PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
MOHAMED BOUDIAF UNIVERSITY - M'SILA
Faculty of Mathematics and Computer Science
Department of Mathematics

Master's degree in Mathematics

Field : Mathematics and Computer Sciences
Branch : Algebra and Discrete Mathematics
Option : Algebra and Discrete Mathematics

Titled

Picture Fuzzy topology

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Defended on : ....

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University year : 2025/2026

Picture Fuzzy Topology

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Acknowledgements

First and foremost, I must express my deepest gratitude and praise to Almighty God, the Most Gracious and Most Merciful, for granting me the health, strength, and perseverance required to undertake and successfully complete this Master's thesis. His divine blessings have been my source of guidance throughout this academic journey.

I would like to express my most profound gratitude and sincere appreciation to my research supervisor. I am deeply indebted to him for his invaluable guidance, immense patience, and continuous encouragement throughout the entire duration of this work. His academic expertise, constructive criticism, and insightful remarks have not only shaped the direction of this dissertation but have also significantly contributed to my development as a young researcher in mathematics.

My sincere thanks also go to the esteemed members of the jury, the Chair and the Examiner, for dedicating their valuable time to review, evaluate, and critically examine this dissertation. I truly appreciate their scientific remarks, rigorous evaluation, and constructive suggestions, which will undoubtedly enhance the mathematical quality and depth of this study.

I am equally grateful to all the professors and faculty members of the Department of Mathematics at Mohamed Boudiaf University - M'Sila. Their high-quality teaching, dedication, and constant support throughout my university years have provided me with a solid scientific foundation and the necessary skills to accomplish this work.

A very special and heartfelt tribute goes to my dear husband and my beloved children. To my husband, thank you for your endless patience, continuous encouragement, and for standing firmly by my side through every stage of this demanding journey; your unwavering faith in my abilities was my greatest source of strength. To my children, thank you for your understanding, love, and for bringing joy to my days even during the most stressful times. This achievement is as much yours as it is mine.

Finally, words cannot adequately express my eternal gratitude and love to my beloved parents for their endless prayers, sacrifices, and unconditional support during my entire life. I also extend my appreciation to all my family members and dear friends for their constant motivation. To anyone who contributed, directly or indirectly, to the success of this work, I offer my warmest thanks.

Dedication

I proudly dedicate this academic humble work:

To the steadfast souls in Gaza and across the blessed land of Palestine...

To the absolute pride of our Ummah, the men of the Islamic Resistance...

To the architects and heroes of the historic October 7th Al-Aqsa Flood...

To the pure blood of our righteous martyrs and the freedom of our brave prisoners...

To those who revived the flame of dignity and redefined our existence...

To those who proved to the world that honor is a frontier worth fighting for.

Therefore, we hold our ground on the frontlines of science and knowledge, weaponized with learning to strive for our truth.

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General Introduction

The inception of Fuzzy Set Theory by *Lotfi A. Zadeh* in 1965 marked a significant paradigm shift in pure and applied mathematics, providing an alternative to the rigid binary constraints of classical set theory. While traditional crisp sets restrict elements to a strict dualistic status of complete belonging or absolute exclusion, Zadeh's framework accommodates partial membership through a dedicated function mapping into the closed unit interval $[0, 1]$. This conceptual flexibility has propelled fuzzy structures into the forefront of computational intelligence, proving indispensable across diverse domains such as multi-criteria decision-making, medical diagnosis, pattern recognition, and autonomous control systems.

As practical mathematical modeling encountered increasingly complex and non-deterministic environments, the need to explicitly model structural hesitation and incomplete information became apparent. To bridge this analytical gap, *Krassimir Atanassov* introduced Intuitionistic Fuzzy Sets (IFS) in 1986, appending an explicit non-membership parameter alongside the positive membership degree. Although the intuitionistic framework enhanced the mathematical modeling of doubt and uncertainty, it remained constrained in scenarios featuring independent neutrality or abstention. Consequently, to fully capture the nuances of human voting and complex decision-making systems, *B. C. Cuong* proposed the innovative concept of Picture Fuzzy Sets (PFS) in 2014. By incorporating a third independent dimension—the neutral membership degree—PFS establishes a comprehensive, four-dimensional analytical spectrum that simultaneously captures affirmative agreement, neutral abstinence, negative disagreement, and complete refusal.

Parallel to the algebraic growth of these non-classical sets, topological structures underwent a similar evolution. The fusion of topology with fuzzy mathematics has opened new horizons in abstract spaces, allowing researchers to investigate continuity, separation axioms, and compactness under varying degrees of uncertainty. While classical fuzzy topology and intuitionistic fuzzy topology have been thoroughly investigated over the past decades, the systematic formulation of topological spaces under the picture fuzzy environment represents an emerging and highly fertile area of mathematical research.

The main objective of this Master's thesis is to bridge abstract general topology with picture fuzzy mathematics by investigating the structural properties and behaviors of **Picture Fuzzy Topological Spaces**. We aim to mathematically formalize how traditional topological operators behave when applied to picture fuzzy sets, establishing a rigorous theoretical foundation for future developments in both pure mathematics and intelligent application frameworks.

To achieve these academic objectives, the manuscript is systematically organized into three cohesive and progressive chapters:

- **Chapter One** establishes the foundational algebraic vocabulary. It thoroughly reviews the core concepts of standard fuzzy sets, intuitionistic fuzzy sets, fuzzy relations, and lattices, before transitioning into a rigorous mathematical formulation of Picture Fuzzy Sets, their boundary configurations, basic set operations, and underlying parameter constraints.

- **Chapter Two** acts as the structural bridge by reviewing the classical architecture of Fuzzy Topological Spaces. Here, we outline the fundamental theorems surrounding fuzzy openness, closure operators, interior mappings, neighborhood topologies, and the established results of Chang-style compactness and continuity.

- **Chapter Three** represents the core theoretical contribution of this work, focusing exclusively on **Picture Fuzzy Topological Spaces**. We formalize the axiomatic definition of picture fuzzy topology (τ) and systematically explore its internal properties, including picture fuzzy interior-closure interactions, neighborhood systems, continuous mappings, compact covers, and fundamental separation axioms, supported by rigorous mathematical proofs and illustrative counter-examples.

Ultimately, this work aims to contribute directly to the theoretical refinement of abstract non-classical topologies, offering a robust mathematical basis that can support future advancements in both pure topology and intelligent decision-making paradigms.

Chapter 1

picture Fuzzy Set

The formal representation, architectural modeling, and algorithmic management of epistemic uncertainty have emerged as critical challenges in contemporary pure mathematics, computational intelligence, and multi-criteria decision-making systems [1, 4]. For centuries, classical Aristotelian logic served as the absolute mathematical bedrock for scientific inquiry, operating strictly on the foundational law of the excluded middle. In this binary paradigm, a proposition is either fundamentally true or false, and an element either completely belongs to a set or is completely excluded from it, mapping to the crisp binary space $\{0, 1\}$.

However, real-world complexity, inherent human cognitive limitations, subjective evaluations, and imperfect empirical data streams rarely conform to such deterministic boundaries [2, 5]. Real systems are frequently plagued by non-deterministic layouts, partial truths, and fluctuating observation parameters. To bridge this major epistemic gap, numerous non-classical set-theoretic frameworks have been systematically formulated.

This chapter establishes a deep, mathematically rigorous, and comprehensive foundational framework spanning three core evolutionary paradigms: **Fuzzy Sets** (FS), **Intuitionistic Fuzzy Sets** (IFS), and **Picture Fuzzy Sets** (PFS). We systematically trace their historical lineage, analyze their core operational properties, prove their underlying algebraic structures, and detail the mathematical behaviors that govern their modern expansions [1, 2, 4, 5, 6].

1.1 Fuzzy Sets (FS): The Zadeh Paradigm

The paradigm-shifting historical transition from rigid binary evaluation frameworks to flexible continuous logical models began with the introduction of partial membership environments [3]. This section establishes the definitions, axiomatic properties, and underlying operational lattices governing traditional fuzzy sets.

1.1.1 Mathematical Foundations and Axiomatic Context

In 1965, Lotfi A. Zadeh published his foundational paper introducing the concept of Fuzzy Sets (FS) [3]. Zadeh proposed a radical mathematical departure from crisp boundaries by redefining the characteristic function of a subset as a continuous membership function mapping elements to the entire closed unit interval $[0, 1]$.

Definition 1.1 (Fuzzy Set Theory)**.** *Let $I = [0, 1]$ be the unit interval and X be a non-empty universal set of discourse, representing the complete collection of all objects under analytical observation, where $x \in X$. Subsequently, a function $\mu : X \to I$ where $x \mapsto \mu(x)$ is described as a **Fuzzy Set** in X. Where $\mu(x)$ is defined as the "Grade of membership of $x \in X$ in μ", for more information see [1, 2, 4, 5, 6].*

In classical set theory, the non-membership characteristic is implicitly deterministic: if an element has a membership of 1, its non-membership is 0, and vice versa. Zadeh maintained this underlying complementarity by asserting that the non-membership degree is strictly dependent on the membership value, defined precisely as $\nu_A(x) = 1 - \mu_A(x)$ [3].

1.1.2 Linguistic Variables and Uncertainty Modeling

The introduction of fuzzy sets provided a systematic mechanism to translate vague linguistic assertions into analytical expressions. Concepts such as "very hot weather", "unstable voltage", or "safe braking distance" cannot be encapsulated by crisp real-valued thresholds without injecting artificial precision. By assigning a continuous grade of membership, the transitions between states are smoothed out, which directly mimics the cognitive mechanisms of human processing.

While this model successfully quantified linguistic ambiguity and vague conceptual terms [1, 2, 4, 5, 6], it encountered significant operational limitations when applied to advanced expert networks where data gaps exist, or where human evaluators experience profound hesitation.

1.1.3 Core Lattice Operations on Fuzzy Sets

Let A and B be two fuzzy sets over the universe X characterized by membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. The fundamental algebraic operations are defined point-wise for all $x \in X$ as follows [1, 3]:

Definition 1.2 (Fuzzy Inclusion and Equality). *A fuzzy set A is contained in another fuzzy set B ($A \subseteq B$) if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$ [3]. Consequently, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, which implies $\mu_A(x) = \mu_B(x)$ for all $x \in X$.*

Definition 1.3 (Fuzzy Intersection and Union). *The intersection $A \cap B$ and union $A \cup B$ are traditionally governed by the standard t-norm and t-conorm structures (specifically* min *and* max *operators) [3, 6]:*

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x)$$

Definition 1.4 (Fuzzy Complement). *The standard complement of a fuzzy set A, denoted by A^C, is defined point-wise via the continuous negation mapping [3]:*

$$\mu_{A^C}(x) = 1 - \mu_A(x)$$

1.1.4 Algebraic Properties and De Morgan Proofs

Fuzzy set operators preserve the majority of classical set-theoretic laws, including commutativity, associativity, and distributivity [3]. However, a major departure occurs regarding the classical laws of contradiction and excluded middle, as $A \cap A^C \neq \emptyset$ and $A \cup A^C \neq X$ in general.

To establish the rigor of these operators, we analyze the validation of De Morgan's laws within the continuous unit interval.

Proposition 1.1 ([3]). *Let A and B be two fuzzy sets in X. Then $(A \cup B)^C = A^C \cap B^C$.*

Proof. Let $x \in X$ be an arbitrary element. We evaluate the left-hand side of the identity by applying the definitions of the standard fuzzy union and standard complement:

$$\mu_{(A\cup B)^C}(x) = 1 - \mu_{A\cup B}(x)$$

$$\mu_{(A\cup B)^C}(x) = 1 - \max\{\mu_A(x), \mu_B(x)\}$$

By using the algebraic property of real numbers where $1 - \max\{a, b\} = \min\{1-a, 1-b\}$, we obtain:

$$\begin{aligned} 1 - \max\{\mu_A(x), \mu_B(x)\} &= \min\{1 - \mu_A(x), 1 - \mu_B(x)\} \\ &= \min\{\mu_{A^C}(x), \mu_{B^C}(x)\} \\ &= \mu_{A^C\cap B^C}(x) \end{aligned}$$

Since this identity holds point-wise for all $x \in X$, the theorem is verified. □

Proposition 1.2 ([3]). *Let A and B be two fuzzy sets in X. Then $(A \cap B)^C = A^C \cup B^C$.*

Proof. Let $x \in X$ be an arbitrary element. We evaluate the left-hand side of the second De Morgan identity:

$$\mu_{(A\cap B)^C}(x) = 1 - \mu_{A\cap B}(x)$$
$$\mu_{(A\cap B)^C}(x) = 1 - \min\{\mu_A(x), \mu_B(x)\}$$

Using the fundamental identity of real numbers where $1-\min\{a,b\} = \max\{1-a, 1-b\}$, we transition as follows:

$$1 - \min\{\mu_A(x), \mu_B(x)\} = \max\{1 - \mu_A(x), 1 - \mu_B(x)\}$$
$$= \max\{\mu_{A^C}(x), \mu_{B^C}(x)\}$$
$$= \mu_{A^C \cup B^C}(x)$$

This completes the point-wise proof for the entire discourse space. □

1.1.5 The Structural Boundedness of Fuzzy Lattices

From a structural perspective, the set of all fuzzy subsets over X, denoted $\mathrm{FS}(X)$, forms a bounded, distributive lattice under the operations of min and max, expressed as $\langle \mathrm{FS}(X), \subseteq, \cap, \cup \rangle$ [3, 4]. The greatest element of this lattice is the universal set $\mathbf{1}_X$ where $\mu(x) = 1$, and the least element is the empty set $\mathbf{0}_X$ where $\mu(x) = 0$.

The failure of the complemented lattice properties (i.e., the law of non-contradiction and the law of excluded middle) underscores why fuzzy mathematical topologists must deploy modified separation axioms and rethink the construction of fuzzy continuous mappings.

1.2 Intuitionistic Fuzzy Sets (IFS): The Atanassov Extension

To overcome the rigid dependency inherent in Zadeh's model, where non-membership is completely restricted by membership, Krassimir Atanassov introduced the theory of intuitionistic fuzzy environments [2].

1.2.1 Decoupling Membership Dimensions

Atanassov's fundamental contribution in 1986 was the complete decoupling of positive and negative evaluation parameters, allowing both components to be chosen independently within the unit interval, subject only to a collective linear boundary condition [2, 6].

Definition 1.5 (Intuitionistic Fuzzy Set Theory)**.** *Let X be a non-empty universe of discourse. An* ***Intuitionistic Fuzzy Set*** *B in X is defined as an object containing triplets of the form [2]:*

$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$$

where $\mu_B : X \to [0,1]$ represents the degree of positive membership and $\nu_B : X \to [0,1]$ represents the degree of negative non-membership. For the set to be mathematically consistent, the following linear constraint must hold for all $x \in X$:

$$0 \leq \mu_B(x) + \nu_B(x) \leq 1$$

Definition 1.6 (Intuitionistic Index)**.** *For any intuitionistic fuzzy set B over X, the* ***intuitionistic index*** *(or hesitation margin) $\pi_B(x)$ represents the remaining unallocated uncertainty or lack of definitive consensus [2, 4]. It is evaluated as:*

$$\pi_B(x) = 1 - \mu_B(x) - \nu_B(x)$$

The introduction of the intuitionistic index significantly enhanced the modeling capability of fuzzy frameworks, proving highly effective in automated control systems, pattern recognition, and medical diagnostics [1, 2, 4, 5].

1.2.2 Standard Intuitionistic Operators and Lattice Structuring

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle\}$ be two intuitionistic fuzzy sets on X. The operations are formalized as follows [2, 5]:

Definition 1.7 (Inclusion and Equality)**.** *$A \subseteq B \iff \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$ [2]. Equality holds when both inclusion conditions are mutually satisfied.*

Definition 1.8 (Lattice Operations and Complement)**.** *The standard meet, join, and complement operators are defined as [2, 5, 6]:*

$$A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\}$$

$$A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\}$$
$$A^C = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$$

1.2.3 Algebraic Proof of the Intuitionistic Intersection Bound

We must verify that executing an intersection operator on two valid intuitionistic fuzzy sets yields an object that strictly adheres to the linear constraints of Atanassov's environment [2].

Theorem 1.1. *Let A and B be two valid intuitionistic fuzzy sets over X. Then $A \cap B$ satisfies the condition $0 \leq \mu_{A\cap B}(x) + \nu_{A\cap B}(x) \leq 1$ for all $x \in X$.*

Proof. Let $x \in X$ be an arbitrary element. By definition, the intersection yields [2]:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

$$\nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\}$$

We test the sum of these parts:

$$S = \min\{\mu_A(x), \mu_B(x)\} + \max\{\nu_A(x), \nu_B(x)\}$$

There are two analytical cases based on the minimality of the positive components:

1. **Case 1:** Assume $\mu_A(x) \leq \mu_B(x)$. Thus, $\min\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$.

$$S = \mu_A(x) + \max\{\nu_A(x), \nu_B(x)\}$$

$$S = \max\{\mu_A(x) + \nu_A(x),\ \mu_A(x) + \nu_B(x)\}$$

   Since A is valid, $\mu_A(x) + \nu_A(x) \leq 1$. Since $\mu_A(x) \leq \mu_B(x)$, it follows that $\mu_A(x) + \nu_B(x) \leq \mu_B(x) + \nu_B(x) \leq 1$. Thus, $S \leq 1$.

2. **Case 2:** Assume $\mu_B(x) < \mu_A(x)$. Thus, $\min\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$.

$$S = \mu_B(x) + \max\{\nu_A(x), \nu_B(x)\}$$

$$S = \max\{\mu_B(x) + \nu_A(x),\ \mu_B(x) + \nu_B(x)\}$$

   Since B is valid, $\mu_B(x) + \nu_B(x) \leq 1$. Since $\mu_B(x) < \mu_A(x)$, it follows that $\mu_B(x) + \nu_A(x) \leq \mu_A(x) + \nu_A(x) \leq 1$. Thus, $S \leq 1$.

In all cases, the linear boundary constraint is satisfied, proving the mathematical consistency of the intuitionistic lattice intersection. □

1.2.4 Geometric Interpretation of Atanassov's Space

Geometrically, while a classical fuzzy set assigns values along a single continuous segment $[0, 1]$, an intuitionistic fuzzy set maps each element to a right-angled triangle in a two-dimensional Euclidean plane. The vertices of this triangle are defined by $(0, 0)$, $(1, 0)$, and $(0, 1)$. Any point (μ, ν) falling within this triangular region represents a valid intuitionistic state. The distance from the point to the bounding hypotenuse $\mu + \nu = 1$ corresponds precisely to the hesitation index π. This geometric structure allows for a more advanced interpretation of containment and proximity metrics compared to standard fuzzy spaces.

1.2.5 The Epistemic Limitations of IFS

Despite the mathematical versatility of Atanassov's IFS, real-world human decision-making often presents cognitive states that cannot be accurately represented by just two independent dimensions [6]. Consider a concrete example involving a democratic voting council or an opinion poll consisting of 100 experts evaluating a complex corporate merger proposal. The responses can be categorized as follows:

1. 45 experts vote **in favor** of the proposal.
2. 25 experts vote **against** the proposal.
3. 20 experts choose to remain **neutral**, intentionally abstaining from selecting a side but participating in the forum.
4. 10 experts completely refuse to participate, presenting an absolute **refusal**.

If we attempt to model this scenario using Atanassov's IFS theory, the positive membership is mapped to $\mu = 0.45$, and the negative non-membership is mapped to $\nu = 0.25$. Consequently, the hesitation margin is automatically calculated as $\pi = 1 - 0.45 - 0.25 = 0.30$ [2].

This mathematical compression merges the 20 neutral voters and the 10 absolute refusers into a single parameter $\pi = 0.30$. This causes a substantial loss of informational granularity, as neutrality (active abstention) and refusal (passive non-participation) represent fundamentally different cognitive states. To preserve these critical distinctions, a more comprehensive framework was required.

1.3 Picture Fuzzy Sets (PFS): The Complete Architecture

In 2013, Cuong addressed the multi-dimensional limitations of previous extensions by introducing the theory of **Picture Fuzzy Sets** (PFS) [6]. PFS expands the evaluation space into four distinct parameters: positive membership, neutral membership, negative non-membership, and refusal degree.

Definition 1.9 (Picture Fuzzy Set). *Let X be a non-empty universal set of discourse. A **Picture Fuzzy Set** A in X is an object structurally defined as [6]:*

$$A = \{\langle x, \mu_A(x), \eta_A(x), \gamma_A(x)\rangle \mid x \in X\}$$

*where $\mu_A(x) : X \to [0, 1]$ represents the degree of **positive membership**, $\eta_A(x) : X \to [0, 1]$ represents the degree of **neutral membership**, and $\gamma_A(x) : X \to [0, 1]$ represents the degree of **negative membership** (non-membership). These continuous functions satisfy the following constraint for all $x \in X$:*

$$0 \le \mu_A(x) + \eta_A(x) + \gamma_A(x) \le 1$$

Definition 1.10 (Refusal Margin). *For every picture fuzzy set A over X, the **degree of refusal membership** of any element $x \in X$ is an explicitly quantified parameter defined as [6]:*

$$\delta_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \gamma_A(x))$$

*The vector $(\mu_A(x), \eta_A(x), \gamma_A(x))$ provides a comprehensive representation of the uncertainty state of x. For analytical convenience, this structural triplet is termed a **Picture Fuzzy Number** (PFN) [4].*

1.3.1 Axiomatic Extremes and Boundary Entities

To establish a stable mathematical topology and lattice configuration, we define the absolute boundary sets within the picture fuzzy domain [6].

Definition 1.11 (Absolute Empty PFS). *The absolute empty picture fuzzy set over X, denoted by $\emptyset$, represents a state of total negative alignment and non-containment. It is defined for all $x \in X$ as:*

$$\emptyset = \{\langle x, 0, 0, 1\rangle \mid x \in X\}$$

Definition 1.12 (Absolute Universal PFS). *The absolute universal picture fuzzy set over X, denoted by X, represents a state of total positive alignment and complete containment. It is defined for all $x \in X$ as:*

$$X = \{\langle x, 1, 0, 0\rangle \mid x \in X\}$$

1.3.2 Standard Algebraic Relations and Set-Theoretic Operators

Let A and B be two picture fuzzy sets over the universe X ($A, B \in \mathrm{PFS}(X)$). Following the core operational baseline proposed by Cuong [6], the relations and operations are defined component-wise as follows:

Definition 1.13 (Partial Order / Containment). *A picture fuzzy set A is a subset of B, denoted by $A \subseteq B$, if and only if for all $x \in X$:*

$$\mu_A(x) \leq \mu_B(x), \quad \eta_A(x) \leq \eta_B(x), \quad \textit{and} \quad \gamma_A(x) \geq \gamma_B(x)$$

Definition 1.14 (Standard Intersection). *The standard intersection of A and B, denoted by $A \cap B$, is defined by taking the point-wise minima of the membership profiles and the maximum of the negative component [6]:*

$$A \cap B = \{\langle x, \min\{\mu_A, \mu_B\}, \min\{\eta_A, \eta_B\}, \max\{\gamma_A, \gamma_B\}\rangle \mid x \in X\}$$

Definition 1.15 (Standard Union). *The standard union of A and B, denoted by $A \cup B$, is defined by taking the point-wise maxima of the positive components and the minima of the remaining coordinates [6]:*

$$A \cup B = \{\langle x, \max\{\mu_A, \mu_B\}, \min\{\eta_A, \eta_B\}, \min\{\gamma_A, \gamma_B\}\rangle \mid x \in X\}$$

Definition 1.16 (Standard Complement). *The standard complement of a picture fuzzy set A, denoted by A^C, is defined by swapping the positive and negative membership components while keeping neutrality invariant [6]:*

$$A^C = \{\langle x, \gamma_A(x), \eta_A(x), \mu_A(x)\rangle \mid x \in X\}$$

1.3.3 Rigorous Algebraic Proofs of Distributive and De Morgan Laws

We now verify the core algebraic properties of these operators to establish their structural consistency within the picture fuzzy domain [6].

Theorem 1.2 (Distributive Laws). *Let $A, B, C \in PFS(X)$. Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

Proof. We prove the distributive law component-wise for an arbitrary element $x \in X$. Let $L = A \cup (B \cap C)$ and $R = (A \cup B) \cap (A \cup C)$. Evaluating the positive membership component $\mu_L(x)$ of the left-hand side:

$$\mu_L(x) = \max\{\mu_A(x), \min\{\mu_B(x), \mu_C(x)\}\}$$

Using the distributivity of the real number lattice under max and min:

$$\mu_L(x) = \min\{\max\{\mu_A(x), \mu_B(x)\}, \max\{\mu_A(x), \mu_C(x)\}\} = \mu_R(x)$$

Evaluating the neutral membership component $\eta_L(x)$:

$$\eta_L(x) = \min\{\eta_A(x), \min\{\eta_B(x), \eta_C(x)\}\} = \min\{\eta_A(x), \eta_B(x), \eta_C(x)\}$$

For the right-hand side component $\eta_R(x)$:

$$\eta_R(x) = \min\{\min\{\eta_A(x), \eta_B(x)\}, \min\{\eta_A(x), \mu_C(x)\}\} = \min\{\eta_A(x), \eta_B(x), \eta_C(x)\}$$

Thus, $\eta_L(x) = \eta_R(x)$ holds identically. Evaluating the negative membership component $\gamma_L(x)$:

$$\gamma_L(x) = \min\{\gamma_A(x), \max\{\gamma_B(x), \gamma_C(x)\}\}$$

Applying the distributive rules for the dual lattice operations:

$$\gamma_L(x) = \max\{\min\{\gamma_A(x), \gamma_B(x)\}, \min\{\gamma_A(x), \gamma_C(x)\}\} = \gamma_R(x)$$

Since all components match identically for all $x \in X$, it follows that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

Theorem 1.3 (De Morgan's Laws). *Let $A, B \in PFS(X)$. Then $(A \cup B)^C = A^C \cap B^C$ and $(A \cap B)^C = A^C \cup B^C$.*

Proof. We prove the identity $(A \cap B)^C = A^C \cup B^C$. Let $M = A \cap B$. The components of M are [6]:

$$M = \{\langle x, \min\{\mu_A, \mu_B\}, \min\{\eta_A, \eta_B\}, \max\{\gamma_A, \gamma_B\}\rangle\}$$

Applying the standard complement definition to M, we swap the first and third coordinates:

$$M^C = \{\langle x, \max\{\gamma_A, \gamma_B\}, \min\{\eta_A, \eta_B\}, \min\{\mu_A, \mu_B\}\rangle\}$$

Now, evaluate the right-hand side. The complements are:

$$A^C = \{\langle x, \gamma_A, \eta_A, \mu_A\rangle\}, \quad B^C = \{\langle x, \gamma_B, \eta_B, \mu_B\rangle\}$$

Computing the standard union $A^C \cup B^C$:

$$A^C \cup B^C = \{\langle x, \max\{\gamma_A, \gamma_B\}, \min\{\eta_A, \eta_B\}, \min\{\mu_A, \mu_B\}\rangle\}$$

The components match M^C identically for every $x \in X$, completing the proof. □

1.3.4 Advanced Arithmetic Operators: Algebraic Sum and Product

Beyond standard lattice-based operators, picture fuzzy numbers can be processed using algebraic sum and product operations [4, 6].

Definition 1.17 (Algebraic Sum). *Let $\alpha = (\mu_\alpha, \eta_\alpha, \gamma_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \gamma_\beta)$ be two picture fuzzy numbers. The* ***algebraic sum*** *$\alpha \oplus \beta$ is defined as:*

$$\alpha \oplus \beta = (\mu_\alpha + \mu_\beta - \mu_\alpha\mu_\beta,\ \eta_\alpha\eta_\beta,\ \gamma_\alpha\gamma_\beta)$$

Definition 1.18 (Algebraic Product)**.** *The* ***algebraic product*** $\alpha \otimes \beta$ *is defined as:*

$$\alpha \otimes \beta = (\mu_\alpha \mu_\beta,\ \eta_\alpha + \eta_\beta - \eta_\alpha \eta_\beta,\ \gamma_\alpha + \gamma_\beta - \gamma_\alpha \gamma_\beta)$$

Theorem 1.4. *The algebraic sum [6] $\alpha \oplus \beta$ of two valid picture fuzzy numbers is itself a valid picture fuzzy number.*

Proof. Let $\alpha = (\mu_\alpha, \eta_\alpha, \gamma_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \gamma_\beta)$ such that $\mu_\alpha + \eta_\alpha + \gamma_\alpha \leq 1$ and $\mu_\beta + \eta_\beta + \gamma_\beta \leq 1$. Let the components of $\alpha \oplus \beta$ be denoted by $\mu_{\text{sum}}, \eta_{\text{sum}}, \gamma_{\text{sum}}$. We must prove that $\mu_{\text{sum}} + \eta_{\text{sum}} + \gamma_{\text{sum}} \leq 1$.

$$\mu_{\text{sum}} + \eta_{\text{sum}} + \gamma_{\text{sum}} = \mu_\alpha + \mu_\beta - \mu_\alpha \mu_\beta + \eta_\alpha \eta_\beta + \gamma_\alpha \gamma_\beta$$

From the baseline constraints, we know that $\eta_\alpha \leq 1 - \mu_\alpha - \gamma_\alpha$ and $\eta_\beta \leq 1 - \mu_\beta - \gamma_\beta$. Substituting these upper bounds yields:

$$\eta_\alpha \eta_\beta \leq (1 - \mu_\alpha - \gamma_\alpha)(1 - \mu_\beta - \gamma_\beta)$$

Expanding the right-hand side product explicitly:

$$\eta_\alpha \eta_\beta \leq 1 - \mu_\beta - \gamma_\beta - \mu_\alpha + \mu_\alpha \mu_\beta + \mu_\alpha \gamma_\beta - \gamma_\alpha + \gamma_\alpha \mu_\beta + \gamma_\alpha \gamma_\beta$$

Rearranging this inequality to isolate the terms matching our sum equation:

$$\mu_\alpha + \mu_\beta - \mu_\alpha \mu_\beta + \eta_\alpha \eta_\beta + \gamma_\alpha \gamma_\beta \leq 1 - \gamma_\beta - \gamma_\alpha + \mu_\alpha \gamma_\beta + \gamma_\alpha \mu_\beta + 2\gamma_\alpha \gamma_\beta$$

Factoring the remaining terms shows that:

$$1 - \gamma_\alpha(1 - \mu_\beta - \gamma_\beta) - \gamma_\beta(1 - \mu_\alpha - \gamma_\alpha) \leq 1$$

Thus, the sum constraint is satisfied, ensuring that $\alpha \oplus \beta$ is structurally valid. $\square$

1.3.5 Comparison Metrics: Score and Accuracy Functions

To establish a total ordering relation over the picture fuzzy space, we introduce comparison metrics [1, 4].

Definition 1.19 (Score Function)**.** *Let $\alpha = (\mu_\alpha, \eta_\alpha, \gamma_\alpha)$ be a picture fuzzy number. The* ***score function*** $S(\alpha)$ *is defined as:*

$$S(\alpha) = \mu_\alpha - \gamma_\alpha$$

Definition 1.20 (Accuracy Function)**.** *The* ***accuracy function*** $H(\alpha)$ *is defined as:*

$$H(\alpha) = \mu_\alpha + \eta_\alpha + \gamma_\alpha$$

Using these metrics, the ranking comparison between any two picture fuzzy numbers α and β is governed by the following rules:

- If $S(\alpha) > S(\beta)$, then $\alpha > \beta$.
- If $S(\alpha) = S(\beta)$, we evaluate their accuracy profiles:
  - If $H(\alpha) > H(\beta)$, then $\alpha > \beta$.
  - If $H(\alpha) = H(\beta)$, then $\alpha = \beta$.

1.3.6 Distance Measures and Geometric Analysis

To quantify the proximity or similarity between two picture fuzzy sets, we define geometric metrics [4].

Definition 1.21 (Normalized Euclidean Distance)**.** *Let $A, B \in PFS(X)$ over a finite universe $X = \{x_1, x_2, \ldots, x_n\}$. The* ***Normalized Euclidean Distance*** *$d_E(A, B)$ is defined as:*

$$d_E(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}[(\mu_A(x_i) - \mu_B(x_i))^2 + (\eta_A(x_i) - \eta_B(x_i))^2 + (\gamma_A(x_i) - \gamma_B(x_i))^2 + (\delta_A(x_i) - \delta_B(x_i))^2}$$

where $\delta(x) = 1 - (\mu(x) + \eta(x) + \gamma(x))$ represents the calculated refusal component.

1.3.7 Comprehensive Numerical Applications

To demonstrate the mathematical operations developed in this chapter, we present an exhaustive numerical application analyzing two alternative technical solutions.

Example 1.1. *Let $X = \{x_1, x_2\}$ be a universe containing two manufacturing systems evaluated under multi-expert criteria. Suppose two expert panels provide their assessments as picture fuzzy sets A and B:*

$$A = \{\langle x_1, 0.6, 0.2, 0.1\rangle,\ \langle x_2, 0.4, 0.3, 0.2\rangle\}$$

$$B = \{\langle x_1, 0.5, 0.3, 0.1\rangle,\ \langle x_2, 0.5, 0.2, 0.2\rangle\}$$

First, we compute the refusal degrees $\delta(x)$ for all components:

$$\delta_A(x_1) = 1 - (0.6 + 0.2 + 0.1) = 0.10$$

$$\delta_A(x_2) = 1 - (0.4 + 0.3 + 0.2) = 0.10$$

$$\delta_B(x_1) = 1 - (0.5 + 0.3 + 0.1) = 0.10$$

$$\delta_B(x_2) = 1 - (0.5 + 0.2 + 0.2) = 0.10$$

Second, we execute the standard lattice operators:

- ***Standard Intersection** ($A \cap B$)**:***

$$A \cap B = \{\langle x_1, 0.5, 0.2, 0.1\rangle,\ \langle x_2, 0.4, 0.2, 0.2\rangle\}$$

- ***Standard Union** ($A \cup B$)**:***

$$A \cup B = \{\langle x_1, 0.6, 0.2, 0.1\rangle,\ \langle x_2, 0.5, 0.2, 0.2\rangle\}$$

- ***Standard Complement*** *(A^C):*

$$A^C = \{\langle x_1, 0.1, 0.2, 0.6\rangle,\ \langle x_2, 0.2, 0.3, 0.4\rangle\}$$

Third, we calculate the continuous algebraic sum $A \oplus B$: For x_1: $\mu_{sum} = 0.80, \eta_{sum} = 0.06, \gamma_{sum} = 0.01$. For x_2: $\mu_{sum} = 0.70, \eta_{sum} = 0.06, \gamma_{sum} = 0.04$.

$$A \oplus B = \{\langle x_1, 0.80, 0.06, 0.01\rangle,\ \langle x_2, 0.70, 0.06, 0.04\rangle\}$$

Fourth, we determine the relative dominance ranking between alternative A and alternative B:

$$S(A(x_1)) = 0.6 - 0.1 = 0.50, \quad S(B(x_1)) = 0.5 - 0.1 = 0.40 \implies A(x_1) > B(x_1)$$

$$S(A(x_2)) = 0.4 - 0.2 = 0.20, \quad S(B(x_2)) = 0.5 - 0.2 = 0.30 \implies B(x_2) > A(x_2)$$

Evaluating the comprehensive mean score vector:

$$S_{mean}(A) = \frac{0.50 + 0.20}{2} = 0.35, \quad S_{mean}(B) = \frac{0.40 + 0.30}{2} = 0.35$$

Since the mean scores are equal, we evaluate their accuracy profiles:

$$H_{mean}(A) = 0.90, \quad H_{mean}(B) = 0.90$$

Since both functions are equal, sets A and B convey structurally equivalent levels of decision certainty.

1.3.8 Comparative Structural Analysis Across Domains

To fully synthesize the theoretical progression presented in this chapter, it is essential to contrast the structural capacities of Fuzzy Sets, Intuitionistic Fuzzy Sets, and Picture Fuzzy Sets. Each iteration expands the descriptive parameters available to model real-world epistemic gaps.

When dealing with simple scenarios involving gray areas or standard linguistic approximations, Zadeh's classic Fuzzy Set framework remains the most efficient choice because it requires only a single parameter. However, when an evaluation system must account for active hesitation or incomplete data streams, Atanassov's Intuitionistic model provides the necessary decoupling of positive and negative bounds.

Finally, for complex modern applications involving voting networks, multi-criteria expert panels, or medical data classification where neutral standpoints and absolute refusals are distinct cognitive decisions, the four-dimensional architecture of Cuong's Picture Fuzzy Sets becomes mandatory.

The algebraic structures, operations, and optimization metrics analyzed in this chapter establish the necessary foundation for developing complex configurations, such as Picture Fuzzy Topological Spaces, which will be detailed in the subsequent parts of this thesis.

Chapter 2

Fuzzy Topology Spaces

Compared to conventional set theory, which generalizes different topological ideas, Fuzzy set theory is perceived as offering us a significantly wider framework for structural analysis. The topological structure and the ordered lattice layout are seamlessly combined within Fuzzy topology. This sophisticated field of modern mathematics was initially introduced from a pure mathematical perspective by the renowned mathematician Ehrenman, who included the two most prominent aspects of topology on lattices that continuously influence one another.

Subsequently, C. L. Chang was the pioneering researcher who initially created the explicit notation and axiomatic foundation of fuzzy topology in 1968. Following this breakthrough, many researchers carried out additional research in this field to establish deeper generalizations. As noted in contemporary literature [13, 14], fuzzy topology is thought of as a specific, highly expressive example of generic point set topology, where classical membership functions are modified from static binary characteristic functions into continuous multi-valued mappings.

The ongoing synthesis of fuzzy structures with topological systems has facilitated major breakthroughs in various applied sciences, including automated system classification, advanced data topology, optimization modeling, and spatial representation algorithms. This chapter provides a rigorous mathematical exposition of Chang's historical framework, thoroughly detailing its interior mechanisms, base dynamics, convergence theories, continuity criteria, and advanced separation metrics up to foundational modern extensions.

2.1 Fuzzy Topologies: Fundamental Definitions and Structural Systems

In this mathematical section, we present the basic foundational notions within a fuzzy topological space. We formally bridge the abstract concept of classical open domains with the paradigm of continuous membership operations. For more details, see [15, 16].

Definition 2.1 (Chang's Fuzzy Topology)**.** *Let X be a non-empty underlying universe of discourse, and let I^X denote the absolute collection of all fuzzy subsets in X mapping into the unit interval $I = [0, 1]$. A family $\mathcal{F} \subseteq I^X$ of fuzzy subsets that complies with the subsequent three structural axioms is said to constitute a valid* ***Fuzzy Topology*** *on X:*

(i) *The constant boundary fuzzy sets* $\mathbf{0}_X$ *(where* $\mathbf{0}_X(x) = 0$ *for all* $x \in X$*) and* $\mathbf{1}_X$ *(where* $\mathbf{1}_X(x) = 1$ *for all* $x \in X$*) must belong entirely to* $\mathcal{F}$*.*

(ii) *The finite intersection of any two fuzzy sets in* $\mathcal{F}$ *must belong to* $\mathcal{F}$*. That is, if* $\mu_1, \mu_2 \in \mathcal{F}$*, then* $(\mu_1 \wedge \mu_2) \in \mathcal{F}$*.*

(iii) *The finite or infinite arbitrary union of any subfamily of fuzzy sets in* $\mathcal{F}$ *must belong to* $\mathcal{F}$*. That is, if* $\{\mu_i : i \in J\} \subseteq \mathcal{F}$ *where* J *denotes an arbitrary index set, then* $\left(\bigvee_{i \in J} \mu_i\right) \in \mathcal{F}$*.*

The family $\mathcal{F}$ *is described as a Fuzzy topology for* X *and the pair* $(X, \mathcal{F})$ *is called a* ***Fuzzy Topological Space*** *(FTS). Every individual member belonging to the family* $\mathcal{F}$ *is referred to as an* $\mathcal{F}$***-open Fuzzy Set****.*

Definition 2.2 (Fuzzy Closed Sets)**.** *Let* $(X, \mathcal{F})$ *be a valid fuzzy topological space. A fuzzy subset* $\lambda \in I^X$ *is defined as an* $\mathcal{F}$***-closed Fuzzy Set*** *if and only if its absolute structural complement* $1 - \lambda$ *is an* $\mathcal{F}$*-open fuzzy set belonging to* $\mathcal{F}$*.*

The "fuzzy" sense in Chang's definition is not always clear because the topology itself is a crisp collection of fuzzy sets; for this, we can also explore how to "fuzzify" this definition of fuzzy topology. Under this fuzzified framework, every fuzzy subset is assigned a degree of openness rather than being strictly open or closed. Symmetrically, let us specify the reference conditions for crisp boundaries: $X = \{\langle x, 1\rangle : x \in X\}$ and $\emptyset = \{\langle x, 0\rangle : x \in X\}$.

Definition 2.3 (Fuzzified Topology)**.** *Let* X *be a non-empty underlying set. A* ***Fuzzified Topology*** *on* X *is defined by a continuous membership function* $\tau : [0,1]^X \to [0,1]$ *where* $A \mapsto \mu_\tau(A)$ *interprets the degree of openness of the fuzzy set* A*, and satisfies the following axiomatic conditions:*

(i) $\mu_\tau(\emptyset) = \mu_\tau(X) = 1$*;*

(ii) $\mu_\tau(A \cap B) \geq \mu_\tau(A) \wedge \mu_\tau(B)$*, for any two fuzzy subsets* $A, B \in [0,1]^X$*;*

(iii) *For any arbitrary family of fuzzy subsets* $\{A_i\}_{i \in J}$*,* $\mu_\tau\left(\bigcup_{i \in J} A_i\right) \geq \bigwedge_{i \in J} \mu_\tau(A_i)$*.*

Example 2.1. *Let* $X = \{a, b\}$*. We can define a fuzzy topology* $\mathcal{F}$ *on* X *by setting* $\mathcal{F} = \{0, 1, \mu\}$*, where the fuzzy set* μ *is explicitly given by its membership values:* $\mu(a) = 0.6$ *and* $\mu(b) = 0.4$*. We verify the axioms step-by-step:*

1. $0, 1 \in \mathcal{F}$ *by definition, satisfying axiom (i) exactly.*
2. *For the intersections:* $0 \wedge \mu = 0 \in \mathcal{F}$*,* $1 \wedge \mu = \mu \in \mathcal{F}$*, and* $\mu \wedge \mu = \mu \in \mathcal{F}$*, verifying axiom (ii).*
3. *For the unions:* $0 \vee \mu = \mu \in \mathcal{F}$*,* $1 \vee \mu = 1 \in \mathcal{F}$*, and* $\mu \vee \mu = \mu \in \mathcal{F}$*, satisfying axiom (iii).*

Thus, $\mathcal{F}$ *is a perfectly valid Chang fuzzy topology on* X*.*

To provide another concrete mathematical realization of these abstract spaces, let us analyze the following parametric family. For a family $\{f_h : 0 < h \leq 1\} \cup \{\emptyset, X\}$ where each function is defined over a bounded interval:

$$f_h(x) = \begin{cases} 2hx & \text{if } x \in [0, \frac{1}{2}] \\ 2h(1-x) & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

The family $\tau = \{f_h : 0 < h \leq 1\} \cup \{\emptyset, X\}$ forms a structured fuzzy topology, and the structured pair (X, τ) represents a valid fuzzy topological space.

Remark 2.1. *Deepening our foundational terminology, we observe that:*

(i) *The indiscrete fuzzy topology contains exclusively the empty fuzzy set* 0 *and the universal reference fuzzy set* 1*;*

(ii) *Conversely, a discrete fuzzy topology is one where every single fuzzy subset of* X *is considered open;*

(iii) *A fuzzy topology* τ_1 *is strictly finer than another fuzzy topology* τ_2 *if and only if the containment relation* $\tau_2 \subseteq \tau_1$ *holds analytically.*

Definition 2.4 (Fuzzy Neighborhood)**.** *A fuzzy set* U *in a fuzzy topological space* $(X, \mathcal{F})$ *is an analytical* ***neighborhood*** *(for short n.b.d.) of a given fuzzy set* A *if and only if there exists an intermediate fuzzy open set* $O \in \mathcal{F}$ *such that the containment chain* $A \subseteq O \subseteq U$ *is satisfied completely.*

Example 2.2. *Let us construct an explicit finite instance. Let* $X = \{u, v, w\}$ *and define the collection* $\mathcal{F} = \{0, 1, \alpha, \beta, \gamma\}$ *where the components of the membership vectors are structured as:* $\alpha = (0.5, 0.1, 1.0)$*,* $\beta = (0.7, 0.1, 0)$*, and* $\gamma = (0.5, 0.1, 0)$*. If we evaluate the partial ordering such that* $\gamma \leq \alpha$*, then* α *acts as a formal Fuzzy n.b.d. of* γ *within* $(X, \mathcal{F})$*.*

Definition 2.5 (Fuzzy Point)**.** *Let* X *be an ordinary reference set and* $x \in X$*. We define the structural* ***fuzzy point*** *denoted by* P_x^λ *(or* x_λ*) by the following explicit localized membership function:*

$$P_x^\lambda(y) = \begin{cases} 0 & \textit{for all } y \in X, y \neq x \\ \lambda & \textit{if } y = x \end{cases}$$

where $\lambda \in (0, 1]$ *represents the non-zero support value or the element's mathematical value.*

Definition 2.6. *Let* X *be a set and* A *a fuzzy subset of* X*. We say that the localized fuzzy point* P_x^λ ***belongs to*** A *if and only if the single-point value satisfies* $P_x^\lambda(x) \leq \mu_A(x)$*. We denote this structural belonging relation by* $P_x^\lambda \in A$*.*

Example 2.3. *Consider the reference set* $X = \{x_1, x_2, x_3\}$ *and a fuzzy subset* A *defined on* X *by the membership vector* $A = (0.8, 0.3, 0.5)$*. Let* $P_{x_1}^{0.7}$ *be a fuzzy point localized at* x_1 *with value* $\lambda = 0.7$*. Since* $P_{x_1}^{0.7}(x_1) = 0.7 \leq A(x_1) = 0.8$*, we conclude that* $P_{x_1}^{0.7} \in A$*. On the other hand, if we take a fuzzy point* $P_{x_2}^{0.6}$ *localized at* x_2*, we see that* $P_{x_2}^{0.6}(x_2) = 0.6 > A(x_2) = 0.3$*, which analytically means that* $P_{x_2}^{0.6} \notin A$*.*

2.2 Properties of Quasi-Coincidence and q-Neighborhood Structural Systems

To establish a more flexible framework for fuzzy separation and continuity, the strict membership relation $\in$ is often replaced or augmented by the concept of quasi-coincidence, originally developed by Pu and Liu. This section details the algebraic properties arising from these structural interactions.

Definition 2.7 (Quasi-Coincidence). *A fuzzy point P_x^λ is said to be **quasi-coincident** with a fuzzy set A, denoted by $P_x^\lambda\ q\ A$, if and only if $\lambda + \mu_A(x) > 1$. Similarly, two fuzzy sets A and B are said to be quasi-coincident, denoted by $A\ q\ B$, if there exists at least one point $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$. If they do not satisfy this condition, we write $A\ \not{q}\ B$, which implies $A \leq B^c$.*

Example 2.4. *Let $X = \{x\}$ and consider two fuzzy subsets A and B defined over X. Let $\mu_A(x) = 0.6$ and $\mu_B(x) = 0.5$. To test for quasi-coincidence, we compute the sum of their memberships at x: $\mu_A(x) + \mu_B(x) = 0.6 + 0.5 = 1.1$. Since $1.1 > 1$, we explicitly have $A\ q\ B$. Now, let us evaluate a fuzzy point $P_x^{0.3}$. We check its interaction with A: $\lambda + \mu_A(x) = 0.3 + 0.6 = 0.9$. Since $0.9 \leq 1$, the fuzzy point is not quasi-coincident with A, denoted by $P_x^{0.3}\ \not{q}\ A$.*

Definition 2.8 (q-Neighborhood). *A fuzzy set U in an f.t.s. $(X, \mathcal{F})$ is called a **q-neighborhood** (for short, q-n.b.d.) of a fuzzy point P_x^λ if and only if there exists a fuzzy open set $O \in \mathcal{F}$ such that $P_x^\lambda\ q\ O \leq U$.*

Theorem 2.1. *A fuzzy point $P_x^\lambda \in \overline{A}$ if and only if every q-neighborhood U of P_x^λ is quasi-coincident with A (i.e., $U\ q\ A$).*

Proof. Suppose $P_x^\lambda \in \overline{A}$ and let U be a q-neighborhood of P_x^λ. Then there exists a fuzzy open set O such that $P_x^\lambda\ q\ O \leq U$. If $O\ \not{q}\ A$, then $A \leq O^c$. Since O is open, O^c is closed, which means $\overline{A} \leq O^c$. Since $P_x^\lambda \in \overline{A}$, we have $P_x^\lambda \in O^c$, which implies $\lambda \leq 1 - O(x)$, or $\lambda + O(x) \leq 1$. This directly contradicts $P_x^\lambda\ q\ O$. Thus, $O\ q\ A$, and since $O \leq U$, we have $U\ q\ A$.

Conversely, suppose every q-neighborhood U of P_x^λ satisfies $U\ q\ A$. Let ρ be any closed fuzzy set containing A ($\rho \geq A$). Then ρ^c is an open fuzzy set. If $P_x^\lambda \notin \rho$, then by definition $\lambda > \rho(x)$, which implies $\lambda + 1 - \rho(x) > 1$, meaning $\lambda + \rho^c(x) > 1$. Thus, ρ^c is a q-neighborhood of P_x^λ. By our initial hypothesis, $\rho^c\ q\ A$, which implies $A \not\leq \rho$, directly contradicting $A \leq \rho$. Therefore, $P_x^\lambda \in \rho$ for all closed sets containing A, which means $P_x^\lambda \in \overline{A}$. □

Theorem 2.2. *Let $(X, \mathcal{F})$ be a fuzzy topological space, P_x^λ a fuzzy point, and $A, B \in [0,1]^X$. The following foundational structural properties hold for the quasi-coincidence relation:*

(i) $A\ q\ B \iff B\ q\ A$.

(ii) $A\ \not{q}\ B \iff A \leq B^c$.

(iii) *If $A \leq B$ and $A\ q\ C$, then $B\ q\ C$.*

(iv) $P_x^\lambda \; q \; (A \vee B) \iff P_x^\lambda \; q \; A$ *or* $P_x^\lambda \; q \; B$.

(v) $A \; q \left(\bigvee_{i \in J} B_i\right) \iff \exists i \in J$ *such that* $A \; q \; B_i$.

Proof. We evaluate each clause step by step:

Proof of (i): By definition, $A \; q \; B \iff \exists x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$. Since addition over the real line is commutative, $\mu_B(x) + \mu_A(x) > 1$, which mathematically indicates that $B \; q \; A$.

Proof of (ii): $A \not{q} \; B \iff \forall x \in X, \mu_A(x) + \mu_B(x) \leq 1 \iff \forall x \in X, \mu_A(x) \leq 1 - \mu_B(x) \iff A \leq B^c$.

Proof of (iii): Given $A \leq B$, we have $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$. Since $A \; q \; C$, there exists $x_0 \in X$ such that $\mu_A(x_0) + \mu_C(x_0) > 1$. Consequently, $\mu_B(x_0) + \mu_C(x_0) \geq \mu_A(x_0) + \mu_C(x_0) > 1$, which explicitly proves $B \; q \; C$.

Proof of (iv): $P_x^\lambda \; q \; (A \vee B) \iff \lambda + \max(\mu_A(x), \mu_B(x)) > 1 \iff \lambda + \mu_A(x) > 1$ or $\lambda + \mu_B(x) > 1 \iff P_x^\lambda \; q \; A$ or $P_x^\lambda \; q \; B$.

Proof of (v): $A \; q \left(\bigvee_{i \in J} B_i\right) \iff \exists x \in X$ such that $\mu_A(x) + \sup_{i \in J} \mu_{B_i}(x) > 1 \iff \exists x \in X, \exists i \in J$ such that $\mu_A(x) + \mu_{B_i}(x) > 1 \iff \exists i \in J$ such that $A \; q \; B_i$.

□

2.3 Core Operations on Fuzzy Topological Spaces: Analytical Operators

In this section, we study the fundamental operators that drive the internal algebraic structure of fuzzy topologies: Closure, Interior, Boundary, and Exterior.

2.3.1 The Closure Operation

The smallest closed Fuzzy set containing μ is known as the closure of μ, denoted by $\overline{\mu}$. It is explicitly defined as:

$$\overline{\mu} = \bigwedge \{\rho : \rho \text{ is } \mathcal{F}\text{-closed and } \rho \geq \mu\}$$

For all $\mu, \lambda \in [0,1]^X$, the closure operator satisfies the standard Kuratowski structural axioms: $\mu \leq \overline{\mu}$ (Extensivity), $\overline{\mu \vee \lambda} = \overline{\mu} \vee \overline{\lambda}$ (Additivity), $\overline{0} = 0$ (Preservation of empty set), and $\overline{\overline{\mu}} = \overline{\mu}$ (Idempotency).

2.3.2 The Interior Operation

The largest open Fuzzy set contained in μ is called the interior of μ, denoted by μ°. It is explicitly defined as:

$$\mu^\circ = \bigvee \{\gamma : \gamma \text{ is } \mathcal{F}\text{-open and } \gamma \leq \mu\}$$

Example 2.5. *Let $X = \{x_1, x_2\}$ and let $\mathcal{F} = \{0, 1, \gamma\}$ be a fuzzy topology on X where $\gamma = (0.6, 0.4)$. Let us choose a fuzzy set $\mu = (0.7, 0.5)$. To compute μ°, we seek all open fuzzy sets contained within μ. The open sets are 0, 1, and γ. We check containment: $0 \leq \mu$ (True), $\gamma \leq \mu$ since $0.6 \leq 0.7$ and $0.4 \leq 0.5$ (True), but $1 \not\leq \mu$. Therefore, the open sets contained in μ are $\{0, \gamma\}$. Taking their supremum gives $\mu^\circ = 0 \vee \gamma = \gamma = (0.6, 0.4)$. To find the closure $\overline{\mu}$, we look for all closed sets containing μ. The closed sets are 1, 0, and $\gamma^c = (0.4, 0.6)$. The only closed set greater than or equal to μ is 1. Thus, $\overline{\mu} = 1$.*

Theorem 2.3. *For any fuzzy sets μ, ν in $(X, \mathcal{F})$, the interior operation satisfies the following structural properties: (i) $0^\circ = 0,\ 1^\circ = 1$; (ii) $\mu^\circ \leq \mu$; (iii) $(\mu^\circ)^\circ = \mu^\circ$; (iv) $(\mu \wedge \nu)^\circ = \mu^\circ \wedge \nu^\circ$.*

Proof. Properties (i) and (ii) follow directly from definitions. Property (iii) is valid because μ° is open, so the largest open set contained within it is itself. For (iv), since $\mu \wedge \nu \leq \mu$ and $\mu \wedge \nu \leq \nu$, by the monotonic nature of the operator we have $(\mu \wedge \nu)^\circ \leq \mu^\circ$ and $(\mu \wedge \nu)^\circ \leq \nu^\circ$, hence $(\mu \wedge \nu)^\circ \leq \mu^\circ \wedge \nu^\circ$. Conversely, since $\mu^\circ \leq \mu$ and $\nu^\circ \leq \nu$, we have $\mu^\circ \wedge \nu^\circ \leq \mu \wedge \nu$. Since the finite intersection of open sets is open, $\mu^\circ \wedge \nu^\circ$ is an open set contained in $\mu \wedge \nu$, meaning it must be less than or equal to the largest open set contained within it, which is $(\mu \wedge \nu)^\circ$. This confirms the identity. $\square$

2.3.3 Duality and Dual Proofs of Core Operations

The fundamental dual relationship connecting the closure and interior operators is expressed through the fuzzy complementation rule.

Theorem 2.4. *Let μribbons be any arbitrary fuzzy set in $(X, \mathcal{F})$. Then: (i) $(\mu^\circ)^c = \overline{(\mu^c)}$ and (ii) $(\overline{\mu})^c = (\mu^c)^\circ$.*

Proof. Let us prove part (i). By definition, the interior operator is given by $\mu^\circ = \bigvee\{O : O \in \mathcal{F} \text{ and } O \leq \mu\}$. Taking the fuzzy complement of both sides yields:

$$(\mu^\circ)^c = \left(\bigvee\{O : O \in \mathcal{F} \text{ and } O \leq \mu\}\right)^c$$

Applying the generalized De Morgan's laws for the supremum/infimum operations, we rewrite this expression as:

$$(\mu^\circ)^c = \bigwedge\{O^c : O \in \mathcal{F} \text{ and } O^c \geq \mu^c\}$$

Since a fuzzy set O is open if and only if its complement O^c is closed, we can introduce a substitution variable $\rho = O^c$, where ρ is a closed fuzzy set. This gives:

$$(\mu^\circ)^c = \bigwedge\{\rho : \rho \text{ is closed and } \rho \geq \mu^c\}$$

By referencing the structural definition of the closure operator, the right-hand side is exactly the closure of the complement. Thus, $(\mu^\circ)^c = \overline{(\mu^c)}$. Property (ii) is established symmetrically by replacing μ with its complement μ^c. $\square$

2.3.4 The Fuzzy Boundary Operation

Definition 2.9 (Fuzzy Boundary)**.** *The* ***fuzzy boundary*** *of a fuzzy set* μ*, denoted by* $\partial\mu$*, represents the region where the set and its complement overlap under the closure operator:* $\partial\mu = \overline{\mu} \wedge \overline{(\mu^c)}$*.*

Theorem 2.5. *For any fuzzy set* $\mu \in [0,1]^X$*, the boundary satisfies the following identity:* $\partial\mu = \overline{\mu} \wedge (\mu^\circ)^c$*.*

Proof. From the structural formulation of the boundary, we have $\partial\mu = \overline{\mu} \wedge \overline{(\mu^c)}$. Applying the duality established in the previous theorem, we replace $\overline{(\mu^c)}$ with $(\mu^\circ)^c$. Substituting this identity directly yields $\partial\mu = \overline{\mu} \wedge (\mu^\circ)^c$. □

Theorem 2.6. *For any two fuzzy subsets* $\mu, \nu \in [0,1]^X$*, the boundary of their union satisfies the following property:* $\partial(\mu \vee \nu) \leq \partial\mu \vee \partial\nu$*.*

Proof. Expanding the definition for the union yields $\partial(\mu\vee\nu) = \overline{(\mu \vee \nu)} \wedge ((\mu\vee\nu)^\circ)^c$. Using the additivity property of the closure operator, we know that $\overline{\mu \vee \nu} = \overline{\mu} \vee \overline{\nu}$. Concurrently, using the monotonic property of the interior operator, we have $(\mu \vee \nu)^\circ \geq \mu^\circ \vee \nu^\circ$, which translates via complements to $((\mu\vee\nu)^\circ)^c \leq (\mu^\circ)^c \wedge (\nu^\circ)^c$. Combining these containment constraints gives $\partial(\mu \vee \nu) \leq (\overline{\mu} \vee \overline{\nu}) \wedge (\mu^\circ)^c \wedge (\nu^\circ)^c$. Distributing the intersection over the logical disjunction yields: $\partial(\mu\vee\nu) \leq (\overline{\mu}\wedge(\mu^\circ)^c\wedge(\nu^\circ)^c)\vee(\overline{\nu}\wedge(\mu^\circ)^c\wedge(\nu^\circ)^c)$. Since $\overline{\mu}\wedge(\mu^\circ)^c\wedge(\nu^\circ)^c \leq \overline{\mu}\wedge(\mu^\circ)^c = \partial\mu$ and $\overline{\nu} \wedge (\mu^\circ)^c \wedge (\nu^\circ)^c \leq \overline{\nu} \wedge (\nu^\circ)^c = \partial\nu$, we obtain $\partial(\mu \vee \nu) \leq \partial\mu \vee \partial\nu$. □

2.3.5 The Fuzzy Exterior Operation

Definition 2.10 (Fuzzy Exterior)**.** *The* ***fuzzy exterior*** *of a fuzzy set* μ*, denoted by* $Ext(\mu)$*, is defined as the interior of its complement:* $Ext(\mu) = (\mu^c)^\circ$*.*

Theorem 2.7. *For any fuzzy sets* $\mu, \nu \in [0,1]^X$*, the exterior operator satisfies the following foundational structural properties: (i)* $Ext(0) = 1$ *and* $Ext(1) = 0$*; (ii)* $Ext(\mu) \leq \mu^c$*; (iii)* $Ext(\mu \vee \nu) = Ext(\mu) \wedge Ext(\nu)$*; (iv)* $Ext(\mu \wedge \nu) \geq Ext(\mu) \vee Ext(\nu)$*.*

Proof. We evaluate each property directly:

Proof of (i): $Ext(0) = (0^c)^\circ = 1^\circ = 1$. Similarly, $Ext(1) = (1^c)^\circ = 0^\circ = 0$.

Proof of (ii): By definition of the interior operator, $(\mu^c)^\circ \leq \mu^c$. Hence, $Ext(\mu) \leq \mu^c$.

Proof of (iii): Using De Morgan's laws and the properties of the interior operator: $Ext(\mu \vee \nu) = ((\mu \vee \nu)^c)^\circ = (\mu^c \wedge \nu^c)^\circ = (\mu^c)^\circ \wedge (\nu^c)^\circ = Ext(\mu) \wedge Ext(\nu)$.

Proof of (iv): Since $\mu\wedge\nu \leq \mu$ and $\mu\wedge\nu \leq \nu$, taking complements gives $\mu^c \leq (\mu\wedge\nu)^c$ and $\nu^c \leq (\mu\wedge\nu)^c$. Thus, $\mu^c \vee \nu^c \leq (\mu \wedge \nu)^c$. Applying the interior operator on both sides establishes the property.

□

2.4 Decomposition Operations: Level Topologies and α-cuts

An elegant decomposition operation used to connect classical point-set topology with fuzzy topology is through the use of α-cuts, which translate fuzzy structures into nested collections of crisp structures.

Definition 2.11 (α-cut of a Fuzzy Set)**.** *Let $\mu \in [0,1]^X$ and $\alpha \in [0,1]$. The strong α-cut $\mu_{\alpha+}$ and weak α-cut μ_α are defined respectively as: $\mu_{\alpha+} = \{x \in X : \mu(x) > \alpha\}$ and $\mu_\alpha = \{x \in X : \mu(x) \geq \alpha\}$.*

Example 2.6. *Let $X = \{x_1, x_2, x_3\}$ and let μ be a fuzzy subset defined by $\mu = (1.0, 0.65, 0.40)$. Let us find the weak and strong α-cuts for $\alpha = 0.65$:*

- *The strong α-cut: $\mu_{0.65+} = \{x \in X : \mu(x) > 0.65\} = \{x_1\}$.*
- *The weak α-cut: $\mu_{0.65} = \{x \in X : \mu(x) \geq 0.65\} = \{x_1, x_2\}$.*

Theorem 2.8. *Let $(X, \mathcal{F})$ be a Chang fuzzy topological space. For each $\alpha \in [0,1)$, the collection $\mathcal{F}_\alpha = \{\mu_{\alpha+} : \mu \in \mathcal{F}\}$ forms a classical crisp topology on the set X.*

Proof. We verify the three standard topological axioms for $\mathcal{F}_\alpha$: (1) Since $0, 1 \in \mathcal{F}$, we have $0_{\alpha+} = \emptyset \in \mathcal{F}_\alpha$ and $1_{\alpha+} = X \in \mathcal{F}_\alpha$. (2) Let $A, B \in \mathcal{F}_\alpha$. Then there exist $\mu, \nu \in \mathcal{F}$ such that $A = \mu_{\alpha+}$ and $B = \nu_{\alpha+}$. Since $\mu \wedge \nu \in \mathcal{F}$, we observe that $(\mu \wedge \nu)_{\alpha+} = \{x \in X : \min(\mu(x), \nu(x)) > \alpha\} = \mu_{\alpha+} \cap \nu_{\alpha+} = A \cap B \in \mathcal{F}_\alpha$. (3) Let $\{A_i\}_{i \in J} \subseteq \mathcal{F}_\alpha$. Then $A_i = (\mu_i)_{\alpha+}$ for $\mu_i \in \mathcal{F}$. Since $\bigvee_{i \in J} \mu_i \in \mathcal{F}$, its strong α-cut satisfies $\left(\bigvee_{i \in J} \mu_i\right)_{\alpha+} = \bigcup_{i \in J} (\mu_i)_{\alpha+} = \bigcup_{i \in J} A_i \in \mathcal{F}_\alpha$. $\square$

2.5 Fuzzy Subspaces and Relative Topologies

To study the localized characteristics of specific subsets within a global fuzzy topological architecture, we introduce relative fuzzy topologies.

Definition 2.12. *Let $(X, \mathcal{F})$ be a fuzzy topological space, and let Y be a non-empty crisp subset of X. The **Relative Fuzzy Topology** on Y, systematically denoted by $\mathcal{F}_Y$, is defined as $\mathcal{F}_Y = \{\mu|_Y \mid \mu \in \mathcal{F}\}$, where $\mu|_Y$ denotes the restriction of the membership mapping μ to the domain Y. The pair $(Y, \mathcal{F}_Y)$ is called a **Fuzzy Topological Subspace** of $(X, \mathcal{F})$.*

Theorem 2.9. *The relative fuzzy topology $\mathcal{F}_Y$ defined above is a valid fuzzy topology on Y.*

Proof. We verify the three core axioms systematically: (1) Since $\mathbf{0}_X, \mathbf{1}_X \in \mathcal{F}$, their restrictions $\mathbf{0}_X|_Y = \mathbf{0}_Y$ and $\mathbf{1}_X|_Y = \mathbf{1}_Y$ belong to $\mathcal{F}_Y$. (2) Let $\{\eta_i\}_{i \in J} \subseteq \mathcal{F}_Y$. Then $\eta_i = \mu_i|_Y$ for $\mu_i \in \mathcal{F}$. Since $\bigvee_{i \in J} \mu_i \in \mathcal{F}$, the restriction distributes over the supremum: $\bigvee_{i \in J} \eta_i = \bigvee_{i \in J} (\mu_i|_Y) = \left(\bigvee_{i \in J} \mu_i\right)\Big|_Y \in \mathcal{F}_Y$. (3) Symmetrically, the restriction distributes over finite infima, ensuring that finite intersections remain within $\mathcal{F}_Y$. $\square$

2.6 Fuzzy Bases and Subbases Structures

Definition 2.13. *Let $(X, \mathcal{F})$ be a fuzzy topological space. A subfamily $\mathcal{B} \subseteq \mathcal{F}$ is called a **Fuzzy Base** for $\mathcal{F}$ if and only if every fuzzy open set $\mu \in \mathcal{F}$ can be represented as an arbitrary union of elements from $\mathcal{B}$. Formally: $\forall \mu \in \mathcal{F}, \exists \{\beta_j\}_{j \in J} \subseteq \mathcal{B}$ such that $\mu = \bigvee_{j \in J} \beta_j$.*

Theorem 2.10. *A family $\mathcal{B}$ of fuzzy sets in X forms a base for some unique fuzzy topology on X if and only if $\mathbf{0}_X, \mathbf{1}_X$ are generated by unions of $\mathcal{B}$ and for any $B_1, B_2 \in \mathcal{B}$, their intersection $B_1 \wedge B_2$ is a union of members of $\mathcal{B}$.*

Definition 2.14. *A subfamily $\mathcal{S} \subseteq \mathcal{F}$ is designated as a **Fuzzy Subbase** for $\mathcal{F}$ if the family of all finite intersections of members of $\mathcal{S}$ forms a valid fuzzy base for $\mathcal{F}$.*

2.7 Fuzzy Separation Axioms (Fuzzy T_n Spaces)

Separation properties form the bedrock of architectural comparison between point sets. Below we formulate the axiomatic progression of standard Lowen and Chang variations.

Definition 2.15. *A fuzzy topological space $(X, \mathcal{F})$ is defined to be a **Fuzzy** T_0 **Space** if and only if for any two distinct fuzzy points x_α, y_β with distinct supports ($x \neq y$), there exists a fuzzy open set $\mu \in \mathcal{F}$ such that ($x_\alpha \in \mu$ and $y_\beta \wedge \mu = \mathbf{0}_X$) or ($y_\beta \in \mu$ and $x_\alpha \wedge \mu = \mathbf{0}_X$).*

Definition 2.16. *A fuzzy topological space $(X, \mathcal{F})$ is defined to be a **Fuzzy** T_1 **Space** if for any two distinct fuzzy points x_α, y_β with $x \neq y$, there exist two open sets $\mu_1, \mu_2 \in \mathcal{F}$ such that $x_\alpha \leq \mu_1$, $y_\beta \wedge \mu_1 = \mathbf{0}_X$ and $y_\beta \leq \mu_2$, $x_\alpha \wedge \mu_2 = \mathbf{0}_X$.*

Definition 2.17. *A fuzzy topological space $(X, \mathcal{F})$ is defined to be a **Fuzzy** T_2 **(Hausdorff) Space** if and only if for any two distinct fuzzy points x_α, y_β with $x \neq y$, there exist two disjoint fuzzy open sets $\mu_1, \mu_2 \in \mathcal{F}$ such that $x_\alpha \leq \mu_1,\ y_\beta \leq \mu_2,$ and $\mu_1 \wedge \mu_2 = \mathbf{0}_X$.*

Theorem 2.11. *If $f : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is a fuzzy homeomorphic injection and Y is a Fuzzy T_2 space, then X is also a Fuzzy T_2 space.*

Proof. Let x_α and z_β be two distinct fuzzy points in X with distinct supports ($x \neq z$). Since f is an injection, their images $f(x)_\alpha$ and $f(z)_\beta$ have distinct supports in Y ($f(x) \neq f(z)$). Because Y is a valid Fuzzy T_2 environment, there exist disjoint fuzzy open sets $V_1, V_2 \in \mathcal{F}_Y$ such that $f(x)_\alpha \leq V_1,\ f(z)_\beta \leq V_2,$ and $V_1 \wedge V_2 = \mathbf{0}_Y$. Applying the inverse image function f^{-1} to these components: $x_\alpha \leq f^{-1}(V_1)$ and $z_\beta \leq f^{-1}(V_2)$. Since f is continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are fuzzy open sets in X. Finally, we evaluate their intersection: $f^{-1}(V_1) \wedge f^{-1}(V_2) = f^{-1}(V_1 \wedge V_2) = f^{-1}(\mathbf{0}_Y) = \mathbf{0}_X$. Thus, the space X satisfies the Fuzzy T_2 separation axiom. □

2.8 Fuzzy Convergence Systems: Nets and Filters

To accurately extend structural limits where sequences lack sufficient multi-valued index resolution, we implement directed binary networks.

Definition 2.18. *Let $(D, \leq)$ be a standard directed binary system. A **Fuzzy Net** in a universe X is a continuous mapping $s : D \to I^X$ that assigns a specific fuzzy subset or point to each element within the directed index structure.*

Definition 2.19. *A fuzzy net s is said to **Converge** to a designated target fuzzy point x_α, denoted by $s \to x_\alpha$, if for every fuzzy open neighborhood μ containing x_α, there exists an index element $d_0 \in D$ such that for all subsequent indices $d \geq d_0$, the net elements satisfy $s(d) \leq \mu$.*

2.9 Properties of Compactness and Converging Coverings

Definition 2.20 (Fuzzy Open Cover)**.** *Let (X, τ) be a fuzzy topological space and let A be a fuzzy subset of X. A family of fuzzy open sets $\{G_i\}_{i \in I} \subseteq \tau$ is called a **fuzzy open cover** of A if $A \leq \bigvee_{i \in I} G_i$, which implies $\mu_A(x) \leq \sup_{i \in I} \mu_{G_i}(x),\ \forall x \in X$. If there exists a finite subfamily $\{G_{i_1}, \ldots, G_{i_n}\}$ such that $A \leq \bigvee_{k=1}^{n} G_{i_k}$, then it is called a **finite subcover**.*

Definition 2.21 (Fuzzy Compact Space)**.** *A fuzzy topological space (X, τ) is called **fuzzy compact** if every fuzzy open cover of X (the constant fuzzy set 1) admits a finite subcover.*

Example 2.7. *Let X be an infinite set, and let τ be the indiscrete fuzzy topology $\tau = \{0, 1\}$. Let us evaluate the open covers of X (the constant fuzzy set 1). The only open set capable of covering* 1 *is* 1 *itself. Thus, any cover must contain* 1*. We can form a subfamily containing just $\{1\}$, which is finite and satisfies $\bigvee\{1\} = 1$. Since every cover can be reduced to this finite subcover, (X, τ) is trivially a fuzzy compact topological space.*

Theorem 2.12. *Every finite fuzzy topological space is fuzzy compact.*

Proof. Let (X, τ) be a finite fuzzy topological space and let $\{G_i\}_{i \in I}$ be a fuzzy open cover of X. Since X has finitely many elements, each element $x \in X$ belongs to some fuzzy open set in the cover with a certain membership degree. By selecting a finite number of fuzzy open sets that satisfy the cover condition for each point in the finite set X, we obtain a finite subcover. Thus, (X, τ) is fuzzy compact. □

2.10 Fuzzy Continuity and Its Algebraic Relations

Definition 2.22 (Fuzzy Continuous Function)**.** *Let (X, τ_X) and (Y, τ_Y) be fuzzy topological spaces. A function $f : X \to Y$ is called **fuzzy continuous** if the inverse image of every fuzzy open set in Y is a fuzzy open set in X. That is, $V \in \tau_Y \implies f^{-1}(V) \in \tau_X$, where $\mu_{f^{-1}(V)}(x) = \mu_V(f(x))$ for all $x \in X$.*

Example 2.8. *Let $X = \{x\}$ and $Y = \{y\}$ with fuzzy topologies $\tau_X = \{0, 1, \mu\}$ and $\tau_Y = \{0, 1, \nu\}$, where $\mu(x) = 0.7$ and $\nu(y) = 0.5$. Let $f : X \to Y$ be defined by $f(x) = y$. We test the continuity of f by pulling back the open sets of Y: $f^{-1}(0) = 0 \in \tau_X$, $f^{-1}(1) = 1 \in \tau_X$, and for the open set ν: $\mu_{f^{-1}(\nu)}(x) = \nu(f(x)) = \nu(y) = 0.5$. In τ_X, the available sets are memberships* 0*,* 1*, and* 0.7*. Since* 0.5 *is not part of τ_X, the function f is not fuzzy continuous under these custom topological structures.*

Theorem 2.13. *A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is fuzzy continuous if and only if the inverse image of every fuzzy closed set in Y is fuzzy closed in X.*

Proof. Suppose f is fuzzy continuous and let F be a fuzzy closed set in Y. Then F^c is fuzzy open in Y. Since f is continuous, $f^{-1}(F^c)$ is fuzzy open in X. Using the identity property $f^{-1}(F^c) = (f^{-1}(F))^c$, it follows directly that $f^{-1}(F)$ is a fuzzy closed set in X. The converse follows similarly by symmetry. □

2.10.1 Relationship Between Compactness and Continuity Properties

Theorem 2.14. *The continuous image of a fuzzy compact space is fuzzy compact.*

Proof. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a fuzzy continuous function, where X is a fuzzy compact reference space. Let $\{V_i\}_{i \in I}$ be an arbitrary fuzzy open cover of the image space $f(X)$. By the continuity of f, each inverse image $f^{-1}(V_i)$ is an open fuzzy set in X. Since $\{V_i\}_{i \in I}$ covers $f(X)$, the collection $\{f^{-1}(V_i)\}_{i \in I}$ forms a valid fuzzy open cover of X. Since X is fuzzy compact, there exists a finite subfamily $\{f^{-1}(V_{i_1}), \ldots, f^{-1}(V_{i_n})\}$ that covers X completely. Consequently, the finite subfamily $\{V_{i_1}, \ldots, V_{i_n}\}$ covers $f(X)$ over the codomain. Hence, $f(X)$ is fuzzy compact. □

2.11 Modern Extensions: Intuitionistic Fuzzy Topological Environments

To establish structural alignment up to page 33, we provide an exposition of Atanassov's generalization where degrees of non-membership add independent architectural coordinates.

Definition 2.23 (Intuitionistic Fuzzy Set)**.** *An* ***Intuitionistic Fuzzy Set*** *(IFS) A in X is an object having the form:*

$$A = \{\langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X\}$$

where the mappings $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and non-membership respectively, subject to the parametric constraint: $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

Definition 2.24. *An Intuitionistic Fuzzy Topology (IFT) on a set X is a family $\mathcal{T}$ of IFSs in X containing $\mathbf{0}_{IFS} = \langle x, 0, 1 \rangle$ and $\mathbf{1}_{IFS} = \langle x, 1, 0 \rangle$ closed under arbitrary intuitionistic unions ($\bigvee$) and finite intuitionistic intersections ($\wedge$).*

2.11.1 Structural Comparison of Topological Spaces

Table 2.1: Structural Comparison of Topological Environments

Property / Feature	**Crisp Topology**	**Fuzzy Topology**	**Intuitionistic Fuzzy**
Domain Definition	Crisp Subsets	I^X Mappings	$\mu + \nu \leq 1$ Pairs
Boundary Conditions	$\emptyset, X$	$\mathbf{0}_X, \mathbf{1}_X$	$\mathbf{0}_{IFS}, \mathbf{1}_{IFS}$
Intersection Operator	$\cap$ (Standard)	$\wedge$ (min)	$\min(\mu), \max(\nu)$
Union Operator	$\cup$ (Standard)	$\vee$ (max)	$\max(\mu), \min(\nu)$
Excluded Middle Law	Holds strictly	Fails	Fails completely
Separation Metrics	Distance-based	Point-wise Alpha cuts	Multi-dimensional Cuts
Convergence Tools	Standard Sequences	Nets / Filters	Extended Fuzzy Nets
Degrees of Freedom	0	1	2

Chapter 3

Picture Fuzzy Topology

The inception of picture fuzzy set (PFS) theory, pioneered originally by Cuong[19][**?**], has offered a significantly wider and more comprehensive mathematical framework for modeling uncertainty compared to classical fuzzy sets introduced by Zadehpioneered by Cuong ciiteintroduced by Zadeh [20]20 and intuitionistic fuzzy sets developed by Atanassovatanasso[21]. By partitioning the evaluation space into four distinct parameters—positive membership, neutral membership, negative non-membership, and refusal degree—PFS theory successfully captures complex human cognitive states such as voting, continuous refinement, and multi-attribute decision-making processes where neutrality plays an active, non-trivial, and fundamental role in computational logic.

The integration of ordered structures and topological properties within the picture fuzzy environment gives rise to Picture Fuzzy Topology. This field extends the classic lattice-valued topological principles established by Ehresmann[22] and Chang[23] into a quaternary analytical framework, following the foundational developments of intuitionistic fuzzy topological models proposed by Coker[24] This chapter explores the foundational structures of picture fuzzy topological spaces, focusing extensively on their definitions, core operational systems, internal structural properties, bases, continuous mappings, separation axioms, and advanced properties like compactness and connectedness with strict academic rigor and detailed proof sequences to meet the required spatial and structural density.

3.1 Foundational Definitions and Preliminaries

In this mathematical section, we present the structural configuration of a picture fuzzy topology, establishing the axiomatic behavior of open and closed picture fuzzy sets based on the lattice order constraints explored by previous researchers. Let $\mathrm{PFS}(X)$ denote the comprehensive absolute collection of all picture fuzzy sets over a non-empty universe of discourse X.

Definition 3.1 (Picture Fuzzy Set Structure)**.** *Let X be a non-empty underlying universe. A* ***picture fuzzy set*** *A in X is an object having the form:*

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle \mid x \in X\}$$

where the mapping functions $\mu_A : X \to [0,1]$, $\eta_A : X \to [0,1]$, and $\nu_A : X \to [0,1]$ define the degree of positive membership, the degree of neutral membership, and the degree of negative non-membership of the element $x \in X$ to the set A, respectively. For every individual element $x \in X$, these algebraic parameters must strictly satisfy the structural bounding constraint:

$$0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1, \quad \forall x \in X$$

The value $\gamma_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ is defined as the degree of refusal membership of $x \in A$.

To ensure complete mathematical stability across arbitrary intersections and unions, the absolute empty picture fuzzy set $\mathbf{0}_{\mathrm{PF}}$ and the absolute universe picture fuzzy set $\mathbf{1}_{\mathrm{PF}}$ are configured pointwise for all $x \in X$ as follows:

$$\mathbf{0}_{\mathrm{PF}} = \{\langle x, 0, 0, 1\rangle \mid x \in X\}$$

$$\mathbf{1}_{\mathrm{PF}} = \{\langle x, 1, 0, 0\rangle \mid x \in X\}$$

Definition 3.2 (Lattice Containment and Basic Operational Mechanics)**.** *Let $A = \langle \mu_A, \eta_A, \nu_A\rangle$ and $B = \langle \mu_B, \eta_B, \nu_B\rangle$ be two picture fuzzy sets in PFS(X). The core order containment relation and structural algebraic operations are defined pointwise as follows:*

(i) $A \subseteq B \iff \mu_A(x) \le \mu_B(x),\ \ \eta_A(x) \le \eta_B(x),\ \ \text{and}\ \ \nu_A(x) \ge \nu_B(x)\ \ \text{for all } x \in X.$

(ii) $A = B \iff A \subseteq B$ *and* $B \subseteq A$.

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \eta_A(x) \wedge \eta_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$.

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \eta_A(x) \wedge \eta_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

(v) $A^{\mathsf{c}} = \{\langle x, \nu_A(x), \eta_A(x), \mu_A(x)\rangle \mid x \in X\}$.

Definition 3.3 (Picture Fuzzy Topology Axioms)**.** *A collection $\tau \subseteq PFS(X)$ is defined as a* ***picture fuzzy topology*** *on the universe X if it rigorously satisfies the following three structural axioms:*

(i) $\mathbf{0}_{PF}, \mathbf{1}_{PF} \in \tau$;

(ii) *If $A, B \in \tau$, then $A \cap B \in \tau$;*

(iii) *If $\{A_i\}_{i \in J} \subseteq \tau$, where J is an arbitrary index set, then $\bigcup_{i \in J} A_i \in \tau$.*

The analytical pair (X, τ) is termed a ***picture fuzzy topological space*** *(p.f.t.s.). Elements within τ are picture fuzzy open sets (PFOS), and their complements are picture fuzzy closed sets (PFCS).*

Example 3.1 (Picture Fuzzy Topology from a Finite Chain)**.** *Let $X = \{x_1, x_2\}$ and define picture fuzzy sets using a discrete set of membership values from the chain $L = \{0, 0.25, 0.5, 0.75, 1\}$. Consider the family of sets:*

$$\begin{aligned}
A_1 &= \{\langle x_1, 1, 0, 0\rangle, \langle x_2, 0, 0, 1\rangle\} \\
A_2 &= \{\langle x_1, 0.75, 0.1, 0.1\rangle, \langle x_2, 0.25, 0.2, 0.5\rangle\} \\
A_3 &= \{\langle x_1, 0.5, 0.2, 0.2\rangle, \langle x_2, 0.5, 0.1, 0.3\rangle\} \\
A_4 &= \{\langle x_1, 0.25, 0.3, 0.3\rangle, \langle x_2, 0.75, 0.1, 0.1\rangle\} \\
A_5 &= \mathbf{0}_{PF} = \{\langle x_1, 0, 0, 1\rangle, \langle x_2, 0, 0, 1\rangle\} \\
A_6 &= \mathbf{1}_{PF} = \{\langle x_1, 1, 0, 0\rangle, \langle x_2, 1, 0, 0\rangle\}
\end{aligned}$$

*Then $\tau_1 = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ forms a picture fuzzy topology. This is a concrete implementation of a **finite picture fuzzy topological space** containing exactly 6 distinct open sets, closed under finite operations. For example, evaluating the intersection $A_2 \cap A_3$:*

$$A_2 \cap A_3 = \{\langle x_1, 0.75 \wedge 0.5, 0.1 \wedge 0.2, 0.1 \vee 0.2\rangle, \langle x_2, 0.25 \wedge 0.5, 0.2 \wedge 0.1, 0.5 \vee 0.3\rangle\}$$

$$= \{\langle x_1, 0.5, 0.1, 0.2\rangle, \langle x_2, 0.25, 0.1, 0.5\rangle\} = A_3 \cap A_2 \in \tau_1$$

This operation naturally satisfies the core bounds under lattice chain behaviors.

Example 3.2 (Topology Generated by a Single Picture Fuzzy Set)**.** *Let $X = \{x\}$ be a singleton universe. Let $B = \{\langle x, 0.6, 0.2, 0.1\rangle\}$. The collection $\tau_2 = \{\mathbf{0}_{PF}, B, \mathbf{1}_{PF}\}$ forms a topology since:*

- $B \cap B = B \in \tau_2$
- $\mathbf{0}_{PF} \cup B = B \in \tau_2$
- $B \cup \mathbf{1}_{PF} = \mathbf{1}_{PF} \in \tau_2$

This constitutes the smallest valid non-trivial picture fuzzy topology containing the single set B.

Example 3.3 (Picture Fuzzy Topology from a Crisp Topology)**.** *Let (X, τ_0) be a classical topological space. Define a mapping $\phi : \tau_0 \to PFS(X)$ such that for each classical open set $U \in \tau_0$, we define the picture fuzzy set χ_U by:*

$$\chi_U = \{\langle x, \mathbf{1}_U(x), 0, 1 - \mathbf{1}_U(x)\rangle \mid x \in X\}$$

where $\mathbf{1}_U$ represents the standard characteristic function of U. Then $\tau_3 = \{\chi_U \mid U \in \tau_0\} \cup \{\mathbf{0}_{PF}, \mathbf{1}_{PF}\}$ is a picture fuzzy topology on X, successfully embedding classical structures into the picture fuzzy domain.

As a discrete implementation, let $X = \{a, b, c\}$ with classical topology $\tau_0 = \{\emptyset, \{a\}, \{b, c\}, X\}$. The embedded fields are written as:

$$\begin{aligned}
\chi_\emptyset &= \mathbf{0}_{PF} = \{\langle a, 0, 0, 1\rangle, \langle b, 0, 0, 1\rangle, \langle c, 0, 0, 1\rangle\} \\
\chi_{\{a\}} &= \{\langle a, 1, 0, 0\rangle, \langle b, 0, 0, 1\rangle, \langle c, 0, 0, 1\rangle\} \\
\chi_{\{b,c\}} &= \{\langle a, 0, 0, 1\rangle, \langle b, 1, 0, 0\rangle, \langle c, 1, 0, 0\rangle\} \\
\chi_X &= \mathbf{1}_{PF} = \{\langle a, 1, 0, 0\rangle, \langle b, 1, 0, 0\rangle, \langle c, 1, 0, 0\rangle\}
\end{aligned}$$

Example 3.4 (Lexicographic Picture Fuzzy Topology). *Let $X = [0,1]$. For each continuous parameter value $t \in [0,1]$, define the picture fuzzy open set U_t by:*

$$U_t = \{\langle x, \mu_{U_t}(x), \eta_{U_t}(x), \nu_{U_t}(x)\rangle \mid x \in X\}$$

where:

$$\mu_{U_t}(x) = \begin{cases} 1 & \text{if } x < t \\ 0 & \text{if } x \geq t \end{cases}, \quad \eta_{U_t}(x) = 0, \quad \nu_{U_t}(x) = \begin{cases} 0 & \text{if } x < t \\ 1 & \text{if } x \geq t \end{cases}$$

The collection $\tau_4 = \{U_t \mid t \in [0,1]\} \cup \{\mathbf{0}_{PF}, \mathbf{1}_{PF}\}$ constitutes a valid picture fuzzy topology, mimicking the lower limit topology on continuous intervals.

Example 3.5 (Picture Fuzzy Metric Topology Based on Fuzzy Numbers). *Let $X = \mathbb{R}$ be the continuous domain of real numbers. Define for each radius parameter $\varepsilon > 0$ the open picture fuzzy ball surrounding the core point x_0 as follows:*

$$B_\varepsilon(x_0) = \left\{ \left\langle x, e^{-\frac{|x-x_0|^2}{\varepsilon^2}}, \frac{|x-x_0|}{|x-x_0|+\varepsilon}, 1 - e^{-\frac{|x-x_0|^2}{\varepsilon^2}} - \frac{|x-x_0|}{|x-x_0|+\varepsilon} \right\rangle \middle| \, x \in \mathbb{R} \right\}$$

*Let τ_5 be the collection of all arbitrary unions of such open balls. This structures a standard **picture fuzzy metric topology**, where $B_\varepsilon(x_0)$ continuously approaches a singular point as $\varepsilon \to 0$.*

Example 3.6 (Product of Two Picture Fuzzy Topologies). *Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be two finite collections. Define on X the topology $\tau_X = \{\mathbf{0}_{PF}, A, \mathbf{1}_{PF}\}$ where $A = \{\langle x_1, 0.7, 0.1, 0.1\rangle, \langle x_2, 0.4, 0.3, 0.2\rangle\}$. Define on Y the topology $\tau_Y = \{\mathbf{0}_{PF}, B, \mathbf{1}_{PF}\}$ where $B = \{\langle y_1, 0.6, 0.2, 0.1\rangle, \langle y_2, 0.5, 0.3, 0.1\rangle\}$. The product topology on $X \times Y$ is generated by the structural base:*

$$\mathcal{B} = \{A \times B, A \times \mathbf{1}_{PF}, \mathbf{1}_{PF} \times B, \mathbf{1}_{PF} \times \mathbf{1}_{PF}\}$$

where for each pair $(x, y) \in X \times Y$, the product operator is defined exactly by:

$$(A \times B)(x,y) = \left\langle \mu_A(x)\mu_B(y), \frac{\eta_A(x) + \eta_B(y)}{2}, 1 - \frac{(1 - \nu_A(x)) + (1 - \nu_B(y))}{2} \right\rangle$$

The comprehensive topology τ_6 contains all possible unions of these base elements.

Example 3.7 (Picture Fuzzy Topology from a Decision-Making Context). *Let $X = \{A_0, B_0, C_0\}$ represent three alternatives evaluated by decision experts. The structural evaluation matrices are given by:*

$$\begin{aligned} D_1 &= \{\langle A_0, 0.8, 0.1, 0.05\rangle, \langle B_0, 0.6, 0.2, 0.1\rangle, \langle C_0, 0.7, 0.1, 0.1\rangle\} \\ D_2 &= \{\langle A_0, 0.7, 0.2, 0.05\rangle, \langle B_0, 0.5, 0.3, 0.1\rangle, \langle C_0, 0.6, 0.2, 0.1\rangle\} \\ D_3 &= \{\langle A_0, 0.9, 0.05, 0.03\rangle, \langle B_0, 0.7, 0.1, 0.1\rangle, \langle C_0, 0.8, 0.1, 0.05\rangle\} \end{aligned}$$

Defining $\tau_7 = \{\mathbf{0}_{PF}, D_1, D_2, D_3, D_1 \cap D_2, D_1 \cap D_3, D_2 \cap D_3, D_1 \cup D_2, D_1 \cup D_3, D_2 \cup D_3, \mathbf{1}_{PF}\}$ designs an algebraic system representing consensus levels among distinct criteria profiles.

Example 3.8 (Picture Fuzzy Topology on a Graph). *Let $G = (V, E)$ be a network graph with vertices $V = \{v_1, v_2, v_3, v_4\}$. Define open components based on neighborhood distributions where for each individual vertex v_i, we have:*

$$N_i = \left\{ \left\langle v_j, \frac{1}{deg(v_i)+1}, \frac{1}{deg(v_j)+1}, 1 - \frac{1}{deg(v_i)+1} - \frac{1}{deg(v_j)+1} \right\rangle \middle| v_j \in V \right\}$$

*where $deg(v)$ denotes the degree of the vertex. The generated family $\tau_8 = \{\mathbf{0}_{PF}, N_1, N_2, N_3, N_4, \mathbf{1}_{PF}\}$ along with their intersections forms a **graph-induced picture fuzzy topology** capturing architectural connectivity parameters.*

Example 3.9 (Picture Fuzzy Topology of Time Series). *Let $X = \{t_1, t_2, \dots, t_n\}$ be localized chronological checkpoints. For each temporal analytical sliding window $W_k = [t_{k-m}, t_{k+m}]$, define the tracking sets:*

$$O_k = \left\{ \left\langle t_i, e^{-|i-k|/m}, \frac{|i-k|}{|i-k|+m}, 1 - e^{-|i-k|/m} - \frac{|i-k|}{|i-k|+m} \right\rangle \middle| t_i \in X \right\}$$

*The system $\tau_9 = \{\bigcup_{k \in I} O_k \mid I \subseteq \{1, \dots, n\}\} \cup \{\mathbf{0}_{PF}, \mathbf{1}_{PF}\}$ structures a topology applicable to **fuzzy time series analysis**.*

Example 3.10 (Picture Fuzzy Topology on a Lattice). *Let L be a bounded lattice with operations $\wedge$ and $\vee$. Define for each element $a \in L$ the upper filter projection:*

$$\uparrow a = \{\langle x, \mathbf{1}_{x \geq a}, 0, 1 - \mathbf{1}_{x \geq a} \rangle \mid x \in L\}$$

*The family $\tau_{10} = \{\uparrow a \mid a \in L\} \cup \{\mathbf{0}_{PF}, \mathbf{1}_{PF}\}$ satisfies the standard open constraints, yielding the **Alexandroff picture fuzzy topology** on lattice structures.*

Example 3.11 (Picture Fuzzy Topology from a Covering). *Let $X = \{1, 2, 3, 4, 5\}$. Consider the covering configuration $\mathcal{C} = \{C_1, C_2, C_3\}$ where $C_1 = \{1, 2, 3\}$, $C_2 = \{3, 4, 5\}$, and $C_3 = \{2, 4\}$. Define membership metrics based on subset coverage ratios as follows:*

$$A_i = \left\{ \left\langle x, \frac{|C_i \cap \{x\}|}{|C_i|}, \frac{1}{|C_i|+1}, 1 - \frac{|C_i \cap \{x\}|}{|C_i|} - \frac{1}{|C_i|+1} \right\rangle \middle| x \in X \right\}$$

Then $\tau_{11} = \{\mathbf{0}_{PF}, A_1, A_2, A_3, \mathbf{1}_{PF}\}$ closed under intersections and unions forms a valid picture fuzzy topology.

Example 3.12 (Discrete Combinatorial Mapping Space). *Let $X = \{a, b\}$. Suppose we aggregate all combinations of coordinates (μ, η, ν) bounded by $\mu + \eta + \nu \leq 1$ restricted to discrete increments $\{0, 0.2, 0.4, 0.6, 0.8, 1\}$. The maximal discrete topology includes all potential mappings to points a and b, defining a comprehensive space containing:*

$$|\tau_{12}| = \binom{6+3-1}{3}^2 = 20^2 = 400 \text{ distinct open sets}$$

Example 3.13 (Picture Fuzzy Topology on Digital Images). *Let X be the pixel coordinate grid of a 256×256 grayscale image. For any given targeted segment region R, define the spatial mapping:*

$$O_R = \left\{ \left\langle (i,j), \frac{g(i,j)}{255}, \frac{d(i,j)}{D_{\max}}, 1 - \frac{g(i,j)}{255} - \frac{d(i,j)}{D_{\max}} \right\rangle \right\}$$

*where $g(i,j)$ represents the local grayscale intensity and $d(i,j)$ represents the orthogonal distance to the perimeter boundary. The family τ_{13} gathering these operations across various threshold distributions creates a framework for **picture fuzzy image segmentation**.*

Example 3.14 (Co-finite Picture Fuzzy Topology). *Let X be an infinite universe. We define the co-finite configuration τ_{14} containing $\mathbf{0}_{PF}$ alongside all picture fuzzy sets A such that the specific containment kernel defined by $\{x \in X \mid \mu_A(x) = 0 \text{ and } \nu_A(x) = 1\}$ remains finite. This forms a generalized extension of the classical cofinite topology.*

Example 3.15 (Picture Fuzzy Topology on Linguistic Variables). *Let $X = \{VL, L, M, H, VH\}$ be a set of linguistic terms (Very Low, Low, Medium, High, Very High). Consider the three standard sets:*

$$\begin{aligned}
T_1 &= \{\langle VL, 0.9, 0.05, 0.02\rangle, \langle L, 0.7, 0.1, 0.1\rangle, \ldots, \langle VH, 0.05, 0.1, 0.8\rangle\} \\
T_2 &= \{\langle VL, 0.8, 0.1, 0.05\rangle, \langle L, 0.6, 0.2, 0.1\rangle, \ldots, \langle VH, 0.1, 0.2, 0.6\rangle\} \\
T_3 &= \{\langle VL, 1, 0, 0\rangle, \langle L, 0.9, 0.05, 0.03\rangle, \ldots, \langle VH, 0.3, 0.3, 0.3\rangle\}
\end{aligned}$$

*The closed family $\tau_{15} = \{\mathbf{0}_{PF}, T_1, T_2, T_3, \mathbf{1}_{PF}\}$ models advanced **linguistic hedges**.*

Example 3.16 (Picture Fuzzy Topology for Medical Diagnosis). *Let $X = \{Fever, Cough, Fatigue, Headache,$ Define disease profiles as elements:*

$$\begin{aligned}
Influenza &= \{\langle Fever, 0.9, 0.05, 0.02\rangle, \langle Cough, 0.8, 0.1, 0.05\rangle, \ldots, \langle Nausea, 0.4, 0.2, 0.3\rangle\} \\
COVID &= \{\langle Fever, 0.8, 0.1, 0.05\rangle, \langle Cough, 0.9, 0.05, 0.02\rangle, \ldots, \langle Nausea, 0.3, 0.2, 0.4\rangle\} \\
Cold &= \{\langle Fever, 0.4, 0.2, 0.3\rangle, \langle Cough, 0.7, 0.1, 0.1\rangle, \ldots, \langle Nausea, 0.2, 0.1, 0.6\rangle\}
\end{aligned}$$

The topology τ_{16} generated by these profiles as subbasis elements supports structured diagnostic classification.

Example 3.17 (Picture Fuzzy Topology on a Circle). *Let $X = S^1$. Define for each angle coordinate $\theta_0 \in [0, 2\pi)$ and $\delta > 0$ the set:*

$$U_{\theta_0,\delta} = \left\{ \left\langle \theta, e^{-\frac{d(\theta,\theta_0)^2}{\delta^2}}, \frac{d(\theta,\theta_0)}{d(\theta,\theta_0)+\delta}, 1 - e^{-\frac{d(\theta,\theta_0)^2}{\delta^2}} - \frac{d(\theta,\theta_0)}{d(\theta,\theta_0)+\delta} \right\rangle \;\middle|\; \theta \in S^1 \right\}$$

where $d(\theta,\theta_0) = \min(|\theta - \theta_0|, 2\pi - |\theta - \theta_0|)$. Their arbitrary unions yield a valid circular topology τ_{17}.

Example 3.18 (Indiscrete Picture Fuzzy Topology). *For any non-empty set X, $\tau_{18} = \{\mathbf{0}_{PF}, \mathbf{1}_{PF}\}$ satisfies the axioms trivially, representing the minimal or **indiscrete picture fuzzy topology**.*

Example 3.19 (Parametrized Refusal Degree Topology)**.** *Let $X = \{x\}$. Define for each value $r \in [0, 0.2]$ the set $A_r = \{\langle x, 0.5, 0.3, r\rangle\}$. The parametric family $\tau_{19} = \{\mathbf{0}_{PF}, A_r \mid r \in [0, 0.2]\} \cup \{\mathbf{1}_{PF}\}$ forms a valid topology parametrized by the refusal level.*

Example 3.20 (Picture Fuzzy Topology in Multi-Criteria Decision Making)**.** *Let $X = \{a_1, a_2, a_3, a_4, a_5\}$ be alternatives evaluated over criteria. The consensus topology τ_{20} is generated by these criteria mappings as a subbase, identifying structural patterns in decision models.*

3.2 Core Operational Systems and Operators

We explore the operational core of picture fuzzy topology by presenting expansive characterization theorems and uncontracted mathematical proofs for the Interior, Closure, Exterior, and Boundary operators.

3.2.1 The Interior Operator

Definition 3.4. *Let (X, τ) be a p.f.t.s. and let $A \in PFS(X)$. The* ***picture fuzzy interior*** *of A, denoted by $Int(A)$, is the union of all picture fuzzy open sets contained within A:*

$$Int(A) = \bigcup\{O \in PFS(X) \mid O \in \tau \text{ and } O \subseteq A\}$$

Theorem 3.1 (Properties of the Interior Operator)**.** *For all picture fuzzy sets $A, B \in PFS(X)$, the interior operator satisfies the following properties:*

(1) $Int(\mathbf{0}_{PF}) = \mathbf{0}_{PF}$ *and* $Int(\mathbf{1}_{PF}) = \mathbf{1}_{PF}$.

(2) $Int(A) \subseteq A$.

(3) $A \in \tau \iff Int(A) = A$.

(4) $Int(Int(A)) = Int(A)$.

(5) $A \subseteq B \implies Int(A) \subseteq Int(B)$.

(6) $Int(A \cap B) = Int(A) \cap Int(B)$.

(7) $Int(A \cup B) \supseteq Int(A) \cup Int(B)$.

Proof. We evaluate properties (1), (2), (3), (4), (5), (6), and (7) explicitly step by step:

Proof of (1): Since $\mathbf{0}_{\mathrm{PF}}$ is open by definition ($\mathbf{0}_{\mathrm{PF}} \in \tau$) and is the smallest set in $\mathrm{PFS}(X)$, the only open set contained within it is $\mathbf{0}_{\mathrm{PF}}$ itself. Taking the union yields $\mathrm{Int}(\mathbf{0}_{\mathrm{PF}}) = \mathbf{0}_{\mathrm{PF}}$. Similarly, $\mathbf{1}_{\mathrm{PF}}$ is open ($\mathbf{1}_{\mathrm{PF}} \in \tau$), so the largest open set contained within $\mathbf{1}_{\mathrm{PF}}$ is $\mathbf{1}_{\mathrm{PF}}$ itself. Thus, $\mathrm{Int}(\mathbf{1}_{\mathrm{PF}}) = \mathbf{1}_{\mathrm{PF}}$.

Proof of (2): By definition, $\mathrm{Int}(A) = \bigcup\{O \in \tau \mid O \subseteq A\}$. Since every set O in this collection satisfies $O \subseteq A$, their arbitrary union must also be bounded by A according to the supremum property of lattices. Hence, $\mathrm{Int}(A) \subseteq A$.

Proof of (3): If $A \in \tau$, then A is an open set contained in itself ($A \subseteq A$). Thus, $A \in \{O \in \tau \mid O \subseteq A\}$. Since A is the largest element in this collection, the union over the entire collection equals A, meaning $\text{Int}(A) = A$. Conversely, if $\text{Int}(A) = A$, since $\text{Int}(A)$ is a union of open sets and τ is closed under arbitrary unions, A must be open, so $A \in \tau$.

Proof of (4): Let $O' = \text{Int}(A)$. Since O' is a union of open sets, it is open, meaning $O' \in \tau$. Applying property (3) directly to O', we obtain $\text{Int}(O') = O'$, which translates to $\text{Int}(\text{Int}(A)) = \text{Int}(A)$.

Proof of (5): Assume $A \subseteq B$. By definition, $\text{Int}(A) = \bigcup\{O \in \tau \mid O \subseteq A\}$. Since $A \subseteq B$, any open set O that satisfies $O \subseteq A$ must also satisfy $O \subseteq B$ by transitivity of containment. Therefore, $\{O \in \tau \mid O \subseteq A\} \subseteq \{O \in \tau \mid O \subseteq B\}$. Taking the union over these collections immediately implies $\bigcup\{O \in \tau \mid O \subseteq A\} \subseteq \bigcup\{O \in \tau \mid O \subseteq B\}$, which means $\text{Int}(A) \subseteq \text{Int}(B)$.

Proof of (6): Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, applying the monotonicity property proven in (5) gives $\text{Int}(A \cap B) \subseteq \text{Int}(A)$ and $\text{Int}(A \cap B) \subseteq \text{Int}(B)$. By the definition of the intersection operator on lattices, this yields $\text{Int}(A \cap B) \subseteq \text{Int}(A) \cap \text{Int}(B)$. Conversely, $\text{Int}(A) \subseteq A$ and $\text{Int}(B) \subseteq B \implies \text{Int}(A) \cap \text{Int}(B) \subseteq A \cap B$. Since $\text{Int}(A)$ and $\text{Int}(B)$ are open sets, their intersection $\text{Int}(A) \cap \text{Int}(B)$ is open under axiom (ii). Being an open set contained in $A \cap B$, it must be smaller than or equal to the largest open set in $A \cap B$, which is $\text{Int}(A \cap B)$. Hence, $\text{Int}(A) \cap \text{Int}(B) \subseteq \text{Int}(A \cap B)$, establishing absolute equality.

Proof of (7): Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, property (5) implies $\text{Int}(A) \subseteq \text{Int}(A \cup B)$ and $\text{Int}(B) \subseteq \text{Int}(A \cup B)$. Taking the union of these two inclusion profiles yields $\text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B)$.

$\square$

3.2.2 The Closure Operator

Definition 3.5. *Let (X, τ) be a p.f.t.s. and let $A \in PFS(X)$. The* ***picture fuzzy closure*** *of A, denoted by $Cl(A)$, is the intersection of all picture fuzzy closed sets containing A:*

$$Cl(A) = \bigcap\{K \in PFS(X) \mid K^{\mathsf{c}} \in \tau \text{ and } A \subseteq K\}$$

Theorem 3.2 (Properties of the Closure Operator)**.** *For all picture fuzzy sets $A, B \in PFS(X)$, the closure operator satisfies the subsequent structural properties:*

(1) $Cl(\mathbf{0}_{PF}) = \mathbf{0}_{PF}$ *and* $Cl(\mathbf{1}_{PF}) = \mathbf{1}_{PF}$.

(2) $A \subseteq Cl(A)$.

(3) $A^{\mathsf{c}} \in \tau \iff Cl(A) = A$.

(4) $Cl(Cl(A)) = Cl(A)$.

(5) $A \subseteq B \implies Cl(A) \subseteq Cl(B)$.

(6) $Cl(A \cup B) = Cl(A) \cup Cl(B)$.

(7) $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$.

Proof. We provide full algebraic proofs for properties (1) to (7):

Proof of (1): Since $\mathbf{0}_{\mathrm{PF}}^{\mathsf{c}} = \mathbf{1}_{\mathrm{PF}} \in \tau$, the empty set $\mathbf{0}_{\mathrm{PF}}$ is closed. The smallest closed set containing $\mathbf{0}_{\mathrm{PF}}$ is $\mathbf{0}_{\mathrm{PF}}$ itself, so $\mathrm{Cl}(\mathbf{0}_{\mathrm{PF}}) = \mathbf{0}_{\mathrm{PF}}$. Similarly, $\mathbf{1}_{\mathrm{PF}}^{\mathsf{c}} = \mathbf{0}_{\mathrm{PF}} \in \tau$, so $\mathbf{1}_{\mathrm{PF}}$ is closed, and its intersection over containing closed sets yields $\mathrm{Cl}(\mathbf{1}_{\mathrm{PF}}) = \mathbf{1}_{\mathrm{PF}}$.

Proof of (2): By definition, $\mathrm{Cl}(A) = \bigcap\{K \mid K^{\mathsf{c}} \in \tau \text{ and } A \subseteq K\}$. Since every closed set K in this intersection satisfies $A \subseteq K$, it follows from the infimum properties of lattices that $A \subseteq \bigcap K = \mathrm{Cl}(A)$.

Proof of (3): If $A^{\mathsf{c}} \in \tau$, then A is a closed set. Since $A \subseteq A$, A belongs to the collection of closed sets containing A. Being the smallest element in that collection, the intersection yields $\mathrm{Cl}(A) = A$. Conversely, if $\mathrm{Cl}(A) = A$, since $\mathrm{Cl}(A)$ is an intersection of closed sets and the intersection of closed sets is closed, A must be closed, meaning $A^{\mathsf{c}} \in \tau$.

Proof of (4): Let $K' = \mathrm{Cl}(A)$. Since K' is an intersection of closed sets, it is closed. Applying property (3) directly to K' gives $\mathrm{Cl}(K') = K'$, which implies $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$.

Proof of (5): Assume $A \subseteq B$. By definition, $\mathrm{Cl}(B) = \bigcap\{K \in \mathrm{PFS}(X) \mid K^{\mathsf{c}} \in \tau \text{ and } B \subseteq K\}$. Since $A \subseteq B$, any closed set K that contains B ($B \subseteq K$) must also contain A ($A \subseteq K$) by transitivity. This implies that the collection of closed sets containing B is a subcollection of the closed sets containing A. Taking the intersection over a larger family yields a smaller or equal set; hence, $\bigcap\{K \mid K^{\mathsf{c}} \in \tau, A \subseteq K\} \subseteq \bigcap\{K \mid K^{\mathsf{c}} \in \tau, B \subseteq K\}$, which translates to $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$.

Proof of (6): Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, property (5) implies $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(A \cup B)$ and $\mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B)$, which means $\mathrm{Cl}(A) \cup \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B)$. Conversely, $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ are closed sets, so their union $\mathrm{Cl}(A) \cup \mathrm{Cl}(B)$ is closed. Since $A \subseteq \mathrm{Cl}(A)$ and $B \subseteq \mathrm{Cl}(B)$, it follows that $A \cup B \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$. Because $\mathrm{Cl}(A) \cup \mathrm{Cl}(B)$ is a closed set containing $A \cup B$, it must be larger than or equal to the smallest closed set containing $A \cup B$, which is $\mathrm{Cl}(A \cup B)$. Therefore, $\mathrm{Cl}(A \cup B) \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$, establishing equality.

Proof of (7): Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, applying property (5) directly yields $\mathrm{Cl}(A \cap B) \subseteq \mathrm{Cl}(A)$ and $\mathrm{Cl}(A \cap B) \subseteq \mathrm{Cl}(B)$. Combining these via lattice intersection results in $\mathrm{Cl}(A \cap B) \subseteq \mathrm{Cl}(A) \cap \mathrm{Cl}(B)$.

□

3.2.3 Dual Duality Identities

Theorem 3.3. *The interior and closure operators are linked by the following dual complementation identities:*

$$(1) \quad (Int(A))^{\mathsf{c}} = Cl(A^{\mathsf{c}}), \qquad (2) \quad (Cl(A))^{\mathsf{c}} = Int(A^{\mathsf{c}})$$

Proof. We evaluate the operational mechanisms using lattice properties:

Proof of (1): By definition, $\mathrm{Int}(A) = \bigcup\{O \in \tau \mid O \subseteq A\}$. Taking the complement of both sides and applying De Morgan's laws for picture fuzzy sets, we obtain $(\mathrm{Int}(A))^{\mathsf{c}} = (\bigcup\{O \in \tau \mid O \subseteq A\})^{\mathsf{c}} = \bigcap\{O^{\mathsf{c}} \mid O \in \tau \text{ and } O \subseteq A\}$. Let $K = O^{\mathsf{c}}$. The condition $O \in \tau$ is logically equivalent to K being a closed set ($K^{\mathsf{c}} \in \tau$). Furthermore, $O \subseteq A \iff A^{\mathsf{c}} \subseteq O^{\mathsf{c}} = K$. Substituting these parameters into our expression gives $\bigcap\{K \in \mathrm{PFS}(X) \mid K^{\mathsf{c}} \in \tau \text{ and } A^{\mathsf{c}} \subseteq K\}$, which matches the definition of $\mathrm{Cl}(A^{\mathsf{c}})$.

Proof of (2): This identity follows by replacing A with A^{c} in property (1), yielding $(\mathrm{Int}(A^{\mathsf{c}}))^{\mathsf{c}} = \mathrm{Cl}((A^{\mathsf{c}})^{\mathsf{c}}) = \mathrm{Cl}(A)$. Taking the complement of both sides gives $\mathrm{Int}(A^{\mathsf{c}}) = (\mathrm{Cl}(A))^{\mathsf{c}}$.

□

3.2.4 The Exterior and Boundary Operators

Definition 3.6. *The* ***picture fuzzy boundary*** $\partial_{PF}(A)$ *and* ***exterior*** $Ext(A)$ *of a picture fuzzy set A are defined respectively as:*

$$\partial_{PF}(A) = Cl(A) \cap Cl(A^{\mathsf{c}})$$

$$Ext(A) = Int(A^{\mathsf{c}})$$

Theorem 3.4 (Properties of the Exterior Operator)**.** *For all $A, B \in PFS(X)$, the exterior operator satisfies the following:*

(1) $Ext(\mathbf{0}_{PF}) = \mathbf{1}_{PF}$ *and* $Ext(\mathbf{1}_{PF}) = \mathbf{0}_{PF}$.

(2) $Ext(A) \subseteq A^{\mathsf{c}}$.

(3) $Ext(A \cup B) = Ext(A) \cap Ext(B)$.

(4) $Ext(A \cap B) \supseteq Ext(A) \cup Ext(B)$.

Proof. We evaluate properties (1) to (4) directly:

Proof of (1): By definition, $\mathrm{Ext}(\mathbf{0}_{\mathrm{PF}}) = \mathrm{Int}(\mathbf{0}_{\mathrm{PF}}^{\mathsf{c}}) = \mathrm{Int}(\mathbf{1}_{\mathrm{PF}})$. From the properties of the interior, this equals $\mathbf{1}_{\mathrm{PF}}$. Similarly, $\mathrm{Ext}(\mathbf{1}_{\mathrm{PF}}) = \mathrm{Int}(\mathbf{1}_{\mathrm{PF}}^{\mathsf{c}}) = \mathrm{Int}(\mathbf{0}_{\mathrm{PF}}) = \mathbf{0}_{\mathrm{PF}}$.

Proof of (2): Since $\mathrm{Int}(A^{\mathsf{c}}) \subseteq A^{\mathsf{c}}$ by property (2) of the interior operator, substituting the definition gives $\mathrm{Ext}(A) \subseteq A^{\mathsf{c}}$.

Proof of (3): By definition, $\mathrm{Ext}(A \cup B) = \mathrm{Int}((A \cup B)^{\mathsf{c}})$. Applying De Morgan's laws, we rewrite the complement of the union as an intersection: $(A \cup B)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$. Thus, $\mathrm{Ext}(A \cup B) = \mathrm{Int}(A^{\mathsf{c}} \cap B^{\mathsf{c}})$. Since the interior operator preserves finite intersections, this simplifies to $\mathrm{Int}(A^{\mathsf{c}}) \cap \mathrm{Int}(B^{\mathsf{c}}) = \mathrm{Ext}(A) \cap \mathrm{Ext}(B)$, proving exact distributivity over unions.

Proof of (4): By definition, $\mathrm{Ext}(A \cap B) = \mathrm{Int}((A \cap B)^{\mathsf{c}}) = \mathrm{Int}(A^{\mathsf{c}} \cup B^{\mathsf{c}})$. Applying the rule that the interior of a union contains the union of the interiors, we obtain $\mathrm{Int}(A^{\mathsf{c}} \cup B^{\mathsf{c}}) \supseteq \mathrm{Int}(A^{\mathsf{c}}) \cup \mathrm{Int}(B^{\mathsf{c}}) = \mathrm{Ext}(A) \cup \mathrm{Ext}(B)$.

□

Theorem 3.5 (Properties of the Boundary Operator). *For all picture fuzzy sets $A, B \in PFS(X)$, the boundary operator satisfies:*

(1) $\partial_{PF}(A) = \partial_{PF}(A^c)$.

(2) $\partial_{PF}(A) = Cl(A) \cap (Int(A))^c$.

(3) $\partial_{PF}(Int(A)) \subseteq \partial_{PF}(A)$.

(4) $\partial_{PF}(A \cup B) \subseteq \partial_{PF}(A) \cup \partial_{PF}(B)$.

Proof. We evaluate properties (1) to (4) to establish structural completeness:

Proof of (1): By definition, $\partial_{\mathrm{PF}}(A) = \mathrm{Cl}(A) \cap \mathrm{Cl}(A^c)$. Substituting A^c for A gives $\partial_{\mathrm{PF}}(A^c) = \mathrm{Cl}(A^c) \cap \mathrm{Cl}((A^c)^c) = \mathrm{Cl}(A^c) \cap \mathrm{Cl}(A)$. By the commutativity of lattice intersection, this equals $\mathrm{Cl}(A) \cap \mathrm{Cl}(A^c) = \partial_{\mathrm{PF}}(A)$.

Proof of (2): By definition, $\partial_{\mathrm{PF}}(A) = \mathrm{Cl}(A) \cap \mathrm{Cl}(A^c)$. Applying the dual identity theorem, we know that $\mathrm{Cl}(A^c) = (\mathrm{Int}(A))^c$. Substituting this expression directly into the formula gives $\partial_{\mathrm{PF}}(A) = \mathrm{Cl}(A) \cap (\mathrm{Int}(A))^c$, which matches the lattice expression for set difference.

Proof of (3): By definition, $\partial_{\mathrm{PF}}(\mathrm{Int}(A)) = \mathrm{Cl}(\mathrm{Int}(A)) \cap (\mathrm{Int}(\mathrm{Int}(A)))^c$. Since $\mathrm{Int}(A) \subseteq A$, by monotonicity of closure, $\mathrm{Cl}(\mathrm{Int}(A)) \subseteq \mathrm{Cl}(A)$. Also, $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$, so $(\mathrm{Int}(\mathrm{Int}(A)))^c = (\mathrm{Int}(A))^c$. Taking the intersection of these inclusions yields $\mathrm{Cl}(\mathrm{Int}(A)) \cap (\mathrm{Int}(\mathrm{Int}(A)))^c \subseteq \mathrm{Cl}(A) \cap (\mathrm{Int}(A))^c = \partial_{\mathrm{PF}}(A)$.

Proof of (4): By definition, $\partial_{\mathrm{PF}}(A \cup B) = \mathrm{Cl}(A \cup B) \cap \mathrm{Cl}((A \cup B)^c) = (\mathrm{Cl}(A) \cup \mathrm{Cl}(B)) \cap \mathrm{Cl}(A^c \cap B^c)$. Since $\mathrm{Cl}(A^c \cap B^c) \subseteq \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c)$, we expand the expression to obtain $(\mathrm{Cl}(A) \cup \mathrm{Cl}(B)) \cap (\mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c))$. Applying the distributive law for lattices yields $(\mathrm{Cl}(A) \cap \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c)) \cup (\mathrm{Cl}(B) \cap \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c))$. Since $\mathrm{Cl}(A) \cap \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c) \subseteq \mathrm{Cl}(A) \cap \mathrm{Cl}(A^c) = \partial_{\mathrm{PF}}(A)$ and $\mathrm{Cl}(B) \cap \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c) \subseteq \mathrm{Cl}(B) \cap \mathrm{Cl}(B^c) = \partial_{\mathrm{PF}}(B)$, we obtain $\partial_{\mathrm{PF}}(A \cup B) \subseteq \partial_{\mathrm{PF}}(A) \cup \partial_{\mathrm{PF}}(B)$.

□

3.3 Structural Bases and Neighborhood Systems

To analyze local spatial behavior, we establish bases and neighborhood systems within the space.

Definition 3.7 (Picture Fuzzy Base). *Let (X, τ) be a p.f.t.s. A subfamily $\mathcal{B} \subseteq \tau$ is defined as a* ***picture fuzzy base*** *for τ if every non-empty picture fuzzy open set $O \in \tau$ can be represented as an arbitrary union of elements from $\mathcal{B}$.*

Definition 3.8 (Picture Fuzzy Subbase). *A subfamily $\mathcal{S} \subseteq \tau$ is called a* ***picture fuzzy subbase*** *for τ if the collection of all finite intersections of elements in $\mathcal{S}$ forms a picture fuzzy base for τ.*

Theorem 3.6 (Characterization of a Base)**.** *A family $\mathcal{B} \subseteq \tau$ is a picture fuzzy base for τ if and only if for every picture fuzzy open set $A \in \tau$ and for each element $x \in X$, there exists a base element $B_x \in \mathcal{B}$ such that $B_x \subseteq A$.*

Proof. Suppose $\mathcal{B}$ is a picture fuzzy base for τ. Then any open set $A \in \tau$ can be expressed as $A = \bigcup_{i \in J} B_i$ where $B_i \in \mathcal{B}$ for all $i \in J$. It follows directly from the definition of lattice unions that $B_i \subseteq A$ for all $i \in J$. Thus, for any point $x \in X$, there exists at least one index i such that the localized parameters are bounded by B_i, meaning $B_i \subseteq A$.

Conversely, assume that for every open set $A \in \tau$, there exists a family of base elements $\{B_i\}_{i \in J} \subseteq \mathcal{B}$ such that $B_i \subseteq A$ and $A = \bigcup_{i \in J} B_i$. Since τ is closed under arbitrary unions, and each $B_i \in \mathcal{B} \subseteq \tau$, the union $\bigcup_{i \in J} B_i$ belongs to τ. This satisfies the identity condition, meaning $\mathcal{B}$ is a valid base for τ. $\square$

Definition 3.9 (Picture Fuzzy Neighborhood System)**.** *Let (X, τ) be a p.f.t.s. A picture fuzzy set $N \in PFS(X)$ is called a **neighborhood** of a picture fuzzy point if there exists an open set $O \in \tau$ such that the point belongs to O and $O \subseteq N$.*

3.4 Picture Fuzzy Continuity and Mapping Mechanics

We extend our topological space into the domain of continuous transformations, providing a link between distinct universes of discourse.

Definition 3.10 (Picture Fuzzy Mapping)**.** *Let X and Y be two non-empty universes and let $f : X \to Y$ be a classical mapping function. The **inverse image** of a picture fuzzy set $B \in PFS(Y)$ under f, denoted by $f^{-1}(B)$, is defined pointwise for all $x \in X$ as:*

$$f^{-1}(B) = \{\langle x, \mu_B(f(x)), \eta_B(f(x)), \nu_B(f(x))\rangle \mid x \in X\}$$

*Similarly, the **direct image** $f(A)$ for $A \in PFS(X)$ is defined pointwise for all $y \in Y$ by:*

$$f(A)(y) = \left\langle \bigvee_{f(x)=y} \mu_A(x), \bigwedge_{f(x)=y} \eta_A(x), \bigwedge_{f(x)=y} \nu_A(x) \right\rangle$$

if $f^{-1}(y) \neq \emptyset$, else $\langle 0, 0, 1\rangle$.

Definition 3.11 (Picture Fuzzy Continuity)**.** *Let (X, τ_X) and (Y, τ_Y) be two separate picture fuzzy topological spaces. A mapping function $f : X \to Y$ is defined as **picture fuzzy continuous** if the inverse image of every picture fuzzy open set in Y is a picture fuzzy open set in X:*

$$\forall V \in \tau_Y \implies f^{-1}(V) \in \tau_X$$

Theorem 3.7 (Characterization of Continuity)**.** *Let (X, τ_X) and (Y, τ_Y) be two spaces. A mapping function $f : X \to Y$ is picture fuzzy continuous if and only if the inverse image of every closed set in Y is a closed set in X.*

Proof. Suppose $f : X \to Y$ is a continuous picture fuzzy function. Let K be an arbitrary closed picture fuzzy set in Y. By definition, its complement K^{c} is an open set in Y, meaning $K^{\mathsf{c}} \in \tau_Y$. Applying the definition of continuity, the inverse image of this open set must be open in X: $f^{-1}(K^{\mathsf{c}}) \in \tau_X$.

Using the algebraic properties of inverse operations on lattices, the inverse image commutes with complementation: $f^{-1}(K^{\mathsf{c}}) = (f^{-1}(K))^{\mathsf{c}}$. Since $(f^{-1}(K))^{\mathsf{c}} \in \tau_X$, its complement $f^{-1}(K)$ is closed in X.

Conversely, assume that the inverse image of every closed set in Y is closed in X. Let $V \in \tau_Y$ be an open set in Y. Its complement V^{c} is closed in Y. By our structural assumption, $f^{-1}(V^{\mathsf{c}})$ is closed in X. Since $f^{-1}(V^{\mathsf{c}}) = (f^{-1}(V))^{\mathsf{c}}$, the set $(f^{-1}(V))^{\mathsf{c}}$ is closed, meaning its complement $f^{-1}(V)$ is open in X. Thus, $f^{-1}(V) \in \tau_X$, proving that f is continuous. □

3.5 Advanced Separation Axioms

To establish structural distinction criteria within our topological spaces, we define the picture fuzzy extensions of the classical T_0, T_1, and T_2 (Hausdorff) separation axioms.

Definition 3.12 (Picture Fuzzy T_0 Separation). *A p.f.t.s. (X, τ) is defined as a **picture fuzzy** T_0 **space** if for any pair of distinct picture fuzzy points with distinct support, there exists a picture fuzzy open set $O \in \tau$ such that one point belongs to O and the other does not.*

Definition 3.13 (Picture Fuzzy T_1 Separation). *A p.f.t.s. (X, τ) is defined as a **picture fuzzy** T_1 **space** if for any pair of distinct picture fuzzy points, there exist two picture fuzzy open sets $O_1, O_2 \in \tau$ such that the first point belongs to O_1 but not O_2, and vice versa.*

Definition 3.14 (Picture Fuzzy T_2 (Hausdorff) Space). *A p.f.t.s. (X, τ) is defined as a **picture fuzzy** T_2 **space** if for any pair of distinct picture fuzzy points, there exist disjoint open sets $O_1, O_2 \in \tau$ containing each point respectively such that $O_1 \cap O_2 = \mathbf{0}_{PF}$.*

Theorem 3.8. *Every picture fuzzy T_2 space is inherently a picture fuzzy T_1 space, and every picture fuzzy T_1 space is inherently a picture fuzzy T_0 space.*

Proof. Let (X, τ) be a picture fuzzy T_2 space. To prove it is T_1, consider two distinct points. Since the space satisfies the T_2 axiom, there exist open sets $O_1, O_2 \in \tau$ containing each point respectively such that $O_1 \cap O_2 = \mathbf{0}_{\mathrm{PF}}$. The condition $O_1 \cap O_2 = \mathbf{0}_{\mathrm{PF}}$ implies that no element can have positive membership in both sets simultaneously. Hence, the first point cannot belong to O_2 and the second point cannot belong to O_1. This satisfies the conditions for a T_1 space.

Next, let the space be T_1. For any distinct points, there exist open sets O_1 and O_2 such that the first point belongs to O_1 and the second does not. This directly satisfies the weaker existential condition for a T_0 space, which requires only one such set to exist. This completes our chain of verification. □

3.6 Relative Topology and Subspace Mechanics

We introduce the concept of induced structures on subsets of our universe of discourse.

Definition 3.15 (Picture Fuzzy Relative Topology). *Let (X, τ) be a p.f.t.s. and let Y be a non-empty subset of X. Let $Y_{\mathbf{1}}$ be the absolute picture fuzzy subset restricted to Y. The **picture fuzzy relative topology** (or subspace topology) on Y is defined as the collection:*

$$\tau_Y = \{O \cap Y_{\mathbf{1}} \mid O \in \tau\}$$

The pair (Y, τ_Y) forms a valid picture fuzzy topological subspace.

3.7 Dense and Nowhere Dense Picture Fuzzy Sets

We analyze spatial density distributions within our quaternary mathematical framework.

Definition 3.16 (Dense Picture Fuzzy Set). *Let (X, τ) be a p.f.t.s. A picture fuzzy set $A \in PFS(X)$ is said to be **dense** in X if its picture fuzzy closure is equal to the absolute universe set:*

$$Cl(A) = \mathbf{1}_{PF}$$

Definition 3.17 (Nowhere Dense Picture Fuzzy Set). *Let (X, τ) be a p.f.t.s. A picture fuzzy set $A \in PFS(X)$ is said to be **nowhere dense** in X if the interior of its closure is equal to the absolute empty set:*

$$Int(Cl(A)) = \mathbf{0}_{PF}$$

Theorem 3.9. *If A is a dense open set in a picture fuzzy topological space (X, τ), then its complement A^{c} is a nowhere dense set.*

Proof. Let $A \in \tau$ be a dense set, which means $\mathrm{Cl}(A) = \mathbf{1}_{\mathrm{PF}}$. We need to compute the interior of the closure of A^{c}, i.e., $\mathrm{Int}(\mathrm{Cl}(A^{\mathsf{c}}))$. From the duality properties, we know that $\mathrm{Cl}(A^{\mathsf{c}}) = (\mathrm{Int}(A))^{\mathsf{c}}$. Since A is given as an open set, we have $\mathrm{Int}(A) = A$. Substituting this gives:

$$\mathrm{Cl}(A^{\mathsf{c}}) = A^{\mathsf{c}}$$

Now, taking the interior of both sides yields $\mathrm{Int}(\mathrm{Cl}(A^{\mathsf{c}})) = \mathrm{Int}(A^{\mathsf{c}})$. Using the second duality property, the interior of a complement is equal to the complement of the closure: $\mathrm{Int}(A^{\mathsf{c}}) = (\mathrm{Cl}(A))^{\mathsf{c}}$. Since A is dense, we substitute $\mathrm{Cl}(A) = \mathbf{1}_{\mathrm{PF}}$ into the expression:

$$\mathrm{Int}(\mathrm{Cl}(A^{\mathsf{c}})) = (\mathbf{1}_{\mathrm{PF}})^{\mathsf{c}} = \mathbf{0}_{\mathrm{PF}}$$

This matches the formal definition of a nowhere dense set, completing the proof. □

3.8 Picture Fuzzy Compactness and Connectedness

We formally present the advanced properties of compactness and connectedness within the space.

Definition 3.18 (Picture Fuzzy Open Cover). *Let (X, τ) be a p.f.t.s. A collection of open sets $\mathcal{U} = \{O_i \mid i \in J\} \subseteq \tau$ is called an **open cover** of X if their arbitrary union equals the absolute universe set:*

$$\bigcup_{i \in J} O_i = \mathbf{1}_{PF}$$

Definition 3.19 (Picture Fuzzy Compact Space)**.** *A p.f.t.s. (X, τ) is defined as* ***compact*** *if every open cover $\mathcal{U}$ of X contains a finite subcollection $\mathcal{U}_0 = \{O_{i_1}, O_{i_2}, \ldots, O_{i_k}\}$ that forms a valid cover:*

$$\bigcup_{m=1}^{k} O_{i_m} = \mathbf{1}_{PF}$$

Theorem 3.10. *The continuous image of a compact picture fuzzy topological space is compact.*

Proof. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous picture fuzzy mapping, and let (X, τ_X) be compact. Let $\mathcal{V} = \{V_i \mid i \in J\}$ be an arbitrary open cover of Y, meaning $\bigcup_{i \in J} V_i = \mathbf{1}_{\mathrm{PF}_Y}$. Taking the inverse image on both sides and using the lattice-preserving properties of mappings yields:

$$f^{-1}\left(\bigcup_{i \in J} V_i\right) = \bigcup_{i \in J} f^{-1}(V_i) = f^{-1}(\mathbf{1}_{\mathrm{PF}_Y}) = \mathbf{1}_{\mathrm{PF}_X}$$

Since f is continuous, each $f^{-1}(V_i)$ is an open set in X. Thus, $\{f^{-1}(V_i) \mid i \in J\}$ forms an open cover of the space X. Because X is given as compact, there must exist a finite subcollection of indices $\{i_1, i_2, \ldots, i_k\}$ such that:

$$\bigcup_{m=1}^{k} f^{-1}(V_{i_m}) = \mathbf{1}_{\mathrm{PF}_X}$$

Applying the direct image mapping function f to both sides results in:

$$f\left(\bigcup_{m=1}^{k} f^{-1}(V_{i_m})\right) = \bigcup_{m=1}^{k} f(f^{-1}(V_{i_m})) \subseteq \bigcup_{m=1}^{k} V_{i_m} = \mathbf{1}_{\mathrm{PF}_Y}$$

This shows that the finite subcollection $\{V_{i_1}, \ldots, V_{i_k}\}$ covers Y. Hence, (Y, τ_Y) is compact. □

Definition 3.20 (Picture Fuzzy Connected Space)**.** *A p.f.t.s. (X, τ) is defined as* ***connected*** *if it cannot be expressed as the union of two non-empty, disjoint picture fuzzy open sets. That is, if $A, B \in \tau$ satisfy $A \cup B = \mathbf{1}_{PF}$ and $A \cap B = \mathbf{0}_{PF}$, then either $A = \mathbf{1}_{PF}$ or $B = \mathbf{1}_{PF}$.*

Compactness in Picture Fuzzy Topological Spaces

1 Definition of Compactness

Definition 1.1 (Picture Fuzzy Open Cover)**.** Let $(X, \mathcal{T}_{PF})$ be a picture fuzzy topological space. A collection $\{A_i\}_{i \in J} \subseteq \mathcal{T}_{PF}$ is called an open cover of X if:

$$\bigcup_{i \in J} A_i = 1_{PF}$$

where $1_{PF} = \{\langle x, 1, 0, 0\rangle \mid x \in X\}$.

Definition 1.2 (Picture Fuzzy Compactness)**.** A picture fuzzy topological space $(X, \mathcal{T}_{PF})$ be said to be compact if every open cover of X has a finite subcover. That is, for any collection $\{A_i\}_{i \in J} \subseteq \mathcal{T}_{PF}$ such that $\bigcup_{i \in J} A_i = 1_{PF}$, there exists a finite subset $J_0 \subseteq J$ such that:

$$\bigcup_{i \in J_0} A_i = 1_{PF}$$

Remark 1.3. Equivalently, a space is compact if it is impossible to find an open cover with no finite subcover. This definition generalizes classical compactness to the picture fuzzy context.

Definition 1.4 (Picture Fuzzy Subspace Compactness)**.** A subset $A \in \mathrm{PFS}(X)$ is called compact if every open cover of A (i.e., a collection of open sets whose union contains A) has a finite subcover.

2 Closed Subset Theorem

Theorem 2.1 (Closed Subset of a Compact Space is Compact)**.** *Let $(X, \mathcal{T}_{PF})$ be a compact picture fuzzy topological space and let C be a picture fuzzy closed set. Then C is compact.*

Proof. Let $\{U_i\}_{i \in J}$ be an open cover of C, i.e., $C \subseteq \bigcup_{i \in J} U_i$. Since C is closed, its complement C^c is open. Consider the collection:

$$\{U_i\}_{i \in J} \cup \{C^c\}$$

This is an open cover of X because:

$$(\bigcup_{i \in J} U_i) \cup C^c \supseteq C \cup C^c = 1_{PF}$$

By compactness of X, there exists a finite subcover. That is, there exist indices $i_1, i_2, \ldots, i_n \in J$ such that:

$$U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n} \cup C^c = 1_{PF}$$

Since $C \cap C^c = 0_{PF}$, the set C^c does not contribute to covering C. Therefore:

$$C \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}$$

Thus $\{U_{i_1}, \ldots, U_{i_n}\}$ is a finite subcover of C. Hence C is compact. □

Corollary 2.2. *In a compact picture fuzzy topological space, every closed picture fuzzy set is compact.*

Remark 2.3. The converse is not true: a compact subset need not be closed in a non-Hausdorff picture fuzzy topology. This mirrors classical topology.

3 Examples in Picture Fuzzy Topological Spaces

Example 3.1 (Finite Set with Discrete Topology)**.** Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set. Consider the discrete picture fuzzy topology $\mathcal{T}_{PFdis} = \text{PFS}(X)$ (all possible picture fuzzy sets). This space is compact.
Proof: Let $\{U_i\}_{i \in J}$ be any open cover of X. For each $x \in X$ there exists some U_{i_x} such that $\mu_{U_{i_x}}(x) > 0$ (since the union equals 1_{PF}). Since X is finite, the collection $\{U_{i_x} : x \in X\}$ is a finite subcover. Therefore X is compact.

Example 3.2 (Finite Set with Indiscrete Topology)**.** Let $X = \{x_1, x_2, \dots, x_n\}$ with $\mathcal{T}_{PFind} = \{0_{PF}, 1_{PF}\}$. The only open cover of X is $\{1_{PF}\}$ itself, which is already finite. Hence the space is compact.

Example 3.3 (Three-Point Space with Specific Topology)**.** Let $X = \{a, b, c\}$. Define picture fuzzy open sets:

$$\begin{aligned}
U_1 &= \{\langle a, 0.8, 0.1, 0.05\rangle, \langle b, 0.5, 0.3, 0.1\rangle, \langle c, 0.3, 0.4, 0.2\rangle\} \\
U_2 &= \{\langle a, 0.7, 0.1, 0.1\rangle, \langle b, 0.6, 0.2, 0.1\rangle, \langle c, 0.4, 0.3, 0.2\rangle\} \\
U_3 &= \{\langle a, 0.9, 0.05, 0.02\rangle, \langle b, 0.4, 0.3, 0.2\rangle, \langle c, 0.2, 0.4, 0.3\rangle\}
\end{aligned}$$

Let $\mathcal{T}_{PF} = \{0_{PF}, U_1, U_2, U_3, U_1 \cup U_2, U_1 \cup U_3, U_2 \cup U_3, U_1 \cap U_2, U_1 \cap U_3, U_2 \cap U_3, 1_{PF}\}$. This space is compact because any open cover must include sets whose union has $\mu = 1$ everywhere. Since there are only finitely many open sets, any cover is already finite or can be reduced to a finite subcover.

Example 3.4 (Two-Point Space with Four Open Sets)**.** Let $X = \{p, q\}$. Define:

$$\begin{aligned}
A &= \{\langle p, 0.6, 0.2, 0.1\rangle, \langle q, 0.4, 0.3, 0.2\rangle\} \\
B &= \{\langle p, 0.5, 0.2, 0.2\rangle, \langle q, 0.5, 0.2, 0.2\rangle\}
\end{aligned}$$

Let $\mathcal{T}_{PF} = \{0_{PF}, A, B, A \cup B, 1_{PF}\}$. Check compactness:

- Any open cover must cover p and q.
- A covers p with $\mu = 0.6$ but covers q with $\mu = 0.4$.
- B covers both with $\mu = 0.5$.
- $A \cup B$ covers with $\mu = \max(0.6, 0.5) = 0.6$ at p and $\mu = \max(0.4, 0.5) = 0.5$ at q.
- To achieve $\mu = 1$ at both points, we need 1_{PF} which is in $\mathcal{T}_{PF}$.

Thus any open cover must include 1_{PF} or a combination whose union equals 1_{PF}. Since there are finitely many sets, the space is compact.

Example 3.5 (Finite Set Always Compact)**. Theorem:** Every picture fuzzy topological space with a finite underlying set X is compact.
Proof: Let X be finite. For any open cover $\{U_i\}_{i \in J}$ consider the collection of distinct open sets in the cover. Since the power set of $\text{PFS}(X)$ on a finite set is finite (though large), there are only finitely many distinct picture fuzzy sets on X. Hence any cover can be reduced to a finite subcover by selecting at most one representative for each distinct open set. Therefore the space is compact.

Example 3.6 (Infinite Set with Indiscrete Topology)**.** Let $X = \mathbb{N}$ with $\mathcal{T}_{PFind} = \{0_{PF}, 1_{PF}\}$. This space is compact because the only open cover is $\{1_{PF}\}$, which is finite.

Example 3.7 (Unit Interval with Specific Picture Fuzzy Topology)**.** Let $X = [0, 1]$. Define picture fuzzy open sets as:

$$U_t\{\langle x, 1, 0, 0\rangle \text{ for } x < t, \langle x, 0, 0, 1\rangle \text{ for } x \geq t\}$$

for each $t \in [0, 1]$. Also include 0_{PF} and 1_{PF}. This topology is compact.
Proof: Let $\{U_{t_i}\}_{i \in J}$ be an open cover. Let $t^* = \inf\{t_i : i \in J\}$. The set U_{t^*} covers all points. More rigorously, consider the set of t values. The cover must include some U_t with $t = 0$ to cover points near 0, and the collection of t's has a minimum. The finite subcover can be constructed using that minimum.

Example 3.8 (Cantor Set with Picture Fuzzy Topology)**.** Let X be the Cantor set. Define a picture fuzzy topology where open sets are generated by clopen intervals in the Cantor set with membership functions $\mu = 1$ on the interval, $\nu = 0$ on the interval, and $\mu = 0$, $\nu = 1$ elsewhere. This space is compact because the Cantor set is compact in the classical sense, and the picture fuzzy topology inherits this property.

Example 3.9 (Product of Compact Spaces (Tychonoff))**.** Let $\{X_i\}_{i\in I}$ be a family of compact picture fuzzy topological spaces. Then the product space $\prod_{i\in I} X_i$ with the product topology is compact. This is the picture fuzzy analogue of the Tychonoff theorem.
Example: Let $X = \{0, 1\}$ with discrete picture fuzzy topology (compact). Then the infinite product $X^{\mathbb{N}}$ (the picture fuzzy Cantor space) is compact.

Example 3.10 (One-Point Compactification)**.** Let $X = \mathbb{N}$ with the discrete picture fuzzy topology (not compact, as we will see). The one-point compactification $X^* = X \cup \{\infty\}$ with neighborhoods of ∞ defined as complements of compact subsets of X is a compact picture fuzzy space.

Example 3.11 (Infinite Set with Discrete Topology)**.** Let $X = \mathbb{N}$ (natural numbers) with the discrete picture fuzzy topology $\mathcal{T}_{PFdis} = \mathrm{PFS}(X)$. This space is NOT compact.
Proof: For each $n \in \mathbb{N}$, define:

$$U_n\{\langle n, 1, 0, 0\rangle\} \cup \{\langle m, 0, 0, 1\rangle \text{ for } m \neq n\}$$

Then $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of X because each point n is covered by U_n with membership 1. However, no finite subcollection can cover all natural numbers because any finite subcollection $\{U_{n_1}, \ldots, U_{n_k}\}$ fails to cover a number m not in $\{n_1, \ldots, n_k\}$ (for that m, $\mu = 0$ in all selected sets). Hence the space is not compact.

Example 3.12 (Real Line with Standard Picture Fuzzy Topology)**.** Let $X = \mathbb{R}$. Define picture fuzzy open intervals:

$$(a, b)_{PF} = \{\langle x, 1, 0, 0\rangle \text{ for } x \in (a, b), \langle x, 0, 0, 1\rangle \text{ for } x \notin (a, b)\}$$

The collection $\{(n, n + 2)_{PF}\}_{n\in\mathbb{Z}}$ is an open cover of $\mathbb{R}$ with no finite subcover. Therefore $\mathbb{R}$ is not compact.

Example 3.13 (Open Unit Interval)**.** Let $X = (0, 1)$ with the subspace picture fuzzy topology inherited from $\mathbb{R}$. The cover $\{(\frac{1}{n+2}, \frac{1}{n})_{PF}\}_{n\in\mathbb{N}}$ has no finite subcover because any finite subcollection leaves out points near 0. Hence $(0, 1)$ is not compact.

Example 3.14 (Infinite Set with Co-finite Topology)**.** Let $X = \mathbb{N}$. Define the co-finite picture fuzzy topology where open sets are 0_{PF}, 1_{PF}, and all picture fuzzy sets U such that $\{x \in X \mid \mu_U(x) = 0 \text{ and } \nu_U(x) = 1\}$ is finite. This space is compact.
Proof: Let $\{U_i\}_{i\in J}$ be any open cover. Pick any U_{i_0} from the cover. The set of points where U_{i_0} does NOT have $\mu = 1$ is finite by the co-finite condition. For each such point, select one open set from the cover that covers it. The union of U_{i_0} with these finitely many additional sets forms a finite subcover.

Example 3.15 (Hilbert Cube with Picture Fuzzy Metric)**.** Let $X = [0, 1]^{\mathbb{N}}$ (Hilbert cube) with the picture fuzzy topology induced by the metric:

$$d_{PF}(x, y) = \left(\sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}, \text{neutral component}, \text{negative component}\right)$$

This space is compact (by the picture fuzzy analogue of the Tychonoff theorem or by showing sequential compactness).

Example 3.16 (Compact Picture Fuzzy Set that is Not Closed)**.** Let $X = \{a, b\}$ with topology $\mathcal{T}_{PF} = \{0_{PF}, U, 1_{PF}\}$ where $U = \{\langle a, 0.8, 0.1, 0.05\rangle, \langle b, 0, 0, 1\rangle\}$. Consider the picture fuzzy set:

$$A = \{\langle a, 0.9, 0.05, 0.02\rangle, \langle b, 0.5, 0.2, 0.2\rangle\}$$

Is A compact? Check any open cover of A. The only open sets are 0_{PF}, U, 1_{PF}. Since A is not contained in U (because at b, $\mu_A(b) = 0.5$ but $\mu_U(b) = 0$) we must include 1_{PF} in any cover. Hence any cover has a finite subcover (namely $\{1_{PF}\}$). So A is compact. But A is not closed because its complement is not open. This shows compact need not imply closed in non-Hausdorff spaces.

Example 3.17 (Finite Union of Compact Sets is Compact)**.** Let $(X, \mathcal{T}_{PF})$ be a picture fuzzy topological space and let A, B be compact picture fuzzy sets. Then $A \cup B$ is compact.
Proof: Let $\{U_i\}_{i\in J}$ be an open cover of $A \cup B$. Then it covers A and B separately. By compactness of A, there exists finite $J_A \subseteq J$ covering A. By compactness of B, there exists finite $J_B \subseteq J$ covering B. Then $J_A \cup J_B$ is a finite subcover of $A \cup B$.

Example 3.18 (Continuous Image of Compact is Compact)**.** Let $f : X \to Y$ be a picture fuzzy continuous function. If X is compact, then $f(X)$ is compact.
Proof: Let $\{V_i\}_{i\in J}$ be an open cover of $f(X)$. Then $\{f^{-1}(V_i)\}_{i\in J}$ is an open cover of X by continuity. By compactness of X, there exists a finite subcover $\{f^{-1}(V_{i_1}), \ldots, f^{-1}(V_{i_n})\}$. Then $\{V_{i_1}, \ldots, V_{i_n}\}$ covers $f(X)$.

Example 3.19 (Closed Unit Disk in Picture Fuzzy Plane)**.** Let $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ with the subspace picture fuzzy topology from $\mathbb{R}^2$. This space is compact because it is closed and bounded in $\mathbb{R}^2$ and the picture fuzzy topology respects the classical compactness.

Example 3.20 (Non-Compact: Infinite Discrete Subspace)**.** Let $X = \mathbb{N}$ with discrete topology. Consider the picture fuzzy set $A = 1_{PF}$ (the whole space). This is not compact as shown earlier. So even the whole space fails compactness.

Example 3.21 (Sequential Compactness in Action)**.** Let $X = [0, 1]$ with the standard picture fuzzy metric. Any sequence $\{x_n\}$ has a convergent subsequence by the Bolzano-Weierstrass theorem, confirming compactness.

4 Comparative Analysis of Spaces

Table 1: Comparison of Compactness between Finite and Infinite Spaces

Property	**Finite X**	**Infinite X**
Discrete Topology	Compact (always)	Not compact (requires infinite cover)
Indiscrete Topology	Compact	Compact
Co-finite Topology	Compact (trivial condition)	Compact (as proved above)
Metric Topology	Compact (finite sets are compact)	Not necessarily compact (e.g., $\mathbb{R}$)
Product Topology	Compact (finite product)	Compact only if each factor compact
Subspace of Compact	Compact (closed subset theorem)	Compact only if closed (in Hausdorff spaces)

5 Characterizations of Compactness

Theorem 5.1 (Finite Intersection Property)**.** *A picture fuzzy topological space $(X, \mathcal{T}_{PF})$ is compact if and only if for every family of closed sets $\{F_i\}_{i\in J}$ with the finite intersection property (every finite subfamily has non-empty intersection), the entire family has non-empty intersection.*

Proof. This follows from De Morgan's laws and the definition of compactness in terms of open covers. □

Theorem 5.2 (Sequential Compactness in Metric Spaces)**.** *In a picture fuzzy metric space, compactness is equivalent to sequential compactness (every sequence has a convergent subsequence).*

6 Summary of Topological Status

Table 2: Summary of Space Compactness Status and Reasons

Space	**Compact?**	**Reason**
Finite set, any topology	Yes	Any cover has finite distinct open sets
$\mathbb{N}$, discrete	No	Infinite cover with no finite subcover
$\mathbb{N}$, indiscrete	Yes	Only open cover is $\{1_{PF}\}$
$\mathbb{N}$, co-finite	Yes	Co-finite condition ensures finite subcover
$[0, 1]$, standard	Yes	Heine-Borel theorem
$(0, 1)$, standard	No	Cover by $(1/n, 1)$ has no finite subcover
$\mathbb{R}$, standard	No	Cover by $(n, n + 2)$ has no finite subcover
Cantor set	Yes	Closed subset of compact $[0, 1]$
Hilbert cube	Yes	Tychonoff theorem
One-point compactification of $\mathbb{N}$	Yes	By construction

Product Picture Fuzzy Topology

Let (X, τ_X) and (Y, τ_Y) be picture fuzzy topological spaces. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projection maps.

For a picture fuzzy set $U \in \tau_X$, the **preimage** $\pi_X^{-1}(U)$ is defined on $X \times Y$ by:

$$\pi_X^{-1}(U)(x, y) = U(x) = \langle \mu_U(x), \eta_U(x), \nu_U(x) \rangle$$

Similarly for $\pi_Y^{-1}(V)$ with $V \in \tau_Y$.

The **product picture fuzzy topology** $\tau_X \times \tau_Y$ on $X \times Y$ is the smallest picture fuzzy topology containing the subbasis:

$$\mathcal{S} = \{\pi_X^{-1}(U) \mid U \in \tau_X\} \cup \{\pi_Y^{-1}(V) \mid V \in \tau_Y\}$$

Thus a basis consists of all finite intersections:

$$\mathcal{B} = \{\pi_X^{-1}(U) \cap \pi_Y^{-1}(V) \mid U \in \tau_X,\ V \in \tau_Y\}$$

and arbitrary unions of basis elements form the topology.

Recall the intersection and union operations of two picture fuzzy sets A, B on $X \times Y$:

$$(A \cap B)(x, y) = \langle \mu_A \wedge \mu_B,\ \eta_A \wedge \eta_B,\ \nu_A \vee \nu_B \rangle$$

$$(A \cup B)(x, y) = \langle \mu_A \vee \mu_B,\ \eta_A \wedge \eta_B,\ \nu_A \wedge \nu_B \rangle$$

Examples

Example 1. *Let $X = \{x\}$, $Y = \{y\}$ be singleton sets. Let $\tau_X = \{0_\sim, 1_\sim, A\}$ where $A = \langle 0.7, 0.2, 0.1\rangle$. Let $\tau_Y = \{0_\sim, 1_\sim, B\}$ where $B = \langle 0.5, 0.3, 0.2\rangle$.*

1. *List all basis elements $\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$ for $U \in \tau_X$, $V \in \tau_Y$.*
2. *Write all open sets in the product topology on $X \times Y$.*

Solution 1. *Since $X \times Y = \{(x, y)\}$ is a singleton, each picture fuzzy set is uniquely determined by a single triple.*
Step 1: Compute preimages.
For any $U \in \tau_X$, $\pi_X^{-1}(U)(x, y) = U(x) = \langle \mu_U, \eta_U, \nu_U\rangle$.
Thus, we have:

$$\pi_X^{-1}(0_\sim) = \langle 0, 0, 1\rangle, \quad \pi_X^{-1}(1_\sim) = \langle 1, 0, 0\rangle, \quad \pi_X^{-1}(A) = \langle 0.7, 0.2, 0.1\rangle$$

$$\pi_Y^{-1}(0_\sim) = \langle 0, 0, 1\rangle, \quad \pi_Y^{-1}(1_\sim) = \langle 1, 0, 0\rangle, \quad \pi_Y^{-1}(B) = \langle 0.5, 0.3, 0.2\rangle$$

Step 2: Basis elements ($\mathcal{B}$)
By intersecting every preimage of τ_X with every preimage of τ_Y, we obtain:

- $\pi_X^{-1}(0_\sim) \cap$ *all preimages* $= \langle 0, 0, 1\rangle = 0_\sim$
- $\pi_X^{-1}(1_\sim) \cap \pi_Y^{-1}(1_\sim) = \langle 1, 0, 0\rangle = 1_\sim$
- $\pi_X^{-1}(1_\sim) \cap \pi_Y^{-1}(B) = \langle 1 \wedge 0.5, 0 \wedge 0.3, 0 \vee 0.2\rangle = \langle 0.5, 0, 0.2\rangle$
- $\pi_X^{-1}(A) \cap \pi_Y^{-1}(1_\sim) = \langle 0.7 \wedge 1, 0.2 \wedge 0, 0.1 \vee 0\rangle = \langle 0.7, 0, 0.1\rangle$
- $\pi_X^{-1}(A) \cap \pi_Y^{-1}(B) = \langle 0.7 \wedge 0.5, 0.2 \wedge 0.3, 0.1 \vee 0.2\rangle = \langle 0.5, 0.2, 0.2\rangle$

Hence, the basis is explicitly written as:

$$\mathcal{B} = \{\, 0_\sim,\ 1_\sim,\ \langle 0.5, 0, 0.2\rangle,\ \langle 0.7, 0, 0.1\rangle,\ \langle 0.5, 0.2, 0.2\rangle \,\}$$

Step 3: Open sets ($\tau_X \times \tau_Y$)
The open sets are formed by taking all possible arbitrary unions of the basis elements. Let us verify the non-trivial combinations using the standard union rule $\langle \mu_1 \vee \mu_2, \eta_1 \wedge \eta_2, \nu_1 \wedge \nu_2\rangle$:

$$\langle 0.7, 0, 0.1\rangle \cup \langle 0.5, 0.2, 0.2\rangle = \langle 0.7 \vee 0.5, 0 \wedge 0.2, 0.1 \wedge 0.2\rangle = \langle 0.7, 0, 0.1\rangle$$

$$\langle 0.5, 0, 0.2\rangle \cup \langle 0.5, 0.2, 0.2\rangle = \langle 0.5 \vee 0.5, 0 \wedge 0.2, 0.2 \wedge 0.2\rangle = \langle 0.5, 0, 0.2\rangle$$

$$\langle 0.7, 0, 0.1\rangle \cup \langle 0.5, 0, 0.2\rangle = \langle 0.7 \vee 0.5, 0 \wedge 0, 0.1 \wedge 0.2\rangle = \langle 0.7, 0, 0.1\rangle$$

Since all combinations result in elements that are already members of the basis, the topology is naturally closed under finite unions. Thus, the product topology contains exactly the same elements as $\mathcal{B}$:

$$\tau_X \times \tau_Y = \{\, 0_\sim,\ 1_\sim,\ \langle 0.5, 0, 0.2\rangle,\ \langle 0.7, 0, 0.1\rangle,\ \langle 0.5, 0.2, 0.2\rangle \,\}$$

Example 2. *Let $X = \{x_1, x_2\}$, $Y = \{y\}$ with τ_X discrete (all picture fuzzy sets are open) and $\tau_Y = \{0_\sim, 1_\sim\}$. Describe the product topology on $X \times Y$.*

Solution 2. *The product space $X \times Y = \{(x_1, y), (x_2, y)\}$ contains two distinct points.*

Because τ_Y is the indiscrete topology $\{0_\sim, 1_\sim\}$, its preimages are simply the global boundaries: $\pi_Y^{-1}(0_\sim) = 0_\sim$ and $\pi_Y^{-1}(1_\sim) = 1_\sim$.

When constructing the basis elements, intersecting any set with $1_\sim$ leaves it unchanged, and intersecting with $0_\sim$ yields $0_\sim$. Thus, the basis $\mathcal{B} = \{\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)\}$ simplifies directly to:

$$\mathcal{B} = \{\pi_X^{-1}(U) \mid U \in \tau_X\}$$

Since τ_X is discrete, it contains all possible picture fuzzy sets on X. Consequently, $\mathcal{B}$ contains every picture fuzzy set on $X \times Y$ whose values are constant across the Y coordinate (which is trivial here since Y is a singleton). Since the arbitrary union of these sets retains independence on the X coordinates, the product topology $\tau_X \times \tau_Y$ consists of all picture fuzzy sets on $X \times Y$.

Example 3. *Let $X = \{x\}$, $Y = \{y_1, y_2\}$ with $\tau_X = \{0_\sim, 1_\sim\}$ (indiscrete) and τ_Y discrete. Find the product topology on $X \times Y$.*

Solution 3. *Symmetrically to Exercise 2, the product space is $X \times Y = \{(x, y_1), (x, y_2)\}$.*

Since X is given the indiscrete topology, its preimages under π_X^{-1} provide no local restrictions on the single element x. On the other hand, τ_Y is discrete, meaning its preimages $\pi_Y^{-1}(V)$ represent all possible picture fuzzy configurations over Y.

Therefore, the basis elements are exactly the collections of all picture fuzzy sets on $X \times Y$. Since any arbitrary union of these sets yields another picture fuzzy set, the resulting product topology is completely discrete, containing all valid picture fuzzy sets on $X \times Y$.

Example 4. *Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$. Define $\tau_X = \{0_\sim, 1_\sim, P\}$ where*

$$P = \{\langle x_1, 0.6, 0.2, 0.1\rangle, \langle x_2, 0.4, 0.3, 0.2\rangle\}$$

Define $\tau_Y = \{0_\sim, 1_\sim, Q\}$ where

$$Q = \{\langle y_1, 0.5, 0.2, 0.2\rangle, \langle y_2, 0.7, 0.1, 0.1\rangle\}$$

Write explicitly the basis element $\pi_X^{-1}(P) \cap \pi_Y^{-1}(Q)$.

Solution 4. *Evaluating the component-wise intersections ($\wedge$ for μ and η, $\vee$ for ν) across all coordinate elements of $X \times Y$:*

For (x_1, y_1):

$$\langle 0.6 \wedge 0.5,\ 0.2 \wedge 0.2,\ 0.1 \vee 0.2\rangle = \langle 0.5, 0.2, 0.2\rangle$$

For (x_1, y_2):

$$\langle 0.6 \wedge 0.7,\ 0.2 \wedge 0.1,\ 0.1 \vee 0.1\rangle = \langle 0.6, 0.1, 0.1\rangle$$

For (x_2, y_1):

$$\langle 0.4 \wedge 0.5,\ 0.3 \wedge 0.2,\ 0.2 \vee 0.2\rangle = \langle 0.4, 0.2, 0.2\rangle$$

For (x_2, y_2):

$$\langle 0.4 \wedge 0.7,\ 0.3 \wedge 0.1,\ 0.2 \vee 0.1\rangle = \langle 0.4, 0.1, 0.2\rangle$$

Thus, the exact basis element is:

$$\pi_X^{-1}(P) \cap \pi_Y^{-1}(Q) = \left\{ \begin{array}{ll} (x_1, y_1): & \langle 0.5, 0.2, 0.2\rangle \\ (x_1, y_2): & \langle 0.6, 0.1, 0.1\rangle \\ (x_2, y_1): & \langle 0.4, 0.2, 0.2\rangle \\ (x_2, y_2): & \langle 0.4, 0.1, 0.2\rangle \end{array} \right\}$$

Example 5. *Prove that the projection map $\pi_X : (X \times Y, \tau_X \times \tau_Y) \to (X, \tau_X)$ is picture fuzzy continuous.*

Solution 5. *By definition, a map between picture fuzzy topological spaces is continuous if the preimage of every open set in the codomain is open in the domain space.*

Let $U \in \tau_X$. By construction, the subbasis $\mathcal{S}$ of the product topology $\tau_X \times \tau_Y$ is defined as:

$$\mathcal{S} = \{\pi_X^{-1}(U) \mid U \in \tau_X\} \cup \{\pi_Y^{-1}(V) \mid V \in \tau_Y\}$$

Since $\pi_X^{-1}(U) \in \mathcal{S}$ and by definition of a generated topology, every subbasis element belongs to the topology ($\mathcal{S} \subseteq \tau_X \times \tau_Y$), it follows directly that $\pi_X^{-1}(U)$ is an open set in the product space. Thus, π_X is picture fuzzy continuous.

Conclusion

In summary, this Master's thesis has successfully established a comprehensive and mathematically consistent exploration of **Picture Fuzzy Topology**[cite: 1]. By synthesizing algebraic picture fuzzy models with abstract geometric concepts, this research demonstrates that expanding the constraints of ordinary fuzzy structures to accommodate independent dimensions of neutrality and refusal significantly enhances the descriptive power of topological configurations[cite: 2]. The formal developments presented throughout this study confirm that the axiomatic foundations of general topology can be gracefully extended into higher-order fuzzy environments without losing structural rigor, thereby providing a more generalized platform for analyzing continuity under multi-layered uncertainty[cite: 3]. The academic trajectory of this investigation was systematically executed through the cohesive structure of three progressive chapters[cite: 4]. The research established its bedrock in the first chapter by conducting a thorough analysis of the algebraic foundations, shifting from classical fuzzy sets and intuitionistic fuzzy sets into the formal definition, axiomatic operations, and principal properties of picture fuzzy sets[cite: 5]. The second chapter provided the necessary mathematical background by examining fuzzy topological spaces under Chang's framework, deeply investigating the behavior of fuzzy open and closed sets, neighborhood systems, compactness, and the structural preservation of continuity[cite: 6]. These operational concepts directly enabled the realization of the third chapter, which constitutes the core contribution of this thesis, where we formally defined picture fuzzy topological spaces and systematically investigated their fundamental properties, picture fuzzy continuous mappings, interior and closure operators, compactness, and underlying separation axioms[cite: 7]. The theoretical and structural outcomes formalized in this work provide a solid foundation that opens several immediate horizons and perspectives for future research[cite: 8]. On a purely theoretical level, the axiomatic environment established herein can be extended to study more advanced topological invariants, such as connectedness, convergence of filters, and product spaces within the picture fuzzy setting[cite: 9]. From an applied perspective, the structural properties of picture fuzzy topology can be utilized to optimize computational models in data analysis, rough set approximations, and multi-criteria decision-making systems where information is affected by hesitation or fragmented opinions[cite: 10]. Ultimately, this thesis contributes a systematic, well-structured reference to the domain of fuzzy mathematics, serving as a reliable starting point for future academic investigations[cite: 11].

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ملخص البحث

تتمحور هذه المذكرة حول دراسة وتطوير البنية الرياضية لـ **التوبولوجيا الضبابية التصويرية** ، كـامتداد حديث ونوعي للتوبولوجيا الضبابية الحدسية. يهدف هذا العمل إلى تحليل الخصائص التوبولوجية الأساسية للمجموعات الضبابية التصويرية المعرفة بواسطة درجات الانتماء، الحياد، واللانتماء. ركزنا في هذا البحث على صياغة المفاهيم المحورية مثل الفضاءات الفرعية، الجوارات، والقواعد التوبولوجية، بالإضافة إلى بحث علاقتها بالبنيات الترتيبية المصاحبة لها، مما يوفر أرضية رياضية صلبة لتطبيقات النمذجة في ظل غياب اليقين.

الكلمات المفتاحية: توبولوجيا ضبابية تصويرية، فضاءات فرعية، جوارات، قواعد التوبولوجيا.

Abstract

This thesis is devoted to studying the mathematical structure of **Picture Fuzzy Topology**, representing a contemporary extension of intuitionistic fuzzy topology. The primary objective is to analyze the essential topological properties of picture fuzzy sets, characterized by membership, neutrality, and non-membership degrees. Our investigation focuses on formulating pivotal concepts such as picture fuzzy subspaces, neighborhood systems, and topological bases, while examining their relationships with associated order structures to provide a rigorous mathematical foundation for modeling under uncertainty.

Keywords: Picture fuzzy topology, Subspaces, Picture fuzzy neighborhoods, Topological bases.

Résumé

Ce mémoire est dédié à l'étude de la structure mathématique de la **Topologie Floue Picturale (Picture Fuzzy Topology)**, qui représente une extension contemporaine majeure de la topologie floue intuitionniste. L'objectif principal est d'analyser les propriétés topologiques essentielles des ensembles flous picturaux, caractérisés par leurs degrés d'appartenance, de neutralité et de non-appartenance. Notre recherche se concentre sur la formulation de concepts pivots tels que les sous-espaces flous picturaux, les systèmes de voisinages et les bases topologiques, offrant ainsi un fondement mathématique rigoureux pour la modélisation sous incertitude.

Mots-clés : Topologie floue picturale, Sous-espaces, Voisinages flous picturaux, Bases topologiques.