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Sujet

Sur Les Principes Elémentaires de Combinatoire

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On Elementary Principles of Combinatorics

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Contents

Introduction

Today a combinatoric has a great reputability between all the branches of mathematics in general form and the discrete mathematics in special form because it help us to solve many different problem , for example the counting of objects or the creation of internet adresses ,a pass words ...

In this memoir , we will talk about the elementory principles of combinatoric ,at first we will talk about elementary notions of combinatoric which implicite two importent principles that become the real base of the other principles "the addition and the multiplication principle ",then we move to explaiathe permutaton and combination .

In the first chapter we will talk about the pigeonhole principle and its applications , after that we will talk about the inclusion exclusion principle and its applications in the second chapter .

In the last chapter we will talk about the invarince principle illustrate by some examples for each one of the previous principles in order to show their applications .

Acknowledgements

Before any thing I want to thanks"ALLAH "

Iwant to thinks my parents ,brathers ,and my sisters because they help me to verify my wishes ,my teachers specially the prof AMROUNE Abd Alaziz ,LADJLAT Laheen ,and BENOUMHANI Moussa for their helps and their directions

thanks for every one of you .

Preliminary

0.1 What'is the combinatorics ?

Combinatorics,its an importent subject in mathematics especially in discrete mathematics ,it has many diffrent definition ,for example we can say it's the study of arrangments of objects or the counting of objects .

This subject was studied as long ago as 17^{th}century when combinatorial questions arose in the study of gambling games. Enumeration, the counting of objects with certain properties, is an important part of combinatorics. We must count objects to solve many different types of problems. For instance, counting is used to determine the complexity of algorithms.

Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand. Recently, it has played a key role in mathematical biology, especially in sequencing DNA. Furthermore, counting techniques are used extensively when probabilities of events are computed .

0.2 The Sum and The Product rules

Introduction

In this section we wll talk about two basic principles of combinatoric The rule of sum and The rule of product which has an inportent role in the creating of the other principles .

First we introduce the sum rule :

0.2.1 The rule of sum

Definition 1 *If a task can be performed in m ways , while another task can be performed in n ways , and the two cannot be performed simultaneously, then performing either task can tasks be accomplished in* $m + n$ *ways .*
Set theoretical version of the rule of sum: If A and B are disjoint sets $(A \cap B = \phi)$ *then :*

$$|A \cup B| = |A| + |B|.$$

More generally, if the sets $A_1, A_2, ..., A_n$ *are pair wise disjoint, then:*

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|.$$

.

Examples

1 - Two sets A and B :$|A| = 4$ elements , $|B| = 7$ elements , so

$$|A + B| = 3 + 7 = 10 elements$$

2- If a class has 30 male students and 25 female students ,then the class has

$$30 + 25 = 45 students.$$

3 - Suppose that either a member of the high school or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative ,if there are 49 members of the high schools and 103 mathematics majors and no one is both a faculty member and a student?

Solution:

There are 49 ways to choose a member of the high school and there are 103 ways to choose a student who is a mathematics major. Choosing a member of the high schools is never the same as choosing a student who is a mathematics major because no one is both a school member and a student. By the sum rule it follows that there are$49 + 103 = 152$ possible ways to pick this representative.

0.2.2 The Rule of Product

Definition 2 *If a task can be performed in m ways and another independent task can be performed in n ways, then the combination of both tasks can be performed in mn ways. Set theoretical version of the rule of product: Let $A \times B$ be the Cartesian product of sets A and B. Then:*

$$|A \times B| = |A|.|B|$$

More generally:

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1|.|A_2| \ldots |A_n|$$

Examples

1 - A license plate contains three letters followed by four digits. How many different license plates can be printed ?

Answer:

Each letter can be printed in 26 ways, and each digit can be printed in 10 ways, so

$$26.26.26.10.10.10.10 = 676000$$

different plates can be printed

2 - There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Solution

The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are $32 \times 24 = 768$ ports.

Extended version of the product rule is often useful. Suppose that a procedure is carried out by performing the tasks$T_1, T_2, ..., T_m$in sequence. If each task $T_i, i = 1, 2, ..., n$, can be done inn_i ways, regardless of how the previous tasks were done, then there are $n_1.n_2....n_m$ ways to carry out the procedure. This version of the product rule can be proved by mathematical induction from the product rule for two tasks.

0.3 Permutation And Combinations

Introduction

We can solve Many problems of counting by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. And ther are other counting problems can be solved by finding the number of ways to select.

0.4 Permutations

Definition 3 *A* ***permutation*** *of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an r-permutation .*

Theorem 1 *If n is a positive integer and r is an integer with* $1 \leq r \leq n$*, then there are*

$$P(n,r) = n(n-1)(n-2)...(n-r+1)$$

r-permutations of a set with n distinct elements.

Proof 1 *We will use the product rule to prove the of this formula is correct. The first element of the permutation can be chosen in n diffrent ways because there are n elements in the set. There are* $n-1$ *ways to choose the second element of the permutation, because there are* $n-1$ *elements left in the set after using the element picked for the first position. Similarly, there are* $n-2$ *ways to choose the third element, and so on, until there are exactly*

$$n-(r-1) = n-r+1$$

ways to choose ther^{th} *element. by the product rule, we have*

$$n(n-1)(n-2)...(n-r+1)$$

r-permutations of the set

Note that $P(n,0) = 1$ when ever n is a non negative integer because there is exactly one way to order zero elements. There is exactly one list with no elements in it, namely the empty list.

Now We state a corollary to this Theorem .

Corollary 1 *If n and r are integers with $0 \leq r \leq n$, then*

$$P(n,r) = \frac{n!}{(n-r)!}$$

Proof 2 *When n and r are integers with $1 \leq r \leq n$, by thelast Theorem we have*

$$\begin{aligned} P(n,r) =& n(n-1)(n-2)\ldots(n-r+1) \\ =& \frac{n!}{(n-r)!} \end{aligned}$$

Because

$$\frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

when n is a non negative integer, we see that the formula

$$P(n,r) = \frac{n!}{(n-r)!}$$

also

$$r = 0.$$

By the last Theorem we know that if n is a positive integer, then$P(n,n) = n!$.We will illustrate this result with some examples.

Example

How many ways are there to select a first student, a second-student , and a third-student from 30 different students the class room?

Solution:

Because it matters which pupil is the first ,the second and the third one, the number of ways to pick the three students is the number of ordered selections of three elements from a set of 30 elements, that is, the number of 3-permutations of a set of 50 elements. so, the answer is

$$P(30,3) = \frac{50!}{3!} = 30.29.28 = 24630$$

0.5 Combinations

Let us start by solving the following question.

Examples :

1. How many different committees of three students can be formed from a group of four students?

   To answer this question, we need only find the number of subsets with three elements from the set which contain the four students. We see that there are four such subsets, one for each of the four students, because the choise of these three students is the same as choising one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter. An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

   We see that $\begin{pmatrix} 6 \\ 2 \end{pmatrix} = 15,$ We can determine the number of r-combinations of a set with n elements using the formula for the number of r-permutations of a set. To do this, note that the r-permutations of a set can be obtained by first forming r-combinations and then ordering the elements in these combinations.

Theorem 2 *The number of r-combinations of a set with n elements, where n is a non-negative integer and r is an integer with* $0 \leq r \leq n$, *equals*

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{r!(n-r)!}$$

Proof 3 *The P(n, r) r-permutations of the set can be obtained by forming the* $\begin{pmatrix} n \\ r \end{pmatrix}$ *r-combinations of the set, and then ordering the elements in each r-combination, which can be done in P(r, r) ways. Consequently, by the product rule,*

$$P(n,r) = \begin{pmatrix} n \\ r \end{pmatrix} .P(r,r).$$

This implies that :

$$\begin{aligned}\binom{n}{r} &= \frac{P(n,r)}{P(r,r)}\\ &= \frac{n!/(n-r)!}{r!/(r-r)!}\\ &= \frac{n!}{r!(n-r)}\end{aligned}$$

which implies as before that

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Some Properties

Let n and r be non negative integers with $r \leq n$. Then

$$\begin{aligned}\binom{n}{r} &= \binom{n}{n-r}.\\ \binom{n}{n} &= \binom{0}{0}\\ \sum_{k=0}^{n}\binom{n}{k} &= 2^n\end{aligned}$$

Proof 4 *From the Theorem 2 it follows that*

$$\begin{aligned}
\binom{n}{r} &= \frac{n!}{r!(n-r)!} \\
\binom{n}{n-r} &= \frac{n!}{(n-r)![n-(n-r)]!} \\
&= \frac{n!}{(n-r)!r!}. \\
Hence, \binom{n}{n-r} &= \binom{n}{n-r} \\
.\binom{n}{n} &= \frac{n!}{(n-n)!n!} = 0! = 1 \\
\binom{0}{0} &= \frac{0!}{(0-0)!0!} = 0! = 1 \\
So \binom{n}{n} &= \binom{0}{0}
\end{aligned}$$

Example :

Suppose that there are 8 colored squares forms and 11 in the colored tringles forms. How many ways are there to select a committee to make toy(small house) if the committee is to consist of three triangle from the colored triangle forms and four from the colored square formss?

Solution :

By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$\binom{8}{3}\cdot\binom{11}{4} = \frac{8!}{3!5!}\cdot\frac{11!}{4!7!} = 56330 = 18480.$$

0.6 *Generalized Permutations*

0.6.1 *Permutations with Repetition*

Counting permutations when repetition of elements is allowed can easily be done using the product rule, as the following example

Example :

How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution:

By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r.
The number of r-permutations of a set with n elements when repetition is allowed is given in next Theorem .

Theorem 3 *The number of r permutations of a set of n objects with repetition allowed is* $n - r$.

Proof 5 *There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed, because for each choice all n objects are available.*
Hence, by the product rule there are nr r-permutations when repetition is allowed.

Part 02 : The most importent principles of combinatorics

1. Chapter 01 : The Pigeonhole Principle .
2. Chapter 02 :The Inclusion Exclusion Principle.
3. Chapter 03: The invariance principle

The Pigeonhole Principle and its applications

The Pigeonhole Principles

The Generalized Pigeonhole Principle

Chapter 1

The Pigeonhole Principle and its applications

Introduction

In this chapter we well talk about the pigeonhole principle and its application. This principle is also called the Dirichlet drawer principle, after the nineteenthcentury German mathematician "G. Lejeune Dirichlet," who used this principle .

1.1 The Pigeonhole Principle

We will illustrate the usefulness of the pigeonhole principle .

Theorem 4 *If k is a positive integer and $k+1$ or more objects are placed into k boxes , then there is at least one box containing two or more objects.*

Proof 6 *We prove the pigeonhole principle using a contradition. Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k.*
This is a contradiction, because there are at least$k+1$ objects.
We introduce it in this chapter because of its many important applications to combinatorics.

Corollary 2 *A function f from a set with $k+1$ or more elements to a set with k elements is not one – to one.*

Proof 7 *Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that*

$$f(x) = y.$$

Because the domain contains $k+1$ or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be one-to-one.

Examples

1. If there are three persons ,is it possible to find two persons with the same sexe ?

   Solution

   Lebel two boxswith the name of sexe(male,female). Put each person in the boxs labeled with his sexe .one box well contain at least two person with the same sexe .

2. Among any 13people ,at least two share a birth month .

   solution

   Lebel 12 boxes with the names of month .Put each person in the boxs labeled with his or her birth month?

   solution

   Some box well contain at least two people ,who share a birth month.

3. Among 366 persons is it possible to find 2 persons born in the same day? Solution In general we have 365 days in the year ,and in this case wehave 366 diffrent persons , by the pigeonhole principle we find two person born in the same day

4. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is grade on a scale from 0 to 100 points?

   Solution There are 101 possible scores on the final exam . any 102 students there must be at least 2 students with the same score. The pigeonhole principle is a useful tool in many proofs, including proofs of surprising results, such as that given in the second example .

1.2 The Generalized Pigeonhole Principle

The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes. However, even more can be said when the number of objects exceeds a multiple of the number of boxes. For instance, among any set of 53 letters must be 3 that are the same.
This follows because when 53 objects are distributed into 26 boxes(because we have only 26letter in the english alphabitic), one box must have more than 2 objects.

Theorem 5 *if N objects are placed into kboxes, then there is at least one box containing at least*$[N/k]$ *objects.*

We will use a proof by contraposition. Suppose that none of the boxes contains more than$N/k1$objects. Then, the total number of objects is at most

$$k\Big(\Big[\frac{a}{b}\Big]\Big\langle k\Big(\Big(\frac{a}{b}-1\Big)+1\Big)=N$$

. Where the inequality $N/k < (N/k)+1$ has been used. This is a contradiction because there are a total of N objects.
A common type of problem asks for the minimum number of objects such that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes. When we have N objects, the generalized pigeonhole principle tells us there must be at least r objects in one of the boxes as long as$N/k \geq r$. The smallest integer N with $N/k > r-1$, namely,$N = k(r-1)+1$, is the smallest integer satisfying the inequalityN/kr.could a smaller value of N suffice?
The answer is no, because if we had$k(r-1)$ objects, we could put $r-1$ of them in each of the k boxes and no box would have at least r objects. When thinking about problems of this type, it is useful to consider how you can avoid having at least r objects in one of the boxes as you add successive objects. To avoid adding a rth object to any box, you eventually end up with $r1$ objects in each box. There is no way to add the next object without putting an rth object in that box.
in this two examples we will explain how the generalized pigeonhole principle is applied.

Example :

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution

The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that$N/5 = 6$.

The smallest such integer is $N = 55 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

1.3 Deep Applications of the Pigeonhole Principle

A few such applications will be described here.

Examples

1. Five points inside a square of side 1.Is it possible to find two points where

$$d(x,y) \leq \frac{\sqrt{2}}{2}$$

Solution : we well explain this situation by the following graph:

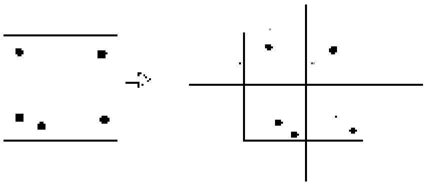

Figure 1.1: distribution of the five points in the squares

There is a square contain five points in the same times ,if we devide this square to 4legal squares .each square of side$\frac{\sqrt{2}}{2}$the diametre of each square is $\frac{\sqrt{2}}{2}$ So there only one square between the 4 squares which contains two points .this verify the folowing $d(x,y) \leq \frac{\sqrt{2}}{2}$

2. Show that among any n + 1 positive integers not exceeding 2n there must be an integer that divides one of the other integers.

   Solution

   Write each of the n + 1 integers$a_1, a_2, ..., a_{n+1}$ as a power of 2 times an odd integer. In other words, let a_j $2^{k_j}q_j$ for $j = 1, 2, ..., n+1$, where k_j is a non negative integer and q_j is odd. The integers $q_1, q_2, ..., q_{n+1}$ are all odd positive integers less than 2n. Because there are only n odd positive integers less than 2n, it follows from the pigeonhole principle that two of the integers $q_1, q_2, ..., q_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of $q_i and q_j$. Then,$a_i = 2^{ki}qanda_j = 2^{kj}q$. It follows that if $k_i < k_j$; then a_i divides a_j ; while if$k_i > k_j$; then a_j divides a_i .

3. Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal? expansion.

   Solution

   Let n be a positive integer. Consider the $(n+1)$ integers$1, 11, 111, ..., 11...1$ (where the last integer in this list is the integer with n + 1$1_s$in itsdecimal expansion). Note that there are n possible remainders when an integer is divided by n. Because there are n + 1 integers in this list, by the pigeonhole principle there must be two with the same remainder when divided by n.

   The larger of these integers less the smaller one is a multiple of n, which has a expansion consisting entirely of 0s and 1s.

4. Five points with integerscoordinatesin orthogonal parametre and homogenous.

   we choose two points randomly .

   We want to create other point called M which located in the midle of this two points.

   is it possible this point has an integers coordinates?

solution

We choose randomly two points among the five points P and Q where $P(x_i, y_i)$and $Q(x_j, y_j)$

The coordinates of the point M giving by

$$x_M = \frac{x_i + x_j}{2}$$
$$y_M = \frac{y_i + y_j}{2}$$

There are 4 diffrents cases .k ,k',s,s'are integers numbers . We say x is a pair if we can write it in this form $x = 2k$. $y = 2s$
we say x is impair if we can write it in this form $x = 2k + 1$,$y = 2s + 1$.
first case:

$$x_i and x_j, y_i and y_j are pairs so$$
$$x_M = \frac{2k + 2k'}{2} = (k + k'); where K + k' is integer$$
$$y_M = \frac{2s + 2s}{2} = 2(s + s') is integer.$$

where s+s' is integer

$$x_i and x_j, y_i and y_j are impairs so$$
$$x_M = \frac{2k + 1 + 2k' + 1}{2} = (k + k' + 1) where K + k' + 1 is integer$$
$$y_M = \frac{2s + 1 + 2s' + 1}{2} = (s + s' + 1) where s + s + 1' is integer$$

$$x_i and y_i are pair, x_j and y_j are imppairs so$$
$$x_M = \frac{2k + 2k' + 1}{2} = 2(s + s') + 1)/2 is'nt integer$$
$$y_M = \frac{2s + 2s' + 1}{2} = 2(s + s') + 1)/2 is'nt integer$$

the fourth case is same case three .

Chapter 02:The Inclusion -Exclusion principle

1. The inclusion - exclusion principle
2. Applications of inclusion - exclusion principle

Chapter 2

The Inclusion -Exclusion

2.1 The inclusion - exclusion principle

How many elements are in the union of two finite sets? In Section 1.2 we showed that the numberof elements in the union of the two sets A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection. That is,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

there is an equivalente between the alternative form and the normal form of inclusion

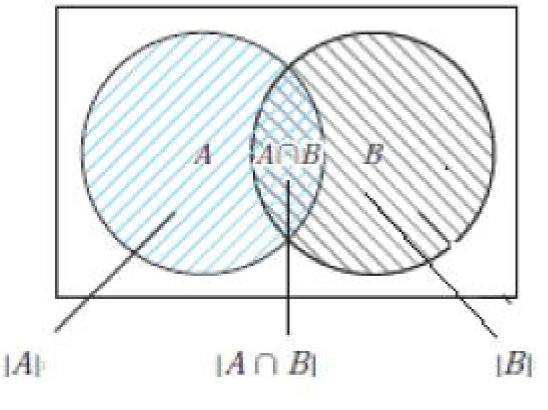

Figure 2.1: union of two sets$|A \cup B| = |A| + |B| - |A \cap B|$

exclusion principle

2.1.1 *An Alternative Form of Inclusion-Exclusion*

There is an alternative form of the principle of inclusion-exclusion which using to solve a counting problem . In particular, this form can be used to solve problems that ask for the number elements in a set that have none of n properties$P_1, P_2, ..., P_n$.

LetA_i be the subset containing the elements that have property P_i . The number of elements with all the properties $P_{i1}, P_{i2}, ..., P_{in}$ will be denoted by $N(P_{i1}.P_{i2}...P_{ik})$.

Writing these quantities in terms of sets, we have

$$|A_{i1} \cap A_{i2} \cap \ldots \cap A_{ik}| = N(P_{i1}P_{i2}.....P_{ik}).$$

If the number of elements with none of the properties $P_1, P_2, ..., P_n i \leq n$ denoted by $N(P_1'P'2....P_n)$ and the number of elements in the set is denoted by N, it follows that

$$N(P_1'P_2'...P_n') = N|A_1 \cup A_2 \cup ... \cup A_n|$$

From the inclusion- exclusion principle, we see that

$$\begin{aligned} N(P_1P_2....P_n) =& N \sum_{1\leq i\leq n} N(P_i) + \sum_{1i<jn} N(P_iP_j) \\ & - \sum_{\leq i<j<k\leq n} N(P_iP_jP_k) + ... + (-1)^n N(P_1P_2...P_n). \end{aligned}$$

Examples:

In a class of midle schoole we ask the pupils about their favorite activity between the two following activities reading , gymnastics every pupil chosen his favorite activity. The result became the number of pupil whos favorite gymnastics activity is 25; The number of pupil whos favorite reading activity is a with 13 ; and the number of pupil whos favorite both activities is 8. How many pupils are in this class?

Solution:

Let A be the set of pupil favorite gymnastics activity and B be the set of pupils favorite reading activity. Then $A \cup B$ is the set of pupils in the class who are joint gymnastics and reading. Because every pupil in the class favorite in either gymnastics or reading (or both), it follows that the number of students in the class is $|A \cup B|$. Then,

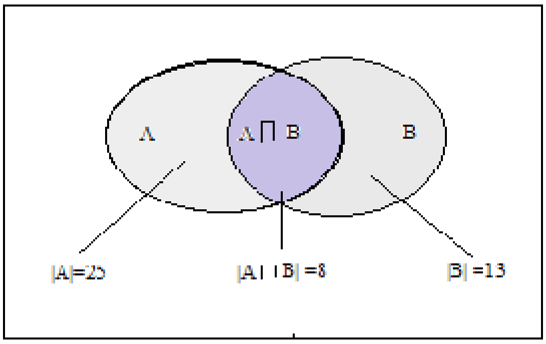

Figure 2.2: $|A \cup B| = |A| + |B| - |A \cap |B| = 25 + 13 - 8 = 30$

$$\begin{aligned} |A \cup B| =& |A| + |B| - |A \cap |B| \\ =& 25 + 13 - 8 \\ =& 30 \end{aligned}$$

Therefore, there are 30 students in the class.

We will now begin our development of a formula for the number of elements in the union of a finite number of sets. The formula we will develop is called the principle of inclusion-exclusion. For concreteness, before we consider unions of n sets, where n is any positive integer, we will derive a formula for the number of elements in the union of three sets A, B, and C. To construct this formula, we note that $|A| + |B| + |C|$ counts each element that is in exactly one of the three sets once, elements that are in exactly two of the sets twice, and elements in all three sets three times. To remove the overcount of elements in more than one of the sets, we subtract the number of elements in the intersections of all pairs of the three sets.We obtain

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|.$$

This expression still counts elements that occur in exactly one of the sets once. An element that occurs in exactly two of the sets is also counted exactly once, because this element will occur in one of the three intersections of sets taken two at a time. However, those elements that occur in all three sets will be counted zero times by this expression, because they occur in all three intersections of sets taken two at a time. A total

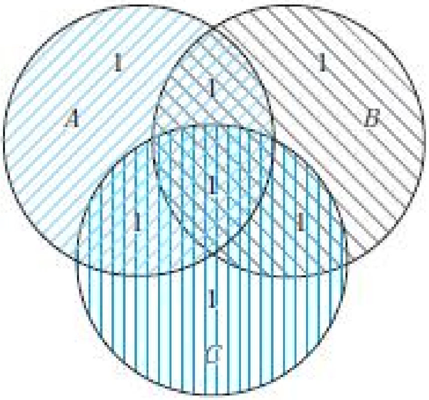

Figure 2.3: Union of 3 sets given by: $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$.

of 400 students have taken a course in arabic, 279 have taken a course in French, and 114 have taken a course in english. Further, 50 have taken courses in both arabic and French, 23 have taken courses in both arabic and english, and 14 have taken courses in both French and english. If 793 students have taken at least one of arabic, French, and english, how many students have taken a course in all three languages?

Solution:

Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and E the set of students who have taken a course in English. Then:

$$|A| = 400, |F| = 279, |E| = 114$$

$$|A \cap F| = 100, |A \cap E| = 23, |F \cap E| = 14$$

, and

$$|S \cup F \cup R| = 793.$$

When we insert these quantities into the equation

$$\begin{aligned} |A \cup F \cup E| &= |A| + |F| + |E| - |A \cap F| - |A \cap E| - |F \cap E| + |A \cap F \cap E| \\ 793 &= 400 + 279 + 114 - 100 - 23 - 14 + |S \cap F \cap R|. \end{aligned}$$

We now solve for $|A \cap F \cap E|$.We find that $|A \cap F \cap E| = 37$. Therefore, there are 37 students who have taken courses in Arabic, French, and English.

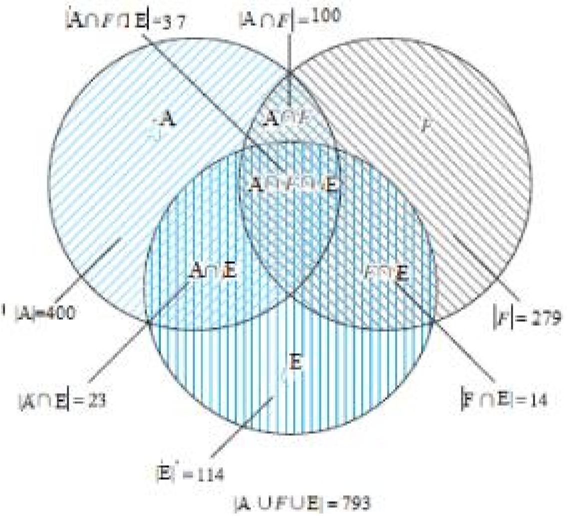

Figure 2.4: Delimitation explain the solution of the problem

Theorem 06 : THE PRINCIPLE OF INCLUSION-EXCLUSION

Let $A_1, A_2, ..., A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{1in} |Ai| - \sum_{1i<jn} |A_i \cap A_j|$$
$$+ \sum_{1\leq i<j<k\leq n} |A_i \cap A_j \cap A_k| - ... + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof:

We will prove the formula by showing that an element in the union is counted exactly once by the right-hand side of the equation. Suppose that a is a member of exactly r of the sets $A_1, A_2, ..., A_n$ where $1 \leq r \leq n$. This element is counted C(r. 1) times by$\sum |A_i|$. It is counted $\begin{pmatrix} r \\ 2 \end{pmatrix}$ times by $\sum |A_i \cap A_j|$. In general, it is counted C(r,m) times by the summation involving m of the setsb A_i . Thus , this element is counted exactly

$$\begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ 1 \end{pmatrix} \begin{pmatrix} r \\ 2 \end{pmatrix} + ... + (1)^{r+1} \begin{pmatrix} r \\ r \end{pmatrix}$$

times by the expression on the right-hand side of this equation. Ourgoal is to evaluate thisquantity.

$$\begin{pmatrix} r \\ 0 \end{pmatrix} - \begin{pmatrix} r \\ 1 \end{pmatrix} \begin{pmatrix} r \\ 2 \end{pmatrix} + ... + (1)^{r+1} = 0.$$

Hence,

$$\binom{r}{0} = \binom{r}{1} \binom{r}{2} + ... + (1)^{r+1} = 1.$$

Therefore, each element in the union is counted exactly once by the expression on the right-hand side of the equation. This proves the principle of inclusion - exclusion. The inclusion - exclusion principle gives a formula for the number of elements in the union of n sets for every positive integer n. There are terms in this formula for the number of elements in the intersection of every non empty subset of the collection of the n sets. Hence, there are $2^n - 1$ terms in this formula.

2.2 Applications of inclusion - exclusion principle

2.2.1 *The Number of Onto Functions*

The principle of inclusion-exclusion can also be used to determine the number of onto functions from a set with m elements to a set with n elements.

Example :

How many onto functions are there from a set with six elements to a set with three elements?

Solution:

Suppose that the elements in the codomain are $b_1, b_2, andb_3.LetP_1, P_2, andP_3$ be the properties that $b_1, b_2, andb_3$ are not in the range of the function, respectively. Note thata function is onto if and only if it has none of the properties P_1, P_2, orP_3. By the inclusion-exclusion principle it follows that the number of onto functions from a set with six elements to a set with three elements is

$$\begin{aligned} N(P_1P_2P_3) =& N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) \\ &+ N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3) \end{aligned}$$

where N is the total number of functions from a set with six elements to one with three elements. We will evaluate each of the terms on the right-hand side of this equation.

Note that $N(P_i)$ is the number of functions that do not have $b_i$in their range. Hence, there are two choices for the value of the function at each element of the domain. Therefore, $N(P_i) = 26$. Furthermore, there are C(3, 1) terms of this kind. Note that $N(P_iP_j)$ is the number of functions that do not have b_i andb_j in their range. Hence, there is only one choice for the value of the function at each element of the domain. Therefore, $N(P_iP_j) = 16$. Furthermore, there are $C(3, 2)$ terms of this kind. Also, note that $N(P_1P_2P_3) = 0$, because this term is the number of functions that have none of $b_1, b_2, andb_3$ in their range. Clearly, there are no such functions. Therefore, the number of onto functions from a set with six elements to one with three elements is

$$36 - \begin{pmatrix} 3 \\ 3 \end{pmatrix} 26 + \begin{pmatrix} 3 \\ 2 \end{pmatrix} 16 = 729 - 192 + 3 = 540.$$

The general result that tells us howmany onto functions there are from a set with m elements to one with n elements will now be stated.

Theorem 07

Let m and n be positive integers with $m \geq n$. Then, there are

$$n^m - \begin{pmatrix} n \\ 1 \end{pmatrix} (n-1)^m + \begin{pmatrix} n \\ 2 \end{pmatrix} (n-2)^m - .... + (-1)^{n-1} \begin{pmatrix} n \\ n-1 \end{pmatrix} 1^m$$

onto functions from a set with m elements to a set with n

Example :

How many ways are there to assign five different jobs to four different employees if every employee is assigned at least one job?

Solution:

Consider the assignment of jobs as a function from the set of five jobs to the set of four employees.
An assignment where every employee gets at least one job is the same as an onto function from the set of jobs to the set of employees. Hence, by Theorem 10 it follows that there are

$$45 - \begin{pmatrix} 4 \\ 4 \end{pmatrix} 3^5 + \begin{pmatrix} 4 \\ 2 \end{pmatrix} 2^5 - \begin{pmatrix} 4 \\ 3 \end{pmatrix} 1^5 = 1024 - 972 + 1924 = 240$$

ways to assign the jobs so that each employee is assigned at least one job.

2.2.2 Derangements

The principle of inclusion-exclusion will be used to count the permutations of n objects that leave no objects in their original positions.
Consider this example .

Examples

The Hatcheck Problem

1. A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats.
   When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat?

Remark:

The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of n hats.We will return to this example after we find the number of permutations of n objects that leave no objects in their original position.

Definition 4 *A derangement is a permutation of objects that leaves no object in its original position.*

To solve the problem posed in the Example we will need to determine the number of derangements of a set of n objects.
The permutation 21453 is a derangement of 12345 because no number is left in its originalposition. However, 21543 is not a derangement of 12345, because this permutation leaves 4 fixed.
Let D_n denote the number of derangements of nobjects. For instance,$D_3 = 2$, because the derangements of 123 are 231 and 312.We will evaluateD_n, for all positive integers n, using the principle of inclusion-exclusion

Theorem 6 *The number of derangements of a set with n elements is*

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n \frac{1}{n!}\right].$$

Proof:

Let a permutation have property P_i if it fixes element i. The number of derangements is the number of permutations having none of the properties $P_i for i = 1, 2, ..., n$. This means that

$$D_n = N(P_1 P_2 ... P_n).$$

Using the principle of inclusion-exclusion, it follows that

$$D_n = N \sum_i N(P_i) + \sum_{i<j} N(P_i P_j) - \sum_{i<j<k} N(P_i P_j P_k) + ... + (1)^n N(P_1 P_2 ... P_n)$$

, where N is the number of permutations of n elements. This equation states that the number of permutations that fix no elements equals the total number of permutations, less the number that fix at least one element, plus the number that fix at least two elements, less the number that fixat least three elements, and so on. All the quantities that occur on the right-hand side of this equation will now be found. First, note that $N = n!$, because N is simply the total number of permutations of n elements. Also,

$$N(P_i) = (n-1)!.$$

This follows from the product rule, because $N(P_i)$ is the number of permutations that fix element i, so the ith position of the permutation is determined, but each of the remaining positions can be filled arbitrarily. Similarly,

$$N(P_i P_j) = (n-2)!$$

, because this is the number of permutations that fix elements i and j , but where the other $n - 2$ elements can be arranged arbitrarily.
In general, note that

$$N(P_{i1} P_{i2} ... P_{im}) = (n-m)!$$

, because this is the number of permutations that fix elements $i1, i2, ..., im$, but where the other $n - m$ elements can be arranged arbitrarily. Because there are $C(n, m)$ ways to choose m . elements from n, it follows that

$$\sum_{1\leq i\leq n} N(P_i) = \left(\begin{array}{c} n \\ 1 \end{array}\right)(n-1)! - \sum_{1\leq i<j\leq n} N(P_iP_j) = \left(\begin{array}{c} n \\ 2 \end{array}\right)(n-2)!$$

, and in general,

$$\sum_{1\leq i1<i2<<im\leq n} N(P_{i1}P_{i2}...P_{im}) = \left(\begin{array}{c} n \\ m \end{array}\right)(n-m)!.$$

Consequently, inserting these quantities into our formula for D_n gives

$$\begin{aligned} D_n =& n! - \left(\begin{array}{c} n \\ 1 \end{array}\right)(n-1)! + \left(\begin{array}{c} n \\ 2 \end{array}\right)(n-2)! - ... + (-1)^n \left(\begin{array}{c} n \\ n \end{array}\right)(n-n)! \\ =& n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n2)! - ... + (-1)^n \frac{n!}{n!0!0!}. \end{aligned}$$

Simplifying this expression gives

$$D_n = n!\left[1\frac{1}{1!} + \frac{1}{2!} - ... + (-1)^n \frac{1}{n!}.\right]$$

[1]

2.2.3 *The Sieve of Eratosthenes*

By the realisation of the principle of inclusion-exclusion, we can find the number of primes not exceeding a specified positive integer with the same reasoning as is used in the sieve of Eratosthenes. Recall that a composite integer is divisible by a prime not exceeding its square root. So, to find the number of primes not exceeding 100, first note that composite integers not exceeding 100 must have a prime factor not exceeding 10. Because the only primes not exceeding 10 are 2, 3, 5, and 7, the primes not exceeding 100 are these four and those positive integers greater than 1 and not exceeding 100 that

---

[1] **HISTORICAL NOTE** In rencontres (matches), an old French card game, the 52 cards in a deck are laid out in a row. The cards of a second deck are laid out with one card of the second deck on top of each card of the first deck. The score is determined by counting the number of matching cards in the two decks.

In 1708 Pierre Raymond de Montmort (1678-1719) posed le problème de rencontres: What is the probability that no matches take place in the game of rencontres? The solution to Montmort's problem is the probability that a randomly selected permutation of 52 objects is a derangement, namely, $D_{52}/52!$, is approximately $1/e$.

are divisible by none of 2, 3, 5, or 7. To apply the principle of inclusion-exclusion, let P_1 be the property that an integer is divisible by 2, let P_2 be the property that an integer is divisible by 3, let P3 be the property that an integer is divisible by 5, and let P_4 be the property that an integer is divisible by 7. Thus, the number of primes not exceeding 100 is:

$$4 + N(P_1P_2P_3P)4)$$

Because there are 99 positive integers greater than 1 and not exceeding 100, the principle of inclusion-exclusion shows that :

$$\begin{aligned}
N(P_1P_2P_3P_4) =& 99 - N(P_1) - N(P_2) - N(P_3) - N(P_4) + N(P_1P_2) + N(P_1P_3) \\
& + N(P_1P_4) + N(P_2P_3) + N(P_2P_4) + N(P_3P_4) - N(P_1P_2P_3) - N(P_1P_2P_4) \\
& - N(P_1P_3P_4) - N(P_2P_3P_4) + N(P_1P_2P_3P_4).
\end{aligned}$$

The number of integers not exceeding 100 (and greater than 1) that are divisible by all the primes in a subset of 2, 3, 5, 7 is 100/N, where N is the product of the primes in this subset. (This follows because any two of these primes have no common factor.) Consequently,

$$\begin{aligned}
N(P_1P_2P_3P_4) =& 99 - \left[\frac{100}{2}\right] - \left[\frac{100}{3}\right] - \left[\frac{100}{5}\right] - \left[\frac{100}{7}\right] \\
& + \left[\frac{100}{2.3}\right] + \left[\frac{100}{2.5}\right] + \left[\frac{100}{2.7}\right] \\
& + \left[\frac{100}{3.5}\right] + \left[\frac{100}{3.7}\right] + \left[\frac{100}{5.7}\right] \\
& - \left[\frac{100}{2.3.5}\right] - \left[\frac{100}{2.3.7}\right] - \left[\frac{100}{2.5.7}\right] - \left[\frac{100}{3.5.7}\right] + \left[\frac{100}{2.3.5.7}\right] \\
=& 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0 \\
=& 21.
\end{aligned}$$

Hence, there are$4 + 21 = 25$ primes not exceeding 100

Chapter 3

The Invariance principle

The principle of invariance is a simple idea in combinatorics .

3.1 Definition

An invariant is a parameter that remains constant under certain operation orindependent choices .
We well explain this concepts by this way :

1. Suppose we have a "pricess" that involve repeted "moves" .Then could be several diffrent possible " moves" .
2. If "something" deos'nt change after every possible " moves" then this something will nerver changeafter sequence of "moves " . This "something "could be any "property "or "value" , and we call it an invariant [1]

Remark

Invariant may not be obvious .But if there is somekind of reoetition we look for what deos'nt change .
We start the invariance principles examples by a simple one :
if $r$is the remaider when n is devuded by$m$then we say "n congruent to r modulo m"and we write

$$"n \equiv r (mod) m"$$

---

[1] the word " invariant " in mathematics can be either a" noun "or "adjective"

We well illustrate the principle by simple examples

Examples

1. An urn contains exacty 2010 white and 2011 black balls . In addition to that ,we have an unlimited supply of black balle .Now we repeatdly draw two balls randomly from the urn .If their colours is tve same , then we remoce them end add a black ball to the urn . If their colours is differnt , we returnthe white ball and remove the blackball tp the urn .

   This is repeated until there is only one ball left .Determine the probability that the last ball is white.

   Solution

   In spite of the formalisation of the problem ,we do actually have to compute any probabilities . Indeed we will prove that the ,last ball has to be black . Notice first that the process always comes to an end since the number of balls in the urn always decreasesby 1.

   Let us now consider the number of white balland prove that it is always even .In the begining ,it is an even number , namely 2010.If the two balls that we draw are both white , the number of white baiis decreases by 2, other wise it remains the same . Hence it stays even throughout the pricess .

   At the end , the number to arrange the numer of ballshas to be even . But there is only one ball left , this means that this number has to be 0, which prpves our assertion.

2. we cut a paper to three parts then we cut every parts ti three parts then we complet by this way .Is possiblie to find 100small papers?
   answer:the qiestion is

$$3^1 + 3^2 + 3^3 + 3^4 .... + 3^n = 100?$$

the invariance coefficiant is 3^n

we cannot to write 100in the form of 3^n

so we cannot to find 100small papers

Conclusion

In this research we talken just once over about some elementary principles of combinatorics which is verry importent in the discrete mathematics because it is the principle basic and the real start of the other principles of this branche which help us to solve many importent problem ,for example the DNA exam ....
As a result the cmbinatorics give us an importent help in the technique and scientifique evolution.

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ملخص

ان الهدف من هذا البحث هو شرح وتبسيط بعض المبادئ الأولية للاحتمالات والتي تمثل بدورها الانطلاقة الحقيقية لهذا الفرع الرياضياتي و الذي يتطور بشكل مستمر

Resumé

Le but de cette mémoire est l’explication et le simplification de quelques principes elementaires de combinatoire qui sont present le debut actuel de ce branche de mathematique qui est developpé ce forme continue

Abstract

The objectif of this research is to explain of some elimentaries principles of combinatorics which is present the real start of this branche that is developing in continus forme