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Theme

Anisotropic Herz and Herz-type Hardy spaces and applications

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Contents

Notations

$\mathbb{R}^n$	The n-dimensional real Euclidean space.
$\mathbb{N}$	The collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
$\mathbb{Z}$	The set of all integer numbers.
x^α	Means that $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$
$f \lesssim g$	Means that $f \leq c\, g$ for some independent constant c (and non-negative functions f and g).
$f \approx g$	Means $f \lesssim g \lesssim f$.
$X \hookrightarrow Y$	The continuous embeddings from X to Y.
$[x]$	The largest integer smaller than or equal to $x \in \mathbb{R}$.
$B(x,r)$	The open ball in $\mathbb{R}^n$ with center x and radius r.
supp$f$	The support of the function f.
$\|E\|$	The (Lebesgue) measure of $E \subset \mathbb{R}^n$.
$\mathcal{S}(\mathbb{R}^n)$	The Schwartz space.
$\mathcal{S}'(\mathbb{R}^n)$	The dual of Schwartz space.
$L^p(\Omega)$	The classical *Lebesgue space*
$p'(\cdot)$	The conjugate exponent of $p(\cdot)$ define by the formula $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$
ℓ^q	The space of all (complex) sequences $\{a_k\}_{k\in\mathbb{Z}}$ equipped with the quasi-norm $\left\\|\{a_k\}_{k\in\mathbb{Z}} \mid \ell^q\right\\| = \left(\sum_{k=-\infty}^{\infty} \|a_k\|^q\right)^{1/q}$ (with the usual modification if $q = \infty$).
$L^1_{\text{loc}}(\mathbb{R}^n)$	The collection of all locally integrable functions on $\mathbb{R}^n$
$\mathcal{M}(f)$	The Hardy-Littlewood maximal operator defined by $\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{\|B(x,r)\|} \int_{B(x,r)} \|f(y)\|\, dy, \forall x \in \mathbb{R}^n$

Introduction

Function spaces with variable exponents has developed since the paper (O. Kovacik and J. Rakosnik, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J.) in 1991. After this, these type of function spaces has been intensively studied in the recent years by a significant number of authors.

In this memory, we study anisotropic Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ and anisotropic Herz-type Hardy spaces $H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ with variable exponent (one parameter is variable) which is a generalization of classical anisotropic Herz and anisotropic Herz-type Hardy spaces. These spaces go back to the authors H. Wang H. Zhao and J. Zhou in the last decade.

In 2015, H. Wang [13] first introduced anisotropic Herz spaces, later in 2018, H. Zhao and J. Zhou [16] presented the characterizations of anisotropic Herz-type Hardy spaces with variable exponent associated with a non-isotropic dilation on $\mathbb{R}^n$ and also established the atomic decomposition of these function spaces.

Our memory consists of three chapters. In the first chapter, we give some basic properties of variable Lebesgue spaces after this we define anisotropic Herz and Herz-type Hardy where one parameter is variable.

In the second chapter, we present some results given in [13, 15] concerning the atomic decomposition of anisotropic Herz and anisotropic Herz-type Hardy

spaces.

In chapter 3, using the decomposition theorems, we present the boundedness of some subilinear operators on these spaces given by

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{\varrho(x-y)}\mathrm{d}y, \quad x \notin \operatorname{supp} f$$

for integrable and compactly supported functions f.

Chapter 1

Anisotropic Herz and Herz-type Hardy spaces with variable exponent

In this chapter, we present the concepts of some spaces used in this memory. Firstly, we mention the variable Lebesque space $L^{p(\cdot)}$ where we present some basic propreties of these spaces. Secondly, we give the definitions of homogeneous and nonhomogeneous anisotropic Herz and Herz-type Hardy spaces with one variable exponent.

1.1 Variable Lebesgue spaces

In this section, we define Lebesque spaces with variable exponent. We begin with the semi-modular space or modular space and the norm.

1.1.1 Basic properties of semi-modular (modular) spaces

Definition 1.1. Let E be a real vector space.
The function ϱ is said to be left-continuous if the mapping $\lambda \to \varrho(\lambda x)$ is left-continuous on $[0, \infty)$ for every $x \in E$, i.e.

$$\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x).$$

($a \to b^-$ means that a tends to b from below, i.e. $a < b$ and $a \to b^-$; $a \to b^+$ is defined analogously).

Definition 1.2. Let E be a real vector space.
A function $\varrho : E \to [0, +\infty]$ is called a semi-modular on E if it satisfies the following conditions:

1- $\varrho(0) = 0$.

2- $\varrho(\lambda x) = \varrho(x)$ for all $x \in E$, and for all scalar λ with $|\lambda| = 1$.

3- ϱ is convex.

4-.ϱ is left-continuous on $[0, \infty)$ for every $x \in E$.

5- $\varrho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$.

A semi-modular ϱ is called continuous if the mapping $\lambda \to \varrho(\lambda x)$ is continuous on $[0, \infty)$ for every $x \in E$.

Example 1.1. Let $1 < p < \infty$, then

$$\varrho_p(f) = \int_\Omega |f(x)|^p \, dx$$

defines a modular on the space of all measurable function on $\Omega \subset \mathbb{R}^n$.

Proposition 1.1 ([5].). *Let E a vector space on $\mathbb{R}$ and ϱ a semi-modular on E.*

1) $\lambda \to \varrho(\lambda x)$ an increasing function on $[0, \infty)$ for every $x \in E$.

2) For every $|\lambda| \leq 1$, we have a $\varrho(\lambda x) = \varrho(|\lambda| x) \leq |\lambda| \varrho(x)$.

3) For every $|\lambda| \geq 1$, we have a $\varrho(\lambda x) = \varrho(|\lambda| x) \geq |\lambda| \varrho(x)$.

Proof. 1) Since ϱ convex ,$\varrho \geq 0$ and $\varrho(0) = 0$, we have for every $0 \leq \lambda < v$

$$\begin{aligned}\varrho(\lambda x) &= \varrho\left(\frac{\lambda}{v}vx\right)\\ &= \varrho\left(\frac{\lambda}{v}vx + \left(1-\frac{\lambda}{v}\right)0\right)\\ &\leq \frac{\lambda}{v}\varrho(vx)\\ &\leq \varrho(vx)\end{aligned}$$

2) By definition 1.2, we have

$$\begin{aligned}\varrho(\lambda x) &= \varrho\left(\frac{\lambda}{|\lambda|}|\lambda|x\right)\\ &= \varrho(|\lambda|x)\end{aligned}$$

If $|\lambda| \leq 1$:

$$\begin{aligned}\varrho(\lambda x) &= \varrho(|\lambda|x)\\ &= \varrho\left(\frac{\lambda}{1}x + \left(1-\frac{\lambda}{1}\right)0\right)\\ &\leq |\lambda|\varrho(x).\end{aligned}$$

For 3) , if $|\lambda| \geq 1$

$$\begin{aligned}\varrho(\lambda x) &= \varrho(|\lambda|x)\\ &= \varrho\left(\frac{\lambda}{|\lambda|}|\lambda|x + \left(1-\frac{\lambda}{|\lambda|}\right)0\right)\\ &\leq \frac{1}{|\lambda|}\varrho(|\lambda|x)\end{aligned}$$

so $\varrho(|\lambda|x) \geq |\lambda|\varrho(\lambda x)$. □

Definition 1.3. Let ϱ is a semi-modular or modular on E, then

$$X_\varrho := \{x \in E,\ \exists\lambda > 0 : \varrho(\lambda x) < \infty\}$$

is called a semi-modular space or modular space.

So X_ϱ a normed vector space on $\mathbb{R}$, include a norm

$$\|x\|_\varrho := \inf\left\{\lambda > 0 : \varrho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

Lemma 1.1 ([5]). ***(Norm-modular unit ball property)*** *Let ϱ be a semi-modular on E.then*

$$\|x\|_{\varrho} \leq 1 \Longleftrightarrow \varrho(x) \leq 1.$$

If ϱ is continuous, then also

$$\|x\|_{\varrho} < 1 \Longleftrightarrow \varrho(x) < 1,$$

and

$$\|x\|_{\varrho} = 1 \Longleftrightarrow \varrho(x) = 1.$$

1.1.2 Variable exponents

In this subsection, we begin with the basic properties and notation of variable exponent. Given an open set $\Omega \subset \mathbb{R}^n$. We put

$$\mathcal{P}_0(\Omega) := \{p \text{ mesurable: } p(\cdot) : \Omega \to [c, \infty[\text{ for some } c > 0\}.$$

The elements of $\mathcal{P}_0(\Omega)$ are called exponent functions or simply exponents.

Notation 1.1. We denote by

$$\mathcal{P}(\Omega) := \{p \text{ mesurable: } p(\cdot) : \Omega \subset \mathbb{R}^n \to [1, \infty[\}.$$

Given $p \in \mathcal{P}_0(\Omega)$ and a set $E \subseteq \Omega$, let

$$p^-(E) = \operatorname*{ess\,inf}_{x \in E} p(x), \quad p^+(E) = \operatorname*{ess\,sup}_{x \in E} p(x).$$

If the domain $E = \Omega = \mathbb{R}^n$ we will simply write

$$p^- = p^-(\mathbb{R}^n), \quad p^+ = p^+(\mathbb{R}^n).$$

Definition 1.4. Let open set $\Omega \subset \mathbb{R}^n$ and $p \in \mathcal{P}_0(\Omega)$. The variable Lebesgue space $L^{p(\cdot)}(\Omega)$ to be the set of all measurable functions f such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$.

$$L^{p(\cdot)}(\Omega) := \left\{ f \text{ measurable} : \exists \lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) = \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \leq 1 \right\},$$

equipped with the following quasi-norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\right\}.$$

If the set on the right-hand side is empty we define $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$. If $\Omega = \mathbb{R}^n$, we will often write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Remark 1.1. If $p(\cdot) = p$ is constant, then the Definition 1.4 is equivalent to the classical norm on $L^p(\Omega)$. If $p < +\infty$ and

$$\int_\Omega \left|\frac{f(x)}{\lambda}\right|^p = 1$$

then

$$\lambda = \|f\|_{L^p(\Omega)}$$

the same is true if $p = \infty$.

Theorem 1.1 ([3]). *Let open set $\Omega \subset \mathbb{R}^n$ and $p \in \mathcal{P}_0(\Omega)$. The function $\|f\|_{L^{p(\cdot)}(\Omega)}$ defines a quasi-norm on $L^{p(\cdot)}(\Omega)$ and $L^{p(\cdot)}(\Omega)$ is quasi Banach spaces.*

Proof. First, we prove that $L^{p(\cdot)}(\Omega)$ is vector spaces.
Since $\varrho_{p(\cdot)}(0) = 0$, then we have $0 \in L^{p(\cdot)}(\Omega)$.
Let $f \in L^{p(\cdot)}(\Omega)$ and $\alpha \in \mathbb{R}^*$. There exists $\lambda > 0$ such that

$$\varrho_{p(\cdot)}(\lambda f) < \infty.$$

We put $\lambda_0 = \frac{\lambda}{|\alpha|}$.

$$\begin{aligned}\varrho_{p(\cdot)}(\lambda_0 \alpha f) &= \varrho_{p(\cdot)}(\lambda_0 |\alpha| f)\\ &= \varrho_{p(\cdot)}(\lambda f) < \infty,\end{aligned}$$

which shows that $\alpha f \in L^{p(\cdot)}(\Omega)$.
It suffices to show that if $f, g \in L^{p(\cdot)}(\Omega)$, then

$$f + g \in L^{p(\cdot)}(\Omega).$$

By the convexity of $\varrho_{p(\cdot)}$,

$$\begin{aligned}\varrho_{p(\cdot)}(\lambda(f+g)) &= \varrho_{p(\cdot)}\left((\frac{1}{2}2\lambda f + (1-\frac{1}{2})2\lambda g)\right)\\ &\leq \frac{1}{2}\varrho_{p(\cdot)}(2\lambda f) + \frac{1}{2}\varrho_{p(\cdot)}(2\lambda g) \to 0 \text{ if } \lambda \to 0.\end{aligned}$$

Now, we show that $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ is a quasi norm, i.e. we will prove that $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ has the following properties:
(1) $\|f\|_{L^{p(\cdot)}(\Omega)} = 0$ if and only if $f \equiv 0$;
(2) for all $\alpha \in \mathbb{R}$, $\|\alpha f\|_{L^{p(\cdot)}(\Omega)} = |\alpha| \, \|f\|_{L^{p(\cdot)}(\Omega)}$
(3) $\|f + g\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)}$.

Let $f \in L^{p(\cdot)}(\Omega)$. There exists $\beta > 0$ such that $\varrho_{p(\cdot)}(\beta f) < 1$. This shows that $\|f\|_{L^{p(\cdot)}(\Omega)} < \infty$ also have $\|0\|_{L^{p(\cdot)}(\Omega)} = 0$.

For $\alpha \in \mathbb{R}$, we have:

$$\begin{aligned}
\|\alpha f\|_{L^{p(\cdot)}(\Omega)} &= \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{\alpha f}{\lambda}\right) \leq 1\right\} \\
&= \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{|\alpha| f}{\lambda}\right) \leq 1\right\} \\
&= \inf\left\{\tau |\alpha| > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\tau}\right) \leq 1\right\} \\
&= |\alpha| \inf\left\{\tau > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\tau}\right) \leq 1\right\} \\
&= |\alpha| \, \|f\|_{L^{p(\cdot)}(\Omega)} .
\end{aligned}$$

We now show the triangular inequality. Let $f, g \in L^{p(\cdot)}(\Omega)$ and $\|f\|_{L^{p(\cdot)}(\Omega)} < \gamma$ and $\|g\|_{L^{p(\cdot)}(\Omega)} < \delta$. Then

$$\left\|\frac{f}{\gamma}\right\|_{L^{p(\cdot)}(\Omega)} \leq 1 \text{ and } \left\|\frac{g}{\delta}\right\|_{L^{p(\cdot)}(\Omega)} \leq 1$$

By the convexity of $\varrho_{p(\cdot)}$, we have:

$$\begin{aligned}
\varrho_{p(\cdot)}\left(\frac{f+g}{\gamma+\delta}\right) &= \varrho_{p(\cdot)}\left(\frac{\gamma}{\gamma+\delta}\frac{f}{\gamma} + \frac{\delta}{\gamma+\delta}\frac{g}{\delta}\right) \\
&\leq \frac{\gamma}{\gamma+\delta}\varrho_{p(\cdot)}\left(\frac{f}{\gamma}\right) + \frac{\delta}{\gamma+\delta}\varrho_{p(\cdot)}\left(\frac{g}{\delta}\right) \\
&\leq \frac{\gamma}{\gamma+\delta} + \frac{\delta}{\gamma+\delta} = 1.
\end{aligned}$$

Therefore,

$$\|f+g\|_{L^{p(\cdot)}(\Omega)} \leq \gamma + \delta.$$

Which implies

$$\|f+g\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)} .$$

If $\|f\|_{L^{p(\cdot)}(\Omega)} = 0$, then $\varrho_{p(\cdot)}(\alpha f) \leq 1$ for all $\alpha > 0$. By the convexity of $\varrho_{p(\cdot)}$,

$$\begin{aligned}
\varrho_{p(\cdot)}(\lambda f) &= \varrho_{p(\cdot)}\left(\tau \frac{\lambda f}{\tau}\right) \\
&= \varrho_{p(\cdot)}\left(\tau \frac{\lambda f}{\tau} + (1-\tau)0\right) \\
&\leq \tau \varrho_{p(\cdot)}\left(\frac{\lambda f}{\tau}\right) \\
&\leq 1,
\end{aligned}$$

for all $\lambda > 0$ and for all $\tau \in (0; 1]$. Then $x = 0$. □

Corollary 1.1 ([3], [5]). *Let open set $\Omega \subset \mathbb{R}^n$ and $p \in \mathcal{P}_0(\Omega)$.*

(1) *If $\|f\|_{p(\cdot)} \leq 1$, so $\varrho(f) \leq \|f\|_{p(\cdot)}$.*

(2) *If $1 < \|f\|_{p(\cdot)}$, so $\|f\|_{p(\cdot)} \leq \varrho(f)$.*

(3) *We have $\|f\|_{p(\cdot)} \leq \varrho(f) + 1$.*

Proof. **(1)** If $\|f\|_{p(\cdot)} = 0$, then $f \equiv 0$ and so $\varrho(f) = 0$.
If $0 < \|f\|_{p(\cdot)} \leq 1$. Since

$$\left\| \frac{f}{\|f\|_{p(\cdot)}} \right\|_{p(\cdot)} = 1$$

by Lemma 1.1, we have

$$\varrho\left(\frac{f}{\|f\|_{\varrho}}\right) = 1.$$

If $\|f\|_{p(\cdot)} \leq 1$, then by the convexity of ϱ, we have

$$\begin{aligned} \varrho(f) &= \varrho(\|f\|_{p(\cdot)} \frac{f}{\|f\|_{p(\cdot)}}) \\ &\leq \|f\|_{p(\cdot)} \varrho(\frac{f}{\|f\|_{p(\cdot)}}) \leq \|f\|_{p(\cdot)} . \end{aligned}$$

(2) We assume that $\|f\|_{p(\cdot)} > 1$ then $\varrho(f) > 1$. So

$$\varrho\left(\frac{f}{\lambda}\right) > 1 \text{ for evry } \|f\|_{p(\cdot)} > \lambda > 1.$$

we have $\varrho(f) > \lambda$ because λ was arbitrary, we deduce that

$$\varrho(f) \geq \|f\|_{p(\cdot)} .$$

(3) This is a consequence of the **(2).** □

1.1.3 Generalization Hölder's inequality

The following theorem is the generalization of the classical Hölder's inequality in variable Lebesgue spaces. The classical Hölder's inequality is that for all p, $1 \leq p \leq \infty$, given $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$

$$\int_\Omega |f(x)g(x)|\, dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)} .$$

The following theorem is the generalization of the classical Hölder's inequality in variable Lebesgue spaces. The classical Hölder's inequality is that for all p, $1 \leq p \leq \infty$, given $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$

$$\int_\Omega |f(x)g(x)|\, dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)} .$$

This inequality is true for variable exponents with a constant on the right-hand side, see for example [4, Theorem 2.33].

Theorem 1.2 ([4]). *Let* Ω *and* $p \in \mathcal{P}(\Omega)$. *Then there exists a constant* K *such that for all* $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{p'(\cdot)}(\Omega)$, $fg \in L^1(\Omega)$ *and*

$$\|fg\|_{L^1(\Omega)} \leq K \|f\|_{L^{p(\cdot)}(\Omega)} \|f\|_{L^{p'(\cdot)}(\Omega)} .$$

where

$$K = (1/p^- + 1 - 1/p^+).$$

1.1.4 Logarithmic Hölder continuity

Definition 1.5. We say that a function $p : \Omega \to \mathbb{R}$ is *locally* log-*Hölder continuous* on Ω, if there exists $c_{\log}(p) > 0$ such that

$$|p(x) - p(y)| \leq \frac{c_{\log}}{\ln(e + 1/|x-y|)}$$

for all $x, y \in \Omega$. If $0 \in \Omega$ and

$$|p(x) - p(0)| \leq \frac{c_{\log}}{\ln(e + 1/|x|)}$$

for all $x \in \Omega$, then we say that p is log-*Hölder continuous at the origin* (or has a log *decay at the origin*). If for some $p_\infty \in \mathbb{R}$ and $c_{\log} > 0$, there holds

$$|p(x) - p_\infty| \leq \frac{c_{\log}}{\ln(e + |x|)}$$

for all $x \in \Omega$, then we say that p is log-*Hölder continuous at infinity* (or has a log *decay at infinity*).

1.2 Anisotropic Herz space with variable exponent

In the following, we introduce some basic notations, definitions of anisotropic spaces associated with general expansive dilations, the following definitions and notaions are from [2] and [13].

Definition 1.6. A dilation is $n \times n$ real matrix A, such that all eigenvalues λ of A satisfy $|\lambda| > 1$.

A set $\triangle \subset \mathbb{R}^n$ is said to be an ellipsoid if

$$\triangle = \{x \in \mathbb{R}^n : |Bx| < 1\}$$

for some nondegenerate $n \times n$ matrix B, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.
In [2, Lemma 2.2], we have for a dilation A, then there exists an ellipsoid $\triangle$ and $r > 1$ such that

$$\triangle \subset r\triangle \subset A\triangle,$$

where $|\triangle|$, the Lebesgue measure of $\triangle$, equals 1.
For convenience, we set

$$B_k = A^k\triangle \quad \text{for } k \in \mathbb{Z},$$

then we have

$$B_k \subset rB_k \subset B_{k+1},$$

and

$$|B_k| = b^k,$$

where $b = |\det A| > 1$.

Definition 1.7. A homogeneous quasi-norm associated with an expansive matrix A is a measurable mapping $\sigma_A : \mathbb{R}^n \to [0, \infty[$, so that
- $\sigma_A(x) > 0$ for $x \neq 0$,
- $\sigma_A(Ax) = b\sigma_A(x)$ for $x \in \mathbb{R}^n$,
- there is $c > 0$ so that $\sigma_A(x+y) \leq c(\sigma_A(x) + \sigma_A(y))$ for $x, y \in \mathbb{R}^n$.

For a fixed dilation A we define the "canonical" quasi-norm σ.

Definition 1.8. Define the step homogeneous quasi-norm on $\mathbb{R}^n$ induced by dilation A as

$$\sigma(x) = \begin{cases} b^j & \text{if } \quad x \in B_{j+1} \backslash B_j \\ 0 & \text{if } \qquad x = 0. \end{cases}$$

For any $x, y \in \mathbb{R}^n$, we have

$$\sigma(x+y) \leq b^\theta (\sigma(x) + \sigma(y)),$$

where θ be the smallest integer so that

$$2B_0 \subset A^\theta B_0 = B_\theta.$$

Also, we use the following notation

$$R_k := B_k \setminus B_{k-1} \text{ and } \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

Definition 1.9. Let $\alpha \in \mathbb{R}, 0 < q \leq \infty$ and $p \in \mathcal{P}_0(\mathbb{R}^n)$. The homogeneous anisotropic Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ associated with the dilation A is defined by

$$\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)} < \infty, \right\}$$

where

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}} := \left\{ \sum_{k=-\infty}^{\infty} \left\| b^{k\alpha} f \chi_k \right\|^q_{p(\cdot)} \right\}^{\frac{1}{q}}.$$

The non-homogeneous anisotropic Herz space $K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ associated with the dilation A is defined by

$$K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n) : \|f\|_{K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)} < \infty, \right\}$$

such that

$$\|f\|_{K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)} := \|f \chi_{B_0}\|_{p(\cdot)} + \left\{ \sum_{k=0}^{\infty} \left\| b^{k\alpha} f \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}^{\frac{1}{q}}.$$

Recall that these function spaces coincid with variable Herz spaces when $A = 2Id$, we refer the reader to the papers [6] and [9] for further details,historical remarks and more references on variable Herz spaces.

1.3 Anisotropic Herz-type Hardy spaces with variable exponent

In this section, we will give the definition of anisotropic homogeneous and nonhomogeneous Herz-type Hardy space with variable exponent.
The nontangential maximal function of f with respect to is defined as

$$M_\varphi(f)(x) := \sup \left\{ b^{-k} |f * \varphi\left(A^{-k}\cdot\right)(y)|, x - y \in B_k, k \in \mathbb{Z} \right\}.$$

The radial maximal function of f with respect to φ is defined as

$$M^0_\varphi(f)(x) := \sup_{k \in \mathbb{Z}} b^{-k} |f * \varphi\left(A^{-k}\cdot\right)(x).$$

Let $\mathcal{M}_{\varphi}(f)$ be the grand maximal function of f defined by

$$\mathcal{M}_N(f)(x) := \sup_{\varphi\in\mathcal{A}_N} M_\varphi(f)(x),$$

the radial grand maximal function of f is

$$\mathcal{M}_N^0(f)(x) := \sup_{\varphi\in\mathcal{A}_N} M_\varphi^0(f)(x),$$

where

$$\mathcal{A}_N = \left\{\varphi\in\mathcal{S} : \|\varphi\|_{\alpha,N} \leq 1 \text{ for } |\alpha|\leq N, m\leq N \text{ and } N \text{ is integer} \geq 0\right\}.$$

The following lemma is from [8, Lemma 1].

Lemma 1.2. *Let $p\in\mathcal{P}^{\log}$. For any cubes (balls) P and Q, such that $P\subset Q$, we have*

$$C\left(\frac{|Q|}{|p|}\right)^{\frac{1}{p^+}} \leq \frac{\|\chi_Q\|_{p(\cdot)}}{\|\chi_P\|_{p(\cdot)}} \leq c\left(\frac{|Q|}{|p|}\right)^{\frac{1}{p^-}}$$

with $c, C>0$ are independent of $|Q|$ and $|P|$.

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 1.3 ([9]). *Let $0<\upsilon<1$ and $0<q\leq\infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers, such that*

$$\left\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} = I < \infty.$$

Then the sequences $\left\{\delta_k : \delta_k = \sum_{j\leq k}\upsilon^{k-j}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ and $\left\{\eta_k : \eta_k = \sum_{j\geq k}\upsilon^{j-k}\varepsilon_j\right\}_{k\in\mathbb{Z}}$ belong to ℓ^q, and

$$\left\|\{\delta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} + \left\|\{\eta_k\}_{k\in\mathbb{Z}}\right\|_{\ell^q} \leq c\, I,$$

with $c>0$ only depending on a and q.

The anisotropic homogeneous and nonhomogeneous Herz-type Hardy spaces are defined in the following way.

Definition 1.10. Let $\alpha\in\mathbb{R}, 0<q\leq\infty$ and $p\in\mathcal{P}_0(\mathbb{R}^n)$.
(i) The *homogeneous anisotropic Herz-type Hardy space* $H\dot{K}_{p(\cdot)}^{\alpha,q}(A;\mathbb{R}^n)$ is defined as the set of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{M}_\varphi(f)\in\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and we put

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha,q}(A;\mathbb{R}^n)} = \|\mathcal{M}_N(f)\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(A;\mathbb{R}^n)}.$$

(ii) The *nonhomogeneous anisotropic Herz-type Hardy space* $HK_{p(\cdot)}^{\alpha,q}(A;\mathbb{R}^n)$ is defined as the set of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{M}_\varphi(f)\in K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$ and we put

$$\|f\|_{HK_{p(\cdot)}^{\alpha,q}(A;\mathbb{R}^n)} = \|\mathcal{M}_N(f)\|_{K_{p(\cdot)}^{\alpha,q}(A;\mathbb{R}^n)}.$$

Recall that, if p, q, and α satisfy the conditions of definition, then the qausi-norms $\|f\|_{H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}$ and $\|f\|_{HK^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}$ does not depend, up to the equivalence of quasi-norms, on the choice of the function φ and, hence, the spaces $H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ and $HK^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ are defined independently of the choice φ.

The anisotropic homogeneous and nonhomogeneous Herz-type Hardy spaces with variable exponent p but fixed α, q were recently studied by H. Zhao and J. Zhou [16]. In [16], we have for $p \in \mathcal{P}^{\log}$ with $1 < p^- \leq p^+ < \infty$, $0 < q < \infty$ and

$$0 < \alpha < n\big(1 - \frac{1}{p^+}\big),$$

then

$$H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n) \cap L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) = \dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n),$$

and

$$HK^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n) \cap L^{p(\cdot)}_{loc}(\mathbb{R}^n) = K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n).$$

Also, if $p \in \mathcal{P}^{\log}$ with $1 < p^- \leq p^+ < \infty$ and

$$n\big(1 - \frac{1}{p^+}\big) < \alpha < \infty,$$

then

$$H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n) \cap L^{p(\cdot)}_{loc}(\mathbb{R}^n\backslash\{0\}) \subsetneqq \dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n),$$

and

$$HK^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n) \cap L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) \subsetneqq K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n).$$

If the dilation $A = 2Id$, then these function space coincides with variable Herz-type Hardy space, we refer the reader to the papers [7], [10], [11] and [14] for further results for these variable function spaces.

CHAPTER 2

DECOMPOSITION THEOREMS OF $\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ AND $H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$

In this chaper, we introduce the bloc decoposition of anisotropic Herz spaces in the first section, and we establish some theory for these space. We present in the secend section of this chapter the atomic decomposition theorems for Herz-type Hardy spaces, to apply it in the application two of third chapter.

2.1 Atom

We begin this section by the definition of block and atoms.

Definition 2.1 ([13],[16]). Let $\alpha \in \mathbb{R}$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.

(i) A mesurable function $a(x)$ is said to be a central $(\alpha, p(\cdot))$-block if it satisfies

(1) $\mathrm{supp} a \subset \mathcal{B}_k$, for some $k \in \mathbb{Z}$,

(2) $\|a\|_{L^{p(\cdot)}} \leq b^{-k\alpha}$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(\alpha, p(\cdot))$-block of restricted type, if it satisfies the conditions (2) above and $\mathrm{supp} a \subset \mathcal{B}_k, \quad$ with $\quad k \geq 0.$

(iii) For $1 - \frac{1}{p^+} \leq \alpha < \infty$, $p \in \mathcal{P}(\mathbb{R}^n)$ and non-negative integer $s \geq \left[\left(\alpha - 1 + \frac{1}{p^+}\right) \ln b / \ln \lambda_-\right]$.

A function a is said to be a central $(\alpha, p(\cdot))$-atom if it satisfies the conditions (1) and (2) above and

(3) $\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad |\beta| \leq s.$

(iv) A function a on $\mathbb{R}^n$ is called a centrale $(\alpha, p(\cdot))$-atom of restricted type, if it satisfies the conditions (2),(3) above and

$$\mathrm{supp} a \subset \mathcal{B}_k \quad \text{with } k \geq 0.$$

2.2 Block decomposition of $\dot{K}^{\alpha,q}_{p(\cdot)}(A; \mathbb{R}^n)$

In this section, we present the decomposition theorems of the homogeneous and non-homogeneous anisotropic variable Herz spaces.

The following two theorems are from [13] when $\alpha(\cdot) = \alpha$ is constant.

Theorem 2.1. *Let $\alpha \in \mathbb{R}, 0 < q \leq \infty$ and $p \in \mathcal{P}_0(\mathbb{R}^n)$, the following two statements are equivalentes*

(i)- $f \in \dot{K}^{\alpha,q}_{p(\cdot)}(A; \mathbb{R}^n)$

(ii)- f can be represented by

$$f(x) = \sum_{k=-\infty}^{+\infty} \lambda_k a_k(x), \tag{2.1}$$

where the series converges in the sense of distributions, $\lambda_k \geq 0$, each b_k is a central $(\alpha, p(\cdot))$-block with support contained in B_k and

$$\left(\sum_{k=-\infty}^{+\infty} |\lambda_k|^q\right)^{\frac{1}{q}} \leq c \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}.$$

Moreover, the norms $\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}$ *and* $\inf\left(\sum_{k=-\infty}^{+\infty}|\lambda_k|^q\right)^{\frac{1}{q}}$ *are equivalent, where the infimum is taken all over all decompositions of* f *as in (2.1).*

Remark 2.1. Since we use Lemma 1.3 then we eliminate the discussions ($0<p\leq 1$ and $1<p<\infty$) given in [13, Proof of Theorem 2.3].

Proof. We first prove (i) implies (ii). For every $f\in\dot{K}^{\alpha,q}_{p(\cdot)}\left(A;\mathbb{R}^n\right)$, write

$$\begin{aligned} f\left(x\right) &= \sum_{k=-\infty}^{+\infty} f\left(x\right)\chi_k\left(x\right) \\ &= \sum_{k=-\infty}^{+\infty} b^{k\alpha}\left\|f\chi_k\right\|_{p(\cdot)}\frac{f\left(x\right)\chi_k\left(x\right)}{b^{k\alpha}\left\|f\chi_k\right\|_{p(\cdot)}} = \sum_{k=-\infty}^{+\infty}\lambda_k a_k\left(x\right), \end{aligned}$$

where $\lambda_k=b^{k\alpha}\left\|f\chi_k\right\|_{p(\cdot)}$ and $a_k\left(x\right)=\frac{f(x)\chi_k(x)}{b^{k\alpha}\|f\chi_k\|_{p(\cdot)}}$. It is obvious that $\mathrm{supp}a_k\subset B_k$ and

$$\left\|a_k\right\|_{p(\cdot)}=b^{-k\alpha}.$$

Thus, each a_k is a central $(\alpha,p(\cdot))$- atoms with the support B_k and

$$\begin{aligned} \left(\sum_{k=-\infty}^{+\infty}|\lambda_k|^q\right)^{\frac{1}{q}} &= \left(\sum_{k=-\infty}^{+\infty}\left\|b^{k\alpha}f\chi_k\right\|_{p(\cdot)}^q\right)^{\frac{1}{q}} \\ &= \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}. \end{aligned}$$

Now we prove (ii) implies (i). Let $f(x)=\sum_{k=-\infty}^{+\infty}\lambda_k a_k\left(x\right)$ be a decomposition of f which satisfies the hypothesis (ii) of Theorem 2.1.

We observe that $\chi_j\cdot a_k=0$ when $2^k<2^{j-1}$ which means that

$$k<j-1.$$

which gives that $\chi_j\cdot a_k\neq 0$ when s that

$$k\geq j.$$

For each $j\in\mathbb{Z}$, by the Minkowski inequality

$$\left\|f\chi_j\right\|_{p(\cdot)}\leq\sum_{k=j}^{\infty}|\lambda_k|\left\|a_k\right\|_{p(\cdot)}. \tag{2.2}$$

By definition of atom and from (2.2), it follows that $\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}$ is bounded by

$$I:=\left(\sum_{k=-\infty}^{+\infty}b^{k\alpha q}\left(\sum_{j=k}^{+\infty}|\lambda_j|\left\|a_j\right\|_{p(\cdot)}\right)^q\right)^{\frac{1}{q}}.$$

Since $0<\alpha<\infty$, then by Lemma 1.3 (with $0<\upsilon=b^{-\alpha}<1$), we have

$$I\leq c\Big(\sum_{k=-\infty}^{+\infty}\Big(\sum_{j=k}^{+\infty}|\lambda_j|b^{-(j-k)\alpha}\Big)^q\Big)^{\frac{1}{q}}\leq c\Big(\sum_{k=-\infty}^{+\infty}|\lambda_k|^q\Big)^{\frac{1}{q}}.$$

This finishes the proof. □

By repeating the same arguments used in the proof of Theorem 2.1, we can obtain the decomposition theorem of the non-homogeneous anisotropic Herz spaces.

Theorem 2.2. *Let $\alpha\in\mathbb{R}, 0<q\leq\infty$ and $p\in\mathcal{P}_0(\mathbb{R}^n)$, the following two statements are equivalentes*
(i)- $f\in K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$
(ii)- f can be represented by

$$f(x)=\sum_{k=0}^{+\infty}\lambda_k a_k(x),\tag{2.3}$$

where the series converges in the sense of distributions, $\lambda_k\geq 0$, each b_k is a central $(\alpha,p(\cdot))$-block with support contained in B_k and

$$\Big(\sum_{k=0}^{+\infty}|\lambda_k|^q\Big)^{\frac{1}{q}}\leq c\,\|f\|_{K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}.$$

Moreover, the norms $\|f\|_{K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}$ and $\inf\Big(\sum_{k=0}^{+\infty}|\lambda_k|^q\Big)^{\frac{1}{q}}$ are equivalent, where the infimum is taken all over all decompositions of f as in (2.3).

2.3 Atomic decomposition of $H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$

In this section, we present the atomic decomposition of anisotropic Herz-type Hardy spaces with one variable exponent.

The following atomic decomposition is given in [16].

Theorem 2.3. *Let $1-\frac{1}{p^+}\leq\alpha<\infty$ and, $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $0<q<\infty$, and non-negative integer $s\geq\Big[\Big(\alpha-1+\frac{1}{p^+}\Big)\ln b\backslash\ln\lambda_-\Big]$. The following two statements are equivalentes*
(i)- $f\in H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$
(ii)- f can be represented by

$$f(x)=\sum_{k=-\infty}^{+\infty}\eta_k a_k(x),\tag{2.4}$$

where the series converges in the sense of distributions, $\eta_k \geq 0$, each b_k is a central $(\alpha, p(\cdot))$-atom with support contained in B_k and

$$\Big(\sum_{k=0}^{+\infty} |\eta_k|^q\Big)^{\frac{1}{q}} \leq C \|f\|_{H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}.$$

Moreover, the norms $\|f\|_{H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}$ and $\inf\Big(\sum_{k=0}^{+\infty}|\eta_k|^q\Big)^{\frac{1}{q}}$ are equivalent, where the infimum is taken all over all decompositions of f as in (2.4).

Proof. First we prove $(i) \Rightarrow (ii)$. Suppose that $f(x) = \sum\limits_{k=-\infty}^{+\infty} \eta_k a_k(x)$, we consider two cases $0 < q \leq 1$ and $1 < q < \infty$.

Case 1: If $0 < q \leq 1$, it suffices to show that

$$\|\mathcal{M}_\varphi(a)\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)} \leq C,$$

holds for any central $(\alpha, p(\cdot))$-atom with minimal value of $s \geq \Big[\Big(\alpha - 1 - \frac{1}{p^+}\Big)\ln b \backslash \ln \lambda_-\Big]$. Since

$$\mathcal{M}_\varphi(a) \approx \sup_{\varphi\in\mathcal{A}_N} \mathcal{M}_\varphi(a) := \mathcal{M}^0_N(a),$$

it suffices to prove that there exists a positive constant C independent of a such that

$$\big\|\mathcal{M}^0_N(a)\big\|_{\dot{K}^{\alpha,q}_{p(\cdot)}} \leq C, \quad \text{for} \quad N \geq N_q, \tag{2.5}$$

where

$$N_q := \begin{cases} [(1/p-1)\ln b/\ln\lambda_-] + 2 & , \quad 0 < p \leq 1 \\ 2 & , \quad p > 1 \end{cases}$$

Suppose that $\operatorname{supp} a \subset \mathcal{B}_{k_0}$ for some $k_0 \in \mathbb{Z}$. Write

$$\begin{aligned}
\big\|\mathcal{M}^0_N(a)\big\|^q_{\dot{K}^{\alpha,q}_{p(\cdot)}} &= \sum_{k=-\infty}^{\infty} b^{k\alpha q}\big\|\big(\mathcal{M}^0_N a\big)\chi_k\big\|^q_{p(\cdot)} \\
&= \sum_{k=-\infty}^{k_0+\omega} b^{k\alpha q}\big\|\big(\mathcal{M}^0_N a\big).\chi_k\big\|^q_{p(\cdot)} + \sum_{k=k_0+\omega+1}^{\infty} b^{k\alpha q}\big\|\big(\mathcal{M}^0_N a\big).\chi_k\big\|^q_{p(\cdot)} \\
&=: I_1 + I_2.
\end{aligned}$$

First, we estimate I_1. Using the boundedness of $\mathcal{M}^0_N$ on $L^{p(\cdot)}(\mathbb{R}^n)$-norm and the size condition (2) of a in Definition 2.1, we have

$$\begin{aligned}
I_1 &\leq \big\|\mathcal{M}^0_N a\big\|^q_{p(\cdot)} \sum_{k=-\infty}^{k_0+\omega} b^{k\alpha q} \\
&\leq C\,\|a\|^q_{p(\cdot)} \sum_{k=-\infty}^{k_0+\omega} b^{k\alpha} \leq C b^{-k\alpha q}.
\end{aligned}$$

Next we estimate I_2. We need a pointwise estimate of $\mathcal{M}^0_N a(x)$ on C_k for $k \geq k_0+\omega+1$. Taking $x \in R_k, \varphi \in \mathcal{S}_N, l \in \mathbb{Z}$ and a polynomiale P of degree $\leq s$, which will be specidied later, then we have

$$\begin{aligned}
|(a*\varphi_l)(x)| &= b^{-l}\left|\int_{\mathbb{R}^n} a(y)\varphi(A^{-l}(x-y))dy\right| \\
&= b^{-l}\left|\int_{\mathcal{B}_{k_0}} a(y)\varphi(A^{-l}(x-y))dy - 0\right| \\
&= b^{-l}\left|\int_{\mathcal{B}_{k_0}} a(y)\left(\varphi(A^{-l}(x-y))dy - P\left(A^{-l}(x-y)\right)\right)dy\right| \\
&\leq b^{-l}\int_{\mathcal{B}_{k_0}} a(y)dy \sup_{y\in A^{-l}x+B_{k_0-l}} |\varphi(y)-P(y)| \\
&=: b^{-l}\int_{\mathbb{R}^n} a(y)\chi_{B_{k_0}}(y)dy \sup_{y\in A^{-l}x+B_{k_0-l}} |\varphi(y)-P(y)| \\
&\leq b^{-l}\|a\|_{p(\cdot)}\left\|\chi_{k_0}\right\|_{p'(\cdot)} \sup_{y\in A^{-l}x+B_{k_0-l}} |\varphi(y)-P(y)| \\
&\leq b^{k_0-l}b^{-k_0(\alpha-1)}\left\|\chi_{k_0}\right\|_{p'(\cdot)} \sup_{y\in A^{-l}x+B_{k_0-l}} |\varphi(y)-P(y)|,
\end{aligned}$$

where we have used Hölder's inequality. Suppose $x \in R_k$ whith $k \geq k_0+\omega+1$. Then $x \in B_{k_0+\omega+m+1}\backslash B_{k_0+\omega+m}$, where integer $m = k-k_0-\omega-1 \geq 0$. Recall that (see [2, (2.11)])

$$t_1 \in B_i \text{ and } t_2 \in B_i \Rightarrow t_1+t_2 \in B_{i+\omega}, \qquad \text{and} \quad t_1 \notin B_{i+\omega} \text{ and } t_2 \in B_i \Rightarrow t_1+t_2 \notin B_i,$$

which gives that

$$\begin{aligned}
A^{-l}x + B_{k_0-l} &\subset A^{-l}\left(B_{k_0+\omega+m+1}\backslash B_{k_0+\omega+m}\right) + B_{k_0-l} \\
&= A^{k_0-l}\left((B_{\omega+m+1}\backslash B_{\omega+m}) + B_0\right) \\
&\subset A^{k_0-l}(B_m)^c = (B_{k_0-l+m})^c.
\end{aligned}$$

We cosider two cases. If $k_0 \geq l$ then we choose $P=0$, and

$$\sup_{y\in A^{-l}x+B_{k_0-l}} |\varphi(y)-P(y)| \leq \sup_{y\in A^{-l}+B_{k_0-l}} \min\left(1,\varrho(y)^{-N}\right) \leq b^{-N(k_0-l+m)}.$$

If $k_0 < l$ then we choose P to be the Taylor expansion of φ at the point $A^{-l}x$ of order s. By the Taylor Remainder Theorem and 2.5, we have

$$\begin{aligned}
\sup_{y\in A^{-l}x+B_{k_0-l}} |\varphi(y)-P(y)| &\leq C \sup_{z\in B_{k_0-l}} \sup_{0\leq\theta\leq 1} \sup_{|\alpha|=s+1} \left|\partial^\alpha \alpha\left(A^{-l}x+\theta z\right)\right| |z|^{s+1} \\
&\leq C\lambda_-^{(s+1)(k_0-l)} \sup_{y\in A^{-l}x+B_{k_0-l}} \min\left(1,\varrho(y)^{-N}\right) \\
&\leq C\lambda_-^{(s+1)(k_0-l)} \min\left(1, b^{-N(k_0-l+m)}\right).
\end{aligned}$$

Then, for $x \in B_{k_0+\omega+m+1} \backslash B_{k_0+\omega+m}, m \geq 0$,

$$\begin{aligned}\mathcal{M}_N^0 a(x) &= \sup_{\varphi \in \mathcal{A}_N} \sup_{l \in \mathbb{Z}} |(a * \varphi_l)(x)| \\ &\leq b^{-k_0(\alpha+1)} \left\| \chi_{k_0} \right\|_{p'(\cdot)} \max \Big(\sup_{l \in \mathbb{Z}, l \leq k_0} b^{(k_0-l)} b^{-N(k_0-l+m)} \\ &+ C \sup_{l \in \mathbb{Z}, l > k_0} b^{(k_0-l)} \lambda_-^{(s+1)(k_0-l)} \min\left(1, b^{-N(k_0-l+m)}\right) \Big).\end{aligned}$$

Notice that the supremum over $l \leq k_0$ is attained when $l = k_0$ and the supremum over $l > k_0$ is attained when $k_0 - l + m = 0$, since $b\lambda_-^{s+1} \leq b^{-N}$ for $N \geq s+2$. Hence, it suffices to check the maximum value for $k_0 < l \leq k_0 + m$ and $l \geq k_0 + m$. For $x \in B_{k_0+\omega+m+1} \backslash B_{k_0+\omega+m}, m \geq 0$, we get

$$\begin{aligned}\mathcal{M}_N^0 a(x) &\leq b^{-k_0(\alpha+1)} \|\chi_{k_0}\|_{p'(\cdot)} \max\left(b^{-Nm}, C\left(b\lambda_-^{s+1}\right)^{-m}\right) \\ &\leq C b^{-k_0(\alpha+1)} \|\chi_{k_0}\|_{p'(\cdot)} \left(b\lambda_-^{s+1}\right)^{-m},\end{aligned}$$

again by $b\lambda_-^{s+1} b^{-N} \leq 1$. Therefor, by $s = \left[\left(\alpha - 1 + \frac{1}{p^+}\right) \ln b \backslash \ln \lambda_-\right]$ and since $\|\chi_k\|_{p(\cdot)} \|\chi_k\|_{p'(\cdot)} \leq cb$, we have

$$\begin{aligned}I_2 &= \sum_{k=k_0+\omega+1}^{\infty} b^{k\alpha q} \left\|\mathcal{M}_N^0 a \chi_k\right\|_{p(\cdot)}^q \\ &\leq C \sum_{k=k_0+\omega+1}^{\infty} b^{k\alpha q} b^{(-k_0(\alpha+1))q} \left\|\chi_{k_0}\right\|_{p'(\cdot)}^q \left(b\lambda_-^{s+1}\right)^{-mq} \|\chi_k\|_{p(\cdot)}^q \\ &\leq C \sum_{k=k_0+\omega+1}^{\infty} b^{k\alpha q} b^{(-k_0(\alpha+1))q} b^{kq} \left(b\lambda_-^{s+1}\right)^{-mq} \left(\frac{\left\|\chi_{k_0}\right\|_{p'(\cdot)}}{\left\|\chi_k\right\|_{p(\cdot)}}\right)^q \\ &\leq C \sum_{k=k_0+\omega+1}^{\infty} \left(b\lambda_-^{-(s+1)} b^{\alpha-1+\frac{1}{p^+}}\right)^{(k-k_2)q} \leq C.\end{aligned}$$

Case 2: If $1 < q < \infty$, we have

$$\begin{aligned}\|\mathcal{M}_\varphi(a)\|_{\dot{K}^{\alpha,q}_{p(\cdot)}}^q &= \sum_{k=-\infty}^{\infty} b^{k\alpha q} \left\|\left(\mathcal{M}_\varphi a\right)\chi_k\right\|_{p(\cdot)}^q \\ &\leq C \sum_{k=-\infty}^{\infty} b^{k\alpha q} \Big(\sum_{l=-\infty}^{\infty} |\eta_l| \left\|\mathcal{M}_N^0 a_l . \chi_k\right\|_{p(\cdot)} \Big)^q \\ &\leq C \sum_{k=-\infty}^{\infty} b^{k\alpha q} \Big(\sum_{l=-\infty}^{k-\omega-1} |\eta_l| \left\|\mathcal{M}_N^0 a_l . \chi_k\right\|_{p(\cdot)} \Big)^q \\ &+ C \sum_{k=-\infty}^{\infty} b^{k\alpha q} \Big(\sum_{l=k-\omega}^{\infty} |\eta_l| \left\|\mathcal{M}_N^0 a_l . \chi_k\right\|_{p(\cdot)} \Big)^q \\ &=: J_1 + J_2.\end{aligned}$$

By Hölder inequality, it follows that

$$\begin{aligned}
J_2 &\leq C\sum_{k=-\infty}^{\infty} b^{k\alpha q}\Big(\sum_{l=k-\omega}^{\infty}|\eta_l|\, b^{-l\alpha}\Big)^q\\
&\leq C\sum_{k=-\infty}^{\infty} b^{k\alpha q}\Big(\sum_{l=k-\omega}^{\infty}|\eta_l|^q\, b^{-l\alpha q/2}\Big)\Big(\sum_{l=k-\omega}^{\infty} b^{-l\alpha q'/2}\Big)^{q/q'}\\
&\leq C\sum_{k=-\infty}^{\infty} b^{k\alpha q/2}\Big(\sum_{l=k-\omega}^{\infty}|\eta_l|^q\, b^{-l\alpha q/2}\Big)\\
&\leq C\sum_{l=-\infty}^{\infty}|\eta_l|^q\, b^{-l\alpha q/2}\Big(\sum_{k=-\infty}^{l+\omega} b^{k\alpha q/2}\Big)\\
&\leq C\sum_{l=-\infty}^{\infty}|\eta_l|^q<\infty.
\end{aligned}$$

Using the same estimate of I_2 for $\mathcal{M}_N^0 a_l$, when $k\geq l+\omega+1$, we have

$$\begin{aligned}
\left\|\left(\mathcal{M}_N^0 a\right)\chi_k\right\| &\leq Cb^{-l(\alpha+1)}\left(b\lambda_-^{s+1}\right)^{l+\omega+1-k}\left\|\chi_{B_l}\right\|_{p'(\cdot)}\left\|\chi_{B_k}\right\|_{p(\cdot)}\\
&\leq Cb^{-l(\alpha+1)+k}b^{(l-k)(1-\frac{1}{p^+})}\left(b\lambda_-^{s+1}\right)^{l+\omega+1-k}.
\end{aligned}$$

Let $z=\lambda_-^{s+1}b^{\alpha-\delta_2}$. Then, by $z<1$,

$$\begin{aligned}
J_1 &= C\sum_{k=-\infty}^{\infty} b^{k\alpha q}\Big(\sum_{l=-\infty}^{k-\omega-1}|\eta_l|\left\|\mathcal{M}_N^0 a_l.\chi_k\right\|_{p(\cdot)}\Big)^q\\
&\leq C\sum_{k=-\infty}^{\infty} b^{k\alpha q}\left\{\sum_{l=-\infty}^{k-\omega-1}|\eta_l|\, b^{-l(\alpha+1)+k}b^{(l-k)(1-\frac{1}{p^+})}\left(b\lambda_-^{s+1}\right)^{l+\omega+1-k}\right\}^q\\
&\leq C\sum_{k=-\infty}^{\infty} z^{kq}\Big(\sum_{l=-\infty}^{k-\omega-1}|\eta_l|\, z^{-l}\Big)^q
\end{aligned}$$

Since $z<1$, then by Lemma 1.3 (with $0<v=z<1$), we have

$$\begin{aligned}
J_1 &\leq C\sum_{k=-\infty}^{\infty}\Big(\sum_{l=-\infty}^{k-\omega-1}|\eta_l|\, z^{k-l}\Big)^q\\
&\leq C\sum_{l=-\infty}^{\infty}|\eta_l|^q<\infty.
\end{aligned}$$

The proof of $(ii)\Rightarrow(i)$ is included in given in the proof of [16, Theorem 2.2]. Hence the proof of this Theorem is complete. □

Remark 2.2. It is easy to see that, for the case $0<q\leq 1$, if we remove the condition $\mathrm{supp} a_k\subset\mathcal{B}_k$, then the conclusion of Theorem 2.3 is also true.

CHAPTER 3

APPLICATIONS

In this chapter we present the boundedness of some bilinear operators on anisotropic variable Herz and anisotropic Herz-type Hardy spaces. Firstly, we see in first section the result of [13, Theorem 3.1] and in the secend section, we present the result of [16, Theorem 4.1].

3.1 Application on $\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$

In the folowing section, we present the result of [13, Theorem 3.1] (when the exponent α is constant) concerning the boundedness of some sublinear operators T satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{\sigma(x-y)} \mathrm{d}y, \quad x \notin \operatorname{supp} f \tag{3.1}$$

for integrable and compactly supported functions f.

Theorem 3.1. *Let $\alpha \in \mathbb{R}, 0 < q \leq \infty$ and $p \in \mathcal{P}(\mathbb{R}^n)$. If p is* log*-Hölder continuous, both at the origin and at infinity and $0 < \alpha < 1 - 1/p^+$. If a sublinear operator T satisfies (3.1) for any integrable function f with a compact support and T is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then T is bounded on $\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$.*

Remark 3.1. Since we use Lemma 1.3 then we eliminate the discussions ($0 < p \leq 1$ and $1 < p < \infty$) given in [13, Proof of Theorem 3.1].

Proof. We must show that

$$\|Tf\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)} \leq c\,\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}$$

for all $f \in \dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$. Using Theorem 2.1, we may assume that

$$f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i$$

where $\lambda_i \geq 0$ and a_i's are $(\alpha, p(\cdot))$- atom with $\operatorname{supp} a_i \subseteq B_i$. We have

$$\begin{aligned}
\|Tf\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)} &\approx \left(\sum_{k=-\infty}^{+\infty} b^{k\alpha q}\left(\sum_{i=-\infty}^{+\infty} \lambda_i a_i \|Ta_i \cdot \chi_k\|_{p(\cdot)}\right)^q\right)^{1/q} \\
&\lesssim \left(\sum_{k=-\infty}^{+\infty} b^{k\alpha q}\left(\sum_{i=-\infty}^{k-\theta-1} \lambda_i \|Ta_i \cdot \chi_k\|_{p(\cdot)}\right)^q\right)^{1/q} \\
&+ \left(\sum_{k=-\infty}^{+\infty} b^{k\alpha q}\left(\sum_{i=k-\theta}^{+\infty} \lambda_i \|Ta_i \cdot \chi_k\|_{p(\cdot)}\right)^q\right)^{1/q} \\
&=: J_1 + J_2.
\end{aligned}$$

Let us first estimate J_1. By the condition (3.1) and the fact if $x \in R_k, y \in B_i$ and $i \leq k - \theta - 1$, then

$$\sigma(x-y) \geq b^{-\theta}\sigma(x) - \sigma(y) \geq b^{-\theta}\sigma(x) - b^{-\theta-1}\sigma(x) = b^{-\theta}\left(1 - \frac{1}{p}\right)\sigma(x),$$

we get

$$|Ta_i(x)| \lesssim \int_{B_i} \frac{|a_i(y)|}{\sigma(x)} \mathrm{d}y$$
$$\leq cb^{-k} \int_{B_i} |a_i(y)| \, \mathrm{d}y,$$

by generalized Hölder inequality with $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, the condition (ii) in Definition 2.1 and Lemma 1.2, we get

$$|Ta_i(x)| \lesssim b^{-k} \|a_i\|_{p(\cdot)} \|\chi_{B_i}\|_{p'(\cdot)}$$
$$\leq cb^{-k-i(\alpha-1+1/p^+)}$$

which gives

$$\|Ta_i \cdot \chi_k\|_{p(\cdot)} \leq cb^{-k-i(\alpha-1+1/p^+)} \|\chi_k\|_{p(\cdot)}$$
$$\leq cb^{k(-1+1/p^+)-i(\alpha-1+1/p^+)},$$

then, J_1 is bounded by

$$J_1 \lesssim \left(\sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-\theta-1} \lambda_i b^{(k-i)(\alpha-1+1/p^+)} \right)^q \right)^{1/q}.$$

By Lemma 1.3 (with $0 < a = b^{\alpha-1+1/p^+} < 1$), we have

$$J_1 \lesssim \left(\sum_{k=-\infty}^{+\infty} \left(\sum_{i=-\infty}^{k-\theta-1} \lambda_i b^{(k-i)(\alpha-1+1/p^+)} \right)^q \right)^{1/q}$$
$$\lesssim \left(\sum_{k=-\infty}^{+\infty} \lambda_i^q \right)^{1/q}$$
$$\leq c \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}.$$

Let us estimate J_2. By the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of T, definition of atom we obtain

$$J_2 \lesssim \left(\sum_{k=-\infty}^{+\infty} b^{k\alpha q} \left(\sum_{i=k-\theta}^{+\infty} \lambda_i \|a_i\|_{p(\cdot)} \right)^q \right)^{1/q}$$
$$\leq \left(\sum_{k=-\infty}^{+\infty} \left(\sum_{i=k-\theta}^{+\infty} \lambda_i b^{-(i-k)\alpha} \right)^q \right)^{1/q},$$

since α is non negative number, then by Lemma 1.3 (with $0<v=b^{-\alpha}<1$), we have

$$\begin{aligned} J_2 &\lesssim \left(\sum_{k=-\infty}^{+\infty}\left(\sum_{i=k-\theta}^{+\infty}\lambda_i b^{-(i-k)\alpha}\right)^q\right)^{1/q} \\ &\lesssim \left(\sum_{k=-\infty}^{+\infty}\lambda_i^q\right)^{1/q} \\ &\leq c\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)}. \end{aligned}$$

A combination of estimations of J_1 and J_2 finish the proof of Theorem 3.1. □

3.2 Application on $H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$

In this section, we give an application of the atomic decomposition of $H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$.

Theorem 3.2 ([16]). *Let $1-\frac{1}{p^+}\leq\alpha<1-\frac{1}{p^+}+\ln b\backslash\ln\lambda_-$, $0<q<\infty$, and $p\in\mathcal{P}(\mathbb{R}^n)$. If a linear operator T satisfies*

$$Tf=\sum_{i\in\mathbb{N}}\lambda_i Ta_i \text{ in } \mathcal{S}', \quad \text{if} \quad f=\sum_{i\in\mathbb{N}}\lambda_i a_i \quad \text{in } \mathcal{S}'. \tag{3.2}$$

for evry central atomic decomposition, is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ and, for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$ whith compact support B_k, $\int_{\mathbb{R}^n}f(x)=0$, T satisfies the following size condition

$$|Tf(x)|\leq C\frac{b^k\|f\|_1}{\sigma^2(x)}, \quad \text{if} \quad \inf_{y\in\operatorname{supp}f}\varrho(x-y)\geq b^{-\theta}\left(1-\frac{1}{b}\right)\varrho(x), \tag{3.3}$$

then T is an operator bounded from $H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ $\left(\text{or } HK^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)\right)$ into $\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$ $\left(\text{or } K^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)\right)$.

Proof. We only prove the homogeneous case, the proof of the nonhomogeneous case is similar. It is necessary to show that

$$\|Tf\|_{\dot{K}^{\alpha,q}_{p(\cdot)}}\lesssim\|f\|_{H\dot{K}^{\alpha,q}_{p(\cdot)}},$$

for all $f\in H\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)$. By using Theorem 2.3, we can write f as

$$f(x)=\sum_{j=-\infty}^{+\infty}\eta_j a_j(x),$$

where each a_j is a central $(\alpha,p(\cdot))$-atom with support contained in B_j, and

$$\|f\|_{H\dot{K}^{\alpha,q}_{p(\cdot)}}\sim\inf\left(\sum_{k=-\infty}^{\infty}|\eta_k|^q\right)^{1/q}.$$

By condition (3.2), we write

$$\begin{aligned}
\|Tf\|^q_{\dot{K}^{\alpha,q}_{p(\cdot)}(A;\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} b^{k\alpha q} \|(Tf)\,\chi_k\|^q_{p(\cdot)} \\
&\lesssim \sum_{k=-\infty}^{\infty} b^{k\alpha q} \left(\sum_{j=-\infty}^{k-\theta-1} |\eta_j| \, \|(Ta_l).\chi_k\|_{p(\cdot)} \right)^q \\
&+ \sum_{k=-\infty}^{\infty} b^{k\alpha q} \left(\sum_{j=k-\theta}^{\infty} |\eta_j| \, \|(Ta_l).\chi_k\|_{p(\cdot)} \right)^q \\
&=: I_1 + I_2.
\end{aligned}$$

First, we estimate I_1. If $j \leq k-\theta-1, x \in R_k$, and $y \in B_j$, then

$$\begin{aligned}
\sigma(x-y) &\geq b^{-\theta}\sigma(x) - \sigma(y) \\
&\geq b^{-\theta}\sigma(x) - b^{-\theta-1}\sigma(x) \\
&= b^{-\theta}\left(1 - \frac{1}{b}\right)\sigma(x).
\end{aligned}$$

By the last estimate and generalized Hölder's inequality with $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, we can obtain the following estimate

$$\begin{aligned}
|Ta_j(x)| &\leq C \frac{b^j \|a_j\|_1}{\sigma^2(x)} \\
&\leq C b^{j+2-2k} \|a_j\|_{p(\cdot)} \left\|\chi_{B_j}\right\|_{p'(\cdot)}.
\end{aligned}$$

By the condition (2) of the atom a and since $\|\chi_{B_k}\|_{p'(\cdot)} \|\chi_{B_k}\|_{p(\cdot)} \approx |B_k|$ and Lemma 1.2, we get

$$\begin{aligned}
\|Ta_j \cdot \chi_k\|_{p(\cdot)} &\leq C b^{j+2-2k} \|a_j\|_{p(\cdot)} \left\|\chi_{B_j}\right\|_{p'(\cdot)} \|\chi_k\|_{p(\cdot)} \\
&\leq C b^{j+2-2k} \|a_j\|_{p(\cdot)} \frac{\left\|\chi_{B_j}\right\|_{p'(\cdot)}}{\|\chi_{B_k}\|_{p'(\cdot)}} |B_k| \\
&\leq C b^{j+2-2k} b^{(j-k)(1-1\backslash p^+)-j\alpha}.
\end{aligned}$$

The case when $0 < q \leq 1$.

If $0 < q \leq 1$, noting that $(1-\frac{1}{p^+})-\alpha+1>0$, we deduce

$$
\begin{aligned}
I_1 &= \sum_{k=-\infty}^{\infty} b^{k\alpha q}\Big(\sum_{j=-\infty}^{k-\theta-1} |\eta_j| \left\|(Ta_j).\chi_k\right\|_{p(\cdot)}\Big)^q \\
&\leq C \sum_{k=-\infty}^{\infty} b^{k\alpha q}\Big(\sum_{j=-\infty}^{k-\theta-1} |\eta_j|\, b^{(j+2-2k)} b^{(j-k)(1-1\backslash p^+)q-j\alpha q}\Big) \\
&\leq C \sum_{j=-\infty}^{\infty} |\eta_j|^q \Big(\sum_{k=j+\theta+1}^{\infty} b^{-(k-j)((1-1\backslash p^+)-\alpha+1)q}\Big) \\
&\leq C \sum_{j=-\infty}^{\infty} |\eta_j|^q . \qquad (3.4)
\end{aligned}
$$

The case when $1 < q < \infty$.

By Lemma 1.3 (with $0 < \upsilon = b^{-((1-1\backslash p^+)-\alpha+1)} < 1$), we have

$$
\begin{aligned}
I_1 &\leq \sum_{k=-\infty}^{\infty} b^{k\alpha q}\Big(\sum_{j=-\infty}^{k-\theta-1} |\eta_j|\, b^{(j+2-2k)} b^{(j-k)(1-1\backslash p^+)-j\alpha}\Big)^q \\
&\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-\theta-1} |\eta_j|\, b^{-(k-j)((1-1\backslash p^+)-\alpha+1)}\Big)^q \\
&\leq C \sum_{k=-\infty}^{\infty} |\eta_k|^q . \qquad (3.5)
\end{aligned}
$$

Next we estimate I_2. By the $L^{p(\cdot)}$-boundedness of T, the condition (2) of the atom a and Lemma 1.3 (with $0 < \upsilon = b^{-\alpha} < 1$), we get

$$
\begin{aligned}
I_2 &= \sum_{k=-\infty}^{\infty} b^{k\alpha q}\Big(\sum_{j=k-\theta}^{\infty} |\eta_j| \left\|(Ta_j).\chi_k\right\|_{p(\cdot)}\Big)^q \\
&\lesssim \sum_{j=-\infty}^{\infty} \Big(\sum_{j=k-\theta}^{\infty} |\eta_j|\, b^{-(j-k)\alpha}\Big)^q \\
&\leq C \sum_{j=-\infty}^{\infty} |\eta_j|^q . \qquad (3.6)
\end{aligned}
$$

Thus, by (3.4), (3.5) and (3.6) we complete the proof of this Theorem. □

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ملخص المذكرة:

في هذه المذكرة قدمنا فضاءات "هارز" متباينة الخواص و فضاءات "هارز–نمط هاردي" متباينة الخواص ذات الدليل المتغير، حيث أعطينا بعض الخصائص الأساسية لهذه الفضاءات، وفي الأخير درسنا محدودية بعض المؤثرات تحت الخطية في فضاءات "هارز" متباينة الخواص و فضاءات "هارز–نمط هاردي" متباينة الخواص باستعمال نظريات التفكيك لهذه الفضاءات.

الكلمات المفتاحية: فضاءات هارز المتباين الخواص، فضاءات هارز–نمط هاردي المتباين الخواص، نظريات التفكيك، مؤثرات تحت الخطية، أدلة متغيرة.

Résumé de thèse:

Dans cette thèse, nous avons présenté les espaces de Herz anisotropes et de Herz-type Hardy anisotropes avec un exposant variable, où nous donnons quelques propriétés de base de ces espaces, et à la fin nous avons étudié la continuité de certains opérateurs sous-linéaires dans les espaces de Herz anisotropes et de Herz-type Hardy anisotropes en utilisant les théorèmes de décomposition de ces espaces.

Mots clés : Espaces de Herz anisotropes, Espaces de Herz-type Hardy anisotropes, Théorèmes de décomposition, Opérateurs sous linéaire, Exposant variable.

Abstract of thesis:

In this thesis, we have presented the anisotropic Herz spaces and anisotropic Herz-type Hardy spaces with variable exponent, where we give some basic properties of these spaces, and finally we studied the boundedness of some sublinear operators in anisotropic Herz and anisotropic Herz – type Hardy spaces by using the decomposition theorems of these spaces.

Key words: Anisotropic Herz spaces, Anisotropic Herz-type Hardy spaces, Decomposition theorems, Sublinear operators, Variable exponent.