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On the range of unbounded linear operators

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Boudina khadidja

Contents

List of symbols

$\langle . \rangle$	Inner product.
H	Hilbert space.
$B(H)$	The set of all bounded linear operator defined on a Hilbert space H.
T	A linear operator defined on a Hilbert space H.
$D(T)$	The domain of T.
$\overline{T}$	The closure of the closable operator T.
I	The identity operator .
$\|T\|$	The norm of T.
$\ker(T)$	The kernel of T.
$Im(T)$	The image of T.
$G(T)$	The graph of T.
T^*	The adjoint operator of T.
$\sigma(T)$	The spectrum of T.

$\rho(T)$	The resolvent set of T.
T^{-1}	The inverse of T.
U	Partial isometry defined on a Hilbert space H.
$\|T\|$	The modulus of T.

Introduction

The study of the range of unbounded linear operators has been a central theme in functional analysis since the early development of operator theory, particularly in the context of Hilbert spaces. These operators naturally arise in diverse areas such as quantum mechanics, differential equations, and spectral theory, where the absence of boundedness necessitates a deeper understanding of their domain, adjoint, and range properties. For further details and additional remarks, we refer to [1, 2, 3, 5, 10].

This thesis is devoted to the detailed analysis of the range of unbounded linear operators, with particular emphasis on structural properties, closedness conditions, and identities involving their range.

We begin by recalling fundamental notions related to linear operators, focusing on unbounded operators and essential tools such as adjoint operators, self-adjointness, and basic spectral concepts. These foundational elements are indispensable for the theoretical developments presented throughout this work.

Subsequently, we investigate conditions that guarantee the closedness of the range of linear operators. In this context, we discuss both classical and contemporary results, analyze the interplay between the closedness of an operator's range and that of its adjoint, and provide illustrative examples and counterexamples to elucidate the theoretical framework.

The final part of the thesis is devoted to specific results and identities concerning the range of unbounded linear operators. In particular, we examine the behavior of fractional powers of positive self-adjoint operators and their implications for the structure and closedness of the range.

The primary objective of this research is to contribute to a deeper understanding of the behavior of the range of unbounded operators, providing a synthesis of theoretical results and practical examples that may serve as a basis for further investigations in this area.

Chapter 1

Essential background

The aim of this chapter is to introduce fundamental definitions and essential properties related to closed linear operators, which will be employed in the coming chapters.

1.1 Unbounded linear operators

Let H be a complex Hilbert space with an inner product noted $\langle \cdot, \cdot \rangle$. Let T be a linear operator defined on a $D(T) \subseteq H$. The subspace $D(T)$ is called the domain of T. The operator T is said to be unbounded if it is not bounded, meaning there does not exist a constant $C > 0$ such that

$$\|Tx\| \leq C\|x\|, \quad \forall x \in D(T).$$

We say that T is a restriction of a linear operator $T^{'}$ defined on $D(T^{'})$, or T is an extension of $T^{'}$, if $D(T) \subseteq D(T^{'})$ and $T(x) = T^{'}(x)$ for all $x \in D(T)$, and we write $T \subseteq T^{'}$. It is not assumed that T is bounded. However, if T is bounded, then it becomes the restriction of a bounded linear operator defined on the whole space H.

1.1.1 Definitions and properties

Definition 1.1.1. *Let T be a linear operator defined on H. The image or the range of T is defined by:*

$$ran(T) = \{T(x) \in H \text{ with } x \in D(T)\} = T(D(T)).$$

The kernel of T or the null space of T is defined as follows:

$$\ker(T) = \{x \in D(T) : T(x) = 0\}.$$

Example 1.1.2. *Let T be the restriction of the identity operator I, with $D(T) = I$. Then* $\ker(T) = \{0\}$ *and* $ran(T) = \{D(T)\}$.

Remark 1.1.3. *The linear operator T is injective if and only if* $\ker(T) = \{0\}$.

Definition 1.1.4. *Let S and T be two linear operators defined on H, with domains $D(S)$ and $D(T)$ respectively. For $\alpha \in \mathbb{C}$, The operator αT is defined by:*

$$(\alpha T)\, x = \alpha\, (Tx) \text{ for all } x \in D\,(\alpha T) = D(T).$$

The operators $S + T$ an ST are defined on their domains:

$$D\,(S + T) = D\,(S) \cap D\,(T)\,,$$

and

$$D\,(ST) = \{x \in D\,(T)\,, \text{ such that } Tx \in D\,(S)\,.\}$$

Example 1.1.5. *For a measurable subset $M \subseteq \mathbb{R}^n$, we define the operator T on $L^2\,(M)$ by $Tf = xf$ where $x \in M$. Then, T is defined on a dense subspace of $L^2\,(M)$ as follows:*

$$D\,(T) = \left\{f \in L^2\,(M)\,, \text{ such that } xf \in L^2\,(M)\right\}.$$

Moreover,

$$D\left(T^2\right) = \left\{f \in L^2\,(M)\,, \text{ such that } Tf \in D(T)\right\},$$

and

$$D\,(\lambda T) = D\,(T + I) = D\,(T)\,,$$

where I is the identity operator and $\lambda \in \mathbb{C}$.

Proposition 1.1.6 ([9])**.** *Let R, S and T be a linear operators defined on H. Then*

1. $RT + ST \subseteq (R + S)T$.
2. $(R + S)T = RT + ST$.

1.2 Closed and closable operators

In this section, we present the definition and the properties of closed operators.

Definition 1.2.1. *For a linear operator T defined on H, we define The graph of T as follows:*

$$G(T) = \{(x, Tx)\,,\ \text{with } x \in D(T)\}\,.$$

Remark 1.2.2. *Clearly, $G(T)$ is a subspace of $H \times H$.*

Definition 1.2.3. *Let T be a linear operator defined on H. T is said to be closed if its graph $G(T)$ is closed subspace of $H \times H$. We say that T is closable if there exists a closed operator S such that $T \subseteq S$, or equivalently $G(T) \subseteq G(S)$.*

Proposition 1.2.4 ([9])**.** *Let T be a linear operator defined on H. Then*

1. *T is closable if and only if $\overline{G(T)}$ is a graph.*
2. *If $D(T) = H$, then T is closed if and only if T is bounded.*

Let T be a closable operator, the only operator $\overline{T}$ who satisfies $G(\overline{T}) = \overline{G(T)}$ is called the closure of T.

Example 1.2.5. *Let S be a bounded operator with $D(T) = H$. If T is a densely defined restriction of S, then $\overline{T} = S$.*

In the next theorem, we will outline both the sufficient and necessary conditions that guarantee the closedness of T.

Theorem 1.2.6 ([9])**.** *Let T be a linear operator with domain $D(T) \subseteq H$. We have:*

1. *T is closed $\Leftrightarrow$ For any sequence $(x_n)_{n\in\mathbb{N}}$ in $D(T)$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} T(x_n) = y$, then $x \in D(T)$ and $Tx = y$.*

2. *$(D(T), \| \,.\, \|_T)$ is a Banach space, with*

$$\| x \|_T = \| x \| + \| Tx \| \, .$$

3. *T is closable $\Leftrightarrow$ For any sequence $(x_n)_{n\in\mathbb{N}}$ in $D(T)$ such that $\lim\limits_{n\to\infty} x_n = 0$ and $\lim\limits_{n\to\infty} T(x_n) = y$, then we obtain $y = 0$.*

4. *The domain of $\overline{T}$ is defined by:*

$$D(\overline{T}) = \{x \in H : \textit{there exists a sequence}\,(x_n)\ \textit{in}\ D(T)\ \textit{such that}\ \lim x_n = x\ \textit{and}$$

$$\lim_{n\to\infty} Tx_n = y\}.$$

In the next result, we will show the relation between the boundedness and the closedness of linear operator.

Theorem 1.2.7 ([8]). *Let T be a linear operator with domain $D(T) \subseteq H$. The following statements are equivalent:*

1. *$D(T)$ is a closed subspace of H and T is closed.*

2. *$D(T)$ is a closed subspace of H and T is bounded.*

3. *T is bounded and closed.*

Definition 1.2.8. *Let T ba a linear operator. T is said to be invertible if there exists an everywhere defined liner operator T^{-1} such that*

$$TT^{-1} = I \quad \textit{and} \quad T^{-1}T \subseteq I,$$

where I is the identity operator on H.

If T is closed and invertible, then T^{-1} must be bounded. Moreover, if T is closed then T is invertible if and only if T is injective and surjective.

Theorem 1.2.9 ([8]). *Let T be a linear operator with domain $D(T) \subseteq H$. The following statements hold to be true.*

1. *If T is closed linear operator, then* $\ker(T)$*is closed.*
2. *If there exists a positive number k and a complex number λ such that*

$$\|(T-\lambda I)x\| \geq k||x|| \quad \textit{for all } x \in D(T)$$

then, $ran(T-\lambda I)$ is closed.

Example 1.2.10. *Let $\{\alpha_n\}_{n\in\mathbb{N}} \in \ell^2$. We set*

$$D(T) = \left\{ \{x_n\}_{n\in\mathbb{N}} \in \ell^2 : \sum_{i=1}^{\infty} |\alpha_i x_i|^2 < \infty \right\} \quad \textit{and} \quad T(\{x_n\}_{n\in\mathbb{N}}) = \{\alpha_n x_n\}_{n\in\mathbb{N}}$$

then, T is closed and $D(T) = H$ if and only if $\{\alpha_n\}_{n\in\mathbb{N}}$ is bounded.

1.3 Adjoint operators

Let T be a densely defined linear operator with domain $D(T) \subseteq H$. We define the linear operator T^* as the operator whose domain consists of all $y \in H$ such that $x \to \langle Tx, y\rangle$ is continuous on $D(T)$. Riesz' theorem guarantees that

$$D(T^*) = \{y \in H : \text{ there exists } z \in H \text{ such that } \langle Tx, y\rangle = \langle x, z\rangle \text{ for all } x \in D(T)\}.$$

Since $D(T)$ is dense, z is unique, indeed

$$\langle x, z\rangle = \left\langle x, z^{'} \right\rangle$$

then $z - z^{'} \in (D(T))^{\perp} = \{0\}$ for all $x \in D(T)$ and so $z = z^{'}$. By setting $T^*y = z$,we establish a well-defined linear operator T^* defined on $D(T^*)$ called the adjoint of T.

Example 1.3.1. *Let T be the restriction of the identity operator on a dense domain of H. Then, $D(T^*) = H$ and $T^* = I$.*

Proposition 1.3.2. *If T is a densely defined linear operator with domain $D(T) \subseteq H$, then $ker(T^*) = (ran(T))^{\perp}$.*

Proof. Let $y \in D(T^*)$, We have

$$\langle Tx, y\rangle = \langle x, T^* y\rangle,$$

for all $x \in D(T)$. Thus $y \in ker(T^*)$ if and only if $y \in (ran(T))^{\perp}$. $\square$

Now, let's explore some fundamental characteristics of adjoint operators.

Theorem 1.3.3 ([9])**.** *Let T and S be two densely defined linear operators with domains $D(T)$ and $D(T)$. Then*

1. $(\lambda T)^* = \overline{\lambda} T^*$.
2. *T^* is closed.*
3. *T is closable if and only if $D(T^*)$ is dense in H. Furthermore, $\overline{T} = T^{**}$ and $(\overline{T})^* = T^*$.*
4. *T is closed if and only if $T = T^{**}$.*
5. *if $T \subseteq S$ then $S^* \subseteq T^*$.*
6. *if $T + S$ has a dense domain, then $T^* + S^* \subseteq (T + S)^*$.*
7. $T^* S^* \subseteq (TS)^*$.
8. *If S is bounded, then $(T + S)^* = T^* + S^*$ and $(TS)^* = T^* S^*$.*
9. *If T is invertible then T^* is also invertible. Moreover, $(T^*)^{-1} = (T^*)^{-1}$.*

1.4 Resolvent and Spectrum of linear Operators

In this section, we define the resolvent and the spectrum of a closed linear operator T accompanying with their properties .

Definition 1.4.1. *Let T be a linear operator. The resolvent set $\rho(T)$ is the set of all complex numbers λ such that $(T - \lambda I)$ is boundedly invertible. The spectrum of T, denoted by $\sigma(T)$, is defined by: $\sigma(T) = \mathbb{C} \backslash \rho(T)$.*

Remark 1.4.2. *If T is not closed then $\sigma(T) = \mathbb{C}$. Indeed, if $\sigma(T) \neq \mathbb{C}$, then there exists a complex $\lambda \in \rho(T)$ such that $(T - \lambda I)$ is boundedly invertible and so $(T - \lambda I)$ is closed, hence T is closed which is impossible, therefore $\sigma(T) = \mathbb{C}$.*

Proposition 1.4.3. *Let T be a closable operator. If T is invertible, then $\sigma(\overline{T}) \neq \mathbb{C}$, and $(\overline{T})^{-1} = \overline{T^{-1}}$.*

Theorem 1.4.4. *If T is a closed densely defined linear operator. Then*

$$\sigma(T^*) = \{\lambda \in \mathbb{C} \textit{ such that } \overline{\lambda} \in \sigma(T)\}$$

and

$$\rho(T^*) = \{\lambda \in \mathbb{C} \textit{ such that } \overline{\lambda} \in \rho(T)\}.$$

Proof. Let $\overline{\lambda} \in \rho(T)$, then $(T - \overline{\lambda} I)$ is invertible. Theorem 1.4.6 ensures that $(T^* - \lambda I)$ is also invertible. Hence $\lambda \in \rho(T^*)$. ☐

Definition 1.4.5. *The set of eigenvalues $\sigma_p(T)$ consists of all complex numbers λ such that $Tx = \lambda x$ for some non-null $x \in D(T)$.*

Theorem 1.4.6 ([9])**.** *Let T be a closed densely defined linear operator. Then*

1. *$\sigma_p(T) \subseteq \sigma(T)$.*
2. *$\sigma(T)$ is a closed subset of $\mathbb{C}$, and if $D(T) = H$, then $\sigma(T)$ is a compact subset of $\mathbb{C}$.*

1.5 Unbounded self-adjoint operators

In this section, we will define and study the symmetric, and self-adjoint operators.

Definition 1.5.1. *Let T be a densely defined linear operator with domain $D(T) \subseteq H$. We say that T is symmetric if $T \subseteq T^*$.*

Remark 1.5.2. *An operator T is said to be symmetric if and only if*

$$\langle Tx, y\rangle = \langle x, Ty\rangle \textit{ for all } x, y \in D(T).$$

In the next result, we will give a characterization of symmetric operators.

Theorem 1.5.3. *Let* T *be a densely defined linear operator with domain* $D(T) \subseteq H$. *Then*

1. T *is symmetric if and only if* $\langle Tx, x\rangle$ *is a real for all* $x \in D(T)$.
2. *The eigenvalues of* T *are real.*
3. *Eigenvectors belonging to different eigenvalues of* T *are mutually orthogonal.*
4. *If* $\langle Tx, x\rangle \geqslant 0$ *for all* $x \in D(T)$, *and if* $\langle Tx, x\rangle = 0$ *for some* $x \in D(T)$, *then* $x \in \ker T$.

Proof.

1. We have,
   $$\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}.$$
   Hence, $\langle Tx, x\rangle$ must be real. The second implication follows immediately from the polarization formula.
2. If λ is an eigenvalue, then there exists an non null x such that $\langle Tx, x\rangle = \lambda\|x\|^2$, and since $\langle Tx, x\rangle$ is real, then λ is real.
3. Let $x \in \ker(T - \lambda I)$ and $y \in \ker(T - \mu I)$, where $\lambda \neq \mu$. We have,
   $$\mu\langle x, y\rangle = \langle x, Ty\rangle = \langle Tx, y\rangle = \lambda\langle x, y\rangle,$$
   and hence $\langle x, y\rangle = 0$.
4. Since $\langle Tx, x\rangle \geqslant 0$, the Cauchy–Schwartz inequality guarantees that
   $$|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle\langle Ty, y\rangle = 0,$$
   for all $y \in D(T)$. Since $D(T)$ is dense, then $Tx = 0$.

□

Definition 1.5.4. *Let* T *be a closed densely defined linear operator. We say that* T *is self-adjoint if* $T = T^*$.

Symmetric operators are self-adjoint but the opposite is not true in general. For example, let T be the restriction of the identity operator on some dense domain, then $T \subsetneq T^* = I_H$.

Definition 1.5.5. *Let T be a closable symmetric operator. T is said to be essentially self-adjoint if $\overline{T} = \overline{T}^*$.*

Proposition 1.5.6. *T is essentially self-adjoint if and only if T^* is self-adjoint.*

Proof. If T is essentially self-adjoint, then

$$T^* = (\overline{T})^* = \overline{T} = T^{**},$$

thus T^* is self-adjoint. Conversely, if T^* is self-adjoint, then

$$\overline{T}^* = T^* = T^{**} = \overline{T}$$

and hence, T is essentially self-adjoint. ☐

Next, we provide some properties of self-adjoint operators.

Theorem 1.5.7 ([9]). *Let T be a densely defined linear operator with domain $D(T) \subseteq H$. Then*

1. *If T is self-adjoint then $\sigma\{T\} \subseteq \mathbb{R}$.*
2. *If T is a closable symmetric operator, and if there exists a real number λ such that*

   $$H = \ker(T - \lambda I) + Im(T - \lambda I),$$

   then T must be self-adjoint.
3. *If T is a symmetric operator, and if there exists a complex number λ such that $ran(T - \lambda I) = \overline{ran(T - \overline{\lambda} I)} = H$, then T must be self-adjoint.*
4. *T is self-adjoint if and only if T is symmetric, and $\sigma(T) \subseteq \mathbb{R}$.*
5. *If T is self-adjoint, then $\lambda \in \sigma_p(T)$ if and only if $\overline{ran(T)} \neq H$.*
6. *If T is self-adjoint. Then, there exists a unique spectral measure E such that*

   $$T = \int_{\mathbb{R}} \lambda dE_T(\lambda).$$

1.5.1 Polar decomposition of a closed linear operators

Definition 1.5.8. *Let T be a densely defined linear operator with domain $D(T) \subseteq H$. We say that T is positive if and only if $\langle Tx, x\rangle \geq 0$.*

Unbounded positive operators are not necessarily self-adjoint, whereas bounded positive operators must be self-adjoint.

Definition 1.5.9. *A bounded linear operator U is said to be a partial isometry, if there exists a closed subspace K such that $||Ux|| = ||x||$ for all $x \in K$, and $Ux = 0$ for all $x \in K^{\perp}$.*

Theorem 1.5.10 ([5])**.** *Let T be a closed densely defined linear operator with domain $D(T) \subseteq H$. Then there is a partial isometry U, such that*

$$T = U\,|T|\,, \tag{1.1}$$

where $|T|$ is an unbounded self-adjoint operator called the modulus of T, and satisfies

1. *$|T|^2 = T^*T$.*
2. *$D(T) = D(|T|)$ and $\|Tx\| = \||T|x\|$, for all $x \in D(T)$.*

The decomposition (1.1) is known as the polar decomposition of T.

Theorem 1.5.11 ([9])**.** *Let T be a closed densely defined linear operator with domain $D(T) \subseteq H$. Then*

1. *$ran(T) = ran(|T|)$.*
2. *T^*T is a positive self-adjoint operator.*
3. *$T^*T \subseteq \mathbb{R}^+$.*

1.6 Unbounded normal operators

In this section, we will introduce and investigate the class of non self-adjoint operators.

Definition 1.6.1. *Let T be a densely defined linear operator with domain $D(T) \subseteq H$. We say that T is normal if*

$$D(T) = D(T^*) \quad \textit{and} \quad ||Tx|| = ||T^*x||, \quad \textit{for all } x \in D(T).$$

Clearly, self-adjoint operators are normal, the converse is not true in general. Moreover, there is not any relation between normal operators and symmetric operators.

Next, we present some properties of normal operators.

Theorem 1.6.2 ([9])**.** *Let T be a densely defined linear operator. T is normal if and only if T is closed and $T^*T = TT^*$.*

Theorem 1.6.3 ([9])**.** *Let T be a densely defined normal operator with domain $D(T) \subseteq H$. Then*

1. *I T is maximal normal (i.e., if there exists a normal operator S such that $T \subseteq S$, then $T = S$).*
2. *T^* is normal.*
3. *$T + \lambda I$ is normal for all complex number λ.*
4. *Then the polar decomposition of T is given by $T = U\,|T|$, where U is unitary and U commutes with T.*
5. *$H = \overline{ran(T - \lambda I)} \oplus \ker(T - \lambda I)$, for all complex numbers λ.*
6. *If $\lambda \in \sigma_p(T)$, then $\overline{\lambda} \in \sigma_p(T^*)$.*
7. *Then there exists a unique spectral measure E such that*

$$T = \int_{\mathbb{C}} \lambda dE_T(\lambda).$$

8. *If* $\sigma(T) \subseteq \mathbb{R}$*, then* T *is self-adjoint.*

9. $\rho(T) = \{\lambda \in \mathbb{C},\ \textit{such that, } ran(T - \lambda I) = H\}$.

10. $\sigma_p(T) = \{\lambda \in \mathbb{C},\ \textit{such that, } \overline{ran(T - \lambda I)} \neq H\}$.

The next theorem is known as Fuglede-Putnam theorem and it plays an essential role on our work.

Theorem 1.6.4 ([9])**.** *Let* S *be a bounded linear operator and* N *and* M *are not necessarily bounded normal operators, then*

$$SN \subset MS \Rightarrow SN^* \subset M^*S.$$

1.6.1 Unbounded hyponormal operators

Definition 1.6.5. *Let* T *be a densely defined linear operator with domain* $D(T) \subseteq H$*. We say that* T *is hyponormal if*

$$D(T) \subset D(T^*) \textit{ and } \|T^*x\| \leq \|Tx\|, \quad \forall x \in D(T).$$

Remark 1.6.6. *Clearly, normal and self-adjoint operators are hyponormal operators.*

Theorem 1.6.7 ([5])**.** *Let* T *be a densely defined hyponormal operator with domain* $D(T) \subseteq H$*. Then*

1. *Symmetric operators are hyponormal.*

2. $\ker(T) \subseteq \ker(T^*)$.

3. $T + \lambda I$ *is hyponormal for all complex number* λ*.*

4. *There exists a contraction operator* K *(i.e.* $\|K\| \leq 1$*), such that*

$$T \subseteq T^*K \quad \textit{and} \quad \ker(K) \subseteq \ker(T^*).$$

5. $ran(T) \subseteq ran(T^*)$.

6. *If $\sigma(T)$ is real, then T must be self-adjoint.*

7. *Symmetric operators are hyponormal.*

Remark 1.6.8. *If T is a densely defined normal operator, then $ran(T) = ran(T^*)$.*

Chapter 2

Conditions implying the closedness of the range of linear operators

In this chapter, we establish the closedness of the range of linear operators under certain conditions. Furthermore, we investigate the additive properties of operator ranges and examine the ranges of unbounded positive self-adjoint operators.

2.1 Conditions implying the closedness of the range of bounded linear operators

In this section, we provide some extra conditions on a bounded linear operator T to ensure the closedness of its range.

Theorem 2.1.1 ([3]). *Let* $T \in B(H)$ *such that* $\ker(T) \subseteq \ker(T^*)$. *If there exists* $n \in \mathbb{N}$, *such that* $n \geq 2$, $ran(T) = ran(T^n)$, *then* $ran(T) = ran(T^*)$ *is closed and* $\ker(T) = \ker(T^*)$.

Proof. For $x \in [\ker(T)]^\perp$. There exists $y \in H$ such that $Tx = T^n y$ and so $x - T^{n-1}y \in \ker(T)$.

For any $z \in \ker(T) \subseteq \ker(T^*) \subseteq \ker(T^{*(n-1)})$, we have

$$\left\langle T^{n-1}y, z \right\rangle = \left\langle y, T^{*(n-1)}z \right\rangle = 0.$$

Hence, $T^{n-1}y \in [\ker(T)]^{\perp}$, and hence $x - T^{n-1}y \in [\ker(T)]^{\perp}$. Thus, we obtain

$$x - T^{n-1}y \in \ker(T) \cap \ker(T)^{\perp}.$$

Thus, $x = T^{n-1}y$ and

$$[\ker(T^*)]^{\perp} \subseteq [\ker(T)]^{\perp} \subseteq ran(T^{n-1}) \subseteq ran(T) \subseteq \ker(T^*)^{\perp},$$

So, $\overline{ran(T)} = ran(T)$. Moreover, $ran(T^*)$ is closed and $\ker(T) = \ker(T^*)$ and hence $ran(T) = ran(T^*)$. $\square$

The following result is evident.

Corollary 2.1.2. *Let $T \in B(H)$. If $\lambda \notin \sigma_p(T)$, then*

$$\lambda \notin \sigma(T) \Longleftrightarrow \exists n \in \mathbb{N} : ran(T - \lambda I) = ran[(T - \lambda I)^n].$$

Since a hyponormal operator T satisfies $\ker(T) \subseteq \ker(T^*)$, the ensuing result is also clear.

Corollary 2.1.3. *Let T be a bounded hyponormal operator. If, for some $n \in \mathbb{N}$, $ran(T) = ran(T^n)$, then $ran(T)$ is closed.*

Obviously, the preceding corollary is true for normal operators. At the same time, it is known that there exists non-normal bounded operator T such that T^n is normal for a certain n (it suffices to assume that T is a bounded nilpotent non-zero matrix). The following result becomes therefore interesting.

Corollary 2.1.4. *Let $T \in B(H)$. Suppose $ran(T) = ran(T^n)$ and T^n is normal for some $n \in \mathbb{N}$, then $ran(T)$ must be closed.*

Proof. Since T^n is normal, then

$$ran(T^n) = ran[(T^n)^*] = ran(T^{*n}) \subseteq ran(T^*).$$

Hence $ranT \subseteq ran(T^*)$ or $\ker T \subseteq \ker(T^*)$. By Theorem 2.1.1, we obtain the closedness of $ran(T)$. $\square$

A bounded linear operator T is said to be posinormal if $TT^* = T^*PT$ for some positive bounded linear operator P. This class was introduced by H. C. Rhaly in [7]. $T \in B(H)$ is posinormal if and only if $ran(T) \subseteq ran(T^*)$. The following theorem deals with bounded posinormal operators.

Theorem 2.1.5 ([3])**.** *Let $T \in B(H)$ such that $\ker(T) \subseteq \ker(T^*)$. If $ran(T^m) = ran(T^n)$ for some $m, n \in \mathbb{N}$. Then for all $k, l \in \mathbb{N}$, we have*

1. *$ran(T^k)$ and $ran(T^{*l})$ are closed.*
2. *$ran(T^k) = ran(T^{*l})$ and $\ker(T^k) = \ker(A^{*l})$.*
3. *T^k and T^{*l} are both posinormal.*

Proof. Since $\ker(T) \subseteq \ker(T^*)$, then $\ker(T) = \ker(T^2)$. From Lemma 2 in [6], it follows that

$$ran(T) = ran(T^k), \quad k \in \mathbb{N}.$$

By applying Theorem 2.1.1, we obtain that $ran(T^k)$ is closed for all $k \in \mathbb{N}$. Moreover, $ran(T) = ran(T^*)$ and hence T and T^* are posinormal. Again by Lemma 2 in [6], we have

$$ran(T^*) = ran(T^{*l}), \quad \text{for all} l \in \mathbb{N}.$$

Hence

$$ran(T^k) = ran(T^{*l})$$

and

$$\ker(T^k) = \ker(T^{*l})$$

for all $k, l \in \mathbb{N}$. This implies the posinormality of all T^k and T^{*l}. □

The following example shows the importance of the condition $\ker(T) \subseteq \ker(T^*)$ and that cannot be replaced by the condition $\ker(A^m) \subseteq \ker(A^{*m})$ for all $m \in \mathbb{N}$, in the above theorem.

Example 2.1.6. *There exists a bounded linear operator* T *such that* $\ker(T^m) \subseteq \ker(T^{*m})$ *and ran*$(T^m) =$ *ran*(T^n) *is closed for all non-null natural numbers* m *and* n *and ran*(T) *is not closed. Indeed, Let* $S \in B(H)$ *such that ran*(S) *is unclosed, then let*

$$T = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.$$

Thus, $T^m = 0_{H\oplus H}$ *for all* m*. It is easy to check that ran*(T) *is unclosed while all of* T^m*, with* $m > 1$*, have a closed range.*

Remark 2.1.7. *There exists a bounded linear operator* T *such that* $\ker(T) \subseteq \ker(T^*)$ *and ran*(T^k) *is closed for all natural numbers* k*, but ran*$(T^m) \neq$ *ran*(T^n) *whenever* $m \neq n$*. To this end, it suffices to consider the usual shift operator on* ℓ^2*.*

Corollary 2.1.8. *Let* $T \in B(H)$ *such that* T^k *injective for some non-null natural number* k*. If ran*$(T^m) =$ *ran*(T^n) *with* $m, n \in \mathbb{N}$ *and* $m < n$*, then* T *must be invertible.*

Proof. Since T^k is injective, then T^{k+1} is also injective. Lemma 2 in [6], ensures that A^m is also injective. From Theorem 2.1.5, deduce that T^m is invertible, thereby T is invertible. $\square$

2.2 Additivity properties of operator ranges

Recall that the spectral theorem allows us to define the powers of an unbounded self-adjoint positive operator T with domain $D(T)$, as follows

$$\langle T^\alpha x, x\rangle = \int_0^\infty \lambda^\alpha d\langle E(\lambda)x, x\rangle$$

where $\alpha > 0$ is real and E stands for the unique spectral measure associated with T.

In the next result, we present some properties of unbounded self-adjoint positive operators.

Theorem 2.2.1 ([3])**.** *Let* T *be an unbounded self-adjoint positive operator with domain* $D(T) \subset H$ *and let* $\alpha, \beta; \gamma, \delta \in]0, \infty[$*. Then*

1. $D(T^\gamma) \subset D(T^\delta)$ *for all* $\gamma > \delta$*. Moreover,* $D(T^\gamma) = D(T^\delta)$ *if and only if* $T \in B(H)$*.*

2. $T^\alpha T^\beta = T^{\alpha+\beta}$.

3. $(T^\alpha)^\beta = T^{\alpha\beta}$.

4. $\ker(T^\alpha) = \ker T$ *for all* $\alpha > 0$.

The following theorem is known, but the proof we present below is fairly simple.

Theorem 2.2.2. *Let T be a closed densely defined linear operator with domain $D(T)$. The following statements are equivalent:*

1. *$ran(TT^*)$ is closed.*

2. *$ran(T)$ is closed.*

3. *$ran(T^*)$ is closed.*

4. *$ran(T^*T)$ is closed.*

Proof. Since $[ran(T)]^\perp = \ker T^* = \ker(TT^*) = [ran(TT^*)]^\perp$, then $\overline{ran(T)} = \overline{ran(TT^*)}$.

1. "(1) $\Rightarrow$ (2)". We have

$$\overline{ran(T)} = \overline{ran(TT^*)} = \overline{ran(|T^*|^2)} = ran(|T^*|^2) \subseteq ran(|T^*|) = ran(T).$$

2. The equivalence "(2) $\Leftrightarrow$ (3)" is already known by the general theory.

3. "(2) $\Rightarrow$ (1)". We solely show that $ranT \subseteq ran(TT^*)$. Since $ran(T)$ is closed, $ran(T^*)$ is closed and so

$$H = \ker(T) \oplus ran(T^*).$$

   For $x \in D(T)$. We have $x = x_1 + T^*x_2$ with $x_1 \in \ker(T)$ and $x_2 \in D(T^*)$. Therefore

$$Tx = TT^*x_2$$

   which implies that that $ranT \subseteq ran(TT^*)$ and hence $ranT = ran(TT^*)$.

4. The equivalence "(3) $\Leftrightarrow$ (4)" holds by replacing T by T^* in the equivalence "(1) $\Leftrightarrow$ (2)".

$\square$

Corollary 2.2.3. *Let T be an unbounded self-adjoint operator and let $n \in \mathbb{N}$. Then $ran(T)$ is closed if and only if $ran(T^{2^n})$ is closed.*

Proposition 2.2.4. *Let T be a closed densely defined linear operator and let $S \in B(H)$. The following statements are equivalent:*

1. $\overline{ran(T^*)} \oplus \overline{ran(S^*)}$ *is closed.*
2. $\ker(T) + \ker(S) = H$.

Proof. Since $\overline{ran(T^*)} + \overline{ran(S^*)}$ is closed if and only if $\ker(T) + \ker(S)$ is closed, the proof of the above theorem follows from the equality

$$\left(\overline{ran(A^*)} \cap \overline{ran(B^*)}\right)^{\perp} = \overline{\ker(A) + \ker(B)}.$$

$\square$

Remark 2.2.5. *It is worth noticing that* $\ker(T)+\ker(S) = H$ *does not imply the closedness of $ran(T^*) \oplus ran(S^*)$. For instance, it suffices to consider an unbounded self-adjoint operator T such that $ran(T)$ is unclosed and $S = 0$ everywhere on H.*

Theorem 2.2.6. *Let T and S be closed densely defined linear operators such that $D(T) \subseteq D(S)$ and $\|Sx\| \leq \|Tx\|$. Then $ran(S^*) \subseteq ran(T^*)$.*

Proof. Let $K^{'}$ be a linear operator defined from $ran(T)$ into $ran(S)$ by $K^{'}Tx = Sx$ for $x \in D(T)$. Since $K^{'}$ defined a contraction on $ran(T)$, then K_0 can be extended to a contraction $K^{''}$ on $\overline{ran(T)}$. Let $K = K^{''}$ on $\overline{ran(T)}$ and $K = 0$ on $\overline{ran(T)}^{\perp}$. Then $KT \subseteq S$ and hence $S^* \subseteq T^*K^*$. Consequently, $ran(S^*) \subseteq ran(T^*)$. $\square$

The order relation between bounded self-adjoint operators in $B(H)$ is easy to describe. However, in the context of unbounded self-adjoint operators, there are at least two definitions (see, [9]), namely:

Definition 2.2.7. *Let T and S be two symmetric operators with domains $D(T)$ and $D(S)$ respectively. We say $T \geq S$ if $D(T) \subseteq D(S)$ and*

$$\langle Tx, x\rangle \geq \langle Sx, x\rangle, \ \forall x \in D(T).$$

Definition 2.2.8. *Let T and S be unbounded self-adjoint positive operators. We say that $T \succeq S$ if $D(T^{\frac{1}{2}}) \subseteq D(S^{\frac{1}{2}})$ and $\|T^{\frac{1}{2}}x\| \geq \|S^{\frac{1}{2}}x\|$ for all $x \in D(T^{\frac{1}{2}})$.*

In general, $T \succeq S$ does not imply that $T \geq S$, even for two self-adjoint and positive operators T and S, mainly because $D(T^{1/2}) \subseteq D(S^{1/2})$ does not necessarily give $D(T) \subseteq D(S)$.

Corollary 2.2.9. *Let T and S be two closed densely defined linear operators such that $TT^* \preceq SS^*$. Then $ran(T) \subseteq ran(S)$.*

Proof. Since $TT^* \preceq SS^*$, then

$$D(S^*) = D(|S^*|) \subseteq D(|T^*|) = D(T^*)$$

and

$$\|T^*x\| = \||T^*|x\| \leq \||S^*|x\| = \|S^*x\|.$$

The above theorem ensures that $ran(T) \subseteq ran(S)$, as needed. $\square$

The next result might regarded as a generalization of Proposition 2.8 in [2].

Theorem 2.2.10. *Let T and S be two unbounded linear operators satisfying $ran(T) \cap ran(S) = \{0\}$. Then,*

$$ran(T+S) = ran(T) + ran(S) \Longleftrightarrow \ker(T) + \ker(S) = D(T) + D(S).$$

Proof. Assume that $ran(T+S) = ran(T) + ran(S)$. Since $\ker(T) + \ker(S) \subseteq D(T) + D(S)$, we only need to prove that $D(T) + D(S) \subseteq \ker(T) + \ker(S)$. Let $x \in D(T) + D(S)$ so that $x = x_1 + x_2$, with $x_1 \in D(T)$ and $x_2 \in D(S)$. Since $ran(T+S) = ran(T) + ran(S)$, then $ran(T) \subseteq ran(T+S)$, and there exists a $y \in D(T)$ such that

$$Tx_1 = Ty + Sy.$$

Hence, $T(x_1 - y) = Sy$, which, and due to $ran(T) \cap ran(S) = \{0\}$, means that $y \in \ker(S)$ and $x_1 - y \in \ker(T)$. Hence, $x_1 \in \ker(T) + \ker(S)$. Reasoning similarly, we obtain $x_2 \in \ker(T) + \ker(S)$, and so $x_1 + x_2 \in \ker(T) + \ker(S)$.

Conversely, suppose $\ker(T) + \ker(S) = D(T) + D(S)$. Let $y = Tx + Sx' \in ran(T) + ran(S)$. Then $x = x_1 + x_2$ and $x^{'} = x_3 + x_4$ where $x_1, x_3 \in \ker T$ and $x_2, x_4 \in \ker S$. We have

$$y = Tx + Sx' = (T + S)(x_2 + x_3).$$

Thus, $ran(T) + ran(S) = ran(T + S)$. □

Remark 2.2.11. *Observe that* $\ker(T) + \ker(S) = D(T) + D(S)$ *implies that* $ran(T+S) = ran(T) + ran(S)$ *even without assuming* $ran(T) \cap ran(S) = \{0\}$.

Corollary 2.2.12. *Let T be a closed densely defined linear operator and let $B \in B(H)$. If* $\ker(T) + \ker(S) = H$, *then*

$$ran(|T|^{\alpha} + |S|^{\beta}) = ran(|T|^{\alpha}) + ran(|S|^{\beta}),$$

for all $\alpha, \beta > 0$.

Proof. Since $\ker(T) + \ker(S) = H$, $\ker(|T|^{\alpha}) + \ker(|S|^{\beta}) = H$. Hence, Theorem 2.2.10 implies that $ran(|T|^{\alpha} + |S|^{\beta}) = ran(|T|^{\alpha}) + ran(|S|^{\beta})$. □

Example 2.2.13. *There exist a closed densely defined linear operator T and $S \in B(H)$ such that*

1. $ran(T + S) = ran(T) + ran(S)$.
2. $ran(T) \cap ran(S) = \{0\}$.
3. $\ker(T) + \ker(S) \neq D(T)$.

It suffices to assume A and $B = 0$ on the entire H.

The next example shows the impossibility to extend Theorem 2.2.10 to three operators.

Example 2.2.14. *There are $T, S, K \in B(H)$ such that*

1. *$ran(T) \cap ran(S) \cap ran(K) = \{0\}$.*

2. *$ran(T + S + K) = ran(T) + ran(S) + ran(K)$.*

3. *$\ker(T) + \ker(S) + \ker(K) \neq H$.*

Let $H := L^2$ the Haar measure on the unit ball. According to Exercise 9 on Page 145 in [8], there exist two closed subspaces M and N of L^2 such that $M \cap N = \{0\}$, $M + N \neq L^2$ and $\overline{M + N} = L^2$. Let $T = I$ on L^2 and let S and K be the orthogonal projections on M and N respectively. Then $ran(T) \cap ran(S) \cap ran(K) = \{0\}$ and

$$ran(T + S + K) = ran(T) + ran(S) + ran(K) = L^2,$$

and yet

$$\ker(T) + \ker(S) + \ker(K) = \ker(S) + \ker(K) \neq L^2$$

On the other hand, if $\ker(B) + \ker(C) = L^2$, then proposition yields that $M + N = L^2$, which is impossible.

Example 2.2.15. *There exist a densely defined closed linear operator T and $S \in B(H)$ such that*

$$ran(T + S) + ran(T - S) \neq ran(T) + ran(S).$$

Let K be a Hilbert space, and let $H = K \oplus K$. Now, take a densely defined closed linear operator T such that $D(T^2) = D(T)$. An explicit example is $A = \begin{pmatrix} I_K & A \\ 0 & 0 \end{pmatrix}$, defined on $K \times D(A)$, where A is closed densely linear operator. Then T is closed and $T^2 = A$. Let $B = I$ on the whole of H. Then

$$ran(T + S) + ran(T - S) \subseteq D(T) \neq ran(T) + ran(S) = H.$$

The following result deals with the additivity properties of unbounded linear operator ranges. Recall that, in general, only the inclusion $ran(T + S) \subseteq ran(T) + ran(S)$ hold, where T and S are linear operators.

Proposition 2.2.16. *Let T and S be densely defined linear operators with domains $D(T)$ and $D(S)$ respectively. Assume that $D(T) \subseteq D(S)$. Consider the following statements:*

1. *$ran(T+S) = ran(T) + ran(S)$.*

2. *$ran(S) \subseteq ran(T+S)$.*

3. *$ran(T) \subseteq ran(T+S)$.*

4. *$ran(T-S) \subseteq ran(T+S)$.*

Then, $(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$.

Proof. Let $x \in D(T)$. The implication "$(1) \Rightarrow (2)$" is plain. To show "$(2) \Rightarrow (1)$", observe that $Tx = Tx + Sx - Sx$. Since $ran(S) \subseteq ran(T+S)$, then $Tx \in ran(T+S)$. Therefore, $ran(T+S) = ran(T) + ran(S)$ holds. By applying similar reasoning, we may show that "$(2) \Rightarrow (3)$".

Now, we show "$(3) \Leftrightarrow (4)$". Since $Sx - Tx = Sx + Tx - 2Tx$, then $ran(T) \subseteq ran(T+S)$. The other implication follows from the fact: $Tx = \frac{Tx-Sx}{2} + \frac{Tx+Sx}{2}$. □

Remark 2.2.17. *In general* (3) *does not imply* (1) *or* (2). *To see that, Let A be a linear operator with domain $D(A)$ and an unclosed range $ran(A)$, then let*

$$T = \begin{pmatrix} 0 & A \\ 0 & I \end{pmatrix},$$

where I is the identity operator.

Let $S = I_{H \times H}$. Clearly, $ran(T) = ran(A) \oplus D(A)$ and $ran(T+S) = H \times D(T)$. Thus, $ran(T) \subseteq ran(T+S)$, and since $ran(T) + ran(S) = H \times H$, then $ran(A+B) \neq ran(A) + ran(B)$ is violated.

2.3 The ranges of unbounded positive self-adjoint operators

In this section, we investigate the closedness of the range of positive self-adjoint operators.

Theorem 2.3.1. *Let T be an unbounded linear positive self-adjoint operator. The following statements are equivalent:*

1. *$ran(T)$ is closed.*

2. $ran(T) = ran(T^\alpha)$ for all $\alpha \in]0,1[$.

3. $ran(T) = ran(T^\alpha)$ for some $\alpha \in]0,1[$.

Proof.

1. "(1) $\Rightarrow$ (2)". By Theorem 2.2.1, we have

$$\overline{ranT^\alpha} = \overline{ran(T)} = ran(T) \subseteq ran(T^\alpha)$$

which implies that

$$ran(T) = ran(T^\alpha).$$

2. Obviously, the assertion "(2) $\Rightarrow$ (3)" is hold.

3. "(3) $\Rightarrow$ (1)". Recall that

$$H = D(T^\alpha) + ran(T^\alpha) = D(T^\alpha) + ran(T).$$

Let $x \in \ker(T)^\perp$ such that $x = x_1 + x_2$ where $x_1 \in D(T^\alpha)$ and $x_2 \in ran(T) \subseteq ran(T^{1-\alpha})$. Since $x \in \ker(T)^\perp$ and $x_2 \in ran(T) \subseteq \ker(A)^\perp$, then $x_1 \in \ker(T)^\perp$.

Next, we show that $x_1 \in ran(T^{1-\alpha})$. Since $x_1 \in D(T^\alpha)$, there exists $y \in D(T)$ such that

$$AT^\alpha x_1 = Ty \Rightarrow T^\alpha x_1 - Ty = T^\alpha(x_1 - T^{1-\alpha}y) = 0,$$

thus, $x_1 - T^{1-\alpha}y \in \ker(T^\alpha) = \ker(T)$.

Besides, $T^{1-\alpha}y \in ran(T^{1-\alpha}) \subseteq \ker(T^{1-\alpha})^\perp = \ker(T)^\perp$. The latter then implies that $x_1 - T^{1-\alpha}y \in \ker(T)^\perp \cap \ker(T) = \{0\}$. Hence

$$x_1 = T^{1-\alpha}y \text{ or } x_1 \in ran(T^{1-\alpha}).$$

Hence

$$\overline{ran(T^{1-\alpha})} = \overline{ran(T)} = \ker(T)^\perp \subseteq ran(T^{1-\alpha}),$$

and hence

$$\overline{ran(T^{1-\alpha})} = ran(T^{1-\alpha}).$$

Using Corollary 2.2.3, and for a suitable n, we obtain

$$ranT^{1-\alpha} = ranT^{2^n(1-\alpha)} \subseteq ran(T) \subseteq ranT^{1-\alpha}.$$

Accordingly, $ran(T) = ranT^{1-\alpha}$ is a closed.

□

Now, we extend the previous theorem may to include real numbers in $]1, \infty[$.

Theorem 2.3.2. *Let T be an unbounded positive self-adjoint operator. The following statements are equivalent:*

1. *$ran(T)$ is closed.*
2. *$ran(T) = ran(T^\alpha)$ for all $\alpha \in]1, \infty[$.*
3. *$ran(T) = ran(T^\alpha)$ for some $\alpha \in]1, \infty[$.*

Proof. We only need to prove the implications, "(1) ⇒ (2)" and "(3) ⇒ (1)".

1. "(1) ⇒ (2)". Let $\alpha > 1$. Since $ran(T)$ is closed, then

   $$ran(T) = ran(T^2) = ran(T^{2^n}),$$

   for all $n \in \mathbb{N}$. Now, for a suitable n, we can show that

   $$ran(T) = ran(T^{2^n}) \subseteq ran(T^\alpha) \subseteq ran(T).$$

2. "(3) ⇒ (1)". Let $S = T^\alpha$. Then the assumption $ran(T) = ran(T^\alpha)$ becomes $ran(S) = ran(S^{1/\alpha})$. Since $\frac{1}{\alpha} < 1$, Theorem 2.3.1 yields the closedness of $ran(T)$.

□

By combining Theorem 2.3.1 and Theorem 2.3.2, we deduce the following result.

Corollary 2.3.3. *Let T be a densely defined closed operator. The following statements are equivalent:*

1. $ran(T)$ *is closed.*

2. $ran(T) = ran(|T^*|^\alpha)$ *for all reals* $\alpha > 0$.

3. $ran(T) = ran(|T^*|^\alpha)$ *for some real* $\alpha > 0$ *with* $\alpha \neq 1$.

Proof. The proof follows immediately from the preceding two results and the relation $ran(T) = ran(|T^*|)$. □

The next result constitutes yet another generalization of the above results.

Corollary 2.3.4. *Let* T *be an unbounded positive self-adjoint operator. If* $ran(T^\alpha) = ran(T^\beta)$ *for some different positive numbers* α *and* β, *then* $ran(T)$ *must be closed.*

Proof. Since $ran(T^\alpha) = ran[(T^\alpha)^{\frac{\beta}{\alpha}}]$, then $ran(T^\alpha)$ is closed as $\alpha \neq \beta$, and so is $ran(T)$ by the preceding results. □

Chapter 3

Identities involving the range of unbounded linear operators

In this chapter, we present several equalities related to the sum of ranges of unbounded linear operators.

3.1 Crimmins' equality

In [4], the authors proved that, If $T, S \in B(H)$, then

$$ran(T) + ran(S) = ran[(TT^* + SS^*)^{\frac{1}{2}}],$$

this equalty is known as Crimmins' equality. In the next result, we assume only one of the operators is bounded.

Proposition 3.1.1. *Let T be a densely defined closed linear operator with domain $D(T) \subseteq H$ and let $B \in B(H)$. Then*

$$ran(T) + ran(S) = ran[(TT^* + SS^*)^{\frac{1}{2}}].$$

Moreover, if $\alpha > 0$ *with* $\alpha \neq 2$, *then*

$$ran(T) + ran(S) = ran[(TT^* + SS^*)^{\frac{1}{\alpha}}].$$

whenever $ran(T) + ran(S)$ *is closed.*

Proof. Let

$$A = \begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}$$

with $D(A) = D(T) \times H$. Then A is densely defined closed operator, and hence AA^* is positive self-adjoint operator. Additionally,

$$ran(A) = (ran(T) + ran(S)) \times \{0\}.$$

Since $ran(A) = ran(|A^*|)$ and

$$|A^*| = \begin{pmatrix} (TT^* + SS^*)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix},$$

it follows that $ran(T) + ran(S) = ran[(TT^* + SS^*)^{\frac{1}{2}}]$.

A similar reasoning implies

$$(AA^*)^{\frac{1}{\alpha}} = \begin{pmatrix} (TT^* + SS^*)^{\frac{1}{\alpha}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $ran(A) = ran(|A^*|)$and $ran(T) + ran(S)$ is closed,then

$$ran(A) = ran(|A^*|) = ran(|A^*|^{\frac{2}{\alpha}}),$$

which implies the desired identity. □

Remark 3.1.2. *The previous result will also be referred to as Crimmins' eqaulity.*

The second Crimmins-like result is given next.

Proposition 3.1.3. *Let* $S_1, \ldots, S_n \in B(H)$ *and let* T *be a densely defined closed operator. Then*

$$ran(T) + ran(S_1) + \cdots + ran(S_n) = ran[(TT^* + S_1S_1^* + \cdots + S_nS_n^*)^{\frac{1}{2}}].$$

Proof. Let

$$A = \begin{pmatrix} T & S_1 & \cdots & S_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

defined on $D(T) \oplus H \oplus H \oplus \cdots \oplus H$. Since

$$|A^*| = \begin{pmatrix} (TT^* + S_1S_1^* + \cdots + S_nS_n^*)^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and as above, we obtain the desired equality. $\square$

Next, we generalize Proposition 3.7 in [1].

Proposition 3.1.4. *Let T be a closed densely defined linear operator and let $B \in B(H)$. Then, for all $\alpha, \beta > 0$ we have*

1. $\overline{ran(|T|^\alpha + |S|^\beta)} = \overline{ran(T^*) + ran(S^*)}$.

2. $\ker\left(|T|^\alpha + |S|^\beta\right) = \ker(T) \cap \ker(S)$.

Proof.

1. We have

$$\begin{aligned}\overline{ran(T^*) + ran(S^*)} = \overline{\overline{ran(T^*)} + \overline{ran(S^*)}} &= \overline{\overline{ran(|T|^{\frac{\alpha}{2}})} + \overline{ran(|S|^{\frac{\beta}{2}})}} \\ &= \overline{(ran(|T|^{\frac{\alpha}{2}})) + ran(|S|^{\frac{\beta}{2}})}\end{aligned}$$

   Crimmins' equality then implies that

$$\overline{ran(T^*) + ran(S^*)} = \overline{ran((|T|^\alpha + |S|^\beta)^{\frac{1}{2}})} = \overline{ran(|T|^\alpha + |S|^\beta)}.$$

2. The second assertion is a consequence of the first one.

$\square$

The following result generalizes both Theorem 3.3 and Corollary 3.6 in [1].

Theorem 3.1.5. *Let T be a closed densely defined linear operator and $B \in B(H)$. The following statements are equivalent:*

1. *$ran(T) + ran(S)$ is closed.*
2. *$ran(|T^*|^\alpha + |S^*|^\beta)$ is closed, for all $\alpha, \beta > 0$.*
3. *$ran(|T^*|^\alpha + |S^*|^\beta)$ is closed, for some $\alpha, \beta > 0$.*

Furthermore, if any of the above conditions holds, then

$$ran(|T^*|^\alpha + |S^*|^\beta) = ran(|T^*|^\alpha) + ran(|S^*|^\beta) = ran(T) + ran(S),$$

for all $\alpha, \beta > 0$.

Proof.

1. "(1) $\Rightarrow$ (2)". Since $ran(T) + ran(S)$ is closed, Proposition 3.1.1 implies that

   $$\begin{aligned} ran(T) + ran(S) = ran(|T^*|^2 + |S^*|^2)^{\frac{1}{2}} &= ran(|T^*|^2 + |S^*|^2) \\ &\subseteq ran(|T^*|^2) + ran(|S^*|^2) \\ &\subseteq ran(A) + ran(B). \end{aligned}$$

   Hence, $ran(|T^*|^2) + ran(|S^*|^2) = ran(T) + ran(S)$ is closed. By induction, $ran(|T^*|^{2^n}) + ran(|S^*|^{2^n}) = ran(T) + ran(S)$ is closed.

   For suitable n, we have

   $$ran(|T^*|^{2^n}) + ran(|S^*|^{2^n}) \subseteq ran(|T^*|^{\frac{\alpha}{2}}) + ran(|S^*|^{\frac{\beta}{2}}) \subseteq ran(|T^*|^\alpha + |S^*|^\beta)^{\frac{1}{2}}$$

   which implies that

   $$ran(|T^*|^\alpha + |S^*|^\beta)^{\frac{1}{2}} \subseteq \overline{ran(|T^*|^\alpha + |S^*|^\beta)} = ran(T) + ran(S),$$

   where we have used Proposition 3.1.4. Therefore,

   $$ran(|T^*|^\alpha + (|S^*|^\beta))^{\frac{1}{2}} = ran(T) + ran(S)$$

is closed, as

$$ran(|T^*|^\alpha + |S^*|^\beta) = ran(|T^*|^\alpha + |S^*|^\beta)^{\frac{1}{2}} = ran(T) + ran(S).$$

On the other hand, we find that

$$ran(T) + ran(S) = ran(|T^*|^{\frac{\alpha}{2}}) + ran(|S^*|^{\frac{\beta}{2}})$$

for all $\alpha, \beta > 0$ and hence

$$ran(|T^*|^\alpha + |S^*|^\beta) = ran(T) + ran(S) = ran(|T^*|^\alpha) + ran(|S^*|^\beta),$$

as required.

2. "(3) $\Rightarrow$ (1)". If $ran(|T^*|^\alpha + |S^*|^\beta)$ is closed for some $\alpha, \beta > 0$, then

$$ran(|T^*|^\alpha + |S^*|^\beta) = ran[(|T^*|^\alpha + |S^*|^\beta)^{\frac{1}{2}}] = ran(|T^*|^{\frac{\alpha}{2}}) + ran(|S^*|^{\frac{\beta}{2}}).$$

Hence, $ran(|T^*|^{\frac{\alpha}{2}}) + ran(|S^*|^{\frac{\beta}{2}})$ is closed. By the first part of the proof, $ran(|T^*|) + ran(|S^*|)$ is closed, thus

$$ran(T) + ran(S) = ran(|T^*|) + ran(|S^*|)$$

is closed.

□

Corollary 3.1.6. *Let T be a closed densely defined linear operator and let $B \in B(H)$. The following assertions are equivalent:*

1. *$ran(T) + ran(S) = H$.*
2. *$ran(|T^*|^\alpha + |S^*|^\beta)$ is invertible for all $\alpha, \beta > 0$.*
3. *$ran(|T^*|^\alpha + |S^*|^\beta)$ is invertible for some $\alpha, \beta > 0$.*

Conclusion

In our work, we establish a collection of identities concerning the closedness of the range for various classes of (possibly unbounded) operators on a Hilbert space, such as self-adjoint, positive, normal, hyponormal, and quasinormal operators as well as their powers and, where applicable, their fractional powers. Indeed, we prove that an unbounded self-adjoint positive operator has closed range if and only if one (equivalently, all) of its fractional powers has closed range. Moreover, we investigate the additivity properties of unbounded linear operators ranges.

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ملخص: تتناول هذه المذكرة دراسة صورةالمؤثرات الخطية الغير محدودة في الفضاءات الهلبرتية مع التركيز على خصائص هذه الصورة وشروط انغلاقها . تهدف هذه الدراسة الى تعميق فهم نظرية المؤثرات غير المحدودة، قد تكون مفيدة في مجالات التحليل الرياضي.

الكلمات المفتاحية: المؤثر الغير محدود ـ المؤثرالمرافق ـ صورةالمؤثرات الخطية الغير محدودة ـ فضاء هلبرت.

Résumé:

Dans ce mémoire, nous avons étudié l'image des opérateurs linéaires non bornés dans les espaces de Hilbert, avec un accent particulier sur les propriétés de cette image et les conditions de sa clôture. Cette étude vise à approfondir la compréhension de la théorie des opérateurs non bornés, laquelle peut s'avérer utile dans divers domaines de l'analyse mathématique.

Mots-Clés: L'opérateur non borné, l'opérateur adjoint, l'image d'opérateur linéaire non bornés, espace de Hilbert.

Abstract:

In this thesis, we studied the the range of unbounded linear operators in Hilbert spaces, with particular emphasis on the properties of the range and the conditions under which it is closed. This study aims to deepen the understanding of unbounded operator theory, which may prove useful in various areas of mathematical analysis.

Keywords: unbounded operator, adjoint operator, the range of unbounded linear operator, Hilbert space.