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جامعة محمد بوضياف - المسيلة
Université Mohamed Boudiaf - M'sila

PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf university of M'sila
Faculty of Mathematics and Informatics
Department of Mathematics

Master memory

Field : Mathematics and Informatics
Branch : Mathematics
Option : Functional Analysis

Theme

The ideal of classical p-compact operators and its injective hull

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Publicly supported on : 28/06/2021.

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University year 2020/2021

Thanks

I cannot begin and finish my work without thanking the greatest and the most powerful " Allah."

I have to thank a lot my supervisor the professor **Achour Dahmane** and to express my warm gratitude for his assistance during the years from my work.

My sincere thanks to the president of the jury professor **Mezrag Lahcène** and the examiner Doctor **Alouani Ahlem**.

I also extend my sincere thanks and gratitude to my dear father and mother for their help to me.

I would also like to thank all my teachers and all my fellow fonctional analysis Master. We also extend our deepest gratitude to all the professors of the University of M'sila, especially those of the Department of Mathematics and Computer Science.

Finally, I owe a debt of gratitude to Dr. Attalah Aldjia for her time and guidance throughout my work.

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Notations

$\mathcal{L}(X,Y)$	The space of all bounded linear operators from X to Y.
$\mathring{B}_X$	The open unite ball of X.
B_X	The closed unite ball of X.
p^*	The conjugate index of p (i.e., $\frac{1}{p}+\frac{1}{p^*}$).
$\mathbb{K}$	The field of real or complex numbers.
$\mathcal{K}(X,Y)$	The space of all compact operators from X to Y.
$\mathcal{F}(X,Y)$	The space o fall finite-rank operators from X to Y.
$\mathcal{K}_p(X,Y)$	The space of all classicl p-compact operators from X to Y.
ℓ_p	The banach space of p-summabl scalar sequences.
J	The natural embedding $J : X \longrightarrow \ell_\infty(B_{X^*})$ is defined as $J(x) = (x^*(x))_{x^*\in B_{X^*}}$.
ℓ_∞	The Banach space of bounded scalar sequences.
c_0	The subspace of ℓ_∞consisting of the scalar sequences.
$\ell_\infty(B_X)$	The Banach spaces of all bounded scalar families(λ_{x^*}) where $x^* \in B_{X^*}$.
T^*	The adjoint linear operator of T.

Introduction

The work of this memory is introduced within the framework of functional analysis. In particular, the theory of p-compact operators was initiated by Pietsch [16] and Fourie and Swart [8]. There is another well-known notion of p-compact operators, which is a recent notion, introduced and studied in 2002 by Sinha and Karn [9]. Following [15], we call classical p-compact operators to the p-compact operators of Fourie and Swart. These classes was extended to the Lipschitz case by Achour et al. [1, 21]. The set of classical p-compact operators from X into Y is denoted by $\mathcal{K}_p(X,Y)$. Fourie and Swart [8] showed that $(\mathcal{K}_p, k_p(\cdot))$ is a Banach operator ideal. Many important properties, it is shown in [9].

Recall that a compact operator is any operator who verifies $T(B_X)$ is relatively compact. The space of compact operators $\mathcal{K}(X,Y)$ is a Banach ideal. If $Y = \ell_p$ we have an important identification with the spaces of uncondemnly p-summable sequences, that is to say

$$\ell_p^u(X^*) = \mathcal{K}(X, \ell_p) \text{ and } \ell_\infty^u(X^*) = \mathcal{K}(X, c_0), \text{ if } Y = c_0.$$

After the definition of the compact operators, we introduce the definition of the classical p-compact operators as being: any linear operator which is factored into two compact operators author of the space ℓ_p, i.e., the following diagram commutes

$$\begin{array}{ccc} X & \underrightarrow{T} & Y \\ R \searrow & & \nearrow S \\ & \ell_p & \end{array}$$

with $R \in \mathcal{K}(X, \ell_p)$ and $S \in \mathcal{K}(\ell_p, X)$. We are going to do a detail study on the article "Banach ideals of p-compact operators" by Jan Fourie and Johan Swart. The memory is based on papers [8] and [9]. In order to achieve this purpose the memory was organized as follows:

The first chapter is an overview of notions and basic concepts and results needed in the following chapters. These include sequence spaces, operator ideals and copmact operators. In the second chapter, we start by introducing the uncondemnly p-summable sequence spaces and we give some properties of these spaces. We defne and analyze the concept $\mathcal{K}_p$ of all classical p-compact operators as an operator ideal. We present a characterization given by a representation; more precisely an operator $T \in \mathcal{K}_p(X, Y)$ if and only if T has a representation of the form $T = \sum_n \langle \cdot, x_n^* \rangle y_n$ where $(x_n^*)_n \in \ell_p^u(X^*)$ and $(y_n)_n \in \ell_{p^*}^u(Y)$. The principal result of the last chapter ([14, 12, 8, 9, 13, 5]), that the injective hull of the classical p-compact operators coincides with the unconditionally quasi p-nuclear operators of Kim [13]. Also we present the related dual result: an operator is unconditionally quasi p-nuclear, if and only if its adjoint is unconditionally p-compact (unconditionally p-compact was introduced and studied by Kim [13]).

Chapter 1

Preliminaries

In this chapter, we present a collection of definitions, properties and basic formulas that will benefit us during this work. for example sequences in Banach spaces (p-summable sequences, weakly absolutely p-summable sequences), operator ideal, finite rank operator and compact linear operator (See [16], [17] and [2]). We will write $\mathbb{K}$ for the real numbers field $\mathbb{R}$ or the complex numbers field $\mathbb{C}$ .the set of all natural numbers $\{0, 1, ...\}$ is denoted by $\mathbb{N}$ Along this letters X and Y denotes Banach spaces with norm$\|\cdot\|$. the open unite ball of X is denoted by B_X that is the set $\{x \in X, \ \|x\| < 1\}$ and the closed unite ball of X is denoted by $\mathring{B}_X$ that is the set $\{x \in X, \ \|x\| \leqslant 1\}$. A linear map or a linear operator T between real (or complex) linear spaces X, Y is a function $T : X \longrightarrow Y$ such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \text{ for all } \alpha, \beta \in \mathbb{R} \text{ (or } \mathbb{C} \text{) and } x, y \in X.$$

If X, Y are normed spaces then we can define the notion of the bounded operators. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces. A linear map $T : X \to Y$ is bounded if there is a constant C such that $\|T(x)\|_Y \leq C \|x\|_X$ for all $x \in X$. If $T : X \to Y$ is a bounded linear operator, then we define the operator norm $\|T\|$ of T by

$$\begin{aligned} \|T\| &= \inf \{C : \text{verifying } \|T(x)\|_Y \leq C \|x\|_X \text{ for all } x \in X.\} \\ &= \sup \left\{ \frac{\|T(x)\|_Y}{\|x\|_X}, x \in X \right\}. \end{aligned}$$

We denote the set of all bounded operators $T : X \to Y$ by $\mathcal{L}(X, Y)$. If X is normed space and $Y = \mathbb{R}$ (or $\mathbb{C}$) then $\mathcal{L}(X, Y) = X^*$, we will say that X^* is the dual topological of X.

Theorem 1.0.1 (Hahn-Banach) *Let be X a normed space, for each $x_0 \in X$, $x_0 \neq 0$, there is a functional lineal $\varphi \in X^*$ such that $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

Definition 1.0.1 *(Dual topological)Let X be a normed space. We call topological dual, and we denote by X^*, the Banach space of continuous linear forms on X, i.e.,*

$$X^* = \mathcal{L}(X, \mathbb{K})$$

such that the norm is given by

$$\begin{aligned} \|x^*\| &= \sup_{\|x\| \leq 1} |x^*(x)| \\ &= \sup_{\|x\| = 1} |x^*(x)|. \end{aligned}$$

1.1 Absolutely p-summable sequence

Let $1 \leq p \leq \infty$. The classical Banach sequence spaces ℓ_p, ℓ_∞ and c_0 are define by

- $\ell_p = \left\{(x_n)_n \subset \mathbb{K}\colon \| (x_n)_n \|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty \right\}, 0 \leq p < \infty.$
- $\ell_\infty = \{(x_n)_n \subset \mathbb{K}\colon \| (x_n)_n \|_\infty = \sup_{n \in I} |x_n| < \infty\}, p = \infty.$
- $c_0 = \{(x_n)_n \subset \mathbb{K}\colon \lim_{n \to \infty} |x_n| = 0\}.$

Definition 1.1.1 *Let $1 \leq p \leq \infty$. A sequence $(x_n)_n$, of elements of X is said to be absolutely p-summable if*

$$\|(x_n)_n\|_p = \begin{cases} (\sum_n \|x_n\|^p)^{\frac{1}{p}} < \infty & , \text{ if } 1 \leq p < \infty \\ \sup_n \|x_n\| < \infty & , \text{ if } p = \infty \end{cases}.$$

When $p = 1$, it is said that $(x_n)_n$ is absolutely summable. We denote by $\ell_p(X)$ the vector space of all absolutely p-summable sequences of elements of X. For $p \geq 1$, $(\ell_p(X), \|\cdot\|_p)$ is a Banach space.

The spaces $c_0(X)$ of norm null sequences in X is Branch spaces with the given by

$$\|(x_n)_n\|_\infty = \sup_n \|x_n\|.$$

Definition 1.1.2 *Let $1 \leqslant p < \infty$. A sequence $(x_n)_n$ in X is said to be weakly p-summable if*

$$\sum_{n=1}^{\infty} |x^*((x_n)| < \infty,$$

for every $x^* \in B_{X^*}$. We denoted by $\ell_p^w(X)$ the Banach spaces of weakly p- summable sequence in X becomes a Banach spaces when equipped with the norm given by

$$\|(x_n)_n\|_p^w = \sup \left\{ \left(\sum_{n=1}^{\infty} |x^*(x_n)|^p \right)^{\frac{1}{p}} : x^* \in B_{X^*} \right\}.$$

Remark 1.1.1 *In the case $p = \infty$. then the spaces $\ell_\infty^w(X)$ of weakly bounded sequences coincide with the spaces $\ell_\infty(X)$,*

$$\|(x_n)_n\|_\infty^w = \|(x_n)_n\|_\infty .$$

The following fact is well known, which is discussed in [6].

Theorem 1.1.1 *Let $1 < p \leq \infty$ be arbitrary Then $\mathcal{L}(\ell_p; X)$ is isometrically isomorphic to $\ell_{p^*}^w(X)$. Hence if $(x_n)_n$ a sequence in X and under the mapping $u_{(x_n)_n} \to u_{(x_n)_n}((\alpha_n)_n) = \sum_{n=1}^{\infty} \alpha_n x_n$ which is converges in X for each $(\alpha_n)_n \in \ell_p$iff $(x_n)_n \in \ell_{p^*}^w(X)$.*

Definition 1.1.3 *[6]A sequence $(x_n)_n$ in X is said to be weakly null iff*

$$\lim_{n\to\infty} |x^*(x_n)| = 0,$$

for evrery $x^* \in X^*$. We denoted by $c_0^w(X)$ the Banach spaces weakly null sequences in X is a closed subspace of $\ell_\infty(X)$. Therefore, it is a Banach space with the supremum norm of $\ell_\infty(X)$.

1.2 Operator ideals

Definition 1.2.1 *(finite rank) Let X and Y be Banach spaces. A linear operator $T \in \mathcal{L}(X,Y)$ is said to be finite rank if and only if there exist functional $x_1^*, x_2^*, ..., x_n^* \in X^*$, and $y_1, y_2, ..., y_n \in Y$ such that*

$$T(x) = \sum_{k=1}^{n} x_k^*(x) y_k, x \in X.$$

The class of all finite rank operators is denoted by $\mathcal{F}$.

Let us recall the definition of a Banach operator ideal, from [6].

Definition 1.2.2 *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces X and Y its components $\mathcal{I}(X,Y) := \mathcal{L}(X,Y) \cap \mathcal{I}$ satisfy:*

(i) $\mathcal{I}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ which contains the finite rank operators.

(ii) The ideal property: if $v \in \mathcal{L}(G,X)$, $u \in \mathcal{I}(X,Y)$ and $w \in \mathcal{L}(Y,H)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(G,H)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies:

(i') $(\mathcal{I}(X,Y), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces X and Y,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') If $v \in \mathcal{L}(G,X)$, $u \in \mathcal{I}(X,Y)$ and $w \in \mathcal{L}(Y,H)$,

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\| \, ,$$

then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ for the sup norm.

1.3 Compact linear operators

Definition 1.3.1 *Let X, Y be Banach spaces and $T : X \to Y$ be a linear operator. We say that T is compact if*

$$T\left(B_X\right) \text{ is relatively compact in } Y.$$

We denote by $\mathcal{K}\left(X,Y\right)$ the space of all compact operators between X and Y. If we provide $\mathcal{K}\left(X,Y\right)$ of the induced norm of $\mathcal{L}\left(X,Y\right)$, it becomes a Banach space.

Theorem 1.3.1 *$\mathcal{K}(X,Y)$ is Banach ideal space.*

Example 1.3.1 *Every finite rank operator between Banach spaces is compact.*

Corollary 1.3.1 *$\mathcal{K}(X,Y)$ is closed subspace of $\mathcal{L}(X,Y)$.*

Example 1.3.2 *Let $a = (a_n)_n \in c_0$ and let $D_a : \ell_p \longrightarrow \ell_p$ such that $D_a(x) = D_a(x_n) = (a_n x_n)_n$. Then D_a is compact. It is easily verified that $\left\| D_{(a_n)_n} \right\| = \|(a_n)_n\|_\infty$. In fact*

1) D_a is linear continuous. For $x, y \in X$ and $\lambda \in \mathbb{K}$

$$\begin{aligned} D_a(x + \lambda y) &= (a_n x_n + \ \lambda a_n y_n)_n \\ &= (a_n x_n)_n + \lambda (a_n y_n)_n \\ &= D_a(x) + \lambda D_a(y), \end{aligned}$$

and

$$\begin{aligned} \|D_a(x)\|_{\ell_p} &= \|(a_n x_n)_n\|_{\ell_p} \\ &= (\sum_{i=1}^{\infty} |a_n x_n|^p)^{\frac{1}{p}} \\ &\leqslant \sup_{n\in\mathbb{N}} |a_n| \, (\sum_{i=1}^{\infty} |x_n|^p)^{\frac{1}{p}} \\ &= \|a_n\|_\infty \|x\|_{\ell_p}. \end{aligned}$$

Hence, D_a is linear continuous, with $\|D_a\|_{\ell_p} \leqslant \|a_n\|_\infty$.

2) Let a^N be a sequence such that

$$a_n^N = \begin{cases} a_n & \text{if} \quad n \leqslant N \\ 0 & \text{otherwise} \end{cases},$$

and let $T_N = D_{a^N}$ such that

$$\begin{array}{rccc} T_N : & \ell_p & \longrightarrow & \ell_p \\ & (x_n) & \longmapsto & (a_n^N x_n)_n = \sum_{n=1}^{N} (a_n^N x_n) e_n \end{array},$$

where e_n canonical basis of ℓ_p. Then finite rank operator, with

$$\begin{aligned}
\|D_a - T_N\| &= \sup_{x\in B_{\ell_p}} \|D_a(x) - T_N(x)\|_p \\
&= \sup_{x\in B_{\ell_p}} \left\|(a_n x_n)_{1\leqslant n\leqslant\infty} - (a_n x_n)_{1\leqslant n\leqslant N}\right\|_p \\
&= \sup_{x\in B_{\ell_p}} \left\|(a_n x_n)_{n\geq N+1}\right\|_p \\
&= \sup_{x\in B_{\ell_p}} \left(\sum_{n=N+1}^{\infty} |a_n|^p\,|x_n|^p\right)^{\frac{1}{p}} \\
&\leq \sup_{x\in B_{\ell_p}} \sup_{n>N+1} |a_n| \left(\sum_{n=N+1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \\
&\leq \sup_{n>N+1} |a_n| \sup_{x\in B_{\ell_p}} \|x\| \\
&\leq \sup_{n>N+1} |a_n| \longrightarrow 0,\ N \longrightarrow \infty \text{ for } n \longrightarrow \infty,
\end{aligned}$$

is a limit of finite rank operator. Hence is compact.

Proposition 1.3.1 *[7]Let X and Y Banach spaces. Then*

a) If $T : X \longrightarrow Y$ is compact and $A \in \mathcal{L}(X,Y)$, then $A \circ T$ is compact.

b) If $T : X \longrightarrow Y$ is compact and $B \in \mathcal{L}(Z,Y)$, then $T \circ B$ is compact.

Theorem 1.3.2 *(Schauder's Theorem) It is well known that if X and Y are normed spaces and $T : X \to Y$ is a linear and compact operator, then also $T^* : Y^* \to X^*$ is compact. The converse is true if Y is complete.*

The convex hull subset

Definition 1.3.2 *The convex hull of a sequence $(x_n)_n \in c_0(X)$ is defined as*

$$conv\{(x_n)_n\} = \left\{\sum_{n=1}^{\infty}\alpha_n x_n : (\alpha_n)_n \in B_{\ell_1}\right\}$$

Definition 1.3.3 *Let $1 \leq p \leq \infty$. The p-convex hull of a sequence $(x_n)_n \in \ell_p(X)$ is defined as*

$$p - conv\{(x_n)_n\} = \left\{\sum_{n=1}^{\infty}\alpha_n x_n : (\alpha_n)_n \in B_{\ell_{p^*}}\right\}.$$

Theorem 1.3.3 *[10, 4]* ***(Grothendieck compactness principle)***. *A subset K of X is relatively compact if and only if there exists $(x_n)_n \in c_0(X)$ such that $K \subset conv\{x_n)_n\}$, the closed convex hull of the sequence $(x_n)_n$.*

Proposition 1.3.2 *Let $K \subset \ell_p$. The set A is relatively compact* ***iff*** *there exists $\lambda = (\lambda_n)_n \in c_0$ such that*

$$K \subset \left\{ \sum_n \lambda_n d_n e_n : d \in B_{\ell_p} \right\}.$$

Definition 1.3.4 *A sequence $(x_n)_n$ is a null sequence if for every open interval containing 0, the sequence is ultimately in that interval. In symbols: $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that $|x_n| < \epsilon$ when $n \geq N$.*

Chapter 2

The ideal of classical p-compact operators

In this chapter we will study the class of p-compact operators. The definition of a c-p-compact operator is essentially based on the decomposition of two compact operators, this decomposition is done around the space ℓ_p. We will start this chapter by defining the unconditionally p-summable sequences, then we will discuss some properties and results concerning the class of classical p-compact operators. This chapter is mainly based on [2], [9], [20], [12] and [8].

2.1 Unconditionally p-summable sequences

We recall the notion of unconditionally p-summabl sequences and we show some new properties of these spaces. Let $(x_i)_i$ a sequence in $\ell_p(X)$. We have $\sum_{i=1}^{+\infty} \|x_i\|^p < \infty$ and, therefore,

$$\lim_{n\to\infty} \|(x_i)_{i=n}^{\infty}\|_p = \lim_{n\to\infty} \left(\sum_{i=n}^{+\infty} \|x_i\|^p \right)^{\frac{1}{p}} = 0. \tag{2.1.1}$$

Is it the same for weakly p-summable sequences? In other words, given a sequence $(x_i)_i \in \ell_p^w(X)$, it is true that $\lim_{n\to\infty} \|(x_i)_{i=n}^{\infty}\|_p^w$? We will see later that this is not always true, which suggests the consideration of the sequences which satisfy this property, again in this sequence, X is a normed space and $1 \leq p < \infty$.

Definition 2.1.1 *Let $1 \leqslant p \leqslant \infty$. A sequence $(x_n)_n \in X$ is said to be unconditionally p-summable if $(x_n)_n \in \ell_p^w(X)$ and*

$$\lim_{k \longrightarrow \infty} \|(x_n)_{n=k}^{\infty}\|_{p,w} = 0$$

and we denoted by

$$\ell_p^u(X) = \left\{ (x_n)_n \in \ell_p^u(X) : \lim_{k \longrightarrow \infty} \|(x_n)_{n=k}^{\infty}\|_{p,w} = 0 \right\}.$$

Definition 2.1.2 *A sequence $(x_i)_{i=n}^{\infty}$ in X is said unconditionally summable if, for all bijection $\eta : \mathbb{N} \longrightarrow \mathbb{N}$, the series $\sum_{i=1}^{\infty} x_{\eta(j)}$ is convergent in X.*

Proposition 2.1.1 *A sequence $(x_i)_{i=n}^{\infty}$ in X is said unconditionally summable if and only if, belongs to $\ell_1^u(X)$.*

Proof. See ([3], Proposition 8.3). ∎

Proposition 2.1.2 *$\ell_p^u(X)$ is a closed subset of $\ell_p^w(X)$ and therefore it is a Banach space, with the norm $\|\cdot\|_{p,w}$ when X is Banach.*

Proof. Let $(x_i)_{i=n}^{\infty}, (y_i)_{i=n}^{\infty} \in \ell_p^u(X)$ and $\lambda \in \mathbb{K}$. Then

$$\lim_{n\to\infty} \|(x_i)_{i=n}^{\infty}\|_{p,w} = \lim_{n\to\infty} \|(y_i)_{i=n}^{\infty}\|_{p,w} = 0$$

and we have

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|(x_i + \lambda y_i)_{i=n}^{\infty}\|_{p,w} \\ &= \lim_{n\to\infty} \|(x_i)_{i=n}^{\infty} + \lambda (y_i)_{i=n}^{\infty}\|_{p,w} \\ &\leq \lim_{n\to\infty} \|(x)_{i=n}^{\infty}\|_{p,w} + |\lambda| \lim_{n\to\infty} \|(y_i)_{i=n}^{\infty}\|_{p,w} = 0, \end{aligned}$$

i.e., $\lim_{n\to\infty} \|(x_i + \lambda y_i)_{i=n}^{\infty}\|_{p,w} = 0$. Therefore $(x_i)_{i=n}^{\infty} + \lambda (y_i)_{i=n}^{\infty} \in \ell_p^u(X)$.

Let's see that $\ell_p^u(X)$ is closed in $\ell_p^w(X)$. Let $\left(x^k\right)_{k=1}^{\infty}$ a sequence in $\ell_p^u(X), x^k \overset{\text{converg}}{\longrightarrow} x$. We put that $x = (x_n)_n$ and, for each $k \in \mathbb{N}$, $x^k = \left(x_n^k\right)_n \in \ell_p^u(X)$. Then,

$$\lim_{n\to\infty} \left\| \left(x_i^k\right)_{i=n}^{\infty} \right\|_{p,w} = 0 \text{ for each } k \in \mathbb{N},$$

i.e., for each $\varepsilon > 0$, $\exists n_0(k) \in \mathbb{N}$ such that $\forall n \geq n_0(k)$

$$\left\| \left(x_i^k\right)_{i=n}^{\infty} \right\|_{p,w} < \frac{\varepsilon}{2} \text{ for each } k \in \mathbb{N}. \tag{2.1.2}$$

Since $x^k \underset{k\to\infty}{\longrightarrow} x$ in $\ell_p^w(X)$, i.e.,, for each $\varepsilon > 0$, $\exists k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$

$$\begin{aligned} \tfrac{\varepsilon}{2} > \left\|x^k - x\right\|_{p,w} &= \left\| \left(x_n^k\right)_n - (x_n)_n \right\|_{p,w} \\ &= \left\| \left(x_n^k - x_n\right)_n \right\|_{p,w} \\ &= \sup_{\varphi \in B_{X^*}} \left(\textstyle\sum_{n=1}^{\infty} \left|\varphi\left(x_n^k - x_n\right)\right|^p \right)^{\frac{1}{p}} \\ &\geq \sup_{\varphi \in B_{X^*}} \left(\textstyle\sum_{i=n}^{\infty} \left|\varphi\left(x_i^k - x_i\right)\right|^p \right)^{\frac{1}{p}} \\ &= \left\| \left(x_i^k\right)_{i=n}^{\infty} - (x_i)_{i=n}^{\infty} \right\|_{p,w}, \end{aligned} \tag{2.1.3}$$

for each $n \in \mathbb{N}$. Hence, for $n \geq n_0(k_0)$, by 2.1.2 and 2.1.3 we have

$$\begin{aligned} \|(x_n)_n\|_{p,w} &= \left\| (x_n)_n - \left(x_i^{k_0}\right)_{i=n}^{\infty} + \left(x_i^{k_0}\right)_{i=n}^{\infty} \right\|_{p,w} \\ &\leq \left\| (x_i)_{i=n}^{\infty} - \left(x_i^{k_0}\right)_{i=n}^{\infty} \right\|_{p,w} + \left\| \left(x_i^{k_0}\right)_{i=n}^{\infty} \right\|_{p,w} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This prove that $x \in \ell_p^w(X)$. Thus, $\ell_p^u(X)$ is closed in $\ell_p^w(X)$. ■

Proposition 2.1.3 *Let* $1 \leq p < \infty$. *Then*

1)$\ell_p(X) \subseteq \ell_p^u(X)$.

2)$\ell_p^u(X) \subseteq c_0(X)$.

Proof. 1)Let $(x_n)_n \in \ell_p(X)$. We know that $\ell_p(X) \subset \ell_p^w(X)$ this implies that $(x_n)_n \in \ell_p^w(X)$. and

$$\|(x_i)_{i=n}^{\infty}\|_{p,w} \leq \|(x_i)_{i=n}^{\infty}\|_p.$$

Hence, by 2.1.1 we have

$$\lim_{n\to\infty} \|(x_i)_{i=n}^{\infty}\|_{p,w} \leq \lim_{n\to\infty} \|(x_i)_{i=n}^{\infty}\|_p = 0.$$

Thus, $\lim_{n\to\infty} \|(x_i)_{i=n}^{\infty}\|_{p,w} = 0$.

2) Let $(x_n)_n \in \ell_p^u(X)$. For each $i \in \mathbb{N}$ choose $\varphi_i \in B_{X^*}$ such that $\|x_i\| = \varphi_i(x_i)$, which we know to be possible by the Hahn-Banach Theorem (Theorem 1.0.1). Thus,

$$\lim_{n\to\infty} \|x_i\| = \lim_{n\to\infty} \varphi_i(x_i) = 0.$$

Since the norm at $c_0(X)$ is the same as $\ell_\infty(X)$. So $(x_n)_n \in c_0(X)$. ∎

We need the following Lemmas

Lemma 2.1.1

(a) For $(a_n)_n \in \ell_p^u(X^*)$, $1 \leq p \leq \infty$ and $\delta > 0$ given, there exist $(b_n)_n \in \ell_p^u(X^*)$ and $\zeta \in c_0$, such that $\|\zeta\|_\infty = 1$, $a_n = \zeta_n b_n$, $\forall n \in \mathbb{N}$ and $\|(b_n)_n\|_{p,w} < \|(a_n)_n\|_{p,w} + \delta$.

(b) For $(x_n)_n \in \ell_p^u(X)$; $1 \leq p \leq \infty$ and $\delta > 0$ given, there exist $(y_n) \in \ell_p^u(X)$ and $\zeta \in \ell_\infty(c_0)$, such that $\|\zeta\| = 1$, $x_n = \zeta_n b_n$, $\forall n \in \mathbb{N}$ and $\|(y_n)\|_{p,w} < \|(x_n)\|_{p,w} + \delta$.

Proof. (a) For $(a_n)_n \in \ell_{p,}^u(X^*)$, $1 \leq p \leq \infty$, and $\delta > 0$ given, there exists an increasing sequence of natural numbers $1 = m_0 < m_1 < m_2 <$ ...such that $\left\|(a_n)_{n \geq m_j}\right\|_{p,w} < \frac{\delta}{(1+j)^3}$.

We put $\zeta_n = \frac{1}{j}$, for $m_{j-1} \leq n < m_j$ and $b_n = \frac{a_n}{\zeta_n}$ for all $n \in \mathbb{N}$. Then $\xi \in c_0$ such that $\|\xi\| = 1$. $a_n = \xi_n b_n$, for $(b_n)_n \in \ell_{p,q}^u(X^*)$

$$\begin{aligned}
\|(b_n)_n\|_{p,w} &= \left\|(b_n)_{n<m_j}\right\|_{p,w} + \left\|(b_n)_{n \geq m_j}\right\|_{p,w} \\
&= \left\|\left(\frac{a_n}{\zeta_n}\right)_{n<m_j}\right\|_{p,w} + \left\|\left(\frac{a_n}{\zeta_n}\right)_{n \geq m_j}\right\|_{p,w} \\
&\leq \frac{1}{\|\zeta\|_\infty}\left\|(a_n)_{n<m_j}\right\|_{p,w} + (j+1)\left\|(a_n)_{n \geq m_j}\right\|_{p,w} \\
&\leq \frac{1}{\|\zeta\|_\infty}\left\|(a_n)_{n<m_j}\right\|_{p,w} + (j+1)\frac{\delta}{(1+j)^3} \\
&\leq \|(a_n)_n\|_{p,w} + (1+j)^{-2}\delta \\
&\leq \|(a_n)_n\|_{p,w} + \delta.
\end{aligned}$$

The proof of (b) is similar to that (a). ∎

By using Lemma 2.1.1 the following characterizations of $\ell_{p,}^u(X^*)$ are easily verified.

Lemma 2.1.2 *Let $1 \leqslant p \leq \infty$. Then*

(a) $(a_n) \in \ell_p^u(X^*)$, iff there exist a $\beta \in c_0$ and $(b_n) \in \ell_p^w(X^*)$ such that $a_n = \beta_n b_n$, $\forall n \in \mathbb{N}$.

(b) $(x_n) \in \ell_p^u(X)$, iff there exist a $\beta \in c_0$ and $(y_n) \in \ell_p^w(X)$ such that $x_n = \beta_n y_n$, $\forall n \in \mathbb{N}$.

the theorem 1.1.1 leads the following lemma.

Lemma 2.1.3

a) Let $1 \leq p \leq \infty$ *and* $(x_n)_n$ *a sequence in* X*. Then* $\sum_n \lambda_n x_n$ *converges in* X *for each* $\lambda \in \ell_p$ *iff* $(x_n)_n \in \ell_{p^*}^w(X)$*.*

b) Let $1 \leq p \leq \infty$ *and* $(x_n)_n$ *a sequence in* X*. Then* $\sum_n \lambda_n x_n$ *converges in* X *for each* $\lambda \in \ell_i$ *iff* $(x_n)_n \in \ell_\infty(X)$*.*

Theorem 2.1.1 *[8]*

a) Let $1 \leq p \leq \infty$*. Then* $\ell_p^u(X^*)$ *is isometrically isomorphic to* $\mathcal{K}(X, \ell_p)$*.*

b) Let $1 \leq p < \infty$*. Then* $\ell_p^u(X)$ *is isometrically isomorphic to* $\mathcal{K}(\ell_{p^*}, X)$*.*

Proof. a) Let $1 \leq p \leq \infty$. We associate with $(a_n)_n \in \ell_p^u(X^*)$ the continuous map $P : X \longrightarrow \ell_p : x \longrightarrow (\langle a_n, x\rangle)_n$. obviously

$$\begin{aligned} \|P\| &= \sup_{x\in B_X} \|P(x)\| \\ &= \sup_{x\in B_X} \|(\langle a_n, x\rangle)_n\|_p \\ &= \|(a_n)_n\|_p^w . \end{aligned}$$

By Lemma 2.1.1, there exist $(b_n)_n \in \ell_p^u(X^*)$ and $(\xi_n)_n \in c_0$ such that $a_n = \xi_n b_n$, $\forall n \in \mathbb{N}$. Hence $R : X \longrightarrow \ell_p : x \longrightarrow (\langle b_n, x\rangle)_n$ is continuous and $P = D_{(\xi_n)_n} \circ R$. Since

$$D_{(\xi_n)_n} : \ell_p \longrightarrow \ell_p : \lambda \longrightarrow \lambda.\xi$$

is compact, by Proposition 1.3.1 it follows that P is compact. Conversely, let $P : X \longrightarrow \ell_p$ be compact. For $x \in X$ we choose any pair $z \in B_X$ and $\zeta > 0$ such that $x = \zeta z$. Since $P(B_X)$ is relatively compact, then by Theorem 1.3.2, there exists a monotone null- sequence $(d_n)_n \in C_0$ such that

$$P(B_X) \subset \left\{ \sum_{n=1}^{\infty} \lambda_n d_n e_n : (\lambda_n)_n \in B_{\ell_p} \right\}.$$

This follows from the well known characterizations of compactness in the spaces ℓ_p. Hence there exists a $(\lambda_n^0)_n \in B_{\ell_p}$ such that $P(z) = \sum_{n=1}^{\infty} \lambda_n^0 d_n e_n$. Define u_n, $n \in \mathbb{N}$, by $\langle u_n, x\rangle = \zeta\lambda_n^0$.

If P_n denotes the ith coordinate projection on ℓ_p, $\begin{array}{rccc} P_n : & \ell_p & \longrightarrow & \mathbb{K} \\ & (\lambda_n) & \longmapsto & \lambda_n \end{array}$,, it is easily verified that u_n is continuous with $\|u_n\| \leq \|P_n\|$ and $(u_n)_n \in \ell_p^w(X^*)$. In fact, we have

$$\begin{aligned} |\langle u_n, x\rangle| &= |\langle u_n, \zeta z\rangle| = |\zeta|\,|\langle u_n, z\rangle| \\ &= |\zeta|\,\left|\lambda_n^0\right| = |\zeta|\,\left|P_n\left(\left(\lambda_0^n\right)_n\right)\right|. \end{aligned}$$

This means $|\langle u_n, z\rangle| = \left|P_n\left(\left(\lambda_0^n\right)_n\right)\right|$, with

$$\|P_n\| = \sup_{(\lambda^n)_n \in B_{\ell_p}} \left|P_n\left(\left(\lambda^n\right)_n\right)\right| \geq \left|P_n\left(\left(\lambda_0^n\right)_n\right)\right| = |\langle u_n, z\rangle|$$

Hence

$$\|u_n\| = \sup_{t \in B_X} |\langle u_n, t\rangle| \leq \|P_n\|,$$

and

$$\begin{aligned} \|(u_n)_n\|_{p,w} &= \sup_{x \in B_X} \left\|\left(\langle u_n, x\rangle\right)_n\right\|_p \\ &= \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} |\langle u_n, x\rangle|^p\right)^{\frac{1}{p}} \\ &\leq \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} \left|\zeta \lambda_n^0\right|^p\right)^{\frac{1}{p}} \\ &\leq |\zeta| \left(\sum_{n=1}^{\infty} \left|\lambda_n^0\right|^p\right)^{\frac{1}{p}} < \infty. \end{aligned}$$

Let $(a_n)_n$ be the sequence defined by $a_n = d_n u_n$. It follows from Lemma 2.1.2 that $(a_n)_n \in \ell_p^u(X^*)$ and further

$$\begin{aligned} \|P\| &= \sup_{x \in B_X} \|P(x)\| = \sup_{x \in B_X} \left\|\sum_{n=1}^{\infty} \langle a_n, x\rangle e_n\right\| \\ &= \sup_{x \in B_X} \left\|\left(\langle a_n, x\rangle\right)_n\right\|_p = \|(a_n)_n\|_p^w. \end{aligned}$$

which gives the desired isometry.

Remark 2.1.1 *The case $p = \infty$ may be treated analogously. we also refer to [19].*

b) Let $(x_n)_n \in \ell_p^u(X)$, with $1 < p < \infty$. We then define $Q : \ell_{p^*} \to X : \lambda \longrightarrow \sum_{n=1}^{\infty} \lambda_n x_n$. By Lemma 2.1.3 Q is well define, continuous and its adjoint $Q^* : X^* \longrightarrow \ell_p$ is given by $Q^*(\varphi) = \left(\langle \varphi, x_n\rangle\right)_n$, $\varphi \in X^*$ and

$$\|Q\| = \|Q^*\| = \sup_{\varphi \in B_{X^*}} \left\|\left(\langle \varphi, x_n\rangle\right)_n\right\| = \|(x_n)_n\|_p^w.$$

By Lemma 2.1.2 there exist a $\xi \in c_0$ and $(y_n)_n \in \ell_p^u(X)$ such that $x_n = \xi_n y_n$ $\forall n \in \mathbb{N}$. Define $S : \ell_{p^*} \to X : \lambda \longrightarrow \sum_{n=1}^{\infty} \lambda_n y_n$. Then $Q = S \circ D_\xi$ which is compact, since $D_\xi : \ell_{p^*} \to \ell_{p^*} : \lambda \longrightarrow \lambda\xi$ is compact by Remark 1.3.2. Conversely, let $Q : \ell_{p^*} \to X$ is compact. Define $x_n := Q(e_n)$. Then $Q(\lambda) = \sum_{n=1}^{\infty} \lambda_n x_n$ for all $\lambda \in \ell_{p^*}$ and by Lemma 2.1.3 it follows that $(x_n)_n \in \ell_p^w(X)$. Now $Q^* : X^* \longrightarrow \ell_p$ is given by $Q^*(\varphi) = (\langle \varphi, x_n \rangle)_n$, $\varphi \in X^*$ is compact by *Schauder's theorem* 1.3.2 and from a) it follows that $(x_n)_n \in \ell_p^u(X)$, where the isometry is again defined by noting that

$$\|Q\| = \|Q^*\| = \sup_{\varphi \in B_{X^*}} \|(\langle \varphi, x_n \rangle)_n\| = \|(x_n)_n\|_p^w .$$

For $p = 1$ the proof follows from Lemma 2.1.2 and proposition 2 of [19]. ■

Corollary 2.1.1 *Let $1 < p < \infty$, be arbitrary. Then*

1) $R : X \longrightarrow \ell_p$ is compact *iff it has a representation of the from $R_{(a_n)_n}(x) = \sum_n \langle x, a_n \rangle e_n = (\langle a_n, x \rangle)_n$ for some $(a_n)_n \in \ell_p^u(X^*)$ and $\|R\| = \|(a_n)_n\|_{\ell_p^w(X^*)}$.*

2) $S : \ell_{p^*} \longrightarrow X$ is compact *iff it has a representation of the from $S_{(x_n)_n}((\xi_n)_n) = \sum_n x_n \xi_n$ for some $(x_n)_n \in \ell_p^u(X)$ and $\|S\| = \|(x_n)_n\|_{\ell_p^w(X)}$.*

Proof. 1) Let $R \in \mathcal{K}(X, \ell_p)$. Then by a) in Theorem 2.1.1, we have $\ell_p^u(X^*)$ is isometrically isomorphic to $\mathcal{K}(X, \ell_p)$, and according there proof, for $(a_n)_n \in \ell_p^u(X^*)$ the continuous map

$$\begin{aligned} R_{(a_n)_n}(x) &= (\langle a_n, x \rangle)_n \\ &= \sum_n \langle x, a_n \rangle e_n, \text{ for all } x \in X. \end{aligned}$$

with $\|R\| = \|(a_n)_n\|_{\ell_p^w(X^*)}$.

2) Let $S \in \mathcal{K}(\ell_{p^*}, X)$. Then by b) in Theorem 2.1.1, we have $\ell_p^u(X^*)$ is isometrically isomorphic to $\mathcal{K}(X, \ell_p)$, and according there proof, for $(x_n)_n \in \ell_p^u(X)$ the continuous map

$$\begin{aligned} S_{(x_n)_n}((\xi_n)_n) &= \sum_n x_n \xi_n \\ &= \sum_n \langle x, a_n \rangle e_n, \text{ for all } (\xi_n)_n \in \ell_{p^*}. \end{aligned}$$

with $\|S\| = \|(x_n)_n\|_{\ell_p^w(X)}$. ■

Proposition 2.1.4 *Let $1 < p < \infty$. Then*

1) For $R \in \mathcal{K}(X, \ell_p)$ and $\varepsilon > 0$ be given. There exist a compact diagonal operator $D_{(\xi_n)_n} : \ell_p \longrightarrow \ell_p$, $(\xi_n)_n \in c_0$, $\left\|D_{(\xi_n)_n}\right\| = 1$ and $\tilde{R} \in \mathcal{K}(X, \ell_p)$ such that $R = D_{(\xi_n)_n} \circ \tilde{R}$, with $\left\|\tilde{R}\right\| < \|R\| + \varepsilon$.

2) For $S \in \mathcal{K}(\ell_p, X)$ and $\varepsilon > 0$ be given. There exist a compact diagonal operator $D_{(\xi_n)_n} : \ell_p \longrightarrow \ell_p$, $(\xi_n)_n \in c_0$, $\left\|D_{(\xi_n)_n}\right\| = 1$ and $\tilde{S} \in \mathcal{K}(\ell_p, X)$ such that $S = \tilde{S} \circ D_{(\xi_n)_n}$, with $\left\|\tilde{S}\right\| < \|S\| + \varepsilon$.

Proof. 1) Let $R \in \mathcal{K}(X, \ell_p)$. Then by Corollary 2.1.1 R has a representation of the from $R_{(a_n)_n}(x) = \sum_n \langle x, a_n \rangle e_n$ for some $(a_n)_n \in \ell_p^u(X^*)$ and $\|R\| = \|(a_n)_n\|_{\ell_p^u(X^*)}$. By Lemma 2.1.1 there exist $(b_n)_n \in \ell_p^u(X^*)$ and $\xi \in C_0$, such that $\|\xi\| = 1$, $a_n = \xi_n b_n$, $\forall n \in \mathbb{N}$ and

$$\|(b_n)_n\|_p^w \leq \|(a_n)_n\|_p^w + \varepsilon. \tag{2.1.4}$$

We put$\tilde{R}(x) = \sum_n \langle x, b_n \rangle e_n$ and $D_{(\xi_n)_n}((\lambda_n)_n) = \xi_n \lambda_n$. By Example 1.3.2 $D_{(\xi_n)_n}$ is compact, by Corollary 2.1.1 again and 2.1.4 $\tilde{R} \in \mathcal{K}(X, \ell_p)$ and $\left\|\tilde{R}\right\| < \|R\| + \varepsilon$.
The proof of (2) is similar to that (1). ■

2.2 The classical p-compact operators

Definition 2.2.1 *Let $1 \leqslant p \leq \infty$. An operator $T : X \longrightarrow Y$ is called classical p-compact (c-p-compact) if there are compact operators $R : X \longrightarrow \ell_p$ and $S : \ell_p \longrightarrow Y$ such that*

$$T = R \circ S.$$

and the vector space of all c-p-compact operator from X to Y is denoted by $\mathcal{K}_p(X, Y)$ with norm defined by

$$\mathrm{k}_p(T) = \inf\{\|R\| \|S\| : R \in \mathcal{K}(X, \ell_p),\ S \in \mathcal{K}(\ell_p, Y),\ T = R \circ S\}.$$

Remark 2.2.1 *If $p = \infty$, T is called ∞-nuclear, ℓ_p is replaced by c_0.*

Theorem 2.2.1 *Let $1 < p < \infty$, and let $T \in \mathcal{L}(X, Y)$. The following are equivalent:*

1) $T \in \mathcal{K}_p(X, Y)$.

2) There exists $R \in \mathcal{K}(X, \ell_p)$ and $S \in \mathcal{L}(\ell_p, X)$ such that $T = S \circ R$. We have

$$\mathrm{k}_p(T) = \inf \|R\| \|S\|,$$

where the infimum runs all over the possible factorizations $T = S \circ R$.

3) There exists a compact diagonal operator $D_{(\xi_n)_n} : \ell_p \longrightarrow \ell_p$, $R \in \mathcal{L}(X, \ell_p)$ and $S \in \mathcal{L}(\ell_p, X)$ such that $T = S \circ D_{(\xi_n)_n} \circ R$. We have

$$\mathrm{k}_p(T) = \inf \|R\| \left\|D_{(\xi_n)_n}\right\| \|S\|,$$

where the infimum runs all over the possible factorizations $T = S \circ R$.

4) Same as 2) except that, $R \in \mathcal{L}(X, \ell_p)$ and $S \in \mathcal{K}(\ell_p, X)$.

Proof. 1) $\Longrightarrow$ 2) is obviously

2) $\Longrightarrow$ 3) Follows from the proof (1) in Proposition 2.1.4. (Let $R \in \mathcal{K}(X, \ell_p)$. Then by Corollary 2.1.1 R has a representation of the from $R_{(a_n)_n}(x) = \sum_n \langle x, a_n \rangle e_n$ for some $(a_n)_n \in \ell_p^u(X^*)$ and $\|R\| = \|(a_n)_n\|_{\ell_p^u(X^*)}$. By Lemma 2.1.1 there exist $(b_n)_n \in \ell_p^u(X^*)$ and $\xi \in C_0$, such that $\|\xi\| = 1$, $a_n = \xi_n b_n$, $\forall n \in \mathbb{N}$ and

$$\|(b_n)_n\|_p^w \leq \|(a_n)_n\|_p^w + \varepsilon.$$

We put $\tilde{R}(x) = \sum_n \langle x, b_n \rangle e_n$ and $D_{(\xi_n)_n}((\lambda_n)_n) = \xi_n \lambda_n$. By Example 1.3.2 $D_{(\xi_n)_n}$ is compact, by Corollary 2.1.1 again and 2.1.4 $R' \in \mathcal{K}(X, \ell_p) \subset \mathcal{L}(X, \ell_p)$ and $\left\|\tilde{R}\right\| < \|R\| + \varepsilon)$. Hence $T = S \circ D_{(\xi_n)_n} \circ \tilde{R}$.

3) $\Longrightarrow$ 4) is obviously

4) $\Longrightarrow$ 1) Suppose that there exists $R \in \mathcal{L}(X, \ell_p)$ and $S \in \mathcal{K}(\ell_p, X)$ such that $T = S \circ R$, with

$$\sigma = \inf \|R\| \|S\|,$$

where the infimum runs all over the possible factorizations $T = S \circ R$. Clearly

$$\sigma \leq k_p(T) = \inf \{\|R\| \|S\| : R \in \mathcal{K}(X, \ell_p),\ S \in \mathcal{K}(\ell_p, Y),\ T = R \circ S\}. \tag{2.2.1}$$

Let $\varepsilon > 0$ there exist $R^0 \in \mathcal{L}(X, \ell_p)$ and $S^0 \in \mathcal{K}(\ell_p, X)$ such that $T = S^0 \circ R^0$ and $\sigma \leq \|S^0\| \|R^0\| < \sigma + \varepsilon$. By (2) in Proposition 2.1.4 there exist a compact diagonal operator

$D_{(\xi_n)_n} : \ell_p \longrightarrow \ell_p$, $(\xi_n)_n \in c_0$, $\left\| D_{(\xi_n)_n} \right\| = 1$ and $P \in \mathcal{K}(\ell_p, X)$ such that $S^0 = P \circ D_{(d_n)_n}$ with $\|P\| < \|S^0\| + \frac{\varepsilon}{\|R^0\|}$. Hence $T = P \circ \left(D_{(\xi_n)_n} \circ R^0 \right)$ with

$$k_p(T) \le \|P\| \left\| D_{(\xi_n)_n} \circ R^0 \right\| \le \|P\| \left\| R^0 \right\| \le \left\| S^0 \right\| \left\| R^0 \right\| + \varepsilon < \sigma + \varepsilon. \tag{2.2.2}$$

Thus $k_p(T) = \sigma$. ■

The next result immediately follows from the above the Corollary 2.1.1 and Theorem 2.2.1.

Corollary 2.2.1 *Let $1 < p < \infty$, and let $T \in \mathcal{L}(X, Y)$. Then the following are equivalent:*

1) $T \in \mathcal{K}_p(X, Y)$.

2) T has representation of the form $T(x) = \sum_n \langle x, x_n^* \rangle y_n$ where $(x_n^*)_n \in \ell_p^u(X^*)$ and $(y_n)_n \in \ell_{p^*}^u(Y)$. In this case

$$\mathrm{k}_p(T) = \inf \|(x_n^*)_n\|_{p,w} \|(y_n)_n\|_{p^*,w}.$$

3) Same as 2) except that $(x_n^*)_n \in \ell_p^u(X^*)$ and $(y_n)_n \in \ell_{p^*}^w(Y)$.

4) Same as 2) except that $(x_n^*)_n \in \ell_p^w(X^*)$ and $(y_n)_n \in \ell_{p^*}^u(Y)$.

Proof. (1) ⇒ (2) Let $T \in \mathcal{K}_p(X, Y)$. then we have $T = S \circ R$ where

$$R \in \mathcal{K}(X, \ell_p) \text{ and } S \in \mathcal{K}(\ell_p, Y).$$

By Corollary 2.1.1 there exist $(x_n^*)_n \in \ell_p^u(X^*)$ such that

$$R(x) = \sum_n x_n^*(x) e_n = (x_n^*(x))_n,$$

and $(y_n)_n \subset \ell_{p^*}^u(Y)$ such that

$$S((a_n)_n) = \sum_n a_n y_n.$$

Then

$$\begin{aligned} T(x) &= S \circ R(x) \\ &= S(R(x)) \\ &= S((x_n^*(x))_n) \\ &= \sum_n x_n^*(x) y_n \end{aligned}$$

that is $T(x) = \sum_n \langle x_n^*, x \rangle y_n$.

(2) $\Rightarrow$ (3) Obviously.

(3) $\Rightarrow$ (4) Let T has representation of the form $T(x) = \sum_n \langle x_n^*, x \rangle y_n$ where $(x_n^*)_n \in \ell_p^u(X^*)$ and $(y_n)_n \in \ell_{p^*}^w(Y)$. By Lemma 2.1.2, we have

$$x_n^* = \lambda_n b_n,$$

where

$$(\lambda_n)_n \in c_0 \text{ and } (b_n)_n \in \ell_p^w(X^*).$$

Then

$$\begin{aligned} T(x) &= \sum_n \langle \lambda_n b_n, x \rangle y_n \\ &= \sum_n \langle b_n, x \rangle \lambda_n y_n \end{aligned}$$

Also by Lemma 2.1.2, we have $(\lambda_n y_n)_n \in \ell_{p^*}^u(Y)$.

(4) $\Rightarrow$ (1) Let T has representation of the form $T(x) = \sum_n \langle x_n^*, x \rangle y_n$, where $(x_n^*)_n \in \ell_p^w(X^*)$ and $(y_n)_n \in \ell_{p^*}^u(Y)$. By Corollary 2.1.1 there exist $R \in \mathcal{L}(X, \ell_p)$ and $S \in \mathcal{K}(\ell_p, Y)$ such that

$$R(x) = \sum_n x_n^*(x) \text{ and } S((a_n)_n) = \sum a_n y_n.$$

Then

$$T = S \circ R.$$

■

Remark 2.2.2 *We have $\mathcal{K}_p(X,Y) \subset \mathcal{L}(X,Y)$, with*

$$\|T\| \leq \mathrm{k}_p(T). \tag{2.2.3}$$

Proof. Let $T \in \mathcal{K}_p(X,Y)$. By Hölder inequality, we have

$$\begin{aligned}\|T(x)\| &= \sup_{\|y^*\|\leq 1}\left\|\left\langle \sum_{n\in J}\langle x, x_n^*\rangle y_n, y^*\right\rangle\right\| = \sup_{\|y^*\|\leq 1}\left|\sum_{n\in J}\langle x, x_n^*\rangle\right| |\langle y_n, y\rangle| \\ &\leq \left(\sum_{n=1}^{\infty}|\langle x, x_n^*\rangle|^p\right)^{\frac{1}{p}} \sup_{\|y^*\|\leq 1}\left(\sum_{n=1}^{\infty}|\langle y^*, y_n\rangle|^{p^*}\right)^{\frac{1}{p^*}} \\ &\leq \sup_{x\in B_X}\left(\sum_{n=1}^{\infty}|\langle x, x_n^*\rangle|^p\right)^{\frac{1}{p}} \sup_{\|y^*\|\leq 1}\left(\sum_{n=1}^{\infty}|\langle y^*, y_n\rangle|^{p^*}\right)^{\frac{1}{p^*}} \\ &= \|(x_n^*)_n\|_{p,q}^w \|(y_n)_n\|_{p^*,q^*}^w .\end{aligned}$$

Hence, $T \in \mathcal{L}(X,Y)$, with

$$\|T\| \leq k_p(T).$$

■

Theorem 2.2.2 *Let $1 \leq p < \infty$. $\left(\mathcal{K}_p(X,Y), \|T\|_{\mathcal{K}_p}\right)$ is a Banach space .*

Proof. 1) Let T_1, $T_2 \in \mathcal{K}_p(X,Y)$ and $\alpha \in \mathbb{K}$. For any $\varepsilon > 0$, T_1 and T_2, By (2) in Corollary 2.2.1, we will consider

$$T_k(x) = \sum_n \left\langle x, x_{k,n}^*\right\rangle y_{k,n}, \text{ for each } x \in X,\ k = 1,2,$$

such that

$$\sum_{n=1}^{\infty}\left|\left\langle x, x_{k,n}^*\right\rangle\right|^p \leq k_p(T_k) + \frac{\varepsilon}{2},\ k = 1,2,\ \forall x \in B_X,$$

and

$$\sum_{n=1}^{\infty}|\langle y_{k,n}, y^*\rangle|^{p^*} \leq k_p(T_k) + \frac{\varepsilon}{2},\ k = 1,2.$$

Hence

$$\begin{aligned}&\left\|\left(\left|\left\langle x, x_{k,n}^*\right\rangle\right|\right)_{1\leq k\leq 2,\ 1\leq n\leq N}\right\|_p^p \\ &\leq \sum_{i=1}^{N}\left|\left\langle x, x_{1,i}^*\right\rangle\right|^p + \sum_{i=1}^{N}\left|\left\langle x^{**}, x_{2,i}^*\right\rangle\right|^p \\ &\leq \left(k_p(T_1) + k_p(T_2) + \frac{\varepsilon}{2}\right).\end{aligned}$$

Therefore

$$\left\|\left(\left|\left\langle x,x_{k,n}^{*}\right\rangle\right|\right)_{1\leq k\leq 2,\ 1\leq n\leq\infty}\right\|_{p}\leq\left(k_{p}\left(T_{1}\right)+k_{p}\left(T_{2}\right)+\varepsilon\right)^{\frac{1}{p}}.$$

In the same way

$$\left\|\left(\left|\left\langle y^{*},y_{k,n}\right\rangle\right|\right)_{1\leq k\leq 2,\ 1\leq n\leq\infty}\right\|_{p^{*}}\leq\left(k_{p}\left(T_{1}\right)+k_{p}\left(T_{2}\right)+\varepsilon\right)^{\frac{1}{p^{*}}}.$$

Hence

$$\begin{aligned}\left(T_{1}+T_{2}\right)(x) &= \sum_{1\leq n\leq N}\left\langle x,x_{1,n}^{*}\right\rangle y_{1,i}+\left\langle x,x_{2,n}^{*}\right\rangle y_{2,n}\\ &= \sum_{1\leq k\leq 2,\ 1\leq n\leq N}\left\langle x,x_{k,n}^{*}\right\rangle y_{k,n},\end{aligned}$$

and

$$\begin{aligned}k_{p}\left(T_{1}+T_{2}\right) &= \inf\left\|\left(\left|\left\langle x,x_{k,n}^{*}\right\rangle\right|\right)_{1\leq k\leq 2,\ 1\leq n\leq N}\right\|_{p}\left\|\left(\left|\left\langle y^{*},y_{k,n}\right\rangle\right|\right)_{1\leq k\leq 2,\ 1\leq n\leq\infty}\right\|_{p^{*}}\\ &\leq k_{p}\left(T_{1}\right)+k_{p}\left(T_{2}\right)+\varepsilon.\end{aligned}$$

Since ε is arbitrary, then $T_1+T_2\in\mathcal{K}_p(X,Y)$ and

$$k_{p}\left(T_{1}+T_{2}\right)\leq\left(k_{p}\left(T_{1}\right)+k_{p}\left(T_{2}\right)\right)<\infty. \tag{2.2.4}$$

Also we have

$$\begin{aligned}k_{p}\left(\alpha T_{1}\right) &= k_{p}\left(\alpha\sum_{n}\left\langle .,x_{1,n}^{*}\right\rangle y_{1,n}\right)\\ &= k_{p}\left(\sum_{n}\left\langle .,\alpha x_{1,n}^{*}\right\rangle y_{1,n}\right)\\ &= \inf\{\left\|\left(\alpha x_{1,n}^{*}\right)_{n}\right\|_{p,q}^{w}\left\|\left(y_{1,n}\right)_{n}\right\|_{p^{*}q^{*}}^{w}\}\\ &= |\alpha|\inf\{\left\|\left(x_{1,n}^{*}\right)_{n}\right\|_{p,q}^{w}\left\|\left(y_{1,n}\right)_{n}\right\|_{p^{*}q^{*}}^{w}\}\\ &= |\alpha|\,k_{p}\left(T_{1}\right).\end{aligned}$$

Therefore, $\alpha T_1\in\mathcal{K}_p(X,Y)$ and

$$k_p(\alpha T_1)=|\alpha|\,k_p(T_1). \tag{2.2.5}$$

By 2.2.4 and 2.2.5 $\mathcal{K}_p(X,Y)$ is a vector subspace.

2)Let $T \in \mathcal{K}_p(X,Y)$ and $\alpha \in \mathbb{K}$, $n \in I$. If $T = 0$, then $k_p(T) = 0$. Conversely, by 2.2.3 we have

$$\|T\| \leq k_p(T) = 0.$$

Thus, $T = 0$. Hence by 2, 2.2.5 and 2.2.4$\mathcal{K}_p(X,Y)$ is a normed space.

3. Let $(T_k)_k$ a sequence in $\mathcal{K}_p(X,Y)$ such that $\sum_k k_p(T_k) < \infty$. Then, by 2.2.3 $\sum_k \|T_k\| \leq \sum_k k_p(T_k) < \infty$. Hence, the series $\sum_k T_k$ is converges to an operator $T = \sum_k T_k$ in $\mathcal{L}(X,Y)$. Now, we show that $T \in \mathcal{K}_p(X,Y)$ and $k_p(T) \leq \sum_k k_p(T_k)$.

Let $\varepsilon > 0$. For each $k \in \mathbb{N}$, choose $\left(x^*_{k,n}\right)_n$ in $\ell^w_p(X^*)$ and $\left(y^*_{k,n}\right)_n$ in $\ell^w_{p^*}(Y)$, such that $T_k(x) = \sum_n \left\langle x, x^*_{k,n}\right\rangle y_{k,n}$ and

$$\left\|\left(x^*_{k,n}\right)_n\right\|^w_p \leq \left(k_p(T_k) + \frac{\varepsilon}{2^k}\right)^{\frac{1}{p}}, \quad \left\|(y_{k,n})_{k,n}\right\|^w_{p^*} \leq \left(k_p(T_k) + \frac{\varepsilon}{2^k}\right)^{\frac{1}{p^*}}.$$

Since, $T = \sum_k T_k = \sum_k \sum_n \left\langle x, x^*_{k,n}\right\rangle y_{k,n} = \sum_{k,n} \left\langle ., x^*_{k,n}\right\rangle y_{k,n}$, then there is $\left(x^*_{k,n}\right)_{k,n}$ in $\ell^w_p(X^*)$, $\left(y^*_{k,n}\right)_{k,n}$ in $\ell^w_{p^*,q^*}(Y)$, such that

$$\left\|\left(x^*_{k,n}\right)_{k,n}\right\|^w_p \leq \left(\sum_k k_p(T_k) + \varepsilon\right)^{\frac{1}{p}}, \quad \left\|(y_{k,n})_{k,n}\right\|^w_{p^*} \leq \left(\sum_k k_p(T_k) + \varepsilon\right)^{\frac{1}{p^*}}.$$

Hence

$$k_p(T) \leq \left\|\left(x^*_{k,n}\right)_{k,n}\right\|^w_p \left\|(y_{k,n})_{k,n}\right\|^w_{p^*} \leq \sum_k k_p(T_k) + \varepsilon.$$

Since ε is arbitrary,

$$k_p(T) \leq \sum_k k_p(T_k) < \infty.$$

Thus, $T \in \mathcal{K}_p(X,Y)$. ■

Proposition 2.2.1 *Let X, Y and G be Banach spaces.*

i) For $T \in \mathcal{K}_p(X,Y)$ and $S \in \mathcal{L}(Y,G)$, we have $ST \in \mathcal{K}_p(X,G)$, with

$$\mathrm{k}_p(ST) \leq \|S\| . \mathrm{k}_p(T).$$

ii) For $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{K}_p(Y,G)$, we have $ST \in \mathcal{K}_p(X,G)$, with

$$\mathrm{k}_p(ST) \leq \mathrm{k}_p(S) . \|T\|.$$

Proof. i) Let $T \in \mathcal{K}_p(X,Y)$ and $S \in \mathcal{L}(Y,G)$, such that

$$T(x) = \sum_n \langle x, x_n^* \rangle y_n, \text{ for each } x \in X.$$

We have

$$S(T(x)) = \sum_n \langle x, x_n^* \rangle S(y_n), \text{ for each } x \in X,$$

with

$$\|(x_n^*)_n\|_p^w < \infty,$$

and

$$\begin{aligned} \|(S(y_n))_n\|_{p^*}^w &= \sup_{z^* \in B_{G^*}} \|(|\langle S(y_n), z^* \rangle|)_n\|_{p^*} \\ &= \sup_{z^* \in B_{G^*}} \|(|\langle y_n, S^*(z^*) \rangle|)_n\|_{p^*} \\ &= \|S\| \sup_{z^* \in B_{Y^*}} \left\| \left(\left| \left\langle y_n, \frac{S^*(z^*)}{\|S\|} \right\rangle \right| \right)_n \right\|_{p^*}. \end{aligned}$$

Then

$$\|(S(y_n))_n\|_{p^*}^w \leq \|S\| \sup_{y^* \in B_{Y^*}} \|(|\langle y_n, y^* \rangle|)_n\|_{p^*}.$$

Thus, $ST \in \mathcal{K}_p(X,G)$ and

$$k_p(ST) \leq \|S\| . k_p(T).$$

2. Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{K}_p(Y,G)$, such that

$$S(x) = \sum_n \langle x, y_n^* \rangle z_n, \text{ for each } x \in X.$$

We have

$$\begin{aligned} S(T(x)) &= \sum_n \langle T(x), y_n^* \rangle z_n \\ &= \sum_n \langle x, T^*(y_n^*) \rangle z_n. \end{aligned}$$

Then

$$S(T(x)) = \sum_n \langle x, T^*(y_n^*) \rangle z_n, \text{ for each } x \in X.$$

We have

$$\begin{aligned} \left\| (T^*(y_n^*))_n \right\|_p^w &= \sup_{x \in B_X} \left(\sum_{n=1}^{\infty} \left| \langle x, T^*(y_n^*) \rangle \right|^p \right)^{\frac{1}{p}} \\ &= \|T\| \sup_{x \in B_X} \left(\sum_{n=1}^{1} \left| \left\langle \frac{T^{**}(x)}{\|T\|}, y_n^* \right\rangle \right|^p \right)^{\frac{1}{p}} \\ &\leq \|T\| \sup_{y \in B_{Y^{**}}} \left\| (|\langle y, y_n^* \rangle|)_n \right\|_p \\ &\leq \|T\| \left\| (y_n^*)_n \right\|_p^w < \infty, \end{aligned}$$

and

$$\left\| (z_n)_n \right\|_{p^*}^w < \infty.$$

Hence, $ST \in \mathcal{K}_p(X, G)$ and

$$k_p(ST) \leq k_p(S) \|T\|.$$

■

Proposition 2.2.2 *Let X and Y be Banach spaces. $\mathcal{L}_f(X, Y)$ is dense in $\mathcal{K}_p(X, Y)$.*

Proof. Let $T \in \mathcal{K}_p(X, Y)$. Then

$$T(x) = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle y_n, \text{ for each } x \in X,$$

with

$$\left\| (x_n^*)_n \right\|_p^w < 1 \text{ and } \left\| (y_n)_n \right\|_{p^*}^w < \infty.$$

We put

$$T_k(x) = \sum_{i=1}^{K} \langle x, x_i^* \rangle y_i,$$

we find

$$T_k \in \mathcal{L}_f(X, Y)$$

and

$$(T - T_k)(x) = \sum_{n=1}^{\infty} \left\langle x, x_{k+n}^* \right\rangle y_{k+n}.$$

Since $\lim_{k\longrightarrow\infty}\sum_{n=1}^{\infty}|\langle x,x_n^*\rangle|^p<\infty$ and $n+k\longrightarrow\infty$ as $k\longrightarrow\infty$, then

$$\sum_{n=1}^{\infty}\left|\left\langle x,x_{n+k}^*\right\rangle\right|^p \underset{k\longrightarrow\infty}{\longrightarrow} 0.$$

On the other hand, we also have

$$\sup_{y^*\in B_{Y^*}\leq 1}\sum_{n=1}^{\infty}|\langle y_{k+n},y^*\rangle|^{p^*}\longrightarrow 0 \text{ as } k\to\infty.$$

Hence

$$\begin{aligned} k_p\left(T-T_k\right) &\leq \sup_{x^{**}\in B_{X^{**}}}\left(\sum_{n=1}^{\infty}\left|\left\langle x,x_{k+n}^*\right\rangle\right|^p\right)^{\frac{1}{p}}\sup_{y^*\in B_{Y^*}\leq 1}\left(\sum_{i=1}^{\infty}|\langle y_{k+n},y^*\rangle|^{p^*}\right)^{\frac{1}{p^*}} \\ &\longrightarrow 0 \text{ as } k\to\infty. \end{aligned}$$

■

Immediate consequence of Theoreme 2.2.2, Proposition 2.2.1 and Proposition 2.2.2 are the following result

Corollary 2.2.2 *Let X and Y be Banach spaces. For $1<p<\infty$, $(\mathcal{K}_p(X,Y),\mathrm{k}_p(.))$ is a Banach ideal.*

Chapter 3

Injective hull of classical p-compact operators

In this chapter, we define the notion of the injective hull and some properties and study the injective hull of $\mathcal{K}_p$ and basic formulas. This chapter is mainly based on [14] [13][9][12]. Recall from [11, p. 421] that

$$X^{\infty} = \ell^{\infty}(B_{X^*}) = \{f : B_{X^*} \to \mathbb{K}\ ;\ f \text{ is bounded}\},$$

There is a linear and isometric embedding $J_{X^{\infty}} : X \to X^{\infty}$ is given by

$$J_{X^{\infty}}(x) = \langle x; \cdot \rangle .$$

Recall from [**?**, Theorem 4.6.9] that an operator ideal $\mathcal{I}$ is *injective* if and only if for every (metric) injection $J \in \mathcal{L}(Y, Y_0)$ and every $T \in \mathcal{L}(X, Y)$ it follows from $J \circ T \in \mathcal{I}(X, Y_0)$ that $T \in \mathcal{I}(X, Y)$. The smallest injective operator ideal $\mathcal{I}^{inj}$ that contains $\mathcal{I}$ is called the *injective hull* of $\mathcal{I}$. Refer to [18, p. 109 and 111] for the following concrete definitions of the *injective hulls* of an operator ideal $\mathcal{I}$, we have

$$\mathcal{I}^{inj}(X, Y) = \{R \in \mathcal{L}(X, Y) : J_Y^{\infty} \circ R \in \mathcal{I}(X, Y^{\infty})\},$$

Where for $R \in \mathcal{I}^{inj}(X, Y)$, we have

$$I^{inj}(R) = I(J_Y^{\infty} \circ R).$$

The space X^{∞} has the *extension property* (or it is an *injective Banach space*), i.e. if Y, G are Banach spaces such that G is a subspace of Y, then for any $T\epsilon\mathcal{L}(G, X^{\infty})$ there exist $\widetilde{T} \in \mathcal{L}(Y, X^{\infty})$ such that $\left\|\widetilde{T}\right\| = \|T\|$ and $\widetilde{T}(x) = T(x)$ for all $x \in G$.

Proposition 3.0.3 *Let $1 \leq p \leq \infty$. Given any pair X, Y of Banach spaces and $T \in \mathcal{L}(X, Y)$, the following are equivalent:*

(a) $T \in \mathcal{K}_p^{inj}(X, Y)$.

(b) There exists a closed subspace Σ of ℓ_p such that $T = S \circ R$ for some $R \in \mathcal{K}(X, \Sigma)$ and $S \in \mathcal{K}(\Sigma, Y)$.

(c) There exists a closed subspace Σ of ℓ_p such that $T = S \circ R$ for some $R \in \mathcal{K}(X, \Sigma)$ and $S \in \mathcal{L}(\Sigma, Y)$.

If $1 < p \leq \infty$, then the above assertions are equivalent to

(d) There exists a closed subspaceΣ of ℓ_p such that$T = S \circ R$ for some $R \in \mathcal{L}(X, \Sigma)$ and $S \in \mathcal{K}(\Sigma, Y)$. We have

$$\mathcal{K}_p^{inj}(T) = \inf\{\|S\|\|R\| : T = S \circ R\},$$

where the infimum is taken over all relevant factorizations.

Proof. a)⇒b) Let $T \in \mathcal{K}_p^{inj}(X, Y)$. Then $J_Y^{\infty} \circ T \in \mathcal{K}_p(X, Y^{\infty})$. This means there exists two operators $R \in \mathcal{K}(X, \ell_p)$ and $Q \in \mathcal{K}(\ell_p, Y^{\infty})$ such that $J_Y^{\infty} \circ T = Q \circ R$. Put $\Sigma = \overline{R(X)}$, define $P : R(X) \to Y$, $R(x) \longmapsto T(x)$ and let S be the continuous linear extension to Σ such that $S_{|R(X)} = P$. Hence $S \circ R(x) = S(R(x)) = T(x)$ for all $x \in X$.

b)⇒c) is trivial.

b)⇒d) is trivial.

c)⇒a) Suppose that there exists a closed subspace Σ of ℓ_p such that $T = S \circ R$ for some $R \in \mathcal{K}(X, \Sigma)$ and $S \in \mathcal{L}(\Sigma, Y)$. We have $J_Y^{\infty} \circ T = (J_{Y^{\infty}} \circ S) \circ R$. Since ℓ_p, Σ are Banach spaces such that Σ is a subspace of ℓ_p, then for $J_Y^{\infty} \circ S \in \mathcal{L}(\Sigma, Y^{\infty})$ there existe $\widetilde{S} \in \mathcal{L}(\ell_p, Y^{\infty})$ such that $\left\|\widetilde{S}\right\| = \|J_Y^{\infty} \circ S\|$ and

$$\widetilde{S}(x) = J_Y^{\infty} \circ S(x)$$

for all $x \in \Sigma$ (Y^{∞} has the *extension property*). By the Theorem 2.2.1

$J_Y^\infty \circ T \in \mathcal{K}_p(X, Y^\infty)$.

d)⇒a) The proof is similar to that c)⇒a).

For each factorization $T = S \circ R$ in all case, there exist $\widetilde{S} \in \mathcal{L}(\ell_p, Y^\infty)$, with $\left\|\widetilde{S}\right\| = \|S\|$ (Y^∞ has the *extension property*).

Clearly that $k_p^{inj}(T) = k_p(J_Y^\infty \circ T) \leq \inf \|R\| \left\|\widetilde{S}\right\| = \inf \|R\| \|S\|$. We have for $\varepsilon > 0$ there exists two operators $R^0 \in \mathcal{K}(X, \ell_{p,q})$ and $\widetilde{S}^0 \in \mathcal{K}(\ell_{p,q}, Y^\infty)$ such that $J_Y^\infty \circ T = \widetilde{S}^0 \circ R^0$, with

$$k_p^{inj}(T) \leq \left\|R^0\right\| \left\|\widetilde{S}^0\right\| \leq k_p^{inj}(T) + \varepsilon.$$

Restrict $\widetilde{S}^0$ to $R^0(X)$ and then consider the continuous linear extension $S : \overline{R^0(X)} \to Y$.

We find $T = S \circ R^0$ (by the above proof of a)⇒b)) and

$$k_p^{inj}(T) \leq \|S\| \left\|R^0\right\| = \left\|\widetilde{S}^0_{|R^0(X)}\right\| \left\|R^0\right\| \leq \left\|\widetilde{S}^0\right\| \left\|R^0\right\| \leq k_p^{inj}(T) + \varepsilon.$$

■

Corollary 3.0.3 *Let X, Y be Banach spaces, where Y is injective (has the* extension property*). Then $\mathcal{K}_p(X,Y) = \mathcal{K}_p^{inj}(X,Y)$ with $\mathcal{K}_p(T) = \mathcal{K}_p^{inj}(T)$.*

Proof. Let $T \in \mathcal{K}_p(X,Y)$. By definition $T \in \mathcal{K}_p^{inj}(X,Y)$, with $k_p^{inj}(T) \leq k_p(T)$. Conversely, let $T \in \mathcal{K}_p^{inj}(X,Y)$. Then by there exist two operators $R \in \mathcal{K}(X, \Sigma)$ and

$S \in \mathcal{K}(\Sigma, Y)$ such that $T = S \circ R$. Since Y is *injective,* the operator S extends to a continuous linear operator $\tilde{S} : \ell_p(\mathbb{K}) \to Y$ such that $\|S\| = \left\|\tilde{S}\right\|$ and

$T = \tilde{S} \circ R$. By Theorem 2.2.1, $T \in \mathcal{K}_p(X,Y)$ and

$$k_p(T) \leq \|R\| \left\|\tilde{S}\right\| = \|R\| \|S\|.$$

Since the factorization $T = S \circ R$ was arbitrary and by Theorem 3.0.3, it follows that $k_p(T) \leq k_p^{inj}(T)$. ■

Theorem 3.0.3 *Let $1 \leq p \leq \infty$ and let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. Then $T \in \mathcal{K}_p^{inj}(X,Y)$ if and only if there is a sequence $(x_n^*)_n \in \ell_p^u(X^*)$ such that*

$$\|T(x)\| \leq \|(x_n^*(x))_n\|_p, \text{ for all } x \in X.$$

In this case,

$$\mathrm{k}_p^{inj}(T) = \inf\left\{\|(x_n^*)_n\|_{p,w^*} : \|T(x)\| \leq \|(x_n^*(x))_n\|_p, \text{ for all } x \in X\right\}.$$

Proof. Assume $T \in \mathcal{K}_p^{inj}(X;Y)$. Then $J_{Y^\infty} \circ T \in \mathcal{K}_p(X, \ell_\infty(B_{Y^*}))$. Therefore, there are $(a_n)_n \in \ell_p^u(X^*)$ and $(y_n)_n \in \ell_{p^*}^u(\ell_\infty(B_{Y^*}))$ such that

$$J_{Y^\infty} \circ T(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n.$$

Set $x_n^* = \|(y_n)_n\|_{p^*}^w a_n$, for all n then $(x_n^*) \in \ell_p^u(X^*)$. Since J_{Y^∞} isometric embedding, we have

$$\begin{aligned}
\|T(x)\| &= \|(J_{Y^\infty} \circ T)(x)\| = \sup_{\|y^*\|_{B_{Y^\infty}} \le 1} |\langle (J_{Y^\infty} \circ T)(x), y^* \rangle| \\
&= \sup_{\|y^*\|_{B_{Y^\infty}} \le 1} \left| \left\langle \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n, y^* \right\rangle \right| \\
&= \sup_{\|y^*\|_{B_{Y^\infty}} \le 1} \left| \sum_{n=1}^{\infty} \langle x, a_n \rangle \langle y_n, y^* \rangle \right| \\
&\overset{\text{I H}}{\le} \left(\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^p \right)^{\frac{1}{p}} \sup_{\|y^*\|_{B_{Y^\infty}} \le 1} \left(\sum_{n=1}^{\infty} \langle y_n, y^* \rangle^{p^*} \right)^{\frac{1}{p^*}} \\
&= \left(\sum_{n=1}^{\infty} \left| \left\langle x, a_n \|(y_n)_n\|_{p^*}^w \right\rangle \right|^p \right)^{\frac{1}{p}} \\
&\le (\sum_{n=1}^{\infty} |\langle x, x_n^* \rangle|^p)^{\frac{1}{p}}.
\end{aligned}$$

Thus, with each representation $J_{Y^\infty} \circ T(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n$ we associate a sequence $(x_n^*)_n \in \ell_p^u(X^*)$ for which the desired inequality holds and which also satisfies

$$\|(x_n^*)_n\|_p^w = \|(y_n)_n\|_{p^*}^w \|(a_n)_n\|_p^w$$

Therefore, $\mathcal{K}_p^{inj}(T) = \mathcal{K}_p(J_{Y^\infty} \circ T) \ge \inf\left\{ \|(x_n^*)\|_p^{w^*} : \|Tx\| \le \|(< x, x_n^* >)\|_p, \forall x \in X \right\}$. Conversely, suppose $\|T(x)\| \le \|(\langle x, x_n^* \rangle)_n\|_p$ $\forall x \in X$ where $(x_n^*) \in \ell_p^u(X^*)$. Define

$P : X \longrightarrow \ell_p$ by $P(x) = (\langle x, x_n^* \rangle)_n$. Then the operator P is compact and $\|P\| = \|(x_n^*)_n\|_p^w$. Clearly, $\|T(x)\| \le \|P(x)\|_p$ for all $x \in X$. Therefore, $S : P(X) \longrightarrow Y : P(x) \longrightarrow T(x)$ defines a bounded linear operator with $\|S\| \le 1$. Let $Q : \overline{P(X)} \longrightarrow Y$ be the continuous linear extension of S, so $\|Q\| \le 1$. By 3.0.3 we have $T \in \mathcal{K}_p^{inj}(X;Y)$ and

$$\mathcal{K}_p^{inj}(T) \le \|P\| = \|(x_n^*)_n\|_p^{w^*}.$$

Since this is true for all $(x_n^*)_n \in \ell_p^u(X^*)$ for which the inequality holds, we have

$$\mathcal{K}_p^{inj}(T) = \inf\left\{\|(x_n^*)\|_p^{w^*} : \|T(x)\| \le \|(\langle x, x_n^*\rangle)\|_p, \forall x \in X\right\}.$$

■

3.1 Unconditionally quasi p-nuclear operators

In [13, p. 136], the author introduce quasi unconditionally p-nuclear operators (quasi u-p-nuclear operators) as follows:

Definition 3.1.1 *Let $1 \le p \le \infty$. A linear operator $T : X \longrightarrow Y$ is called quasi u-p-nuclear if there exists $(x_n^*) \in \ell_p^u(X^*)$ such that*

$$\|T(x)\| \le \|(x_n^*(x))\|_p, \ \forall x \in X.$$

The collection of all quasi u-p-nuclear operators from X to Y is denoted by $\mathcal{N}_{up}^Q(X;Y)$. The norm

$$v_{up}^Q(T) = \inf\left\{\|(x_n^*)\|_p^w : \|T(x)\| \le \|(x_n^*(x))\|_p, \forall x \in X\right\},$$

turns the vector space $\mathcal{N}_{up}^Q(X;Y)$ into a Banach space.

Comparing Definition 3.1.1 with Theorem 3.0.3 yields:

Theorem 3.1.1 *Let $1 \le p \le \infty$. Then $\mathcal{N}_{up}^Q(X;Y) = \mathcal{K}_p^{inj}(X;Y)$ and $v_{up}^Q(T) = \|T\|_{\mathcal{K}_p^{inj}}$ for all $T \in \mathcal{N}_{up}^Q(X;Y)$*

The Theorem 3.1.1 and Corollary 3.0.3 leads the following result

Corollary 3.1.1 *Let $1 \le p \le \infty$. Suppose Y is injective. Then $T \in \mathcal{N}_{up}^Q(X;Y)$ if and only if $T \in \mathcal{K}_p(X,Y)$ and $v_{up}^Q(T) = \mathcal{K}_p^{inj}(T)$.*

Unconditionally p-compact operators

In [13] the concept of relatively unconditionally p-compact set is defined as follows:

Definition 3.1.2 *A subset A of X is called relatively unconditionally p-compact (or relatively u-p-compact) if there exists $(x_n) \in \ell_p^u(X)$ such that*

$$A \subseteq p - co\left(\{x_n\}\right) := \left\{ \sum_n a_n x_n : (a_n) \in B_{\ell_{p^*}} \right\}.$$

Using this definition, it is natural to introduce the concept of unconditionally p-compact operator as in [13, p. 135]

Definition 3.1.3 *A linear operator $T : X \longrightarrow Y$ is said to be u-p-compact if $T(B_X)$ is a relatively u-p-compact subset of Y. The collection of all u-p-compact operators from X to Y is denoted by $\mathcal{U}_p(X;Y)$. A norm $u_p$is defined on $\mathcal{U}_p(X;Y)$ by*

$$u_p(T) = \inf \left\{ \|(y_n)\|_p^w : (y_n)_n \in \ell_p^u(Y) \text{ and } T(B_X) \subseteq p - co\left(\{y_n\}\right) \right\}.$$

Proposition 3.1.1 *[13, Theorem 2.1] For $1 \leq p < \infty$, $(\mathcal{U}_p,u_p)$ is a Banach operator ideal.*

Proof. The proof of the following Proposition is similar to that Corollary 2.2.2. ■

Proposition 3.1.2 *Let $1 \leq p < \infty$, $T \in \mathcal{L}(X,Y)$ and $(y_n)_n \in \ell_p^w(Y)$. The following statements are equivalent:*

a) $\|T^*(y^*)\| \leq \|(y^*(y_n))_n\|_p , \forall y^* \in Y^*$.

b) $T(B_X) \subseteq p - co\left(\{y_n\}\right)$.

Proof. $a \implies b$.By contradiction, assume that there exists $x_0 \in B_X$ so that $T(x_0) \notin \overline{p - co\{y_n\}}$. We put $A = \{T(x_0)\}$ and $B = \overline{p - co\{y_n\}}$. As A and B are convex, A is compact and B is closed. By Hanh-Banach, we can separate A and B strictly by a closed hyperplane; that is to say, there exist $\alpha > 0$ and $y^* \in Y^*$ such that $H = [\langle \cdot, y^* \rangle \equiv \alpha]$, $\langle T(x_0), y^* \rangle > \alpha$ and $\langle y, y^* \rangle < \alpha$ for all $y \in B$. Then

$$\begin{aligned}
\alpha &< \langle T(x_0), y^* \rangle = \langle x_0, T^*(y^*) \rangle \\
&\leq \sup_{x \in B_X} \langle x, T^*(y^*) \rangle = \|T^*(y^*)\| \\
&\leq \left(\sum_{n=1}^{\infty} |\langle y_n, y^* \rangle|^p \right)^{\frac{1}{p}} \\
&= \sup_{\alpha_n \in \ell_{p^*}} \left| \sum_{n=1}^{\infty} \alpha_n \langle y_n, y^* \rangle \right| \\
&= \sup_{\alpha_n \in \ell_{p^*}} \left| \left\langle \sum_{n=1}^{\infty} \alpha_n y_n, y^* \right\rangle \right| \leq \alpha,
\end{aligned}$$

a contradiction.

$(b) \Rightarrow (a)$ Given $\varepsilon > 0$ and $y^* \in B_{Y^*}$, choose $x \in B_X$ such that $\|T^*(y^*)\| < |\langle x, T^*(y^*)\rangle| + \frac{\varepsilon}{2}$. Now, take $(a_n) \in B_{\ell_{p^*}}$ so that $\|T(x) - \sum_{n=1}^{\infty} \alpha_n y_n\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
\|T^*(y^*)\| &< |\langle x, T^*(y^*)\rangle| + \frac{\varepsilon}{2} = |\langle T(x), y^*\rangle| + \frac{\varepsilon}{2} \\
&\leq \left|\left\langle T(x) - \sum_{n=1}^{\infty} \alpha_n y_n, y^* \right\rangle\right| + \left|\left\langle \sum_{n=1}^{\infty} \alpha_n y_n, y^* \right\rangle\right| + \frac{\varepsilon}{2} \\
&< \sum_{n=1}^{\infty} |\alpha_n| \, |\langle y_n, y^*\rangle| + \varepsilon \leq \|(\alpha_n)_n\|_{p^*} \left(\sum_{n=1}^{\infty} |\langle y_n, y^*\rangle|^p\right)^{\frac{1}{p}} + \varepsilon \\
&\leq \left(\sum_{n=1}^{\infty} |\langle y_n, y^*\rangle|^p\right)^{\frac{1}{p}} + \varepsilon.
\end{aligned}$$

and letting $\varepsilon \to 0$ we obtain

$$\|T^*(y^*)\| \leq \|(x_n^*(y^*))\|_p .$$

■

Arguing in a similar way, we obtain the dual version of the above result:

Proposition 3.1.3 *Let $1 \leq p < \infty$, $T \in \mathcal{L}(X, Y)$ and $(x_n^*)_n \in \ell_p^w(Y)$. The following statements are equivalent:*

a) $\|T(x)\| \leq \|(x_n^*(x))\|_p , \forall x \in X.$

b) $T^*(B_{Y^*}) \subseteq p - co(\{x_n^*\})$.

Proposition 3.1.4 *Let $1 \leq p < \infty$. For Banach spaces X and Y we have:*

$$T \in \mathcal{N}_{up}^Q(X;Y) \Longleftrightarrow T^* \in \mathcal{U}_p(Y^*;X^*).$$

In this case

$$v_{up}^Q(T) = u_p(T^*).$$

Proof. This is immediate from Proposition 3.1.3. ■

Proposition 3.1.5 *Let $1 \leq p < \infty$. For Banach spaces X and Y we have:*

$$T \in \mathcal{U}_p(X;Y) \Longleftrightarrow T^* \in \mathcal{N}_{up}^Q(Y^*;X^*).$$

In this case

$$v_{up}^{Q}(T^{*}) = u_{p}(T).$$

Proof. Let $T \in \mathcal{U}_p(X;Y)$ and let $(y_n)_n \in \ell_p^u(Y)$ be such that $T\left(B_X\right) \subseteq p - co\left(\{y_n\}\right)$. Then by Proposition 3.1.2,

$$\|T^{*}\left(y^{*}\right)\| \leq \left\|\left(i_Y(y_n\left(y^{*}\right))\right)\right\|_p ,$$

for every $y^{*} \in Y^{*}$. Note that $(i_Y(y_n) \in \ell_p^u(Y)$, where $i_Y : Y \longrightarrow Y^{**}$ is the natural isometry, and

$$\left\|(i_Y(y_n)\right\|_{p;w} = \left\|(y_n)\right\|_{p,w} .$$

Hence $T^{*} \in \mathcal{N}_{up}^{Q}(Y^{*};X^{*})$ and $v_{up}^{Q}(T^{*}) \leq u_p(T)$. ∎

Bibliography

[1] D. Achour, E. Dahia and P. Turco. Lipschitz p-compact mapping, Monatshefte fuïr Mathematik, 189 (2019), 595–609.

[2] K. Ain, R. Lillemets and E. Oja. Compact operators which are defined by ℓ_p-spaces,Quaestiones Math. 35 (2012), 145-159

[3] J. B. Conway. A Course in Functional Analysis, 2nd ed., Springer-Verlag, 1990.

[4] J. M. Delgado, C. Pñieiro and E. Serrano. Operators whose adjoints are quasi p-nuclear, Studia Math. 197 (2010), 291-304

[5] J. Diestel. Sequences and Series in Banach Spaces, Grad. Texts in Math. 92, Springer, New York, 1984

[6] J. Diestel, H. Jarchow and A. Tonge. Absolutely Summing Operators, Cambrigde University Press, Cambridge, 1995.

[7] N. Dunford and J. Schwartz. Linear Operators, Part I, Interscience, New York, 1964.

[8] J. H. Fourie and J. Swart. Banach ideals of p-compact operators, Manuscripta Math.26 (1979), 349-362

[9] J .H. Fourie. Injective and surjective hulls of classical p-compact operators with application to unconditionally p-compact operators,Studia Math. 240 (2018),np.2,147-159

[10] A. Grothendieck. Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc, 140 (16) 1955.

[11] H. Jarchow. Locally Convex Spaces, B.G. Teubner, Stuttgart, 1981.

[12] W.B. Johnson. Factoring compact operrators. Is real J.Math .9,337-345 (1971)

[13] J. M. Kim. Unconditionally p-null sequences and unconditionally p-compact operators, Studia Math. 224 (2014), 133-142

[14] F. Muñoz, E. Oja and C. Piñeiro. On α-nuclear operators with applications to vector-valued function spaces, J. Funct. Anal. 269 (2015), 2871-2889

[15] E. Oja. A remark on the approximation of p-compact operators by finite-rank operators, J. Math. Anal. Appl. 387 (2012), 949{952.

[16] A. Pietsch. Operator Ideals, Deutsch. Verlag Wiss, Berlin, 1978, North Holland Publishing Company, Amsterdam -New York-Oxford, 1980.

[17] A. Pietsch. History of Banach Spaces and Linear Operators, Birkhäuser Boston, Inc,Boston, MA, 2007.

[18] A. Pietsch. The ideal of p-compact operators and its maximal hull, Proc. Amer. Math. Soc. 142 (2014), 519–530.

[19] D. Randtke. Representation theorems for compact operators. Proc.Amer.Math. Soc.37, 481-485 (1973).

[20] D. P. Sinha and A. K. Karn. Compact operators whose adjoints factor through subspaces of ℓ_p, Studia Math. 150 (2002), 17-33

[21] T. Tiaiba. The ideal of Lipschitz classical p-compact operators and its injective hull. Moroccan J. of Pure and Appl. Anal. 8(1), (2022),28–43

الملخص .

الهدف من هذا العمل هو دراسة تفصيلية لمثاليات باناخ للمؤثرات الخطيه المستمره بين فضاءات باناخ المفككة بواسطة مؤثرات متراصه من خلال الفضاء ℓ_p. نقدم مناقشة لبعض خصائص الهياكل المتباينه لهذا المثالي وربط هذه المثل العليا بفئات المؤثرين p متراص غير المشروط و p-نيكليار غير المشروط.

الكلمات المفتاحية: مثالي المؤثرات الخطية، مؤثر متراص، متتاليات بناخ p جمعية غير المشروطة، مؤثرات p متراصة العادية.

Abstract.

The aim of this work is a detail study of the Banach ideal of continuous linear operators, between Banach spaces, factorized compactly through ℓ_p, called classical p-compact. We present a discussion of some properties for the injective hull of this ideal and relate this work to the classes of unconditionally p-compact and quasi unconditionally p-nuclear operators.

Keywords: Linear operator ideals, compact operator, Unconditionally p-summable sequences, Classical p-compact operator.

Résumé.

Le but de ce travail est d'étudier en détail les idéaux d'opérateurs linéaires continus entre espaces de Banach factorisés par des opérateurs compacts à travers ℓ_p, appelés p-compacts classiques. Nous présentons une discussion de quelques propriétés de l’injective hull de cet idéal, et de relier ce travail à la classe d'opérateurs inconditionnellement p-compacts et la classe d'opérateurs quasi inconditionnellement p-nucléaires.

Mots clés: L’idéal linéaire, Opérateur compact, Suite inconditionnelle p-sommable, Opérateur p-compact classique.