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Exact and approximate solutions for deformable fractional differential equations

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I dedicate this thesis to:

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INTRODUCTION

Fractional calculus, an extension of traditional calculus, involves derivatives and integrals of non-integer orders [7, 21]. This field has gained significant attention due to its ability to model complex systems exhibiting anomalous behavior, such as viscoelastic materials [14], fractal processes [13], and anomalous diffusion [20]. Traditionally, fractional calculus has been studied in the context of fixed, immutable structures. However, the concept of deformable fractional calculus introduces a dynamic aspect, where the fractional order itself can change in response to varying conditions.

Deformable fractional calculus extends the versatility of traditional fractional models by allowing the fractional order to be a function of time, space, or other variables. This adaptability provides a more accurate representation of systems where properties evolve dynamically. For instance, in materials science, the deformation of materials under stress can be better described using a variable-order fractional derivative, capturing the changes in viscoelastic properties over time.

The introduction of deformable fractional calculus has opened new avenues for research and application. By accommodating the inherent variability in complex systems, it offers more precise modeling tools across various scientific and engineering domains. This innovative approach is particularly useful in fields such as biomechanics, where biological tissues exhibit time-dependent and nonlinear behaviors, and in finance, where market volatility may vary with different economic conditions.

In this study, we explore the basic concepts of the deformed fractional derivative Calculus, its mathematical formula, and its applications. We also discuss direct and numerical methods for solving differential equations involving a fractional derivative of the distorted type of fractional order α whose value is between 0 and 1.

In the first chapter, preliminary concepts are given about the distorted fractional derivative, as well as the distorted fractional integral, and some properties related to the distorted fractional derivative and integration.

In the second chapter, we give the concept of the deformable fractional Laplace transform related to the ordinary (classical) Laplace transform, where we use this transform to find exact solutions to differential equations with a deformable fractional derivative of the α order, where: $0 < \alpha \leq 1$. Finding solutions is also related to the inverse deformable fractional Laplace transform. We give the inverse Laplace transform of some common functions.

In the third chapter, we searched for approximate solutions to deformable fractional differential equations of order α, where $0 < \alpha \leq 1$, using two methods: Euler's method and the Range-Kutta4 method. We applied these two methods to several examples in this chapter.

CHAPTER 1

PRELIMINARIES

In this chapter, we collect various definitions and theorems which are key tools for proving our main theorems. Some proofs are omitted and may be found in [3, 16] or [15, 11].

1.1 Some Basic definition and Tools

1.1.1 Deformable derivative

Definition 1.1 *[2]Given a real-valued function $h(t)$ defined on interval (a,b),then for arbitrary order α deformable derivative of a function h is defined as:*

$$\mathcal{D}^{\alpha}h(t) = \lim_{\epsilon \longrightarrow 0} \frac{(1+\epsilon\beta)h(t+\epsilon\alpha) - h(t)}{\epsilon}, \tag{1.1}$$

☞ *where $\alpha + \beta = 1$.if the aforementioned limit exists $\forall 0 \leq \alpha \leq 1$,then,deformable derivative of the real-valued function $h(t)$ is denoted by the symbol $\mathcal{D}^{\alpha}h(t)$.*

Remark 1.1 *In this definition.* (1.1)*we note:*

1. *for $\alpha = 0 \Rightarrow \mathcal{D}^{0}h(t) = h(t)$*
2. *for $\alpha = 1 \Rightarrow \mathcal{D}^{1}h(t) = \frac{dh}{dt} = h'(t)$*

The deformable derivative can be considered as α-derivative as well.
the following theorems are useful results of the aforementioned definition.

Theorem 1.1 *Let h be α-differentiable at a point t_0 for some α.then h is continuous at t_0.*
It can be easily shown that operator $\mathcal{D}^{\alpha}$ has the following properties:

Theorem 1.2 *Let h_1 and h_2 be α-differentiable at t.Then*

1. *Linearity:* $\mathcal{D}^\alpha(ah_1 + bh_2) = a\mathcal{D}^\alpha h_1 + b\mathcal{D}^\alpha h_2, \forall a, b \in \mathbb{R}$.
2. *Commutativity:* $\mathcal{D}^{\alpha_1}.\mathcal{D}^{\alpha_2} = \mathcal{D}^{\alpha_2}.\mathcal{D}^{\alpha_1}$.
3. $\mathcal{D}^\alpha(s) = \beta s$, $\forall$ *constant functions* $h(t) = s$.
4. $\mathcal{D}^\alpha(h_1.h_2) = (\mathcal{D}^\alpha h_1).h_2 + \alpha h_1.\mathcal{D}h_2$, *hence* $\mathcal{D}^\alpha$ *does not follow the Leibniz rule.*
5. $\mathcal{D}^\alpha h(t) = \beta h(t) + \alpha \mathcal{D}h(t)$.

Following are the arbitrary order deformable derivatives of well-known elementary functions :

- ☞ $\mathcal{D}^\alpha(t^K) = \beta t^K + K\alpha t^{K-1}, K \in \mathbb{R}$.
- ☞ $\mathcal{D}^\alpha(e^t) = \beta e^t + \alpha e^t$.
- ☞ $\mathcal{D}^\alpha(\sin wt) = \beta \sin wt + \alpha w \cos wt$.

1.1.2 Deformable integral

This section introduced deformable integral,the inverse operator for deformable integral,the inverse operator for deformable derivative.Besides discussing some basic properties of this deformable fractional integral,we list out deformable fractional integral of some elementary functions.Also,we introduced deformable form of exponential function (See: [4]).

Definition 1.2 *[2, 15] Let* h *be a continuous function defined on* $[a, b]$*.we define* α*-fractional integral of* h*,denoted by* $I_a^\alpha h$*,by the integral*

$$I_a^\alpha h(t) = \int_a^t e^{-\frac{\beta}{\alpha}(t-x)} h(x) d_\alpha x, \tag{1.2}$$

- ☞ *Where* $\alpha + \beta = 1, \alpha \in]0, 1]$*, and* $d_\alpha x = \frac{1}{\alpha} dx$.

Next theorem explains some basic properties of this fractional integral.

Theorem 1.3 *The operator* I_a^α *possesses the following properties:*

1. *Linearity:* $I_a^\alpha(bh_1 + ch_2) = bI_a^\alpha h_1 + cI_a^\alpha h_2$.
2. *Commutativity:* $I_a^{\alpha_1} I_a^{\alpha_2} = I_a^{\alpha_2} I_a \alpha_1$*, where* $\alpha_i + \beta i = 1, \quad i = 1, 2$.

Following are the fractional integral of some well-known elementary functions:

Proposition 1.1 1. $I_a^\alpha \sin t = \frac{1}{\alpha^2+\beta^2}(\beta \sin t - \alpha \cos t + e^{\frac{\beta}{\alpha}(a-t)}(\alpha \cos a - \beta \sin a))$.

2. $I_a^\alpha e^t = (e^t - e^{\frac{(a-\beta t)}{\alpha}})$.
3. $I_a^\alpha \lambda = \frac{\lambda}{\beta}(1 - e^{\frac{\beta}{\alpha}(a-t)})$*, Where* λ *is a constant.*

4. $I_0^{\alpha} t^m = \frac{1}{\beta}(\sum_{r=0}^{m} \frac{(-1)^r (m)!}{(m-r)!}(\frac{\alpha}{\beta})^r t^{m-r} + (-1)^{m+1} m!(\frac{\alpha}{\beta})^m e^{-\frac{\beta}{\alpha}t})$.
Furthermore the deformable exponential function is defined as follows:

Definition 1.3 (Deformable exponential function) *For some point* $s, t \in \mathbb{R}$ *with* $s \leq t$,*the exponential function with respect to* $\mathcal{D}^{\alpha}$ *in* (1.1) *is defined as:*

$$e_{\alpha}(t,s) = e^{-\int_s^t \beta d_{\alpha} u}, \tag{1.3}$$

☞ *where,*$\alpha + \beta = 1$, $\alpha \in]0.1]$,*and* $d_{\alpha} u = \frac{1}{\alpha} du$.

Theorem 1.4 *A differentiable function* f *at a point* $t \in (a, b)$ *is always* α-*differentiable at that point for any* α.*Moreover,we have* $\mathcal{D}^{\alpha} f(t) = \beta f(t) + \alpha \mathcal{D} f(t)$,

☞ *Where* $\alpha + \beta = 1$ *and* $\mathcal{D} := \frac{d}{dt}$ *is the usual derivative.*

Theorem 1.5 *Let* f *be differentiable at a point* t *for some* α,*then, it is continuous there.*

Theorem 1.6 *Let* f*be defined in* (a, b).*For any* α,f *is* α-*differentiable if and only if it is differentiable.*
The operators $\mathcal{D}_{\alpha}$ *and* $\mathcal{I}_a^{\alpha}$ *possess the following properties.*

Theorem 1.7 *(Taylor's theorem,) Suppose* f *is* n-*times* α *differentiable and such that all* α-*derivatives are continuous on* $[a, a + h]$.*then,*

$$f(a+h) = \sum_{k=0}^{n-1} \frac{h^k}{k!\alpha^k}\left[\mathcal{D}_k^{\alpha} f(a) - \beta \frac{(1-\theta)^{(k-n+1)} h}{\alpha n} \mathcal{D}_k^{\alpha} f(a+\theta h)\right] + \frac{h^n}{n!\alpha} \mathcal{D}_n^{\alpha} f(a+\theta h),$$

☞ *Where* $\mathcal{D}_k^{\alpha} = \mathcal{D}^{\alpha}\mathcal{D}^{\alpha}...\mathcal{D}^{\alpha}$*(k times) and* $0 < \theta < 1$,$0 < k < 1$.

We state and prove the following result.

Theorem 1.8 *the operator* $\mathcal{D}^{\alpha}$*possesses also the following property*

$$\mathcal{D}^{\alpha}(\frac{f}{g}) = \frac{g\mathcal{D}^{\alpha}(f) - \alpha f}{g^2}.$$

Proof. . *we have*

$$\begin{aligned}
\mathcal{D}^{\alpha}(\frac{f}{g}) &= \beta(\frac{f}{g}) + \alpha \mathcal{D}(\frac{f}{g}) \\
&= \beta(\frac{f}{g}) + \alpha\left[\mathcal{D}\frac{f}{g} + f\mathcal{D}\frac{1}{g}\right] \\
&= \beta\frac{f}{g} + \alpha\mathcal{D}\frac{f}{g} + \alpha f \mathcal{D}\frac{1}{g} \\
&= [\beta f + \alpha \mathcal{D} f]\frac{1}{g} + \alpha f \mathcal{D}\frac{1}{g} \\
&= \frac{g\mathcal{D}^{\alpha}(f) - \alpha f}{g^2}.
\end{aligned}$$

the proof is complete. □

Theorem 1.9 *Suppose f and g are α-differentiable.then,*

$$\mathcal{D}^{\alpha}(fog)(t) = \beta(fog)(t) + \alpha\mathcal{D}(fog)(t) = \beta(fog)(t) + \alpha f'(g(t))g'(t).$$

Proof. *Since*

$$\mathcal{D}^{\alpha} f(t) = \lim_{\epsilon\longrightarrow 0} \frac{(1+\epsilon\beta)f(t+\epsilon\alpha) - f(t)}{\epsilon} = \lim_{\epsilon\longrightarrow 0}\left[\frac{f(t+\epsilon\alpha)-f(t)}{\epsilon} + \beta f(t+\epsilon\alpha)\right],$$

we have

$$\begin{aligned}\mathcal{D}^{\alpha} f(g(t)) &= \lim_{\epsilon\longrightarrow 0}\left[\frac{f(g(t+\epsilon\alpha)) - f(g(t))}{\epsilon} + \beta f(g(t+\epsilon\alpha))\right]\\ &= \lim_{\epsilon\longrightarrow 0}\left[\frac{f(g(t+\epsilon\alpha)) - f(g(t))}{g(t+\epsilon\alpha)-g(t)}.\frac{g(t+\epsilon\alpha)-g(t)}{\epsilon} + \beta f(g(t+\epsilon\alpha))\right]\\ &= \lim_{\epsilon\longrightarrow 0}\left[\frac{f(g(t)+\epsilon_0) - f(g(t))}{\epsilon_0}.\frac{g(t+\epsilon\alpha)-g(t)}{\epsilon} + \beta f(g(t+\epsilon\alpha))\right],\end{aligned}$$

where $\epsilon_0 \longrightarrow 0$ as $\epsilon \longrightarrow 0$.we obtain

$$\begin{aligned}\mathcal{D}^{\alpha} f(g(t)) &= \lim_{\epsilon_0\longrightarrow 0}\frac{f(g(t)+\epsilon_0 - f(g(t))}{\epsilon_0}.\lim_{\epsilon\longrightarrow 0}\frac{g(t+\epsilon\alpha)-g(t)}{\epsilon} + \lim_{\epsilon\longrightarrow 0}\beta f(g(t+\epsilon\alpha))\\ &= f'(g(t))\alpha g'(t) + \beta f(g(t))\\ &= \alpha\mathcal{D}\left[f(g(t))\right] + \beta f(g(t))\\ &= \beta f(g(t)) + \alpha\mathcal{D}\left[f(g(t))\right].\end{aligned}$$

the proof is now complete. □

Theorem 1.10 *Let f be continious on $[a,b]$.Then,$I_a^{\alpha} f$ is α-differentiable in (a,b),and we have*

- $\mathcal{D}^{\alpha}(I_a^{\alpha} f)(t) = f(t)$
- $I_a^{\alpha}(\mathcal{D}^{\alpha f}(t) = f(t) - e^{\frac{\beta}{\alpha}(a-t)} f(a).$

CHAPTER 2

EXACT SOLUTIONS OF DEFORMABLE FRACTIONAL EQUATIONS USING THE LAPLACE TRANSFORM

In this chapter, we give a definition of the Laplace transform in terms of the used (deformable fractional derivative).See:[22],[17],[18] and [9], we use this transformation in order to find exact solutions to several deformable fractional differential equations.

2.1 Deformable Laplace transform

in this section ,we introduce the Laplace transform and the deformable Laplace transform (DLT) in very natural way. we also discuss existence theorem of DLT.

Definition 2.1 *[1] If f is a function (locally integrable), defined on $\mathbb{R}_+$, with values in $\mathbb{C}$, we call the* **classical Laplace transform** *of f the function*

$$\mathcal{L}(f)(s) = \int_0^{+\infty} e^{-st} f(t)dt = \lim_{x \to +\infty} \int_0^x e^{-st} f(t)dt, \tag{2.1}$$

for the values of s for which this integral converges.

Definition 2.2 *[2] Let $f : [0, \infty[\longrightarrow \mathbb{R}$.For $\alpha \in]0, 1]$,we define DLT of order α,denoted by $L^\alpha f$,by the integral:*

$$L^\alpha f(t) = F^\alpha(p) = \int_0^\infty e^{-st} f(t) e_\alpha(t, 0) d_\alpha t, \tag{2.2}$$

☞ *where $\alpha + \beta = 1$,$p = s + \frac{\beta}{\alpha}$,and $d_\alpha t = \frac{1}{\alpha} dt$.*

Theorem 2.1 *[2] A continuous function f is of exponential order a_*,there exists a positive real number C s.t. $|f(t)| \leq Ce^{a_* t}$,*

☞ *where* $a_* \in \mathbb{R}, t \geq 0$,*then DLT of* f *exists for* $s > a_*$.

Proof.

$$\begin{aligned} |L^\alpha f(t)| &= \left| \int_0^\infty e^{-st} f(t) e_\alpha(t,0) d_\alpha t \right| \\ &\leq \int_0^\infty \left| e^{-st} f(t) e_\alpha(t,0) \right| d_\alpha t \\ &\leq \int_0^\infty e^{-st} \left| f(t) \right| e_\alpha(t,0) d_\alpha t \\ &\leq \int_0^\infty e^{-st} C e^{a_* t} e_\alpha(t,0) d_\alpha t, s \succ a_* \\ &= \frac{C}{\alpha} (\frac{1}{p - a_*}). \end{aligned}$$

□

Remark 2.1 *It should however be kept in mind that the aforementioned condition is sufficient but not necessary.*
For example,$L^\alpha(\frac{1}{\sqrt{t}})$ *exists though* $\frac{1}{\sqrt{t}}$ *is discontinuous at* $t = 0$.
Now,we list out DLT of some certain functions in the following preposition:

Proposition 2.1 *this table*

	The function	Deformable Laplace Transform
1.	1	$\frac{1}{\alpha}(\frac{1}{p})$,$p > 0$,
2.	t^m	$\frac{1}{\alpha}(\frac{m!}{p^{m+1}})$,$p > 0$,$m = 0, 1, 2, ...$
3.	e^{at}	$\frac{1}{\alpha}(\frac{1}{p-a})$,$p > a$.
4.	$\sin at$	$\frac{1}{\alpha}(\frac{a}{p^2+a^2})$
5.	$\sinh at$	$\frac{1}{\alpha}(\frac{a}{p^2-a^2})$
6.	$\cos at$	$\frac{1}{\alpha}(\frac{p}{p^2+a^2})$
7.	$\cosh at$	$\frac{1}{\alpha}(\frac{p}{p^2-a^2})$

Table 2.1: Deformable Laplace Transform

2.1.1 Basic properties of deformable Laplace transform

Apart from discussing fundamental properties of DLT like linearity and change of scale property ,the section deals with fundamental theorems:First shifting and second shifting property.

Theorem 2.2 *The operator* L^α *possesses the following properties:*

1. *Lineraty:* $L^\alpha(a_* f + b_* g) = a_* L^\alpha f + b_* L^\alpha g$.

2. *First translation or shifting property: If* $L^\alpha g(t) = G^\alpha(p)$,*then* $L^\alpha(e^{at}g(t)) = G^\alpha(p-a)$.

3. *Second translation or shifting property:* $L^\alpha g(t) = G^\alpha(p)$ *and*

$$F(t) = \begin{cases} g(t-a), t > a, \\ 0, t < a. \end{cases}$$

then, $L^\alpha F(t) = e^{-ap}G^\alpha(p)$

4. *Change of scale property : if* $L^\alpha g(t) = G^\alpha(p)$,*then* $L^\alpha g(at) = \frac{1}{a}G^\alpha(\frac{p}{a})$

5. *DLT of successive derivatives: If* $L^\alpha g(t) = G^\alpha(p)$,*then*

$$L^\alpha(\underbrace{(D^\alpha D^\alpha ... D^\alpha)}_{n\ times} g(t) = (\alpha p+\beta)^n G^\alpha(p) - (\alpha p+\beta)^{n-1}g(0)$$
$$- (\alpha p+\beta)^{n-2}\mathcal{D}^\alpha g(0) - (\alpha p+\beta)^{n-3}\mathcal{D}^\alpha\mathcal{D}^\alpha g(0) - ... - \underbrace{(\mathcal{D}^\alpha\mathcal{D}^\alpha ... \mathcal{D}^\alpha)}_{n-1\ times} g(0).$$

6. *DLT of tg(t):*
If $L^\alpha g(t) = G^\alpha(p)$ *then* $L^\alpha(t^n g(t)) = (-1)^n \frac{d^n}{dp^n}G^\alpha(p)$.

7. *DLT of* $\frac{g(t)}{t}$*:*
If $L^\alpha(g(t)) = G^\alpha(p)$,*then* $L^\alpha(\frac{g}{t}) = \int_p^\infty G^\alpha(p)dp$.

Proof. Linearity is obvious from definition,parts(2),(4),(8)can be established easily.For part(3),we have

$$\begin{aligned}
L^\alpha F(t) &= \int_0^\infty e^{-st}F(t)e_a(t,0)d_\alpha t \\
&= \frac{1}{\alpha}(\int_0^a e^{-pt}F(t)dt + \int_a^\infty e^{-pt}F(t)dt) \\
&= \frac{1}{\alpha}(0 + \int_a^\infty e^{-pt}g(t-a)dt) \\
&= \frac{1}{\alpha}\int_a^\infty e^{-pt}g(t-a)dt) \\
&= e^{-ap}\frac{1}{\alpha}\int_0^\infty e^{-pu}g(u)du \\
&= e^{-ap}G^\alpha(p).
\end{aligned}$$

To prove part(5),we have

$$\begin{aligned} L^{\alpha}(\mathcal{D}^{\alpha}g(t)) &= \int_0^{\infty} e^{-st}\mathcal{D}^{\alpha}(g(t))e_{\alpha}(t,0)d_{\alpha}t \\ &= \frac{1}{\alpha}\int_0^{\infty} e^{-st}[\beta g+\alpha g]e_{\alpha}(t,0)dt. \end{aligned}$$

Therefore,

$$L^{\alpha}(\mathcal{D}^{\alpha}g(t)) = (\alpha p+\beta)G^{\alpha}(p) - g(0). \tag{2.3}$$

Now replacing g by $\mathcal{D}^{\alpha}g$ and $\mathcal{D}^{\alpha}g$ by $\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}g$ in(5),we obtain

$$L^{\alpha}(\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}g(t)) = (\alpha p+\beta)^2G^{\alpha}(p) - (\alpha p+\beta)g(0) - \mathcal{D}^{\alpha}g(0). \tag{2.4}$$

with the same process we have,

$$L^{\alpha}(\underbrace{\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}...\mathcal{D}^{\alpha}}_{n\ times})g(t) = (\alpha p+\beta)^nG^{\alpha}(p) - (\alpha p+\beta)^{n-1}g(0) - (\alpha p+\beta)^{n-2}\mathcal{D}^{\alpha}g(0)- \tag{2.5}$$

$$(\alpha p+\beta)^{n-3}\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}g(0) - ... - (\underbrace{\mathcal{D}^{\alpha}\mathcal{D}^{\alpha}...\mathcal{D}^{\alpha}}_{n-1\ times})g(0). \tag{2.6}$$

For part(7),we have

$$\begin{aligned} L^{\alpha}(g(t)) = G^{\alpha}(p) &= \frac{1}{\alpha}\int_0^{\infty} e^{-pt}g(t)dt \\ &= \frac{d}{dp}G^{\alpha}(p) = \frac{d}{dp}(\frac{1}{\alpha}\int_0^{\infty} e^{-pt}g(t)dt) \\ &= \frac{1}{\alpha}\int_0^{\infty} \frac{\partial}{\partial p}(e^{-pt})g(t)dt \\ &= \frac{1}{\alpha}\int_0^{\infty} e^{-pt}(-tg(t))dt \\ &= L^{\alpha}(-tg(t)) \longrightarrow L^{\alpha}(tg(t)) = (-1)^1\frac{d}{dp}G^{\alpha}(p) \end{aligned}$$

similary,

✍ $L^{\alpha}(t^2g(t)) = (-1)^2\frac{d^2}{dp^2}(G^{\alpha}(p))$,

✍ $L^{\alpha}(t^3g(t)) = (-1)^3\frac{d^3}{dp^3}(G^{\alpha}(p))$,

✍ $L^{\alpha}(t^ng(t)) = (-1)^n\frac{d^n}{dp^n}(G^{\alpha}(p))$.

□

2.2 Deformable Inverse Laplace Transform

This section explains about DILT and further Heaviside expansion formula and convolution theorem for DILT are also discussed.DILT is a process for determining the function which generates given DLT.If $G^{\alpha}(p)$ is the DLT of $g(t)$,then $g(t)$ is called DILT of $G^{\alpha}(p)$.The operator for DILT is $(L^{\alpha})^{-1}$.

Theorem 2.3 *(Heaviside expansion formula for DILT)If $G_1(p)$ and $G_2(p)$ are two polynomials and $G_2(p)$ has degree less than degree of $G_1(p)$,then*

$$(L^{\alpha})^{-1}(\frac{G_2(p)}{G_1(p)}) = \alpha \sum_{r=1}^{m} \frac{G_2(a_r)}{G_1'(a_r)} e^{a_r t},$$

☞ *Where $G_1(p)$ has distinct zeroes $a_1, a_2, ..., a_m$.*

Proof. We have

$$\begin{aligned}\frac{G_2(p)}{G_1(p)} &= \frac{G_2(p)}{(p-a_1).(p-a_2)...(p-a_m)} \\ &= \frac{A_1}{(p-a_1)} + \frac{A_2}{(p-a_2)} + ... + \frac{A_m}{(p-a_m)}.\end{aligned}$$

Multiplying both sides by $(p - a_r)$ and taking the limit as $p \longrightarrow a_r$

$$A_r = \lim_{p \longrightarrow a_r} \frac{G_2(p)(p-a_r)}{G_1(S)} = \frac{G_2(a_r)}{G_1'(a_r)},$$

i.e.,

$$\frac{G_2(p)}{G_1(p)} = \frac{G_2(a_1)}{G_1'(a_1)}(\frac{1}{p-a_1}) + \frac{G_2(a_2)}{G_1'(a_2)}(\frac{1}{p-a_2}) + ... + \frac{G_2(a_m)}{G_1'(a_m)}(\frac{1}{p-a_m})$$

Taking DILT on both sides,we have

$$(L^{\alpha})^{-1}(\frac{G_2(p)}{G_1(p)}) = \alpha \sum_{r=1}^{m} \frac{G_2(a_r)}{G_1'(a_r)} e^{a_r t}.$$

□

Theorem 2.4 *[2] (Convolution theorem) If $(L^{\alpha})^{-1}(G_1^{\alpha}(p)) = g_1(t)$ and $(L^{\alpha})^{-1}(G_2^{\alpha}(p)) = g_2(t)$ then*

$$(L^{\alpha})^{-1}(G_1^{\alpha}(p)G_2^{\alpha}(p)) = g_1 * g_2 = \int_0^t g_1(x)g_2(t-x)d_{\alpha}x. \quad (2.7)$$

2.3 Applications of deformable Laplace transform to arbitrary order deferential equations

In first example,we discuss method of solving homoge-neous linear differential equation of arbitrary order ,whereas in second and third examples non-homogeneous linear arbitrary order differential equations are solved

Example 2.1 *Let a homogeneous deformable fractional differential equation:*

$$\begin{cases} \mathcal{D}^{\alpha}y(t) - \lambda y(t) = 0, \\ y(0) = y_0. \end{cases} \tag{2.8}$$

with λ is a constant.
We Take the DLT on both the sides using equation(7),

$$\mathcal{L}_{\alpha}\left(\mathcal{D}^{\alpha}y(t) - \lambda y(t)\right) = 0 \Leftrightarrow (\alpha p + \beta)Y^{\alpha}(p) - y(0) - \lambda Y^{\alpha}(p) = 0.$$
$$\Longrightarrow Y^{\alpha}(p) = \frac{y_0}{\alpha p + \beta - \lambda}.$$

Using DILT,general solution is given by:

$$y(t) = y_0 e^{(\frac{\lambda-\beta}{\alpha})t}. \tag{2.9}$$

Example 2.2 *Let non-homogeneous deformable fractional equation:*

$$\begin{cases} \mathcal{D}^{\alpha}y(t) = \sin t, \\ y(0) = 0. \end{cases} \tag{2.10}$$

We take the DLT on both the sides ,then we have the expression

$$(\alpha p + \beta)Y^{\alpha}(p) = \frac{1}{\alpha(p^2+1)} \Longrightarrow Y^{\alpha}(p) = \frac{1}{\alpha(p^2+1)(\alpha p + \beta)},$$

by the inverse deformable Laplace transform, we have

$$L_{\alpha}^{-1}\left(Y^{\alpha}(p)\right) = L_{\alpha}^{-1}\left(\frac{1}{\alpha(p^2+1)(\alpha p + \beta)}\right), \tag{2.11}$$

and the general solution is given by:

$$y(t) = (\frac{\beta}{\alpha^2+\beta^2})\sin t - (\frac{\alpha}{\alpha^2+\beta^2})\cos t + (\frac{\alpha}{\alpha^2+\beta^2})e^{-\frac{\beta}{\alpha}t} \tag{2.12}$$

Example 2.3 *Consider the problem*

$$\begin{cases} (\mathcal{D}^{\frac{1}{4}}\mathcal{D}^{\frac{1}{4}})u(t) = e^{3t}; \\ u(0) = 1, \\ \mathcal{D}^{\alpha}u(0) = 0. \end{cases} \tag{2.13}$$

Taking DLT both by using Eq.we obtain

$$(\alpha p+\beta)L^{\alpha}(u) - (\alpha p+\beta)u(0) - \mathcal{D}^{\alpha}u(0) = \frac{4}{p-3} \Longleftrightarrow L^{\alpha}(u) = \frac{4}{(p-3)(\alpha p+\beta)^2} + \frac{1}{\alpha p+\beta},$$
$$\Longrightarrow u = (L^{\alpha})^{-1}\left[\frac{4}{(p-3)(\alpha p+\beta)^2} + \frac{1}{\alpha p+\beta}\right].$$

Here $\alpha = \frac{1}{4}$ *,*$\beta = \frac{3}{4}$*,therefore*

$$u = (L^{\frac{1}{4}})^{-1}\left[\frac{64}{(p-3)(p+3)^2} + \frac{4}{p+3}\right]$$

Using convolution theorem for DILT

$$(L^{\frac{1}{4}})^{-1}\left[\frac{64}{(p-3)(p+3)^2}\right] = 4\int_0^t xe^{3t-6x}d_{\alpha}x = 16e^{3t}\int_0^t xe^{-6x}dx = -\frac{8}{3}te^{-3t} - \frac{4}{9}e^{-3t} + \frac{4}{9}e^{3t}$$

and $(L^{\frac{1}{4}})^{-1}\left[\frac{1}{(p-3)}\right] = \frac{1}{4}e^{-3t}$. *Therefore,complete solution is*

$$u(t) = \frac{5}{9}e^{-3t} - \frac{8}{3}te^{-3t} + \frac{4}{9}e^{3t}. \tag{2.14}$$

Remark 2.2 *It should be noted that all the aforementioned solutions coincide with* $\alpha = 1$

CHAPTER 3

APPROXIMATE SOLUTIONS OF DEFORMABLE FRACTIONAL EQUATIONS

On the third chapter, we find the approximate solutions of fractional deformable differential equations by two methods. First method is the Euler method, and second method is the Rang-Kutta method order 4.

3.1 Approximate solution of deformable fractional equation by Euler method

3.1.1 Euler's method

In mathematics and computational science, the Euler method (also called the forward Euler method) is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. It is the most basic explicit method for numerical integration of ordinary differential equations. The Euler method is named after Leonhard Euler, who first proposed it in his book Institution-um calculi integrals (published 1768–1770).(See:[10, 11, 5]) The Euler method is a first-order methods, which means that the local error (errors per step) is proportional to the square of the step size, and the global error (errors at a given time) is proportional to the step size. Euler method often serves as the basis to construct more complex methods.

The Euler method is a simple numerical methods used to approximate the solution of ordinary differential equations (ODEs) with an initial value. It works by approximating the curve defined by the ODE with a sequence of straight line segments. (See:[6],[12]).

3.2 Problem Position

In this section, we take the problem for a deformable fractional differential equation of degree α as follows:

$$\begin{cases} \mathcal{D}^{\alpha} y(t) = F(t, y), \\ y(0) = y_0. \end{cases} \tag{3.1}$$

In order to solve this problem (3.1), we transform it into a problem with a classical derivative, where we use Theorem (1.2), specifically in step (5). We obtain the equivalent problem with an ordinary classical problem.

3.2.1 The equivalent problem

The equivalent problem with a normal classical form of problem (3.1) is as follows:

$$\begin{cases} y'(t) = -\frac{\beta}{\alpha} y_i + \frac{F(t,y)}{\alpha}, \ 0 < \alpha \leq 1, \ \alpha + \beta = 1. \\ y(0) = y_0. \end{cases} \tag{3.2}$$

In order to find the approximate solution to problem (3.2), we use Euler's method, According to problem (3.2), we find:

$$\frac{y_{i+1} - y_i}{h} = -\frac{\beta}{\alpha} y_i + \frac{F(t_i, y_i)}{\alpha}. \tag{3.3}$$

From the statement (3.3) we extract the formula y_{i+1},which is as follows:

$$y_{i+1} = \left(1 - \frac{\beta}{\alpha} h\right) y_i + \frac{F(t_i, y_i)}{\alpha} h. \tag{3.4}$$

with initial condition $y(t_0) = y_0$, the Euler method computes an approximation to $t_0, t_1, t_2, ...$
The steps of Euler program can be summarized as follows:

Algorithm 1 (Euler's Algorithm)

1: **Initializations:**

1. Choose α and β values.
2. Start at the initial point (t_0, y_0).
3. For each step i:
   (a) Compute the slope $f(t_i, y_i)$.
   (b) Use this slope to approximate the value of y at the next point:
   $$y_{i+1} = y_i + h.f(t_i, y_i),$$
   where h is the step size.
4. **Update the time:**
   (a) For the time $t_{i+1} = t_i + h$.
   (b) Set $i = i + 1$ and go to step ((3b)).

The accuracy of the Euler method depends on the step size h. Smaller step sizes generally lead to more accurate results but require more computation. The method is a first-order method, meaning that the error decreases linearly with the step size.

Example 3.1 *Consider the initial value problem*

$$\begin{cases} \mathcal{D}^\alpha y(t) = y(t), 0 < \alpha \leq 1, \\ y(0) = 1. \end{cases} \tag{3.5}$$

This is a linear problem and can be solved exactly to yield the solution

$$y(t) = e^{\frac{1-\beta}{\alpha}t}, \text{ with } \alpha + \beta = 1. \tag{3.6}$$

Using Euler's algorithm, we can find approximate solutions to problem (3.5) *with the initial condition.*

The following graphical representations show the real and approximate solutions for different values of: α and β. And also N.

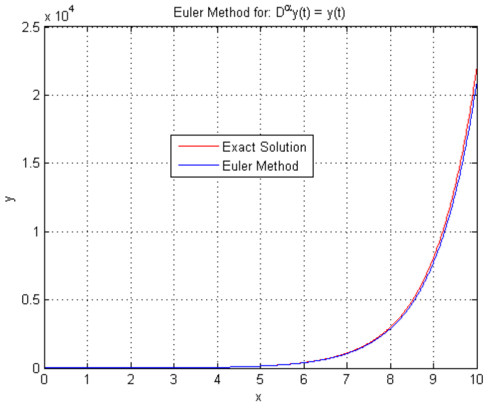

Figure 3.1: Exact solutions and numerical solutions for $\alpha = \beta = 0.5$ and $h = 0.01$

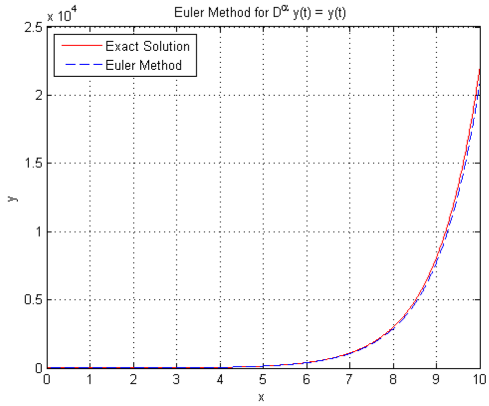

Figure 3.2: Exact solutions and numerical solutions for $\alpha = 0.7$, $\beta = 0.3$ and $h = 0.01$

Figure 3.3: Comparison of the numerical and exact solutions of the example **??**.

Example 3.2 *Consider the initial value problem*

$$\begin{cases} \mathcal{D}^\alpha y(t) = y(t) + [(\beta - 1)t + \alpha]cos(t) - \alpha sin(t), \ \ \alpha \in \,]\,0, 1\,]\,, \\ y(0) = 0, with\ \alpha + \beta = 1. \end{cases} \tag{3.7}$$

The equivalent problem for Problem (3.7) *is as follows:*

$$\begin{cases} y'(t) = (\frac{1-\beta}{\alpha})y(t) + [(\frac{\beta-1}{\alpha})t + 1]cos(t) - sin(t), \ \ \alpha \in]0, 1], \ \ \alpha + \beta = 1, \\ y(0) = 0. \end{cases} \tag{3.8}$$

The exact solution of (3.10) *is:*

$$y(t) = t\cos(t). \tag{3.9}$$

Using Euler's algorithm, we can find approximate solutions to problem (3.10) *with the initial condition.*

The following graphical representations show the real and approximate solutions for different values of: α and β. And also $N =$.
we consider the fractional problem

$$\begin{cases} \mathcal{D}^\alpha y = (\beta - \alpha)y + \alpha x, \\ y(0) = 1. \end{cases} \tag{3.10}$$

the equivalent problem for problem (3.10)*is as follows:*

$$\begin{cases} y' = -y + x \\ y(0) = 1 \end{cases} \tag{3.11}$$

This is a linear problem and can be solved exactly to yield the solution

$$Y(x) = x - 1 + 2e^{-x}. \tag{3.12}$$

Hence, there is no need to solve (1.11) numerically. We will proceed numerically for illustrative purposes only. For Euler's method, $L = 1$ and $h = 0.2$. Then, $x_0 = 0.0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$ and differential equation (1.11) is approximated by the difference equation

$$\frac{(y_{i+1} - y_i)}{0.2} + y_i = x_i, i = 0, 1, 2, 3, 4,$$

or, equivalently, by

$$y_{i+1} = (0.8)y_i + (0.2)x_i, i = 0, 1, 2, 3, 4. \tag{3.13}$$

Since $y_0 = 1$, (1.13) yields, to three decimal places

$$\begin{aligned}
y_1 &= (0.8)y0 + (0.2)x0 = (0.8)(1.000) + (0.2)(0.0) = 0.800\\
y_2 &= (0.8)y1 + (0.2)x1 = (0.8)(0.800) + (0.2)(0.2) = 0.680\\
y_3 &= (0.8)y2 + (0.2)x2 = (0.8)(0.680) + (0.2)(0.4) = 0.624\\
y_4 &= (0.8)y3 + (0.2)x3 = (0.8)(0.624) + (0.2)(0.6) = 0.619\\
y_5 &= (0.8)y4 + (0.2)x4 = (0.8)(0.619) + (0.2)(0.8) = 0.655
\end{aligned}$$

Thus, the numerical approximation with $h = 0.2$ is

$$\begin{aligned}
y(0.0) &= 1.000\\
y(0.2) &= 0.800\\
y(0.4) &= 0.680\\
y(0.6) &= 0.624\\
y(0.8) &= 0.619\\
y(1.0) &= 0.655
\end{aligned}$$

However, from (3.12), the exact solution, rounded to three decimal places, at the grid points is given by

$$\begin{aligned}
Y(0.0) &= 1.000\\
Y(0.2) &= 0.837\\
Y(0.4) &= 0.741\\
Y(0.6) &= 0.698\\
Y(0.8) &= 0.699\\
Y(1.0) &= 0.736
\end{aligned}$$

Through the results for the approximate solution and the exact solution for time periods for: t. We can write the following table in order to show the absolute error.

The time t	Exact solution $y(t)$	Approximate solution $Y(t)$	Absolute error
0.0	1.000	1.000	0.000
0.2	0.800	0.837	0.037
0.4	0.680	0.741	0.061
0.6	0.624	0.698	0.074
0.8	0.619	0.699	0.080
1.0	0.655	0.736	0.081

Table 3.1: The absolute error between the exact solution and the approximate solution in the example (3.2)

Comparison of the numerical and the exact solutions then yields the precise amount of error that results at each grid point when employing Euler's method. It is essential then to know, a priori,that the unknown error at each grid point is arbitrarily small if h is arbitrarily small, that is, that the error at each grid point decreases to zero as h decreases to zero. If this were valid, then one would have the assurance that the error generated by Euler's method is negligible for all sufficiently small grid sizes h.

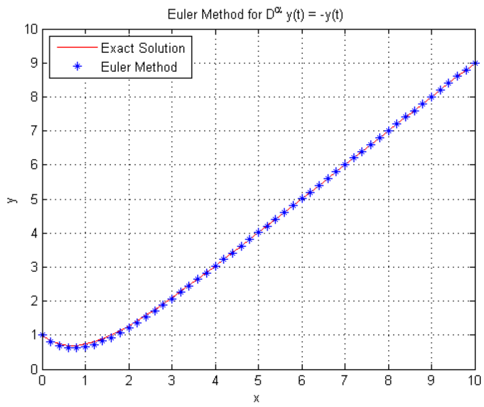

Figure 3.4: Exact solutions and numerical solutions for $t \in [0, 10]$ and $h = 0.2$

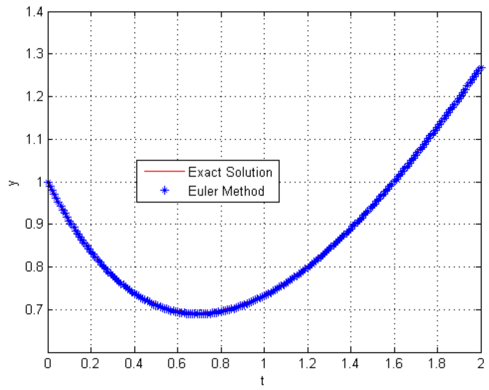

Figure 3.5: Exact solutions and numerical solutions for $t \in [0, 2]$ and $h = 0.02$

Figure 3.6: Comparison of the numerical and exact solutions of the example 3.2.

Example 3.3

$$\begin{cases} \mathcal{D}^{\alpha} y(t) = sint \\ y(0) = 0 \end{cases} \tag{3.14}$$

we transform this equation as follow:

$$\begin{cases} y'(t) = \frac{-\beta}{\alpha} y(t) + \frac{\sin(t)}{\alpha} \\ y(0) = 0 \end{cases} \tag{3.15}$$

Through the algorithm on the Matlab program, we obtain graphical representations for the values ??of: α,β, and the step h and also for different periods of time t. The following two graphs illustrate this

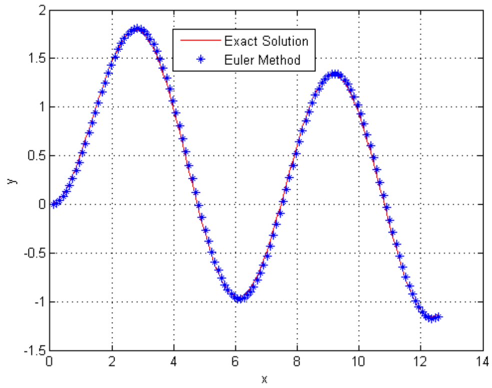

Figure 3.7: Exact solutions and numerical solutions for $\alpha = 0.8$, $\beta = 0.2$, $t \in [0, 4\pi]$ and $h = \pi/30$.

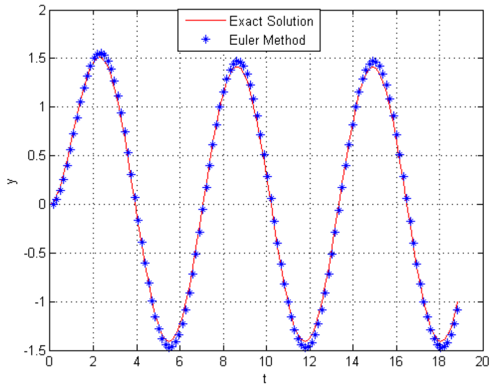

Figure 3.8: Exact solutions and numerical solutions for $\alpha = \beta = 0.5$, $t \in [0, 6\pi]$ and $h = \pi/20$.

Figure 3.9: Comparison of the numerical and exact solutions of the example 3.2.

Example 3.4

$$\begin{cases} \mathcal{D}^{\alpha} y(t) = ty(t) + (-1 - \frac{\alpha}{(t-\beta)^2}) \\ y(0) = \frac{-1}{\beta} \end{cases} \tag{3.16}$$

we transform equation (3.16)as

$$\begin{cases} y' = \frac{(t-\beta)}{\alpha} y + (\frac{-1}{\alpha} - \frac{1}{(t-\beta)^2}) \\ y(0) = \frac{-1}{\beta} \end{cases} \tag{3.17}$$

the exact solution of equation(3.17) is

$$y(t) = \frac{1}{t - \beta} \tag{3.18}$$

Example 3.5

$$\begin{cases} \mathcal{D}^{\alpha} y(t) = e^t y(t) + (\frac{\alpha}{(1-\beta)}) \cos t + \frac{(\beta - e^t)}{(1-\beta)} \sin t \\ y(0) = 0 \end{cases} \tag{3.19}$$

we transform equation (3.19)as

$$\begin{cases} y'(t) = \frac{(e^t - \beta)}{\alpha} y(t) + \frac{1}{(1-\beta)} \cos t + \frac{(\beta - e^t)}{\alpha(1-\beta)} \sin t \\ y(0) = 0 \end{cases} \tag{3.20}$$

the exact solution of equation (3.20)is

$$y(t) = \frac{1}{(1-\beta)} sin(t) \tag{3.21}$$

Example 3.6

$$\begin{cases} \mathcal{D}^{\alpha} y(t) = y(t) + \frac{\alpha.\beta}{t} \sin t + \frac{(\beta-1)}{t} y(t) \\ y(0) = \beta \end{cases} \tag{3.22}$$

we transform equation (3.22)as

$$\begin{cases} y'(t) = \frac{(1-\beta)}{\alpha} y(t) + t.\beta. \sin t - \frac{(1-\beta)}{\alpha} t.y(t) \\ y(0) = \beta \end{cases} \tag{3.23}$$

the exact solution of equation (3.23)is

$$y(t) = \beta cost \tag{3.24}$$

3.3 Approximate solution of deformable fractional equation by Rang-Kutta-4 method

We can employ a new set of numerical techniques known as Runge-Kutta methods to solve differential equations. These methods offer enhanced accuracy without requiring additional computations. They are designed to approximate Taylor series solutions up to the term proportional to h_i, where i denotes the method's order. One notable advantage of Runge-Kutta formulas is their reliance on function values at specific points.

3.3.1 Fourth Order RK Method

The most commonly used Runge Kutta method to find the solution of a differential equation is the RK4 method, i.e., the fourth-order Runge-Kutta method [8]. The Runge-Kutta method provides the approximate value of y for a given point x. Only the first order ODEs can be solved using the Runge Kutta RK4 method.

3.3.2 Runge-Kutta Fourth Order Method Formula

Let the problem,

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(0) = y_0. \end{cases} \tag{3.25}$$

The formula for the fourth-order Runge-Kutta method is given by:

$$y_1 = y_0 + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}.$$

Here,

☞ $k_1 = hf(x_0, y_0)$,

☞ $k_2 = hf[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1]$,

☞ $k_3 = hf[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2]$,

☞ $k_4 = hf(x_0 + h, y_0 + k_3)$.

So,Press et al. 1992 (See:[19]) sometimes known as RK4. This method is simple and reasonably robust and is a good general candidate for the numerical solution of differential equations when combined with an intelligent adaptive step size routine. In this section, we find approximate solutions to a deformable fractional differential problem with an initial condition using the Range Kutta 4 method. We transform the problem into a classic problem and solve it as we did in Euler's method.
Here is the RK4 algorithm in a general form:

Algorithm 2 (Range Kutta 4 Algorithm)

1: **Initializations:**

1. We take the problem as the initial value (3.25) after it was a fractional problem with the fractional values α and β.
2. Choose α and β values.
3. Start with the initial condition: y_0 at t_0.
4. For each time step:
   (a) Calculate the four intermediate slopes k_1, k_2, k_3 and k_4. Using:
   - ☞ $k_1 = hf(x_0, y_0)$,
   - ☞ $k_2 = hf[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1]$,
   - ☞ $k_3 = hf[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2]$,
   - ☞ $k_4 = hf(x_0 + h, y_0 + k_3)$.

   (b) Calculate the weighted average of these slopes to get the next value of y.
   $$y_{i+1} = y_i + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}.$$
   (c) Update the time: $t_{i+1} = t_i + h$.
5. Repeat steps 3 and 4 until the desired time is reached.

Example 3.7 *Consider a deformable fractional ordinary differential equation with initial condition*

$$\begin{cases} \mathcal{D}^{\alpha}y(t) = 2\beta cos(t) - sin(t) + \frac{\beta^2}{\alpha}sin(t), \\ y(0) = 0 \end{cases} \tag{3.26}$$

The exact solution to the problem with initial condition A is given as follows:

$$y_e(t) = (\alpha/\beta)sin(t). \tag{3.27}$$

Using Euler's algorithm, we can find approximate solutions to problem (3.26) *with the initial condition.*

The following graphical representations show the real and approximate solutions for different values of: α *and* β*. And also* N*.*

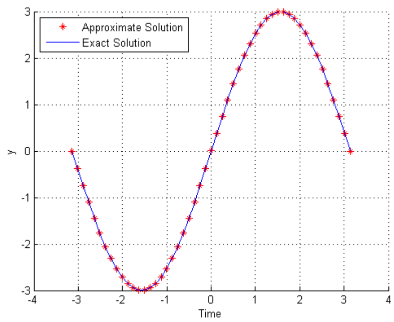

Figure 3.10: Exact solutions and numerical solutions for $\alpha = 0.75, \beta = 0.25$ and $N = 50$.

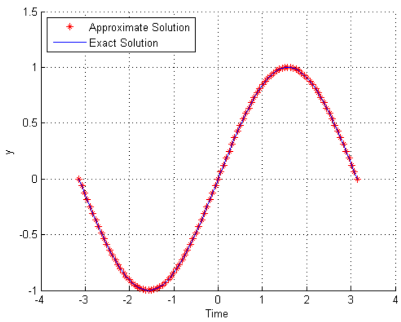

Figure 3.11: Exact solutions and numerical solutions for $\alpha = \beta = 0.5$, and $N = 100$

Figure 3.12: Comparison of the numerical and exact solutions of the example 3.7.

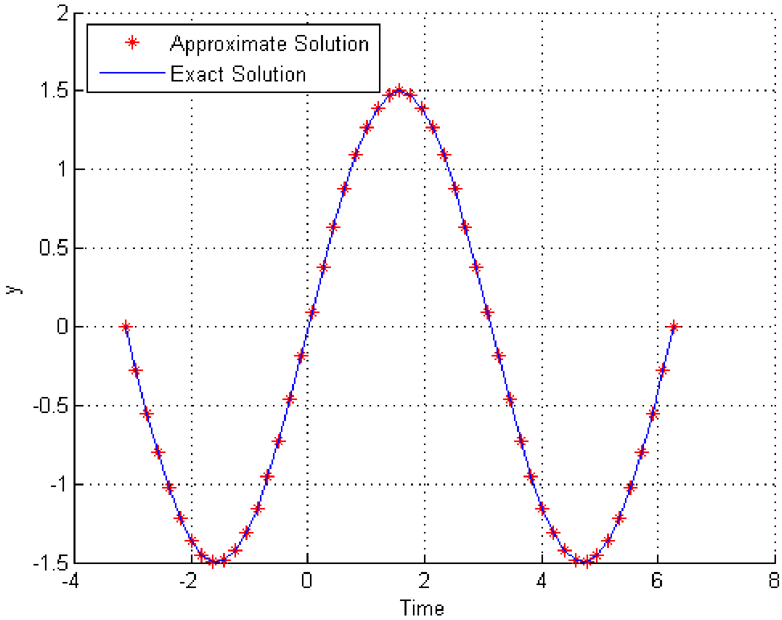

Figure 3.14: Exact solutions and numerical solutions for $\alpha = 0.6$,$\beta = 0.4$, and $N = 50$, with $t \in [-\pi, 4\pi]$

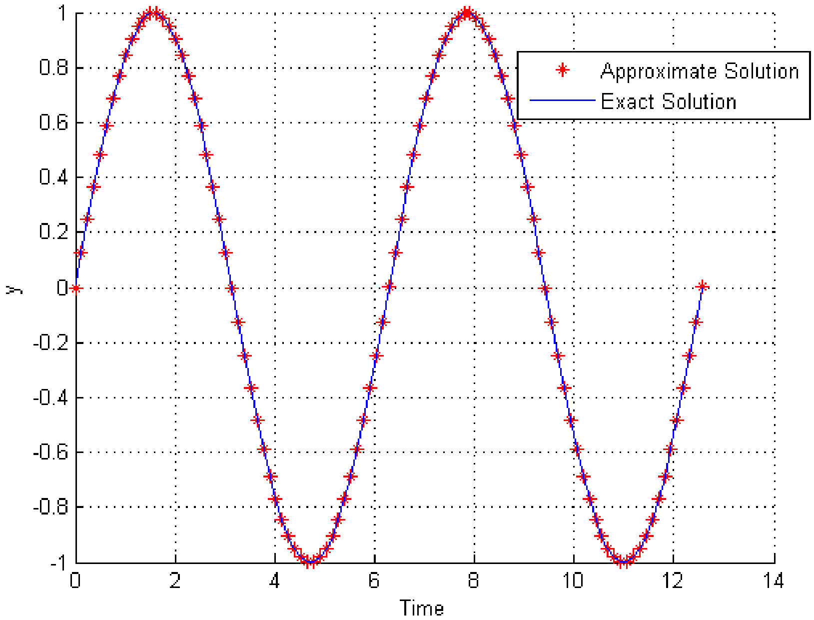

Figure 3.13: Exact solutions and numerical solutions for $\alpha = \beta = 0.5$, and $N = 100$, with $t \in [0, 4\pi]$.

Example 3.8 *Consider the initial value problem*

$$\begin{cases} \mathcal{D}^{\alpha}y(t) = \beta t y(t) - \beta t^4 + \beta t^3 + (3\alpha + \beta)t^2 - \beta t - \alpha, \\ y(0) = 0. \end{cases} \tag{3.28}$$

With $0 < \alpha \leq 1$*, and* $\alpha + \beta = 1$*. The exact solution for this problem is*

$$y_e(t) = t^3 - t. \tag{3.29}$$

The equivalent problem to Problem (3.28) *is as follows*

$$\begin{cases} y'(t) = \frac{\beta}{\alpha}(t-1)y(t) - \frac{\beta}{\alpha}t^4 + \frac{\beta}{\alpha}t^3 + (3 + \frac{\beta}{\alpha})t^2 - \frac{\beta}{\alpha}t - 1, \\ y(0) = 0. \end{cases} \tag{3.30}$$

Using RK-4 algorithm, we can find approximate solutions to problem (3.28) *with the initial condition.*

The following graphical representations show the real and approximate solutions for different values of: α *and* β*. And also* N*.*

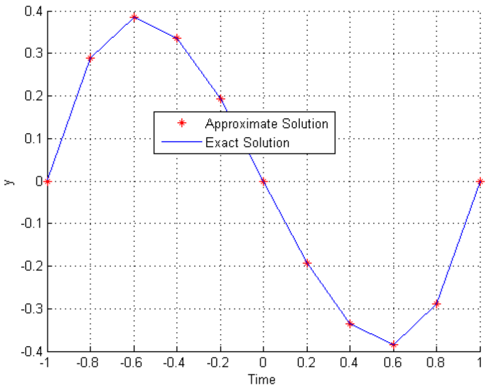

Figure 3.15: Exact solutions and numerical solutions for $\alpha = 0.9, \beta = 0.1$ and $N = 10$.

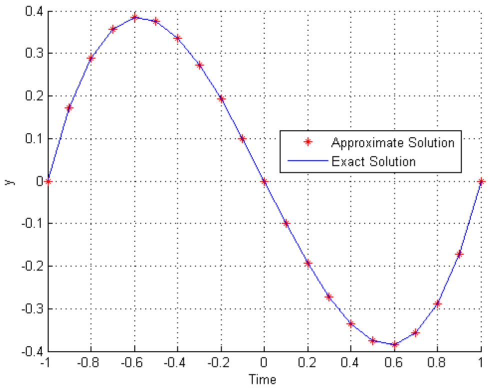

Figure 3.16: Exact solutions and numerical solutions for $\alpha = \beta = 0.5$, and $N = 20$

Figure 3.17: Comparison of the numerical and exact solutions of the example 3.8.

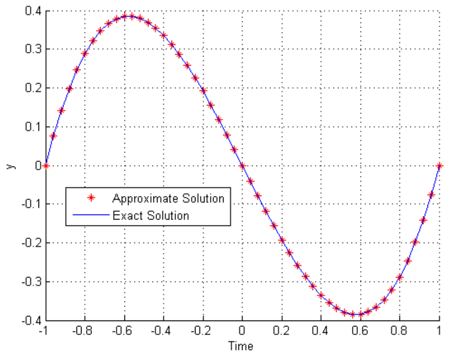

Figure 3.18: Exact solutions and numerical solutions for $\alpha = 0.7$,$\beta = 0.3$, and $N = 50$

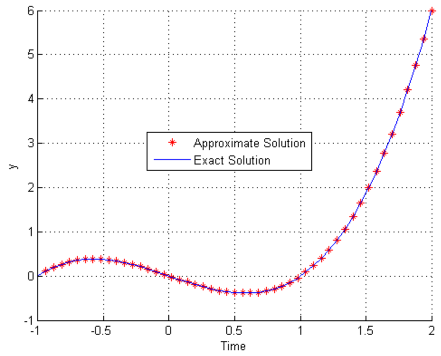

Figure 3.19: Exact solutions and numerical solutions for $\alpha = \beta = 0.5$, and $N = 50$ in $-1 \leq t \leq 2$.

Figure 3.20: Comparison of the numerical and exact solutions of the example 3.8.

Example 3.9 *Consider the initial value problem*

$$\begin{cases} \mathcal{D}^{\alpha}y(t) = \beta y(t) + \beta cos(t), \\ y(0) = 0. \end{cases} \tag{3.31}$$

With $0 < \alpha \leq 1$, *and* $\alpha + \beta = 1$. *The exact solution for this problem is*

$$y_e(t) = \frac{\beta}{\alpha} sin(t). \tag{3.32}$$

The equivalent problem to Problem (3.31) *is as follows*

$$\begin{cases} y'(t) = \frac{\beta}{\alpha} cos(t), \\ y(0) = 0. \end{cases} \tag{3.33}$$

Using RK-4 algorithm, we can find approximate solutions to problem (3.28) with the initial condition.

The following graphical representations show the real and approximate solutions for different values of: α and β. And also N.

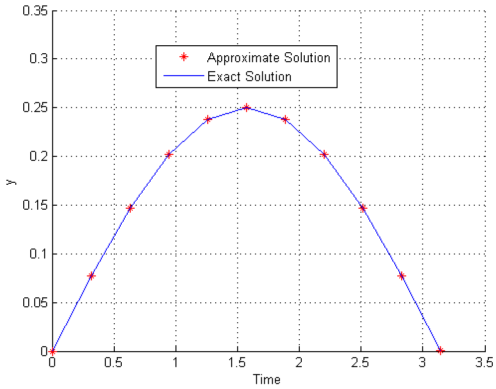

Figure 3.21: Exact solutions and numerical solutions for $\alpha = 0.8, \beta = 0.2$ and $N = 10$.

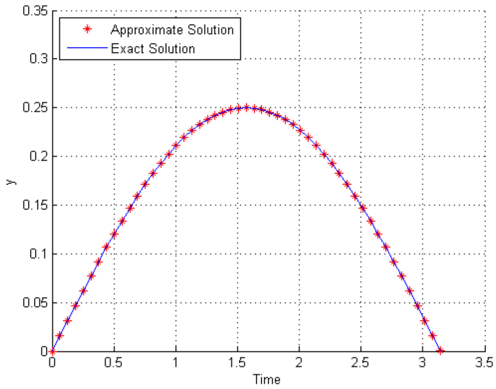

Figure 3.22: Exact solutions and numerical solutions for $\alpha = 0.8, \beta = 0.2$, and $N = 50$

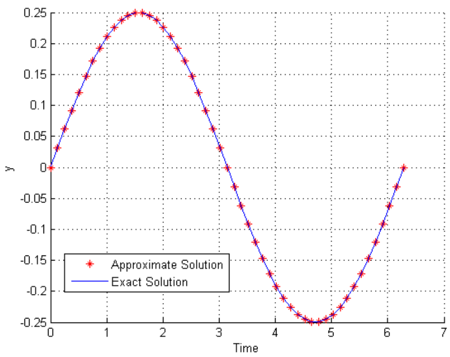

Figure 3.23: Exact solutions and numerical solutions for $\alpha = 0.7$,$\beta = 0.3$, and $N = 50$

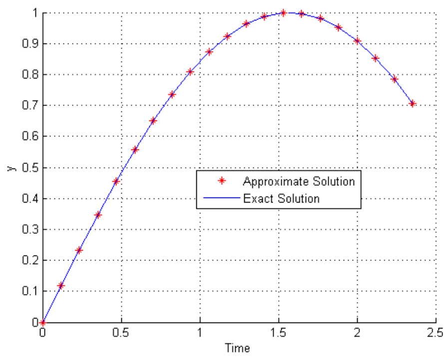

Figure 3.24: Exact solutions and numerical solutions for $\alpha = \beta = 0.5$, and $N = 20$ in $0 \leq t \leq 3\frac{\pi}{4}$.

Figure 3.25: Comparison of the numerical and exact solutions of the example 3.9.

CONCLUSION

Fractional arithmetic of all kinds, including the study we conducted in this memorandum, is a very important aspect in modeling various different phenomena in reality.

Calculating exact and approximate solutions to various differential equations, especially fractional derivatives, opens the way for other studies.

In conclusion:

- ☞ Deformable fractional calculus represents a major advance In the field of differential equations, providing more accurate modeling tools And analyze systems with dynamic and complex behaviors.
- ☞ Fractional differential equations with the distorted fractional derivative of order α, where: $0 < \alpha \leq 1$ are the equations that we can solve and find their solutions, whether analytically or using numerical methods.
- ☞ The scope remains open to solve this type of equation for $\alpha > 1$.

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ملخص:

في هذه المذكرة، قمنا بإيجاد الحلول الدقيقة والحلول التقريبية للمعادلات التفاضلية الكسرية ذات مشتق كسري يسمى المشتق الكسري المشوه من الرتبة α حيث: $0<\alpha\leq 1$.

لإيجاد الحلول الدقيقة قمنا باستعمال تحويل لابلاس وفق المشتق الكسري المستعمل.

أما الحلول التقريبية فقمنا باستخدام طريقتين عدديتين وهما طريقة أويلر وطريقة رونج كوتا-4.

كلمات مفتاحية: المشتق الكسري المشوه – تحويل لابلاس – طريقة أويلر – طريقة رونج كوتا -4.

Abstract:

In this work, we find exact solutions and approximate solutions for fractional differential equations with a fractional derivative called the distormable fractional derivative of order α, where : $0<\alpha\leq 1$.

To find the exact solutions, we used the Laplace transform depending on the fractional derivative used.

As for the approximate solutions, we used two numerical methods: the Euler method and the Runge-Kutta-4 method.

Keywords: *Deformable fractional derivative - Laplace transform - Euler method - Runge-Kutta method -4.*

Résumé:

Dans ce travail, nous trouvons des solutions exactes et des solutions approchées pour des équations différentielles fractionnaires avec une dérivée fractionnaire appelée dérivée fractionnaire déformable d'ordre α, où : $0<\alpha\leq 1$.

Pour trouver les solutions exactes, nous avons utilisé la transformée de Laplace en fonction de la dérivée fractionnaire utilisée.

Quant aux solutions approchées, nous avons utilisé deux méthodes numériques : la méthode d'Euler et la méthode Runge-Kutta-4.

Mots clés : *La dérivée fractionnaire déformable - Transformée de Laplace - Méthode d'Euler - Méthode de Runge-Kutta-4.*