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Differential quadrature in ordinary differential equations

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IN THE NAME OF GOD MERCIFUL THE MOST MERCIFUL

… Praise be to **Allah**, who hath guided us to this (thesis) never could we have found guidance , had it not been for the guidance of **Allah**… .

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Finally, I hope that this modest work will benefit you all and you may find it useful, and i hope someone will do better and more successful.

Contents

Chapter 1

INTRODUCTION

The Differential Quadrature Method (DQM) is a numerical solution technique for initial and boundary value problems. It was developed by the late Richard Bellman and his associates in the early 70s and since then, The method has been projected by its proponents as a potential alternative to the conventional numerical solution techniques such as the fnite difference and finite element methods. (Bert,Malik (1996)).

The DQ method, akin to the convential integral quadrature method, approximates the derivative of a function at any location by a linear summation of all the functional values along a mesh (grid) line. The key procedure in the DQ application lies in the determination of the weighting coefficients. The DQ method and its applications were rapidly developed after the late 1980s, thanks to the innovative work in the computation of the weighting coefficients by other researchers. As a result, the DQ method has emerged as a powerful numerical discretization tool in the past decade.(Shu and Richards(1990), Shu(2000)).

At the first time the Differential Quadrature Method was mentioned in a book written by Bellman and Roth in 1986. There are many innovative ideas contained in this book. In 1996, Bert and Malik presented a comprehensive review of the chronological development and the application of the DQ method. The textbook of Shu(2000) represents the first comprehensive work on the DQ method and applications. However, there is no abundant book which systematically describes both the theoretical analysis and the application of the DQ method. Since there are many achievements in the DQ method, the number of reference books on the DQ method and its applications will increase.

In seeking an efficient discretization technique to obtain accurate numerical solutions using a considerably small number of grid points, Bellman(1971, 1972) introduced the method of DQ where a partial derivative of a function with respect to a coordinate direction is expressed as a linear weighted sum of all the functional values at all grid points along that direction. Bellman(1972) suggested two methods to determine the weighting coefficients of the first order derivative. The first method solves an algebraic equation system. The second uses a simple algebraic formulations, but with the coordinates of grid points chosen as the roots of the Legendre Polynomials. Unfortunately, when the order of the algebraic equation system is large, its matrix is ill-conditioned. Thus, it is difficult to obtain the weighting coefficients.

To further improve the computation of, Quann and Chang (1989a,b) applied Lagrange Interpolated polynomials as test functions and obtained explicit formulations to calculate the weighting coefficients for the discretization of the first and second order derivatives.

Shu and Richards(1990) generalized all the current methods for determination of the weighting coefficients under the analysis of a high order polynomial approximation and the analysis of a linear vector space. The weighting coefficients of the first order derivative are determined by a simple algebraic formulation whereas the weighting coefficients of the second and higher order derivatives are determined by a recurrence relationship.

the powerful technique DQ has been succesfully employed in a variety of practical problems their applications can be found in engineering and biosciences ,phsical sciences like transport processes fluid mechanics, chemical reactor design,static and dynamic structural mechanics, static aeroelasticity and lubrication mechanics ect...

1.1 Integral Quadrature

One problem which frequently arises in science and engineering is the evaluation of $\int_b^a f(x)dx$ over a finite interval $[a.b]$. If a function F exists such that $dF/dx = f$, then the value of this integral is $F(b) - F(a)$. Unfortunately. in practical problems. it is extremely difficult. if not impossible. to obtain an explicit expression for F . Indeed. the values of f may only be known at a discrete set of points and in this situation. a numerical approach is essential.

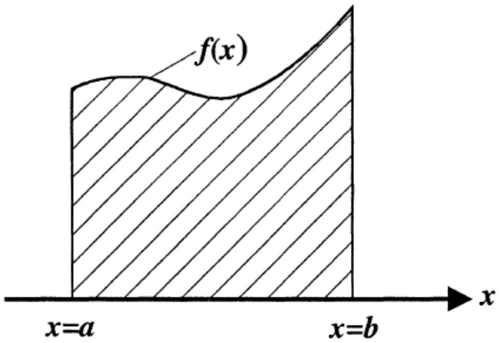

Figure 1: Integral of $f(x)$ over an interval $[a,b]$

On the other hand . it was found that the integral $\int_b^a f(x)dx$ represents the area under the curve $f(x)$ as shown in Figure 1. Thus. evaluating the integral is equivalent to the approximation of the area. Using this principle. many numerical techniques were developed. In general. the integral $\int_b^a f(x)dx$ can be approximated by,

$$\int_b^a f(x)dx = w_1 f_1 + w_2 f_2 + ... + w_n f_n = \sum_{k=1}^{n} w_k f_k$$

where $w_1, w_2, ..., w_n$ are the weighting coefficients, $f_1, f_2, ..., f_n$ are the functional values at the discrete points $a = x_1, x_2, ..., x_n = b$. Previous equation is called the integral quadrature, which uses all the functional values in the whole integral domain to approximate an integral over a finite interval.

In general, the discrete points are selected so as to give a uniform distribution . i.e, $x_i = x_{i-1} + h$, $(i = 2, 3, ..., n)$, where h is called the step size. All the conventional quadrature rules can be written in the form Previous equation . The following lists two special cases of previous equation :

(1) Trapezoidal rule : for this case , $w_1, w_2, ..., w_n$ are taken as,

$$w_1 = \frac{h}{2} \ , \qquad w_2 = w_3 = ... = w_{n-1} = h \ , \qquad w_n = \frac{h}{2}$$

(2) Simpson's one-third rule : for this case , $w_1, w_2, ..., w_n$ are taken as,

$$w_k = \begin{cases} \frac{h}{3} \ , & \text{when } k = 1, n \\ \frac{4h}{3} \ , & \text{when } k = 2, 4, 6, ..., n-2 \\ \frac{2h}{3} \ , & \text{when } k = 3, 5, 7, ..., n-1 \end{cases}$$

1.2 Analysis of A Linear Vector Space

In this section, we will review a number of concepts and properties in a linear vector space, which will be used in the following chapters to compute the weighting coefficients in the DQ approximation.

1.2.1 Definition of A Linear Vector Space

A linear vector space is defined over a field. The formal definition of a field is given as follows.

★**definition of field :** A field, denoted by F, consists of a set of elements called scalars and two operations called addition "+" and multiplication "×" or "·". The two operations are defined to satisfy the following conditions :

(1) To every pair of elements a and b in F , there is a corresponding element $a+b$ called the sum of a and b, and an element $a{\cdot}b$ or ab in F, called the product of a and b.

(2) Addition and multiplication are respectively commutative : for any a and b in F,

$$a+b=b+a\ ,\quad a\cdot b=b\cdot a$$

(3) Addition and multiplication are respectively associative : for any a, b and c in F,

$$(a+b)+c=a+(b+c)\quad ,\quad (a\cdot b)\cdot c=a\cdot(b\cdot c)$$

(4) Multiplication is distributive with respect to addition : for any a, b and c in F,

$$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$$

(5) F contains two elements, denoted by 0 and 1 respectively, such that,

$$a+0=a\quad ,\quad 1\cdot a=a\qquad \text{for every } a \text{ in } F.$$

(6) To every a in F, there is an element b in F such that,

$$a+b=0$$

(7) To every a in F which is not the element 0, there is an element c in F such that,

$$a\cdot c=1$$

Before we introduce the concept of vector spaces, let us consider a two dimensional geometric plane. When the origin is chosen, every point in the plane can be represented by a vector. The vector has direction as well as magnitude, and a vector can be extended. Any two vectors can be added, but the product of two vectors is not defined. Such a plane, using mathematical terminology, is called a linear space, or a vector space, or a linear vector space.

The formal definition of a linear vector space is given as follows.

★**Definition of a linear vector space :** A linear vector space, denoted by V, consists of a set of elements called vectors, a field F, and two operations called vector addition and scalar multiplication. The two operations are defined such that the following conditions are satisfied:

(1) To every pair of vectors α and β in V, there is a corresponding vector $\alpha + \beta$ called the sum of α and β.

(2) Addition is commutative : For any α and β in V,

$$\alpha + \beta = \beta + \alpha$$

(3) Addition is associative : For any α,β and γ in V,

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

(4) V contains a vector, denoted by 0, such that $0 + \alpha = \alpha$, for every α in V. The vector 0 is called the zero vector or the origin.

(5) To every α in V, there is a vector β in V such that ,

$$\alpha + \beta = 0$$

(6) To every c in F, and every α. in V, there is a corresponding vector $c\alpha$ in V called the scalar product of c and α.

(7) Scalar multiplication is associative : For any a,b in F and any γ in V,

$$a(b\gamma) = (ab)\gamma$$

(8) Scalar multiplication is distributive with respect to vector addition: For any a in F, and any β, γ in V,

$$a(\beta + \gamma) = a\beta + a\gamma$$

(9) Scalar multiplication is distributive with respect to scalar addition: For any a, b in F, and any γ in V,

$$(a+b)\gamma = a\gamma + b\gamma$$

(10) For any α in V, $1\alpha = \alpha$ where 1 is element in F.

1.2.2 Properties of A Linear Vector Space

★**Linear independence:** A set of vectors, $\alpha_1, \alpha_2, ..., \alpha_n$ in a linear vector space V over a field F, is said to be linearly independent if and only if the equation

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + ... + \lambda_n\alpha_n = 0$$

implies $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$, where $\lambda_1, \lambda_2, ..., \lambda_n$ are the elements in a field F.

From the above definition, it can be seen that the linear independence depends not only on the set of vectors but also on the field. It is also clear from the definition of linear independence that if the vectors $\alpha_1, \alpha_2, ..., \alpha_n$ are linearly dependent, then at least one of them can be written as a linear combination of the others.

★**Dimension of a linear vector space:** The maximum number of linearly independent vectors in a linear vector space V is called the dimension of the linear vector space.

★**Basis or base vectors:** A set of linearly independent vectors in a linear vector space V is said to be a basis of V if every vector in V can be expressed as a unique linear combination of these vectors

When the dimension of the linear vector space is n, the above property can be stated as: in an n-dimensional linear vector space V_n , any set of linearly independent vectors qualifies as a basis.

★**Change of basis:** In an n-dimensional linear vector space V_n , there exist many sets of basis, and each set of basis (base vectors) can be uniquely expressed by another set of basis (base vectors). This property is obvious, and can be easily derived from the property of basis.

Example: Consider two sets of base vectors $\alpha_1, \alpha_2, ..., \alpha_n$ and $\beta_1, \beta_2, ..., \beta_n$ in V_n . This property indicates that,

$$\begin{aligned} \alpha_i &= \sum_{j=1}^{n} c_{i,j} \cdot \beta_j \quad , \quad i = 1, 2, ..., n \\ \beta_i &= \sum_{j=1}^{n} d_{i,j} \cdot \alpha_j \quad , \quad i = 1, 2, ..., n \end{aligned}$$

where $c_{i,j}$ and $d_{i,j}$ are the elements in a field F.

1.3 Analysis of Normed vector spaces

It is an important class of metric spaces, of which Euclidean spaces are the model basic. In general, a vector normed space is a vector space in which there is a metric compatible with the vector space structure.

Definition of Normed vector spaces

Let E be a vector space on the field $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$, we say that E is a normed vector space if it has a norm $\|.\|$ who checks:

1. $\forall x \in E$, $\|x\| \geqslant 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.

2. $\forall \lambda \in \mathbb{k}$, $x \in E$, $\|\lambda x\| = |\lambda| \, \|x\|$ or $|\lambda|$ respectively denotes the absolute value if $\mathbb{k} = \mathbb{R}$ or the module $\mathbb{k} = \mathbb{C}$.

3. $\forall x, y \in E$, $\|x + y\| \leq \|x\| + \|y\|$ (triangular inequality).

Example 1 *if* $(E, \|.\|)$ *is a normed vector space, we define the distance associated with a norm by ,*

$$d(x, y) = \|x - y\|$$

1. In $\mathbb{R}^n$ we can define several normed :

- $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
- $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
- $\|x\|_\infty = \max_{1 \leq i \leq n} \{|x_i|\}$

2. Vector space $C([0,1],\mathbb{R})$ can be provided with standards :

- $\|f\|_1 = \int\limits_0^1 |f(t)|\,dt$
- $\|f\|_2 = \sqrt{\int\limits_0^1 (f(t))^2 dt}$
- $\|f\|_\infty = \max_{t\in[0,1]} |f(t)|$

3. In the space of bounded numerical sequences (with value in $\mathbb{R}$ or $\mathbb{C}$), we can define norm by,

$$\|u\| = \sup_{n\geqslant 0} |u_n|$$

4. Product Norm: if $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are two normed spaces, we can define a norm on the vector space $E \times F$ by,

$$\forall (x,y) \in E \times F, \|(x,y)\| = \max\{\|x\|_E, \|y\|_F\}$$

Definition 2 *Let E be a normalized vector space. Two norms $\|.\|_1$ and $\|.\|_2$ of E*

are said to be equivalent if there is $c_1, c_2 > 0$,such that, for all $x \in E$:

$$c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1$$

a. In $\mathbb{R}^n$, norms $\|.\|_1, \|.\|_2$ and $\|.\|_\infty$ are equivalent.

b. In $C([0,1],\mathbb{R})$,norms $\|.\|_1, \|.\|_2$ and $\|.\|_\infty$ are not equivalent.

c. Two equivalent norms induce two equivalent distances.

Suites de Cauchy

Let x_n be a sequence of elements of a norm space $(E, \|.\|)$, we say that the sequence x_n is Cauchy if, we have the following relation

$$\forall \varepsilon > 0,\ \exists N_\varepsilon,\ \forall p,q \geq N_\varepsilon, \quad \text{we have } \|x_p - x_q\| < \varepsilon$$

Lemma 3 *Let x_n be a Cauchy sequence in a normed space $(E, \|.\|)$ contains a subsequence x_{nk} converging to x then the sequence x_n is also convergent to the same element x .*

Weierstrass approximation theorem

Let $f(x)$ be a continuous function on the interval $[a,b]$. Then for any $\varepsilon > 0$, there exists an integer n and a polynomial p_n ,such that,

$$\max_{x\in[a,b]} |f(x) - p_n(x)| < \varepsilon$$

Weierstrass theorems (and in fact their original proofs) postulate existence of some sequence of polynomials converging to a prescribed continuous function uniformly on bounded closed intervals.

1.4 Banach spaces

Definition 4 *We call Banach space* $(E, \|.\|)$ *any normalized and complete vector space for the distance deduced from its norm.*

Throughout the suite, we are in the space of continuous functions

$$C([a,b]) = \{f : [a,b] \to \mathbb{R}/ \text{ continue}\},$$

Equipped with the norm of uniform convergence

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)| = \max_{x\in[a,b]} |f(x)|,$$

since $[a,b]$ is closed, bounded.
Therefore the space

$$C([a,b], \|f\|_\infty), \text{ is a BANACH space.}$$

Chapter 2

Ordinary differential equations

General definitions: differential equation is called an equation establishing a relationship between the independent variable x and the unknown function $y = \varphi(x)$ and its derived $y, y^{'}, ...y^{(n)}$ Symbolically.

Definition 5 *the differential order equation n is represented as follows,*

$$f(x, y, y^{'}, ..y^{(n)}) = 0 \qquad \text{(ODE)}$$

where, f is a function of $(n+2)$ variables.We only consider the case where, x and y have values in $\mathbb{R}$.

the solution for such a differential equation over the interval $I \subset \mathbb{R}$ is a function $y \in C^n(I, \mathbb{R})$ ($y : I \to \mathbb{R}$ which is n times continuously differentiable).

for every $x \in I$ we have,

$$f(x, y(x), y^{'}(x), y^{''}(x), ..., y^{(n)}(x)) = 0$$

2.0.1 Linear Differential Equations

A differential equation of the order n it is linear if and only if on the form,

$$l(y) = f(x) \qquad \text{(D.E)}$$

With,

$$l(y) = a_0(x)y + a_1(x)y^{'} + a_2(x)y^{''} + ... + a_n(x)y^{(n)}$$

The application $l : C^n \to C^0$ which associates the new function $l(y)$ with the function y, is a linear application.

Indeed,

$$\begin{aligned} l(y+z) &= \sum_{i=0}^{n} a_i(x)(y+z)^{(i)} \\ &= \sum_{i=0}^{n} a_i(x)y^{(i)} \sum_{i=0}^{n} a_i(x)z^{(i)}) \\ &= l(y)+l(z) \end{aligned}$$

and for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} l(\lambda y) &= \sum_{i=0}^{n} a_i(x)(\lambda y)^{(i)} \\ &= \lambda \sum_{i=0}^{n} a_i(x)y^{(i)} \\ &= \lambda l(y) \end{aligned}$$

The following differential equation called homogeneous equation,associated with linear differential equation.

$$l(y) = 0 \qquad \text{(H.E)}$$

Proposition 6 *The set S_0 of the solutions of **(H.E)** is the core of the linear application l . it is therefore a vector subspace of $C^n(\mathbb{R})$. The set S of the solutions to **(D.E)** given by,*

$$\begin{aligned} S &= y_p + S_0 \\ &= \{y_p + y_h; y_h \in S_0\} \end{aligned}$$

with,

$$l(y_p) = f(x)$$

So, the solutions are the form,

$$y = y_p + y_h$$

where, y_p is the particular solution of Eq (**D**.**E**), y_h is the solution of the Eq (**H**.**E**)

Theorem 7 *(existence of Peano's)*

let $R(a, b)$ is a rectangle in rectangle in xy plane and the point (x_0, y_0) is inside such that,

$$R(a, b) = \{(x, y) : |x - x_0| < a. |y - y_0| < b\}$$

if $f(x, y)$ is continous and $|f(x, y)| < M$ at all point $(x, y) \in \mathbb{R}$ then the problem with initial valeus has $y(x)$ a solution that is defined for all x over the interval $|x - x_0| < c$ where, $c = \min\{a, b/M\}$.

Theorem 8 *(uniqueness of Picard and Lindelöf)*

Under the hypotheses of theorem Peano's and if $\frac{\partial f}{\partial y}(x, y)$ is continuous and bounded for all points (x, y) in R, then the problem with initial values has a unique solution $y(x)$ which is defined for all x over an interval $|x - x_0| < c$.

2.0.2 Linear Differential Equations of the First Order

Definition 9 *A linear differential equation (LDE) of the first order, which can be write on the form,*

$$a(x)y^{'} + b(x)y = c(x)$$

where, a, b, c continuous functions in $I \subset \mathbb{R}$ and $\forall x \in I : a(x) \neq 0$.

To this differential equation, we can associate the same equation with $c = 0$,

$$a(x)y^{'} + b(x)y = 0$$

This is the homogeneous equation associated with linear (EDO),without a second member.

2.0.3 Linear Differential Equations of the Second Order

Definition 10 *A Linear differential equations of the second order with coefficients constants is a differential equation on the form*

$$ay^{''} + by^{'} + cy = f(x)$$

where $a, b, c \in \mathbb{R}$ $(a \neq 0)$ and $f \in C^0(I)$ (I open from $\mathbb{R}$), So The general linear homogeneous second order ODE associated is given by,

$$ay^{''} + by^{'} + cy = 0$$

Theorem 11 *the function $f(x, y, y')$ and its partial derivatives with respect to y,y' are continuous in a domain containing the values $x = x_0$, $y = y_0$, $y = y'$ there is one solution and only one $y = y(x)$.*

Chapter 3

Differential Quadrature method in one-dimensional

Differential quadrature method (DQM) is a numerical solution technique to solve ordinary and partial differential equations. By this elegant method, we approximate the spatial derivatives of unknown function at any grid points using weighted linear sum of the function values at certain points in whole domain

By following the idea of integral quadrature , Bellman et al. (1972) suggested that the first order derivative of the function $f(x)$ with respect to x at a grid point x_i,

$$f^{'}\left(x_i\right)=\sum_{j=1}^{N} a_{ij}\cdot f(x_j) \qquad (i=1,2,...,N) \tag{1.1}$$

where $f^{'}\left(x_i\right)$ indicates the first order derivative of $f(x)$ at x_i ,x_j is discrete points (nodes), $f\left(x_j\right)$ is a function values at x_j, and a_{ij} is weighting coefficients to be determined.

3.1 Computation of Weighting Coefficients for the First Order Derivative

Without loss of generality, Consider a one-dimensional function $f(x)$ that is continuous and differentiable with respect to x in a closed interval $[a,b]$. It is supposed that there are N grid points with coordinates including the two end points, that is, $a = x_1, x_2, ..., x_N = b$.

The determination of weighting coefficients a_{ij} in Equation (1.1) is a key procedure in the DQ approximation. Once the weighting coefficients are determined, the bridge to link the derivatives in the governing differential equation and the functional values at the mesh points is established. In other words, with the weighting coefficients, one can easily use the functional values to compute the derivatives.

In order to determine the weighting coefficients in must be approximated by some test functions. To select a suitable test function, one needs to satisfy the following conditions:

- **Differentiability:** The test function of the differential equation must be differentiable at least up to the nth derivative (here n is the highest order of the differential equation).
- **Smoothness:** $f(x)$ must be sufficiently smooth to be satisfied the condition of the differentiability.

In the following, we shall show that the weighting coefficients can be efficiently computed by employing some explicit formulations.

3.1.1 Bellman's Approaches

Bellman et al. (1972) proposed two approaches to compute the weighting coefficients $a_{ij} \quad (i, j = 1, ..., N)$ in Equation (1.1). The two approaches are based on the use of two different test functions.

Bellman's first approach:

In this approach, the test functions are chosen as,

$$f_k(x) = x^k \quad , \ (k = 0, 1, ..N - 1) \qquad (2.1)$$

Obviously, Equation (2.1) gives N test functions. For the weighting coefficients a_{ij} in Equation (1.1),the total number of weighting coefficients is $N \times N$. To obtain these weighting coefficients, the N test functions should be applied at N grid points $x_1, x_2, ..., x_N$ As a consequence, we get the following algebraic system of equations $N \times N$ for a_{ij} are obtained

for $k = 0$ we find,

$$f_0(x_j) = x_j^0 = 1 \ , \quad f_0^{'}(x_i) = 0$$

substuting these values in Equation (1.1) we obtain,

$$\begin{aligned} f_0^{'}(x_i) &= \sum_{j=1}^{N} a_{ij} \cdot f_0(x_j) \\ 0 &= \sum_{j=1}^{N} a_{ij} \ \cdot \ 1 \quad , \quad (i = 1, 2, ..., N) \\ 0 &= \sum_{j=1}^{N} a_{ij} \end{aligned}$$

for $k = 1$ we find,

$$f_1(x_j) = x_j^1 \ , \quad f_1^{'}(x_i) = 1 \cdot x_i^0 = 1$$

substuting these values in Equation (1.1) we obtain,

$$\begin{aligned} f_1^{'}(x_i) &= \sum_{j=1}^{N} a_{ij} \cdot f_1(x_j) \\ 1 &= \sum_{j=1}^{N} a_{ij} \cdot x_j \quad , \ (i = 1, 2, ..., N) \end{aligned}$$

for generalisation we have,

$$k \cdot x_i^{k-1} = \sum_{j=1}^{N} a_{ij} \cdot x_j^k, \quad (k = 2, 3, ..., N-1), \quad (i = 1, 2, ..., N)$$

$$\begin{cases} i = 1 & k \cdot x_1^{k-1} = \sum_{j=1}^{N} a_{1j} \cdot x_j^k \\ i = 2 & k \cdot x_2^{k-1} = \sum_{j=1}^{N} a_{2j} \cdot x_j^k \\ & \quad \cdot \\ & \quad \cdot \\ & \quad \cdot \\ i = N & k \cdot x_N^{k-1} = \sum_{j=1}^{N} a_{Nj} \cdot x_j^k \end{cases} \qquad (k = 0, 1, ..., N-1)$$

$$\begin{cases} k=0 & \sum_{j=1}^{N} a_{ij}\cdot x_j^0=\sum_{j=1}^{N} a_{ij}\ \cdot\ 1=a_{i1}+a_{i2}+...+a_{iN} \\ k=1 & \sum_{j=1}^{N} a_{ij}\cdot x_j^1=\sum_{j=1}^{N} a_{ij}\cdot x_j=a_{i1}\cdot x_1+a_{i2}\cdot x_2+...+a_{iN}\cdot x_N \\ & . \\ & . \\ & . \\ k=N-1 & \sum_{j=1}^{N} a_{ij}\cdot x_j^{N-1}=a_{i1}\cdot x_1^{N-1}+a_{i2}\cdot x_2^{N-1}+...+a_{iN}\cdot x_N^{N-1} \end{cases}$$

so we obtain,

$$\begin{bmatrix} k\cdot x_1^{k-1} \\ k\cdot x_2^{k-1} \\ . \\ . \\ . \\ k\cdot x_N^{k-1} \end{bmatrix}=\begin{bmatrix} a_{i1} \\ a_{i2} \\ . \\ . \\ . \\ a_{iN} \end{bmatrix}\begin{bmatrix} 1 & x_1 & . & . & . & x_1^{N-1} \\ 1 & x_2 & . & . & . & x_2^{N-1} \\ . & . & . & & & . \\ . & . & & & . & . \\ . & . & & & . & . \\ 1 & x_N & . & . & . & x_N^{N-1} \end{bmatrix},(k=0,1,...,N-1)\ ,(i=1,2,...,N) \quad (2.2)$$

In matrix form,Equation System (2.2) can be written as,

$$G=AV$$

where A is the weighting coefficient matrix of the first-order derivatives with respect to x, and

$$G=\begin{bmatrix} 0 & 1 & 2x_1 & ... & (N-1)x_1^{N-2} \\ 0 & 1 & 2x_2 & ... & (N-1)x_2^{N-2} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 1 & 2x_N & ... & (N-1)x_N^{N-2} \end{bmatrix} \qquad V=\begin{bmatrix} 1 & x_1 & ... & x_1^{N-1} \\ 1 & x_2 & ... & x_2^{N-1} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ 1 & x_N & ... & x_N^{N-1} \end{bmatrix}$$

Since V is the Vandermonde matrix, its inverse V^{-1} exists. One has,

$$A=GV^{-1}$$

In this way, the $N\times N$ weighting coefficients a_{ij} can be obtained.

Remark 12 *Equation System* (2.2) *has a unique solution because its matrix is of Vandermonde form. Unfortunately, when N is large, the matrix is iII-conditioned and its inversion is difficult. In the practical application of this approach, N is usually chosen to be less or equal than* 13 , $(N\leq 13)$.

Bellman's second approach:

In this approach, the test functions are chosen as,

$$f_k(x) = \frac{p_N^*(x)}{(x - x_k) \cdot p_N^{*(1)}(x_k)}, \qquad (k = 1, 2, ...N) \qquad (2.3)$$

where $p_N^*(x)$ is defined in terms of the Legendre N polynomials by the relation,

$$p_N^*(x) = p_N(1 - 2x)$$

where $P_N(x)$ is the Nth order Legendre polynomial for $-1 \leq x \leq 1$. and $p_N^{*(1)}(x_k)$ is the first order derivative of $p_N^*(x)$. By choosing x_i to be the roots of the shifted Legendre polynomial and applying Equation (2.3) at N grid points $x_1, x_2, ..., x_N$ Bellman et al.(1972) obtained a simple algebraic formulation to compute a_{ij} we see that

$$a_{ij} = \frac{p_N^{*(1)}(x_i)}{(x_i - x_j) \cdot p_N^{*(1)}(x_j)}, \text{ for } i \neq j \qquad (2.4a)$$

For the case when $i = k$, use of L'Hospital's rule plus the fact that the Legendre polynomial satisfies the differential equation,

$$x\left(1 - x^2\right) p_N^{*(2)}(x) + (1 + 2x)\, p_N^{*(1)}(x) + N(N + 1)\, p_N^{*(1)}(x) = 0$$

gives,

$$a_{ii} = \frac{1 - 2x_i}{2x_i(x_i - 1)} \qquad (2.4b)$$

Therefore, by choosing N, the order of the approximation, the N roots of the shifted Legendre polynomial are known. This ,in turn, defines the required coefficients which make the differential quadrature possible.

Remark 13 *Using Equation* (2.4)*the computation of the weighting coefficients is a simple task. However, this approach is not as flexible as the first approach because the coordinates of the grid points in this approach cannot be chosen arbitrarily. Instead, they should be chosen as the roots of the Legendre polynomial of degree N.*

Remark 14 *Equation* (2.4) *only reflects a special case. Due to the inflexibility associated with the second approach in selecting the grid points, the first approach is usually adopted in practical applications.*

3.1.2 Quan and Chang's Approach

To improve Bellman's approaches in computing the weighting coefficients, many attempts have been made by researchers. One of the most useful approaches is the one introduced by (Quan and Chang,1989a,b). Quan and Chang used the following Lagrange interpolation polynomials as the test functions,

$$f_k(x) = \frac{M\left(x\right)}{\left(x - x_k\right) \cdot M^{(1)}\left(x_k\right)}, k = 1, 2, ...N \qquad (2.5)$$

where,

$$M(x) = (x - x_1)(x - x_2)...(x - x_N) = \prod_{k=1}^{N}(x - x_k) \qquad (2.6)$$

and,

$$M^{(1)}(x_i) = \prod_{k=1,k\neq i}^{N}(x_i - x_k) \qquad (2.7)$$

Although there can be many choices of the test functions, a convenient and most commonly used choice in one-dimensional problems is the Lagrangian interpolation shape functions $f_k(x)$, where,

$$\psi\left(x\right) = \sum_{j=1}^{N} f_k(x)\psi_j$$

Note that the Lagrangian interpolation shape functions $f_k\left(x\right)$ have the following properties,

$$f_k\left(x_i\right) = \delta_{ij} \begin{cases} 1 & i = k = j \\ 0 & i \neq k = j \end{cases} (k = 1, ..., N)$$

Subsequently, by applying Equation (2.5) at N grid points, Quan and Chang (1989) and Shu and Richards (1992) obtained the following weighting coefficients a_{ij} ,

$$a_{ij} = a_j\left(x_i\right) = \frac{df_k\left(x_i\right)}{dx} = \frac{M^{(1)}(x_i)}{\left(x_i - x_k\right) \cdot M^{(1)}(x_k)}, \quad \left(i, j = 1, 2, ..., N\right), \quad i \neq j \qquad (2.8a)$$

$$a_{ij} = \frac{1}{x_j - x_i} \prod_{k=1,k\neq i,j}^{N} \frac{(x_i - x_k)}{(xj - x_k)}, \quad (i, j = 1, 2, ..., N), \quad i \neq j \qquad (2.8b)$$

$$a_{ii} = \sum_{k=1,k\neq i}^{N} \frac{1}{x_i - x_k}, (i = 1, 2, ..., N) \qquad (2.8c)$$

Remark 15 *When equation* (2.8) *is used, there is no restriction on the choice of the grid points.*

3.1.3 Shu's General Approach

Shu's general approach was inspired from Bellman's approaches. It covers all the approaches. including Quan and Chang's approach.

Starting from Bellman's two approaches. Shu raised two questions. The first one is why we can use two approaches to compute the weighting coefficients. The second is whether these two approaches give the same weighting coefficients. If so.there may exist some other approaches to compute the weighting coefficients. It was found that these questions could be answered by polynomial approximation and linear vector space analysis.

Now. we suppose that the degree of the approximated polynomial is $(N-1)$.The polynomial of degree less than or equal to $(N-1)$ constitutes an N dimensional linear vector space V_N with respect to the operation of vector addition and scalar multiplication. and can be expressed in different forms. One popular form is

$$f(x) = \sum_{k=0}^{N-1} c_k.x^k \qquad (2.9)$$

where c_k's are constants , also there exist many sets of base vectors in the linear vector space V_N in this case. the vector is actually the polynomial. So. the base vectors are also called the base polynomials. Four typical sets of the base,polynomials are listed as follows.

$$r_k(x) = x^k, \quad (k = 1, 2, ..., N-1) \qquad (2.10a)$$

$$r_k(x) = \frac{L_N(x)}{(x - x_k) \cdot L_N^{(1)}(x_k)}, \quad (k = 1, 2, ..., N) \tag{2.10b}$$

$$r_k(x) = \frac{M(x)}{(x - x_k) \cdot M^{(1)}(x_k)}, \quad (k = 1, 2, ..., N) \tag{2.10c}$$

$$r_k(x) = (x - x_{k-l}) \cdot r_{k-l}(x), \quad r_1(x) = 1, \quad (k = 2.3..., N) \tag{2.10d}$$

Among the four sets of base polynomials. Equations $(2.10b)$ and $(2.10c)$ are from the Lagrange interpolation polynomials while Equation $(2.10d)$ is from the Newton interpolation polynomials. The difference between Equation $(2.10b)$ and $(2.10c)$ lies in the distribution of the grid points. Equation $(2.10b)$ is a special case of Equation $(2.10c)$ since it is only valid at the Legendre collocation points.

It can be seen that Equation $(2.10a)$ is the same as the test functions of Bellman's first approach while Equation $(2.10b)$ is the same as the test functions of Bellman's second approach. In other words, the test functions of Bellman's two approaches are actually two sets of base polynomials in V_N Note that Equation (1.1) is a linear operator. Then from the properties of a linear vector space, we know that if one set of base polynomials satisfies a linear operator such as Equation (1.1), so do other sets of base polynomials. This means that every set of base polynomials would give the same weighting coefficients. Hence, the weighting coefficients do not depend on the choice of the test functions.

Remark 16 *the difference in Bellman's'two approaches only lies in the use of different test functions. The test functions are equivalent to the base polynomials.*

Remark 17 *the use of different sets of base polynomials will result in different approaches to compute the weighting coefficients.Since there are many sets of base polynomials in the linear vector space V_N, we have many approaches to compute the weighting coefficients.*

So, for generality, we use two sets of base polynomials. The Lagrange interpolation polynomials (Equation (2.10c)) are taken as the first set of base polynomials. For simplicity, we set

$$M(x) = N(x, x_k) \cdot (x - x_k), \quad (k = 1, 2, ..., N) \qquad (2.11)$$

with,

$$N(x_i, x_j) = M^{(1)}(x_i) \cdot \delta_{ij} \qquad (2.12)$$

where δ_{ik} is the Kronecker operator.

Using Equation (2.11), Equation (2.10c) can be simplified,

$$r_k(x) = \frac{N(x, x_k)}{M^{(1)}(x_k)}, \quad (k = 1, 2, ..., N) \qquad (2.13)$$

and at the point x_i ,

$$r_k(x_i) = \frac{N(x_i, x_k)}{M^{(1)}(x_k)}, \quad (i = 1, 2, ..., N) \quad (k = 1, 2, ..., N) \qquad (2.14)$$

From Equation (2.12), we can obtain the following expression as,

$$N(x_i, x_k) = M^{(1)}(x_i) \cdot \delta_{ik} = \begin{cases} 0 & if \quad i \neq k \\ M^{(1)}(x_i) & if \quad i = k \end{cases} \qquad (2.15)$$

giving,

$$r_k(x_i) = \begin{cases} 0 & if \quad i \neq k \\ 1 & if \quad i = k \end{cases} \qquad (2.16)$$

Here, $r_k(x)$, $k = 1, 2, ..., N$ are the base polynomials in V_N ,$f(x)$ can then be expressed by,

$$f(x) = \sum_{k=1}^{N} d_k \cdot r_k(x) \qquad (2.17)$$

Using this property of $r_k(x)$ when $i = k$ in the Equation (2.17) at the point x_i we obtain,

$$f(x_i) = \sum_{k=1}^{N} d_k \cdot r_k(x_i)$$

Thus the first order derivative of $f(x)$ with respect to x at the point x_i are,

$$f^{'}(x_i) = \sum_{k=1}^{N} d_k \cdot r_k^{'}(x_i) \qquad (2.18)$$

From Equation (1.1), the coefficients a_{ij} in the first order derivatives of $f(x)$ at the point x_i become,

$$r_k^{'}(x_i) = a_{ik}$$

Thus the coefficients aij can be computed by taking first order derivatives of $r_k(x)$ as follows,

$$r_j^{'}(x) = \frac{N^{(1)}(x_i, x_j)}{M^{(1)}\,(x_j)} = a_{ij} \quad , \quad (i, j = 1, 2, ..., N) \qquad (2.19)$$

where $N^{(1)}(x_i, x_j)$ is the first order derivative of the function $N(x_i, x_j)$.

We successively differentiate Equation (2.11) with respect to x and obtain the following recurrence formulation

$$\begin{aligned} M^{(m)}(x) \quad &= \quad N^{(m)}(x, x_k) \cdot (x - x_k) + mN^{(m-1)}(x, x_k), \\ &\quad \text{for} \quad (k = 1, 2, ..., N) \quad ; \quad (m = 1, 2, ..., N-1) \qquad (2.20) \end{aligned}$$

where $M^{(m)}(x)$ and $N^{(m)}(x, x_k)$ indicate the m^{th} order derivative of $M(x)$ and$N(x, x_k)$ respectively.

From the Equation (2.20) , we can easily obtain for $m = 1$ gives,

$$N^{(1)}(x_i, x_j) = \frac{M^{(1)}(x_i)}{(x_i - x_j)}, \quad i \neq j \qquad (2.21)$$

$$N^{(1)}(x_i, x_i) = \frac{M^{(1)}(x_i)}{2}, \quad i = j \qquad (2.22)$$

Subsituting Equation (2.21) into Equation (2.19), we finally obtain the coefficients a_{ij}

$$a_{ij} = \frac{M^{(1)}(x_i)}{(x_i - x_j) \cdot M^{(1)}(x_j)} \qquad (2.23a)$$

$$a_{ii} = \frac{M^{(2)}(x_i)}{2M^{(1)}(x_i)} \qquad (2.23b)$$

It is observed from Equation (2.23) that, if x_i is given, it is easy to compute $M^{(1)}(x_i)$ from Equation (2.7), and hence a_{ij} for $i \neq j$. However, the calculation of a_{ii} is based on the computation of the second order derivative

$M^{(2)}(x_i)$ which is not an easy task. This difficulty can be eliminated by using the second set of base polynomials.

According to the property of a linear vector space, if one set of base polynomials satisfies a linear operator, say Equation (1.1), so does another set of base polynomials. As a consequence, the equation system for determination of a_{ij} derived from the Lagrange interpolation polynomials (Equation (2.10c)) should be equivalent to that derived from another set of base polynomials x^k , $(k = 0, 1, ..., N-1)$ (Equation (2.10a)). Thus a_{ij} satisfies the following equation which is obtained by the base polynomial x^k.

when $k = 0$,

$$\sum_{j=1}^{N} a_{ij} = 0 \quad \text{or} \quad a_{ii} = -\sum_{j=1, j \neq i}^{N} a_{ij} \qquad (2.24)$$

Remark 18 *Equations* (2.23) *and* (2.24) *are two formulations to compute the weighting coefficients* a_{ij} *. It is noted that in the development of these two formulations, two sets of base polynomials were used in the linear polynomial vector space* V_N*.*

3.2 Computation of Weighting Coefficients for the second and Higher Order Derivatives

In this section, we show that the weighting coefficients of the second and higher order derivatives can be computed from a recurrence relationship.

3.2.1 Weighting Coefficients of the Second Order Derivative

For the discretization of the second order derivative, we introduce a similar approximation form given by,

$$f^{(2)}(x_i) = \sum_{j=1}^{N} b_{ij} \cdot f(x_j) \qquad (i = 1, 2, ..., N) \qquad (2.25)$$

where $f''(x_i)$ is the second order derivative of $f(x)$ at x_i ,b_{ij} is the weighting coefficient of the second order derivative. Obviously, Equation (2.25) is a linear operator. Note that Equation (2.25) has the same form as Equation (1.1).

Remark 19 *The only difference between Equation* (2.25) *and Equation* (1.1) *is the use of different weighting coefficients. The following two approaches can be used to determine* b_{ij} *.*

Quan and Chang's approach

In this approach, Quan and Chang (Quan and Chang,1989a,b) used the Lagrange interpolation polynomials as the test functions and then derived

Hence the analytical expressions as following,

$$b_{ij} = \frac{d^2 f_k\left(x_i\right)}{dx} = \frac{d}{dx}\left(\frac{M^{(1)}(x_i)}{(x_i - x_k)\cdot M^{(1)}(x_k)}\right) \tag{2.26a}$$

$$b_{ij} = \frac{2}{x_j - x_i}\left(\prod_{k=1,k\neq i,j}^{N} \frac{x_i - x_k}{x_j - x_k}\right)\left(\sum_{l=1,l\neq i,j}^{N} \frac{1}{x_i - x_l}\right), \quad \text{for} \quad i \neq j \tag{2.26b}$$

$$b_{ii} = 2\sum_{k=1,k\neq i}^{N-1}\left[\frac{1}{x_i - x_k}\left(\sum_{l=k+1,l\neq i}^{N} \frac{1}{x_i - x_l}\right)\right] \tag{2.26c}$$

Shu's general approach

Similar to the first order derivative. Shu's general approach is also based on polynomial approximation and linear vector space analyses. Two sets of base polynomials (Equations (2.10c) and (2.10a)) are used. Substituting Equation (2.13) into Equation (2.25) gives,

$$b_{ij} = \frac{N^{(2)}(x_i, x_j)}{M^{(1)}\left(x_j\right)}, \qquad (i, j = 1, 2, ..., N) \tag{2.27}$$

On the other hand, from Equation (2.20) for $m = 2$. we obtain,

$$N^{(2)}(x_i, x_j) = \frac{M^{(2)}(x_i) - 2N^{(1)}(x_i, x_j)}{(x_i - x_j)}, \quad i \neq j \tag{2.28a}$$

$$N^{(2)}(x_i, x_j) = \frac{M^{(3)}(x_i)}{3}, \qquad i = j \qquad (2.28\text{b})$$

Substituting Equation (2.28) into Equation (2.27) yields,

$$b_{ij} = \frac{M^{(2)}(x_i) - 2N^{(1)}(x_i, x_j)}{(x_i - x_j) \cdot M^{(1)}(x_j)}, \qquad i \neq j \qquad (2.29\text{a})$$

$$b_{ii} = \frac{M^{(3)}(x_i)}{3M^{(1)}(x_i)}, \qquad i = j \qquad (2.29\text{b})$$

Finally, by substituting Equation (2.23) into Equation ($2.29a$), we obtain,

$$b_{ij} = 2a_{ij}\left(a_{ii} - \frac{1}{x_i - x_j}\right), \qquad i \neq j \qquad (2.30)$$

When $i \neq j$. b_{ij} can be easily computed from Equation (2.30). However, from Equation ($2.29b$), it can be seen that the computation of b_{ii} involves the third orderderivative $M^{(3)}(x_i)$ which cannot be computed easily. This difficulty can be eliminated by employing the properties of a linear vector space. Similar to the analysis for the case of the first order derivative, the equation system for b_{ij} derived from the Lagrange interpolation polynomials (a set of base polynomials) is equivalent to that derived from another set of base polynomials x^k , $(k = 0, 1, ..., N-1)$. Thus b_{ij} should also satisfy the following formulation derived from the base polynomial x^k .

when $k = 0$,

$$\sum_{j=1}^{N} b_{ij} = 0 \quad \text{or} \quad b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij} \qquad (2.31)$$

Remark 20 *for the application of Shu's general approach,* b_{ij} *is firstly computed from Equation* (2.30) *when* $i \neq j$. *Subsequently,* b_{ii} *is computed from Equation* (2.31).

3.2.2 Weighting Coefficients of the Higher Order Derivative

Shu's Recurrence Formulation

For the discretization of higher order derivatives, the following two linear operators are applied

$$f^{(m-1)}\left(x_i\right) = \sum_{j=1}^{N} e_{ij}^{(m-1)} \cdot f(x_j) \qquad (2.32)$$

$$\begin{aligned} f^{(m)}\left(x_i\right) &= \sum_{j=1}^{N} e_{ij}^{(m)} \cdot f(x_j) \\ &\text{for} \quad (i = 1, 2, ..., N) \;\; ; \;\; (m = 2, 3, ..., N-1) \end{aligned} \qquad (2.33)$$

where $f^{(m-1)}\left(x_i\right)$, $f^{(m)}\left(x_i\right)$ indicate the $(m-I)$th and mth order derivatives of $f(x)$ with respect to x at x_i . $e_{ij}^{(m-1)}$ $e_{ij}^{(m)}$ are the weighting coefficients related to $f^{(m-1)}\left(x_i\right)$ and $f^{(m)}\left(x_i\right)$. Two sets of base polynomials will also be used to derive explicit formulations for $e_{ij}^{(m)}$. The first set of base polynomials is given by Equation (2.13). Substituting Equation (2.13) into Equations (2.32) and (2.33) gives,

$$e_{ij}^{(m-1)} = \frac{N^{(m-1)}(x_i, x_j)}{M^{(1)}\left(x_j\right)}, \qquad (i, j = 1, 2, ..., N) \qquad (2.34)$$

$$e_{ij}^{(m)} = \frac{N^{(m)}(x_i, x_j)}{M^{(1)}\left(x_j\right)}, \qquad (i, j = 1, 2, ..., N) \qquad (2.35)$$

By rewriting Equation (2.34), we obtain,

$$N^{(m-1)}(x_i, x_j) = e_{ij}^{(m-1)} \cdot M^{(1)}\left(x_j\right) \qquad (2.36)$$

Equation (2.36) is valid for any i and j. On the other hand, from the recurrence formulation (2.20), we have,

$$N^{(m-1)}(x_i, x_i) = \frac{M^{(m)}\left(x_i\right)}{m} \qquad (2.37)$$

$$N^{(m)}(x_i, x_j) = \frac{M^{(m)}(x_i) - mN^{(m-1)}(x_i, x_j)}{(x_i - x_j)}, \quad i \neq j \qquad (2.38)$$

$$N^{(m)}(x_i, x_i) = \frac{M^{(m+1)}(x_i)}{m+1} \qquad (2.39)$$

Substituting Equation (2.37) into Equation (2.38) leads to,

$$N^{(m)}(x_i, x_j) = \frac{m\left[N^{(m-1)}(x_i, x_i) - N^{(m-1)}(x_i, x_j)\right]}{(x_i - x_j)}, \quad i \neq j \qquad (2.40)$$

Equation (2.40) can be further simplified using Equation (2.36)

$$N^{(m)}(x_i, x_j) = \frac{m\left[e_{ii}^{(m-1)} M^{(1)}(x_i) - e_{ij}^{(m-1)} M^{(1)}(x_j)\right]}{(x_i - x_j)}, \quad i \neq j \qquad (2.41)$$

Substituting Equation (2.41) into Equation (2.35), and using Equation (2.23a), a recurrence formulation is obtained as follows

$$e_{ij}^{(m)} = m\left(a_{ij} e_{ii}^{(m-1)} - \frac{e_{ij}^{(m-1)}}{x_i - x_j}\right), \quad (i, j = 1, 2, ..., N) \quad ; \quad (m = 2, 3, ..., N-1) \qquad (2.42)$$

where, a_{ij} is the weighting coefficient of the first order derivative described above. The formulation for $e_{ii}^{(m)}$ can be obtained by substituting Equation (2.39) into Equation (2.35), which gives,

$$e_{ii}^{(m)} = \frac{M^{(m+1)}(x_i)}{(m+1) \cdot M^{(1)}(x_i)}, \quad (i = 1, 2, ..., N) \quad ; \quad (m = 2, 3, ..., N-1) \qquad (2.43)$$

Obviously, Equation (2.42) offers an easy way to compute the weighting coefficients $e_{ij}^{(m)}$ for $j \neq i$. However, it is very difficult to apply Equation (2.43) to compute the weighting coefficients $e_{ii}^{(m)}$. Again, this difficulty can be overcome by the properties of a linear vector space. In terms of the analysis of the N-dimensional linear vector space, the equation system for $e_{ij}^{(m)}$ derived from the Lagrange interpolation polynomials should be equivalent to that derived from the base polynomials x^k, $(k = 0, 1, ..., N-1)$. Thus $e_{ij}^{(m)}$ should satisfy the following equation obtained from the base polynomial x^k .

when $k = 0$,

$$\sum_{j=1}^{N} e_{ij}^{(m)} = 0 \qquad \text{or} \qquad e_{ii}^{(m)} = -\sum_{j=1, j\neq i}^{N} e_{ij}^{(m)}, \quad (i = 1, 2, ..., N\ ); \quad m \geq 1 \tag{2.44}$$

Remark 21 *from this formulation,* $e_{ii}^{(m)}$ *can be determined from* $e_{ij}^{(m)}$ $(i \neq j)$.

3.2.3 Matrix Multiplication Approach

From the definition of the differential operator, we have,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial}\left(\frac{\partial f}{\partial x}\right) \tag{2.45}$$

For simplicity, $\partial^2 f/\partial x^2$ is represented by $f^{(2)}$ while $\partial f/\partial x$ is denoted by $f^{(1)}$. When the DQ approximation is applied to the left side of Equation (2.45), we obtain Previous Equation (2.25)

$$f^{(2)}(x_i) = \sum_{j=1}^{N} b_{ij} \cdot f(x_j) \qquad (i = 1, 2, ..., N)$$

and when the DQ method is applied to the right side of Equation (2.45) twice, we get,

$$\begin{aligned} f^{(2)}(x_i) &= \left(\frac{d^2 f(x)}{dx^2}\right)_{x=x_i} = \left(\frac{d}{dx}\left(\frac{df(x)}{dx}\right)\right)_{x=x_i} = \sum_{k=1}^{N} a_{ik}\left(\frac{df(x)}{dx}\right)_{x=x_k} \qquad (2.46) \\ &= \sum_{k=1}^{N} a_{ik} \cdot f^{(1)}(x_k) = \sum_{k=1}^{N} a_{ik} \sum_{j=1}^{N} a_{kj} \cdot f(xj) = \sum_{j=1}^{N}\left[\sum_{k=1}^{N} a_{ik} \cdot a_{kj}\right] \cdot f(x_j), (i = 1, 2, .., \end{aligned}$$

By comparing Equations (2.25) and (2.46), we obtain,

$$b_{ij} = \sum_{k=1}^{N} a_{ik} \cdot a_{kj} \qquad (i, j = 1, 2, ..., N) \tag{2.47}$$

Now, if we define two matrices by,

$$\left[A^{(1)}\right] = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1N} \\ a_{21} & a_{22} & . & . & . & a_{2N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ a_{N1} & a_{N2} & . & . & . & a_{NN} \end{bmatrix}, \qquad \left[A^{(2)}\right] = \begin{bmatrix} b_{11} & b_{12} & . & . & . & b_{1N} \\ b_{21} & b_{22} & . & . & . & b_{2N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ b_{N1} & b_{N2} & . & . & . & b_{NN} \end{bmatrix}$$

then Equation (2.47) gives,

$$\left[A^{(2)}\right]=\left[A^{(1)}\right]\left[A^{(1)}\right] \qquad (2.48)$$

Equation (2.48) indicates that the weighting coefficients for the second order derivative can be computed by the matrix multiplication of the weighting coefficient matrix for the first order derivative. Similarly, the mth order derivative of $f(x)$ can be written as

$$\frac{\partial^{(m)} f}{\partial x^{(m)}}=\frac{\partial}{\partial x}\left(\frac{\partial^{(m-1)} f}{\partial x^{(m-1)}}\right)=\frac{\partial^{(m-1)}}{\partial x^{(m-1)}}\left(\frac{\partial f}{\partial x}\right) \qquad (2.49)$$

Let $\left[A^{(m-1)}\right]$,$\left[A^{(m)}\right]$ be the weighting coefficient matrices of the $(m-1)$th and mth order derivatives respectively. Then the application of DQ approximation to Equation (2.49) results in the following recurrence relationship

$$\left[A^{(m)}\right]=\left[A^{(1)}\right]\left[A^{(m-1)}\right]=\left[A^{(m-1)}\right]\left[A^{(1)}\right] \quad ,(m=2,3,...,N-1) \qquad (2.50)$$

On the other hand , When we apply the previous rule as an example in the second order derivative .

we have a_{ij} are known, As a result , a relationship between first and second ordre coefficients so b_{ij} can be alternatively computed by Equations (2.46) and (2.47)

$$f^{(2)}\left(x_i\right)=\sum_{j=1}^{N}\left[\sum_{k=1}^{N} a_{ik}\cdot a_{kj}\right].f\left(x_j\right) \ , \qquad (i,j=1,2,...,N)$$

$$b_{ij}=\sum_{k=1}^{N} a_{ik}\cdot a_{kj} \qquad (i,j=1,2,...,N)$$

in matrix notation,

$$B=A^2$$

where,

$$A=\left(a_{ij}\right)_{n\times n} \qquad B=\left(b_{ij}\right)_{n\times n}$$

Remark 22 *from the matrix multiplication approach the , the coefficients b_{ij} can be determined by two alternative procedures (but equivalent),i.e. they can be obtained by directly solving , or squaring the first order matrix A .*

3.3 Some Properties of DQ Weighting Coefficient Matrices

It has been shown before that the DQ approximation for a derivative of any order has a similar form. The difference in the approximation for the respective derivatives lies only in the weighting coefficients. Consider a one-dimensional problem.

$$f^{(n)}(x_i) = \sum_{j=1}^{N} e_{ij}^{(n)} \cdot f(x_j), \quad (i = 1, 2, ..., N), (n = 1, 2, ..., N-1) \text{ or } (n = 2, 3, ..., N) \tag{2.51}$$

When Equation (2.51) is applied at all the grid points, we can obtain an equation system which, when written in the matrix form, becomes

$$\left\{f^{(n)}\right\} = \left[E^{(n)}\right] \cdot \{f\} \tag{2.52}$$

where,

$$\left\{f^{(n)}\right\} = \begin{bmatrix} f^{(n)}(x_1) \\ f^{(n)}(x_2) \\ \cdot \\ \cdot \\ \cdot \\ f^{(n)}(x_N) \end{bmatrix}, \quad \left[E^{(n)}\right] = \begin{bmatrix} e_{1,1}^{(n)} & e_{1,2}^{(n)} & \cdot & \cdot & \cdot & e_{1,N}^{(n)} \\ e_{2,1}^{(n)} & e_{2,2}^{(n)} & \cdot & \cdot & \cdot & e_{2,N}^{(n)} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ e_{N,1}^{(n)} & e_{N,2}^{(n)} & \cdot & \cdot & \cdot & e_{N,N}^{(n)} \end{bmatrix}, \quad \{f\} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \cdot \\ \cdot \\ \cdot \\ f(x_N) \end{bmatrix}$$

In order to apply the DQ method more proficiently and meaningfully, it is necessary to understand the basic properties of the DQ weighting coefficient matrices, $\left[E^{(n)}\right]$, $n = 1, 2, ..., N-1$. For example, if the determinant of $\left[E^{(n)}\right]$ is zero, then its rank should be less than N. We need to know the rank of $\left[E^{(n)}\right]$. If the rank of $\left[E^{(n)}\right]$ is $(N-n)$, then Equation (2.52) can only be applied at $(N-n)$ grid points in order to ensure that the resultant DQ discretization matrix is not singular , some efficient approaches can be used to simplify the solution process for the DQ algebraic equations or the DQ ordinary differential equations. The determinant, rank, so we will discuss the rank of the DQ weighting coefficient matrices from the computation of their determinants.

3.3.1 Definition and Properties of Determinant and Rank

Definition 23 *the determinant is a scalar function of a square matrix. It is conventional to indicate a determinant with the same symbol as the matrix, but with a bar "$|$" on both sides. Thus, $|A|$ is the determinant of the matrix $[A]$, which must be square. The determinant of a N by N matrix $[A]$ can be computed by the following recursive formula,*

$$|A| = \sum_{i=1}^{N} (-1)^{i+j} \cdot a_{i,j} \cdot |A_{i,j}| \qquad (2.53)$$

where $a_{i,j}$ is the element of $[A]$ at the ith row and jth column , $|A_{i,j}|$ is the determinant of the $(N - I)$ by $(N - I)$ matrix , which is formed by removing the ith row and jth column from $[A]$. We have the following four basic properties for matrix $[A]$.

Proposition 24 *If a scalar multiple of one column of a square matrix is added to another column, then the determinant of the resulting matrix is the same as the determinant of the original matrix.*

Proposition 25 *If a column of a matrix consists entirely of zeros, then the determinant of that matrix is zero.*

Proposition 26 *When the scalar multiples of all columns of a square matrix are used to replace a column, the determinant of the resulting matrix is the same as the determinant of the original matrix. Mathematically, it is written as ,*

$$\begin{vmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,N} \\ a_{2,1} & \cdots & a_{2,k} & \cdots & a_{2,N} \\ . & \cdots & . & \cdots & . \\ . & \cdots & . & \cdots & . \\ . & \cdots & . & \cdots & . \\ a_{N,1} & \cdots & a_{N,k} & \cdots & a_{N,N} \end{vmatrix} = \begin{vmatrix} a_{1,1} & \cdots & \sum_{j=1}^{N} \beta_j \cdot a_{1,j} & \cdots & a_{1,N} \\ a_{2,1} & \cdots & \sum_{j=1}^{N} \beta_j \cdot a_{2,j} & \cdots & a_{2,N} \\ . & \cdots & . & \cdots & . \\ . & \cdots & . & \cdots & . \\ . & \cdots & . & \cdots & . \\ a_{N,1} & \cdots & \sum_{j=1}^{N} \beta_j \cdot a_{N,j} & \cdots & a_{N,N} \end{vmatrix} \qquad (2.54)$$

Using Equation (2.54) *and the first proposition, we can arrive at the following proposition, which will be used to analyze the DQ weighting coefficient matrices.*

Proposition 27 *Let* β_j , $(j = 1, 2, ..., N)$ *be scalars, which have at least one component with non-zero value. If the following condition*

$$\sum_{j=1}^{N} \beta_j \cdot a_{i,j} = 0 \qquad \text{(2.55)}$$

is satisfied for all $(i = 1, 2, ..., N)$, then the determinant of $[A]$ is zero.

Remark 28 *The determinant is a very useful parameter for matrix analysis. If the determinant of a* N *by* N *matrix is zero, then the inversion of the matrix does not exist, and its rank is less than* N*.*

Definition 29 *The rank of a matrix is the order of the highest order square submatrix with non-zero determinant. The submatrix is formed from the original matrix by removing certain rows and columns.*

From the above definition, it is known that for a N by N matrix, it has an inverse if and only if its rank is N. Now, we can study the determinant and the rank of the DQ weighting coefficient matrices by using the above properties and definition.

3.3.2 Some Characteristics of the DQM

1. DQM can be extended to solve boundary-value and initial-value differential equations with a linear or nonlinear nature.

2. The classical DQM is polynomial-based, and it is well known that the number of grid points involved is usually restricted to be below 30 .

3. the weighting coefficients of the differential quadrature method depend on the test functions and distribution of grid points but are independent of some specific problems.

4. The key procedure in the DQ application lies in the determination of the weighting coefficients.

5. DQM have higher accuracy, good convergence, reasonable stability and less computation.

3.3.3 Choice of Grid Points distributions

The choice of the grid points plays an important role in the accuracy of the solution of the differential equations, Sherbourne and Pandey are the first ones to report that the DQ solution is very sensitive to grid spacing .

There are four typical grid points' distributions: Legendre grid points, Chebyshev grid points, Chebyshev-Gauss-Lobatto grid points, and Uniform grid points (also called equally spaced grid points),We will mention just two types of commonly used grid points, which is defined as follows:

★ **Uniformly distributed grid points**

Using uniform grids can be considered to be a convenient and easy selection method.it is the convenient and natural choice for the grid points is that of the equally spaced ,due to their obvious convenience, have been in use by most investigators. which is given by,

$$x_k = a + \frac{k-1}{N-1}(b-a) \ , \quad (k = 1, 2, ..., N)$$

★ **Chebyshev grid points**

These are the roots of the Chebyshev polynomial of the first kind of degree N in the interval $[a, b]$. which the most frequently employed in various DQ formulations and we will use it in the application . it is given by,

$$x_k = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\left(\frac{(k-1)}{N-1}\pi\right), \quad (k = 1, 2, ..., N)$$

For convenience, it is rewritten here in the form of,

$$x_k = x_1 + \frac{1}{2}(1 - \cos\frac{k-1}{N-1}\pi)(x_N - x_1), \qquad (k = 1, 2, ..., N)$$

Remark 30 *the Differential Quadrature solutions usually deliver more accurate results with unequally spaced grid points. A rational basis for the grid points is provided by the zeros of the orthogonal polynomials like Legendre and Chebyshev.*

Remark 31 *polynomials usually give more accurate solutions then the uniform grid points (equally spaced grid points).*

3.3.4 Error Analysis

is interesting to analyze the errors resulting from the approximation of the function and its derivatives. Shu (1991) presented a comprehensive error analysis in his doctoral thesis, he would begin with error estimation to approximate the function, followed by an error estimation to approximate the derivative.The error estimated is determined by,

$$Error \leq \frac{1}{N!} \left| \prod_{j=1}^{N} (x - x_j) \right| \max \left| f^{(N)}(x) \right|$$

For more understanding and clarification see [8] page 38 .

Chapter 4

Application of DQM for ordinary differential equations

In this chapter we will apply the differential quadrature method to ordinary differential equations. to understand the method well, we will explain it with the following example.

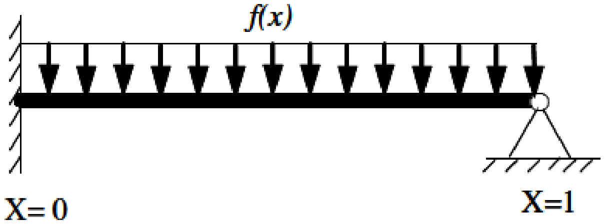

Figure 2 : Euler beam under uniform loading.

Example 32 *Bending of Euler beam*

A uniform Euler beam under pure bending shown in Fig is governed by the following fourth-order Euler-beam equation:

$$EI\frac{d^4\omega}{dx^4} + f(x) = 0 \qquad 0 < x < L \qquad (3.1)$$

where, EI is the flexural rigidity of the beam, $f(x)$ the external distributed load, and L the length of the beam. Equation (3.1) may be further transformed to a dimensionless form for the convenience of calculation. With non-dimensionalisation procedures neglected, we obtain

$$\frac{d^4W}{dX^4} + F(x) = 0 \qquad 0 < X < 1 \qquad (3.2)$$

where, $X = x/L,\ \ W = \omega/a, \quad a = f_0L/EI, \quad F(x) = f(x)/f_0, \quad f_0$ is a constant for non-dimensionalisation.

As shown in Figure. 2, the beam is clamped at the left end and simply supported at the right end. The boundary conditions are then,

$$\begin{aligned} W &= \frac{dW}{dX} = 0 \qquad \text{at} \ \ X = 0 \qquad (3.3a) \\ W &= \frac{d^2W}{dX^2} = 0 \qquad \text{at} \ \ X = 1 \qquad (3.3b) \end{aligned}$$

The exact solution to this problem is

$$W(X) = \frac{1}{48}X^2(5X - 2X^2 - 3) \qquad 0 \le X \le 1 \qquad \text{into } N = 21$$

where $[0, 1]$ divide the beam domain into $N = 21$, nodes distributed of Chebyshev grid points. Writing the deflection $W(x)$ in the form of Eq (2.33)

$$\frac{d^4W}{dX^4} = W^{(4)}(X_i) = \sum_{j=1}^{N} e_{ij}^{(4)} \cdot W(X_j)$$

and substituting it in Eq (3.2)

$$\sum_{j=1}^{N} e_{ij}^{(4)} \cdot W(X_j) = -F(X_i) \qquad \text{for} \ \ (i = 1, 2, ..., N) \qquad (3.4)$$

where, $a_{ij}^{(4)}$ are given in Eq. (2.42).

Inserting Eq. (1.1) and Eq (2.25) into Eq. (3.3) we obtain the boundary conditions in the discrete form of,

$$\begin{aligned} W_1 &= 0, \qquad \sum_{j=1}^{N} e_{1j}^{(1)} \cdot W_j = 0 \qquad (3.5a) \\ W_N &= 0, \qquad \sum_{j=1}^{N} e_{Nj}^{(2)} \cdot W_j = 0 \qquad (3.5b) \end{aligned}$$

Consider a uniform load with the value $f(x) = f_0$. Then $F(X) = 1$. The deflections of the beam at the nodes is then governed by the system of linear equations

$$\sum_{j=3}^{N-2} C_{ij} \cdot W_j = -1, \qquad \text{for } (i = 3, 4, ..., N-2) \qquad (3.6)$$

where,

$$C_{ij} = e_{ij}^{(4)} + \frac{e_{i,2}^{(4)}(a_{1,j} \cdot b_{N,N-1} - a_{1,N-1} \cdot b_{N,j}) + e_{i,N-1}^{(4)}(a_{1,2} \cdot b_{N,j} - a_{1,j} \cdot b_{N,2})}{b_{N,2} \cdot a_{1,N-1} - a_{1,2} \cdot b_{N,N-1}} \qquad (3.7)$$

where ,a_{ij} , the weighting coefficients of Quan and Chang's Approach for the First Order Derivative see Eq. $(2.8b)$ .and b_{ij} , the weighting coefficients of Quan and Chang's Approach for the second Order Derivative see Eq. $(2.26b)$.

In summary, direct DQ method is composed of the following procedures:

(1) The function to be determined is replaced by a group of function values at a group of selected nodes. Chebyshev nodes are strongly recommended for numerical stability.

(2) Approximate derivatives in a differential equation by these N unknown function values.

(3) Form a system of linear equations

(4) Solving the system of linear equation yields the desired unknowns.

It should be noted that these are basic procedures, thus allowing for suitable adaptations for a particular problem.

Solving Eq.(3.6) see [13] using Gauss elimination technique yields the numerical result of the problem.

TABLE : Deflection of a beam under uniformly distributed load

X	$W\ (Exact)$	$W\ (DQ,\ N=21)$	$Error$
0	$0.0000000000E+0$	$0.0000000000E+0$	0.0
0.00616	$-2.34415033E-6$	$-2.34415033E-6$	$-3.45620757E-11$
0.02447	$-3.59174775E-5$	$-3.59174775E-5$	$-3.35252396E-11$
0.0545	$-1.69126555E-4$	$-1.69126555E-4$	$-3.28543093E-11$
0.09549	$-4.82675398E-4$	$-4.82675398E-4$	$-3.32891966E-11$
0.14645	$-1.03241288E-3$	$-1.03241288E-3$	$-3.43403416E-11$
0.20611	$-1.81817534E-3$	$-1.81817534E-3$	$-3.57907255E-11$
0.273	$-2.77015180E-3$	$-2.77015180E-3$	$-3.73696574E-11$
0.34549	$-3.75816531E-3$	$-3.75816531E-3$	$-3.90387931E-11$
0.42178	$-4.62129499E-3$	$-4.62129499E-3$	$-4.10473715E-11$
0.5	$-5.20833314E-3$	$-5.20833314E-3$	$-4.38982201E-11$
0.57822	$-5.41611948E-3$	$-5.41611948E-3$	$-4.73707426E-11$
0.65451	$-5.21393933E-3$	$-5.21393933E-3$	$-5.09543520E-11$
0.727	$-4.64733857E-3$	$-4.64733857E-3$	$-5.42551513E-11$
0.79389	$-3.82187643E-3$	$-3.82187643E-3$	$-5.74855657E-11$
0.85355	$-2.87383765E-3$	$-2.87383765E-3$	$-5.94420824E-11$
0.90451	$-1.93844959E-3$	$-1.93844959E-3$	$-6.07303242E-11$
0.9455	$-1.12560121E-3$	$-1.12560121E-3$	$-5.93151190E-11$
0.97553	$-5.08927378E-4$	$-5.08927378E-4$	$-4.78693501E-11$
0.99384	$-1.28232166E-4$	$-1.28232166E-4$	$1.05687423E-11$
1	$0.00000000E+0$	$0.00000000E+0$	0

•The numerical and exact solutions are listed in Table for comparison. It can be seen from the table that the results obtained from direct DQ method are very close to the exact solutions and up to 11 digits accuracy can be achieved by using only 21 nodes.

•The numerical results obtained from direct DQ method are also compared with the exact solutions in Figure.3 , No difference is observed between the DQ results denoted by the solid line and the exact solutions denoted by open circles. The computational effort is small due to the small number of nodes used .

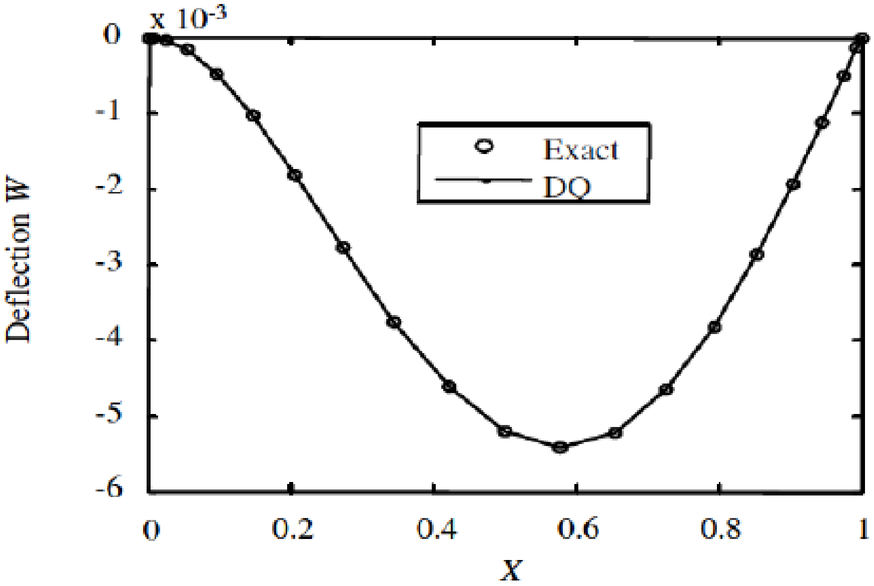

Figure 3: Deflection of the beam under bending.

Remark 33 *In the graph, the solution seemed equal to us,and the approximate solution DQ is applicable to the exact solution.*

Chapter 5

Conclusion

The applicability of the direct DQ method in its original form is limited. It has been known to fail for problems with strong nonlinearity and material discontinuity as well as for problems involving singularity, irregularity and multiple scales. Researchers working in applied mathematics, computational mechanics and engineering have developed a variety of DQ-based methods to overcome these shortcomings. Although these methods have different for mulations and may even look completely different from one another, all DQ methods share a common objective. they are tools for performing numerical differentiation.

The results in this thesis represent the latest important developments of DQ methods in recent years. In addition to gaining an insight into the dynamic changes in this field, the reader will quickly master the use of DQ methods to solve complex problems.

There is no necessity for the reader to be familiar with the physical problems used in the example in this thesis. The only prerequisite is an understanding of the fundamentals of calculus, ordinary and partial differential equations and numerical methods.

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Abstract :

In this memory investigate the Differential Quadrature Method, to approximate the solution of Differential Ordinary Equations. Numerical examples are presented in Hammerstein integral equations.

Keywords :

Differential Quadrature Method, ODE, error of approximation, Norms, Banach space.

Résumé :

Dans ce mémoire, on a étudié les méthodes des Quadratures Différentielles pour approcher les solutions d’une équation Différentielle Ordinaires. Dans le but d’une comparaison, des exemples numériques sont donnés.

Mots clés :

Quadratures différentielles, EDO, erreur d’approximation, Normes, espace de Banach.

الخلاصة:

في هذه المذكرة قمنا بدراسة خاصيتي محدودية وتراص مؤثرات التكامل الغير خطية في الفضاءات الدالية. ثم قدمنا أمثلة عددية في المعادلات التكاملية من نوع هامرشتاين.

الكلمات المفتاحية:

المحدودية، التراص، المؤثرات، الفضاءات الدالية، معادلات هامرشتاين.