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Dedication and Acknowledgments

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NOTATION

Ω A open bounded domain

$\mathcal{D}(\Omega)$ Space of functions indefinitely derivables

$\mathcal{D}'(\Omega)$ Space of distribution

$\mathbf{V}(Q_T)$ Space of weak solution

$\rightharpoonup$ Weak convergence

$\rightarrow$ Strong convergence

$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N})$

$\Delta u = \sum\limits_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$

CONTENTS

INTRODUCTION

Quasi-linear Singular Parabolic Problem arise in the study of non-Newtonian fluids (in particular pseudoplastic fluids), boundary-layer phenomena for viscous fluids, in the Langmuir-Hinshelwood model of chemical heterogeneous catalyst kinetics, as well as in the theory of heat conduction in electrically conduction materials and in the study of guide modes of an electromagnetic field in non medium . Problem (Pt) with p=2 arises specifically in the study of turbulent flow of a gas in porous media. We refer to the survey Hernández -Mancebo-Vega, the book Ghergu-Radulescu and the bibliography therein for more details about corresponding models .

We are particularly interested to discuss existence of weak solution, so in the first chapter we recall some preliminaries and functional spaces that will be using later. In the second chapter, we prove the existence and uniqueness of weak solution using method of semi-discretization in time in which we construct iterative scheme to define approximated solution, then we passe to the limit to obtain the weak solution. In the third chapter we study the stationary problem associated to the Quasi-linear Singular Parabolic Problem, using strongly the construction of suitable "uniformly " sub- and super-solution which control the singular term along the flow after that we passe to the limit as t tends to infinity to show that the weak solution of the Quasi-linear Singular Parabolic Problem converge to the unique solution of the stationary problem .

CHAPTER 1

PRELIMINARIES AND FUNCTIONAL SPACES

In this chapter, we recall some basic tools that will be used in what follows.

1.1 Definition and elementary properties of L^p spaces

Throughout this chapter Ω will denote a bounded domain in $\mathbb{R}^n$. By a measurable function on Ω we shall mean an equivalence class of measurable function on Ω which differ only on a subset of measure zero. Any point-wise property attributed to measurable function will thus understood to hold in the usual sens for some function in the same equivalence class. The super-mum and inf-mom of a measurable function will then be understood as the essential super-mum and inf-mom.

Definition 1.1. *let* $p \in \mathbb{R}$ *with* $1 \leq p < \infty$ *, we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}, f \textit{is measurable and } \int_\Omega |f|^p \, dx < \infty \right\}$$

with

$$\| f \|_{L^p(\Omega)} = \left[\int_\Omega |f(x)|^p \, dx \right]^{1/p}.$$

Definition 1.2. *We set*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \middle| \begin{matrix} f \textit{ is measurable and there is constant } C \\ \textit{such that} |f(x)| \leq C \quad \textit{a.e. on } \Omega \end{matrix} \right\}$$

with

$$\| f \|_{L^\infty(\Omega)} = inf \left\{ C : |f(x)| \leq C \ a.e. \ on \ \Omega \right\}.$$

Notation Let $1 \leq p \leq \infty$, we denote by p' the conjugate exponent satisfying:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proposition 1.1. ***(Young's inequality)*** *Assume that a and b are a positive real number and $1 < p < \infty$ then*

$$ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'}.$$

Moreover for $\varepsilon > 0$

$$ab \leq \varepsilon a^p + C(\varepsilon) b^{p'}.$$

Proposition 1.2. ***(Hölder's inequality)*** *Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$ then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg| \, dx \leq \| f \|_{L^p(\Omega)} \| g \|_{L^{p'}(\Omega)} .$$

Theorem 1.3. ***(Interpolation inequality for L^p-norms)*** *Assume $1 \leq p \leq s \leq q \leq \infty$, and,*

$$\frac{1}{s} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \alpha \in \left]0, 1\right[.$$

Suppose also $f \in L^p(\Omega) \cap L^q(\Omega)$, then

$$f \in L^s(\Omega),$$

and

$$\| f \|_{L^s(\Omega)} \leq \| f \|^{\alpha}_{L^p(\Omega)} \| f \|^{1-\alpha}_{L^q(\Omega)} .$$

Theorem 1.4. ***(Lebesgue dominated convergence theorem)*** *Let (f_n) be a sequence of function in $L^p(\Omega)$ such that*

- *$f_n \to f$ a.e. on Ω,*
- *there is a function $g \in L^p(\Omega)$ such that for all $n : |f_n(x)| \leq g(x)$ a.e. in Ω.*
  Then $f \in L^p(\Omega)$ and $\| f_n - f \|_{L^p(\Omega)} \to 0$.

Theorem 1.5. ***(inverse Lebesgue theorem)*** *If $f_n \to f$ in $L^p(\Omega)$ then*

- *there exist subsequence $(f_{n,k})$, such that $f_{n,k} \to f$ a.e. in Ω,*
- *$(f_{n,k})$ is upper bounded by $g \in L^p(\Omega)$ i.e.,*

$$|f_{n,k}(x)| \leq g(x).$$

Definition 1.3. *Let E be a Banach space and $f : E \to \mathbb{R}$*

- *we say that f is weakly lower semi continuous on E, if*

  $$f(u) \leq \underline{\lim} f(u_k),$$

  whenever

  $$u_k \rightharpoonup u \quad in \quad E.$$

- *We say that f is coercive if*

  $$E(u) \to \infty \quad as \quad \|u\| \to \infty.$$

Theorem 1.6. *Suppose E is a reflexive Banach space, let f: $E \to \mathbb{R} \cup +\infty$ a weakly lower semi continuous and a coercive function. Then f is bounded from below on E and attains its infimum on E .*

Theorem 1.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then f is convex if and only if for all $x, y \in \mathbb{R}$,*

$$f(x) - f(y) \leq \nabla f(x).(x - y).$$

Proposition 1.8. *Let $p > 1$, and $a, b \in \mathbb{R}$, than*

$$(|a|^{p-2} a - |b)|^{p-2} b).(a - b) \geq 0.$$

1.2 Sobolev spaces

1.2.1 Definition

The $W^{k,p}(\Omega)$ spaces are Banach spaces analogous in certain sens to the $C^{k,\alpha}(\bar{\Omega})$. In the $W^{k,p}(\Omega)$ spaces differentiability is replaced by weak differentiability.

Definition 1.4. *For p≥1 and k a non negative integer, we set*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega) \ \ for\ all\ \alpha \in \mathbb{N}^n \ \ such\ that \ \ |\alpha| \leq k\}$$

with

$$\| f \|_{W^{k,p}(\Omega)} = \left[\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha f(x)|^p \, dx \right]^{1/p}.$$

Definition 1.5. *Given $1 \leq p < \infty$ denote by $W_0^{1,p}(\Omega)$the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$.*

Proposition 1.9. ***(Poincaré's inequality)*** *Suppose Ω is a open bounded domain . Then there exist a constant C (depending on $|\Omega| < \infty$) such that ,*

$$\| f \|_{L^p(\Omega)} \leq C \| \nabla f \|_{L^p(\Omega)}, \forall f \in W_0^{1,p}(\Omega).$$

Theorem 1.10. ***(Hardy 's inequality)*** *Let Ω be bounded open set of class C^1 and let $1 < p < \infty$. There exists a constant C such that*

$$\|\frac{u}{d}\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \qquad \forall u \in W_0^{1,p}(\Omega).$$

Conversely

$$u \in W^{1,p}(\Omega) \text{ and } (u/d) \in L^p(\Omega) \Rightarrow u \in W_0^{1,p}(\Omega),$$

$d = d(x, \partial\Omega)$.

1.2.2 Sobolev embeddings

Our goal in this subsection is to discover embedding of various Sobolev spaces into others. The crucial analytic tools here will be certain so-called "Sobolev type inequalities ".

Definition 1.6. *If $1 \leq p < n$, the Sobolev conjugate of p is*

$$p^* = \frac{np}{n-p}.$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Theorem 1.11. ***(Gagliardo-Nirenberg-Sobolev embedding)*** *Let $1 \leq p < \infty$. Assume that Ω is a bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1 .*

(i) If $1 \leq p < n$, then $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$,

(ii) if $p = n$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \forall q \in [1, +\infty[$,

(iii) if $p > n$, then $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$,

with continuous embedding.

In the next theorem we will give a Sobolev compact embedding.

Theorem 1.12. *Let $1 \leq p < \infty$. Assume that Ω is a bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1. We have*

(i) If $1 \leq p < n$, then $W^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega), \forall q \in [1, p^*[$,

(ii) if $p = n$, then $W^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega), \forall q \in [1, +\infty[$,

(iii) if $p > n$, then $W^{1,p}(\Omega) \hookrightarrow_c C(\bar{\Omega})$.

1.3 Vector value spaces

We study some other sorts of Sobolev spaces, these comprising functions mapping time into Banach spaces. These will prove essential in our constructions of weak solutions to linear parabolic Partial Differential Equation . Let X denote a real Banach space, with norm $\| \, . \, \|$.

Definition 1.7. . *The space*

$$L^p(0,T;X),$$

consists of all strongly measurable function $f : [0,T] \to X$, *with*

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p dt \right)^{1/p} < \infty,$$

for $1 < p < \infty$ *and*

$$\|f\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0 \le t \le T} \|f(t)\|_X < \infty.$$

The space $W^{1,p}(0,T;E)$ *is defined as follow*

$$W^{1,p}(0,T;E) = \{u(0,T) \to E : u, \partial_t u \in L^p(0,T;E)\}$$

Definition 1.8. *The space*

$$C\left([0,T];X\right),$$

comprise all continuous function $f : [0,T] \to X$, *with*

$$\max_{0 \le t \le T} \|f(t)\|_X < \infty.$$

The space $W^{1,p}(0,T;X)$ *is defined as follow*

$$W^{1,p}(0,T;X) = \{u : (0,T) \to X : u, \partial_t u \in L^p(0,T;X)\}$$

with the norm

$$\|u\|_{W^{1,p}(0,T;X)} = \|u\|_{L^p(0,T;X)} + \|\partial_t u\|_{L^p(0,T;X)}.$$

These spaces are reflexive and Banach spaces with their associated norms (since X is reflexive and Banach space).

Theorem 1.13. *We have the following embedding*

$$W^{1,p}(0,T;X) \hookrightarrow C\left([0,T];X\right).$$

Theorem 1.14. ***(Aubin-Simon)*** *Consider* $p \in \,]1,+\infty[$, $q \in [1,+\infty]$*and* V, E *and* F *three Banach spaces such that* $V \hookrightarrow_c E \hookrightarrow F$.*Then , if* A *is a bounded subset of* $W^{1,p}(0T;F)$ *and of* $L^q(0,T;V)$, A *is relatively compact in* $C([0,T],F)$ *and in* $L^p(0,T;F)$.

CHAPTER 2

EXISTENCE AND UNIQUENESS

In the present we investigate the following quasi-linear and singular parabolic problem ,

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \;\text{ in }\; (0,T)\times\Omega, \\ u = 0, \;\text{ on }\; (0,T)\times\partial\Omega, \;\; u > 0 \;\text{ in }\; (0,T)\times\Omega, \\ u(0,x) = u_0(x) \;\text{ in }\; \Omega. \end{cases} \tag{Pt}$$

Where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^N$, $1 < p < \infty, 0 < \delta$ and $T > 0$. We assume that $(x,s) \in \Omega \times \mathbb{R}^+ \to f(x,s)$ is bounded below Caratheodory function, asymptotically sub-homogeneous, i.e.,

$$\begin{cases} \text{if }\; p \le 2, 0 \le \limsup\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{p-1}} = \alpha_f < \lambda_1(\Omega), \\ \text{if }\; p > 2, 0 \le \limsup\limits_{t\to+\infty} \dfrac{f(x,t)}{t} = \alpha_f < \infty. \end{cases} \tag{2.1}$$

(Where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_p$ in Ω , with homogeneous Dirichlet boundary condition) and $u_0 \in W_0^{1,p}(\Omega)$. Then for any $\delta \in (0,1)$, we prove for any $T > 0$ the existence of a weak solution u

Definition 2.1. *We define the following space*

$$\boldsymbol{V}(Q_T) = \left\{ u \in L^2(Q_T) : u \in L^\infty(0,T; W_0^{1,p}(\Omega)), u_t \in L^2(Q_T) \right\}.$$

Then we give the definition of weak solution

Definition 2.2. *A weak solution to* (Pt) *is a function* $u \in \boldsymbol{V}(Q_T)$ *satisfying:*

1. *for any compact* $K \subset [0,T]\times\Omega, \operatorname*{ess\,inf}_K u > 0$,
2. *for every test function* $\Phi \in \mathcal{D}([0,T]\times\Omega)$,

$$\int_{Q_T} \left(\phi \frac{\partial u}{\partial t} + |\nabla u|^{p-2}\nabla u \nabla \phi - \phi\Big(\frac{1}{u^\delta} + f(t,u)\Big) \right) dxdt = 0, \tag{2.2}$$

3. $u(0,x) = u_0(x)$ *a.e. in* Ω

Definition 2.3. *We define* ϕ_1 *as the normalized positive eigenfunction associated to* λ_1*, the first eigenvalue of* $-\Delta_p$ *in* Ω*. with homogeneous Dirichlet boundary condition, that is*

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1}, x \in \Omega, \\ \phi_1 \qquad = 0, \qquad x \in \partial\Omega. \end{cases}$$

Theorem 2.1. *Let* $T > 0$ *,* $p > 2N/(N+2)$*,* $0 < \delta < 1$*. suppose that* $u_0 \in W_0^{1,p}(\Omega)$*such that for any compact* $K \subset \Omega$*,* $\operatorname*{ess\,inf}_K u_0 > 0$*. and that* f *is a bounded below caratheodory function, satisfying* (2.1)*. Then, there exist a weak solution to* (Pt) *.*

To prove existence of weak solution, we use a semi-discretization in time. Precisely taking advantage of the study an auxiliary quasi-linear and singular elliptic equation and energy estimate, we are able to prove existence of weak solution to (Pt) for $u_0 \in W_0^{1,p}(\Omega)$ and positive. We state this result below

2.1 Method of semi-discretization in time

We use the following iterative scheme to define approximated solution, namely $u_{\Delta_t}$and $\tilde{u}_{\Delta_t}$. More specifically, let N $\in \mathbb{N}$ and denote $\Delta_t = T \backslash N$. We construct a sequence $(u^n)_{n\in\mathbb{N}^*} \subset W_0^{1,p}(\Omega)$, verifying

$$\begin{cases} \dfrac{u^n - u^{n-1}}{\Delta_t} - \Delta_p u^n - \dfrac{1}{(u^n)^\delta} = f(x, u^{n-1}) \text{ in } \Omega, \\ u^n = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

and we define $u^0 = u_0 \in W_0^{1,p}(\Omega)$. Let us show the existence of u^n for any $n \in \mathbb{N}^*$ satisfying (2.3) that means that for $\phi \in W_0^{1,p}(\Omega)$

$$\int_\Omega \left(\phi(\frac{u^n - u^{n-1}}{\Delta_t}) + |\nabla u^n|^{p-2}\nabla u^n \nabla\phi - \phi(\frac{1}{(u^n)^\delta} + f(x, u^{n-1})) \right) dx = 0. \tag{2.4}$$

2.1.1 Existance of u^n

Proposition 2.2. *Suppose that* $u_0 \in W_0^{1,p}$ *and* $u_0 \geq \eta\phi_1$ *in* Ω *and* f *is bounded below, for* $\eta = \eta(\Omega) > 0$ *small enough,* ϕ_1 *satisfying*

$$\frac{\eta\phi_1 - u_0}{\Delta_t} - \Delta_p \eta\phi_1 - \frac{1}{(\eta\phi_1)^\delta} - f(x, u_0) < 0 \quad in\ \Omega. \tag{2.5}$$

Proof. For the first term we have

$$u_0 \geq \eta\phi_1,$$

then

$$\frac{\eta\phi_1 - u_0}{\Delta_t} < 0. \tag{2.6}$$

For the second term and since f is bounded below, we get

$$\begin{aligned} -\Delta_p \eta\phi_1 - f &= -div(|\nabla \eta\phi_1|^{p-2}\nabla(\eta\phi_1)) - f \\ &= -\eta^{p-1} div(|\nabla\phi_1|^{p-2}\nabla\phi_1) - f(x,u_0) \\ &= -\eta^{p-1}\Delta_p\phi_1 - f(x,u_0) \\ &= \eta^{p-1}\lambda_1\phi_1^{p-1} - f(x,u_0) \\ &< \eta^{p-1}\lambda_1\phi_1^{p-1} - C, \end{aligned}$$

since $\phi_1 < C$ and η small enough, we obtain

$$\eta^{p-1}\lambda_1\phi_1^{p-1} - C < C'. \tag{2.7}$$

For the third term and (2.7), we get

$$C' - \frac{1}{(\eta\phi_1)^\delta} < 0, \tag{2.8}$$

from (2.6) end(2.8) we get (2.5) .
Now we wont to prove the existence of u^n, for this purpose, we use a variational method. We consider the following energy functional E_n defined in $W_0^{1,p}(\Omega)$ by

$$\begin{aligned} E_n = \frac{1}{\Delta_t}\left(\int_\Omega \frac{u^2}{2}dx - \int_\Omega uu^{n-1}dx\right) + \frac{1}{p}\int_\Omega |\nabla u|^p dx \quad &- \int_\Omega \frac{(u^+)^{1-\delta}}{1-\delta}dx \\ &- \int_\Omega f(x,u^{n-1})\, u^+ dx. \end{aligned}$$

Proposition 2.3. *$E_n(u)$ is coercive, weakly lower semi continuous and strictly convex.*

Proof.

- We show that $E_n(u)$ is coercive, that is $E_n(u) \to +\infty$, as $\|u\|_{W_0^{1,p}(\Omega)} \to +\infty$.
  For the first term
  $$\frac{1}{\Delta_t}\int_\Omega \frac{u^2}{2}dx = \frac{1}{2\Delta_t}\|u\|^2_{L^2(\Omega)}. \tag{2.9}$$
  For the second term, from Hölder's inequality 1.2,
  $$\begin{aligned} \int_\Omega uu^{n-1}dx &\leq \|u\|_{L^2(\Omega)}\|u^{n-1}\|_{L^2(\Omega)} \\ &\leq C\|u\|_{L^2(\Omega)}, \end{aligned}$$
  from Sobolev embedding, we get
  $$\int_\Omega uu^{n-1}dx \leq C\|u\|_{W_0^{1,p}(\Omega)}. \tag{2.10}$$

For the third term

$$\frac{1}{p}\int_\Omega |\nabla u|^p dx = \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}. \tag{2.11}$$

For the fourth term, from Hölder's inequality 1.2 and Young's inequality 1.1,

$$\begin{aligned}\frac{1}{1-\delta}\int_\Omega (u^+)^{1-\delta}dx &\leq C\left(\int_\Omega (u^+)^2 dx\right)^{1-\delta/2}\\ &\leq C\|(u^+)\|_{L^2(\Omega)} + C',\end{aligned}$$

from Sobolev embedding, we get

$$\frac{1}{1-\delta}\int_\Omega (u^+)^{1-\delta}dx \leq C\|(u^+)\|_{W_0^{1,p}(\Omega)} + C'. \tag{2.12}$$

Four the fifth term, from Hölder's inequality 1.2,

$$\begin{aligned}\int_\Omega f(x,u^{n-1})u^+ dx &\leq \left(\int_\Omega f(x,u^{n-1})^2 dx\right)^{1/2}\left(\int_\Omega (u^+)^2 dx\right)^{1/2}\\ &\leq C\|(u^+)\|_{L^2(\Omega)},\end{aligned}$$

from Sobolev embedding, we get

$$\int_\Omega f(x,u^{n-1})u^+ dx \leq C\|(u^+)\|_{W_0^{1,p}(\Omega)}. \tag{2.13}$$

Therefore,from(2.9),(2.10),(2.11),(2.12)and (2.13), thus

$$\begin{aligned}\lim_{\|u\|_{W_0^{1,p}(\Omega)}\to+\infty} E_n(u) &\geq \lim_{\|u\|_{W_0^{1,p}(\Omega)}\to+\infty}\Big(\frac{1}{\Delta_t}\|u\|^2_{L^2(\Omega)} - C\|u\|_{W_0^{1,p}(\Omega)} + \frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}\\ &\quad - C\|(u^+)\|_{W_0^{1,p}(\Omega)} - C' - C\|(u^+)\|_{W_0^{1,p}(\Omega)}\Big).\end{aligned}$$

Since $p > 1$, we get

$$\begin{aligned}\lim_{\|u\|_{W_0^{1,p}(\Omega)}\to+\infty} E_n(u) &\geq \lim_{\|u\|_{W_0^{1,p}(\Omega)}\to+\infty}\frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}\\ &\geq \infty.\end{aligned}$$

- $E_n(u)$ is weakly lower semi continuous in $W_0^{1,p}(\Omega)$, from theorem 1.3, we show that

$$E_n(u) \leq \varliminf_{k\to+\infty} E_n(u_k),$$

Let (u_k) be a sequence in $W_0^{1,p}(\Omega)$ converges weakly to u. Then since $p > 2N/N+2$, we have $W_0^{1,p}(\Omega)$ inject compactly in $L^2(\Omega)$. Then we have for the first and second term

$$\frac{1}{\Delta_t}\int_\Omega \frac{u^2}{2}dx = \frac{1}{\Delta_t}\varliminf_{k\to+\infty}\int_\Omega \frac{u_k^2}{2}dx, \tag{2.14}$$

$$\int_\Omega u u^{n-1}dx = \varliminf_{k\to+\infty}\int_\Omega u_k u^{n-1}dx. \tag{2.15}$$

For the the third term

$$\frac{1}{p}\int_{\Omega}|\nabla u|^{p}dx \leq \frac{1}{p}\varliminf_{k\to+\infty}\int_{\Omega}|\nabla u_k dx|^{p}dx. \tag{2.16}$$

For the fourth and fifth term, we have from the Sobolev embedding,

$$\int_{\Omega}\frac{u^{1-\delta}}{1-\delta}dx = \varliminf_{k\to+\infty}\int_{\Omega}\frac{(u_k)^{1-\delta}}{1-\delta}dx, \tag{2.17}$$

$$\int_{\Omega}f(x,u^{n-1})u^{+}dx = \varliminf_{k\to+\infty}\int_{\Omega}f(x,u^{n-1})u_k^{+}dx. \tag{2.18}$$

Therefor,(2.14)together with (2.15)and (2.16), (2.17) and (2.18) yield

$$E_n(u) \leq \varliminf_{k\to+\infty} E_n(u_k).$$

- We now show that $E_n(u)$ is convex
  It suffices to apply theorem 1.7, let u, v in $W_0^{1,p}(\Omega)$, then
  for the first therm

$$(2u-2v)(u-v) \geq 0. \tag{2.19}$$

For the second term , which is convex and concave

$$(u^{n-1}-u^{n-1})(u-v) = 0. \tag{2.20}$$

For the third term, by proposition 1.8, we have

$$(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v).(|\nabla u| - |\nabla v|) \geq 0. \tag{2.21}$$

For the fourth and fifth terms, we apply theorem 1.7, we obtain

$$-((u^{+})^{-\delta}-(v^{+})^{-\delta})(u-v) \geq 0, \tag{2.22}$$

$$(f(x,u^{n-1})-f(x,u^{n-1}))(u-v) = 0. \tag{2.23}$$

Therefore from (2.19), (2.20), (2.21), (2.22) and (2.23) $E_n(u)$ is convex .

Proposition 2.4. *For all $n \in N^*$, (2.3) admit weak solution u^n and satisfying*

$$u^n \in W_0^{1,p}(\Omega) \quad \textit{and} \quad u^n \geq \eta\phi_1 \quad \textit{on} \quad \Omega.$$

Proof. We prove by induction, for that assuming that

$$u^{n-1} \in W_0^{1,p}(\Omega) \quad \text{and} \quad u^{n-1} \geq \eta\phi_1 \quad \text{on} \quad \Omega.$$

Notice that from Sobolev embeddings and since E_n is bounded below, coercive, weakly lower semi continuous in $W_0^{1,p}(\Omega)$and strictly convex in the positive cone of $W_0^{1,p}(\Omega)$. Then from theorem 1.6, E_n admits a unique global minimizer, we denote by u^n, in $W_0^{1,p}(\Omega)$ and $u^n \geq 0$ a.e. in Ω. We show

$$u^n \geq (\eta\phi_1) \quad \text{on } \Omega.$$

Let us consider

$$\begin{array}{rcl} \psi & = & (\eta\phi_1 - u^n)^+ \in W_0^{1,p}(\Omega), \\ \psi & = & \begin{cases} \eta\phi_1 - u^n, & \eta\phi_1 \geq u^n, \\ 0\ , & \eta\phi_1 \leq u^n. \end{cases} \end{array}$$

Let us notice that ψ has a compact support included in Ω. Since u^n is the global minimizer of E_n,

$$\xi(t) = E_n(u^n + t\psi) \geq E(u^n) \quad \forall t \geq 0.$$

Moreover, since ϕ_1 satisfies $\phi_1 \geq \eta_0 \operatorname{dist}(x, \partial\Omega)$ for some $\eta_0 > 0$ small enough and $\delta < 1$, we have by Lebesgue theorem that

$$\lim_{t\to t_0} \frac{1}{t-t_0}\left[\int_\Omega \frac{(u^n+t\psi)^{1-\delta}}{1-\delta}dx - \int_\Omega \frac{(u^n+t\psi)^{1-\delta}}{1-\delta}dx\right] = \int_\Omega (u^n + t_0\psi)^{-\delta}\psi dx, \tag{2.24}$$

for $t_0 \in (0, 1]$, and then $\xi(t)$ is differentiable in $(0, 1]$. From the convexity of $\xi(t)$ and the varational nature of u^n, we obtain that

$$\forall t \in (0,1), 0 \leq \xi^{'}(t) \leq \xi^{'}(1). \tag{2.25}$$

Furthermore,

$$\begin{array}{rcl} \xi^{'}(t) & = & D\xi(t).1 \\ & = & DE_n\left(u^n + t\psi\right).\left(D(u^n+t\psi).1\right) \\ & = & DE_n\left(u^n + t\psi\right).\psi, \end{array}$$

for t=1,

$$\begin{array}{rcl} \xi^{'}(1) & = & DE_n\left(u^n + 1\psi\right).\psi \\ & = & DE_n\left(\eta\phi_1\right).\psi \\ & = & \dfrac{1}{\Delta_t}\displaystyle\int_{\Omega^+}\eta\phi_1\psi dx - \dfrac{1}{\Delta_t}\int_{\Omega^+}u^{n-1}\psi dx + \int_{\Omega^+}|\nabla\eta\phi_1|^{p-2}\nabla\eta\phi_1\nabla\psi dx \\ & & -\displaystyle\int_{\Omega^+}\dfrac{1}{(\eta\phi_1)^\delta}\psi dx - \int_{\Omega^+}f(x,u^{n-1})\psi dx + \dfrac{1}{\Delta_t}\int_{\Omega^-}\eta\phi_1\psi dx - \dfrac{1}{\Delta_t}\int_{\Omega^-}u^{n-1}\psi dx \\ & & +\displaystyle\int_{\Omega^-}|\nabla\eta\phi_1|^{p-2}\nabla\eta\phi_1\nabla\psi dx - \int_{\Omega^-}\dfrac{1}{(\eta\phi_1)^\delta}\psi dx - \int_{\Omega^-}f(x,u^{n-1})\psi dx, \end{array}$$

we have ψ=0 on Ω^-, then

$$\xi^{'}(1) = \langle\frac{\eta\phi_1 - u^{n-1}}{\Delta_t} - \Delta_p(\eta\phi_1) - \frac{1}{(\eta\phi_1)^\delta} - f(x, u^{n-1}), \psi\rangle.$$

If the measure of the support ψ is different of zero, from (2.5) we get $\xi^{'}(1) < 0$ and there by a contradiction with (2.25). Thus

$$u^n + \psi = \begin{cases} \eta\phi_1, & \eta\phi_1 \geq u^n, \\ u^n\ , & \eta\phi_1 \leq u^n. \end{cases}$$

$$\text{i.e., } \eta\phi_1 \le u^n \text{in } \Omega \text{ for every n} \ge 0,$$

for any u $\in W_0^{1,p}(\Omega)$. Then, for $\phi \in \mathcal{D}(\Omega)$,

$$\lim_{t\to 0}\frac{E_n(u^n + t\phi) - E_n(u^n)}{t} = 0.$$

Consequently, u^n satisfies the Euler-Lagrange equation, namely (2.3), in the sense of distribution, that is (2.4) is satisfied for any $\phi \in \mathcal{D}(\Omega)$. By density argument and since u^n in $W_0^{1,p}(\Omega)$, we get (2.4) is satisfied for any ϕ in $W_0^{1,p}(\Omega)$.

2.1.2 A priori estimates

We define for all $n \in \{1, ..., N\}$, the following functions

$$\forall t \in [(n-1)\Delta_t, n\Delta_t), \begin{cases} u_{\Delta_t} = u^n, \\ \tilde{u}_{\Delta_t} = \dfrac{(t-(n-1)\Delta_t)}{\Delta_t}(u^n - u^{n-1}) + u^{n+1}. \end{cases} \tag{2.26}$$

Then $u_{\Delta_t}, \tilde{u}_{\Delta_t}$ are well defined and satisfied in addition $u_{\Delta_t}, \tilde{u}_{\Delta_t} \ge \eta\phi_1$ on Ω. In addition , we have that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \Delta_p u_{\Delta_t} - \frac{1}{(u_{\Delta_t})^\delta} = f(x, u_{\Delta_t}(. - \Delta_t)). \tag{2.27}$$

Which implies that for any $\phi \in C_c^\infty([0,T]\times\Omega)$

$$\int_0^T\int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t}\phi dxdt - \int_0^T\int_\Omega |\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}\nabla\phi dxdt - \int_0^T\int_\Omega \frac{1}{(u_{\Delta_t})^\delta}\phi dxdt$$
$$= \int_0^T\int_\Omega f(x, u_{\Delta_t}(. - \Delta_t))dxdt. \tag{2.28}$$

We have that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \in L^\infty(0,T; W_0^{1,p}(\Omega))$$

and from $p > 2N/(N+2)$ and (2.1), we obtain that

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t}, \Delta_p u_{\Delta_t}, f(x, u_{\Delta_t}) \text{ belong to } L^{p'}(0,T; W^{-1,p'}(\Omega)).$$

Then, (2.27) hold on in $(L^p(0T; W_0^{1,p}(\Omega)))'$. Therefore, for any $\phi \in L^p(0,T; W_0^{1,p}(\Omega))$

$$\int_0^T\int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} dxdt - \int_0^T\int_\Omega |\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}\nabla\phi dxdt - \int_0^T\int_\Omega \frac{1}{(u_{\Delta_t})^\delta} dxdt$$
$$= \int_0^T\int_\Omega f(x, u_{\Delta_t}(. - \Delta_t))dxdt. \tag{2.29}$$

We now derive some energy estimates on $u_{\Delta_t}\tilde{u}_{\Delta_t}$.

Proposition 2.5.

$$u_{\Delta_t}, \tilde{u}_{\Delta_t} \quad \textit{is bounded in} \quad L^{\infty}(0,T;L^2(\Omega)), \tag{2.30}$$

$$u_{\Delta_t}, \tilde{u}_{\Delta_t} \quad \textit{is bounded in} \quad L^{p}(0,T;W_0^{1,p}(\Omega)). \tag{2.31}$$

Proof. Taking $\varepsilon > 0$ small enough such that $\alpha + 2\varepsilon < \lambda_1(\Omega)$, and multiplying (2.3) by $\Delta_t u^n$, summing from $n = 1$ to N and integrating over Ω we obtain

$$\Delta_t \sum_{n=1}^{N} \int_{\Omega} \frac{u^n - u^{n-1}}{\Delta_t} u^n dx + \Delta_t \sum_{n=1}^{N} \int_{\Omega} |\nabla u^n|^p dx - \Delta_t \sum_{n=1}^{N} \int_{\Omega} {u^n}^{1-\delta} dx$$
$$= \Delta_t \sum_{n=1}^{N} \int_{\Omega} f(x, u^{n-1}) u^n dx.$$

From proposition **??**, the above expression implies

$$\sum_{n=1}^{N} \frac{1}{2} \int_{\Omega} (|u^n|^2 - |u^{n-1}|^2 + |u^n - u^{n-1}|^2) dx + \Delta_t \sum_{n=1}^{N} \int_{\Omega} |\nabla u^n|^p dx - \sum_{n=1}^{N} \int_{\Omega} (u^n)^{1-\delta} dx$$
$$= \Delta_t \sum_{n=1}^{N} \int_{\Omega} f(x, u^{n-1}) u^n dx.$$

Now since f is bounded below and satisfies (2.1), $\exists C = C(\alpha)$ >0 large enough such that

- **Case 1 :** $p \leq 2$,

  $$0 \leq \limsup_{t \to +\infty} \frac{f(x,t)}{t^{p-1}} = \alpha_f < \lambda_1(\Omega).$$

  For t $> C(\alpha)$,

  $$\frac{f(x,t)}{t^{p-1}} \leq \alpha,$$
  $$f(x,t) \leq \alpha t^{p-1}. \tag{2.32}$$

  For $0 \leq t \leq C(\alpha)$,

  $$f(x,t) \leq C. \tag{2.33}$$

  From (2.32) and (2.33), thus

  $$f(x,t) \leq \alpha t^{p-1} + C. \tag{2.34}$$

- **Case 2 :** p > 2,

  $$0 \leq \limsup_{t \to +\infty} \frac{f(x,t)}{t} = \alpha_f < \infty.$$

  By Young's inequality 1.1, we have

  $$\begin{aligned} 0 \leq f(x,t) = \alpha t \quad &\leq \quad \alpha\left(\frac{1}{p-1}(t)^{p-1} + \frac{p-2}{p-1}(1)^{(p-1/p-2)}\right) \\ &\leq \quad \alpha\left(C(t)^{p-1} + C'\right), \end{aligned}$$

  then

  $$f(x,t) \leq \quad \alpha t^{p-1} + C. \tag{2.35}$$

Where $\alpha < \lambda_1(\Omega)$. Then, the term $\Delta_t \sum_{n=1}^{N} \int_\Omega f(x,u^{n-1})u^n dx$ in the right hand side can be estimated as follows :

$$\Delta_t \sum_{n=1}^{N} \int_\Omega f(x,u^{n-1})u^n dx \leq \Delta_t \sum_{n=1}^{N} \left[\alpha \int_\Omega (u^{n-1})^{p-1} u^n dx + C\int_\Omega u^n dx\right].$$

Now, applying Young's inequality 1.1, on the two last terms in the right hand side we obtain for any $\varepsilon > 0$

$$\begin{aligned}
\Delta_t \sum_{n=1}^{N} \int_\Omega f(x,u^{n-1})u^n dx \quad &\leq \quad \Delta_t \sum_{n=1}^{N} \left[\alpha \int_\Omega (u^{n-1})^{p-1} u^n dx + C\int_\Omega u^n \frac{\varepsilon^{1/p}}{\varepsilon^{1/p}} dx\right] \\
&\leq \quad \alpha\ \Delta_t \sum_{n=1}^{N} \left[\frac{1}{p/p-1}\int_\Omega (u^{n-1})^p dx + \frac{1}{p}\int_\Omega (u^n)^p dx\right] \\
&\quad + \Delta_t \sum_{n=1}^{N} \left[\int_\Omega \frac{1}{p}\varepsilon (u^n)^p dx + \frac{1}{p'}\frac{1}{\varepsilon^{p/p'}}\int_\Omega C^{p'} dx\right] \\
&\leq \quad \alpha\ \Delta_t \sum_{n=1}^{N} \left[\frac{p-1}{p}\int_\Omega (u^{n-1})^p dx + \frac{1}{p}\int_\Omega (u^n)^p dx\right] \\
&\quad + \Delta_t \sum_{n=1}^{N} \int_\Omega \varepsilon (u^n)^p dx + C_\varepsilon T|\Omega| \\
&\leq \quad \alpha\ \Delta_t \sum_{n=0}^{N-1} \frac{p-1}{p}\int_\Omega (u^n)^p dx + \Delta_t \sum_{n=1}^{N} (\frac{\alpha}{p} + \varepsilon)\int_\Omega (u^n)^p dx + C_\varepsilon T|\Omega| \\
&\leq \quad \alpha\ \Delta_t \frac{p-1}{p}\int_\Omega (u^0)^p dx + \alpha\ \Delta_t \sum_{m=1}^{N-1} \frac{p-1}{p}\int_\Omega (u^n)^p dx \\
&\quad + \alpha\ \Delta_t \frac{p-1}{p}\int_\Omega (u^N)^p dx + \Delta_t \sum_{n=1}^{N} (\frac{\alpha}{p} + \varepsilon)\int_\Omega (u^n)^p dx + C_\varepsilon T|\Omega| \\
&\leq \quad \alpha\ \Delta_t \frac{p-1}{p}\int_\Omega (u^0)^p dx + \alpha\Delta_t \sum_{n=1}^{N} \frac{p-1}{p}\int_\Omega (u^n)^p dx \\
&\quad + \Delta_t \sum_{n=1}^{N} (\frac{\alpha}{p} + \varepsilon)\int_\Omega (u^n)^p dx + C_\varepsilon T|\Omega| \\
&\leq \quad \alpha\ \Delta_t \frac{p-1}{p}\int_\Omega (u^0)^p dx + \ \Delta_t \sum_{n=1}^{N} (\alpha + \varepsilon)\int_\Omega (u^n)^p dx + C_\varepsilon T|\Omega|.
\end{aligned}$$

For a constant C_ε large enough depending on ε and α.
Using again Young's inequality 1.1, we estimate the last term in the left hand side in the following way

$$\Delta_t \sum_{n=1}^{N} \int_\Omega (u^n)^{1-\delta} dx \ \leq \ \Delta_t \sum_{n=1}^{N} \int_\Omega \varepsilon (u^n)^p dx + C'_\varepsilon.$$

Where C'_ε >0 is large enough depends on ε and δ.

Now using Poincaré's inequality 1.9, we get

$$\begin{aligned}
\sum_{n=1}^{N}\frac{1}{2}\int_{\Omega}(|u^n|^2-|u^{n-1}|^2+|u^n-u^{n-1}|^2)dx+\Delta_t\sum_{n=1}^{N}\int_{\Omega}|\nabla u^n|^p dx \\
\le \alpha\Delta_t\frac{p-1}{p}\int_{\Omega}(u^0)^p dx+\Delta_t\sum_{n=1}^{N}(\alpha+\varepsilon)\int_{\Omega}(u^n)^p dx+C_{\varepsilon}T|\Omega| \\
+\Delta_t\sum_{n=1}^{N}\int_{\Omega}\varepsilon(u^n)^p dx+C'_{\varepsilon}T|\Omega| \\
\le \alpha\Delta_t\frac{p-1}{p}\int_{\Omega}(u^0)^p dx+\Delta_t\sum_{n=1}^{N}(\frac{\alpha+2\varepsilon}{\lambda_1(\Omega)})\int_{\Omega}|\nabla u^n|^p dx+\tilde{C}_{\varepsilon}T|\Omega|,
\end{aligned}$$

then

$$\begin{aligned}
\sum_{n=1}^{N}\frac{1}{2}\int_{\Omega}(|u^n|^2-|u^{n-1}|^2+|u^n-u^{n-1}|^2)dx+\Delta_t(1-\frac{\alpha+2\varepsilon}{\lambda_1(\Omega)})\sum_{n=1}^{N}\int_{\Omega}|\nabla u^n|^p dx \\
\le \alpha\Delta_t\frac{p-1}{p}\int_{\Omega}(u^0)^p dx+\tilde{C}_{\varepsilon}T|\Omega| \\
\le C.
\end{aligned}$$

Where $\tilde{C}$ is large constant depending on ε and δ.
Taking $\varepsilon > 0$ small enough such that $\alpha+2\varepsilon<\lambda_1(\Omega)$, it follows that

$$\Delta_t\sum_{n=1}^{N}\int_{\Omega}|\nabla u^n|^p dx\le C,$$

then u_{Δ_t} is bounded in $L^p(0,T;W_0^{1,p}(\Omega)$.
In the other hand, we have

$$\begin{aligned}
\sum_{n=1}^{N}\frac{1}{2}\int_{\Omega}(|u^n|^2-|u^{n-1}|^2dx &\le C \\
\int_{\Omega}|u^N|^2dx-\int_{\Omega}|u^0|^2dx &\le C \\
\int_{\Omega}|u^N|^2dx &\le C+\int_{\Omega}|u^0|^2dx \\
\int_{\Omega}|u^N|^2dx &\le C \\
\sup_{0\le t\le T}\int_{\Omega}|u_{\Delta_t}|^2dx &\le C.
\end{aligned}$$

Then u_{Δ_t} is bounded in $L^\infty(0,T;L^2(\Omega))$. And we have

$$\begin{aligned}
|\nabla\tilde{u}_{\Delta_t}| &= |\frac{\nabla u^n-\nabla u^{n-1}}{\Delta_t}(t-t_{n-1})+\nabla u^{n-1}| \\
&\le |\frac{\nabla u^n-\nabla u^{n-1}}{\Delta_t}||(t-t_{n-1})|+|\nabla u^{n-1}| \\
&\le |\nabla u^n|+2|\nabla u^{n-1}| \\
|\nabla\tilde{u}_{\Delta_t}|^p &\le (|\nabla u_{\Delta_t}(t)|+2|\nabla u_{\Delta_t}(t-\Delta_t)|)^p \\
\int_\Omega|\nabla\tilde{u}_{\Delta_t}|^p dx &\le C\left(\int_\Omega|\nabla u_{\Delta_t}(t)|^p dx+\int_\Omega|\nabla u_{\Delta_t}(t-\Delta_t)|^p dx\right) \\
\int_0^T\int_\Omega|\nabla\tilde{u}_{\Delta_t}|^p dxdt &\le C\left(\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\int_\Omega|\nabla u_{\Delta_t}(t)|^p dxdt+\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\int_\Omega|\nabla u_{\Delta_t}(t-\Delta_t)|^p dxdt\right) \\
&\le C\Delta_t\left(\sum_{n=1}^N\int_\Omega|\nabla u_{\Delta_t}(t)|^p dx+\sum_{n=1}^N\int_\Omega|\nabla u_{\Delta_t}(t-\Delta_t)|^p dx\right)<\infty.
\end{aligned}$$

Then $\tilde{u}_{\Delta_t}$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$. And also we have

$$\begin{aligned}
|\tilde{u}_{\Delta_t}| &= |\frac{u^n-u^{n-1}}{\Delta_t}(t-t_{n-1})+u^{n-1}| \\
&\le |\frac{u^n-u^{n-1}}{\Delta_t}||(t-t_{n-1})|+|u^{n-1}| \\
&\le |u^n|+2|u^{n-1}| \\
|\tilde{u}_{\Delta_t}|^2 &\le (|u_{\Delta_t}(t)|+2|u_{\Delta_t}(t-\Delta_t)|)^2 \\
\int_\Omega|\tilde{u}_{\Delta_t}|^2 dx &\le C\left(\int_\Omega|u_{\Delta_t}(t)|^2 dx+\int_\Omega|u_{\Delta_t}(t-\Delta_t)|^2 dx\right) \\
\sup_{0\le t\le T}\int_\Omega|\tilde{u}_{\Delta_t}|^2 dx &\le C\left(\sup_{0\le t\le T}\int_\Omega|u_{\Delta_t}(t)|^2 dx+\sup_{0\le t\le T}\int_\Omega|u_{\Delta_t}(t-\Delta_t)|^2 dx\right)<\infty.
\end{aligned}$$

Then $\tilde{u}_{\Delta_t}$ is bounded in $L^\infty(0,T;L^2(\Omega))$.
We now derive the second energy estimates.

Proposition 2.6.

$$\frac{\partial\tilde{u}_{\Delta_t}}{\partial t}\ \ \textit{is bounded in}\ \ L^2(Q_T)\ \ \textit{uniformly in}\ \ \Delta_t. \tag{2.36}$$

Proof. Using (2.31) and multiplying (2.3) by (u^n-u^{n-1}) summing from $n=1$ to N and interring over Ω, we get

$$\Delta_t\sum_{n=1}^N\int_\Omega(\frac{u^n-u^{n-1}}{\Delta_t})^2dx+\sum_{n=1}^N\int_\Omega|\nabla u^n|^{p-2}\nabla u^n.\nabla(u^n-u^{n-1})dx$$

$$-\sum_{n=1}^N\int_\Omega\frac{u^n-u^{n-1}}{(u^n)^\delta}dx=\sum_{n=1}^N\int_\Omega f(x,u^{n-1})(u^n-u^{n-1})dx.$$

For the second hand, we have

$$\sum_{n=1}^{N}\int_\Omega f(x,u^{n-1})(u^n-u^{n-1})\ \frac{\Delta_t}{\Delta_t}dx$$
$$=\frac{\Delta_t}{2}\sum_{n=1}^{N}\int_\Omega\left[f(x,u^{n-1})^2+(\frac{u^n-u^{n-1}}{\Delta_t})^2\right]dx,$$

then, we get

$$\sum_{n=1}^{N}\int_\Omega f(x,u^{n-1})(u^n-u^{n-1})\ \frac{\Delta_t}{\Delta_t}dx$$
$$=\frac{\Delta_t}{2}\sum_{n=1}^{N}\int_\Omega f(x,u^{n-1})^2dx+\frac{\Delta_t}{2}\sum_{n=1}^{N}\int_\Omega(\frac{u^n-u^{n-1}}{\Delta_t})^2dx. \qquad (2.37)$$

Therefore, (2.37) together with (2.30) and (2.1) yield

$$\frac{\Delta_t}{2}\sum_{n=1}^{N}\int_\Omega(\frac{u^n-u^{n-1}}{\Delta_t})^2dx$$
$$+\sum_{n=1}^{N}\int_\Omega|\nabla u^n|^{p-2}\nabla u^n.\nabla(u^n-u^{n-1})dx$$
$$-\sum_{n=1}^{N}\int_\Omega\frac{u^n-u^{n-1}}{(u^n)^\delta}dx=\frac{\Delta_t}{2}\sum_{n=1}^{N}\int_\Omega f(x,u^{n-1})^2\le C. \qquad (2.38)$$

This gives the result.

Proposition 2.7.

$$\tilde u_{\Delta_t},u_{\Delta_t}\quad \textit{is bounded in}\quad L^\infty(0,T;W_0^{1,p}(\Omega))\quad \textit{uniformly in}\quad \Delta_t. \qquad (2.39)$$

Proof. From the convexity of the expression $\int_\Omega|\nabla u|^pdx$ and $-\dfrac{1}{1-\delta}\int_\Omega u^{1-\delta}dx$ we get the following inequalities :

$$\frac{1}{2}\left[\int_\Omega|\nabla u^n|^pdx-\int_\Omega|\nabla u^{n-1}|^pdx\right]\le\int_\Omega|\nabla u^n|^{p-2}\nabla u^n\nabla(u^n-u^{n-1})dx,$$

and

$$-\frac{1}{1-\delta}\left[\int_\Omega(u^{n-1})^{1-\delta}dx-\int_\Omega(u^n)^{1-\delta}dx\right]\le-\int_\Omega\frac{u^n-u^{n-1}}{(u^n)^\delta}dx\ ,$$

using the above inequalities and from (2.36), (2.38), multiplying (2.3) by (u^n-u^{n-1}) and summing from $n=1$ to $N'<N$, such that $t\in]t_{N'}-t_{N'+1}]$,we get

$$\frac{\Delta_t}{2}\sum_{n=1}^{N'}\int_\Omega(\frac{u^n-u^{n-1}}{\Delta_t})^2dx+\sum_{n=1}^{N'}\int_\Omega|\nabla u^n|^{p-2}\nabla u^n.\nabla(u^n-u^{n-1})dx$$
$$-\sum_{n=1}^{N'}\int_\Omega\frac{u^n-u^{n-1}}{(u^n)^\delta}dx\le C$$
$$\sum_{n=1}^{N'}\int_\Omega|\nabla u^n|^{p-2}\nabla u^n.\nabla(u^n-u^{n-1})dx\le C+C'+\sum_{n=1}^{N'}\frac{1}{1-\delta}\left[\int_\Omega(u^{n-1})^{1-\delta}dx-\int_\Omega(u^n)^{1-\delta}dx\right]$$
$$\sum_{n=1}^{N'}\int_\Omega|\nabla u^n|^{p-2}\nabla u^n.\nabla(u^n-u^{n-1})dx\le C+\frac{1}{1-\delta}\left[\int_\Omega(u^0)^{1-\delta}dx-\int_\Omega(u^{N'})^{1-\delta}dx\right],$$

for $p > 2N/(N+2)$ and since $0 < 1-\delta < 1$,

$$\begin{aligned}\sum_{n=1}^{N'}\int_\Omega |\nabla u^n|^{p-2}\nabla u^n.\nabla(u^n-u^{n-1})dx &\leq C+\frac{1}{1-\delta}\left[\int_\Omega (u^0)^{1-\delta}dx-\int_\Omega (u^{N'})^{1-\delta}dx\right]\\ &\leq C+C'\ ,\end{aligned}$$

using the last inequality of convexity, we get

$$\begin{aligned}\frac{1}{2}\left[\int_\Omega |\nabla u^n|^p dx-\int_\Omega |\nabla u^{n-1}|^p dx\right] &\leq \int_\Omega |\nabla u^n|^{p-2}\nabla u^n\nabla(u^n-u^{n-1})dx,\\ \frac{1}{2}\sum_{n=1}^{N'}\left[\int_\Omega |\nabla u^n|^p dx-\int_\Omega |\nabla u^{n-1}|^p dx\right] &\leq \sum_{n=1}^{N'}\int_\Omega |\nabla u^n|^{p-2}\nabla u^n\nabla(u^n-u^{n-1})dx < C\\ \int_\Omega |\nabla u^{N'}|^p dx-\int_\Omega |\nabla u^0|^p dx &< C\\ \int_\Omega |\nabla u^{N'}|^p dx &\leq C+\int_\Omega |\nabla u^0|^p dx\\ \sup_{0\leq t\leq T}\int_\Omega |\nabla u_{\Delta_t}(t)|^p dx &\leq C.\end{aligned}$$

We prove know that $\tilde{u}_{\Delta_t}$ is bounded in $L^\infty(0,T;W_0^{1,p}(\Omega))$, we have

$$\begin{aligned}|\nabla\tilde{u}_{\Delta_t}| &= |\frac{\nabla u^n-\nabla u^{n-1}}{\Delta_t}(t-t_{n-1})+\nabla u^{n-1}|\\ &\leq |\frac{\nabla u^n-\nabla u^{n-1}}{\Delta_t}||(t-t_{n-1})|+|\nabla u^{n-1}|\\ &\leq |\nabla u^n|+2|\nabla u^{n-1}|\\ |\nabla\tilde{u}_{\Delta_t}|^p &\leq (|\nabla u_{\Delta_t}(t)|+2|\nabla u_{\Delta_t}(t-\Delta_t)|)^p\\ \int_\Omega |\nabla\tilde{u}_{\Delta_t}|^p dx &\leq C\left(\int_\Omega |\nabla u_{\Delta_t}(t)|^p dx+\int_\Omega |\nabla u_{\Delta_t}(t-\Delta_t)|^p dx\right)\\ \sup_{0\leq t\leq T}\int_\Omega |\nabla\tilde{u}_{\Delta_t}|^p dx &\leq C\left(\sup_{0\leq t\leq T}\int_\Omega |\nabla u_{\Delta_t}(t)|^p dx+\sup_{0\leq t\leq T}\int_\Omega |\nabla u_{\Delta_t}(t-\Delta_t)|^p dx\right)<\infty.\end{aligned}$$

Which complete the proof .

Proposition 2.8.

$$\max_{[0,T]}\|\tilde{u}_{\Delta_t}(t)-u_{\Delta_t}(t)\|_{L^2(\Omega)}\overset{\Delta_t\to 0}{\longrightarrow}0.$$

Proof. From (2.36) and (2.31), we have

$$\begin{aligned}
\|\tilde{u}_{\Delta_t}(t)-u_{\Delta_t}(t)\|^2_{L^2(\Omega)} &= \int_\Omega |\frac{u^n-u^{n-1}}{\Delta_t}(t-t_{n-1})+u^{n-1}-u^n|^2dx \\
&= \int_\Omega |\frac{u^n-u^{n-1}}{\Delta_t}(t-t_{n-1})+(u^{n-1}-u^n)\frac{\Delta_t}{\Delta_t}|^2dx \\
&\le \int_\Omega |\frac{u^n-u^{n-1}}{\Delta_t}\left[(t-t_{n-1})-\Delta_t\right]|^2dx \\
&\le \int_\Omega |\frac{u^n-u^{n-1}}{\Delta_t}|^2\left[|(t-t_{n-1})-\Delta_t|\right]^2dx \\
&\le \int_\Omega |\frac{u^n-u^{n-1}}{\Delta_t}|^2\left[\ |t-t_{n-1}|+|\Delta_t|\ \right]^2dx \\
&\le \int_\Omega |\frac{u^n-u^{n-1}}{\Delta_t}|^2\left[\ |\Delta_t|+|\Delta_t|\ \right]^2dx \\
&\le C\Delta_t\Delta_t\int_\Omega |\frac{u^n-u^{n-1}}{\Delta_t}|^2dx \\
&\le C\Delta_t\int_{t_{n-1}}^{t_n}\int_\Omega |\frac{u^n-u^{n-1}}{\Delta_t}|^2dxdt \\
&\le C\Delta_t,
\end{aligned}$$

then

$$\max_{[0,T]}\|\tilde{u}_{\Delta_t}(t)-u_{\Delta_t}(t)\|_{L^2(\Omega)} \le C(\Delta_t)^{1/2}\longrightarrow 0,\ \text{as } \Delta_t\to 0.$$

And this complete the proof.

2.1.3 Passage to the limit

We now use the compactness result of Aubin-Simon (see [?]).
From (2.39), for $p>2N/(N+2)$ then

$$\tilde{u}_{\Delta_t}, u_{\Delta_t} \ \text{ is bounded in }\ L^2(0,T;L^2(\Omega)). \tag{2.40}$$

From (2.36) and (2.40), it follows that

$$\tilde{u}_{\Delta_t}, u_{\Delta_t} \ \text{ is bounded in }\ W^{1,2}(0,T;L^2(\Omega)). \tag{2.41}$$

Taking V $=W_0^{1,p}(\Omega)$, E $=L^q(\Omega)$ for any $2\le q<Np/(N-p)$, and F $=L^2(\Omega)$ from Aubin-Simon 1.14, (2.39) and (2.41), it follows that there exists u $\in C([0,T],L^2(\Omega))$ such that up to a subsequence

$$\tilde{u}_{\Delta_t}, u_{\Delta_t} \text{ converge to } w,u \text{ (respectively) in }\ L^\infty(0,T;L^2(\Omega)). \tag{2.42}$$

Proposition 2.9. *We have up to a subsequence, as* $\Delta_t\to 0$

$$\tilde{u}_{\Delta_t}, u_{\Delta_t} \overset{*}{\rightharpoonup} u \quad \mathit{in} \quad L^\infty(0,T;W_0^{1,p}(\Omega)).$$

Proof. From (2.40), for subsequence and as $\Delta_t \to 0$, we have

$$\begin{aligned} \tilde{u}_{\Delta_t} &\rightharpoonup w \text{ in } L^2(0,T;L^2(\Omega)) \equiv L^2(\Omega_T), && (2.43)\\ u_{\Delta_t} &\rightharpoonup u \text{ in } L^2(0,T;L^2(\Omega)) \equiv L^2(\Omega_T). && (2.44)\end{aligned}$$

From (2.44), (2.43) and proposition 2.8, we get

$$\begin{aligned} \tilde{u}_{\Delta_t} - u_{\Delta_t} &\rightharpoonup w-u \text{ in } L^2(\Omega_T),\\ \tilde{u}_{\Delta_t} - u_{\Delta_t} &\to 0 \quad \text{ in } L^2(\Omega_T).\end{aligned}$$

Therefor, for the uniqueness of the limit $w-u=0 \Longrightarrow w=u$.

Proposition 2.10.

$$\textit{as } \Delta_t \to 0, \quad \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \textit{in} \ \ L^2(\Omega_T). \tag{2.45}$$

Proof. From (2.43), (2.36) and for subsequence as $\Delta_t \to 0$, we have

$$\begin{aligned} \tilde{u}_{\Delta_t} &\rightharpoonup u \text{ in } L^2(\Omega_T) \Longrightarrow \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } \mathcal{D}'(\Omega),\\ \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} &\rightharpoonup w \text{ in } L^2(\Omega_T) \Longrightarrow \frac{\partial \tilde{u}_{\Delta_t}}{\partial t} \to w \quad \text{ in } \mathcal{D}'(\Omega).\end{aligned}$$

Thus, for the uniqueness of limit we get $\dfrac{\partial u}{\partial t} = w$.

Proposition 2.11.

$$\textit{as } \Delta_t \to 0, \quad f(x, u_{\Delta_t}(t-\Delta_t)) \to f(x,u) \ \ \textit{in} \ \ L^2(\Omega_T). \tag{2.46}$$

Proof. From (2.42), we have

$$u_{\Delta_t} \overset{\Delta_t \to 0}{\longrightarrow} u \ \text{ in } \ L^2(0,T;L^2(\Omega)) \equiv L^2(\Omega_T). \tag{2.47}$$

From (2.47), and inverse lebesgue theorem 1.5, there exist subsequence such that $|u_{\Delta_t}| \leq g \in L^2(\Omega_T)$.
Since f bounded below function, and from (2.1) we get

- **Case 01:** p $<$ 2 ,
  from (2.34)

$$\begin{aligned} f(x, u_{\Delta_t}) &\leq \alpha(u_{\Delta_t}^{p-1}) + C,\\ &\leq \alpha(g^{p-1}) + C.\end{aligned}$$

In the other hand for p $<$ 2 we get 2(p-1) $<$ 2 ,thus

$$\begin{aligned} \left(\int_{\Omega_T} (g^{(p-1)}(x,t))^2 dxdt\right)^{1/2} &= \left(\int_{\Omega_T} (g^{2(p-1)}(x,t))dxdt\right)^{1/2}\\ &\leq C\left(\int_{\Omega_T} (g^2(x,t))dxdt\right)^{1/2}\\ &\leq \infty.\end{aligned}$$

Taking $h = \alpha(g^{p-1}) + C \in L^2(\Omega_T)$ .From Lebesgue dominated convergence theorem 1.4, thus

$$\int_{\Omega_T} |f(x, u_{\Delta_t})|^2 dxdt \leq \int_{\Omega_T} |h(x,t)|^2 dxdt \leq C.$$

Therefore

$$f(x, u_{\Delta_t}(t - \Delta_t)) \overset{\Delta_t \to 0}{\longrightarrow} f(x, u) \quad \text{in} \quad L^2(\Omega_T). \tag{2.48}$$

- **Case 02:** $p \geq 2$

$$\begin{aligned} |f(x, u_{\Delta_t})| &\leq \alpha u_{\Delta_t} + C, \\ &\leq \alpha g + C. \end{aligned}$$

Taking $h = \alpha g + C \in L^2(\Omega_T)$. From Lebesgue dominated convergence theorem 1.4, thus

$$\int_{\Omega_T} |f(x, u_{\Delta_t})|^2 dxdt \leq \int_{\Omega_T} |h(x,t)|^2 dxdt \leq C.$$

Therefore

$$f(x, u_{\Delta_t}(t - \Delta_t)) \overset{\Delta_t \to 0}{\longrightarrow} f(x, u) \quad \text{in} \quad L^2(\Omega_T). \tag{2.49}$$

Proposition 2.12.

$$as\ \Delta_t \to 0^+, u_{\Delta_t} \to u\ in\ L^p(0, T, W_0^{1,p}(\Omega)).$$

Proof. We now show that u is indeed solution in the weak sens given in the definition 2.2. From (2.29) , multplying (2.27) by $(u_{\Delta_t} - u)$, we get

$$\int_0^T \int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t}(u_{\Delta_t} - u)dxdt - \int_0^T \langle \Delta_p u_{\Delta_t}, (u_{\Delta_t} - u)\rangle dt - \int_0^T \int_\Omega \frac{1}{(u_{\Delta_t})^\delta}(u_{\Delta_t} - u)dxdt$$
$$= \int_0^T \int_\Omega f(x, u_{\Delta_t}(. - \Delta_t))(u_{\Delta_t} - u)dxdt,$$

and from (2.47) and (2.45) , for the first term

$$\begin{aligned}
\int_0^T \int_\Omega \frac{\partial \tilde{u}_{\Delta_t}}{\partial t}(u_{\Delta_t} - u)dxdt &= \int_0^T \int_\Omega \left[\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t}\right](u_{\Delta_t} - \tilde{u}_{\Delta_t} + \tilde{u}_{\Delta_t} - u)dxdt \\
&= \int_0^T \int_\Omega \left[\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t}\right](\tilde{u}_{\Delta_t} - u)dxdt + \int_0^T \int_\Omega \left[\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t}\right](u_{\Delta_t} - \tilde{u}_{\Delta_t})dxdt \\
&\quad + \int_0^T \int_\Omega \frac{\partial u}{\partial t}(u_{\Delta_t} - u)dxdt \\
&= \int_0^T \int_\Omega \left[\frac{\partial}{\partial t}(\tilde{u}_{\Delta_t} - u)\right](\tilde{u}_{\Delta_t} - u)dxdt + o_{\Delta_t}(1) \\
&= \int_0^T \frac{d}{dt}\int_\Omega |u_{\Delta_t}(s) - u(s)|^2 dxds + o_{\Delta_t}(1) \\
&= \int_\Omega |u_{\Delta_t}(T) - u(T)|^2 dx + o_{\Delta_t}(1),
\end{aligned}$$

such that

$$o_{\Delta_t}(1) = \int_0^T \int_\Omega \left[\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \frac{\partial u}{\partial t}\right](u_{\Delta_t} - \tilde{u}_{\Delta_t})dxdt + \int_0^T \int_\Omega \frac{\partial u}{\partial t}(u_{\Delta_t} - u)dxdt,$$

and $o_{\Delta_t}(1) \to 0$ as $\Delta_t \to 0$.
For the second term, from (2.31), using $\Delta_p u \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, we get

$$\begin{aligned}\int_0^T \langle \Delta_p u_{\Delta_t}, (u_{\Delta_t} - u)\rangle dt &= \int_0^T \langle \Delta_p u_{\Delta_t} - \Delta_p u + \Delta_p u, (u_{\Delta_t} - u)\rangle dt \\ &= \int_0^T \langle \Delta_p u_{\Delta_t} - \Delta_p u, (u_{\Delta_t} - u)\rangle dt + o_{\Delta_t}(1),\end{aligned}$$

such that

$$o_{\Delta_t}(1) = \int_0^T \langle \Delta_p u_{\Delta_t} dt + \Delta_p u, (u_{\Delta_t} - u)\rangle dt,$$

and $o_{\Delta_t}(1) \to 0$ as $\Delta_t \to 0$.
Therefor from the using inequalities of previous terms, and from convexity of the term $\int_\Omega u^{1-\delta}dx$, we get that

$$\begin{aligned}\int_\Omega |u_{\Delta_t}(T) - u(T)|^2 dx - \int_0^T \langle \Delta_p u_{\Delta_t} &- \Delta_p u + \Delta_p u, (u_{\Delta_t} - u)\rangle dt \\ &- \frac{1}{1-\delta}\int_0^T \int_\Omega \left(u_{\Delta_t}^{1-\delta} - u^{1-\delta}\right) dxdt \\ &\quad \leq \int_0^T \int_\Omega f(x, u_{\Delta_t}(. - \Delta_t))(u_{\Delta_t} - u)dxdt + o_{\Delta_t}(1),\end{aligned}$$

and from (2.47) and (2.46), we have from Hölder's inequality 1.2,

$$\begin{aligned}\int_0^T \int_\Omega f(x, u_{\Delta_t}(. - \Delta_t))(u_{\Delta_t} - u)dxdt &\leq \|f(x, u_{\Delta_t}\|_{L^2(\Omega_T)} \|u_{\Delta_t} - u\|_{L^2(\Omega_T)} \\ &\leq \|f(x, u)\|_{L^2(\Omega_T)} \|u_{\Delta_t} - u\|_{L^2(\Omega_T)} \to 0, \text{ as } \Delta_t \to 0^+,\end{aligned}$$

then

$$\int_0^T \int_\Omega f(x, u_{\Delta_t}(. - \Delta_t))(u_{\Delta_t} - u)dxdt = o_{\Delta_t}(1).$$

From (2.47) and inverse lebesgue theorem 1.5, there exist subsequence such that

$$u_{\Delta_t} \overset{\Delta_t \to 0}{\longrightarrow} u \text{ a.e. in } \Omega_T,$$

and

$$|u_{\Delta_t}| \leq g \in L^2(\Omega_T).$$

Since $\delta < 1$,thus

$$u_{\Delta_t}^{1-\delta} \overset{\Delta_t \to 0}{\longrightarrow} u^{1-\delta} \quad \text{a.e. in } \Omega_T,$$

and

$$|u_{\Delta_t}^{1-\delta}| \leq g^{1-\delta}.$$

We have from Hölder's inequality 1.2,

$$\begin{aligned}
\int_0^T\int_\Omega |g^{1-\delta}|dxdt &\leq \left(\int_0^T\int_\Omega (1)^{2/1+\delta}dxdt\right)^{1+\delta/2}\left(\int_0^T\int_\Omega (g^{1-\delta})^{2/1-\delta}dxdt\right)^{1-\delta/2}\\
&\leq C\left(\int_0^T\int_\Omega (g)^2dxdt\right)^{1-\delta/2}\\
&\leq C(C')^{1-\delta/2} \quad <\infty.
\end{aligned}$$

From Lebesgue dominated convergence theorem 1.4, thus

$$u_{\Delta_t}^{1-\delta}\overset{\Delta_t\to 0}{\longrightarrow}u^{1-\delta}\quad\text{in}\quad L^1(\Omega_T)$$

and

$$\int_0^T\int_\Omega |u_{\Delta_t}^{1-\delta}-u^{1-\delta}|dxdt\to 0,\quad\text{as}\quad \Delta_t\to 0^+,$$

then

$$\int_0^T\int_\Omega |u_{\Delta_t}^{1-\delta}-u^{1-\delta}|dxdt=o_{\Delta_t}(1).$$

Therefore

$$\begin{aligned}
\int_\Omega |u_{\Delta_t}(T)-u(T|^2dx-\int_0^T\langle\Delta_p u_{\Delta_t}-\Delta_p u,(u_{\Delta_t}-u)\rangle dt &= o_{\Delta_t}(1)\\
\int_\Omega |u_{\Delta_t}(T)-u(T|^2dx+\int_0^T\int_\Omega\left[|\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}-|\nabla u|^{p-2}\nabla u\right].\nabla(u_{\Delta_t}-u)dxdt &= o_{\Delta_t}(1).
\end{aligned}$$

From the convexity of the term $\int_\Omega |\nabla u|^p dx$, and since $u_{\Delta_t}\rightharpoonup u$ in $L^p(0,T,W_0^{1,p}(\Omega))$ as $\Delta_t\to 0^+$, we get

$$\frac{1}{p}\left[\int_\Omega |\nabla u_{\Delta_t}|^pdx-\int_\Omega |\nabla u|^pdx\right]\leq\int_\Omega\left(|\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}-|\nabla u|^{p-2}\nabla u\right).\nabla(u_{\Delta_t}-u)dx.$$

Thus

$$u_{\Delta_t}\overset{\Delta_t\to 0}{\longrightarrow}u\text{ in }L^p(0,T,W_0^{1,p}(\Omega)),$$

and consequently

$$\nabla u_{\Delta_t}\overset{\Delta_t\to 0}{\longrightarrow}\nabla u\text{ in }L^p(\Omega).$$

Then

$$\nabla u_{\Delta_t}\overset{\Delta_t\to 0}{\longrightarrow}\nabla u,\ \text{a.e. in }\Omega_T,$$

witch imply

$$|\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}\overset{\Delta_t\to 0}{\longrightarrow}|\nabla u|^{p-2}\nabla u\text{ a.e. in }\Omega_T,$$

and we have from Lebesgue inverse theorem 1.5, there exist subsequence such that $|u_{\Delta_t}| \leq g \in L^p(\Omega)$, then $||\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}| \leq g^{p-2}g$.
Such that

$$\begin{aligned} \int_\Omega \left(g^{p-2}g\right)^{p'} dx &= \int_\Omega \left(g^{p-1}\right)^{p/p-1} dx \\ &\leq \int_\Omega g^p dx < \infty. \end{aligned}$$

There for from Lebesgue dominated convergence theorem 1.4, we get

$$|\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t} \overset{\Delta_t \to 0}{\longrightarrow} |\nabla u|^{p-2}\nabla u \text{ in } L^{p'}(\Omega)$$

Notice that u$\geq \eta\phi_1$ in Ω, then $u^{-\delta} \leq (\eta\phi_1)^{-\delta}$. And since $\delta < 1$

$$\int_\Omega (\eta\phi_1)^{-\delta} dx < \infty.$$

Therefor from (2.42) and Lebesgue dominated convergence theorem 1.4, we get

$$\int_0^T \int_\Omega \frac{1}{(u_{\Delta_t})^\delta} dxdt \to \int_0^T \int_\Omega \frac{1}{(u)^\delta} dxdt, \quad \text{as} \quad \Delta_t \to 0^+,$$

for any $w \in C_0^\infty([0,T] \times \Omega)$. Then, passing to the limit as $\Delta_t \to 0^+$ in (2.28), we get from above compactness properties of $\{\tilde{u}_{\Delta_t}\}_{\Delta_t}$ and $\{u_{\Delta_t}\}$ that

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial \tilde{u}}{\partial t} w dxdt - \int_0^T \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w dxdt - \int_0^T \int_\Omega \frac{1}{(u)^\delta} w dxdt \\ = \int_0^T \int_\Omega f(x,u) w dxdt, \end{aligned}$$

for any w$\in C_0^\infty([0,T] \times \Omega)$. We have also

$$u(0) = \lim_{\Delta_t \to 0} \tilde{u}_{\Delta_t}(0) = u_0.$$

This completes the proof .

2.2 Uniqueness of solution

Theorem 2.13. *In addition theorem 2.1 , we suppose also f is locally lipchtiz with respect to the second variable uniformly in $x \in \Omega$, then the weak solution is unique.*

Proof. let u and v two solution of (Pb), by subtraction we get

$$\frac{\partial (u-v)}{\partial t} - (\Delta_p u - \Delta_p v) = f(x,u) - f(x,v) + \frac{1}{u^\delta} - \frac{1}{v^\delta} \quad ,$$

since f lipchtiz with respect to the second variable uniformly in $x \in \Omega$, thus

$$\frac{\partial (u-v)}{\partial t} - (\Delta_p u - \Delta_p v) \leq C(u-v) + \frac{1}{u^\delta} - \frac{1}{v^\delta} \quad ,$$

multiplying by $(u-v)$ and integrating over Ω, by parties we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u-v)^2dx \quad + \quad &\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla(u-v)dx \\ &\leq \int_\Omega (u-v)^2dx + \int_\Omega \left(\frac{1}{u^\delta}-\frac{1}{v^\delta}\right)(u-v)dx,\end{aligned}$$

from proposition (1.8), $\int_\Omega(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla(u-v)dx \geq 0$, and $\int_\Omega(\frac{1}{u^\delta}-\frac{1}{v^\delta})(u-v) \leq 0$, then by Gronwall lemma, we get

$$\frac{d}{dt}\int_\Omega (u-v)^2dx \leq \int_\Omega C(u-v)^2dx,$$

for $t=0$, thus

$$\frac{d}{dt}\int_\Omega (u-v)^2dx \leq \int_\Omega C(u-v)^2(0)=0,$$

therefore

$$u-v=0 \Rightarrow u=v.$$

CHAPTER 3

STABILIZATION

3.1 Stationary problem

Definition 3.1. *Let $\mathscr{C}$ be the set of function $v \in L^{\infty}(\Omega)$ such that there exists $C_1 > 0$ and $C_2 > 0$ satisfying*

$$C_1 d(x) \leq v \leq C_2 d(x) \qquad \text{if } \delta < 1 \ ,$$

where $d(x) = dist(x, \partial\Omega)$ then we have

Theorem 3.1. *Let $0 < \delta < 1$ and $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be a bounded below Caratheodory function, locally Lipschtiz with respect to the second variable uniformly in $x \in \Omega$, satisfying*

$$0 \leq \limsup_{t \to +\infty} \frac{f(x,t)}{t^{p-1}} < \lambda_1(\Omega),$$

and such that $f(x,s)/s^{p-1}$ is decreasing function in $\mathbb{R}^+$ for a.e. $x \in \Omega$. There exists a unique u_∞ in $W_0^{1,p}(\Omega) \cap \mathscr{C} \cap C_0(\bar{\Omega})$ satisfying

$$\begin{cases} -\Delta_p u_\infty - \dfrac{1}{u_\infty^\delta} = f(x, u_\infty) & \text{in } \ \Omega, \\ \qquad\qquad u_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{Q}$$

Proof. We consider the following energy functional E_n defined in $W_0^{1,p}(\Omega)$ by

$$E_n(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx \ - \int_\Omega \frac{(u^+)^{1-\delta}}{1-\delta} dx - \int_\Omega f(x, u^{n-1}) \ u^+ dx.$$

for the connectivity , weakly lower semi continuity and the strict convex of E_n we refer to proposition 2.3, therefore from theorem 1.6, E_n it has a unique global minimizer which is the solution of (Q) .

Theorem 3.2. *Let hypothesis in theorem 2.1 satisfied and assume that*

$$\frac{f(x,s)}{s^{p-1}} \text{ is decreasing in } (0,\infty) \text{ for a.e. } x \in \Omega.$$

Then the solution to (Q) is defined in $(0,\infty)\times\Omega$ *and satisfies*

$$u(t) \to u_\infty \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty. \tag{3.1}$$

Where u_∞ *is defined in theorem 3.1 .*

3.2 A priori estimates

In this section prove additional a priori estimates on $u_{\Delta_t}$and $\tilde{u}_{\Delta_t}$ defined in chapter 2 in $[0,\infty[\times\Omega$.

Lemma 3.3. *Let assumption in theorem 2.1 be satisfied. Let u be the weak solution and global solution to (Pt), prescribed at = 0 by* u_0*, given by theorem 2.1. Then :*

1. *there exist* $\underline{u} = c\phi_1$ *and* $\overline{u} = \gamma\phi_1$ *belong to* $\mathscr{C}$*, independent of* Δ_t *such that for all* $t \geq 0$, $\underline{u} \leq u_{\Delta_t}(t)$, $\tilde{u}_{\Delta_t}(t) \leq \overline{u}$ *;*
2. $1/u^\delta_{\Delta_t}$ *and* $1/\tilde{u}^\delta_{\Delta_t}$ *are bounded in* $L^\infty(0,\infty;W^{-1,p'}(\Omega))$ *;*
3. $\dfrac{\partial \tilde{u}_{\Delta_t}}{\partial t}$ *is bounded in* $L^2(0,\infty;L^2(\Omega))$ *independently of* Δ_t *;*
4. $u_{\Delta_t}, \tilde{u}_{\Delta_t}$ *are bounded in* $L^\infty(0,\infty;W^{1,p}_0(\Omega))$.

Proof. We prove assertion (1).
We show that $u_{\Delta_t} \leq \overline{u}$ and $\tilde{u}_{\Delta_t} \leq \overline{u}$, we have $u^n \in \mathscr{C}^1(\bar{\Omega})$, then

$$\begin{aligned} |u_{\Delta_t}(x,t) - u_{\Delta_t}(y,t)| &\leq C|u^n(x) - u^n(y)| \\ &\leq C|x-y|, \end{aligned}$$

for $y \in \partial\Omega$, taking C=γ, we get

$$\begin{aligned} |u_{\Delta_t}(x,t)| &\leq \gamma d(x) \sim \gamma\phi_1(x) \\ u_{\Delta_t}(x,t) &\leq \overline{u}(x). \end{aligned}$$

We show that $\tilde{u}_{\Delta_t} \leq \overline{u}$, for $3C \ll \gamma$, we have

$$\begin{aligned} |\tilde{u}_{\Delta_t}| &= |\frac{u^n - u^{n-1}}{\Delta_t}(t - t_{n-1}) + u^{n-1}| \\ &\leq |u^n| + 2|u^{n-1}| \\ &\leq u_{\Delta_t}(t) + 2u_{\Delta_t}(t - \Delta_t) \\ &\leq C\phi_1(x) + 2C\phi_1(x) \\ &\leq \gamma\phi_1(x) \\ &\leq \overline{u}. \end{aligned}$$

We prove the second part of assertion (1), we show that $u_{\Delta_t}(t) \geq \underline{u}$ and $\tilde{u}_{\Delta_t}(t) \geq \underline{u}$, we have from proposition 2.4, for $\eta = c$

$$\begin{aligned} u^n &\geq \eta\phi_1 \\ u_{\Delta_t}(t) &\geq c\phi_1 \\ u_{\Delta_t}(t) &\geq \underline{u}. \end{aligned}$$

Know we show that $\tilde{u}_{\Delta_t} \geq \underline{u}$,

$$\frac{\partial \underline{u}}{\partial t} - \Delta_p \underline{u} \leq f(x, \underline{u}) + \frac{1}{\underline{u}^{\delta}}, \tag{Pb}$$

we have

$$\begin{aligned} \frac{\partial \underline{u}}{\partial t} - \Delta_p \underline{u} &= -\Delta_p \underline{u} \\ &= -\Delta_p(c\phi_1) \\ &= c^{p-1}\lambda_1\phi_1^{p-1}, \end{aligned}$$

in the other hands

$$\underline{u}^{-\delta} = c^{-\delta}\phi_1^{-\delta} \geq c^{p-1}\lambda_1\phi_1^{p-1} + C',$$

where C' is constant large enough . Since f bounded below, thus

$$\underline{u}^{-\delta} + f(x, \underline{u}) \geq c^{p-1}\lambda_1\phi_1^{p-1} + C' + C,$$

taking $C' + C \geq 0$, then

$$\underline{u}^{-\delta} + f(x, \underline{u}) \geq -\Delta_p \underline{u} = \frac{\partial \underline{u}}{\partial t} - \Delta_p \underline{u}.$$

Therefore $\underline{u}$ is a subsolution of (Pb). we have

$$\frac{\partial \tilde{u}_{\Delta_t}}{\partial t} - \Delta_p u_{\Delta_t} = f(x, u_{\Delta_t}) + \frac{1}{(u_{\Delta_t})^{\delta}}, \tag{3.2}$$

from (Pb) and (3.2), we get

$$\frac{\partial\left(\underline{u} - \tilde{u}_{\Delta_t}\right)}{\partial t} - (\Delta_p \underline{u} - \Delta_p u_{\Delta_t}) \leq f(x, \underline{u}) - f(x, u_{\Delta_t}) + \frac{1}{(\underline{u})^{\delta}} - \frac{1}{(u_{\Delta_t})^{\delta}} \quad ,$$

since f bounded below, thus

$$\frac{\partial\left(\underline{u} - \tilde{u}_{\Delta_t}\right)}{\partial t} - (\Delta_p \underline{u} - \Delta_p u_{\Delta_t}) \leq C - C' + \frac{1}{(\underline{u})^{\delta}} - \frac{1}{(u_{\Delta_t})^{\delta}} \quad ,$$

taking $C - C' \leq 0$, and since $u_{\Delta_t}(t) \geq \underline{u}$, we get

$$\frac{\partial\left(\underline{u} - \tilde{u}_{\Delta_t}\right)}{\partial t} - (\Delta_p \underline{u} - \Delta_p u_{\Delta_t}) \leq 0,$$

multiplying by $(\underline{u}-\tilde{u}_{\Delta_t})^+$ and integrating over Ω, by parties we get

$$\frac{d}{dt}\int_\Omega \left((\underline{u}-\tilde{u}_{\Delta_t})^+\right)^2 dx + \int_\Omega \left(|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}\right)\nabla(\underline{u}-\tilde{u}_{\Delta_t})^+ dx \leq 0.$$

From proposition (1.8), $\int_\Omega \left(|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u_{\Delta_t}|^{p-2}\nabla u_{\Delta_t}\right)\nabla(\underline{u}-\tilde{u}_{\Delta_t})^+ dx \geq 0$, then

$$\frac{d}{dt}\int_\Omega \left((\underline{u}-\tilde{u}_{\Delta_t})^+\right)^2 dx \leq 0.$$

Therefore $(\underline{u}-\tilde{u}_{\Delta_t})(t)$ is decreasing function, then $\forall t \in [0,T]$

$$\begin{array}{rcl}
(\underline{u}-\tilde{u}_{\Delta_t})(t) & \leq & (\underline{u}-\tilde{u}_{\Delta_t})(0)\\
\underline{u}(t)-\tilde{u}_{\Delta_t}(t) & \leq & \underline{u}-u^0,
\end{array}$$

we have $u^0 \geq \eta\phi_1$, for $c=\eta$

$$\begin{array}{rcl}
\underline{u}(t)-\tilde{u}_{\Delta_t}(t) & \leq & \underline{u}-\eta\phi_1\\
 & \leq & c\phi_1-\eta\phi_1\\
 & \leq & \eta\phi_1-\eta\phi_1\\
 & \leq & 0\\
\underline{u}(t) & \leq & \tilde{u}_{\Delta_t}(t).
\end{array}$$

We prove the assertion (2)
from the assertion (1), we have $(u_{\Delta_t})^{-\delta} \leq (\underline{u})^{-\delta}$ and we show that $(\underline{u})^{-\delta} \in W^{-1,p'}(\Omega)$ i.e., for $\varphi \in W_0^{1,p}(\Omega)$

$$\langle(\underline{u})^{-\delta},\varphi\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)} \leq C\|\varphi\|_{W_0^{1,p}(\Omega)}.$$

we have

$$\begin{array}{rcl}
\langle(\underline{u})^{-\delta},\varphi\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)} & = & \displaystyle\int_\Omega (\underline{u})^{-\delta}\varphi dx\\
 & = & \displaystyle\int_\Omega (c\phi_1)^{-\delta}\varphi dx\\
 & = & c^{-\delta}\displaystyle\int_\Omega \phi_1(\phi_1)^{-\delta}\frac{\varphi}{\phi_1}dx,
\end{array}$$

Hölder's inequality 1.2,

$$\langle(\underline{u})^{-\delta},\varphi\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)} \leq C\left(\int_\Omega \phi_1^{p(1-\delta)}dx\right)^{1/p'}\left(\int_\Omega (\frac{\varphi}{\phi_1})^p dx\right)^{1/p},$$

Hardy 's inequality 1.10,

$$\begin{array}{rcl}
\langle(\underline{u})^{-\delta},\varphi\rangle_{W^{-1,p'}(\Omega)\times W_0^{1,p}(\Omega)} & \leq & CC'\|\nabla\varphi\|_{L^p(\Omega)}\\
 & \leq & C\ \|\varphi\|_{W_0^{1,p}(\Omega).}
\end{array}$$

Thus $(\underline{u})^{-\delta}$ is bounded in $L^\infty(0,\infty;W^{-1,p'}(\Omega))$,
then $(u_{\Delta_t})^{-\delta}$ is bounded in $L^\infty(0,\infty;W^{-1,p'}(\Omega))$, by the same way we show that $(\tilde{u}_{\Delta_t})^{-\delta}$ is bounded i

We prove assertions (3) and (4) to complete the proof. Multiplying (2.3) by $(u^n - u^{n-1})$ and summing from 1 to N' and integrating over Ω, we get

$$\frac{1}{\Delta_t}\sum_{n=1}^{N'}\|u^n - u^{n-1}\|^2_{L^2(\Omega)} - \sum_{n=1}^{N'}\langle\Delta_p u^n, u^n - u^{n-1}\rangle$$
$$-\sum_{n=1}^{N'}\int_\Omega \frac{1}{(u^n)^\delta}(u^n - u^{n-1})dx$$
$$= \sum_{n=1}^{N'}\int_\Omega f(x, u^{n-1})\left(u^n - u^{n-1}\right)dx \quad (3.3)$$

Set F(x,s)$\overset{def}{=}\int_0^s f(x,\tau)d\tau$. Since f is locally lipchtiz with respect to the second variable uniformly in $x \in \Omega$, there exists R > 0 such that $t \to \chi(t) = F(x,t) + \dfrac{Rt}{2}$ is convex in $\left[0, \|\overline{u}\|_{L^\infty(\Omega)}\right]$ uniformly in $x \in \Omega$ .Then we have from the theorem 1.7,

$$\begin{aligned}
\left(F(x,u^{n-1}) + \frac{R}{2}u^{n-1} - F(x,u^n) - \frac{R}{2}u^n\right) &\leq (\chi'(u^n)).(u^n - u^{n-1})\\
\left(F(x,u^{n-1}) - F(x,u^n) + \frac{R}{2}(u^{n-1} - u^n)\right) &\leq \left(f(x,u^{n-1}) + Ru^{n-1}\right).(u^{n-1} - u^n)\\
-\left(F(x,u^n) - F(x,u^{n-1}) + \frac{R}{2}(u^n - u^{n-1})\right) &\leq -\left(f(x,u^{n-1}) + Ru^{n-1}\right).(u^n - u^{n-1})\\
\left(F(x,u^n) - F(x,u^{n-1}) + \frac{R}{2}(u^n - u^{n-1})\right) &\geq \left(f(x,u^{n-1}) + Ru^{n-1}\right).(u^n - u^{n-1}).
\end{aligned}$$

From the above inequalities and the second hand of (3.3), we get

$$\begin{aligned}
&\sum_{n=1}^{N'}\int_\Omega f(x,u^{n-1})\left(u^n - u^{n-1}\right)dx\\
&\quad= \sum_{n=1}^{N'}\int_\Omega\left[f(x,u^{n-1}) + Ru^{n-1}\right]\left(u^n - u^{n-1}\right)dx - \sum_{n=1}^{N'}\int_\Omega Ru^{n-1}\left(u^n - u^{n-1}\right)dx\\
&\quad\leq \left[F(x,u^n) - F(x,u^{n-1}) + \frac{R}{2}(u^n - u^{n-1})\right] - \sum_{n=1}^{N'}\int_\Omega Ru^{n-1}\left(u^n - u^{n-1}\right)dx
\end{aligned}$$

Using proposition **??**, we have

$$\sum_{n=1}^{N'}\int_\Omega Ru^{n-1}\left(u^n - u^{n-1}\right)dx = \frac{R}{2}\int_\Omega\left[|u^0|^2 - |u^{N'}|^2\right]dx + \frac{R}{2}\sum_{n=1}^{N'}\int_\Omega|u^n - u^{n-1}|^2dx.$$

From the of convexity of the therms$\int_\Omega|\nabla u|^p dx$ and $\int_\Omega -\frac{1}{1-\delta}dx$ we derive the following estimates :

$$\frac{1}{p}\left[\int_\Omega|\nabla u^n|^p dx - \int_\Omega|\nabla u^{n-1}|^p dx\right] \leq \int_\Omega|\nabla u^n|^{p-2}\nabla u^n\nabla(u^n - u^{n-1})dx,$$

and

$$-\frac{1}{1-\delta}\left[\int_\Omega (u^{n-1})^{1-\delta}dx - \int_\Omega (u^n)^{1-\delta}dx\right] \leq -\int_\Omega \frac{u^n-u^{n-1}}{(u^n)^\delta}dx\ .$$

Gathering the above inequalities, we deduce that

$$\begin{aligned}
\frac{1}{\Delta_t}\sum_{n=1}^{N'}\|u^n-u^{n-1}\|^2_{L^2(\Omega)} - \sum_{n=1}^{N'}\langle\Delta_p u^n, u^n-u^{n-1}\rangle + \int_\Omega\left[\frac{|\nabla u^{N'}|^p}{p}dx - \frac{|\nabla u^0|^p}{p}\right]dx& \\
\leq \frac{1}{1-\delta}\int_\Omega\left[|u^{N'}|^{1-\delta} - |u^0|^{1-\delta}\right]dx + \int_\Omega\left[F(x,u^{N'}) - F(x,u^0)\right]dx& \\
-\sum_{n=1}^{N'}\int_\Omega\frac{R}{2}\int_\Omega|u^n-u^{n-1}|^2dx.&
\end{aligned}$$

From the definition of $\mathscr{C}$ and the fact that $\delta < 1$, from assertion (1), notice that

$$\frac{1}{1-\delta}\int_\Omega (u^{N'})^{1-\delta}dx \leq \frac{1}{1-\delta}\int_\Omega (\underline{u})^{1-\delta}dx < \infty,$$

consequently, rearranging the terms, we have

$$\begin{aligned}
\frac{1}{\Delta_t}\sum_{n=1}^{N'}\|u^n-u^{n-1}\|^2_{L^2(\Omega)} + \int_\Omega\left[\frac{|\nabla u^{N'}|^p}{p}dx - \frac{|\nabla u^0|^p}{p}\right]dx& \\
\leq \frac{1}{1-\delta}\int_\Omega (u^{N'})^{1-\delta}dx + \int_\Omega\left[F(x,u^{N'}) - F(x,u^0)\right]dx& \\
\leq \int_\Omega\left[F(x,\overline{u}) - R\overline{u}^2\right]dx + \frac{1}{1-\delta}\int_\Omega(\underline{u})^{1-\delta}dx& \\
\leq C,&
\end{aligned}$$

where C=$(\underline{u},\overline{u},\delta) > 0$ is independent of N' and Δ_t. Thus, we have from the above estimation that $\tilde{u}_{\Delta_t}$ and $\frac{\partial\tilde{u}_{\Delta_t}}{\partial t}$ are bounded respectively in $L^\infty(0,T;W_0^{1,p})$ and in $L^2(Q_T)$ independently of Δ_t and T . therefore, $\tilde{u}_{\Delta_t}$ and $\frac{\partial\tilde{u}_{\Delta_t}}{\partial t}$ are bounded respectively in $L^\infty(0,\infty;W_0^{1,p})$ and in $L^2(0,\infty;L^2(\Omega))$ independently of Δ_t.

Now we passe to the limit as $\Delta_t \longrightarrow 0$. By the same way of the section 3.13 Priori estimates, we can prove that $u(t) \overset{t\to\infty}{\longrightarrow} u$ solution of the parabolic problem

$$\begin{cases}
u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \ \text{ in } \ (0,T)\times\Omega, \\
u = 0, \ \text{ on } \ (0,T)\times\partial\Omega, \ \ u>0 \ \text{ in } \ (0,T)\times\Omega, \\
u(0,x) = u_0(x) \ \text{ in } \ \Omega.
\end{cases} \tag{Pt}$$

3.3 Passage to the limit

In this section we prove theorem 3.2, for that we prove that the solution u converges as $t \longrightarrow \infty$, to the solution of the stationary problem u_∞ :

$$\begin{cases} -\Delta_p u_\infty - \dfrac{1}{u_\infty^\delta} = f(x, u_\infty) & \text{in } \Omega, \\ \qquad\qquad u_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{Q}$$

From the previous section, we have $u(t) \in L^\infty(0, \infty; W_0^{1,p}(\Omega))$, then

$$\text{as } t \to \infty, u(t) \rightharpoonup w \text{ in } W_0^{1,p}(\Omega).$$

By the same way of proposition 2.12, we can prove that

$$u(t) \overset{t\to\infty}{\longrightarrow} w, \text{ in } W_0^{1,p}(\Omega),$$

and consequently

$$\nabla u(t) \overset{t\to\infty}{\longrightarrow} \nabla w, \text{ in } L^p(\Omega).$$

Witch implies

$$|\nabla u(t)|^{p-1} \overset{t\to\infty}{\longrightarrow} |\nabla w|^{p-1} \text{ in } L^{p'}(\Omega),$$

and we have from Lebesgue inverse theorem 1.5, up to a subsequence we have

$$|\nabla u(t)|^{p-1} \le g \in L^{p'}(\Omega), \text{ for all } t > 0,$$

then

$$|\nabla u(t)|^{p-2} \nabla u(t) \nabla v \le g \nabla v \in L^{p'}(\Omega),$$

From Lebesgue dominated convergence theorem 1.4, thus

$$\int_\Omega |\nabla u(t)|^{p-2} \nabla u(t) \nabla v dx \overset{t\to\infty}{\longrightarrow} \int_\Omega |\nabla w|^{p-2} \nabla w \nabla v dx \text{ in } L^{p'}(\Omega),$$

for all test function $v \in D(\Omega)$.

Since $p > 2N/(N+2)$, we have $W_0^{1,p}(\Omega) \hookrightarrow_c L^p(\Omega)$, then

$$u(t) \overset{t\to\infty}{\longrightarrow} w \text{ in } L^p(\Omega),$$

we have from assertion (1) of lemma 3.3, and since $\delta < 1$, we get

$$u^{-\delta}(t) \le \overline{u}^{-\delta} = (\gamma\phi_1)^{-\delta} \in L^1(\Omega),$$

from Lebesgue dominated convergence theorem 1.4, thus

$$\int_\Omega u^{-\delta}(t) v dx \overset{t\to\infty}{\longrightarrow} \int_\Omega w^{-\delta} v dx,$$

for all test function $v \in D(\Omega)$.

From inverse lebesgue theorem 1.5, up to a subsequence we have

$$|u(t)| \le g \in L^p(\Omega),$$

then from (2.34) and (2.35) in both cases we have

$$|f(x,u(t))| \leq Cg^{p-1} + C' \in L^{p'}(\Omega),$$

therefore from Lebesgue dominated convergence theorem 1.4, thus

$$f(x,u(t)) \overset{t\to\infty}{\longrightarrow} f(x,w),$$

for all test function $v \in D(\Omega)$.
Then

$$\int_\Omega f(x,u(t))vdx \overset{t\to\infty}{\longrightarrow} \int_\Omega f(x,w)vdx \text{ in } L^{p'}(\Omega).$$

We obtain that w satisfies the problem:

$$\begin{cases} -\Delta_p w - \dfrac{1}{w^\delta} = f(x,w) & \text{in } \ \Omega, \\ \qquad\qquad w_\infty = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense of distribution . By the uniqueness of the solution of the stationary problem (Q) we deduce that $w = u_\infty$. This completes the proof of theorem 3.2 .

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ملخص

نقوم بدراسة المسالة الشبه خطية الشاذة من النوع المكافئ التالية,

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \ \text{في} \ Q_T, \\ u = 0, \ \text{على} \ \Sigma_T, \ \ u > 0 \ \text{في} \ Q_T, \\ u(0,x) = u_0(x) \ \text{في} \ \Omega. \end{cases} \qquad (Pt)$$

حيث Ω هو ميدان محدود بحدود متجانسة سلسة في $\mathbb{R}^N$ (مع $N \geq 2$), $1 < p < \infty$, $Q_T = (0,T) \times \Omega$, $0 < \delta, T > 0$, و $\Sigma_T = (0,T) \times \partial\Omega$. نفرض ان f تابع كارثدوري محدود من الاسفل و $u_0 \in W_0^{1,p}(\Omega)$. في هذه المذكرة سوف ندرس وجود ووحدانية الحل الضعيف للمسألة (Pt) باستعمال طريقة نصف تقسيم الزمن وندرس الاستقرار .

كلمات مفتاحية:
مسائل شبه خطية شاذة من النوع المكافئ, وجود ووحدانية الحل الضعيف، المؤثر لابلاس-p , طريقة نصف تقسيم الزمن، حل فوقي وحل تحتي.

Abstract

We investigate the following quasi-linear and singular parabolic equation ,

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \ \text{in} \ Q_T, \\ u = 0, \ \text{on} \ \Sigma_T, \ \ u > 0 \ \text{in} \ Q_T, \\ u(0,x) = u_0(x) \ \text{in} \ \Omega. \end{cases} \qquad (Pt)$$

Where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^N$(with $N \geq 2$), $1 < p < \infty$, $0 < \delta, T > 0$, $Q_T = (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \partial\Omega$. We assume that f is bounded below Caratheodory function and $u_0 \in W_0^{1,p}(\Omega)$. In this memory we will study the existence and uniqueness of the weak solution of (Pt) using method of semi- discretization in time and we study the stabilization.

key words :
Quasi-linear and singular parabolic equation, existence and uniqueness of the weak solution, p-Laplacian, method of semi- discretization in time, sub- and super-solution .

Résumé

Nous étudions l'équation parabolique quasi-linéaire et singuliére suivante,

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \ \text{dans} \ Q_T, \\ u = 0, \ \text{sur} \ \Sigma_T, \ \ u > 0 \ \text{dans} \ Q_T, \\ u(0,x) = u_0(x) \ \text{dans} \ \Omega. \end{cases} \qquad (Pt)$$

Où Ω est un domaine ouvert borné avec une limite lisse dans $\mathbb{R}^N$(avec $N \geq 2$), $1 < p < \infty$, $0 < \delta, T > 0$, $Q_T = (0,T) \times \Omega$ et $\Sigma_T = (0,T) \times \partial\Omega$. Nous supposons que f est bornée inferieurement de la fonction Caratheodory et $u_0 \in W_0^{1,p}(\Omega)$. Dans ce mémoire nous étudierons l'existence et l'unicité de la solution faible de (Pt) utilisant la méthode de semi-discrétisation en temps et nous étudions la stabilisation.

Mots clés :
Équation parabolique quasi-linéaire et singuliére, existence et l'unicité de la solution faible, p-Laplacian, méthode de semi-discrétisation en temps, sous- et sur-solution .