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Nuclear operators and its applications

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Dedicace

To my family.

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ملخص

في هذه الدراسة، هدفنا الأساسي هو توسيع نظرية المثاليات المؤثرات الخطية لتشمل المؤثرات متعددة الخطية وكثيرات الحدود. وفي هذا السياق، نوجه اهتمامنا نحو دراسة مفهومين متميزين لمؤثر $p-$ النووي. تتضمن الدراسة الأولى توسيعًا لمفهوم المؤثر $p-$ النووي الضعيف، الذي قدمه كيم، ليشمل كلا من المؤثرات المتعددة الخطية وكثيرات الحدود. لقد أثبتنا أن هذه الفئة تشكل فضاء باناخ المتعدد الخطية (على التوالي، كثير حدود). في السعي للبحث عن فئة من فضاء الأشكال خطية التي تمثل فضاء المؤثرات $p-$ النووية الضعيفة المتعددة الخطية (على التوالي، كثير حدود) أدى بنا إلى تقديم مفهوم شبه كوهين $p-$ النووي المتعددة الخطية (على التوالي، كثير حدود). لقد أثبتنا أن هذه المؤثرات تحقق نظرية الهيمنة لـ بيتيش، كما أثبتنا أنه في ظل شروط إعتيادية يوجد تماثل متساوي القياس بين ثنوي فضاء المؤثرات $p-$ النووية الضعيفة المتعددة الخطية (على التوالي، كثير حدود) وفضاء مؤثرات شبه كوهين $p-$ النووية المتعددة الخطية (على التوالي، كثير حدود). الدراسة الثانية هي توسيع مفهوم مؤثر كوهين $p-$ النووي الخطي التي قدمها كوهين إلى كثيرات الحدود بين فضاءات مشبكية، أثبتنا أن كثير الحدود يكون كوهين $p-$ النووي موجب إذا وفقط إذا كان المؤثر متعدد الخطية المتناظر المساعد له كوهين $p-$ النووي موجب. بالإضافة إلى ذلك، عرفنا كثير حدود موجب كوهين $p-$ النووي كتركيب كثير الحدود كوهين $p-$ جمعي قوى موجب ومؤثر $p-$ جمعي موجب خطي، ولقد سلطنا الضوء على علاقتها مع الفئات الأخرى.

الكلمات والجمل المفتاحية: $p-$ النووي الضعيف، شبه كوهين $p-$ النووي، بناخ مشبكي، $p-$ جمعي موجب، كوهين $p-$ جمعي قوى موجب، كوهين $p-$ النووي موجب، نظرية الهيمنة لـ بيتيش .

Abstract

In the present study, our primary main is extend operator ideal theory to multilinear operators and polynomials. In this context, we direct our attention towards the study of two distinct concepts to p-nuclear operators. The first study involves an extension of the notion of weakly p-nuclear operators, which was introduced by J.M. Kim, in (J. Korean Math. Soc **56** (2019), 225-237), to encompass multilinear operators and polynomials. We show that this class forms a Banach multi-ideal (respectively, polynomials), In the quest to look for a class of operators that represent bounded linear functionals on the space of weakly p-nuclear multilinear operators (respectively, polynomials) led us to the introduction of the class of quasi Cohen p-nuclear multilinear operators (respectively, polynomials). We show that such operators realise a Pietsch domination theorem. Moreover, we prove that, under the usual conditions, there exists an isometric isomorphism between the dual of the space of weakly p-nuclear multilinear operators (respectively, polynomials) and the space of quasi Cohen p-nuclear multilinear operators (respectively, polynomials). The second study is the extension of the concept of Cohen p-nuclear operators introduced by J. S. Cohen in (Math. Ann. **201**(1973) 177-201) to polynomials between Banach lattices, we show that a polynomial is positive Cohen p-nuclear if, and only if, its associated symmetric multilinear operator is positive Cohen p-nuclear. Additionally, the study defines positive Cohen p-nuclear polynomials as a combination of positive Cohen strongly p-summing polynomials and positive absolutely p-summing linear operators, and shedding light on their relationship with other classes.

Keywords and phrases: Weakly p-nuclear, Quasi Cohen p-nuclear, Banach lattice, Positive p-summing, Positive Cohen strongly p-summing, Positive Cohen p-nuclear, Pietsch domination theorem.

Résumé

Dans l'étude actuelle, notre objectif principal est d'étendre la théorie des opérateurs idéaux aux opérateurs multilinéaires et aux polynômes. Dans ce contexte, nous concentrons notre attention sur l'étude de deux concepts distincts concernant les opérateurs p-nucléaires. La première étude concerne une extension de la notion d'opérateurs faiblement p-nucléaires, introduite par J. M. Kim, afin d'inclure les opérateurs multilinéaires et les polynômes. Nous montrons que cette classe forme un multi-idéal de Banach (respectivement, les polynômes). Dans notre quête pour trouver une classe d'opérateurs représentant des fonctionnelles linéaires bornées dans l'espace des opérateurs multilinéaires faiblement p-nucléaires (respectivement, les polynômes), nous avons introduit la classe des opérateurs multilinéaires quasi Cohen p-nucléaires (respectivement, les polynômes). Nous montrons que de tels opérateurs satisfont un théorème de domination de Pietsch. De plus, nous prouvons que, dans les conditions habituelles, il existe un isomorphisme isométrique entre le dual de l'espace des opérateurs multilinéaires faiblement p-nucléaires (respectivement, les polynômes) et l'espace des opérateurs multilinéaires quasi Cohen p-nucléaires (respectivement, les polynômes). La deuxième étude concerne l'extension du concept d'opérateurs Cohen p-nucléaires introduit par Cohen aux polynômes entre les espaces de Banach réticulés. Nous montrons qu'un polynôme est positif Cohen p-nucléaire si et seulement si son opérateur multilinéaire symétrique associé est positif Cohen p-nucléaire. De plus, cette étude définit les polynômes positifs Cohen p-nucléaires comme une combinaison de polynômes positif Cohen fortement p-sommant et d'opérateurs linéaires positif absolument p-sommant, mettant en lumière leur relation avec d'autres classes.

Mots clés et phrases: Faiblement p-nucléaire, Quasi Cohen p-nucléaire, Banach réticulé, Positivement p-sommant, Positivement fortement p-sommant, Positivement Cohen p-nucléaire, Théorème de domination de Pietsch.

Notations

General Symbols

$\mathbb{N}$	Natural numbers.
$\mathbb{K}$	The field of real or complex numbers.
p^*	The conjugate of the number $p(1 \leq p \leq \infty)$, that is $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$.
$\langle x, x^* \rangle$	Image of the function x^* at x.
π	The projective norm.
ε	The injective norm.

Let $X, X_1, ..., X_m$ and Y be Banach spaces

X^*	Topological dual of the Banach space X.
B_X	The closed unit ball of X.
$L(X;Y)$	The set of all linear operators.
$\mathcal{L}(X;Y)$	The set of all continuous linear operators.
$\mathcal{L}_f(X;Y)$	The set of all finite type linear operators.
$X \otimes Y$	The tensor product between Banach spaces.
$X \otimes_\alpha Y$	The space $X \otimes Y$ equipped with a cross-norm α.
$X \widehat{\otimes}_\alpha Y$	The completion of $X \otimes_\alpha Y$.
$\otimes_s^m X$	The m-fold symmetric tensor product of X.

Let $E, E_1, ..., E_m$ and F be Banach lattices

E^+	The positive cone of E.
$\mathcal{L}^r(E;F)$	The set of all regular linear operators.
$E \overline{\otimes} F$	The tensor product between Banach lattices .
$E \widehat{\otimes}_{\lvert\pi\rvert} F$	The positive projective tensor product of E, F.

Associated operators

T_L	The linearization of the multilinear operator T.
P_L	The linearization of the polynomial T.
$P^\otimes$	The linearization of the regular polynomial P.
$\widehat{P}$	The associated symmetric m-linear operators of polynomial P.

Spaces of multilinear operators and polynomials

$L(X_1,...,X_m;Y)$	The set of all multilinear operators.
$\mathcal{L}(X_1,...,X_m;Y)$	The set of all continuous multilinear operators.
$\mathcal{L}_f(X_1,...,X_m;Y)$	The set of all finite type multilinear operators.
$\mathcal{L}^r(E_1,...,E_m;F)$	The set of all regular multilinear operators.
$\mathcal{P}(^mX;Y)$	The set of all continuous m-homogeneous polynomials.
$\mathcal{P}_f(^mX;Y)$	The set of all finite type m-homogeneous polynomials.
$\mathcal{P}^r(^mE;F)$	The set of all regular m-homogeneous polynomials.

Ideals of operators

$\mathcal{N}_p(X;Y)$	The set of all Cohen p-nuclear linear operators.
$\mathcal{N}_{wp}(X;Y)$	The set of all weakly p-nuclear linear operators.
$\mathcal{N}_\sigma(X;Y)$	The set of all σ-nuclear linear operators.
$\Pi_p^+(E;Y)$	The set of all positive p-summing operators.
$\mathcal{N}_p^+(E;F)$	The set of all positive Cohen p-nuclear operators.

Ideals of multilinear operators and polynomials

$\mathcal{N}_{wp}(X_1,...,X_m;Y)$	The set of all weakly p-nuclear multilinear operators.
$\mathcal{N}_{q(p)}(X_1,...,X_m;Y)$	The set of all quasi Cohen p-nuclear multilinear operators.
$\mathcal{P}_{\mathcal{N}_{wp}}(^mX;Y)$	The set of all weakly p-nuclear polynomials.
$\mathcal{P}_{\mathcal{QN}_p}(^mX;Y)$	The set of all quasi Cohen p-nuclear polynomials.
$\mathcal{P}_{coh^+-p}(^mX;F)$	The set of all positive Cohen strongly p-summing polynomials.
$\mathcal{P}_{N-p}^{c^+}(^mE;F)$	The set of all positive Cohen p-nuclear polynomials.
$\mathcal{P}_{d,p}^+(^mE;F)$	The set of all positive p-dominated polynomials.

List of publications

1. **Hammou Asma; Belacel Amar.** ***Generalization of some properties of ideal operators to Lipschitz situation.*** **Palest. J. Math. 11, Spec. Iss. II, (2022) 57-62.**

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Introduction

In 1955, A. Grothendieck introduced the notions of 1-nuclear (or, simply, nuclear) and summing operators in his famous work titled "*Résumé de la théorie métrique des produits tensoriels topologiques*" commonly known as "*Résumé*," [30]. Later on, in 1969, A. Persson and A. Pietsch expanded on this notion by introducing the concept of p-nuclear operators for $1 < p \leq \infty$ [49]. Pietsch made fundamental contributions to the generalization of operator ideals, which are documented in his influential book titled *"Operator Ideals"* [45], which is regarded as a crucial milestone in the field. This book greatly affected the subject and provides an extensive treatment of operator ideals, encompassing not only absolutely p-summing and p-nuclear operators but also a wide range of operator ideals and their relationships. Among the classes discussed in this context, we find the class of σ-nuclear linear operators between Banach spaces (see. [45, Definition 23.2.1]).

The theory of operator ideals has been extended to ideals of nonlinear operators, for example as, multilinear operators, homogeneous polynomials, Lipschitz operators..., and the different ways of generating such non-linear operator ideals as well as the study of their properties. In 1983, Pietsch [47] made leading contributions through his work titled "*Ideals of Multilinear Functionals*" by extending the concept of linear summing and nuclear operators to nonlinear ones, particularly in multilinear and polynomial cases, which advanced the understanding of p-summing operators and solidified their importance. R. Alencar promptly contributed with a supportive paper [6] in this regard, he studied the properties of expanding the notion of nuclear and integral operators to encompass multilinear cases.

Following the release of seminal papers by Pietsch and R. Alencar [47, 6] on summing and nuclear multilinear operators, various forms of summability in nonlinear cases have emerged, as we have mentioned before. Numerous authors have devoted their research efforts to investigate the ideals of multilinear operators, we refer to the monographs (see e.g. [1, 2, 4, 5, 7, 13, 14, 18, 24]).

In another approach, J. M. Kim introduced a more extensive ideal of weakly p-nuclear as an extension of the existing notion of p-nuclear operators in his publication [37]. This novel concept of weakly p-nuclear is relatively recent and is a special subclass of class σ-nuclear.

In 1967, Pietsch [46] noticed that the identity operator from ℓ_1 into ℓ_2 is absolutely 2-summing, while its conjugate, from ℓ_2 into ℓ_∞, is not absolutely 2-summing. This observation spurred the interest of J. S. Cohen, where he introduced a novel category of linear operators known as strongly p-summing operators in his work titled " *Absolutely p-summing, p-nuclear operators and their conjugates*"[20]. This category characterizes the conjugate of the class of absolutely p^*-summing linear operators, with $1 = 1/p + 1/p^*$. In his work, also, remarked that every operator $T : X \to Y$, such that $I \otimes T : \ell_p \otimes_\varepsilon X \to \ell_p \otimes_\pi Y$ is continuous, is also absolutely p-summing $(p > 1)$, however, the converse is not true. which led him to explicitly define a subset of the absolutely p-summing such that $I \otimes T$ is continuous. These specific operators are called " Cohen p-nuclear operators" (see. Definition 1.1.7).

Cohen's work further demonstrates that if T is a p-summing linear operator and S is a strongly p-summing operator, then composition ST is Cohen p-nuclear operator. Moreover, he provied several results for this new class such as inclusion relations and a Pietsch domination theorem.

The concepts strongly p-summing and Cohen p-nuclear linear operators had many extensions to nonlinear cases and highlighted to verify the generalization of the properties of the original linear ideal to the nonlinear ideal (see e.g. [4, 5, 14, 15, 24, 43]).

In recent years, significant contributions were made by D. Achour et al [4, 5], where they introduced one of the important generalizations of the concepts of strongly p-summing linear operators to multilinear and polynomial cases.

In 2010, an extension of the concept Cohen p-nuclear operator was witnessed through the introduction of Cohen p-nuclear multilinear operators (see. Definition 1.2.20) between Banach spaces by D. Achour and A. Alouani [1]. Later, in 2018 [2], the Cohen p-nuclear polynomial (see. Definition 3.1.10) between Banach spaces were defined by D. Achour et al, as a natural polynomial extension of the class ideal of p-nuclear linear operators. Moreover, they investigated many results such as Pietsch Domination Theorem, Kwapien's factorization, etc.

Historically, many examples of multilinear operators and polynomial ideals develop as extensions of operator ideals. Our objective is to explain the relationship between an operator's ideal and its likely extension to nonlinear. In this context, we focus on common properties shared by nonlinear and linear ideals. In alignment with the above-mentioned foundational approach to nonlinear theory, we present this thesis with the aim of generalizing and extending certain findings and properties associated with linear weakly p-nuclear operators to their nonlinear counterparts.

This thesis has a twofold purpose:

In the first chapter of this thesis, we will give a thorough overview by presenting essential aspects related to sequences in Banach spaces. Additionally, we will introduce and explain fundamental definitions and properties associated with operator ideals and multilinear ideals.

In the second chapter, Our inspiration and main motivation come from very recent results of J. M. Kim [37], we expand the concept of weakly p-nuclear linear operator (see. Definition 1.1.8) to weakly p-nuclear multilinear operators, $(p \geq 1)$(see. Definition 2.1.1). We show that the class of weakly p-nuclear multilinear operators is an ideal of multilinear operators. Additionally, we prove, under usual conditions on the underlying spaces, a simpler formula for the weakly p-nuclear norm of a finite type operator. The objective of this chapter to look for a class of operators that represent continuous linear functionals on the space of weakly p-nuclear multilinear operators, for this purpose, we introduce the class of quasi Cohen p-nuclear multilinear operators (see. Definition 2.2.1). We show that such operators realise a Pietsch domination theorem and we show that, under usual conditions, the dual of the space of weakly p-nuclear multilinear operators from $X_1 \times \cdots \times X_m$ into Y is isometric isomorphism to the space of quasi Cohen p-nuclear multilinear operators from $X_1^* \times \cdots \times X_m^*$ into Y^*(see. Theorem 2.2.6).

In the third chapter of our study, we take the step to extend the results presented in the previous chapter, with the aim of applying the obtained results to the case of m- homogeneous polynomials.

The end of this work is the fourth chapter, our focus continues on homogeneous polynomials. We introduce the concept of extending the positive Cohen p-nuclear operator to case polynomials (see. Definition 4.2.1). Our aim is to prove that both classes coincide in the sense that a polynomial is positive Cohen p-nuclear if, and only if, its associated symmetric multilinear operator is positive Cohen p-nuclear. Moreover, we can define the set of positive Cohen p-nuclear polynomials as a combination of positive Cohen strongly p-summing polynomials and positive absolutely p-summing linear operators (see. Theorem 4.2.11), we shed light on the relationship between positive Cohen p-nuclear m-homogeneous polynomials and some classes (see. Section 4.3).

Chapter 1

Preliminaries

In this introductory chapter, we provide an overview of the fundamental concepts regarding linear operator ideals and multilinear operator ideals, and we give some basic definitions and properties used throughout the thesis. We also state, mostly without proofs.

1.1 Linear operators

We begin by explaining essential notations and terminology.

1.1.1 Notation and background

We denote X for a Banach space , $B_X = \{x \in X : \|x\| \leq 1\}$ stands for its closed unit ball of Banach space X, whereas . By $\mathbb{N}$ we represent the set of all natural numbers, and by $\mathbb{K}$ the scalar field (real or complex). For $p \geq 1$, we denote by p^* its conjugate, that is, $\frac{1}{p} + \frac{1}{p^*} = 1$. If Y is a Banach space, we use the notation $\mathcal{L}(X;Y)$ for the space of all continuous linear (bounded linear) operators from X into Y, If $X = Y$, we write id_X the identity operator on X. the Banach space of all continuous linear operators from X into Y, under the norm,

$$\|T\| = \sup_{x \in B_X} \|T(x)\| \quad \text{for } T \in \mathcal{L}(X;Y).$$

A linear operator $T : X \to Y$ between Banach spaces X and Y is an isomorphism if T is a continuous bijection whose inverse T^{-1} is also continuous. In this case, the spaces X and Y are isomorphic. In addition, if $\|T(x)\| = \|x\|$, for all $x \in X$, then T be isometric isomorphism. In this case, X and Y are said to be isometrically isomorphic.

The continuous dual of a Banach space X is $X^* := \mathcal{L}(X;\mathbb{K})$, its typical member will be denoted by x^*, and for $x \in X$, we shall write $\langle x, x^* \rangle$ for the action of the functional x^* on x. The bidual of X is the space $X^{**} = (X^*)^*$.

Let $T : X \to Y$ be continuous linear operator. Then the continuous linear operator $T^* : Y^* \to X^*$ defined as

$$\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle ,$$

for every $y^* \in Y^*$ and $x \in X$ is called the adjoint of T and has the property that $\|T^*\| = \|T\|$.

A linear operator $T \in \mathcal{L}(X;Y)$ is said to have finite type if $T(X)$ is finite dimensional. The class of all finite type linear operators between Banach spaces is denoted by $\mathcal{L}_f(X;Y)$. Furthermore, this space is generated by the operators of the special form

$$x^* \otimes y : x \mapsto \langle x, x^* \rangle \, y$$

i.e. if $T \in \mathcal{L}_f(X;Y)$ we have

$$T = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$ (see [45]).

Spaces of p-summable sequences

We denote by $\ell_p(X)$ the Banach space of all absolutely p-summable sequences $(x_n)_n$ in X with the norm

$$\begin{cases} \|(x_n)_n\|_{\ell_p(X)} = \left(\sum\limits_{n=1}^{\infty} \|x_n\|^p \right)^{1/p}, & \text{if} \quad 1 \leq p < +\infty, \\ \|(x_n)_n\|_{\ell_\infty(X)} = \sup\limits_n \|x_n\|, & \text{if} \quad p = +\infty \end{cases}.$$

We denote by $\ell_{p,w}(X)$ the Banach space of all weakly p-summable sequences $(x_n)_n$ in X with the norm

$$\begin{cases} \|(x_n)_n\|_{\ell_{p,w}(X)} = \sup\limits_{x^* \in B_{X^*}} \left(\sum\limits_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p \right)^{1/p}, & \text{if} \quad 1 \leq p < +\infty, \\ \|(x_n)_n\|_{\ell_{\infty,w}(X)} = \sup\limits_{x^* \in B_{X^*}} \sup\limits_n |\langle x_n, x^* \rangle|, & \text{if} \quad p = +\infty \end{cases}.$$

If $p = \infty$, we are restricted to the case of bounded sequences and in $\ell_\infty(X)$ we use the sup norm. Then the spaces $\ell_\infty(X)$ and $\ell_{\infty,w}(X)$ coincide [22, Page 33]. For the particular case $X = \mathbb{K}$, we denote $\ell_p(\mathbb{K})$ by ℓ_p.

We shall frequently make use of the formula, for $1 \leq p \leq \infty$ and let $(x_n)_n \subset X$.

$$\|(x_n)_n\|_{\ell_{p,w}(X)} = \sup_{\|(\lambda_n)\|_{\ell_{p^*}} = 1} \sup_{x^* \in B_{X^*}} \left| \sum_{n=1}^{\infty} \lambda_n \langle x_n, x^* \rangle \right|. \tag{1.1}$$

This formula shows that for each $(x_n^*)_n \subset X^*$, we have

$$\|(x_n^*)_n\|_{\ell_{p,w}(X^*)} = \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p \right)^{1/p}.$$

For more detailed information, refer to [8, 19].

Approximation property

The concept of the approximation property, initially introduced by S. Banach in [10], plays a fundamental role in the structure theory of Banach spaces. The first systematic study of the variants of the approximation property and the relations between them appeared by Grothendieck in [31]. In this section, we review the definitions of the approximation property.

Definition 1.1.1 *A Banach space X is said to have the approximation property if for each compact set $K \subseteq X$ and $\epsilon > 0$ there is an operator $T \in \mathcal{L}_f(X;X)$ such that*

$$\|Tx - x\| \leq \epsilon, \ \ for\ all\ x \in K.$$

The approximation property definition imposes no limitations on the norm of a finite type operator that approximates the identity of a compact set. Now, we introduce a more restrictive definition.

Definition 1.1.2 *A Banach space X is said to have the λ-bounded approximation property if there exists a $\lambda \geq 1$ such that for every compact set K in X and every $\epsilon > 0$, there is a $T \in \mathcal{L}_f(X;X)$ such that $\|Tx - x\| < \epsilon$ and $\|T\| \leq \lambda$ for every $x \in K$. If $\lambda = 1$ we say X has metric approximation property.*

Definition 1.1.3 *A Banach space is said to have bounded approximation property, if it has the λ-bounded approximation property for some λ.*

Lemma 1.1.4 [45, Lemma 10.2.6] *Suppose that X^* has the metric approximation property. Let $S \in \mathcal{L}_f(X;Y)$ and let $\epsilon > 0$. Then there exists an operator $T \in \mathcal{L}_f(X;X)$ such that $\|T\| \leq 1 + \epsilon$ and $S \circ T = S$.*

1.1.2 Operator ideals

Recall the definition of an operator ideal, we refer the reader to [45] for the linear case.

Definition 1.1.5 *A linear ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces X and Y its components $\mathcal{I}(X;Y) := \mathcal{L}(X;Y) \cap \mathcal{I}$ satisfy the following conditions:*

(i) *$\mathcal{I}(X;Y)$ is a linear subspace of $\mathcal{L}(X;Y)$ which contains the finite type operators.*

(ii) *The ideal property: If $u \in \mathcal{I}(X;Y)$, $v \in \mathcal{L}(G;X)$ and $w \in \mathcal{L}(Y;F)$ then $w \circ u \circ v$ is in $\mathcal{I}(G;F)$. If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies*

(i') *$[\mathcal{I}(X;Y), \|.\|_{\mathcal{I}}]$ is a normed space for all Banach spaces X and Y,*

(ii') $\|id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} : id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}} = 1,$

(iii') *If $u \in \mathcal{I}(X;Y)$, $v \in \mathcal{L}(G;X)$ and $w \in \mathcal{L}(Y;F)$ then*

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\|\|u\|_{\mathcal{I}}\|v\|$$

then $[\mathcal{I}, \|.\|_{\mathcal{I}}]$ is called a normed operator ideal.

Proposition 1.1.6 *Let $[\mathcal{I}, \|.\|_{\mathcal{I}}]$ be a normed operator ideal. Then $\|u\| \leq \|u\|_{\mathcal{I}}$ for all $u \in \mathcal{I}$.*

Basic examples

The ideal $\mathcal{L}_f$ of finite type linear operators is the least operator ideal and $\mathcal{L}$ the greatest one. This assertion is proved by the results elucidated in [45, Theorem 1.2.2], confirming the creation importance of these two operator ideals within the field of functional analysis.

1) The ideal of Cohen p-nuclear operators

The notion of Cohen p-nuclear operators, $(1 \leq p \leq \infty)$, was initiated by J. S. Cohen. He showed that every Cohen p-nuclear linear operator on Banach spaces is strongly p-summing and absolutely p-summing. Additionally, the composition of an absolutely p-summing operator with a strongly p-summing one results in a Cohen p-nuclear operator. For more detailed information, refer to [20].

Definition 1.1.7 *We say that a linear operator $T : X \to Y$ between Banach spaces is Cohen p-nuclear, $1 \leq p \leq \infty$, if there is a constant $C > 0$ such that*

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p \right)^{1/p} \sup_{y^{**} \in B_{Y^{**}}} \left(\sum_{i=1}^{n} |\langle y_i^*, y^{**} \rangle|^{p^*} \right)^{1/p^*}, \qquad (1.2)$$

for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$.

The least constant C for which this inequality (1.2) holds is denoted by $n_p(\cdot)$. We write $\mathcal{N}_p(X;Y)$ for the Banach space set of all Cohen p-nuclear operators from X into Y.

2) The ideal of weakly p-nuclear operators

In [37], J. M. Kim introduced the concept of weakly p-nuclear operators a special subclass of class the σ-nuclear operators [45, Definition 23.2.1], and a larger class of the ideal of p-nuclear operators.

Definition 1.1.8 *A linear operator $T : X \to Y$ between Banach spaces is weakly p-nuclear $(1 \leq p \leq \infty)$, if can be written in the form*

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n,$$

where $(x_n^*)_n \in \ell_{p,w}(X^*)$ and $(y_n)_n \in \ell_{p^*,w}(Y)$.

We denote by $\mathcal{N}_{wp}(X;Y)$ the space of all weakly p-nuclear operators from X into Y endowed with the weakly p-nuclear norm

$$\|T\|_{\mathcal{N}_{wp}} = \inf \|(x_n^*)_n\|_{\ell_{p,w}(X^*)} \|(y_n)_n\|_{\ell_{p^*,w}(Y)},$$

where the infimum is taken over all such weakly p-nuclear representations of T as above.

1.2 Multilinear operators

Before proceeding any further, we recall some basic concepts and notations.

Definition 1.2.1 *Let $m \in \mathbb{N}$, $X_1, ..., X_m$ and Y are Banach spaces. An operator $T : X_1 \times \cdots \times X_m \to Y$ is called multilinear (or, simply, m-linear) if it is*

$$\begin{aligned} T(x_1, ..., x_{j-1}, \lambda x_j' + x_j'', x_{j+1}, ..., x_m) =& \lambda T(x_1, ..., x_{j-1}, x_j', x_{j+1}, ..., x_m) \\ &+ T(x_1, ..., x_{j-1}, x_j'', x_{j+1}, ..., x_m) \end{aligned}$$

for all $\lambda \in \mathbb{K}$ and $x_j, x_j', x_j'' \in X_j (1 \leq j \leq m)$.

Noted by $L(X_1, ..., X_m; Y)$ (if $Y = \mathbb{K}$, we write $L(X_1, ..., X_m)$) the space of all m-linear operators from $X_1 \times \cdots \times X_m$ into Y. In the case $X_1 = \cdots = X_m = X$, we simply write $L(^m X; Y)$.

Definition 1.2.2 *An m-linear operator $T : X_1 \times \cdots \times X_m \to Y$ is continuous if it is continuous as a function between normed spaces.*

We write $\mathcal{L}(X_1, ..., X_m; Y)$ for the vector space of all continuous m-linear operators. If $Y = \mathbb{K}$, we write $\mathcal{L}(X_1, ..., X_m)$. In addition if $X_j = X$ for $j = 1, ..., m$ then we denote this space by $\mathcal{L}(^m X)$.

Proposition 1.2.3 *Let $T \in L(X_1, ..., X_m; Y)$. The following assertions are equivalent.*

1) *T is continuous.*

2) *T is continuous in 0.*

3) *There is a constant $C > 0$ with*

$$\|T(x_1, ..., x_m)\| \leq C\|x_1\| \times \cdots \times \|x_m\|, \tag{1.3}$$

for all $(x_1, ..., x_m) \in X_1 \times \cdots \times X_m$.

4)
$$\begin{aligned} \|T\| &= \sup_{\substack{x_j \in B_{X_j} \\ 1 \leq j \leq m}} \|T(x_1, ..., x_m)\| \\ &= \inf\{C,\ C \text{ verifying the inequality } (1.3)\}. \end{aligned}$$

Definition 1.2.4 *An m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be finite type, if it is generated by operators of the form*

$$\begin{aligned} X_1 \times \cdots \times X_m &\to Y \\ (x_1, ..., x_m) &\mapsto x_1^*(x_1) \cdots x_m^*(x_m) y \end{aligned}$$

for some non-zero $x_j^* \in X_j^* (1 \leq j \leq m)$ and $y \in Y$.

The vector space of all multilinear operators of finite type is denoted by $\mathcal{L}_f(X_1, ..., X_m; Y)$.

Now, we present the definition of an adjoint of an m-linear operator which was due to M. S. Ramanujan and E. Schock [50].

Definition 1.2.5 *The adjoint of an m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is a unique linear operator defined by*

$$\begin{aligned} T^* : Y^* &\to \mathcal{L}(X_1, \ldots, X_m) \\ y^* &\mapsto T^*(y^*)(x_1, ..., x_m) = y^* T(x_1, ..., x_m). \end{aligned}$$

1.2.1 Tensor products of Banach spaces

In this subsection, several outcomes can be viewed as natural extensions of the case when $m = 2$. For further details, we recommend that the reader [51].

The m-fold tensor product $X_1 \otimes \cdots \otimes X_m$ of the Banach spaces $X_1, ..., X_m$ can be constructed from the elements of the space $L(X_1, ..., X_m)^*$. For given $x_j \in X_j (1 \leq j \leq m)$ we define a linear functional $x_1 \otimes \cdots \otimes x_m \in L(X_1, ..., X_m)^*$ by the formula

$$(x_1 \otimes \cdots \otimes x_m)(A) = A(x_1, ..., x_m),\ A \in L(X_1, ..., X_m).$$

Definition 1.2.6 *The tensor product* $X_1 \otimes \cdots \otimes X_m$ *is the subspace of* $L(X_1, ..., X_m)^*$ *spanned by all elements* $x_1 \otimes \cdots \otimes x_m$*, where* $x_j \in X_j (1 \leq j \leq m)$.

Thus, the typical element of $X_1 \otimes \cdots \otimes X_m$ is of the form

$$u = \sum_{i=1}^{n} \lambda_i x_{1i} \otimes \cdots \otimes x_{mi},$$

where $m \in \mathbb{N},\ \lambda_i \in \mathbb{K},\ x_{ji} \in X_j\ (1 \leq i \leq n)(1 \leq j \leq m)$. Note that the representation of u is not unique.

Tensor products satisfy the following universal property.

Theorem 1.2.7 (Linearization of multilinear operators) *Let* $X_1, \ldots, X_m$ *and* Y *be Banach spaces. For every* m*-linear operator* $T : X_1 \times \cdots \times X_m \longrightarrow Y$ *there exists a unique linear operator* $T_L : X_1 \otimes \cdots \otimes X_m \longrightarrow Y$ *satisfying*

$$T_L\left(x_1 \otimes \cdots \otimes x_m\right) = T\left(x_1, \ldots, x_m\right),$$

for every $x_j \in X_j (1 \leq j \leq m)$.

The space $L\left(X_1, \ldots, X_m; Y\right)$ is isomorphic to $L\left(X_1 \otimes \cdots \otimes X_m; Y\right)$ through the correspondence $T \longleftrightarrow T_L$.

Reasonable crossnorms and tensor norms

A. Grothendieck introduced the notion of reasonable crossnorm, and defined the greatest crossnorm and the least reasonable crossnorm: the projective and the injective tensor norms, respectively.

Definition 1.2.8 *Let* $X_1, \ldots, X_m$ *be Banach spaces. We call that a norm,* α *on* $X_1 \otimes \cdots \otimes X_m$ *is a reasonable crossnorm if it has the following properties:*

1) *For any* $x_j \in X_j (1 \leq j \leq m)$,

$$\alpha\left(x_1 \otimes \cdots \otimes x_m\right) \leq \|x_1\| \cdots \|x_m\|.$$

2) *For each* $x_j^* \in X_j^* (1 \leq j \leq m)$ *the linear functional* $x_1^* \otimes \cdots \otimes x_m^* \in \left(X_1 \otimes \cdots \otimes X_m, \alpha\right)^*$ *defined by*

$$x_1^* \otimes \cdots \otimes x_m^*(u) := \sum_{i=1}^{n} \langle x_{1i}, x_1^* \rangle \cdots \langle x_{mi}, x_m^* \rangle,$$

for $u = \sum_{i=1}^{n} x_{1i} \otimes \cdots \otimes x_{mi} \in X_1 \otimes \cdots \otimes X_m$ *is continuous and*

$$\|x_1^* \otimes \cdots \otimes x_m^*\| \leq \|x_1^*\| \cdots \|x_m^*\|.$$

We denote by $X_1 \otimes_\alpha \cdots \otimes_\alpha X_m$ the m-fold tensor product of Banach spaces $X_1, \ldots, X_m$ with the norm α, and $X_1 \widehat{\otimes}_\alpha \cdots \widehat{\otimes}_\alpha X_m$ for the completed $X_1 \otimes_\alpha \cdots \otimes_\alpha X_m$.

The greatest and least crossnorms

Of special interest are two reasonable crossnorms: the least reasonable crossnorm and the greatest reasonable crossnorm. We will now present an overview of each.

1) The projective tensor product

Definition 1.2.9 *Let $X_1, \ldots, X_m$ be Banach spaces. The projective norm on $X_1 \otimes \ldots \otimes X_m$ is defined by*

$$\pi(u) = \inf \left\{ \sum_{i=1}^{n} \|x_{1i}\| \cdots \|x_{mi}\| : u = \sum_{i=1}^{n} x_{1i} \otimes \cdots \otimes x_{mi} \in X_1 \otimes \cdots \otimes X_m \right\},$$

where the infimum is taken over all possible representations of u as above.

Obviously π is a norm on $X_1 \otimes \cdots \otimes X_m$. Furthermore, π is a reasonable crossnorm and is the greatest reasonable crossnorm.

Proposition 1.2.10 *Let $X_1, \ldots, X_m$ be Banach spaces. Then π is a norm on $X_1 \otimes \cdots \otimes X_m$. Moreover:*

1) $\pi(x_1 \otimes \ldots \otimes x_m) = \|x_1\| \ldots \|x_m\|$ *for all* $x_j \in X_j (1 \leq j \leq m)$.

2) *If* $x_j^* \in X_j^* (1 \leq j \leq m)$ *then, the linear functional* $x_1^* \otimes \cdots \otimes x_m^* \in (X_1 \otimes \cdots \otimes X_m, \pi)^*$ *is continuous and*
$$\|x_1^* \otimes \cdots \otimes x_m^*\| = \|x_1^*\| \cdots \|x_m^*\|.$$

we denote by $X_1 \otimes_\pi \cdots \otimes_\pi X_m$ the m-fold tensor product of Banach spaces $X_1, \ldots, X_m$ endowed with the projective norm π, and $X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m$ for the completed projective tensor product of Banach spaces $X_1, \ldots, X_m$.

2) The injective tensor product

Definition 1.2.11 *Let $X_1, \ldots, X_m$ be Banach spaces. The injective norm on $X_1 \otimes \cdots \otimes X_m$ is defined by*

$$\varepsilon(u) = \sup_{\substack{x_j^* \in B_{X_j^*} \\ 1 \leq j \leq m}} \left| \sum_{i=1}^{n} \langle x_{1i}, x_1^* \rangle \cdots \langle x_{mi}, x_m^* \rangle \right|,$$

where $\sum\limits_{i=1}^{n} x_{1i} \otimes \cdots \otimes x_{mi}$ is any representation of $u \in X_1 \otimes \cdots \otimes X_m$.

Obviously, ε is a norm on $X_1 \otimes \cdots \otimes X_m$. Furthermore, ε is a reasonable crossnorm and is the least reasonable crossnorm.

Proposition 1.2.12 *Let $X_1, \ldots, X_m$ be Banach spaces.*

1) *$\varepsilon(u) \leq \pi(u)$ for every $u \in X_1 \otimes \cdots \otimes X_m$.*

2) *$\varepsilon(x_1 \otimes \cdots \otimes x_m) = \|x_1\| \cdots \|x_m\|$ for every $x_j \in X_j (1 \leq j \leq m)$.*

3) *If $x_j^* \in X_j^* (1 \leq j \leq m)$ then, the linear functional $x_1^* \otimes \cdots \otimes x_m^* \in (X_1 \otimes \cdots \otimes X_m, \varepsilon)^*$ is continuous and*
$$\|x_1^* \otimes \cdots \otimes x_m^*\| = \|x_1^*\| \cdots \|x_m^*\|.$$

We denote by $X_1 \otimes_\varepsilon \cdots \otimes_\varepsilon X_m$ the m-fold tensor product of Banach spaces $X_1, \ldots, X_m$ with the injective norm ε, and $X_1 \widehat{\otimes}_\varepsilon \cdots \widehat{\otimes}_\varepsilon X_m$ for the completed injective tensor product of Banach spaces $X_1, ..., X_m$.

Proposition 1.2.13 [51, Proposition 6.1] *A norm α on $X_1 \otimes \cdots \otimes X_m$ is a reasonable crossnorm if, and only if,*
$$\varepsilon(u) \leq \alpha(u) \leq \pi(u),$$
for all $u \in X_1 \otimes \cdots \otimes X_m$.

1.2.2 Multilinear operator ideals

Definition 1.2.14 *An ideal of multilinear operators (or multi-ideal) is a subclass $\mathcal{M}$ of all continuous multilinear operators between Banach spaces such that for all $m \in \mathbb{N}$ and Banach spaces $X_1, ..., X_m$ and Y, the components*
$$\mathcal{M}(X_1, ..., X_m; Y) := \mathcal{L}(X_1, ..., X_m; Y) \cap \mathcal{M}$$
satisfy the following conditions:

(i) *$\mathcal{M}(X_1, ..., X_m; Y)$ is a linear subspace of $\mathcal{L}(X_1, ..., X_m; Y)$ which contains the m-linear operators of finite type.*

(ii) *The ideal property: If $T \in \mathcal{M}(G_1, ..., G_m; F)$, $u_j \in \mathcal{L}(X_j; G_j)$ for $j = 1, ..., m$ and $v \in \mathcal{L}(F; Y)$ then $v \circ T \circ (u_1, ..., u_m)$ is in $\mathcal{M}(X_1, ..., X_m; Y)$.*
If $\|.\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^+$ satisfies

  (i') *$[\mathcal{M}(X_1, ..., X_m; Y), \|.\|_{\mathcal{M}}]$ is a normed (Banach) space for all Banach spaces $X_1, ..., X_m$ and Y and all m;*

  (ii') *$\|T^m : \mathbb{K}^m \to \mathbb{K} : T^m(x_1, ..., x_m) = x_1 \cdots x_m\|_{\mathcal{M}} = 1$ for all m,*

  (iii') *if $T \in \mathcal{M}(G_1, ..., G_m; F)$, $u_j \in \mathcal{L}(X_j, G_j)$ for $j = 1, ..., m$ and $v \in \mathcal{L}(F; Y)$ then*
$$\|v \circ T \circ (u_1, ..., u_m)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| \cdots \|u_m\|,$$

then $[\mathcal{M}, \|.\|_{\mathcal{M}}]$ is called a normed (Banach) multi-ideal.

The representation of multi-ideals by tensor norms

Definition 1.2.15 [12] *We say that a tensor norm α of order $m+1$ represents the multi-ideal $\mathcal{M}$ if $\mathcal{M}(X_1,\ldots,X_m;Y^*)$ and $(X_1\otimes_\alpha\cdots\otimes_\alpha X_m\otimes_\alpha Y)^*$ are isometrically isomorphic under the canonical operator defined by*

$$\Psi:\mathcal{M}(X_1,\ldots,X_m;Y^*)\longrightarrow(X_1\otimes_\alpha\cdots\otimes_\alpha X_m\otimes_\alpha Y)^*$$
$$T\mapsto\Psi(T)(x_1\otimes\cdots\otimes x_m\otimes y)=T(x_1,\ldots,x_m)(y)$$

for all $m\in\mathbb{N}$ and all Banach spaces $X_1,\ldots,X_m,Y$.

Theorem 1.2.16 (Uniqueness of the representation) [12, Theorem 2.5] *The tensor norm that represents a given multi-ideal, if any, is unique.*

Proposition 1.2.17 *Let $X_1,\ldots,X_m$ and Y be Banach spaces. Then we have the isometric isomorphism identification.*

$$\mathcal{L}(X_1,\ldots,X_m;Y^*)=(X_1\otimes_\pi\cdots\otimes_\pi X_m\otimes_\pi Y)^*.$$

A few illustrative examples of tensor norms representing multi-ideals are provided below.

1) σ-Nuclear multilinear operators

The multi-ideal space of σ-nuclear multilinear operators between Banach spaces was introduced by G. Botelho and X. Mujica in [13] as a natural multilinear extension of the classical ideal of σ-nuclear linear operators [45, Definition 23.2.1].

Definition 1.2.18 *An operator $T:X_1\times\cdots\times X_m\to Y$ is called σ-nuclear if T can be written in the form*

$$T=\sum_{n=1}^{\infty}\lambda_n x_{1n}^*\otimes\cdots\otimes x_{mn}^*\otimes y_n,$$

where $(\lambda_n)_{n=1}^\infty\in\ell_\infty$, $(x_{jn}^*)_{n=1}^\infty\subset X_j^*(1\leq j\leq m)$ and $(y_n)_{n=1}^\infty\subset Y$, satisfy

$$\sup_{\substack{x_j\in B_{X_j}\\ y^*\in B_{Y^*}}}\left(\sum_{n=1}^{\infty}\left|\langle x_1,x_{1n}^*\rangle\cdots\langle x_m,x_{mn}^*\rangle\langle y_n,y^*\rangle\right|\right)<\infty$$

and

$$\lim_{k\to\infty}\sup_{\substack{x_j\in B_{X_j}\\ y^*\in B_{Y^*}}}\left(\sum_{n=k}^{\infty}\left|\langle x_1,x_{1n}^*\rangle\cdots\langle x_m,x_{mn}^*\rangle\langle y_n,y^*\rangle\right|\right)=0.$$

The σ-nuclear norm is defined by

$$\|T\|_{\sigma} := \inf\left\{ \|(\lambda_n)_{n=1}^{\infty}\|_{\ell_\infty} \sup_{\substack{x_j \in B_{X_j} \\ y^* \in B_{Y^*}}} \left(\sum_{n=1}^{\infty} |\langle x_1, x_{1n}^* \rangle \cdots \langle x_m, x_{mn}^* \rangle \langle y_n, y^* \rangle| \right) \right\},$$

where the infimum being taken over all the σ-nuclear representations of T. The Banach space set of all σ-nuclear multilinear operators is denoted by $\mathcal{L}_\sigma(X_1, \ldots, X_m; Y)$.

If $X_1^*, ..., X_m^*$ have the bounded approximation property, the multi-ideal of σ-nuclear multilinear operators from $X_1 \times \cdots \times X_m$ into Y is isometric to $X_1^* \widehat{\otimes}_\sigma \cdots \widehat{\otimes}_\sigma X_m^* \widehat{\otimes}_\sigma Y$, such that

$$\sigma(u) = \inf\left\{ \|(\lambda_i)_{i=1}^{n}\|_{\ell_\infty} \sup_{\substack{x_j \in B_{X_j} \\ y^* \in B_{Y^*}}} \left(\sum_{i=1}^{n} |\langle x_1, x_{1i}^* \rangle \cdots \langle x_i, x_{mn}^* \rangle \langle y_i, y^* \rangle| \right) \right\}$$

where the infimum is taken over all representations of $u = \sum_{i=1}^{n} \lambda_i x_{1i}^* \otimes \ldots \otimes x_{mi}^* \otimes y_i \in X_1^* \otimes \cdots \otimes X_m^* \otimes Y$.

2) p-Semi-integral multilinear operators

R. Alencar and M. Matos [7] introduced semi-integral multilinear operators as a natural extension of the class concept of p-summing linear operators to the multilinear case. This concept was immediately extended to p-semi-integral multilinear operators,($1 < p < \infty$), by E. Çaliskan and D. M. Pellegrino in [17].

Definition 1.2.19 *We say that $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is p-semi-integral if there exists a constant $C \geq 0$ and a regular probability measure μ on the Borel σ-algebra of $B_{X_1^*} \times \cdots \times B_{X_m^*}$ endowed with the weak* topologies $\sigma(X_j^*, X_j)$, $(1 \leq j \leq m)$, such that*

$$\|T(x_1, \cdots, x_m)\| \leq C \left(\int_{B_{X_1^*} \times \cdots \times B_{X_m^*}} |\langle x_1, x_1^* \rangle \cdots \langle x_m, x_m^* \rangle|^p \, d\mu(x_1^*, ..., x_m^*) \right)^{1/p}$$

for every $x_j \in X_j (1 \leq j \leq m)$.

The infimum of all such constants C is denoted by $\|T\|_{si,p}$. We denote this class of operators by $\mathcal{L}_{si,p}(X_1, \cdots, X_m; Y)$.

It is well known that $\mathcal{L}_{si,p}$ is a multi-ideal (see[17]). Whenever $p \geq 1$, the representation theorem ensures that the space of p-semi-integral multilinear operators between the Banach spaces $X_1 \times \cdots \times X_m$ and Y is isometrically isomorphic to $(X_1 \otimes \cdots \otimes X_m \otimes Y, \alpha_p)^*$, where

$$\alpha_p(u) = \|(\lambda_i)_{i=1}^{n}\|_{\ell_{p^*}} \sup_{\substack{x_j^* \in B_{X_j^*} \\ 1 \leq j \leq m}} \left(\sum_{i=1}^{n} |\langle x_{1i}, x_1^* \rangle \cdots \langle x_{mi}, x_m^* \rangle|^p \right)^{1/p} \|(y_i^*)_{i=1}^{n}\|_{\ell_\infty(Y^*)}$$

and the infimum is taken over all representations of $u = \sum_{i=1}^{n} \lambda_i x_{1i} \otimes \ldots \otimes x_{mi} \otimes y_i \in X_1 \otimes \cdots \otimes X_m \otimes Y$.

3) Cohen p-nuclear Multilinear operators

D. Achour and A. Alouani extended in [1] the notion of Cohen p-nuclear linear operators introduced by J. S. Cohen in [20] to the case of the multilinear operators.

Definition 1.2.20 *An m-linear operator $T : X_1 \times \cdots \times X_m \longrightarrow Y$ is Cohen p-nuclear $(1 < p \leq \infty)$ if there is a constant $C > 0$ such that for any $x_{j1}, \ldots, x_{jn} \in X_j, (1 \leq j \leq m)$, and any $y_1^*, \ldots, y_n^* \in Y^*$, we have*

$$\left| \sum_{i=1}^{n} \langle T(x_{1i}, \ldots, x_{mi}), y_i^* \rangle \right| \leq C \sup_{\substack{x_j^* \in B_{X_j^*} \\ 1 \leq j \leq m}} \left(\sum_{i=1}^{n} |\langle x_{1i}, x_1^* \rangle \cdots \langle x_{mi}, x_m^* \rangle|^p \right)^{1/p} \sup_{y^{**} \in B_{Y^{**}}} \left(\sum_{i=1}^{n} |\langle y_i^*, y^{**} \rangle|^{p^*} \right)^{1/p^*} \tag{1.4}$$

The class of all Cohen p-nuclear m-linear operators from $X_1 \times \cdots \times X_m$ into Y, which is denoted by $\mathcal{N}_p^m(X_1, \ldots, X_m; Y)$, is a multi-ideal (see [1]) with the norm $\|\cdot\|_{p,N}$, which is the smallest constant C such that the inequality (1.4) holds. It is worth to observe that, in particular,

$$\mathcal{N}_p^m(X_1, \ldots, X_m; \mathbb{K}) = \mathcal{L}_{si,p}(X_1, \ldots, X_m; \mathbb{K}).$$

The representation theorem ensures that the space of Cohen p-nuclear multilinear operators between the Banach spaces $X_1 \times \cdots \times X_m$ and Y^* is isometrically isomorphic to $(X_1 \otimes \cdots \otimes X_m \otimes Y, \omega_p)^*$, such that

$$\omega_p(u) = \|(\lambda_i)_{i=1}^n\|_{\ell_\infty} \sup_{\substack{x_j^* \in B_{X_j^*} \\ 1 \leq j \leq m}} \left(\sum_{i=1}^{n} |\langle x_{1i}, x_1^* \rangle \cdots \langle x_{mi}, x_m^* \rangle)|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*},$$

where the infimum is taken over all representations of $u = \sum_{i=1}^{n} \lambda_i x_{1i} \otimes \ldots \otimes x_{mi} \otimes y_i \in X_1 \otimes \cdots \otimes X_m \otimes Y$.

Chapter 2

Weakly p-nuclear multilinear operators

Introduction

The introduction of tensor products into the field of functional analysis took place in the late 1930s, through the contributions of Murray, von Neumann, and Schatten [40, 54]. However, it was Grothendieck who truly found the extensive set of properties associated with tensor products. His seminal work, titled "*Résumé de la théorie métrique des produits tensoriels topologiques*" [30], which involves studying Banach spaces in terms of their finite-dimensional subspaces. Furthermore, Grothendieck's work confirmed the content of employing tensor products in the theory of normed spaces, contributing greatly to the development of a fruitful theory of duality.

In 1968, Lindenstrauss and Pełczyński, in their work "*Absolutely summing operators in L_p spaces and their applications*"[39] introduced some important connections and applications to the theory of absolutely p-summing operators, formulating results presented in the terms of certain the tensor norms by Grothendieck into properties of operator ideals.

The theory of tensor norms evolved into an interestingly and significant field in itself. Notably, Defant and Floret, through their renowned monograph "*Tensor Norms and Operator Ideals*"[25] established a crucial connection between the theory of tensor products and the theory of operator ideals. In their work, they illuminated the idea that these two theories are closely related, as they were two sides of the same coin. This realization had a lasting impact, leading authors to smoothly utilize both tools. This approach has been further explored in various other books [23, 51].

These studies allowed for more effective solutions to certain problems, particularly in the theory of duality. For instance, in the work of M.C. Matos [41], a tensor norm is established such that linear operators on the tensor product that are continuous with respect to this norm correspond to the class of nuclear multilinear operators.

2.1 Characterizations of the space of weakly p-nuclear multilinear operators

A large number of research has been done on multilinear generalizations of operator ideals. Now, we introduce the following concept as a version of the extension of the ideals of weakly p-nuclear operators to the multilinear setting and then show that the normed space that has been constructed is a Banach ideal of multilinear operators. The findings presented in this section constitute a significant aspect of the results outlined in our research paper [34].

Definition 2.1.1 *For $1 \leq p < \infty$, we say that a multilinear operator $T : X_1 \times \cdots \times X_m \to Y$ is weakly p-nuclear if there are sequences $\left(x_{jn}^*\right)_{n=1}^{\infty} \subset X_j^*$, for $j = 1, ..., m$, and $(y_n)_{n=1}^{\infty} \in \ell_{p^*,w}(Y)$, such that*

$$T = \sum_{n=1}^{\infty} x_{1n}^* \otimes \cdots \otimes x_{mn}^* \otimes y_n,$$

and

$$\sup_{\substack{x_j \in B_{X_j} \\ 1 \leq j \leq m}} \left(\sum_{n=1}^{\infty} \left| \langle x_1, x_{1n}^* \rangle \cdots \langle x_m, x_{mn}^* \rangle \right|^p \right)^{1/p} < \infty.$$

In this case we say that $\sum_{n=1}^{\infty} x_{1n}^* \otimes \cdots \otimes x_{mn}^* \otimes y_n$ is a weakly p-nuclear representation of T and define

$$\|T\|_{\mathcal{N}_{wp}} := \inf \left\{ \sup_{\substack{x_j \in B_{X_j} \\ 1 \leq j \leq m}} \left(\sum_{n=1}^{\infty} \left| \langle x_1, x_{1n}^* \rangle \cdots \langle x_m, x_{mn}^* \rangle \right|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} \left| \langle y_n, y^* \rangle \right|^{p^*} \right)^{1/p^*} \right\},$$

with the infimum taken over all weakly p-nuclear representations of T.
We write $\mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right)$ to denote all weakly p-nuclear operators from $X_1 \times \cdots \times X_m$ into Y.

Theorem 2.1.2 *For $1 \leq p < \infty$, $[\mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right), \|.\|_{\mathcal{N}_{wp}}]$ is a Banach ideal of m-linear operators.*

Proof. First, let us show that $[\mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right), \|.\|_{\mathcal{N}_{wp}}]$ is always a normed space. Whereas this is obvious for $p = 1$, the statement that $T_1 + T_2 \in \mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right)$ and $\|T_1 + T_2\|_{\mathcal{N}_{wp}} \leq \|T_1\|_{\mathcal{N}_{wp}} + \|T_2\|_{\mathcal{N}_{wp}}$ whenever $T_1, T_2 \in \mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right)$ requires proof if $1 < p < \infty$.

Consequently, fix $\epsilon > 0$ and choose representation $T_1 = \sum_{n=1}^{\infty} x_{2n-1}^{*(1)} \otimes \cdots \otimes x_{2n-1}^{*(m)} \otimes y_{2n-1}$ and $T_2 = \sum_{n=1}^{\infty} x_{2n}^{*(1)} \otimes \cdots \otimes x_{2n}^{*(m)} \otimes y_{2n}$ such that

$$\sup_{\substack{x_j \in B_{X_j} \\ 1 \le j \le m}} \left(\sum_{n=1}^{\infty} \left| \left\langle x_1, x_{2n-1}^{*(1)} \right\rangle \cdots \left\langle x_m, x_{2n-1}^{*(m)} \right\rangle \right|^p \right)^{1/p} \le \|T_1\|_{\mathcal{N}_{wp}} + \epsilon$$

$$\sup_{\substack{x_j \in B_{X_j}, \\ 1 \le j \le m}} \left(\sum_{n=1}^{\infty} \left| \left\langle x_1, x_{2n}^{*(1)} \right\rangle \cdots \left\langle x_m, x_{2n}^{*(m)} \right\rangle \right|^p \right)^{1/p} \le \|T_2\|_{\mathcal{N}_{wp}} + \epsilon,$$

and $\|(y_{2n})\|_{\ell_{p^*,w}(Y)} = \|(y_{2n-1})\|_{\ell_{p^*,w}(Y)} = 1$. Fix $r, s > 0$ arbitrarily and define $(\tilde{y}_n) \in \ell_{p^*,w}(Y)$ by $\tilde{y}_{2n} = s^{-1} y_{2n}$ and $\tilde{y}_{2n-1} = r^{-1} y_{2n-1}$, by

$$\tilde{x}_{2n}^{*(1)} \otimes \cdots \otimes \tilde{x}_{2n}^{*(m)} = s x_{2n}^{*(1)} \otimes \cdots \otimes x_{2n}^{*(m)}$$

and

$$\tilde{x}_{2n-1}^{*(1)} \otimes \cdots \otimes \tilde{x}_{2n-1}^{*(m)} = r x_{2n-1}^{*(1)} \otimes \cdots \otimes x_{2n-1}^{*(m)}$$

$$\begin{aligned} \sup_{\substack{x_j \in B_{X_j} \\ 1 \le j \le m}} \left(\sum_{n=1}^{\infty} \left| \left\langle x_1, \tilde{x}_n^{*(1)} \right\rangle \cdots \left\langle x_m, \tilde{x}_n^{*(m)} \right\rangle \right|^p \right) &= r^p \sup_{\substack{x_j \in B_{X_j} \\ 1 \le j \le m}} \left(\sum_{n=1}^{\infty} \left| \left\langle x_1, x_{2n-1}^{*(1)} \right\rangle \cdots \left\langle x_m, x_{2n-1}^{*(m)} \right\rangle \right|^p \right) + \\ &\quad s^p \sup_{\substack{x_j \in B_{X_j} \\ 1 \le j \le m}} \left(\sum_{n=1}^{\infty} \left| \left\langle x_1, x_{2n}^{*(1)} \right\rangle \cdots \left\langle x_m, x_{2n}^{*(m)} \right\rangle \right|^p \right) \\ &\le r^p (\|T_1\|_{\mathcal{N}_{wp}} + \epsilon)^p + s^p (\|T_2\|_{\mathcal{N}_{wp}} + \epsilon)^p \end{aligned}$$

and

$$\begin{aligned} \left(\|(\tilde{y}_n)_n\|_{\ell_{p^*,w}(Y)} \right)^{p^*} &\le \left(\frac{1}{r} \|(y_{2n-1})_n\|_{\ell_{p^*,w}(Y)} \right)^{p^*} + \left(\frac{1}{s} \|(y_{2n})_n\|_{\ell_{p^*,w}(Y)} \right)^{p^*} \\ &= r^{-p^*} + s^{-p^*}. \end{aligned}$$

Writing $T_1 + T_2 = \sum_{n=1}^{\infty} \tilde{x}_n^{*(1)} \otimes \cdots \otimes \tilde{x}_n^{*(m)} \otimes \tilde{y}_n^*$, we see that $T_1 + T_2$ is weakly p-nuclear with

$$\begin{aligned} \|T_1 + T_2\|_{\mathcal{N}_{wp}} &\le \sup_{\substack{x_j \in B_{X_j} \\ 1 \le j \le m}} \left(\sum_{n=1}^{\infty} \left| \left\langle x_1, \tilde{x}_n^{*(1)} \right\rangle \cdots \left\langle x_m, \tilde{x}_n^{*(m)} \right\rangle \right|^p \right)^{1/p} \|(\tilde{y}_n)_n\|_{\ell_{p^*,w}(Y)} \\ &\le \frac{1}{p} \sup_{\substack{x_j \in B_{X_j} \\ 1 \le j \le m}} \left(\sum_{n=1}^{\infty} \left| \left\langle x_1, \tilde{x}_n^{*(1)} \right\rangle \cdots \left\langle x_m, \tilde{x}_n^{*(m)} \right\rangle \right|^p \right) + \frac{1}{p^*} \left(\|(\tilde{y}_n)_n\|_{\ell_{p^*,w}(Y)} \right)^{p^*} \\ &\le \frac{r^p}{p} (\|T_1\|_{\mathcal{N}_{wp}} + \epsilon)^p + \frac{r^{-p^*}}{p^*} + \frac{s^p}{p} (\|T_2\|_{\mathcal{N}_{wp}} + \epsilon)^p + \frac{s^{-p^*}}{p^*}. \end{aligned}$$

With, $r = (\|T_1\|_{\mathcal{N}_{wp}} + \epsilon)^{-1/p^*}$ and $s = (\|T_2\|_{\mathcal{N}_{wp}} + \epsilon)^{-1/p^*}$ we obtain

$$\|T_1 + T_2\|_{\mathcal{N}_{wp}} \leq \|T_1\|_{\mathcal{N}_{wp}} + \|T_2\|_{\mathcal{N}_{wp}} + 2\epsilon.$$

To show that $[\mathcal{N}_{wp}(X_1, ..., X_m; Y),\ \|.\|_{\mathcal{N}_{wp}}]$ is complete, let (T_k) be a sequence in $\mathcal{N}_{wp}(X_1, ..., X_m; Y)$ such that $\sum\limits_{k=1}^{\infty} \|T_k\|_{\mathcal{N}_{wp}} < \infty$. Then $\sum\limits_{k=1}^{\infty} \|T_k\| < \infty$ too, and such that $T = \sum\limits_{k=1}^{\infty} T_k$ exists in $\mathcal{L}(X_1, ..., X_m; Y)$. We shall show that $T \in \mathcal{N}_{wp}(X_1, ..., X_m; Y)$.

We consider (J_M) a sequences of real numbers, given by

$$J_M = \left\| \sum_{k=1}^{M} T_k \right\|_{\mathcal{N}_{wp}}, \quad M \in \mathbb{N}^*.$$

Thus,

$$J_M = \inf \sup_{\substack{x_j \in B_{X_j} \\ 1 \leq j \leq m}} \left(\sum_{k=1}^{M} \sum_{n=1}^{\infty} \left| \left\langle x_1, x_{k,n}^{*(1)} \right\rangle \cdots \left\langle x_m, x_{k,n}^{*(m)} \right\rangle \right|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{k=1}^{M} \sum_{n=1}^{\infty} |\langle y_{k,n}, y^* \rangle|^{p^*} \right)^{1/p^*}.$$

Then, there exists a sequence $(a_{k,n})$ in ℓ_{p^*}, and $(b_{k,n})$ in ℓ_p. Moreover, $\| (a_{k,n}) \|_{p^*} = 1$ and $\| (b_{k,n}) \|_p = 1$. Hence

$$J_M = \inf \sup_{\substack{x_j \in B_{X_j} \\ 1 \leq j \leq m}} \left(\sum_{k=1}^{M} \left| \sum_{n=1}^{\infty} a_{k,n} \left\langle x_1, x_{k,n}^{*(1)} \right\rangle \cdots \left\langle x_m, x_{k,n}^{*(m)} \right\rangle \right| \right) \sup_{y^* \in B_{Y^*}} \left(\sum_{k=1}^{M} \left| \sum_{n=1}^{\infty} b_{k,n} \langle y_{k,n}, y^* \rangle \right| \right).$$

We can see that (J_M) is a bounded increasing sequence then (J_M) is convergent.

letting $M \to \infty$, we conclude that

$$\left\| \sum_{k=1}^{\infty} T_k \right\|_{\mathcal{N}_{wp}} = \|T\|_{\mathcal{N}_{wp}} < \infty.$$

This completes the proof. ■

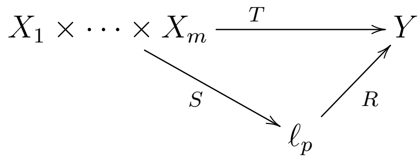
Proposition 2.1.3 *Let $1 \leq p$. A multilinear operator $T : X_1 \times \cdots \times X_m \to Y$ is weakly p-nuclear, then T has a factorization $T = R \circ S$, such that $S \in \mathcal{L}(X_1, ..., X_m; \ell_p)$ and $R \in \mathcal{L}(\ell_p; Y)$. Moreover, $\|T\|_{\mathcal{N}_{wp}} \geq \inf \|S\| \|R\|$.*

Proof. Assume that T is weakly p-nuclear and let $\sum\limits_{n=1}^{\infty} x_{1n}^* \otimes \cdots \otimes x_{mn}^* \otimes y_n$, with $\left(x_{jn}^*\right)_{n=1}^{\infty} \subset X_j^*$, for $j = 1, ..., m$, and $(y_n)_{n=1}^{\infty} \in \ell_{p^*,w}(Y)$ be a weakly p-nuclear representation of T . Let

$$S: \quad X_1 \times \cdots \times X_m \to \ell_p, \quad (x_1, ..., x_m) \mapsto (x_{1n}^*(x_1) \cdots x_{mn}^*(x_m))_n\,,$$

$$R: \qquad \ell_p \to Y, \qquad (s_n)_n \mapsto \sum_{n=1}^{\infty} s_n y_n.$$

Then, we see that $\|S\| \leq \sup_{\substack{x_j \in B_{X_j} \\ 1\leq j\leq m}} \|(x_{1n}^*(x_1)\cdots x_{mn}^*(x_m))_n\|_{\ell_p}$ and $\|R\| = \|(y_n)_n\|_{\ell_{p^*,w(Y)}}$, and the following diagram is commutative



This completes the proof. ■

2.1.1 Connection with tensor products

The aim of this subsection is to obtain characterizations of the space weakly p-nuclear multilinear operators as the space of the tensor product $\left(X_1^*\widehat{\otimes}\cdots\widehat{\otimes}X_m^*\widehat{\otimes}Y, \omega_p\right)$ up to an isometric.

We consider tensor norm and associate it with an operator ideal. Let us define a cross norm $\omega_p(.)$, ($1 \leq p < \infty$), on the tensor product $X_1 \otimes \cdots \otimes X_m \otimes Y$ as follows: If $u \in X_1 \otimes \cdots \otimes X_m \otimes Y$ then

$$\omega_p(u) = \inf\left\{ \|(\lambda_i)_{i=1}^n\|_{\ell_\infty} \sup_{\substack{x_j^* \in B_{X_j^*} \\ 1\leq j\leq m}} \left(\sum_{i=1}^n |\langle x_{1i}, x_1^*\rangle \cdots \langle x_{mi}, x_m^*\rangle|^p\right)^{1/p} \sup_{y^*\in B_{Y^*}} \left(\sum_{i=1}^n |\langle y_i, y^*\rangle|^{p^*}\right)^{1/p^*} \right\},$$

where the infimum is taken over all representations of u of the form

$$u = \sum_{i=1}^n \lambda_i x_{1i} \otimes \cdots \otimes x_{mi} \otimes y_i,$$

with $(x_{ji})_{i=1}^n \subset X_j$ $(1 \leq j \leq m)$ and $(y_i)_{i=1}^n \subset Y$.

Proposition 2.1.4 *If $X_1, ..., X_m$ are finite dimensional, then*

$$\|T\|_{\mathcal{N}_{wp}} \geq \omega_p(T),$$

for every $T \in \mathcal{L}_f(X_1, ..., X_m; Y)$.

Proof. By supposition there is a constant $C \geq 0$ such that $\omega_p(T) \leq C\|T\|_{\mathcal{N}_{wp}}$, let $T \in \mathcal{L}_f(X_1, ..., X_m; Y)$ and $\epsilon > 0$, we choose a representation

$$T = \sum_{i=1}^\infty x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i,$$

such that

$$\sup_{\substack{x_j \in B_{X_j} \\ 1\leq j\leq m}} \left(\sum_{i=1}^{n-1} |\langle x_1, x_{1i}^*\rangle \cdots \langle x_m, x_{mi}^*\rangle|^p\right)^{1/p} \sup_{y^*\in B_{Y^*}} \left(\sum_{i=1}^{n-1} |\langle y_i, y^*\rangle|^{p^*}\right)^{1/p^*} \leq (1+\frac{\epsilon}{2})\|T\|_{\mathcal{N}_{wp}}.$$

In particular, for each $n \in \mathbb{N}$,

$$\omega_p\left(\sum_{i=1}^{n-1} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right) \leq \sup_{\substack{x_j \in B_{X_j} \\ 1\leq j\leq m}} \left(\sum_{i=1}^{n-1} |\langle x_1, x_{1i}^*\rangle \cdots \langle x_m, x_{mi}^*\rangle|^p\right)^{1/p} \sup_{y^*\in B_{Y^*}} \left(\sum_{i=1}^{n-1} |\langle y_i, y^*\rangle|^{p^*}\right)^{1/p^*}$$

$$\leq (1+\frac{\epsilon}{2})\|T\|_{\mathcal{N}_{wp}}.$$

For a sufficiently large $n \in \mathbb{N}$ we can write

$$\left\|\sum_{i=n}^{\infty} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right\|_{\mathcal{N}_{wp}} \leq \sup_{\substack{x_j \in B_{X_j} \\ 1\leq j\leq m}} \left(\sum_{i=n}^{\infty} |\langle x_1, x_{1i}^*\rangle \cdots \langle x_m, x_{mi}^*\rangle|^p\right)^{1/p} \sup_{y^*\in B_{Y^*}} \left(\sum_{i=n}^{\infty} |\langle y_i, y^*\rangle|^{p^*}\right)^{1/p^*}$$

$$\leq \frac{\epsilon}{2C}\|T\|_{\mathcal{N}_{wp}}.$$

It follows that

$$\begin{aligned}
\omega_p(T) &\leq \omega_p\left(\sum_{i=1}^{n-1} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right) + \omega_p\left(\sum_{i=n}^{\infty} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right) \\
&\leq \left(1+\frac{\epsilon}{2}\right)\|T\|_{\mathcal{N}_{wp}} + C\left\|\sum_{i=n}^{\infty} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right\|_{\mathcal{N}_{wp}} \\
&\leq \left(1+\frac{\epsilon}{2}\right)\|T\|_{\mathcal{N}_{wp}} + \frac{\epsilon}{2}\|T\|_{\mathcal{N}_{wp}} = (1+\epsilon)\|T\|_{\mathcal{N}_{wp}},
\end{aligned}$$

and as this holds for every $\epsilon > 0$, the result follows. ■

Proposition 2.1.5 *If $T \in \mathcal{N}_{wp}(X_1, ..., X_m; Y)$ and $S_j \in \mathcal{L}_f(D_j; X_j)$, for $j = 1, ..., m$, then*

$$\omega_p(T \circ (S_1, ..., S_m)) \leq \|T\|_{\mathcal{N}_{wp}} \|S_1\| \cdots \|S_m\|.$$

Proof. If J_j denotes the natural injection of $S_j(D_j)$ into X_j, we can write $S_j = J_j \circ \tilde{S}_j$, with $\left\|\tilde{S}_j\right\| = \|S_j\|$. Hence,

$$T \circ (J_1, ..., J_m) \in \mathcal{L}_f(S_1(D_1), ..., S_m(D_m); Y).$$

Now, we apply Proposition 2.1.4 and property ideal in order to have the result proved. ■

Proposition 2.1.6 *If $X_1^*, ..., X_m^*$ have the bounded approximation property, then*

$$\|T\|_{\mathcal{N}_{wp}} \geq \omega_p(T),$$

for every $T \in \mathcal{L}_f(X_1, ..., X_m; Y)$.

Proof. We note that the operator $T_j \in \mathcal{L}\left(X_j; \mathcal{L}\left(X_j; \mathcal{L}\left(X_1, ..., X_{j-1}, X_{j+1}, ...X_m; Y\right)\right)\right)$, defined by

$$T_j(x_j)\left(x_1, ..., x_{j-1}, x_{j+1}, ..., x_m\right) = T\left(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_m\right)$$

is finite type. Since X_j^* has the λ_j-approximation property for some $\lambda_j > 1$, for each $\epsilon > 0$, we can find $S_j \in \mathcal{L}_f\left(X_j; X_j\right)$, such that $T_j = T_j \circ S_j$ and $\|S_j\| \leq (1+\epsilon)\lambda_j$. Therefore, for all $x_j \in X_j$, with $j = 1, ..., m$, we have

$$T\left(x_1, ..., x_{j-1}, S_j\left(x_j\right), x_{j+1}, ..., x_m\right) = T\left(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_m\right).$$

Now, we can write

$$T\left(x_1, ..., x_m\right) = T\left(S_1\left(x_1\right), \ldots, S_m\left(x_m\right)\right).$$

for all $x_j \in X_j (1 \leq j \leq m)$. Thus, by Proposition 2.1.6, we have

$$\begin{aligned} \omega_p(T) &\leq \|T\|_{\mathcal{N}_{wp}} \|S_1\| \cdots \|S_m\| \\ &\leq \|T\|_{\mathcal{N}_{wp}} \lambda_1 ... \lambda_m. \end{aligned}$$

Hence

$$\omega_p(T) \leq \lambda_1 ... \lambda_m \|T\|_{\mathcal{N}_{wp}}.$$

With the same argument used in the proof of Proposition 2.1.4 , we finally have:

$$\omega_p(T) \leq \|T\|_{\mathcal{N}_{wp}}.$$

This completes the proof. ■

Corollary 2.1.7 *If $X_1^*, ..., X_m^*$ have the bounded approximation property, then*

$$\mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right)$$

and $\left(X_1^* \widehat{\otimes} \cdots \widehat{\otimes} X_m^* \widehat{\otimes} Y, \omega_p\right)$ are isometric.

2.2 The dual of $\mathcal{N}_{wp}(X_1, ..., X_m; Y)$

In this section, our main goal is to find a class of bounded linear functionals that represent the space of weakly p-nuclear multilinear operators. To get started, let us identify the following class of multilinear operators that will fit our purpose.

2.2.1 Quasi Cohen p-nuclear multilinear operators

Now, we present the class of quasi Cohen p-nuclear multilinear operators for $(1 \leq p)$.

Definition 2.2.1 *A multilinear operator $T : X_1 \times \cdots \times X_m \to Y^*$ is quasi Cohen p-nuclear $(1 \leq p)$ if there is a constant $C > 0$ such that for any $x_{j1}, \ldots, x_{jn} \in X_j (1 \leq j \leq m)$ and any $y_1, \ldots, y_n \in Y$, we have*

$$\left| \sum_{i=1}^{n} \langle y_i, T(x_{1i}, ..., x_{mi}) \rangle \right| \leq C \sup_{\substack{x_j^* \in B_{X_j^*} \\ 1 \leq j \leq m}} \left(\sum_{i=1}^{n} |\langle x_{1i}, x_1^* \rangle \cdots \langle x_{mi}, x_m^* \rangle|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \tag{2.1}$$

The least constant C for which this inequality (2.1) always holds is denoted by $\|T\|_{\mathcal{N}_{q(p)}}$. We shall write $\mathcal{N}_{q(p)}(X_1, ..., X_m; Y^*)$ for the Banach space set of all quasi Cohen p-nuclear multilinear operators.

Remark 2.2.2 *It is easy to see that $\mathcal{N}_p^m(X_1, ..., X_m; Y^*) \subseteq \mathcal{N}_{q(p)}(X_1, ..., X_m; Y^*)$ with $\|\cdot\|_{\mathcal{N}_{q(p)}} \leq \|\cdot\|_{p,N}$ for every Y and that $\mathcal{N}_{q(p)}(X_1, ..., X_m; Y^*) = \mathcal{N}_p^m(X_1, ..., X_m; Y^*)$ isometrically for reflexive Y.*

Domination Theorem

The main feature of the class quasi Cohen p-nuclear multilinear operators is that it enjoy a Pietsch domination theorem. For the proof, we will use Ky Fan's lemma [22].

Lemma 2.2.3 (KY FAN's) *Let E be a Hausdorff topological vector space, and let $\mathcal{C}$ be a compact convex subset of X. Let M be a set of functions on $\mathcal{C}$ with values in $(-\infty, \infty]$ having the following properties:*

(a) *each $f \in M$ is convex and lower semicontinuous.*

(b) *if $g \in \mathrm{conv}(M)$, then there is an $f \in M$ with $g(x) \leq f(x)$ for every $x \in \mathcal{C}$.*

(c) *there is an $r \in \mathbb{R}$ such that each $f \in M$ has a value not greater than r. Then there is an $x_0 \in \mathcal{C}$ such that $f(x_0) \leq r$ for all $f \in M$.*

Theorem 2.2.4 *For $T \in \mathcal{L}(X_1, ..., X_m; Y^*)$ and $1 < p$, the following conditions are equivalent:*

(i) *The operator T is quasi Cohen p-nuclear.*

(ii) *For all $x_{j1},\ldots,x_{jn}$ in X_j $(1\leq j\leq m)$ and $y_1,\ldots,y_n$ in Y, we have*

$$\sum_{i=1}^{n}\left|\left\langle y_i,T\left(x_{1i},\ldots,x_{mi}\right)\right\rangle\right|\leq C\sup_{\substack{x_j^*\in B_{X_j^*}\\ 1\leq j\leq m}}\left(\sum_{i=1}^{n}\left|\langle x_{1i},x_1^*\rangle\cdots\langle x_{mi},x_m^*\rangle\right|^p\right)^{1/p}\times$$
$$\sup_{y^*\in B_{Y^*}}\left(\sum_{i=1}^{n}\left|\langle y_i,y^*\rangle\right|^{p^*}\right)^{1/p^*}. \tag{2.2}$$

(iii) *There exist Radon probability measures $\mu_j\in\mathcal{C}\left(B_{X_j^*}\right)^*$ $(1\leq j\leq m)$ and $\lambda\in\mathcal{C}\left(B_{Y^*}\right)^*$ such that for all $(x_1,...,x_m)\in X_1\times\cdots\times X_m$ and $y\in Y$*

$$\left|\left\langle y,T\left(x_1,...,x_m\right)\right\rangle\right|\leq C\prod_{j=1}^{m}\left\|x_j\right\|_{L_p\left(B_{X_j^*},\mu_j\right)}\left\|y\right\|_{L_{p^*}\left(B_{Y^*},\lambda\right)}. \tag{2.3}$$

Proof. The implication (i) $\Rightarrow$ (ii)Easily results from equality (1.1).

The implication (ii) $\Rightarrow$ (iii). The proof was inspired by [2]. Let the sets $P\left(B_{X_j^*}\right)$, $(1\leq j\leq m)$, and $P\left(B_{Y^*}\right)$ of probability measures in $\mathcal{C}\left(B_{X_j^*}\right)^*$ and $\mathcal{C}\left(B_{Y^*}\right)^*$, respectively. These are convex sets which are compact when we endow $\mathcal{C}\left(B_{X_j^*}\right)^*$ and $\mathcal{C}\left(B_{Y^*}\right)^*$ with their weak* topologies. By apply Ky Fan's lemma with $E=\mathcal{C}\left(B_{X_1^*}\right)^*\times\cdots\times\mathcal{C}\left(B_{X_m^*}\right)^*\times\mathcal{C}\left(B_{Y^*}\right)^*$ and $\mathcal{C}=P\left(B_{X_1^*}\right)\times\cdots\times P\left(B_{X_m^*}\right)\times P\left(B_{Y^*}\right)$.

Consider the set M of all functions $f:\mathcal{C}\to\mathbb{R}^+$ for which there exist $x_{j1},\ldots,x_{jn}\in X_j(1\leq j\leq m)$ and $y_1,\ldots,y_n\in Y$ such that

$$f\left(\mu_1,...,\mu_m,\lambda\right):=\sum_{i=1}^{n}\left|\left\langle y_i,T\left(x_{1i},...,x_{mi}\right)\right\rangle\right|-\frac{C}{p}\sum_{i=1}^{n}\prod_{j=1}^{m}\int_{B_{X_j^*}}\left|\langle x_{ji},x_j^*\rangle\right|^p d\mu_j\left(x_j^*\right)$$
$$-\frac{C}{p^*}\sum_{i=1}^{n}\int_{B_{Y^*}}\left|\langle y_i,y^*\rangle\right|^{p^*}d\lambda\left(y^*\right)$$

for all $(\mu_1,...,\mu_m,\lambda)\in\mathcal{C}$. It is clear that all such f are continuous and affine and that the set M is a convex cone and consequently conditions (a) and (b) of Ky Fan's lemma are satisfied.

For condition (c), since $B_{X_j^*}$ and B_{Y^*} are weak* compact and norming, there exist for $f\in M$ elements $x_0^{*j}\in B_{X_j^*}$ and $y_0^*\in B_{Y^*}$ such that and

$$\sup_{y^*\in B_{Y^*}}\left(\sum_{i=1}^{n}\left|\langle y_i,y^*\rangle\right|^{p^*}\right)=\sum_{i=1}^{n}\left|\langle y_i,y_0^*\rangle\right|^{p^*}$$

Using the elementary identity

$$\alpha\beta=\inf_{\epsilon>0}\left\{\frac{1}{p}\left(\frac{\alpha}{\epsilon}\right)^p+\frac{1}{p^*}(\epsilon\beta)^{p^*}\right\},\quad\forall\alpha,\beta\in\mathbb{R}_+^*,$$

we find by taking

$$\alpha=\left(\sup_{\substack{x_j^*\in B_{X_j^*}\\1\le j\le m}}\sum_{i=1}^{n}\left|\langle x_{1i},x_1^*\rangle\cdots\langle x_{mi},x_m^*\rangle\right|^p\right)^{1/p},\quad \beta=\sup_{y^*\in B_{Y^*}}\left(\sum_{i=1}^{n}\left|\langle y_i,y^*\rangle\right|^{p^*}\right)^{1/p^*}$$

and $\epsilon=1$ that

$$\begin{aligned}
f\left(\delta_{x_0^{*1}},...,\delta_{x_0^{*m}},\delta_{y_0}\right)&=\sum_{i=1}^{n}\left|\langle y_i,T\left(x_{1i},...,x_{mi}\right)\rangle\right|-\frac{C}{p}\sup_{\substack{x_j^*\in B_{X_j^*}\\1\le j\le m}}\left(\sum_{i=1}^{n}\left|\langle x_{1i},x_1^*\rangle\cdots\langle x_{mi},x_m^*\rangle\right|^p\right)\\
&-\frac{C}{p^*}\sup_{y^*\in B_{Y^*}}\left(\sum_{i=1}^{n}\left|\langle y_i,y^*\rangle\right|^{p^*}\right)\\
&\le\sum_{i=1}^{n}\left|\langle y_i,T\left(x_{1i},...,x_{mi}\right)\rangle\right|\\
&-C\sup_{\substack{x_j^*\in B_{X_j^*}\\1\le j\le m}}\left(\sum_{i=1}^{n}\left|\langle x_{1i},x_1^*\rangle\cdots\langle x_{mi},x_m^*\rangle\right|^p\right)^{1/p}\sup_{y^*\in B_{Y^*}}\left(\sum_{i=1}^{n}\left|\langle y_i,y^*\rangle\right|^{p^*}\right)^{1/p^*},
\end{aligned}$$

where δ_x is the Dirac measure at x. The last quantity is less than or equal to zero (by hypothesis (ii)) and hence condition (c) is satisfied with $r=0$. By Ky Fan's lemma, there is $(\mu_1,...,\mu_m,\lambda)\in\mathcal{C}$ with $f(\mu_1,...,\mu_m,\lambda)\le 0$ for all $f\in M$. Then, if f is generated by the single elements $(x_1,...,x_m)\in X_1\times\cdots\times X_m$ and $y\in Y$

$$\left|\langle y,T\left(x_1,...,x_m\right)\rangle\right|\le\frac{C}{p}\prod_{j=1}^{m}\int_{B_{X_j^*}}\left|\langle x_j,x_j^*\rangle\right|^p d\mu_j\left(x_j^*\right)+\frac{C}{p^*}\int_{B_{Y^*}}\left|\langle y,y^*\rangle\right|^{p^*}d\lambda\left(y^*\right).$$

Fix $\epsilon>0$. Replacing x_j by $\epsilon^{-1/m}x_j$, y by ϵy and taking the infimum over all $\epsilon>0$, we find

$$\begin{aligned}
\left|\langle y,T\left(x_1,...,x_m\right)\rangle\right|&\le C\left[\frac{1}{p}\left(\left(\prod_{j=1}^{m}\int_{B_{X_j^*}}\left|\langle x_j,x_j^*\rangle\right|^p d\mu_j\left(x_j^*\right)\right)^{1/p}/\epsilon\right)^p\right.\\
&\left.+\frac{1}{p^*}\left(\epsilon\left(\int_{B_{Y^*}}\left|\langle y,y^*\rangle\right|^{p^*}d\lambda\left(y^*\right)\right)^{1/p^*}\right)^{p^*}\right]\\
&\le C\prod_{j=1}^{m}\left(\int_{B_{X_j^*}}\left|\langle x_j,x_j^*\rangle\right|^p d\mu_j\left(x_j^*\right)\right)^{1/p}\left(\int_{B_{Y^*}}\left|\langle y,y^*\rangle\right|^{p^*}d\lambda\left(y^*\right)\right)^{1/p^*}.
\end{aligned}$$

Now, we prove that (iii) implies (i). Let $(x_{1i},...,x_{mi})\in X_1\times\cdots\times X_m$ and $y_i\in Y$. By inequality (2.3), we have for all $1\le i\le n$, and

$$\left|\sum_{i=1}^{n}\langle y_i,T\left(x_{1i},...,x_{mi}\right)\rangle\right|\le\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_{ji}\right\|_{L_p\left(B_{X_j^*},\mu_j\right)}\left\|y_i\right\|_{L_{p^*}(B_{Y^*},\lambda)}.$$

Using Holder's inequality, we get

$$\left|\sum_{i=1}^{n}\langle y_i, T(x_{1i},...,x_{mi})\rangle\right|$$
$$\leq C\left(\sum_{i=1}^{n}\prod_{j=1}^{m}\|x_{ji}\|_{L_p\left(B_{X_j^*},\mu_j\right)}\right)^{1/p}\left(\sum_{i=1}^{n}\|y_i^*\|_{L_{p^*}(B_{Y^*},\lambda)^{p^*}}\right)^{1/p^*}$$
$$= C\left(\sum_{i=1}^{n}\int_{B_{X_1^*}\times\cdots\times B_{X_m^*}}|\langle x_{1i},x_1^*\rangle\cdots\langle x_{mi},x_m^*\rangle|^p\, d(\mu_1\otimes\cdots\otimes\mu_m)(x_1^*,...,x_m^*)\right)^{1/p}\times$$
$$\left(\sum_{i=1}^{n}\int_{B_{Y^*}}|\langle y_i,y^*\rangle|^{p^*}\, d\lambda(y^*)\right)^{1/p^*}$$
$$\leq C\left(\sup_{\substack{x_j^*\in B_{X_j^*}\\ 1\leq j\leq m}}\sum_{i=1}^{n}|\langle x_{1i},x_1^*\rangle\cdots\langle x_{mi},x_m^*\rangle|^p\right)^{1/p}\sup_{y^*\in B_{Y^*}}\left(\sum_{i=1}^{n}|\langle y_i,y^*\rangle|^{p^*}\right)^{1/p^*}.$$

Thus, T is quasi Cohen p-nuclear and $\|T\|_{\mathcal{N}_{q(p)}}\leq C$. ■

In [13], G. Botelho and X. Mujica introduced the notation $\mathcal{L}_{q\tau(p)}(X_1,...,X_m;Y^*)$ to represent the space of quasi $\tau(p)$-summing multilinear operators. We are able to use the Pietsch Domination Theorem for quasi $\tau(p)$-summing multilinear operators, to establish a comparison between the classes of quasi $\tau(p)$-summing multilinear operators and quasi Cohen p-nuclear multilinear operators.

Corollary 2.2.5 *Let $X_1,...,X_m$ be Banach spaces . For every Banach space Y, we have*

$$\begin{cases}\mathcal{L}_{q\tau(p)}(X_1,...,X_m;Y^*)\subset\mathcal{N}_{q(p)}(X_1,...,X_m;Y^*), & 1\leq p\leq 2\\ \mathcal{N}_{q(p)}(X_1,...,X_m;Y^*)\subset\mathcal{L}_{q\tau(p)}(X_1,...,X_m;Y^*), & 2\leq p\end{cases}$$

The following result shows the topological dual of space of weakly p-nuclear multilinear operators can be characterized as the space of quasi Cohen p-nuclear multilinear operators, up to an isometric isomorphism.

Theorem 2.2.6 *If $X_1^*,...,X_m^*$ have the bounded approximation property, then, for every Banach space Y and $1\leq p<\infty$, the space $[\mathcal{N}_{wp}(X_1,...,X_m;Y)]^*$ is isometrically isomorphic to $\mathcal{N}_{q(p)}(X_1^*,...,X_m^*;Y^*)$.*

Proof. Given $\varphi\in[\mathcal{N}_{wp}(X_1,...,X_m;Y)]^*$,

$$\varphi\colon\mathcal{N}_{wp}(X_1,...,X_m;Y)\to\mathbb{K}$$
$$a=\sum_{i=1}^{\infty}x_{1i}^*\otimes\cdots\otimes x_{mi}^*\otimes y_i\mapsto\varphi(a)=\varphi\left(\sum_{i=1}^{\infty}x_{1i}^*\otimes\cdots\otimes x_{mi}^*\otimes y_i\right)$$

we define

$$S_\varphi : X_1^* \times \cdots \times X_m^* \longrightarrow Y^*$$
$$(x_1^*, ..., x_m^*) \mapsto S_\varphi(x_1^*, ..., x_m^*) : Y \longrightarrow \mathbb{K}$$
$$y \mapsto S_\varphi(x_1^*, ..., x_m^*)(y) := \varphi(x_1^* \otimes \cdots \otimes x_m^* \otimes y)$$

In order to prove that $S_\varphi \in \mathcal{N}_{q(p)}(X_1^*, ..., X_m^*; Y^*)$, let $n \in \mathbb{N}$, $x_{j1}^*, \ldots, x_{jn}^* \in X_j^*$, $(1 \leq j \leq m)$, $y_1, \ldots, y_n \in Y$. So,

$$\begin{aligned}\left|\sum_{i=1}^{n} S_\varphi(x_{1i}^* \otimes \cdots \otimes x_{mi}^*)(y_i)\right| &= \left|\sum_{i=1}^{n} \varphi(x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i)\right| \\ &= \left|\varphi\left(\sum_{i=1}^{n} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right)\right| \\ &\leq \|\varphi\| \cdot \left\|\sum_{i=1}^{n} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right\|_{\mathcal{N}_{wp}} \\ &\leq \|\varphi\| \sup_{\substack{x_j \in B_{X_j} \\ 1 \leq j \leq m}} \left(\sum_{i=1}^{n} \prod_{j=1}^{m} |\langle x_j, x_{ji}^* \rangle|^p\right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{\infty} |\langle y_i, y^* \rangle|^{p^*}\right)^{1/p^*}\end{aligned}$$

proving that S_φ is quasi Cohen p-nuclear and $\|S_\varphi\|_{\mathcal{N}_{q(p)}} \leq \|\varphi\|$.

Conversely, given $S \in \mathcal{N}_{q(p)}(X_1^*, ..., X_m^*; Y^*)$,

$$S : X_1^* \times \cdots \times X_m^* \longrightarrow Y^*$$
$$(x_1^*, ..., x_m^*) \mapsto S(x_1^*, ..., x_m^*) : Y \longrightarrow \mathbb{K}$$
$$y \mapsto S(x_1^*, ..., x_m^*)(y)$$

We define

$$T_S : X_1^* \times \cdots \times X_m^* \times Y \longrightarrow \mathbb{K},\ T_S(x_1^*, ..., x_m^*, y) := S(x_1^*, ..., x_m^*)(y).$$

It is plain that T_S is $(m+1)$-linear, so, having in mind that $X_1^* \otimes \cdots \otimes X_m^* \otimes Y = \mathcal{L}_f(X_1, ..., X_m; Y)$, by the universal property of the tensor product there exists a linear operator $\mathcal{T}_S : \mathcal{L}_f(X_1, ..., X_m; Y) \longrightarrow \mathbb{K}$ such that

$$\mathcal{T}_S(x_1^* \otimes \cdots \otimes x_m^* \otimes y) = T_S(x_1^*, ..., x_m^*, y) = S(x_1^*, ..., x_m^*)(y),$$

for all $x_j^* \in X_j^* (1 \leq j \leq m)$, $y \in Y$. Now we shall prove that $\mathcal{T}_S$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{N}_{wp}}$. Given $\epsilon > 0$ and $T \in \mathcal{L}_f(X_1, ..., X_m; Y)$, by definition of the norm $\omega_p(.)$ we can choose a representation $T = \sum\limits_{i=1}^{n} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i$ such that

$$\sup_{\substack{x_j \in B_{X_j} \\ 1 \leq j \leq m}} \left(\sum_{i=1}^{n} \prod_{j=1}^{m} |\langle x_j, x_{ji}^* \rangle|^p\right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p^*}\right)^{1/p^*} \leq (1+\epsilon)\omega_p(T).$$

Therefore,

$$\begin{aligned}
|\mathcal{T}_{\mathcal{S}}(T)| &= \left| \mathcal{T}_{\mathcal{S}} \left(\sum_{i=1}^{n} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i \right) \right| \\
&= \left| \sum_{i=1}^{n} S\left(x_{1i}^*, ..., x_{mi}^*\right)(y_i) \right| \\
&\leq \|S\|_{\mathcal{N}_{q(p)}} \cdot \sup_{\substack{x_j \in B_{X_j} \\ 1 \leq j \leq m}} \left(\sum_{i=1}^{n} \prod_{j=1}^{m} |\langle x_j, x_{ji}^* \rangle|^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \\
&\leq \|S\|_{\mathcal{N}_{q(p)}} (1+\epsilon) \omega_p(T).
\end{aligned}$$

As this holds for arbitrary $\epsilon > 0$ and the spaces $X_1^*, ..., X_m^*$ have the bounded approximation property, by invoking Proposition 2.1.6 we conclude that

$$|\mathcal{T}_{\mathcal{S}}(T)| \leq \|S\|_{\mathcal{N}_{q(p)}} \cdot \omega_p(T) = \|S\|_{\mathcal{N}_{q(p)}} \cdot \|T\|_{\mathcal{N}_{wp}}.$$

So, $\mathcal{T}_{\mathcal{S}} \in \left[\mathcal{L}_f\left(X_1, ..., X_m; Y\right), \|\cdot\|_{\mathcal{N}_{wp}}\right]^*$ and $\|\mathcal{T}_{\mathcal{S}}\| \leq \|S\|_{\mathcal{N}_{q(p)}}$. As $\mathcal{L}_f\left(X_1, ..., X_m; Y\right)$ is $\|\cdot\|_{\mathcal{N}_{wp}}$-dense in $\mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right)$, there is a unique norm-preserving continuous linear extension φ_S of $\mathcal{T}_S$ to the whole of $\mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right)$. In particular, $\|\varphi_S\| \leq \|S\|_{\mathcal{N}_{q(p)}}$ and for $T = \sum\limits_{i=1}^{\infty} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i \in \mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right)$,

$$\begin{aligned}
\varphi_S(T) = \varphi_S\left(\sum_{i=1}^{\infty} x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i \right) &= \sum_{i=1}^{\infty} \varphi_S\left(x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right) \\
= \sum_{i=1}^{\infty} \mathcal{T}_S\left(x_{1i}^* \otimes \cdots \otimes x_{mi}^* \otimes y_i\right) &= \sum_{i=1}^{\infty} S\left(x_{1i}^*, ..., x_{mi}^*\right)(y_i).
\end{aligned}$$

From the expression above it follows easily that the correspondences $\varphi \mapsto S_\varphi$ and $S \mapsto \varphi_S$ are each other's inverse in the sense that $\varphi_{S_\varphi} = \varphi$ and $S_{\varphi_S} = S$ for $\varphi \in \left[\mathcal{N}_{wp}\left(X_1, ..., X_m; Y\right)\right]^*$ and $S \in \mathcal{N}_{q(p)}(X_1^*, ..., X_m^*; Y^*)$. The equality $\|S_\varphi\|_{\mathcal{N}_{q(p)}} = \|\varphi\|$ completes the proof. ■

The forthcoming corollary is just a combination of the theorem above with Definition 1.2.20. In the previous theorem if we take $Y = \mathbb{K}$. Then

Corollary 2.2.7 *If $X_1^*, ..., X_m^*$ have the bounded approximation property, then*

$$\left[\mathcal{N}_{wp}\left(X_1, ..., X_m\right)\right]^* = \mathcal{L}_{si,p}\left(X_1^*, ..., X_m^*\right) \textit{ isometrically isomorphic.}$$

Chapter 3

Weakly p-nuclear polynomials

Introduction

Much research has been done on the notion of m-homogeneous polynomials generalizations of well-established operator ideals, (see, for instance, [2, 5, 18, 32]). Many previous studies have followed a similar approach, focusing on the same properties found in the space of multilinear operators. Thus, this leads us to think in this direction, following the natural approach, in this chapter, we propose to construct the class of the weakly p-nuclear m-homogeneous polynomials, we prove that the class of weakly p-nuclear m-homogeneous polynomials is a Banach ideal of polynomials.

3.1 Homogeneous polynomials

We begin by studying, the relationship between symmetric m-linear forms and m-homogeneous polynomials.

Definition 3.1.1 (Symmetric multilinear operator)
Let Σ_m the Banach space set of all permutations over $\{1, ..., m\}$. We say $T \in \mathcal{L}(^m X; Y)$ is a symmetric m-linear operator if

$$T \circ \sigma(x_1, ..., x_m) := T(x_{\sigma(1)}, ..., x_{\sigma(m)}) = T(x_1, ..., x_m)$$

for all $\sigma \in \Sigma_m$ and $x_1, ..., x_m \in X$.

Definition 3.1.2 *An operator $P : X \to Y$ is an m-homogeneous polynomial if there exists a unique symmetric m-linear operator $\widehat{P} : X \times \cdots \times X \to Y$ such that*

$$P(x) = \widehat{P}(x, \overset{(m)}{\ldots}, x), \quad (x \in X).$$

We denote by $\mathcal{P}(^m X;Y)$ the Banach space of all continuous m-homogeneous polynomials from X into Y endowed with the norm

$$\begin{aligned} \|P\| &= \sup\{\|P(x)\| : \|x\| \leq 1\} \\ &= \inf\{C : \|P(x)\| \leq C\|x\|^m, x \in X\}. \end{aligned}$$

The next proposition shows that there is a relationship between the norm of an m-homogeneous polynomial and the symmetric m-linear operator associated.

Proposition 3.1.3 [44, Theorem 2.2] *For each $\widehat{P} \in \mathcal{L}(^m X;Y)$, let $P \in \mathcal{P}(^m X;Y)$ defined by $P(x) = \widehat{P}(x, \overset{(m)}{\ldots}, x)$ for every $x \in X$. Then*

(a) *The operator $\widehat{P} \leftrightarrow P$ induces an isomorphism between $\mathcal{L}(^m X;Y)$ and $\mathcal{P}(^m X;Y)$.*

(b) *We have the inequalities*

$$\|P\| \leq \|\widehat{P}\| \leq \frac{m^m}{m!}\|P\|$$

for every $\widehat{P} \in \mathcal{L}(^m X;Y)$.

According to [44, Theorem 1.10], we have the polarization formula

$$\widehat{P}(x_1, \ldots, x_m) = \frac{1}{m!2^m} \sum_{\substack{\epsilon_j = \pm 1 \\ 1 \leq j \leq m}} \epsilon_1 \ldots \epsilon_m P\left(\sum_{j=1}^{m} \epsilon_j x_j\right). \tag{3.1}$$

Example 3.1.4 Let X and Y be Banach spaces, $\varphi \in X^*$, $u \in \mathcal{L}(X;Y)$ and $m \in \mathbb{N}$. Consider the operator

$$\begin{aligned} P\colon X &\to Y \\ x &\mapsto P(x) = (\varphi(x))^{m-1}u(x) \end{aligned}$$

Obviously P is well defined. Let us verify that $P \in \mathcal{P}(^m X;Y)$, simply take the operator $A : X \times \overset{(m)}{\cdots} \times X \longrightarrow Y$ given by

$$A(x_1, \ldots, x_m) = \varphi(x_1) \ldots \varphi(x_{m-1})\, u(x_m),$$

for any $x_1, \ldots, x_m \in X$.

Example 3.1.5 Let X and Y be Banch spaces, let $a \in X^*$ and $y \in Y$. The operator $P : X \to Y$ defined by

$$P(x) = a^m(x)y$$

is clearly polynomial. A finite linear combination of polynomials of this type is called a polynomials of finite type. The vector space of all polynomials of finite type is denoted by $\mathcal{P}_f(^m X;Y)$.

The adjoint of a continuous homogeneous polynomial in the literature by Aron and Schottenloher in [9], as follows.

Definition 3.1.6 *Let $P \in \mathcal{P}(^m X; Y)$, the adjoint of P is the linear operator $P^* : Y^* \to \mathcal{P}(^m X)$ given by*

$$P^*(y^*)(x) := y^*(P(x)),$$

for $y^* \in Y^*$ and $x \in X$.

It is well known that, $(u \circ P)^* = P^* \circ u^*$ for all $u \in \mathcal{L}(Y; Z)$, where Z is a Banach space.

3.1.1 Symmetric tensor product

R. Ryan, in [51], introduced the projective tensor norm on the symmetric tensor product of Banach spaces to study homogeneous polynomials.

We use the notation $\otimes^m X := X \otimes \overset{(m)}{\cdots} \otimes X$ for the m-fold tensor product of X. By $\otimes_s^m X := X \otimes_s \overset{(m)}{\cdots} \otimes_s X$ we denote the m-fold symmetric tensor product of X, which is the set of all elements $u \in \otimes^m X$ of the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i,$$

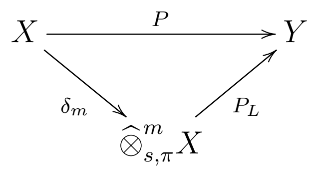
where $(\lambda_i)_{i=1}^n \subset \mathbb{K}$ and $(x_i)_{i=1}^n \subset X$.

Let $\widehat{\otimes}_{s,\pi}^m X$ denote the completed m-fold symmetric projective tensor product of X with the symmetric projective tensor norm

$$\|u\|_{s,\pi} = \inf\left\{\sum_{i=1}^{n} |\lambda_i| \|x_i\|^m,\ n \in \mathbb{N}^*,\ u = \sum_{i=1}^{n} \lambda_i x_i \otimes \cdots \otimes x_i\right\}.$$

It's worth noting the Universal Property of tensor products as follows.

Proposition 3.1.7 *Let X, Y be Banach spaces. For every m-homogeneous polynomial $P \in \mathcal{P}(^m X; Y)$ there is a unique linear operator $P_L \in \mathcal{L}(\widehat{\otimes}_{s,\pi}^m X; Y)$, such that the following diagram commutes:*



where $\delta_m : X \to \widehat{\otimes}_{s,\pi}^m X$ the canonical polynomial defined by

$$\delta_m(x) = x \otimes \overset{(m)}{\cdots} \otimes x.$$

The operator P_L is called the linearization of P.

3.1.2 Polynomial ideals

The theory of ideals of homogeneous polynomials started in the 1980's, mainly with the works of A. Pietsch [47].

Definition 3.1.8 *A polynomials ideal $\mathcal{Q}$ is a subclass of the class $\mathcal{P}$ of all continuous homogeneous polynomials between Banach spaces such that for all $m \in \mathbb{N}$ and Banach spaces X and Y its components $\mathcal{Q}(^m X; Y) := \mathcal{P}(^m X; Y) \cap \mathcal{Q}$ satisfy the following conditions:*

(i) *$\mathcal{Q}(^m X; Y)$ is a linear subspace of $\mathcal{P}(X; Y)$ which contains the m-homogeneous polynomials of finite type.*

(ii) *The ideal property: If $P \in \mathcal{Q}(^m X; Y)$, $u \in \mathcal{L}(G, X)$ and $v \in \mathcal{L}(Y, F)$ then $v \circ P \circ u$ is in $\mathcal{Q}(^m G, F)$. If $\|.\|_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{R}^+$ satisfies*

  (i') *$[\mathcal{Q}(^m X; Y), \|.\|_{\mathcal{Q}}]$ is a normed space for all Banach spaces X and Y and all m,*

  (ii') *$\|I_m : \mathbb{K} \to \mathbb{K} : I_m(x) = x^m\|_{\mathcal{Q}} = 1$ for all m,*

  (iii') *if $P \in \mathcal{Q}(^m X; Y)$, $u \in \mathcal{L}(G, X)$ and $v \in \mathcal{L}(Y, F)$ then*

$$\|v \circ P \circ u\|_{\mathcal{Q}} \leq \|v\| \|P\|_{\mathcal{Q}} \|u\|^m,$$

then $[\mathcal{Q}, \|.\|_{\mathcal{Q}}]$ is called ideal of m-homogeneous polynomials.

The case $m = 1$ recovers the classical theory of Banach operator ideals.

Some examples

Some examples are fundamental and will show up in all of our coming discussions.

1) Cohen strongly p-summing polynomials

The ideal of Cohen strongly p-summing polynomial was introduced by D. Achour and K. Saadi [5].

Definition 3.1.9 *Suppose that $1 < p < \infty$, and that $P : X \to Y$ is an m-homogeneous polynomial between Banach spaces. We say that P is Cohen strongly p-summing, if there is a constant $C > 0$ such that*

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^{n} \|x_i\|^{mp} \right)^{1/p} \sup_{y^{**} \in B_{Y^{**}}} \left(\sum_{i=1}^{n} |\langle y_i^*, y^{**} \rangle|^{p^*} \right)^{1/p^*} \tag{3.2}$$

for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$.

The least constant C for which the inequality (3.2) holds is denoted by $d_p^m(T)$. We use $\mathcal{P}_{coh-p}(^m X; Y)$ to denote the Banach space set of all Cohen strongly p-summing polynomials from X into Y.

2) Cohen p-nuclear polynomials

The class of Cohen p-nuclear operators $1 \leq p \leq \infty$ was initiated by J. S. Cohen in [20] and generalized to Cohen p-nuclear polynomials $1 \leq p \leq \infty$ by D. Achour et al in [2].

Definition 3.1.10 *We say that a polynomial $P : X \to Y$ between Banach spaces is Cohen p-nuclear, $1 \leq p \leq \infty$, if there is a constant $C > 0$ such that*

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^{mp} \right)^{1/p} \sup_{y^{**} \in B_{Y^{**}}} \left(\sum_{i=1}^{n} |\langle y_i^*, y^{**} \rangle|^{p^*} \right)^{1/p^*}, \tag{3.3}$$

for every $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset Y^*$.

The least constant C for which the inequality (3.3) holds is denoted by $\|T\|_{p,N}$. We use $\mathcal{P}^c_{p,N}(^m X; Y)$ to denote the Banach space set of all Cohen p-nuclear polynomials from X into Y.

3.2 Weakly p-nuclear polynomials

In this section, we present details of the findings from our article [35]. We start by introducing the concept of weakly p-nuclear polynomials, which extends the concept of weakly p-nuclear operators introduced in [37].

Definition 3.2.1 *Given $1 \leq p < \infty$. A polynomial $P \in \mathcal{P}(^m X; Y)$ is weakly p-nuclear if it can be written in the form*

$$P(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle^m y_n, \qquad (x \in X)$$

where $(a_n) \subset X^*$ and $(y_n) \in \ell_{p^*,w}(Y)$ such that

$$\sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^{mp} \right)^{1/p} < \infty.$$

We use $\mathcal{P}_{\mathcal{N}_{wp}}(^m X; Y)$ to denote the Banach space set of all weakly p-nuclear polynomials from X into Y and define a norm on $\mathcal{P}_{\mathcal{N}_{wp}}(^m X; Y)$ by

$$\|P\|_{\mathcal{N}_{wp}} = \inf \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle y_n, y^* \rangle|^{p^*} \right)^{1/p^*},$$

where the infimum is taken over all such representations of P as above.

Theorem 3.2.2 *For $1 \leq p < \infty$, $[\mathcal{P}_{\mathcal{N}_{wp}}, \|.\|_{\mathcal{N}_{wp}}]$ is a Banach polynomial ideal.*

Proof.

Let $P_1, P_2, \ldots \in \mathcal{P}_{\mathcal{N}_{wp}}(^mX;Y)$ such that $\sum\limits_{k=1}^{\infty} \|P_k\|_{\mathcal{N}_{wp}} < \infty$, and consider such representations that for each k, $P_k = \sum\limits_{n=1}^{\infty} a_{k,n}^m \otimes y_{k,n}$ such that

$$\sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle y_{k,n}, y^* \rangle|^{p^*} \right)^{1/p^*} \leq \left[(1+\epsilon) \|P_k\|_{\mathcal{N}_{wp}} \right]^{1/p^*}$$

$$\sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_{k,n} \rangle|^{mp} \right)^{1/p} \leq \left[(1+\epsilon) \|P_k\|_{\mathcal{N}_{wp}} \right]^{1/p}.$$

It follows that

$$\begin{aligned}
\sup_{y^* \in B_{Y^*}} \left\| (\langle y_{k,n}, y^* \rangle)_{n,k=1}^{\infty} \right\|_{\ell_{p^*,w}} &= \sup_{y^* \in B_{Y^*}} \left(\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle y_{k,n}, y^* \rangle|^{p^*} \right)^{1/p^*} \\
&= \sup_{y^* \in B_{Y^*}} \left[\sum_{k=1}^{\infty} \left(\left[\sum_{n=1}^{\infty} |\langle y_{k,n}, y^* \rangle|^{p^*} \right]^{1/p^*} \right)^{p^*} \right]^{1/p^*} \\
&\leq \left[\sum_{k=1}^{\infty} \left(\left[(1+\epsilon) \|P_k\|_{\mathcal{N}_{wp}} \right]^{1/p^*} \right)^{p^*} \right]^{1/p^*} \\
&= (1+\epsilon)^{1/p^*} \left[\sum_{k=1}^{\infty} \|P_k\|_{\mathcal{N}_{wp}} \right]^{1/p^*}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{x \in B_X} \left(\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle x, a_{k,n} \rangle|^{mp} \right)^{1/p} &= \left[\sup_{x \in B_X} \sum_{k=1}^{\infty} \left(\left[\sum_{n=1}^{\infty} |\langle x, a_{k,n} \rangle|^{mp} \right]^{1/p} \right)^{p} \right]^{1/p} \\
&\leq \left[\sum_{k=1}^{\infty} \left(\sup_{x \in B_X} \left[\sum_{n=1}^{\infty} |\langle x, a_{k,n} \rangle|^{mp} \right]^{1/p} \right)^{p} \right]^{1/p} \\
&\leq \left[\sum_{k=1}^{\infty} \left(\left[(1+\epsilon) \|P_k\|_{\mathcal{N}_{wp}} \right]^{1/p} \right)^{p} \right]^{1/p} \\
&= (1+\epsilon)^{1/p} \left[\sum_{k=1}^{\infty} \|P_k\|_{\mathcal{N}_{wp}} \right]^{1/p}.
\end{aligned}$$

Then, $P = \sum\limits_{k=1}^{\infty}\sum\limits_{n=1}^{\infty} a_{k,n}^m \otimes y_{k,n}$ and

$$\begin{aligned}
&\sup_{x\in B_X}\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|\langle x, a_{k,n}\rangle|^{mp}\right)^{1/p}\sup_{y^*\in B_{Y^*}}\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|\langle y_{k,n}, y^*\rangle|^{p^*}\right)^{1/p^*}\\
&\qquad\le (1+\epsilon)^{1/p^*}\left[\sum_{k=1}^{\infty}\|P_k\|_{\mathcal{N}_{wp}}\right]^{1/p^*}(1+\epsilon)^{1/p}\left[\sum_{k=1}^{\infty}\|P_k\|_{\mathcal{N}_{wp}}\right]^{1/p}\\
&\qquad= (1+\epsilon)\sum_{k=1}^{\infty}\|P_k\|_{\mathcal{N}_{wp}}.
\end{aligned}$$

For every $\epsilon > 0$, follows that

$$\|P\|_{\mathcal{N}_{wp}} \le \sum_{k=1}^{\infty}\|P_k\|_{\mathcal{N}_{wp}} < \infty$$

Let $Q : X_1 \longrightarrow X$, $O : Y \longrightarrow Y_1$ be a bounded linear operators and a polynomial $P \in \mathcal{P}_{\mathcal{N}_{wp}}({}^mX;Y)$. We want to show that: $OPQ \in \mathcal{P}_{\mathcal{N}_{wp}}({}^mX_1;Y_1)$, we have

$$\begin{aligned}
OPQ(x) &= O\left(\sum_{n=1}^{\infty} a_n(Q(x))^m y_n\right)\\
&= \sum_{n=1}^{\infty} a_n(Q(x))^m \cdot O(y_n)\\
&= \sum_{n=1}^{\infty} (Q^*a_n)(x)^m \cdot O(y_n).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{x\in B_{X_1}}\left(\sum_{n=1}^{\infty}|\langle x, Q^*a_n\rangle|^{mp}\right)^{1/p}\sup_{y^*\in B_{Y_1^*}}\left(\sum_{n=1}^{\infty}|\langle O(y_n), y^*\rangle|^{p^*}\right)^{1/p^*}\\
&\quad= \sup_{x\in B_{X_1}}\left(\sum_{n=1}^{\infty}|\langle Qx, a_n\rangle|^{mp}\right)^{1/p}\sup_{y^*\in B_{Y_1^*}}\left(\sum_{n=1}^{\infty}|\langle y_n, O^*y^*\rangle|^{p^*}\right)^{1/p^*}\\
&\quad= \|Q\|^m\|O^*\|\sup_{x\in B_{X_1}}\left(\sum_{n=1}^{\infty}\left|\left\langle \frac{Qx}{\|Q\|}, a_n\right\rangle\right|^{mp}\right)^{1/p}\sup_{y^*\in B_{Y_1^*}}\left(\sum_{n=1}^{\infty}\left|\left\langle y_n, \frac{O^*y^*}{\|O^*\|}\right\rangle\right|^{p^*}\right)^{1/p^*}\\
&\quad= \|Q\|^m\|O\|\sup_{x\in B_{X_1}}\left(\sum_{n=1}^{\infty}\left|\left\langle \frac{Qx}{\|Q\|}, a_n\right\rangle\right|^{mp}\right)^{1/p}\sup_{y^*\in B_{Y_1^*}}\left(\sum_{n=1}^{\infty}\left|\left\langle y_n, \frac{O^*y^*}{\|O^*\|}\right\rangle\right|^{p^*}\right)^{1/p^*}\\
&\quad\le \|Q\|^m\|O\|\sup_{u\in B_X}\left(\sum_{n=1}^{\infty}|\langle u, a_n\rangle|^{mp}\right)^{1/p}\sup_{\varphi\in B_{Y^*}}\left(\sum_{n=1}^{\infty}|\langle y_n, \varphi\rangle|^{p^*}\right)^{1/p^*}.
\end{aligned}$$

Then, OPQ is weakly p-nuclear and

$$\|OPQ\|_{\mathcal{N}_{wp}} \le \|Q\|^m \cdot \|P\|_{\mathcal{N}_{wp}} \cdot \|O\|.$$

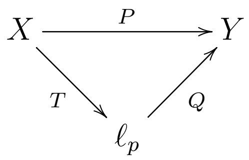
Then, $\mathcal{P}_{\mathcal{N}_{wp}}$ with the norm $\|\cdot\|_{\mathcal{N}_{wp}}$ is Banach ideal of polynomials. ■

The following theorem shows that weakly p-nuclear polynomial has factorization through ℓ_p.

Theorem 3.2.3 *Let X and Y be Banach spaces, and let $P : X \to Y$ be an m-homogeneous polynomial. Then the following are equivalent:*

(a) *P is weakly p-nuclear.*

(b) *There exists $T \in \mathcal{L}(X; \ell_p)$ and $Q \in \mathcal{P}(^m\ell_p; Y)$ such that its associated m-linear symmetric application $\widehat{Q} \in \mathcal{L}(^m\ell_p; Y)$ is diagonal. the following diagram commutes:*



In this case,

$$\|P\|_{\mathcal{N}_{wp}} = \inf \|Q\| \cdot \|T\|^m$$

where the infimum is taken over all such factorizations of P.

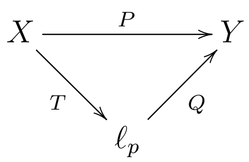
Proof. ($\Rightarrow$) Assume that P is weakly p-nuclear. Let $(a_n)_n \in \ell_{mp,w}(X^*)$ and $(y_n)_n \in \ell_{p^*,w}(Y)$ such that

$$P = \sum_{n=1}^{\infty} a_n^m \otimes y_n.$$

Consider

$$T: \quad X \to \ell_p, \quad x \mapsto (a_n(x))_n$$
$$Q: \quad \ell_p \to Y, \quad (s_n)_n \mapsto \sum_{n=1}^{\infty} s_n^m y_n.$$

Then, we see that $\|T\|^m \leq \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^{mp} \right)^{1/p}$ and $\|Q\| \leq \|(y_n)_n\|_{\ell_{p^*,w}(Y)}$, and the following diagram is commutative



Thus

$$\|P\|_{\mathcal{N}_{wp}} \geq \inf \|Q\| \|T\|^m.$$

$(\Leftarrow) P = Q \circ T$. Let $(f_n)_{n=1}^{\infty}$ be the sequence of coordinate functional a Schauder basis $\{e_n\}_{n=1}^{\infty}$ for ℓ_p, we have

$$T(x) = \sum_{n=1}^{\infty} f_n(T(x)) \cdot e_n = \sum_{n=1}^{\infty} \left(T^* f_n\right)(x) \cdot e_n,$$

and

$$\begin{aligned} P(x) &= Q(T(x)) \\ &= Q\left(\sum_{n=1}^{\infty} \left(T^* f_n\right)(x) \cdot e_n\right) \\ &= \widehat{Q}\left(\sum_{n=1}^{\infty} \left(T^* f_n\right)(x) \cdot e_n, \overset{(m)}{\ldots}, \sum_{n=1}^{\infty} \left(T^* f_n\right)(x) \cdot e_n\right) \\ &= \sum_{n=1}^{\infty} \left(T^* f_n\right)(x)^m \widehat{Q}\left(e_n, \overset{(m)}{\ldots}, e_n\right) \\ &= \sum_{n=1}^{\infty} \left(T^* f_n\right)(x)^m Q\left(e_n\right) \end{aligned}$$

We set $a_n = (T^* f_n) \in X^*$ and $y_n = Q\left(e_n\right) \in Y$, we have the representation

$$P = \sum_{n=1}^{\infty} a_n^m \otimes y_n,$$

where $(y_n)_{n=1}^{\infty} \in \ell_{p^*,w}(Y)$, and $\sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n\rangle|^{mp}\right)^{1/p} < \infty$.

Showing that P is weakly p-nuclear. In addition, we have:

$$\begin{aligned} \|P\|_{\mathcal{N}_{wp}} &\leq \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n\rangle|^{mp}\right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle y_n, y^*\rangle|^{p^*}\right)^{1/p^*} \\ &\leq \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, T^* f_n\rangle|^{mp}\right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle Q\left(e_n\right), y^*\rangle|^{p^*}\right)^{1/p^*} \\ &= \|Q\| \, \|T\|^m \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} \left|\left\langle \frac{Tx}{\|T\|}, f_n\right\rangle\right|^{mp}\right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} \left|\left\langle e_n, \frac{Q^* y^*}{\|Q^*\|}\right\rangle\right|^{p^*}\right)^{1/p^*} \\ &\leq \|Q\| \, \|T\|^m. \end{aligned}$$

Then,

$$\|P\|_{\mathcal{N}_{wp}} \leq \inf \|Q\| \|T\|^m.$$

Thus completing the proof. ■

3.2.1 Connection with polynomials of finite type

We consider a cross norm $\omega_p(.)$, ($1 \le p < \infty$), on the tensor product $\otimes^m X \otimes Y$ as follows: if $u \in \otimes^m X \otimes Y$ then

$$\omega_p(u) = \inf \left\{ \|(\lambda_i)_{i=1}^n\|_{\ell_\infty} \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^n |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \right\}$$

where the infimum is taken over all representations of u of the form

$$u = \sum_{i=1}^n \lambda_i x_i \otimes \cdots x_i \otimes y_i$$

with $(\lambda_i)_{i=1}^n \subset \mathbb{K}$, $(x_i)_{i=1}^n \subset X$ and $(y_i)_{i=1}^n \subset Y$.

Proposition 3.2.4 *ω_p is a reasonable crossnorm on $\otimes^m X \otimes Y$ and $\epsilon \le \omega_p$, where ϵ denotes the injective tensor norm on $\otimes^m X \otimes Y$.*

Lemma 3.2.5 *If the norms $\|.\|_{\mathcal{N}_{wp}}$ and $\omega_p(.)$ are equivalent on $\mathcal{P}_f\left({}^m X; Y\right)$, then they coincide on our space.*

Proof. Assume that there is a constant $c > 0$ such that $\omega_p(.) \le c\| \cdot \|_{\mathcal{N}_{wp}}$ on $\mathcal{P}_f\left({}^m X; Y\right)$. Given $P \in \mathcal{P}_f\left({}^m X; Y\right)$ and $\epsilon > 0$, take an infinite weakly p-nuclear representation

$$P = \sum_{i=1}^\infty a_i^m \otimes y_i$$

such that

$$\sup_{x \in B_X} \left(\sum_{i=1}^\infty |\langle x, a_i \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^\infty |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \le \left(1 + \frac{\epsilon}{2}\right) \|P\|_{\mathcal{N}_{wp}}.$$

In particular, for each $n \in \mathbb{N}$,

$$\begin{aligned} \omega_p\left(\sum_{i=1}^{n-1} a_i^m \otimes y_i \right) &\le \sup_{x \in B_X} \left(\sum_{i=1}^{n-1} |\langle x, a_i \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n-1} |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \\ &\le \left(1 + \frac{\epsilon}{2}\right) \|P\|_{\mathcal{N}_{wp}} \end{aligned}$$

for a sufficiently large $n \in \mathbb{N}$ we get

$$\begin{aligned} \left\| \sum_{i=n}^\infty a_i^m \otimes y_i \right\|_{\mathcal{N}_{wp}} &\le \sup_{x \in B_X} \left(\sum_{i=n}^\infty |\langle x, a_i \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=n}^\infty |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \\ &\le \sup_{x \in B_X} \left(\sum_{i=1}^\infty |\langle x, a_i \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^\infty |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \\ &\le \frac{\epsilon}{2c} \|P\|_{\mathcal{N}_{wp}}. \end{aligned}$$

It follows that

$$\begin{aligned}\omega_p(P) &\leq \omega_p\left(\sum_{i=1}^{n-1} a_i^m \otimes y_i\right) + \omega_p\left(\sum_{i=n}^{\infty} a_i^m \otimes y_i\right)\\ &\leq \left(1+\frac{\epsilon}{2}\right)\|P\|_{\mathcal{N}_{wp}} + c\left\|\sum_{i=n}^{\infty} a_i^m \otimes y_i\right\|_{\mathcal{N}_{wp}}\\ &\leq \left(1+\frac{\epsilon}{2}\right)\|P\|_{\mathcal{N}_{wp}} + \frac{\epsilon}{2}\|P\|_{\mathcal{N}_{wp}} = (1+\epsilon)\|P\|_{\mathcal{N}_{wp}}.\end{aligned}$$

And as this holds for every $\epsilon > 0$, the result follows. ∎

Proposition 3.2.6 *If $P \in \mathcal{P}_{\mathcal{N}_{wp}}({}^mX;Y)$ and $Q \in \mathcal{L}_f(D;X)$, then*

$$\omega_p(P \circ Q) \leq \|P\|_{\mathcal{N}_{wp}}\|Q\|^m.$$

Proof. Let $J : Q(D) \to X$ be the formal inclusion and $\tilde{Q} : D \to Q(D)$ be defined by $\tilde{Q} : u \mapsto \tilde{Q}(u) := Q(u)$ we can write $Q = J \circ \tilde{Q}$. Since each, $Q \circ J \in \mathcal{L}(Q(D);Y)$, where $\dim Q(D) < \infty$, by Lemma 3.2.5 we get

$$\omega_p(P \circ J) = \|P \circ J\|_{\mathcal{N}_{wp}} \leq \|P\|_{\mathcal{N}_{wp}}\|J\|^m = \|P\|_{\mathcal{N}_{wp}}$$

from which it follows that

$$\omega_p(P \circ Q) = \omega_p(P \circ J \circ \tilde{Q}) \leq \omega_p(P \circ J)\|\tilde{Q}\|^m = \|P\|_{\mathcal{N}_{wp}}\|Q\|^m.$$

Thus completing the proof. ∎

Proposition 3.2.7 *If X^* has the bounded approximation property, then $\omega_p(P) = \|P\|_{\mathcal{N}_{wp}}$ on $\mathcal{P}_f({}^mX;Y)$ regardless of the Banach space Y.*

Proof.

We give the proof for $m = 2$, as for other values of m it is similar. Let $P \in \mathcal{P}_f({}^2X;Y)$. We know that $\mathcal{L}_f(X,X;Y)$ is isometrically isomorphic to $\mathcal{L}_f(X, \mathcal{L}_f(X;Y))$, by the application that associates $\bar{S} \in \mathcal{L}_f(X; \mathcal{L}_f(X;Y)$ to $S \in \mathcal{L}_f(X,X;Y)$ by

$$\bar{S}(x_1)(x_2) := S(x_1, x_2).$$

So, $S = \widehat{P}$ is the symmetric bilinear operator associate a P, note that

$$\bar{S}(x_1)(x_2) = S(x_1,x_2) = S(x_2,x_1) = \bar{S}(x_2)(x_1).$$

Since X^* has the bounded property approximation, by Lemm 1.1.4, given $\epsilon > 0$, there is $\bar{T} \in \mathcal{L}_f(X;X)$ such that $\|\bar{T}\| \leq (1+\epsilon)\lambda$, let $\lambda \geq 1$ and $\overline{ST} = \bar{S}$. So, we have

$$S(\bar{T}x_1, x_2) = \bar{S}(\bar{T}x_1)(x_2) = \overline{ST}(x_1)(x_2) = \bar{S}(x_1)(x_2) = S(x_1,x_2)$$

And by the symmetry of S,

$$S\left(x_1, \bar{T}x_2\right) = S\left(\bar{T}x_2, x_1\right) = S\left(x_2, x_1\right) = S\left(x_1, x_2\right).$$

So, $S \circ (\bar{T}, \bar{T}) = S$, for all $x \in X$ we have

$$P(x) = S(x, x) = S \circ (\bar{T}x, \bar{T}x) = P(\bar{T}x) = P \circ \bar{T}(x).$$

for all $x \in X$, proving that $S = S \circ (\bar{T}, \bar{T})$. Calling on Proposition 3.2.6 we have

$$\begin{aligned} \omega_p(P) &= \omega_p(P \circ \bar{T}) \\ &\leq \|P\|_{\mathcal{N}_{wp}} \|\bar{T}\|^2 \\ &\leq (1+\epsilon)^2 \lambda^2 \|P\|_{\mathcal{N}_{wp}} \end{aligned}$$

For each $\epsilon > 0$, follows that $\omega_p(P) \leq \lambda^2 \|P\|_{\mathcal{N}_{wp}}$. The result follows from Lemma 3.2.5. ■

Proposition 3.2.8 *If X^* has the bounded approximation property, then $\mathcal{P}_f\left({}^mX; Y\right)$ dense in $\mathcal{P}_{\mathcal{N}_{wp}}\left({}^mX; Y\right)$ by the norm $\|\cdot\|_{\mathcal{N}_{wp}}$.*

Proof.

Let $P \in \mathcal{P}_{\mathcal{N}_{wp}}\left({}^mX; Y\right)$ and given $\epsilon > 0$ consider a representation of P such that

$$\sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{n=1}^{\infty} |\langle y_n, y^* \rangle|^{p^*} \right)^{1/p^*} \leq (1+\epsilon)\|P\|_{\mathcal{N}_{wp}}$$

Consider $P_n = \sum_{i=1}^{n} a_i^m \otimes y_i \in \mathcal{P}_f\left({}^mX; Y\right)$, then, P_n converges to P in the norm $\|\cdot\|_{\mathcal{N}_{wp}}$. ■

3.2.2 Dual of $\mathcal{P}_{\mathcal{N}_{wp}}({}^mX; Y)$

The representation of bounded linear functionals on spaces of polynomials is a classical and very useful topic in Functional Analysis and Operator Theory. In this section, we focus on finding a class of operators that can represent bounded linear functionals on the space of weakly p-nuclear polynomials. For this purpose, the following definition is well-suited.

Definition 3.2.9 *We say that an m-homogeneous polynomial $P : X \to Y^*$ is quasi Cohen p-nuclear, $1 \leq p < \infty$, if there is a constant $C > 0$ such that for any $(x_i)_{i=1}^n \subset X$ and any $(y_i)_{i=1}^n \subset Y$, we have*

$$\sum_{i=1}^{n} |\langle y_i, P(x_i) \rangle| \leq C \sup_{a \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, a \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*}. \tag{3.4}$$

The class of all quasi Cohen p-nuclear m-homogeneous polynomials from X into Y^* is denoted by $\mathcal{P}_{\mathcal{QN}_p}(^mX;Y^*)$. Our space is a Banach space with the norm $\|\cdot\|_{\mathcal{QN}_p}$, which is the smallest constant C such that the inequality (3.4) holds.

Remark 3.2.10 *It is straightforward to see that* $\mathcal{P}^c_{p,N}(^mX;Y^*) \subset \mathcal{P}_{\mathcal{QN}_p}(^mX;Y^*)$ *with* $\|\cdot\|_{\mathcal{QN}_p} \le \|\cdot\|_{p,N}$ *for every* Y *and that* $\mathcal{P}^c_{p,N}(^mX;Y^*) = \mathcal{P}_{\mathcal{QN}_p}(^mX;Y^*)$ *isometrically for reflexive* Y.

Theorem 3.2.11 *If* X^* *has the bounded approximation property, then, for every Banach space* Y *and* $1 \le p < \infty$, *the space* $\mathcal{P}_{\mathcal{QN}_p}(^mX^*;Y^*)$ *is isometrically isomorphic to* $\left[\mathcal{P}_{\mathcal{N}_{wp}}(^mX;Y)\right]^*$.

Proof.

Given $\varphi \in \left[\mathcal{P}_{\mathcal{N}_{wp}}(^mX;Y)\right]^*$,

$$\begin{array}{rccc} \varphi: & \mathcal{P}_{\mathcal{N}_{wp}}(^mX;Y) & \longrightarrow & \mathbb{K} \\ & a = \sum\limits_{n=1}^{\infty} a_n^m \otimes y_n & \mapsto & \varphi(a) = \varphi\left(\sum\limits_{n=1}^{\infty} a_n^m \otimes y_n\right) \end{array}$$

we define

$$\begin{aligned} P_\varphi : X^* &\longrightarrow Y^* \\ a \mapsto P_\varphi(a) &: \; Y \longrightarrow \mathbb{K} \\ & \qquad y \mapsto P_\varphi(a)(y) := \varphi(a^m \otimes y) \end{aligned}$$

In order to prove that $P_\varphi \in \mathcal{P}_{\mathcal{QN}_p}(^mX^*;Y^*)$, let $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in X^*$, $y_1, \ldots, y_n \in Y$. So,

$$\begin{aligned} \left|\sum_{i=1}^n P_\varphi(a_i)(y_i)\right| &= \left|\sum_{i=1}^n \varphi(a_i^m \otimes y_i)\right| \\ &= \left|\varphi\left(\sum_{i=1}^n a_i^m \otimes y_i\right)\right| \\ &\le \|\varphi\| \cdot \left\|\sum_{i=1}^n a_i^m \otimes y_i\right\|_{\mathcal{N}_{wp}} \\ &\le \|\varphi\| \cdot \sup_{x \in B_X}\left(\sum_{i=1}^n |\langle x, a_i\rangle|^{mp}\right)^{1/p} \sup_{y^* \in B_{Y^*}}\left(\sum_{i=1}^\infty |\langle y_i, y^*\rangle|^{p^*}\right)^{1/p^*} \end{aligned}$$

proving that P_φ is quasi Cohen p-nuclear and $\|P_\varphi\|_{\mathcal{QN}_p} \le \|\varphi\|$.

Conversely, given $P \in \mathcal{P}_{\mathcal{QN}_p}(^mX^*;Y^*)$, define

$$\begin{aligned} P : X^* &\longrightarrow Y^* \\ a \mapsto P(a) &: \; Y \longrightarrow \mathbb{K} \\ & \qquad y \mapsto P(a)(y) \end{aligned}$$

having in mind that $\otimes^m X^* \otimes Y = \mathcal{P}_f({}^m X; Y)$, by the universal property of the tensor product there exists a linear operator $\mathcal{T}_P : \mathcal{P}_f({}^m X; Y) \longrightarrow \mathbb{K}$ such that

$$\mathcal{T}_P(a^m \otimes y) = P(a)(y)$$

for all $a \in X^*$ and $y \in Y$.

Now, we shall prove that $\mathcal{T}_P$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{N}_{wp}}$. Given $\epsilon > 0$ and $A \in \mathcal{P}_f({}^m X; Y)$, by definition of the norm $\omega_p(.)$ we can choose a representation $A = \sum_{i=1}^{n} a_i^m \otimes y_i$ such that

$$\sup_{x \in B_X} \left(\sum_{i=1}^{n} |\langle x, a_i \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \leq (1+\epsilon)\omega_p(A).$$

Therefore,

$$\begin{aligned} |\mathcal{T}_P(A)| &= \left| \mathcal{T}_P \left(\sum_{i=1}^{n} a_i^m \otimes y_i \right) \right| \\ &= \left| \sum_{i=1}^{n} P(a_i)(y_i) \right| \\ &\leq \|P\|_{\mathcal{QN}_p} \cdot \sup_{x \in B_X} \left(\sum_{i=1}^{n} |\langle x, a_i \rangle|^{mp} \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left(\sum_{i=1}^{n} |\langle y_i, y^* \rangle|^{p^*} \right)^{1/p^*} \\ &\leq \|P\|_{\mathcal{QN}_p}(1+\epsilon)\omega_p(A). \end{aligned}$$

As this holds for arbitrary $\epsilon > 0$ and the spaces X^* has the bounded approximation property, by Proposition 3.2.7 we conclude that

$$|\mathcal{T}_P(A)| \leq \|S\|_{\mathcal{QN}_p} \cdot \omega_p(A) = \|P\|_{\mathcal{QN}_p} \cdot \|T\|_{\mathcal{N}_{wp}}.$$

So, $\mathcal{T}_P \in \left[\mathcal{P}_f({}^m X; Y), \|\cdot\|_{\mathcal{N}_{wp}}\right]^*$ and $\|\mathcal{T}_P\| \leq \|P\|_{\mathcal{QN}_p}$. As $\mathcal{P}_f({}^m X; Y)$ is $\|\cdot\|_{\mathcal{N}_{wp}}$-dense in $\mathcal{P}_{\mathcal{N}_{wp}}({}^m X; Y)$, there is a unique norm-preserving continuous linear extension φ_S of $\mathcal{T}_P$ to the whole of $\mathcal{P}_{\mathcal{N}_{wp}}({}^m X; Y)$. In particular, $\|\varphi_P\| \leq \|P\|_{\mathcal{QN}_p}$ and for $A = \sum_{i=1}^{\infty} a_i^m \otimes y_i \in \mathcal{P}_{\mathcal{N}_{wp}}({}^m X; Y)$,

$$\begin{aligned} \varphi_P(A) = \varphi_P \left(\sum_{i=1}^{\infty} a_i^m \otimes y_i \right) &= \sum_{i=1}^{\infty} \varphi_P(a_i^m \otimes y_i) \\ = \sum_{i=1}^{\infty} \mathcal{T}_p(a_i^m \otimes y_i) &= \sum_{i=1}^{\infty} P(a_i)(y_i). \end{aligned}$$

From the expression above it follows easily that the correspondences $\varphi \mapsto P_\varphi$ and $S \mapsto \varphi_S$ are each other's inverse in the sense that $\varphi_{P_\varphi} = \varphi$ and $P_{\varphi_S} = S$ for $\varphi \in \left[\mathcal{P}_{\mathcal{N}_{wp}}({}^m X; Y)\right]^*$ and $P \in \mathcal{P}_{\mathcal{N}_{wp}}({}^m X; Y)$. The equality $\|P_\varphi\|_{\mathcal{QN}_p} = \|\varphi\|$ completes the proof. ■

Chapter 4

Positive Cohen p-nuclear polynomials

Introduction

The seeds of positive p-summing, $(1 \leq p \leq \infty)$, linear operators come from O. Blasco in 1987, in the paper [11] titled "*Positive p-summing operators on L_p-spaces.*" This concept is an extension of the ideal of p-absolutely summing operators. Worth noting is the fact that the notion of positive p-summing operators had previously been introduced by Schaefer in the case where $p = 1$. These operators were denoted as "cone absolutely summing operators" [53]. In 1974, Schaefer's contributions were instrumental not only in pioneering the concept of cone absolutely summing operators but also in establishing the connections between these operators and majorizing operators. A linear operator T between Banach lattice E and Banach space Y is called positive p-summing, $(1 \leq p \leq \infty)$, if there exists a constant $C > 0$ such that for every $x_1, ..., x_n$ positive elements in E, we have

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p \right)^{1/p} \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle x_i, x^* \rangle^{p^*} \right)^{1/p^*} \tag{4.1}$$

The smallest constant C for which this inequality (4.1) holds is denoted by $\pi_p^+(T)$. The Banach space set of all positive p-summing operators from E into Y is denoted as $\Pi_p^+(E; Y)$.

Recently, in 2021, H. Hamdi et al. [32] generalized certain results from D. Achour and A. Belacel's work [3] to the polynomial case. A polynomial P between Banach space X and Banach lattice F is called positive Cohen strongly p-summing, $(1 \leq p \leq \infty)$, if there exists a constant $C > 0$ such that for any sequences $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset F^{*+}$, the following inequality holds:

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^{n} \|x_i\|^{mp} \right)^{1/p} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,\omega}^n(F^{*+})} . \tag{4.2}$$

The least constant C for which this inequality (4.2) holds is denoted by $d_p^{m+}(u)$. We shall write $\mathcal{P}_{coh^+-p}(^m X; F)$ for the Banach space set of all positive Cohen strongly p-summing m-

homogeneous polynomials from X into F. For $m = 1$, it is the space of positive strongly p-summing linear operators $(\mathcal{P}_{coh^+-p}(^1X; F) = \mathcal{D}_p^+(X; F))$.

The cornerstones of the theory of positive Cohen strongly p-summing polynomials are the following theorems:

- Inclusion theorem. If $1 < p < \infty$, then every Cohen strongly p-summing polynomial is positive Cohen strongly p-summing polynomial.
- Pietsch-Domination theorem.
- Factorization theorem.

Similarly, in 2021, A. Bougoutaia et al. introduced the concept of positive Cohen p-nuclear linear and multilinear operators in their work [15], and provided several results for this new class such as inclusion relations and a Pietsch domination type theorem. A multilinear operator $T : E_1 \times \cdots \times E_m \to F$ between Banach lattices is called positive Cohen p-nuclear $1 \leq p \leq \infty$ if there is a positive constant C such that for all $(x_{ij})_{i=1}^n \subset E_j^+$, $(1 \leq j \leq m)$, and any $(y_i^*)_{i=1}^n \subset F^{*+}$, we have

$$\left| \sum_{i=1}^{n} \left\langle T\left(x_i^1, \cdots, x_i^m\right), y_i^* \right\rangle \right| \leq C \sup_{\substack{x_j^* \in B_{E_j^*}^+ \\ 1 \leq j \leq m}} \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \left\langle x_{ij}, x_j^* \right\rangle^p \right)^{1/p} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,w}(F^{*+})}. \tag{4.3}$$

The least constant C for which this inequality (4.3) holds is denoted by $n_p^+(T)$. We use $\mathcal{N}_p^+(E_1 \times \cdots \times E_m; F)$ to denote the Banach space set of all positive Cohen p-nuclear operators from $E_1 \times \cdots \times E_m$ into F.

Our purpose in this chapter is to extend the class of positive Cohen p-nuclear linear operators to the case of polynomials between Banach lattices. Building upon the foundations and previous studies, we aim to show that a polynomial is positive Cohen p-nuclear if, and only if, its associated symmetric multilinear operator is Cohen positive p-nuclear, thus generalizing a linear result of Pietsch's domination theorem. We characterize the class of Cohen p-nuclear polynomial as the composition of positive p-summing operator and positive Cohen strongly p-summing polynomial. Finally, we obtain certain connections between other classes. The results in this chapter are among the most important in our research paper [33].

4.1 Banach Lattice

In this section, we provide a summary of fundamental concepts and terminology related to Banach lattices.

4.1.1 Ordered sets

Definition 4.1.1 *Let E be an arbitrary set. A partial order (or simply, an order) on E is a binary relation, denoted here by $(\leqslant)$, which is reflexive, transitive, and antisymmetric, that is*

1. $x \leqslant x \qquad (x \in E)$,
2. *if* $x \leqslant y$ *and* $y \leqslant z$*, then* $x \leqslant z \qquad (x, y, z \in E)$,
3. *if* $x \leqslant y$ *and* $y \leqslant x$*, then* $x = y \qquad (x, y \in E)$.

Definition 4.1.2 *An ordered Banach space is a Banach space E equipped with partial order which is compatible with its vector structure in the sense that*

1. $x \leqslant y$ *implies* $x + z \leqslant y + z$ *for all* $x, y, z \in E$
2. $x \leqslant y$ *implies* $\lambda x \leqslant \lambda y$ *for any* $x, y \in E$ *and* $\lambda \in \mathbb{R}^+$.

Definition 4.1.3 *An element x in an ordered Banach space E is called positive whenever $0 \leqslant x$ holds.*

The set of all positive elements of E will be denoted by E^+ (i.e. $E^+ = \{x \in E,\ 0 \leqslant x\}$) and will be referred to as the positive cone of E. We write $B^+_{E^*} = B_{E^*} \cap E^{*+} = \{x^* \in B_{E^*}, 0 \leqslant x^*\}$. Of course, here $B^+_{E^*}$ is considered as a compact subset with respect to the weak* topology.

Definition 4.1.4 *We say that an ordered space E is a lattice if for any $x, y \in E$ both $\inf\{x, y\}$ and $\sup\{x, y\}$ exist.*

For an element x in a lattice space E we can define its positive and negative part, and its absolute value, respectively, by

$$x^+ := \sup\{x, 0\},\ x^- := \sup\{-x, 0\}.$$

The functions $(x, y) \to \sup\{x, y\}$, $(x, y) \to \inf\{x, y\}$, $x \to x\pm$ and $x \to |x|$ are collectively referred to as the lattice operations of a Riesz space. The relation between them is given in the next proposition.

Proposition 4.1.5 *If x is an element of a Riesz space, then*

$$x = x^+ - x^- \text{ and } |x| = x^+ + x^-.$$

Thus, in particular, the positive cone in a lattice space is generating.

Example 4.1.6 Here are some familiar examples of normed Riesz spaces and Banach lattices.

1. The Euclidean spaces $\mathbb{R}^n$ with their Euclidean norms are all Banach lattices.

2. If K is a compact space, then the Riesz space $C(K)$ of all continuous real functions on K is a Banach lattice under the norm

$$\|f\|_\infty = \sup_{x \in K} |f(x)|.$$

3. The Riesz spaces $L_p(\mu), 1 \leq p < \infty$, (and hence the ℓ_p-spaces) are all Banach lattices when equipped with their L_p-norms

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}.$$

   Similarly, the $L_\infty(\mu)$-spaces are all Banach lattices with their essential sup norms.

4. If E is Banach lattice. The space of absolutely weakly p-summable sequences define by

$$\ell_{p,|w|}(E) := \left\{ (x_n)_{n=1}^\infty : (|x_n|)_{n=1}^\infty \in \ell_{p,w}(E) \right\},$$

   under norm

$$\begin{cases} \|(x_n)_{n=1}^\infty\|_{\ell_{p,|w|}(E)} = \sup\limits_{x^* \in B_{E^*}^+} \left(\sum\limits_{n=1}^\infty \langle |x_n|, x^* \rangle^p \right)^{1/p}, & \text{if} \quad 1 \leq p < +\infty, \\ \|(x_n)_{n=1}^\infty\|_{\ell_{p,|w|}(E)} = \sup\limits_{x^* \in B_{E^*}^+} \sup\limits_n \langle |x_n|, x^* \rangle, & \text{if} \quad p = +\infty \end{cases}$$

   is Banach lattice, [38]

Definition 4.1.7 *A Banach space is called* ***Dedekind complete*** *whenever every nonempty subset that is bounded above has a supremum.*

Definition 4.1.8 *Let $E_1, ..., E_m$ and F be Banach lattices. An m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is positive if $0 \leqslant T(x_1, ..., x_m)$ whenever $x_1, ..., x_m$ lie in the positive cones of $E_1, ..., E_m$ respectively. T is called regular if T is the difference of two positive multilinear operators.*

Let $\mathcal{L}^r(E_1, ..., E_m; F)$ denote the space of all regular m-linear operators from $E_1 \times \cdots \times E_m$ into F. If F is Dedekind complete then $\mathcal{L}^r(E_1, ..., E_m; F)$ is a Banach lattice, with the regular operator norm $\|T\|_r = \||T|\|$ for every $T \in \mathcal{L}^r(E_1, ..., E_m; F)$.

Definition 4.1.9 *Let $E_1, \ldots, E_m, E$ and F be Banach lattices. An m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is called a lattice m-morphism if $|T(x_1, \ldots, x_m)| = T(|x_1|, \ldots, |x_m|)$.*

Definition 4.1.10 *Let E and F be Banach lattices. An m-homogeneous polynomial $P \in \mathcal{P}(^mE; F)$ is positive if its the associated symmetric m-linear operator is positive. P is called regular if P is the difference of two positive polynomials.*

Let $\mathcal{P}^r(^mE; F)$ denote the space of all regular m-homogeneous polynomials from E into F. If F is Dedekind complete then $\mathcal{P}^r(^mE; F)$ is a Dedekind complete Banach lattice, see [16].

Remark 4.1.11 *It is easy to see that P is regular if, and only if, $\widehat{P}$ is regular.*

Definition 4.1.12 *We call a polymorphism to be an homogeneous polynomial $P : E \to F$ that satisfies $|P(x)| = P(|x|)$ for all $x \in E$.*

4.1.2 Tensor Products of Banach Lattices

The study of the tensor product of vector lattices, along with various structures, has garnered considerable attention. This concept has evolved through the contributions of several authors such as H. H. Schaefer, J. J. Grobler, C. C. A. Labuschagne, Q. Bu and G. Buskes [53, 29, 16] since first introduced by D. H. Fremlin as the Riesz tensor product. In [27], D. H. Fremlin established a theorem demonstrating the existence and uniqueness of the tensor product.

Now, we recall the main results on the product tensor of Banach lattices. For Banach lattices $E_1, \ldots, E_m$, D. H. Fremlin in [28], defined the positive projective tensor norm $\|\cdot\|_{|\pi|}$ on the Archimedean Banach lattice tensor product $E_1 \bar{\otimes} \cdots \bar{\otimes} E_m$ for every $u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_m$ by

$$\|u\|_{|\pi|} = \inf \left\{ \sum_{i=1}^{n} \|x_{1i}\| \cdots \|x_{mi}\| : x_{ji} \in E_j^+,\ |u| \leqslant \sum_{i=1}^{n} x_{1i} \otimes \cdots \otimes x_{mi} \right\}.$$

Let $E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$ denote the completion of $E_1 \bar{\otimes} \cdots \bar{\otimes} E_m$ under the lattice norm $\|\cdot\|_{|\pi|}$. Then $E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m$ is a Banach lattice, called the positive m-fold projective tensor product of $E_1, \ldots, E_m$.

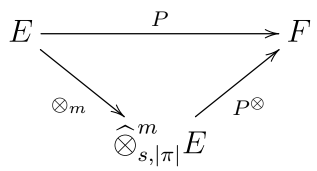
Q. Bu and G. Buskes [16] extended the linearization method adapted to the case of Banach lattices by considering regular multilinear operators and polynomials and defining their linearizations on positive projective tensor products in the sense of Fremlin [27, 28].

Proposition 4.1.13 *Let $E_1, \ldots, E_m, F$ be Banach lattices such that F is Dedekind complete. Then $\mathcal{L}^r(E_1, \ldots, E_m; F)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r\left(E_1 \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} E_m; F\right)$.*

For a Banach lattice E, the positive symmetric projective tensor norm on $\widehat{\otimes}^m_{s,|\pi|} E$ is defined by

$$\|u\|_{s,|\pi|} = \inf \left\{ \sum_{i=1}^{n} \|x_i\|^m : x_i \in E^+,\ |u| \leqslant \sum_{i=1}^{n} x_i \otimes \cdots \otimes x_i \right\}$$

Proposition 4.1.14 *Let E and F be Banach lattices such that F is Dedekind complete. Then for any regular m-homogeneous polynomial $P : E \to F$, there exists a unique regular linear operator $P^{\otimes} : \widehat{\otimes}^m_{s,|\pi|} E \to F$, called the linearization of P, such that the following diagram is commutative*



that is,

$$P(x) = P^{\otimes}(x \otimes \cdots \otimes x)$$

for all $x \in E$. Moreover, the correspondence $P \mapsto P^{\otimes}$ is isometrically isomorphic and lattice homomorphic between the Banach lattices $\mathcal{P}^r(^m E; F)$ and $\mathcal{L}^r(\widehat{\otimes}^m_{s,|\pi|} E; F)$.

4.2 Domination and factorization theorems

First of all, we present the definition of positive Cohen p-nuclear m-homogeneous polynomials.

Definition 4.2.1 *We say that an m-homogeneous polynomial $P : E \to F$ between Banach lattices is positive Cohen p-nuclear, $1 \leq p < \infty$, if there is a constant $C > 0$ such that*

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \leq C \sup_{x^* \in B^+_{E^*}} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{1/p} \sup_{y^{**} \in B^+_{F^{**}}} \left(\sum_{i=1}^{n} \langle |y_i^*|, y^{**} \rangle^{p^*} \right)^{1/p^*} \tag{4.4}$$

for every $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset F^*$.

The class of all positive Cohen p-nuclear m-homogeneous polynomials from E into F is denoted by $\mathcal{P}^{c+}_{N-p}(^m E; F)$. Our space is a Banach space with the norm $n_p^{m+}(\cdot)$, which is the smallest constant C such that the inequality (4.4) holds. For $p=\infty$, we have $\mathcal{P}^{c+}_{N-\infty}({}^mE;F)=\mathcal{P}_{coh^+-\infty}({}^mE;F)$.

Below, we provide an example of a positive Cohen p-nuclear polynomial:

Example 4.2.2 Let $1<p\leq\infty$ and let $u:E\to F$ be an positive Cohen p-nuclear operator, where $E,\ F$ are Banach lattices. For $\varphi\in E^*$ the operator

$$\begin{aligned} P:&E\longrightarrow F\\ &x\longmapsto P(x)=\varphi^{m-1}(x)u(x), \end{aligned}$$

is positive Cohen p-nuclear polynomial, moreover $n_p^{m+}(P)\leq n_p^+(u)\|\varphi\|^{m-1}$.

Indeed, for $x_1,...,x_n\in E$ and $y_1^*,...,y_n^*\in F^*$, we have :

$$\begin{aligned} \sum_{i=1}^n|\langle P(x_i),{y_i}^*\rangle| &= \sum_{i=1}^n\left|\left\langle\varphi^{m-1}(x_i)u(x_i),{y_i}^*\right\rangle\right| \\ &= \sum_{i=1}^n\left|\left\langle u(\varphi^{m-1}(x_i)(x_i)),{y_i}^*\right\rangle\right| \\ &\leq n_p^+(u)\sup_{x^*\in B^+_{E^*}}\left(\sum_{i=1}^n\langle|\varphi^{m-1}(x_i)||x_i|,x^*\rangle^{mp}\right)^{1/p}\sup_{y^{**}\in B^+_{F^{**}}}\left(\sum_{i=1}^n\langle|y_i^*|,y^{**}\rangle^{p^*}\right)^{1/p^*} \\ &\leq n_p^+(u)\sup_{x^*\in B^+_{E^*}}\left(\sum_{i=1}^n|\varphi^{m-1}(x_i)|x^*(|x_i|)^{mp}\right)^{1/p}\sup_{y^{**}\in B^+_{F^{**}}}\left(\sum_{i=1}^n\langle|y_i^*|,y^{**}\rangle^{p^*}\right)^{1/p^*} \\ &\leq n_p^+(u)\|\varphi\|^{m-1}\sup_{x^*\in B^+_{E^*}}\left(\sum_{i=1}^n\langle|x_i|,x^*\rangle^{mp}\right)^{1/p}\sup_{y^{**}\in B^+_{F^{**}}}\left(\sum_{i=1}^n\langle|y_i^*|,y^{**}\rangle^{p^*}\right)^{1/p^*}. \end{aligned}$$

Then, P is positive Cohen p-nuclear. Moreover, $n_p^{m+}(P)\leq n_p^+(u)\|\varphi\|^{m-1}$.

Proposition 4.2.3 *Let $P\in\mathcal{P}({}^mE;F)$ and let $v:\ell_p^n\to F^*$ be positive linear operator. Then the polynomial P is positive Cohen p-nuclear if*

$$\sum_{i=1}^n|\langle P(x_i),v(e_i)\rangle|\leq C\left\|(x_i)\right\|^m_{\ell^n_{p,|w|}(E)}\left\|v\right\|. \tag{4.5}$$

Proof. Let $v:\ell_p^n\to F^*$ be positive linear operator such that

$$v=\sum_{i=1}^n e_i\otimes y_i^*,$$

where e_i is the canonical base of $\ell^n_{p^*}$. Since there is an isometric between the spaces $\ell^n_{p^*,|w|}(F^*)$ and $\mathcal{L}^r(\ell_p^n,F^*)$, and $\|v\|=\|(y_i^*)\|_{\ell^n_{p^*,|w|}(F^*)}$. ∎

The following theorem investigates the relationship between positive Cohen p-nuclear m-homogeneous polynomials and m-linear positive Cohen p-nuclear operators.

Theorem 4.2.4 *Let P be an m-homogeneous polynomial between Banach lattices E and F. P is positive Cohen p-nuclear if, and only if, its associated symmetric m-linear operator $\widehat{P} \in \mathcal{L}(^mE;F)$ is positive Cohen p-nuclear, and*

$$n_p^{m+}(P) = n_p^{m+}(\widehat{P}).$$

Proof. Let P be positive Cohen p-nuclear m-homogeneous polynomial, and let $(x_i)_{i=1}^n \subset E$ such that $\|(x_i)_{i=1}^n\|_{\ell_{mp,|w|}(E)} \leq 1$ and $y_i^* \in F^*(1 \leq i \leq n)$

$$\left\|\left(\epsilon_1 x_i^1 + \cdots + \epsilon_m x_i^m\right)_{i=1}^n\right\|_{\ell_{mp,|w|}(E)} \leq \left\|\left(\epsilon_1 x_i^1\right)_{i=1}^n\right\|_{\ell_{mp,|w|}(E)} + \cdots + \|(\epsilon_m x_i^m)_{i=1}^n\|_{\ell_{mp,|w|}(E)} \leq m$$

for every $\epsilon_1, \cdots, \epsilon_m = \pm 1$

Using the polarization formula (3.1), we obtain

$$\begin{aligned}
\sum_{i=1}^n \left|\left\langle \widehat{P}\left(x_{1i}, ..., x_{mi}\right), y_i^*\right\rangle\right| &= \sum_{i=1}^n \left|\left\langle \frac{1}{m!2^m} \sum_{\epsilon_1,...,\epsilon_m=\pm 1} \epsilon_1 ... \epsilon_m P\left(\sum_{j=1}^m \epsilon_j x_{ji}\right), y_i^*\right\rangle\right| \\
&\leq \frac{1}{m!2^m} \sum_{\epsilon_1,...,\epsilon_m=\pm 1} \sum_{i=1}^n \left|\left\langle P\left(\sum_{j=1}^m \epsilon_j x_{ji}\right), y_i^*\right\rangle\right| \\
&\leq \frac{1}{m!2^m} n_p^{m+}(P) \sum_{\epsilon_1,...,\epsilon_m=\pm 1} \left\|\sum_{i=1}^n \epsilon_1 x_{ji}\right\|_{\ell_{mp,|w|}(E)}^m \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|w|}(F^*)} \\
&\leq \frac{1}{m!2^m} n_p^{m+}(P) \left(\sum_{\epsilon_1,...,\epsilon_m=\pm 1} m^m\right) \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|w|}(F^*)} \\
&\leq \frac{\left(2^{\frac{1}{p}-1} m\right)^m}{m!} n_p^{m+}(P) \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|w|}(F^*)}
\end{aligned}$$

and for $(x_{ji})_{i=1}^n \in \ell_{p,|w|}^n(E)$ with $x_{ji} \neq 0$ and $j = 1, ..., m$

$$\sum_{i=1}^n \left|\left\langle \widehat{P}\left(\frac{x_{1i}}{\|(x_{1i})_{i=1}^n\|_{\ell_{mp,|w|}^n(E)}}, ..., \frac{x_{mi}}{\|(x_{mi})_{i=1}^n\|_{\ell_{mp,|w|}^n(E)}}\right), y_i^*\right\rangle\right| \leq \frac{\left(2^{\frac{1}{p}-1} m\right)^m}{m!} n_p^{m+}(P) \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|w|}(F^*)}.$$

Then

$$\sum_{i=1}^n \left|\left\langle \widehat{P}(x_{1i}, ..., x_{mi}), y_i^*\right\rangle\right| \leq \frac{\left(2^{\frac{1}{p}-1} m\right)^m}{m!} n_p^{m+}(P) \prod_{j=1}^m \|(x_{ji})_{i=1}^n\|_{\ell_{mp,|w|}^n(E)} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|w|}(F^*)}$$

we fin that, $\widehat{P}$ is positive Cohen p-nuclear and

$$n_p^{m+}(\widehat{P}) \leq \frac{\left(2^{\frac{1}{p}-1} m\right)^m}{m!} n_p^{m+}(P) \leq \frac{m^m}{m!} n_p^{m+}(P).$$

Conversely. Suppose that $\widehat{P}$ is positive Cohen p-nuclear m-linear operator by definition we have, for all $x_1, ..., x_n \in E$ and $y_1^*, ..., y_n^* \in F^*$, then

$$\begin{aligned}\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| &= \sum_{i=1}^{n} \left| \left\langle \widehat{P}(x_i, ..., x_i), y_i^* \right\rangle \right| \\ &\leq n_p^{m+}(\widehat{P}) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^p \overset{(m)}{\cdots} \langle |x_i|, x^* \rangle^p \right)^{1/p} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|w|}(F^*)} \\ &\leq n_p^{m+}(\widehat{P}) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{1/p} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|w|}(F^*)}\end{aligned}$$

This implies that P is positive Cohen p-nuclear polynomial, and $n_p^{m+}(P) \leq n_p^{m+}(\widehat{P})$. ■

We can obtained the next result as an application of Theorem 4.2.4 and [15, Proposition 2.3].

Corollary 4.2.5 *Let $P \in \mathcal{P}(^m E; F)$, then P is positive Cohen p-nuclear m-homogeneous polynomial if, and only if, there is a constant $C > 0$, for all $(x_i)_{i=1}^n \subset E^+$ and $(y_i^*)_{i=1}^n \subset F^{*+}$, we have*

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle x_i, x^* \rangle^{mp} \right)^{1/p} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,w}^n(F^{*+})}. \tag{4.6}$$

Our next result establishes that in the case of positive Cohen p-nuclear polynomials, the ideal property also holds.

Proposition 4.2.6 *Let $P \in \mathcal{P}(^m E; F)$, R an operator in $\mathcal{L}(F; G)$ and S a positive operator in $\mathcal{L}(G; E)$.*

1. *If P is positive Cohen p-nuclear, then $R \circ P$ is positive Cohen p-nuclear from E into G and $n_p^{m+}(R \circ P) \leq n_p^{m+}(P)\|R\|$.*
2. *If P is positive Cohen p-nuclear, then $P \circ S$ is positive Cohen p-nuclear from G into F and $n_p^{m+}(P \circ S) \leq n_p^{m+}(P)\|S\|^m$.*

Proof. Let $P \in \mathcal{P}_{N-p}^{c+}(^m E; F)$. Let $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset F^{*+}$, we have

$$
\begin{aligned}
\sum_{i=1}^{n} |\langle R \circ P(x_i), y_i^* \rangle| &= \sum_{i=1}^{n} |\langle P(x_i), R^*(y_i^*) \rangle| \\
&\leq n_p^{m+}(P) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{1/p} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^{n} \langle R^*(y_i^*), y^{**} \rangle^{p^*} \right)^{1/p^*} \\
&\leq n_p^{m+}(P) \|R\| \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{1/p} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^{n} \left\langle y_i^*, \tfrac{R^{**}(y^{**})}{\|R^{**}\|} \right\rangle^{p^*} \right)^{1/p^*} \\
&\leq n_p^{m+}(P) \|R\| \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{1/p} \sup_{\varphi \in B_{G^{**}}^+} \left(\sum_{i=1}^{n} \langle y_i^*, \varphi \rangle^{p^*} \right)^{1/p^*}.
\end{aligned}
$$

So, $R \circ P \in \mathcal{P}_{N-p}^{c+}(^m E; G)$ and $n_p^{m+}(R \circ P) \leq n_p^{m+}(P) \|R\|$.

$$
\begin{aligned}
\sum_{i=1}^{n} |\langle P \circ S(x_i), y_i^* \rangle| &\leq n_p^{m+}(P) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle (|Sx_i|), x^* \rangle^{mp} \right)^{1/p} \sup_{y^{**} \in B_{F^{**}}^+} \|y_i^*(y^{**})\|_{\ell_{p^*}^n} \\
&\leq n_p^{m+}(P) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle S(|x_i|), x^* \rangle^{mp} \right)^{1/p} \sup_{y^{**} \in B_{F^{**}}^+} \|y_i^*(y^{**})\|_{\ell_{p^*}^n} \\
&\leq n_p^{m+}(P) \|S\|^m \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \left\langle |x_i|, \frac{S^*(x^*)}{\|S\|} \right\rangle^{mp} \right)^{1/p} \sup_{y^{**} \in B_{F^{**}}^+} \|y_i^*(y^{**})\|_{\ell_{p^*}^n} \\
&\leq n_p^{m+}(P) \|S\|^m \sup_{\varphi \in B_{G^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, \varphi \rangle^{mp} \right)^{1/p} \sup_{y^{**} \in B_{F^{**}}^+} \|y_i^*(y^{**})\|_{\ell_{p^*}^n}.
\end{aligned}
$$

Which means that $P \circ S$ is positive Cohen p-nuclear m-homogeneous polynomial and

$$n_p^{m+}(P \circ S) \leq n_p^{m+}(P) \|S\|^m.$$

This completes the proof. ■

4.2.1 Pietsch's Domination Theorem

Next, we present a version of Pietsch's Domination Theorem, for positive Cohen p-nuclear polynomials. For the proof of the Domination Theorem we use the full general Pietsch Domination Theorem presented by Pellegrino et al. in [48]. Let $X_1, ..., X_m, Y$ and E be (arbitrary) non-void sets, $\mathcal{H}$ be a family of operators from $X_1 \times \cdots \times X_m$ into Y, G be a Banach spaces and $K_1, ..., K_t$ be a compact Hausdorff topological spaces, $G_1, ..., G_t$ be Banach spaces and suppose that the maps

$$\begin{cases} R_j : K_j \times E_1 \times \cdots \times E_r \times G_j \to [0, \infty), \ j = 1, ..., t, \\ S : \mathcal{H} \times E_1 \times \cdots \times E_r \times G_1 \times \cdots \times G_t \to [0, \infty) \end{cases}$$

satisfy:

(1) $(R_j)_{x^{(1)},...,x^{(r)},b} : K_j \to [0,\infty)$ defined by $(R_j)_{x^{(1)},...,x^{(r)},b}(\varphi) = R(\varphi, x^{(1)}, ..., x^{(r)}, b)$ is continuous for every $x^{(l)} \in E_l$ and $b \in G_j$, with $(j,l) \in 1,...,t \times 1,...,r$

(2) The following inequalities hold:

$$\begin{cases} R_j(\varphi, x^{(1)}, ..., x^{(r)}, \eta_j b^{(j)}) \leq \eta_j R(\varphi, x^{(1)}, ..., x^{(r)}, b^{(j)}) \\ S(f, x^{(1)}, ..., x^{(r)}, \alpha_1 b^{(1)}, ..., \alpha_t b^{(t)}) \geq \alpha_1 ... \alpha_t S(f, x^{(1)}, ..., x^{(r)}, b^{(1)}, ..., b^{(t)}) \end{cases}$$

for any $\varphi \in K_j$, $x^{(l)} \in E_l$, $0 \leq \eta,\ \alpha_j \leq 1$, $b^{(j)} \in G_j$, with $j = 1, ..., t$ and $f \in \mathcal{H}$.

Definition 4.2.7 *If $0 < p_1, ..., p_t, p < \infty$, with $\dfrac{1}{p} = \dfrac{1}{p_1} + \cdots \dfrac{1}{p_t}$, an operator $f : X_1 \times \cdots \times X_m \to Y$ in $\mathcal{H}$ is said to be $R_1, ..., R_t - S-$ abstract $(p_1, ..., p_t)-$summing if there is a constant $C > 0$ such that*

$$\left(\sum_{j=1}^{m} S(f, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(1)}, ..., b_j^{(t)})^p\right)^{1/p} \leq C \prod_{k=1}^{t} \sup_{\varphi \in K_j} \left(\sum_{j=1}^{m} R_k(\varphi, x_j^{(1)}, ..., x_j^{(r)}, b_j^{(k)})^{p_k}\right)^{\frac{1}{p_k}}, \tag{4.7}$$

for all $x_1^{(s)}, ..., x_m^{(s)} \in E_s$, $b_1^{(l)}, ..., b_m^{(l)} \in G_l$ and $m \in \mathbb{N}$ and $(s,l) \in \{1, ..., r\} \times \{1, ..., t\}$.

Theorem 4.2.8 *A map $f \in \mathcal{H}$ is $R_1, ..., R_t - S-$ abstract $(p_1, ..., p_t)-$summing if, and only if, there is a constant $C > 0$ and a Borel probability measures μ_j on K_j, such that*

$$S(f, x_1, ..., x_r, b_1, ..., b_t) \leq C \prod_{j=1}^{t} \left(\int_{K_j} R_j(\varphi, x_1, ..., x_r, b_j)^{p_j} d\mu\right)^{1/p_j}, \tag{4.8}$$

for all $x_{(l)} \in E_l, l = 1, ..., r$ and $b_{(j)} \in G_j$, with $j = 1, ..., t$.

Theorem 4.2.9 *Let $1 < p < \infty$. The following are equivalent for a polynomial $P \in \mathcal{P}(^mE; F)$:*

(a) *P is positive Cohen p-nuclear.*

(b) *There exists a positive constant $C > 0$ and regular Borel probability measures μ_1 on $B^+_{E^*}$, μ_2 on $B^+_{F^{**}}$ and $C > 0$ such that, for all $x \in E^+$ and $y^* \in F^{*+}$, we have*

$$|\langle P(x), y^* \rangle| \leq C \left(\int_{B^+_{E^*}} \langle x, x^* \rangle^{mp} d\mu_1(x^*)\right)^{1/p} \left(\int_{B^+_{F^{**}}} \langle y^*, y^{**} \rangle^{p^*} d\mu_2(y^{**})\right)^{1/p^*}. \tag{4.9}$$

Proof. (a)⇒(b) Assume that P is positive Cohen p-nuclear polynomial. Note that by choosing the parameters

$$\begin{cases} t = 2 \quad \text{and } r = m-1 \\ G_1 = E^+ \text{ and } G_2 = F^{*+} \\ K_1 = B^+_{E^*}, K_2 = B^+_{F^{**}} \\ q = 1, q_1 = p \quad \text{and} \quad q_2 = p^* \\ \mathcal{H} = \mathcal{P}(^m E; F) \\ S(P, x, y^*) = |\langle P(x), y^* \rangle| \\ R_1(\varphi, x) = \langle x, x^* \rangle^m \\ R_2(\varphi, x, y^*) = \langle \varphi, y^* \rangle . \end{cases}$$

we have $P : E \to F$ is positive Cohen p-nuclear if, and only if, P is $R_1, R_2 - S$ abstract (p, p^*)-summing. For definition of $R_1, R_2 - S$ abstract (p, p^*)-summing. It is well-known by Theorem 4.2.8 that P is $R_1, R_2 - S$abstract (p, p^*)-summing if, and only if, there is a $C > 0$ and there are probability measures μ_k on K_k, $k = 1, 2$, such that

$$S(P, x, y^*) \leq C \Big(\int_{K_1} R_1(\varphi, x)^p d\mu_1 \Big)^{1/p} \Big(\int_{K_2} R_2(\varphi, y^*)^{p^*} d\mu_2 \Big)^{\frac{1}{p^*}},$$

then

$$|\langle P(x), y^* \rangle| \leq C \Bigg(\int_{B^+_{E^*}} \langle x, x^* \rangle^{mp} d\mu_1 \Bigg)^{1/p} \Bigg(\int_{B^+_{F^{**}}} \langle y^*, y^{**} \rangle^{p^*} d\mu_2 \Bigg)^{1/p^*}.$$

(b)$\Rightarrow$ (a)) Let $(x_i)_{i=1}^n \subset E^+$ and $(y_i^*)_{i=1}^n \subset F^{*+}$. We have

$$\sum_{i=1}^n |\langle P(x_i), y_i^* \rangle| \leq C \sum_{i=1}^n \Bigg(\Bigg(\int_{B^+_{E^*}} \langle x_i, x^* \rangle^{mp} d\mu_1(x^*) \Bigg)^{1/p} \Bigg(\int_{B^+_{F^{**}}} \langle y_i^*, y^{**} \rangle^{p^*} d\mu_2(y^{**}) \Bigg)^{1/p^*} \Bigg)$$

for all $1 \leq i \leq n$. Thus, using Hölder's inequality we obtain that

$$\begin{aligned} \sum_{i=1}^n |\langle P(x_i), y_i^* \rangle| &\leq C \Bigg(\sum_{i=1}^n \Bigg(\int_{B^+_{E^*}} \langle x_i, x^* \rangle^{mp} d\mu_1(x_i^*) \Bigg) \Bigg)^{1/p} \Bigg(\sum_{i=1}^n \int_{B^+_{F^{**}}} \langle y_i^*, y^{**} \rangle^{p^*} d\mu_2(y^{**}) \Bigg)^{1/p^*} \\ &\leq C \sup_{x^* \in B^+_{E^*}} \Bigg(\sum_{i=1}^n \langle x_i, x^* \rangle^{mp} \Bigg)^{1/p} \Bigg(\sup_{y^{**} \in B^+_{F^{**}}} \sum_{i=1}^n \langle y_i^*, y^{**} \rangle^{p^*} \Bigg)^{1/p^*} \\ &\leq C \sup_{x^* \in B^+_{E^*}} \Bigg(\sum_{i=1}^n \langle x_i, x^* \rangle^{mp} \Bigg)^{1/p} \|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,w}(F^{*+})}. \end{aligned}$$

This implies that $P \in \mathcal{P}^{c+}_{N-p}(^m E; F)$ and $n_p^{m+}(P) \leq C$. ■

4.2.2 Kwapien's Factorization Theorem

The main objective of this section is to demonstrate that a polynomial P is positive Cohen p-nuclear if, and only if, it can be expressed in the form $P = Q \circ u$, where Q is a positive Cohen strongly p-summing m-homogeneous polynomial and u is a positive mp-summing operator.

In order to prove this result, we will need to utilize the following lemma, which is stated below. Let i_E be the embedding of E into $C\left(B_{E^*}^+\right)$ define by $i_E(x) = \langle x, \cdot\rangle$. For $f \in i_E(E) \subset C\left(B_{E^*}^+\right)$, we define the seminorm,

$$|||f||| = \inf\left\{\left(\int_{B_{E^*}^+} \langle |z|, \cdot\rangle^p d\mu(\cdot)\right)^{1/p} \ : \ \langle |z-f|, \cdot\rangle = 0 \ \mu- \ \text{a.e.}\right\},$$

where μ is the regular probability measure on $B_{E^*}^+$.

Let R be the closed subspace of $i_E(E)$ given by $R = \{f \in i_E(E), |||f||| = 0\}$. We write $L_0^p(\mu)$ the completion of the quotient space $i_E(E)/R$ with the norm

$$\|[f]\| = |||f|||,$$

where $[f]$ is the equivalence class of $f \in i_E(E)$ [3].

Lemma 4.2.10 [3] *The operator $J_{p,0} \circ i_E : E \to i_E(E) \to L_0^p(\mu)$ is positive p-summing, and $\pi_p^+(J_{p,0} \circ i_E) \leq 1$, such that for all $x \in E$*

$$\|\langle x, \cdot\rangle\|_{L_0^p(\mu)} = \|J_{p,0} \circ i_E\|_{L_0^p(\mu)}.$$

Where μ is a probability measure on the set $B_{E^*}^+$.

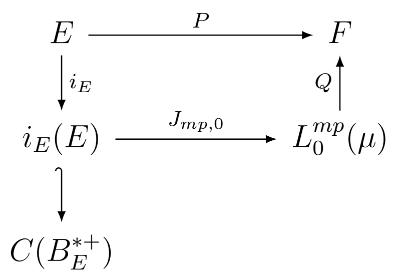
Theorem 4.2.11 *Let $1 < p < \infty$. A polynomial $P : E \to F$ is positive Cohen p-nuclear if, and only if, there exist Banach space G, positive mp-summing linear operator $u \in \mathcal{L}(E; G)$ and a positive Cohen strongly p-summing m-homogeneous polynomial $Q \in \mathcal{P}(^mG; F)$ such that $P = Q \circ u$, (i.e. $\mathcal{P}_{N-p}^{c+} = \mathcal{P}_{coh^+-p} \circ \prod_{mp}^+$, isometrically)*

$$n_p^{m+}(P) = \inf\left\{d_p^{m+}(Q)\left(\pi_{mp}^+(u)\right)^m : \quad P = Q \circ u\right\}.$$

Proof. Suppose that $P \in \mathcal{P}_{N-p}^{c+}(^mE; F)$. Then, through Theorem 4.2.9, there exist Radon probability measures μ on $B_{E^*}^+$ and λ on $B_{F^{**}}^+$ such that for all $x \in E$ and $y^* \in F^{*+}$.

$$|\langle P(x), y^*\rangle| \leq C\, |||x|||_{L_{mp}(B_{E^*}^+, \mu)}^m \, \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \lambda)}$$

And we consider the diagram



Where $i_E : E \to C(B^+_{E^*})$ is the canonical injection. If we denote the range of $J_{p,0} \circ i_E$ by G, and the closure of G by $L_0^{mp}(\mu)$, the map $G \to F : J_{mp,0} \circ i_E(x) \mapsto P(x)$ is well defined operator.

So, we apply the Lemma 4.2.10, we find that the operator $u = J_{mp,0} \circ i_E : E \to i_E(E) \to L_0^{mp}(\mu)$ are positive mp-summing, with $\pi^+_{mp}(u) \leq 1$, and the polynomial Q is defined on $u(E)$, where $u(x) = (J_{mp,0} \circ i_E)(x)$, by

$$Q(u(x)) := P(x),$$

this definition makes sense because

$$|\langle Q(u(x)), y^* \rangle| \leq C \, \|u(x)\|^m_{L_{mp}(\mu)} \left(\int_{B^+_{F^{**}}} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}}.$$

Then, from [32, Theorem 1] we deduce Q is a Cohen positive strongly p-summing polynomial and $d_p^{m+}(Q) \leq n_p^{m+}(P)$.

$$|\langle Q(u(x)), y^* \rangle| \leq n_p^{m+}(P) \|u(x)\|^m_{L_{mp}(\mu)} \left(\int_{B^+_{F^{**}}} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{1/p^*}.$$

Conversely, let $x \in E$ and $y^* \in F^{*+}$, we have

$$\begin{aligned} |\langle Q(u(x)), y^* \rangle| &\leq d_p^{m+}(Q) \|u(x)\|^m \left(\int_{B^+_{F^{**}}} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{1/p^*} \\ &\leq d_p^{m+}(Q) (\pi^+_{mp}(u))^m \left(\int_{B^+_{E^*}} \langle |x|, x^* \rangle^{mp} d\mu(x^*) \right)^{1/p} \left(\int_{B^+_{F^{**}}} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{1/p^*}, \end{aligned}$$

this implies that $Q \circ u \in \mathcal{P}^+_{N-p}(^mE; F)$ and $n_p^{m+}(Q \circ u) \leq d_p^{m+}(Q)(\pi^+_{mp}(u))^m$. ■

So, in view of the previous result, it is obvious that we can derive the following result.

Proposition 4.2.12 *For $1 < p < \infty$.*

$$\mathcal{P}^c_{p,N}(^mE; F) \subseteq \mathcal{P}^{c+}_{N-p}(^mE; F).$$

Proof. Suppose that $P \in \mathcal{P}^c_{p,N}(^mE; F)$. It is well-known [15, Proposition 3.4] that there exists Banach space G such that $P = Q \circ u$ where $Q \in \mathcal{P}_{coh^+-p}(^mG, F)$ and $u \in \Pi_{mp}(E; G)$. It follows from [32, Proposition 4] that $Q \in \mathcal{P}_{coh^+-p}(^mG; F)$ and it follows from [11, Proposition 3] that $u \in \Pi^+_{mp}(E; G)$. Theorem 4.2.11 clearly implies that $P \in \mathcal{P}^{c+}_{N-p}(^mE; F)$. ■

4.3 Relationships between some classes

Theorem 4.3.1 *Let $1 < p < \infty$, let E and F be Banach lattices such that F is Dedekind complete. If $P \in \mathcal{P}^r(^mE;F)$ is positive Cohen p-nuclear polynomial, then its linearization operator $P^{\otimes}$ belongs to $\mathcal{N}_p^+(\widehat{\otimes}_{s,|\pi|}^m E;F)$.*

Proof. Let $P \in \mathcal{P}(^mE;F)$. By Theorem 4.2.9 we obtain that

$$|\langle P(x), y^*\rangle| \leq n_p^{m+}(P)\, \|x\|^m_{L_{mp}(B^+_{E^*},\mu)}\, \|y^*\|_{L_{p^*}(B^+_{F^{**}},\lambda)},$$

for all $x \in E^+$ and $y^* \in F^{*+}$, we take $z \in \widehat{\otimes}_{s,|\pi|}^m E$, for all $\epsilon > 0$, z admits a representation $z = \sum_{i=1}^n x_i \otimes \overset{(m)}{\cdots} \otimes x_i$ such that

$$\sum_{i=1}^n \|x_i\|^m_{L_{mp}(B^+_{E^*},\mu)} \leq \|z\|_{L_p(B^+_{(\widehat{\otimes}_{s,|\pi|}^m E)^*},\widehat{\mu})} + \epsilon \tag{4.10}$$

where $\widehat{\mu} = \mu \otimes \overset{(m)}{\cdots} \otimes \mu$. Then

$$\begin{aligned}
\left|\left\langle P^{\otimes}(z), y^*\right\rangle\right| &= \left|\left\langle P^{\otimes}\left(\sum_{i=1}^n x_i \otimes \overset{(m)}{\cdots} \otimes x_i\right), y^*\right\rangle\right| \\
&= \left|\sum_{i=1}^n \left\langle P^{\otimes}(x_i \otimes \overset{(m)}{\cdots} \otimes x_i), y^*\right\rangle\right| \\
&\leq \sum_{i=1}^n |\langle P(x_i), y^*\rangle| \\
&\leq n_p^{m+}(P) \sum_{i=1}^n \|x_i\|^m_{L_{mp}(B^+_{E^*},\mu)}\, \|y^*\|_{L_{p^*}(B^+_{F^{**}},\lambda)} \\
&\leq n_p^{m+}(P) \left(\|z\|_{L_p(B^+_{(\widehat{\otimes}_{s,|\pi|}^m E)^*},\widehat{\mu})} + \epsilon\right) \|y^*\|_{L_{p^*}(B^+_{F^{**}},\lambda)}
\end{aligned}$$

Thus, $P^{\otimes} \in \mathcal{N}_p^+(\widehat{\otimes}_{s,|\pi|}^m E;F)$ and $n_p^{m+}(P) \leq n_p^+(P^{\otimes})$. ■

Problem 4.3.2 Does $P^{\otimes} \in \mathcal{L}^r(\widehat{\otimes}_{s,|\pi|}^m E;F)$ positive Cohen p-nuclear imply $P \in \mathcal{P}^r(^mE;F)$ is positive Cohen p-nuclear polynomial?

Proposition 4.3.3 *Let $m \in \mathbb{N}$. Assume that $\mathcal{P}^r_{coh^+-p}(^mE;F) \subset \mathcal{P}^{c+r}_{N-p}(^mE;F)$. Then $\mathcal{D}_p^{+r}(E;F) \subset \mathcal{N}_p^{+r}(E;F)$.*

Proof. Let $u \in \mathcal{D}_p^{+r}(E;F)$, we show that $u \in \mathcal{N}_p^{+r}(E;F)$. Fix $e \in B_{E^+}$ and $x_0^* \in B^+_{E^*}$ such that $x_0^*(e) = 1$. Define, see [36] , the operator

$$\widehat{K}_j : \widehat{\otimes}_{s,|\pi|}^{j+1} E \to \widehat{\otimes}_{s,|\pi|}^{j} E \ (1 \leq j \leq m-1)$$

by

$$\widehat{K}_j\left(\sum_{i=1}^{n} x_i \otimes \overset{(j+1)}{\cdots} \otimes x_i\right) = \sum_{i=1}^{n} x^*(x) x_i \otimes \overset{(j)}{\cdots} \otimes x_i.$$

Let $\otimes_m(x) = x \otimes \overset{(m)}{\cdots} \otimes x$. It follows from [32, Proposition 6], we obtain

$$P := u \circ \widehat{K}_1 \circ \cdots \circ \widehat{K}_{m-1} \circ \otimes_m : E \to F$$

Then, P is positive Cohen strongly p-summing, it follows from this and our hypotheses that $P \in \mathcal{P}^{c+r}_{N-p}({}^mE; F)$. By the decomposition $P = P^{\otimes} \circ \otimes_m$ we obtain $P^{\otimes} = u \circ \widehat{K}_1 \circ \cdots \circ \widehat{K}_{m-1}$ is positive Cohen p-nuclear operator.

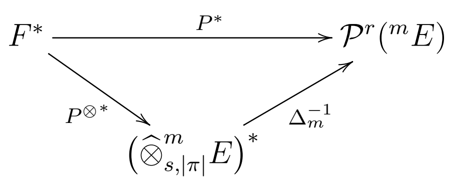
Now, as it has been proven in the proof of [36, Theorem 4.1], that there are operators $\widehat{J}_j : \widehat{\otimes}^j_{s,|\pi|} E \to \widehat{\otimes}^{j+1}_{s,|\pi|} E$, $(1 \leq j \leq m-1)$ defined in terms of x_0^* and e such that $\widehat{K}_j \circ \widehat{J}_j$ is the identity operator on $\widehat{\otimes}^j_{s,|\pi|} E$. We obtain

$$u = u \circ \widehat{K}_1 \circ \cdots \circ \widehat{K}_{m-1} \circ \widehat{J}_{m-1} \circ \cdots \circ \widehat{J}_1 : E \to F,$$

thanks to the ideal property, u is positive Cohen p-nuclear operator. ■

Theorem 4.3.4 *An m-homogeneous polynomial $P \in \mathcal{P}^r({}^mE; F)$ is positive Cohen p-nuclear polynomial imply the adjoint $P^* : F^* \to \mathcal{P}^r({}^mE)$ is positive Cohen p^*-nuclear operator.*

Proof. Assume first that $P : E \to F$ is positive Cohen p-nuclear polynomial. By Theorem 4.3.1, the linearization $P^{\otimes} : \widehat{\otimes}^m_{s,|\pi|} E \to F$ is positive Cohen p-nuclear, then its adjoint $P^{\otimes *} : F^* \to (\widehat{\otimes}^m_{s,|\pi|} E)^*$ is a positive Cohen p^*-nuclear operator. Consider the isometric isomorphism $\triangle_m : \mathcal{P}^r({}^mE) \to (\widehat{\otimes}^m_{s,|\pi|} E)^*$ given by $\triangle_m(P) = P^*$. Since $P = P^{\otimes} \circ \otimes_m$, by duality we get $P^* = \otimes_m^* \circ P^{\otimes *} = \triangle_m^{-1} \circ P^{\otimes *}$.



The ideal property ensures that P^* is positive Cohen p^*-nuclear. ■

Problem 4.3.5 Does the adjoint $P^* \in \mathcal{L}^r(F^*; \mathcal{L}^r({}^mE))$ positive Cohen p^*-nuclear operator imply $P \in \mathcal{P}^r({}^mE; F)$ is positive Cohen p-nuclear polynomial ?

In 1983, Pietsch introduced the concept of p-dominated homogeneous polynomials between Banach spaces as an important extension of the concept of absolutely p-summing linear operators to the nonlinear case. This concept has since attracted attention from several authors. For example, it was recently investigated by Hamdi et al. in [32], where they introduced the concept of positive p-dominated homogeneous polynomials.

Definition 4.3.6 *Let $1 \leq p < \infty$. An m-homogeneous polynomial $P : E \to Y$ from a Banach lattice E into a Banach space Y is said to be positive p-dominated if there exists a constant $C > 0$ such that*

$$\left(\sum_{i=1}^{n} \| P(x_i) \|^{p/m} \right)^{m/p} \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle x_i, x^* \rangle^p \right)^{m/p}, \tag{4.11}$$

for every $m \in \mathbb{N}^*$ and every $(x_i)_{i=1}^n \subset E^+$.

We denote $\mathcal{P}_{d,p}^{+}(^m E; Y)$ the space of positive p-dominated polynomials from E into Y, and by $\delta_p^+(.)$ the norm defined by the infimum of all constants verifying the inequality (4.11), for $p \geq m$, but for $p < m$ it is only a quasi-norm.

This class satisfies a Pietsch domination theorem which is the principal tool of the next theorem.

Theorem 4.3.7 *Let $1 \leq p < \infty$ and let E be a Banach lattice and Y be a Banach space. An m-homogeneous polynomial $P : E \to Y$ is positive p-dominated if, and only if, there exist a constant $C > 0$ and a probability measure μ on $B_{E^*}^+$ such that*

$$\|P(x)\| \leq C \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^p \, d\mu(x^*) \right)^{m/p} \tag{4.12}$$

for every $x \in E^+$. Moreover, $\delta_p^+(P) = \inf\{C \setminus C$ as inequality (4.12)$\}$.

In the next theorem, we study the relationship between positive Cohen p-nuclear m-homogeneous polynomials and positive p-dominated m-homogeneous polynomials.

Theorem 4.3.8 *Let $1 < p < \infty$, let $E,\ F$ be Banach lattices and let $P : E \to F$ be an m-homogeneous polynomials.*

(a) *If $P \in \mathcal{P}_{N-p}^{c+}(^m E; F)$ then $P \in \mathcal{P}_{coh^+-p}(^m E; F)$ and $d_p^{m+}(P) \leq n_p^{m+}(P)$.*

(b) *If $P \in \mathcal{P}_{N-p}^{c+}(^m E; F)$ then $P \in \mathcal{P}_{d,p}^{+}(^m E; F)$ for all $p \geq m$ and $\delta_p^+(P) \leq n_p^m(P)$.*

Proof. (a) If P is positive Cohen p-nuclear, for all $x \subset E^+$ and $y^* \subset F^{*+}$, we have

$$\begin{aligned} |\langle P(x), y^* \rangle| &\leq n_p^{m+}(P) \left(\int_{B_{E^*}^+} \langle x, x^* \rangle^{mp} \, d\mu_j(x^*) \right)^{1/p} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} \, d\lambda(y^{**}) \right)^{1/p^*} \\ &\leq n_p^{m+}(P) \left(\sup_{x^* \in B_{E^*}^+} \langle x, x^* \rangle \right)^m \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} \, d\mu(y^{**}) \right)^{1/p^*} \\ &\leq n_p^{m+}(P) \, \|x\| \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} \, d\mu(y^{**}) \right)^{1/p^*}. \end{aligned}$$

Thus, $P \in \mathcal{P}_{coh^{+}-p}\left({}^{m}E;F\right)$ and $d_{p}^{m+}(P) \leq n_{p}^{m+}(P)$.
(b) If P is positive Cohen p-nuclear, for all $x \subset E^{+}$ and $y^{*} \subset F^{*+}$, we have

$$\begin{aligned}
\|P(x)\| &= \sup_{y^{*}\in B_{F^{*}}^{+}} \left|\left\langle P(x), y^{*}\right\rangle\right| \\
&\leq n_{p}^{m+}(P)\,\|x\|_{L_{p}\left(B_{E^{*}}^{+},\mu\right)}^{m} \sup_{y^{*}\in B_{F^{*}}^{+}} \left(\int_{B_{F^{**}}^{+}} \left\langle y^{*}, y^{**}\right\rangle^{p^{*}} d\lambda\left(y^{**}\right)\right)^{1/p^{*}} \\
&\leq n_{p}^{m+}(P)\,\|x\|_{L_{p}\left(B_{E^{*}}^{+},\mu\right)}^{m} \sup_{y^{*}\in B_{F^{*}}^{+}} \|y^{*}\| \\
&\leq n_{p}^{m+}(P)\,\|x\|_{L_{p}\left(B_{E^{*}}^{+},\mu\right)}^{m}.
\end{aligned}$$

Therefore, P belongs to $\mathcal{P}_{d,p}^{+}\left({}^{m}E;F\right)$ and $\delta_{p}^{+}(P) \leq n_{p}^{m+}(P)$. ■

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