

The vector space of all such mappings is denoted by $\mathcal{N}_p^2(E_1, E_2; F)$, and the smallest constant C for which (2.1) holds is denoted by $\eta_p^2(T)$.

For $p = \infty$, inequality (2.1) takes the form

$$\sum_{j=1}^{m} \langle T(x_j, y_j), a_j^* \rangle \leq C \left(\sup_{j=1} \leq m \|x_j\|_{E_1} \|y_j\|_{E_2} \right) \sup_{a \in B_F} \left\| (a_j^*(a))_{j=1}^m \right\|_{\ell_1^m}.$$

It follows immediately that every $T \in \mathcal{N}_p^2(E_1, E_2; F)$ is continuous and satisfies $\|T\| \leq \eta_p^2(T)$.

The Lipschitz extension of the notion of Cohen p-nuclear linear operators was introduced by Tallab and Mezrag [15].

Definition 2.1.3.

Let X be a pointed metric space and E a Banach space. A Lipschitz operator $T : X \to E$ is said to be Cohen Lipschitz p-nuclear $(1 < p < \infty)$ *if there exists a constant $C > 0$ such that, for every $m \in \mathbb{N}$, every $(x_j)_{1 \leq j \leq m}, (y_j)_{1 \leq j \leq m} \subset X$, every $(a_j^*)_{j=1}^m \subset E^*$ and every $(\lambda_j)_{1 \leq j \leq m} \subset \mathbb{R}_+$, one has*

$$\sum_{j=1}^{m} \lambda_j \langle T(x_j) - T(y_j), a_j^* \rangle \leq C \sup_{f \in B_{X^\#}} \left(\sum_{j=1}^{m} |\lambda_j (f(x_j) - f(y_j))|^p \right)^{1/p} \sup_{a \in B_E} \left(\sum_{j=1}^{m} |\langle a_j^*, a \rangle|^{p^*} \right)^{1/p^*}.$$

The least constant C satisfying the above inequality is denoted by $\eta_p^L(T)$ and is called the *Cohen Lipschitz p-nuclear norm.* Equipped with this norm, the space $\mathcal{N}_p^L(X, E)$ of all Cohen Lipschitz p-nuclear operators from X into E is a Banach space.

Theorem 2.1.4. *[15]*

If $T \in \mathcal{N}_p^L(X, E)$, then there exist Radon probability measures μ_1 on $B_{X^\#}$ and μ_2 on $B_{E^{**}}$

such that, for all $x, xy \in X$ and all $a^* \in E^*$, one has

$$\left|\langle T(x) - T(y), a^* \rangle\right| \leq C \left(\int_{B_{X\#}} Apiola |f(x) - f(y)|^p \, d\mu_1(f) \right)^{1/p} \left(\int_{B_{E^{**}}} |a^{**}(a^*)|^{p^*} \, d\mu_2(a^{**}) \right)^{1/p^*}. \tag{2.2}$$

In this case,

$$\eta_p^L(T) = \inf\{ C > 0 : C \text{ satisfies inequality (2.2)} \}.$$

2.2 Two-Lipschitz Cohen p-Nuclear Operators

2.2.1 Definition and Basic Properties

Definition 2.2.1.

Let $1 < p < \infty$. An two-Lipschitz operator $A : X \times E \to F$ is said to be two-Lipschitz Cohen p-nuclear operator if there is a constant $C > 0$ such that for any $(x_j)_{j=1}^m, (y_j)_{j=1}^m \subset X, (e_j)_{j=1}^m \subset E$ and $\left(a_j^*\right)_{j=1}^m \subset F^*$ we have

$$\begin{aligned} &\left| \sum_{j=1}^m \left\langle A(x_j, e_j) - A(y_j, e_j), a_j^* \right\rangle \right| \\ &\leq C \sup_{\substack{f \in B_{X\#} \\ \varphi \in B_{E^*}}} \left(\sum_{j=1}^m |f(x_j) - f(y_j)|^p |\varphi(e_j)|^p \right)^{\frac{1}{p}} \sup_{a \in B_E} \left(\sum_{j=1}^m |\langle a, a_j^* \rangle|^{p^*} \right)^{1/p^*}. \end{aligned} \tag{2.3}$$

We denote this class of two-Lipschitz operators by $\mathcal{N}_p^{\mathrm{BL}}(X, E; F)$, In this case, we define

$$\eta_p^{\mathrm{BL}}(A) = \inf\{C : \textit{ satisfying } (2.3)\}.$$

For $p = \infty$, (2.3) becomes

$$\left| \sum_{j=1}^m \left\langle A(x_j, e_j) - A(y_j, e_j), a_j^* \right\rangle \right| \leq C \sup_{1 \leq j \leq m} d(x_j, y_j) \|e_j\| \sup_{a \in B_E} \sum_{j=1}^m |\langle a, a_j^* \rangle|^\cdot$$

Remark 2.2.2.

The following statements come immediately from Definition **??***:*

1. *It is obvious that in the case $A \in \mathrm{Lip}_0(X, F)$ we get the well-known concept of Cohen Lipschitz p-nuclear operators.*

2. *It is clear that every* $A \in \mathcal{N}_p^{\mathrm{BL}}(X, E; F)$ *is two-Lipschitz and* $\mathrm{BLip}(A) \leq \eta_p^{\mathrm{BL}}(A)$.

Proposition 2.2.3.

1. *For* $p = \infty$, *we have* $\mathcal{N}_\infty^{\mathrm{BL}}(X, E; F) = \mathcal{D}_\infty^{\mathrm{BL}}(X, E; F)$, *the class of strongly two-Lipschitz* ∞*-summing operators.*

2. *For* $p = 1$, *we have* $\mathcal{N}_1^{\mathrm{BL}}(X, E; F) \subset \mathrm{BL}_{as,(1;p,p^*)}(X, E; F)$, *the class of two-Lipschitz* $(1; p, p^*)$*-summing operators.*

Proof. (1)-is obvious.

(2)- Let $A \in \mathcal{N}_1^{\mathrm{BL}}(X, E; F)$, then for any $(x_j)_{j=1}^m, (y_j)_{j=1}^m \subset X, (e_j)_{j=1}^m \subset E$ and $\left(a_j^*\right)_{j=1}^m \subset F^*$ we have

$$\begin{aligned}
&\left|\sum_{j=1}^m \left\langle A(x_j, e_j) - A(y_j, e_j), a_j^* \right\rangle\right| \\
&\leq \eta_1^{\mathrm{BL}}(A) \sup_{\substack{f \in B_{X^\#} \\ \varphi \in B_{E^*}}} \left(\sum_{j=1}^m |f(x_j) - f(y_j)| \, |\varphi(e_j)|\right) \sup_{a \in B_E} \left\| \left(a_j^*(a)\right)_{j=1}^m \right\|_{l_\infty^m} \\
&\leq \eta_1^{\mathrm{BL}}(A) \sup_{\substack{f \in B_{X^\#} \\ \varphi \in B_{E^*}}} \left(\sum_{j=1}^m |f(x_j) - f(y_j)| \, |\varphi(e_j)|\right) \sup_j \left\|a_j^*\right\|.
\end{aligned}$$

On the other hand, we have

$$\sum_{j=1}^m \|A(x_j, e_j) - A(y_j, e_j)\| = \sup\left\{\left|\sum_{j=1}^m \left\langle A(x_j, e_j) - A(y_j, e_j), a_j^* \right\rangle\right| : \sup_j \left\|a_j^*\right\| \leq 1\right\}.$$

This implies

$$\sum_{j=1}^m \|A(x_j, e_j) - A(y_j, e_j)\| \leq \eta_1^{\mathrm{BL}}(T) \sup_{\substack{f \in B_{X^\#} \\ \varphi \in B_{E^*}}} \left(\sum_{j=1}^m |f(x_j) - f(y_j)| \, |\varphi(e_j)|\right).$$

Therefore, by by Hölder's inequality we have

$$\sum_{j=1}^m \|A(x_j, e_j) - A(y_j, e_j)\| \leq \eta_1^{\mathrm{BL}}(T) \sup_{f \in B_{X^\#}} \left(\sum_{j=1}^m \left|f(x_j) - f\left(x_j'\right)\right|^p\right)^{\frac{1}{p}} \left(\sup_{\varphi \in B_{E^*}} |\varphi(e_j)|^{p^*}\right)^{\frac{1}{p^*}}.$$

Thus from Definition 4.15 in [13], A is two-Lipschitz $(1; p, p^*)$-summing and

$$\|A\|_{\mathrm{BL}_{as,(1;p,p^*)}} \leq \eta_1^{\mathrm{BL}}(A).$$

□

We are unsure whether a two-Lipschitz Cohen p-nuclear mapping implies that the corresponding bilinear operator is Cohen p-nuclear when the mapping A is bilinear. However, the converse is certainly true. The proof follows from the inclusions $B_{X^*} \in B_{X^\#}$.

Proposition 2.2.4.

If $A : X \times E \to F$ is a bilinear Cohen p-nuclear operator between Banach spaces $X, E,$ and F, then A is also a two-Lipschitz Cohen p-nuclear operator. Furthermore, $\eta_p^{\mathrm{BL}}(A) \leq \eta_p^2(A)$.

Proposition 2.2.5.

$\left(\mathcal{N}_p^{\mathrm{BL}}(X, E; F), \eta_p^{\mathrm{BL}}(\cdot)\right)$ *is a Banach two-Lipschitz ideal.*

The next theorem gives an interesting relationship between the integrability of a two-Lipschitz operator and the integrability of its associated operator.

Theorem 2.2.6.

Let $1 < p < \infty$, X be a pointed metric space, E and F be a Banach spaces and let $A \in \mathrm{BLip}_0(X, E; F)$. If $A^{\boxtimes} \in \mathcal{N}_p(X \widehat{\boxtimes}_\pi E, F)$ then $A \in \mathcal{N}_p^{\mathrm{BL}}(X, E; F)$. In this case, $\eta_p^{\mathrm{BL}}(A) \leq \eta_p(A^{\boxtimes})$.

2.2.2 Domination Theorem and example

Now, we establish a version of Pietsch's domination theorem for the class of two-Lipschitz p-nuclear operators. The result utilizes the general Pietsch domination theorem recently introduced by Pellegrino et al. [17].

Theorem 2.2.7.

Let $A \in \mathrm{BLip}_0(X, E; F)$ and C a positive constant. Then the following assertions are equivalent.

1- The two-Lipschitz operator A is Cohen two-Lipschitz p-nuclear.

2- There exist Radon probability measures μ_1 on $B_{X^\#}$, μ_2 on B_{E^*} and ν on $B_{F^{**}}$, such that *for all x, y in X, e in E and a^* in F^*, we have*

$$|\langle A(x,e) - A(y,e), a^* \rangle|$$

$$\leq C \left(\int\limits_{B_{X^\#}} |f(x) - f(y)|^p \, d\mu_1(f) \right)^{\frac{1}{p}} \left(\int\limits_{B_{E^*}} |\varphi(e)|^p \, d\mu_2(\varphi) \right)^{\frac{1}{p}} \left(\int\limits_{B_{F^{**}}} |a^*(a^{**})|^{p^*} \, d\nu(a^{**}) \right)^{\frac{1}{p^*}}. \tag{2.4}$$

Moreover, in this case $\eta_p^{\mathrm{BL}}(A) = \inf C$ for all C, verifying the inequality (2.4).

Example 2.2.8.

Let $1 < p < \infty$, and let $B : X \to F$ be a Lipschitz p-summing mapping and $T : E \to F$ be a Cohen p-nuclear mapping. Consider the mapping

$$A : X \times E \to F, \qquad A(x,e) = B(x)\, T(e)$$

Then, A is a two-Lipschitz Cohen p-nuclear operator with $\eta_p^{\mathrm{BL}}(A) \leq \pi_p^L(B)\, \eta_p(T)$. Indeed, for any $x, y \in X, e \in E$, and $a^* \in F^*$, we have

$$\begin{aligned} \left|\langle A(x,e) - A(y,e), a^* \rangle\right| &= \left|\langle B(x)\, T(e) - B(y)\, T(e), a^* \rangle\right| \\ &= \|B(x) - B(y)\| \cdot \left|\langle T(e), a^* \rangle\right|. \end{aligned}$$

By Theorem 1 in [11] and Theorem 2.3.2 in [8], there exist Radon probability measures μ_1 on $B_{X^\#}$, μ_2 on B_{E^*}, and λ on $B_{F^{**}}$, such that for all $x, y \in X$, $e \in E$, and $a^* \in F^*$, we have

$$\begin{aligned} &\|B(x) - B(x')\| \left|\langle T(e), a^* \rangle\right| \\ &\leq \pi_p^L(B)\, \eta_p(T) \left(\int_{B_{X^\#}} |f(x) - f(y)|^p \, d\mu_1 \right)^{\frac{1}{p}} \cdot \left(\int_{B_{E^\#}} |\varphi(e)|^p \, d\mu_2 \right)^{\frac{1}{p}} \\ &\cdot \left(\int_{B_{F^{**}}} |a^*(a^{**})|^{p^*} \, d\lambda(a^{**}) \right)^{\frac{1}{p^*}}. \end{aligned}$$

it follows that from theorem (2.2.7) $A \in \mathcal{N}_p^{\mathrm{BL}}(X, E; F)$ and

$$\eta_p^{\mathrm{BL}}(A) \leq \pi_p^L(B)\, \eta_p(T).$$

2.2.3 Relationships between the various classes of two-Lipschitz operators

From the domination theorem of strongly p-summing two-Lipschitz operator and a p-summing operator, we infer the following proposition

Proposition 2.2.9.

Let C be a positive constant. A two-Lipschitz operator $A : X \times E \to F$ is Cohen two-Lipschitz p-nuclear and $\eta_p^B(A) \leq C$ if, and only if, there exist a Banch G, s strongly p-summing two-Lipschitz operator $B : X \times E \to G^*$ and a linear p^*-summing operator $T : F^* \longrightarrow G$ such that $A = T^* \circ B$ and $\pi_p^B(B)\pi_{p^*}(T) \leq C$.

Theorem 2.2.10.

We have for a two-Lipschitz operator $A : X \times E \to F$.

1- $\mathcal{N}_p^{\mathrm{BL}}(X, E; F) \subseteq \mathcal{D}_{st,p}^{\mathrm{BL}}(X, E; F)$ and $d_{st,p}^{\mathrm{BL}}(A) \leq \eta_p^{\mathrm{BL}}(A)$ for $1 < p \leq \infty$.

2- $\mathcal{N}_p^{\mathbb{D}}(X, E; F) \subseteq \mathbb{D}_{p,p_1,p_2}^{\mathrm{BL}}(X, E; F)$ and $\mathrm{d}_{p,p_1,p_2}^{\mathrm{BL}}(A) \leq \eta_p^{\mathrm{BL}}(A)$ for $1 \leq p < \infty$.

Proof. (1) Let $A \in \mathcal{N}_p^{\mathrm{BL}}(X, E; F)$. Consider x, y in $X, e \in E$ and $a^* \in F^*$. Then from (2.4) we have

$$
\begin{aligned}
&|\langle A(x,e) - A(y,e), a^* \rangle| \\
&\leq \eta_p^{\mathrm{BL}}(A) \left(\int_{f \in B_{X^\#}} |f(x_j) - f(y_j)|^p \, d\mu_1(f) \right)^{\frac{1}{p}} \left(\int_{B_{E^*}} |\varphi(e)|^p \, d\mu_2(\varphi) \right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}} |a^*(a^{**})|^{p^*} \, d\nu(a^{**}) \right)^{\frac{1}{p^*}} \\
&\leq \eta_p^{\mathrm{BL}}(A) \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} d(x,y)^p \, d\mu_1(g) \right)^{\frac{1}{p}} \left(\int_{B_{E^*}} \|e\|^p \, d\mu_2(\varphi) \right)^{\frac{1}{p}} \left(\int_{B_{X^{**}}} |a^*(a^{**})| \, d\nu(x^{**}) \right)^{\frac{1}{p^*}} \\
&\leq \eta_p^{\mathrm{BL}}(A) \, d(x,y) \, \|e\| \left(\int_{B_{X^{**}}} |a^*(a^{**})| \, d\nu(x^{**}) \right)^{\frac{1}{p^*}}.
\end{aligned}
$$

Hence by[14, Theorem 3.3], A is strongly two-Lipschitz p-summing and $d_{st,p}^{\mathrm{BL}}(A) \leq \eta_p^{\mathrm{BL}}(A)$.
(2) Let A be a two-Lipschitz in $\mathcal{N}_p^{\mathrm{BL}}(X, E; F)$. We have by (2.4)

$$
\begin{aligned}
&|\langle a^*(A(x,e) - A(y,e)) \rangle| \\
&= \sup_{a^* \in B_{F^*}} |\langle A(x,e) - A(y,e), a^* \rangle| \\
&\leq \sup_{a^* \in B_{F^*}} \eta_p^{\mathrm{BL}}(A) \left(\int_{B_{X^\#}} |f(x_j) - f(y_j)|^p \, d\mu_1(f) \right)^{\frac{1}{p}} \left(\int_{B_{E^*}} |\varphi(e)|^p \, d\mu_2(\varphi) \right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}} |a^*(a^{**})|^{p^*} \, d\nu(a^{**}) \right)^{\frac{1}{p^*}} \\
&\leq \eta_p^{\mathrm{BL}}(A) \left(\int_{B_{X^\#}} |f(x_j) - f(y_j)|^p \, d\mu_1(f) \right)^{\frac{1}{p}} \left(\int_{B_{E^*}} |\varphi(e)|^p \, d\mu_2(\varphi) \right)^{\frac{1}{p}}.
\end{aligned}
$$

By Pietsch domination [12, Theorem 1], A is two-Lipschitz p-dominated and $d_p^{\mathrm{BL}}(A) \leq \eta_p^{\mathrm{BL}}(A)$.

$\square$

From the results obtained above we get.

Corollary 2.2.11.

Consider $1 \leq p \leq \infty$.with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ Let $A \in \mathrm{BLip}_0(X, E; F)$ and $T \in \mathcal{L}(F, G)$, If T is strongly p-summing operator, and A is two-Lipschitz (p, p_1, p_2)-summing operator, then $T \circ A$ is Cohen two-Lipschitz p-nuclear operator and $\eta_p^{\mathrm{BL}}(A)(T \circ A) \leq d_{st,p}(T)\, \|A\|_{\mathrm{BL}^{\mathrm{BL}}_{p,p_1,p_2}}$

Proof. Let $x, y \in X$, $e \in E$, and $a^* \in F^*$. Then, we have

$$\begin{aligned}
&|\langle (T \circ A)(x, e) - (T \circ A)(x, e), a^* \rangle| \\
&= |\langle T(A(x, e) - A(y, e)), a^* \rangle| \\
&\leq d_{st,p}(T) \| A(x, e) - A(y, e) \| \left(\int_{B_{F^{**}}} |a^*(a^{**})|^{p^*} \, d\nu_2(a^{**}) \right)^{\frac{1}{p^*}},
\end{aligned}$$

$$\begin{aligned}
&|\langle (T \circ A)(x, e) - (T \circ A)(x, e), a^* \rangle| \\
&\leq d_{st,p}(T)\, \|A\|_{\mathrm{BL}^{\mathrm{BL}}_{p,p_1,p_2}} \left(\int_{B_{X^{\#}}} |f(x_j) - f(y_j)|^{p_1} \, d\mu_1(f) \right)^{\frac{1}{p_1}} \left(\int_{B_{E^*}} |\varphi(e)|^{p_2} \, d\mu_2(\varphi) \right)^{\frac{1}{p_2}} \\
&\qquad \left(\int_{B_{F^{**}}} |a^*(a^{**})|^{p^*} \, d\nu_2(a^{**}) \right)^{\frac{1}{p^*}}.
\end{aligned}$$

This gives that $T \circ A \in \mathcal{N}_p^{\mathrm{BL}}(X, E; G)$ and $\eta_p^{\mathrm{BL}}(T \circ A) \leq d_{st,p}(T)\, \|A\|_{\mathrm{BL}^{\mathrm{BL}}_{p,p_1,p_2}}$. $\square$

Corollary 2.2.12.

Consider $1 \leq p \leq \infty$.with $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ Let $A \in \mathrm{BLip}_0(X, E; F)$ and there exist pointed metric space Y, Banach spaces G and $B \in \mathrm{BLip}_0(Y, G, F)$, a Lipschitz p-summing operators $R \in \mathrm{Lip}_0(X, Y)$, absolutly p-summing linear operators $T \in \mathcal{L}(E, G)$ and a strongly p-summing two-Lipschitz mapping $B \in \mathrm{BLip}_0(Y, G; F)$ such that $A = B \circ (R, T)$, then $B \circ (R, T)$ is Cohen two-Lipschitz p-nuclear operator and $\eta_p^{\mathrm{BL}}(A) \leq d_{st,p}^{\mathrm{BL}}(B) \pi_p^L(R)\, \pi_p(T)$

Proof. Let $x, y \in X,\ e \in E$, and $a^* \in F^*$. Then, it follows that

$$\begin{aligned}
&\left|\left\langle \left(B\circ(R,T)\right)(x,e) - \left(B\circ(R,T)\right)(y,e), a^*\right\rangle\right| \\
&= \left|\left\langle B\left(R\left(x\right), T\left(e\right)\right) - B\left(R\left(y\right), T\left(e\right)\right), a^*\right\rangle\right| \\
&\leq d_{st,p}^{\mathrm{BL}}(B) d\left(R\left(x\right), R\left(y\right)\right)\left\|T\left(e\right)\right\| \left(\int\limits_{B_{F^{**}}} \left|a^*(a^{**})\right|^{p^*} d\nu(a^{**})\right)^{\frac{1}{p^*}},
\end{aligned}$$

and by Pietsch domination theorem

$$\begin{aligned}
&\left|\left\langle \left(B\circ(R,T)\right)(x,e) - \left(B\circ(R,T)\right)(y,e), a^*\right\rangle\right| \\
&\leq d_{st,p}^{\mathrm{BL}}(B)\pi_p^L\left(R\right)\pi_p\left(T\right)\left(\int\limits_{B_{X^\#}} \left|f\left(x\right)-f\left(y\right)\right|^p d\mu_1(f)\right)^{\frac{1}{p}} \left(\int\limits_{B_{E^*}} \left|\varphi(e)\right|^p d\mu_2(\varphi)\right)^{\frac{1}{p}} \\
&\qquad \left(\int\limits_{B_{F^{**}}} \left|a^*(a^{**})\right|^{p^*} d\nu(b^{**})\right)^{\frac{1}{p^*}}.
\end{aligned}$$

This gives that $T\circ A \in \mathcal{N}_p^{\mathrm{BL}}(X,E;G)$ and $\eta_p^{\mathrm{BL}}(T\circ A) \leq d_{st,p}(T)\left\|A\right\|_{\mathrm{BL}_{p,p_1,p_2}^{\mathrm{BL}}}$. $\square$

Chapter 3

Tensor Representations

3.1 Tensor Norms and Two-Lipschitz Structures

In this section, we introduce the tensor norm, μ_p^{BL}, of order 3 represent two-Lipschitz ideal $\mathcal{N}_p^{\mathrm{BL}}$ i.e., the topological dual of the normed space $(X\widehat{\boxtimes}_\pi E\otimes F, \mu_p^{\mathrm{BL}})^*$ is isometric to $\left(\mathcal{N}_p^{\mathrm{BL}}\left(X, E; F^*\right), \eta_p^{\mathrm{BL}}(\cdot)\right)$.

Let $u \in X\widehat{\boxtimes}_\pi E \otimes F$, for $1 < p < \infty$, we consider

Definition 3.1.1.

Let $1 < p < \infty$, X be a pointed metric space and E and F be a Banach spaces. for each $u \in X\widehat{\boxtimes}_\pi E \otimes F$, we set

$$\mu_p^{\mathrm{BL}}(u) = \inf\left\{ \sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}} \left(\sum_{j=1}^m \left|f\left(x_j\right) - f\left(y_j\right)\right|^p \left|\varphi\left(e_j\right)\right|^p\right)^{\frac{1}{p}} \cdot \sup_{a\in B_F} \left(\sum_{j=1}^m \left|\langle a, a_j^*\rangle\right|^{p^*}\right)^{1/p^*} \right\},$$

where the infimum is taken over all representations of u of the form $u = \sum_{j=1}^m \delta_{x_j y_j} \boxtimes e_j \otimes a_j$ with $x_j, y_j \in X, e_j \in E$ and $a_j \in F, j = 1, ..., m$ and $m \in \mathbb{N}$.

Inspired by the corresponding concept in the theory of tensor product spaces, we give the following definition.

Definition 3.1.2.

Let X be a pointed metric space and let E and F be Banach spaces. A norm α on $X\widehat{\boxtimes}_\pi E\otimes F$ is called a two-Lipschitz reasonable cross-norm *if it satisfies the following conditions:*

1- For every $x, y \in X$, $e \in E$, and $b \in F$,

$$\alpha\big(\delta_{x,y} \boxtimes e \otimes b\big) \leq \|\delta_{x,y} \boxtimes e\| \, \|b\|.$$

2- For each $\varphi \in \mathrm{BLip}_0(X, E)$ and $b^* \in F^*$, the linear functional $\varphi \otimes b^* : X \widehat{\boxtimes}_\pi E \otimes F \longrightarrow \mathbb{C}$ defined by

$$(\varphi \otimes b^*)\big(\delta_{x,y} \boxtimes e \otimes b\big) = \big(\varphi(x, e) - \varphi(y, e)\big)\, b^*(b)$$

is bounded on $X \widehat{\boxtimes}_\pi E \otimes_\alpha F$, and satisfies $\|\varphi \otimes b^*\| \leq \mathrm{BLip}(\varphi)\, \|b^*\|$.

Theorem 3.1.3.

Let $1 < p < \infty$. Then μ_p^{BL} is a reasonable crossnorm.

Proof. We prove the result for $1 < p < \infty$. . Let $u = \sum_{j=1}^m \delta_{x_j y j} \boxtimes e_j \otimes a_j \in X \widehat{\boxtimes}_\pi E \otimes F$. Clearly, $\mu_p^{\mathrm{BL}}(u) > 0$. Given $\lambda \in \mathbb{K}$, since $u = \sum_{j=1}^m \lambda \delta_{x_j y_j} \boxtimes e_j \otimes a_j$ is a representation of λu, we have

$$\begin{aligned} \mu_p^{\mathrm{BL}}(\lambda u) &\leq \sup_{\substack{f \in B_{X^\#} \\ \varphi \in B_{E^*}}} \left(\sum_{j=1}^m \lambda \left| f\left(x_j\right) - f\left(y_j\right) \right|^p \left| \varphi\left(e_j\right) \right|^p \right)^{\frac{1}{p}} \cdot \sup_{a \in B_F} \left(\sum_{j=1}^m \left| \langle a, a_j^* \rangle \right|^{p^*} \right)^{1/p^*} \\ &= |\lambda| \sup_{\substack{f \in B_{X^\#} \\ \varphi \in B_{E^*}}} \left(\sum_{j=1}^m \lambda \left| f\left(x_j\right) - f\left(y_j\right) \right|^p \left| \varphi\left(e_j\right) \right|^p \right)^{\frac{1}{p}} \cdot \sup_{a \in B_F} \left(\sum_{j=1}^m \left| \langle a, a_j^* \rangle \right|^{p^*} \right)^{1/p^*}. \end{aligned} \tag{3.1}$$

Moreover, if $\lambda = 0$ we get $\mu_p^{\mathrm{BL}}(\lambda u) = 0 = |\lambda|\, \mu_p^{\mathrm{BL}}(u)$. For $\lambda \neq 0$, since (3.1) holds for every representation of u, we conclude that $\mu_p^{\mathrm{BL}}(\lambda u) \leq |\lambda|\, \mu_p^{\mathrm{BL}}(u)$.

For the converse inequality, in the same way, we have

$$\mu_p^{\mathrm{BL}}(u) = \mu_p^{\mathrm{BL}}\left(\frac{1}{\lambda} \lambda u\right) \leq \frac{1}{|\lambda|} \mu_p^{\mathrm{BL}}(\lambda u),$$

giving $|\lambda|\, \mu_p^{\mathrm{BL}}(u) \leq \mu_p^{\mathrm{BL}}(\lambda u)$. Therefore, $\mu_p^{\mathrm{BL}}(\lambda u) = |\lambda|\, \mu_p^{\mathrm{BL}}(u)$.

We now prove the triangle inequality. Let $u_1, u_2 \in X \boxtimes E \otimes F$, and let $\varepsilon > 0$. Choose representations of u_1 and u_2 in the form:

$$u_1 = \sum_{j=1}^{m_1} \delta_{x_j^1 y_j^1} \boxtimes z_j^1 \otimes a_j^1, \qquad u_2 = \sum_{j=1}^{m_2} \delta_{x_j^2 y_j^2} \boxtimes z_j^2 \otimes a_j^2,$$

so that

$$\mu_p^{\mathrm{BL}}(u_1)+\varepsilon \geq \sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_1}\left|f\left(x_j^1\right)-f\left(y_j^1\right)\right|^p\left|\varphi\left(e_j^1\right)\right|^p\right)^{\frac{1}{p}}\cdot\sup_{a^1\in B_F}\left(\sum_{j=1}^{m}|\langle a^1,a_j^{1*}\rangle|^{p^*}\right)^{1/p^*}$$

$$\mu_p^{\mathrm{BL}}(u_2)+\varepsilon \geq \sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_2}\left|f\left(x_j^2\right)-f\left(y_j^2\right)\right|^p\left|\varphi\left(e_j^2\right)\right|^p\right)^{\frac{1}{p}}\cdot\sup_{a^2\in B_F}\left(\sum_{j=1}^{m}|\langle a^2,a_j^{2*}\rangle|^{p^*}\right)^{1/p^*}.$$

Fix arbitrary $r,s\in\mathbb{C}^+$, and define:

$$\delta_{x_j^3y_j^3}\boxtimes e_j^3 = \begin{cases} r^{-1}\delta_{x_j^1y_j^1}\boxtimes z_j^1, & , \\ s^{-1}\delta_{x_j^2y_j^2}\boxtimes z_j^2, & \text{if } m_1+1\leq j\leq m_1+m_2, \end{cases}$$

$$b_j^3 = \begin{cases} r^{-1}b_j^1, & \text{if } 1\leq j\leq m_1, \\ s^{-1}b_j^2, & \text{if } m_1+1\leq j\leq m_1+m_2. \end{cases}$$

It is clear that

$$u_1+u_2=\sum_{j=1}^{m_1+m_2}\delta_{x_j^3y_j^3}\boxtimes z_j^3\otimes b_j^3,$$

and thus we have:

$$\mu_p^{\mathrm{BL}}(u_1+u_2)\leq \sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_1+m_2}\left|f\left(x_j^3\right)-f\left(y_j^3\right)\right|^p\left|\varphi\left(e_j^3\right)\right|^p\right)^{\frac{1}{p}}\cdot\sup_{a^3\in B_F}\left(\sum_{j=1}^{m_1+m_2}|\langle a^3,a_j^{3*}\rangle|^{p^*}\right)^{1/p^*}$$

An easy verification gives

$$\begin{aligned}&\left(\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_1+m_2}\left|f\left(x_j^3\right)-f\left(y_j^3\right)\right|^p\left|\varphi\left(e_j^3\right)\right|^p\right)^{\frac{1}{p}}\right)^p\\ \leq &\left(r^{-1}\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_1}\left|f\left(x_j^1\right)-f\left(y_j^1\right)\right|^p\left|\varphi\left(e_j^1\right)\right|^p\right)^{\frac{1}{p}}\right)^p \\ &+\left(s^{-1}\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_2}\left|f\left(x_j^2\right)-f\left(y_j^2\right)\right|^p\left|\varphi\left(e_j^2\right)\right|^p\right)^{\frac{1}{p}}\right)^p\end{aligned} \tag{3.2}$$

and

$$\left\|(b_j^3)_{j=1}^{m_1+m_2}\right\|_{p^*,\omega}^{p^*} = r^{p^*}\left\|(b_j^1)_{j=1}^{m_1}\right\|_{p^*,\omega}^{p^*} + s^{p^*}\left\|(a_j^2)_{i=1}^{m_2}\right\|_{p^*,\omega}^{p^*}, \tag{3.3}$$

Moreover from (3.2) (3.3) it follows that

$$
\begin{aligned}
&\mu_p^{\mathrm{BL}}(u_1+u_2)\\
&\le \tfrac{1}{p}\left(\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_1+m_2}\left|f\left(x_j^3\right)-f\left(y_j^3\right)\right|^p\left|\varphi\left(e_j^3\right)\right|^p\right)^{\frac{1}{p}}\right)^p+\tfrac{1}{p^*}\left\|(a_j^3)_{j=1}^{m_1+m_2}\right\|_{p^*,\omega}^{p^*}\\
&\le \tfrac{r^{-p}}{p}\left(\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_1}\left|f\left(x_j^1\right)-f\left(y_j^1\right)\right|^p\left|\varphi\left(e_j^1\right)\right|^p\right)^{\frac{1}{p}}\right)^p+\tfrac{r^{p^*}}{p^*}\left\|(a_j^1)_{j=1}^{m_1}\right\|_{p^*,\omega}^{p^*}\\
&+\tfrac{s^{-p}}{p}\left(\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_2}\left|f\left(x_j^2\right)-f\left(y_j^2\right)\right|^p\left|\varphi\left(e_j^2\right)\right|^p\right)^{\frac{1}{p}}\right)^p+\tfrac{s^{p^*}}{p^*}\left\|(a_j^2)_{i=1}^{m_2}\right\|_{p^*,\omega}^{p^*}
\end{aligned}
$$

Since r,s were arbitrary in $\mathbb{R}^+$, taking above

$$
\begin{aligned}
r= &\quad \left(\mu_p^{\mathrm{BL}}(u_1)+\epsilon\right)^{-\frac{1}{p}}\left(\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_1}\left|f\left(x_j^1\right)-f\left(y_j^1\right)\right|^p\left|\varphi\left(e_j^1\right)\right|^p\right)^{\frac{1}{p}}\right)\\
s= &\quad \left(\mu_p^{\mathrm{BL}}(u_2)+\epsilon\right)^{-\frac{1}{p}}\left(\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^{m_2}\left|f\left(x_j^2\right)-f\left(y_j^2\right)\right|^p\left|\varphi\left(e_j^2\right)\right|^p\right)^{\frac{1}{p}}\right)
\end{aligned}
$$

we obtain that $\mu_p^{\mathrm{BL}}(u_1+u_2)\le\mu_p^{\mathrm{BL}}(u_1)+\mu_p^{\mathrm{BL}}(u_2)+2\epsilon$, and since ϵ is arbitan we obtian $\mu_p^{\mathrm{BL}}(u_1+u_2)\le\mu_p^{\mathrm{BL}}(u_1)+\mu_p^{\mathrm{BL}}(u_2)$.

Let $f\in B_{X^\#,}$ $\varphi\in B_{E^*}$ and $\psi\in B_{F^*}$ and let $(x_j,e_j)_{i=1}^m\subset X\times E$, and $(a_i)_{j=1}^m\subset F$. It follows directly from Hölder's inequality

$$
\begin{aligned}
&\left|\sum_{j=1}^m\left(f\left(x_j\right)-f\left(y_j\right)\right)\varphi\left(e_j\right)\psi(a_j)\right|\\
&\le\sum_{j=1}^m\left\|\left(f\left(x_j\right)-f\left(y_j\right)\right)\varphi\left(e_j\right)\right\|\sum_{j=1}^m\left|\langle\psi,a_j^*\rangle\right|\\
&\le\left(\sum_{j=1}^m\left\|\left(f\left(x_j\right)-f\left(y_j\right)\right)\varphi\left(e_j\right)\right\|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^m\left|\langle\psi,a_j^*\rangle\right|^{p^*}\right)^{1/p^*}\\
&=\sup_{\substack{f\in B_{X^\#}\\ \varphi\in B_{E^*}}}\left(\sum_{j=1}^m\left\|\left(f\left(x_j\right)-f\left(y_j\right)\right)\varphi\left(e_j\right)\right\|^p\right)^{\frac{1}{p}}\sup_{\psi\in B_F}\left(\sum_{j=1}^m\left|\langle\psi,a_j^*\rangle\right|^{p^*}\right)^{1/p^*}.
\end{aligned}
$$

that taking the infimum over all representations of u we deduce that

$$
\left|\sum_{j=1}^m\left(f\left(x_j\right)-f\left(y_j\right)\right)\varphi\left(e_j\right)\psi(a_j)\right|\quad\le\mu_p^{\mathrm{BL}}(u)
$$

Then, if ϵ is the injective norm, we have

$$\begin{aligned}
\epsilon(u) &= \sup_{\substack{\xi \in B_{\left(X\widehat{\boxtimes}_\pi E\right)^*} \\ \psi \in B_{F^*}}} \left|\sum_{j=1}^m \xi(\delta_{x_j y_j} \boxtimes e_j)\psi(a_i)\right| \\
&= \sup_{\substack{f \in B_{X^\#}, \varphi \in B_{E^*} \\ \psi \in B_{F^*}}} \left|\sum_{j=1}^m \left(f\left(x_j\right) - f\left(y_j\right)\right)\varphi\left(e_j\right)\psi(a_j)\right| \\
&\leq \sup_{\substack{f\in B_{X^\#} \\ \varphi\in B_{E^*}}} \left(\sum_{j=1}^m \left\|\left(f\left(x_j\right) - f\left(y_j\right)\right)\varphi\left(e_j\right)\right\|^p\right)^{\frac{1}{p}} \left\|(a_j)_{j=1}^m\right\|_{p^*,\omega}.
\end{aligned}$$

Since it holds for every representation of u, consequently $\epsilon(u) \leq \mu_p^{\mathrm{BL}}(u)$. Thus $\mu_p^{\mathrm{BL}}(u) = 0$ imply $u = 0$. Hence μ_p^{BL} is a norm on $X\widehat{\boxtimes}_\pi E \otimes F$ and $\epsilon \leq \mu_p^{\mathrm{BL}}$.

On the other hand we have

$$\begin{aligned}
\mu_p^{\mathrm{BL}}(u) &\leq \sup_{\substack{f\in B_{X^\#} \\ \varphi\in B_{E^*}}} \left(\sum_{j=1}^m \left|\left(f\left(x_j\right) - f\left(y_j\right)\right)\varphi\left(e_j\right)\right|^p\right)^{\frac{1}{p}} \left\|(a_i)_{i=1}^n\right\|_{p^*,\omega} \\
&\leq \left(\sum_{j=1}^m \left|\left(f\left(x_j\right) - f\left(y_j\right)\right)\varphi\left(e_j\right)\right|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|a_i\|^{p^*}\right)^{\frac{1}{p^*}}.
\end{aligned}$$

By replacing in the representation of u the $\delta_{x_j y_j} \boxtimes e_j$ by

$$\frac{\left(\left|\left(f\left(x_j\right) - f\left(y_j\right)\right)\varphi\left(e_j\right)\right| \|a_j\|\right)^{\frac{1}{p}}}{d\left(x_j, y_j\right)\|e_j\|}\left(\delta_{x_j y_j} \boxtimes e_j\right),$$

and b_i by

$$\frac{\left(\left|\left(f\left(x_j\right) - f\left(y_j\right)\right)\varphi\left(e_j\right)\right| \|a_i\|\right)^{\frac{1}{p^*}}}{\|a_i\|} a_i,$$

a simple calculation gives

$$\mu_p^{\mathrm{BL}}(u) \leq \sum_{j=1}^n d\left(x_j, y_j\right)\|e_j\|\|a_j\|.$$

Taking the infimum over all representations of u, we find $\mu_p^{\mathrm{BL}}(u) \leq \pi(u)$. And we have shown that μ_p^{BL} is a reasonable crossnorm.

$\square$

3.2 Tensor Representation of Two-Lipschitz Cohen p-Nuclear Operators

In this section We investigate the tensor representation of the space of two Lipschitz factorable strongly p-summing operators from $X \times E$ to a Banach space F.

Theorem 3.2.1. *The space* $\left(\mathcal{N}_p^{\mathrm{BL}}(X, E; F), \eta_p^{\mathrm{BL}}(\cdot)\right)$ *is isometrically isomorphic to* $(X \widehat{\boxtimes}_\pi E \otimes F^*, \mu_p^{\mathrm{BL}})^*$ *through the mapping*

$$\Psi : \left(\mathcal{N}_p^{\mathrm{BL}}(X, E; F), \eta_p^{\mathrm{BL}}(\cdot)\right) \longrightarrow \left(X \widehat{\boxtimes}_\pi E \otimes F^*, \mu_p^{\mathrm{BL}}\right)^*,$$

defined by

$$\Psi(A)\left(\delta_{xy} \boxtimes_\pi e \otimes a^*\right) = \langle A(x, e) - A(y, e), a^* \rangle,$$

for every $A \in \mathcal{N}_p^{\mathrm{BL}}(X, E; F), x, y \in X, e \in E$ *and* $a^* \in F^*$.

Proof. It is easy to see that the correspondence Ψ defined as above is linear. It remains to show the surjectivity and that

$$\|\Psi(A)\|_{\left(X \widehat{\boxtimes}_\pi E \otimes F^*, \mu_p^{\mathrm{BL}}\right)^*} = \eta_p^{\mathrm{BL}}(A),$$

for all A in $\mathcal{N}_p^{BL}(X, E; F)$.

Take $A \in \mathcal{N}_p^{\mathrm{BL}}(X, E; F)$ and let $u \in X \widehat{\boxtimes}_\pi E \otimes F^*$ with the representation $u = \sum_{j=1}^{m} \delta_{x_j y_j} \boxtimes e_j \otimes a_j^*$.

We have

$$\begin{aligned}
&|\Psi(A)(u)| \\
&= \left| \sum_{j=1}^{m} \left\langle A(x_j, e_j) - A(y_j, e_j), a_j^* \right\rangle \right| \\
&\leq \sum_{j=1}^{m} \left| \left\langle A(x_j, e_j) - A(y_j, e_j), a_j^* \right\rangle \right| \\
&\leq \eta_p^{\mathrm{BL}}(A) \sup_{\substack{f \in B_{X^\#} \\ \varphi \in B_{E^*}}} \left(\sum_{j=1}^{m} |f(x_j) - f(y_j)|^p |\varphi(e_j)|^p \right)^{\frac{1}{p}} \cdot \sup_{a \in B_F} \left(\sum_{j=1}^{m} |\langle a, a_j^* \rangle|^{p^*} \right)^{1/p^*}.
\end{aligned}$$

Since it holds for every representation of u we get

$$|\Psi(A)(u)| \leq \eta_p^{\mathrm{BL}}(A) \cdot \mu_p^{\mathrm{BL}}(u).$$

It follows that

$$\|\Psi(A)\|_{\left(X \widehat{\boxtimes}_\pi E \otimes F^*, \mu_p^{\mathrm{BL}}\right)^*} \leq \eta_p^{\mathrm{BL}}(A).$$

In order to establish the reverse inequality, let $\phi \in \left(X \widehat{\boxtimes}_\pi E \otimes F^*, \mu_p^{\mathrm{BL}}\right)^*$, define the two-Lipschitz mapping $A \in \mathrm{BLip}_0(X, E; F)$ by

$$\langle A(x,e) - A(y,e), a^* \rangle = \phi(\delta_{x,y} \boxtimes e \otimes a^*).$$

Let $(x_j, y_j)_{j=1}^m \subset X \times X, (e_j)_{j=1}^m \subset E$ and $\left(a_j^*\right)_{j=1}^m \subset F^*$, then we can write

$$\begin{aligned}
&\sum_{j=1}^m \left|\left\langle A(x_j,e_j) - A(y_j,e_j), a_j^*\right\rangle\right| \\
&= \sum_{j=1}^m \left|\phi\left(\delta_{x_j y_j} \boxtimes e_j \otimes a_j^*\right)\right| \\
&\leq \|\phi\| \, \mu_p^{\mathrm{BL}} \left(\sum_{j=1}^m \delta_{x_j y_j} \boxtimes e_j \otimes a_j^*\right) \\
&\leq \|\phi\| \sup_{\substack{f \in B_{X^\#} \\ \varphi \in B_{E^*}}} \left(\sum_{j=1}^m |f(x_j) - f(y_j)|^p \, |\varphi(e_j)|^p\right)^{\frac{1}{p}} \cdot \sup_{a \in B_F} \left(\sum_{j=1}^m |\langle a, a_j^*\rangle|^{p^*}\right)^{1/p^*}.
\end{aligned}$$

this becomes that $A \in \mathcal{N}_p^{\mathrm{BL}}(X, E; F)$ and

$$\eta_p^{\mathrm{BL}}(A) \leq \|\phi\| = \|\Psi(A)\|_{\left(X \widehat{\boxtimes}_\pi E \otimes F^*, \mu_p^{\mathrm{BL}}\right)^*}.$$

□

Conclusion

In this thesis, we introduced and studied a new class of two-Lipschitz operators, namely the two-Lipschitz Cohen p-nuclear operators, as a nonlinear extension of the classical theory of p-nuclear operators. We proved that this class forms a Banach two-Lipschitz operator ideal and established a Pietsch-type domination theorem, providing an integral representation with respect to Radon probability measures. The relationships between this new class and other classes of two-Lipschitz operators, such as strongly summing and dominated operators, were systematically investigated. Finally, we developed a tensor representation theory for two-Lipschitz Cohen p-nuclear operators by introducing a new tensor norm μ_p^{BL}. A canonical isometric isomorphism was established between the space $\mathcal{N}_p^{\mathrm{BL}}(X, E; F)$ and the dual space $(X \widehat{\otimes}_\pi E \otimes F^*, \mu_p^{\mathrm{BL}})^*$. The main objective of this work was to extend the theory of operator ideals to the nonlinear two-Lipschitz setting while preserving the fundamental structural properties of the linear theory. This study opens new perspectives for the interaction between metric geometry, tensor products, and nonlinear functional analysis.

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TENSOR REPRESENTATIONS OF COHEN P-NUCLEAR TWO- LIPSCHITZ OPERATOR IDEALS

1. RESEARCH CORE OBJECTIVE

The central framework of this research focuses on the intersection between nonlinear analysis and the structural. theory of Banach spaces. Specifically, it aims to introduce and formalize new examples of two-Lipschitz operator ideals, with a rigorous emphasis on the class of two-Lipschitz Cohen p-nuclear operators. Using modern composition methods, the study investigates the deep structural properties of these mathematical objects and successfully establishes an isometric tensor representation for this specific ideal. The ultimate goal is to extend classical linear ideal frameworks into nonlinear Lipschitz analysis, providing robust structural tools for future research.

1. OBJECTIF CENTRAL DE LA RECHERCHE

Le cadre central de cette recherche se situe à l'intersection entre l'analyse non linéaire et la théorie structurelle des espaces de Banach. Plus précisément, elle vise à introduire et à formaliser de nouveaux exemples d'idéaux d'opérateurs two-Lipschitz, en mettant un accent rigoureux sur la classe des **opérateurs de Cohen -nucléaires two-Lipschitz**. En utilisant des méthodes de composition modernes, l'étude examine les propriétés structurelles profondes de ces objets mathématiques et réussit à établir une **représentation tensorielle** isométrique pour cet idéal spécifique. L'objectif ultime est d'étendre les cadres classiques des idéaux linéaires à l'analyse non linéaire de Lipschitz, fournissant ainsi des outils structurels robustes pour les recherches futures.

1- الهدف الأساسي للبحث

يتمحور الإطار المركزي لهذا البحث حول نقطة التقاطع بين التحليل غير الخطي والنظرية البنيوية لفضاءات باناخ. وبشكل أكثر دقة، يهدف البحث إلى تقديم وصياغة أمثلة جديدة لمثاليّات المؤثرات ثنائية اللبسشتز (two-Lipschitz operator ideals)، مع التركيز الصارم على فئة **مؤثرات كوهين النووية ثنائية اللبسشتز**. وباستخدام طرق التركيب الحديثة، تدرس هذه الدراسة الخصائص البنيوية العميقة لهذه الكائنات الرياضية، وتنجح في إثبات **تمثيل تينسوري** تقايسي (isometric tensor representation) لهذا المثالي المحدد. ويكمن الهدف الأسمى في توسيع أطر المثاليّات الخطية الكلاسيكية لتشمل تحليل لبسشتز غير الخطي، مما يوفر أدوات بنيوية قوية للأبحاث المستقبلية.