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Type-2 topology generated by type-2 fuzzy relation

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Contents

Introduction

Topology is one of the branches of mathematics that received great importance in studies due to its enormous applications in limited sets, discrete sets, and geometric spaces.
Lotfi Zadeh introduced in 1965 [14] the concept of the fuzzy set to treat mysterious phenomena by generalizing the concept to ordinary sets, the concept to of the fuzzy set was extended to fuzzy topology in 1968 by Chang [3], of general topology to the fuzzy topological spaces.
In 2018, Mishra [11] presented a fuzzy topology resulting from a fuzzy relation by giving some related concepts and properties, for a fuzzy topology generated by a fuzzy relation, a fuzzy topology generated by a fuzzy interval order [11].
The concept of fuzzy topology has been extended to the type-2 fuzzy topology with the proof of topological spaces and many related theorems [1]. The aim of this work, is to study how to construct a type-2 fuzzy topology by means of a type-2 fuzzy relation, based on the extension of concepts previously introduced by Mishra.
This memory contains four chapters:
In the first chapter, we mention the fuzzy sets and their operations(union, intersection) presented by Zadeh [14], and the fuzzy relations and their operations(union, intersection and composition). In addition to the fuzzy topology by Chang [3].
In the second chapter, we give the concept of the type-2 fuzzy sets, type-2 fuzzy relations and the type-2 fuzzy topology, and the operations on them.
In the third chapter, we present Mishra's(2018) work on the topology resulting from a fuzzy relation.
In the fourth chapter, based on concepts introduced by Mishra, we attempt to create a type-2 fuzzy topology generated by type-2 fuzzy relation.

Chapter 1

Preliminary on fuzzy sets, α-cuts, t-norms, fuzzy relations, fuzzy topology

In this chapter, we address the concept presented by Lotfi Zadeh about fuzzy sets and the operations on them, where we find some definitions and operation in the fuzzy set that are an extension of the normal set, such as simple operations (intersection, union) and other concepts that do not apply, such as the principle of non-contradiction. We also mention the definition of fuzzy relation and fuzzy topology and the processes that place on them.

1.1 Fuzzy sets

In this section, we define a fuzzy set, giving some properties and the operations that take place on it.

Definition 1.1 *[14] A fuzzy set in X is Characterized by a membership (Characteristic) function* $\mu_A : X \longrightarrow [0,1]$, *called the membership function of A.*

$$\mu_A : \quad X \longrightarrow [0,1]$$
$$x \longrightarrow \mu_A(x).$$

Example 1.2 *Let* $X = \{0,1,2\}$

a fuzzy set $A = \{(0,0.2),(1,0.4),(2,0.6)\}$.

1.1.1 Operations on Fuzzy Sets

Definitions and operations that include fuzzy sets are obvious extensions of the definitions corresponding to the normal set. The basic operations in the fuzzy set provided by Zadeh are containment, equality, union, intersection and complement.

Definition 1.3 *[14]*

-Containment: Let A and B two fuzzy sets. We say that set A contained in B ($A \subseteq B$) if

$$\mu_A(x) \leq \mu_B(x),\ \forall x \in X.$$

-Equality: Two fuzzy sets A and B are equal, $A = B$ if and only if

$$\mu_A(x) = \mu_B(x),\ \forall x \in X.$$

-Union: Two fuzzy sets A and B, whose membership functions is related to those of A and B, is the union of two fuzzy sets A and B with respective membership functions $\mu_A(x)$ and $\mu_B(x)$.

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\},\ \forall x \in X.$$

or, in abbreviated form

$$\mu_{A\cup B} = \mu_A \vee \mu_B$$

-Intersection: The intersection of two fuzzy sets A and B with respective functions $\mu_A(x)$ and $\mu_B(x)$ is a fuzzy set $A \cap B$, whose membership function is related to those of A and B

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\},\ \forall x \in X.$$

or, in abbreviated form

$$\mu_{A\cap B} = \mu_A \wedge \mu_B$$

-Complement: The complement of a fuzzy set A is denoted by $\tilde{A}$ and is defined by

$$\mu_{\tilde{A}}(x) = 1 - \mu_A(x),\ \forall x \in X.$$

Example 1.4 *Let A and B two fuzzy sets, where $X = \{x_1, x_2, x_3\}$.*

$$A = \{(x_1, 0.3), (x_2, 0.6), (x_3, 0)\}.$$
$$B = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}.$$

The union two fuzzy sets A and B, for $x = x_1$:

$$\begin{aligned} \mu_{A\cup B}(x_1) &= \max\{\mu_A(x_1), \mu_B(x_1)\} \\ &= \max\{0.3, 0.2\} \\ &= 0.3 \end{aligned}$$

For $x = x_2$:

$$\begin{aligned} \mu_{A\cup B}(x_2) &= \max\{\mu_A(x_2), \mu_B(x_2)\} \\ &= \max\{0.6, 0.8\} \\ &= 0.8 \end{aligned}$$

For $x = x_3$:

$$\begin{aligned} \mu_{A\cup B}(x_3) &= \max\{\mu_A(x_3), \mu_B(x_3)\} \\ &= \max\{0, 0.4\} \\ &= 0.4 \end{aligned}$$

$$A \cup B = \{(x_1, 0.3), (x_2, 0.8), (x_3, 0.4)\}.$$

In the same way we calculate the intersection: $\mu_{A\cap B}(x_1) = 0.2, \mu_{A\cap B}(x_2) = 0.6,$

$\mu_{A\cap B}(x_3) = 0.$

Hence, $A \cap B = \{(x_1, 0.2), (x_2, 0.6), (x_3, 0)\}.$

$-\tilde{A}$ *is complement of a fuzzy set A then*

$$\begin{aligned} \mu_{\tilde{A}}(x_1) &= 1 - \mu_A(x_1) \\ &= 1 - 0.3 \\ &= 0.7 \end{aligned}$$

$$\mu_{\tilde{A}}(x_2) = 0.4 \text{ and } \mu_{\tilde{A}}(x_3) = 1$$

$$\tilde{A} = \{(x_1, 0.7), (x_2, 0.4), (x_3, 1)\}.$$

1.1.2 Cartesian product

Definition 1.5 *[8] Cartesian product applied to multiple fuzzy sets can be defined as follows.*

Let $\mu_{A_1}(x), \mu_{A_2}(x), ..., \mu_{A_n}(x)$ *as membership functions of* $A_1, A_2, ..., A_n$*,*

for $\forall x_1 A_1, ..., x_n \in A_n$*. then, the probability for n-tuple* $(x_1, x_2, ..., x_n)$ *to be involved in fuzzy set* $A_1 \times A_2 \times ... \times A_n$ *is,* $\mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) = \min[\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)]$ *.*

Example 1.6

Lets $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2\}$ *and lets* A_1, A_2*, are two fuzzy subsets respectively defined on X and Y given by :*

$A_1 = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.1)\}$; $A_2 \{(y_1, 0.1), (y_2, 0.6)\}$.

$$\begin{aligned} \mu_{A_1 \times A_2} &= \{\langle (x_1, y_1), 0.1\rangle ; \langle (x_1, y_2), 0.2\rangle ; \langle (x_2, y_1), 0.1\rangle ; \\ &\quad \langle (x_2, y_2), 0.5\rangle ; \langle (x_3, y_1), 0.1\rangle ; \langle (x_3, y_2), 0.1\rangle\} . \end{aligned}$$

1.1.3 $\alpha - Cutset$

Definition 1.7 *[8] The* $\alpha - cutset$ A_α *is made up of members whose membership is not less then* α*.*

$$A_\alpha = \{x \in X | \ \mu_A(x) \geq \alpha\} .$$

Proposition 1.8 *[7] Lets A and B two fuzzy sets on X. then,*

1. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
2. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.
3. $(\overline{\tilde{A}})_\alpha \neq \tilde{A}_\alpha$.
4. *For any* $\alpha \leq \lambda$*, where* $0 \leq \lambda \leq 1$*, it is true that* $A_\lambda \subseteq A_\alpha$*, where* $A_0 = X$*.*

Example 1.9

Let $X = \{1, 2, 3, 4, 5\}$ *and A be a fuzzy subset of X where :*

$A = \{(1, 0.2), (2, 0.5), (3, 0.4), (4, 0.9), (5, 1)\}$, $B = \{(1, 0.1), (2, 0.2), (3, 0.3), (4, 0.4), (5, 0.5)\}$.

The $\alpha-$ cuts of the fuzzy set A are:

$A_{0.2} = \{x \in X \mu_A(x) \geq 0.2\} = \{1, 2, 3, 4, 5\}$.

$A_{0.5} = \{x \in X, \mu_A(x) \geq 0.5\} = \{2, 4, 5\}$.

$A_{0.4} = \{x \in X, \mu_A(x) \geq 0.4\} = \{2, 3, 4, 5\}$.

$A_{0.9} = \{x \in X, \mu_A(x) \geq 0.9\} = \{4, 5\}$.

$A_1 = \{x \in X, \mu_A(x) \geq 1\} = \{5\}$.

The union is :

$$\begin{aligned} (A \cup B)_{0.4} &= A_{0.4} \cup B_{0.4} \\ &= \{2, 3, 4, 5\} \cup \{4, 5\} \\ &= \{2, 3, 4, 5\} \end{aligned}$$

The intersection is:

$$\begin{aligned} (A \cap B)_{0.4} &= A_{0.4} \cap B_{0.4} \\ &= \{2, 3, 4, 5\} \cap \{4, 5\} \\ &= \{4, 5\} \end{aligned}$$

1.1.4 T-norms and T-conorms

Definition 1.10 *[12] A triangular norm or "t-norm" is binary operation on [0,1], and a common practice is to denote them by T, and write*

$$T : [0, 1] \times [0, 1] \longrightarrow [0, 1], \ \forall x, y \in [0, 1]$$

satisfying following conditions:

1. *Commutativity* $T(x, y) = T(y, x)$
2. *Associativity* $T(x, T(y, z)) = T(T(x, y), z)$
3. *Monotonicity* $T(x, y) \leq T(x, z) \quad (y \leq z)$
4. *Identity* $T(x, 1) = x$

Example 1.11 $T_M(x,y) = min(x,y) = x \wedge y$

$T_M : [0,1] \times [0,1] \longrightarrow [0,1]$

Let $x, y \in [0,1] \Rightarrow T_M(x,y) = min(x,y) = min(y,x) = T_M(y,x).$

Let $x \in [0,1] \Rightarrow x \leq 1 \Rightarrow T_M(x,1) = min(x,1) = x.$

$T_L(x,y) = max(x+y-1,0).$

$T_L : [0,1] \times [0,1] \longrightarrow [0,1]$

Let $x, y \in [0,1] \Rightarrow T_L(x,y) = max(x+y-1,0) = max(0,x+y-1) = T_L(y,x).$

let $x \in [0,1] \Rightarrow x \geq 0 \Rightarrow T_L(x,1) = max(x+1-1,0) = max(x,0) = x.$

Definition 1.12 *[12] A triangular cnorm (t-conorm) is a binary operation on [0,1] if only if*

1. *Commutativity* $T(x,y) = T(y,x)$
2. *Associativity* $T(x,T(y,z)) = T(T(x,y),z)$
3. *Monotonicity* $T(x,y) \leq T(x,z) \quad (y \leq z)$
4. *Identity* $T(x,0) = x$

Example 1.13 *Let* $T_M(x,y) = max(x,y)$

$T_M : [0,1] \times [0,1] \longrightarrow [0,1]$

Let $x, y \in [0,1] \Rightarrow T_M(x,y) = max(x,y) = max(y,x) = T_M(y,x).$

Let $x \in [0,1] \Rightarrow x \geq 0 \Rightarrow T_M(x,0) = x.$

1.2 Fuzzy relations

Definition 1.14 *[15] fuzzy relation from X to Y is a fuzzy subset of $X \times Y$ characterized by a membership function $\mu_{\mathcal{R}} : X \times Y \longrightarrow [0,1]$, which associates with each pair (x,y) its "grade of membership " $\mu_{\mathcal{R}}(x,y)$ in $\mathcal{R}$*

$$\mathcal{R} = \{(x,y), \mu_{\mathcal{R}}(x,y) | (x,y) \in X \times Y\}.$$

1.2.1 Operations on fuzzy relations

Definition 1.15 *(Containment)[9] The containment of a fuzzy relation $\mathcal{R}$ in a fuzzy relation Q is denoted by $\Re \subset Q$ and is defined $\mu_{\mathcal{R}} \leq \mu_Q$, which means*

$$\mu_{\mathcal{R}}(x,y) \leq \mu_Q(x,y), \forall (x,y) \in X \times Y.$$

Definition 1.16 *(Union)[9] The union of $\mathcal{R}$ and Q is denoted by $\mathcal{R} \cup Q$ and is defined by $\mu_{\mathcal{R}\cup Q} = \mu_{\mathcal{R}} \vee \mu_Q$, that is*

$$\mu_{\mathcal{R}\cup Q}(x,y) = \max\{\mu_{\mathcal{R}}(x,y), \mu_Q(x,y)\}, (x,y) \in X \times Y.$$

Definition 1.17 *(Intersection)[9] The intersection of $\mathcal{R}$ and Q is denoted by $\mathcal{R} \cap Q$ and is defined by $\mu_{\mathcal{R}\cap Q} = \mu_{\mathcal{R}} \wedge \mu_Q$, that is*

$$\mu_{\mathcal{R}\cap Q}(x,y) = \min\{\mu_{\mathcal{R}}(x,y), \mu_Q(x,y)\}, (x,y) \in X \times Y.$$

Definition 1.18 *(complement)[9] The complement of $\mathcal{R}$ is denoted by $\tilde{\mathcal{R}}$ is defined by*

$$\mu_{\tilde{\mathcal{R}}} = 1 - \mu_{\mathcal{R}}.$$

Example 1.19 *Lets $\mathcal{R}$ and Q two fuzzy relations on $X \times X$, where $X = \{x,y,z\}$, represented by its matrices $\mathcal{M}_{\mathcal{R}}$ and $\mathcal{M}_Q$ which are defined by the arrays :*

$\mathcal{R}$	x	y	z
x	1	0.2	0.6
y	0.8	1	0.5
z	0.3	0.7	1

and

Q	x	y	z
x	1	0.3	0.5
y	0.7	0.9	0.6
z	0.8	0.2	0

The matrices $\mathcal{M}_{\mathcal{R}}$ and $\mathcal{M}_{Q}$ correspond to the two fuzzy relations $\mathcal{R} \cup Q$ and $\mathcal{R} \cap Q$

$\mathcal{R} \cup Q$	x	y	z
x	1	0.3	0.6
y	0.8	1	0.6
z	0.8	0.7	1

and

$\mathcal{R} \cap Q$	x	y	z
x	1	0.2	0.5
y	0.7	0.9	0.5
z	0.3	0.2	0

$\tilde{\mathcal{R}}$ is complement of a fuzzy relation $\mathcal{R}$, so

$\tilde{\mathcal{R}}$	x	y	z
x	0	0.8	0.4
y	0.2	0	0.5
z	0.7	0.3	0

Definition 1.20 *composition of fuzzy relation(max-min composition)[16]*
Let $\mathcal{R}(x, y), (x, y) \in X \times Y$ and $Q(y, z), (y, z) \in Y \times Z$ be two fuzzy relations. The max-min composition $\mathcal{R}$ max-min Q is then fuzzy set

$$\mathcal{R} \circ Q = \{(x, z) \max \{\min(\mu_{\mathcal{R}}(x, y), \mu_{Q}(y, z))\}, x \in X, y \in Y, z \in Z\}.$$

Example 1.21 *Let $\mathcal{R} \subseteq X \times Y$ and $Q \subseteq Y \times Z$ two fuzzy relations as follows :*

$\mathcal{R}$	y_1	y_2	y_3
x_1	0.1	0.8	0.3
x_2	0.4	0.2	0.6

and

Q	z_1	z_2	z_3
y_1	0.2	0.6	0.7
y_2	0.9	0.3	0.5
y_3	0.4	0.5	0.8

The composition is $\mathcal{R} \circ Q$

$\mathcal{R} \circ Q$	z_1	z_2	z_3
x_1	0.1	0.3	0.5
x_2	0.4	0.5	0.6

1.3 Fuzzy topology

In order to give an overview of a fuzzy topology, we present the following definition, which includes a set of illustrative examples.

Definition 1.22 *[3] A fuzzy topology is a family τ of fuzzy sets in X which satisfies the following conditions:*

1. $\emptyset, X \in \tau$.

2. $A, B \in \tau$, *then* $A \cap B \in \tau$.

3. *If* $A_i \in \tau$ *for each* $i \in I$, *then* $\cup_I A_i \in \tau$.

τ *is called a fuzzy topology for X, and the pair* (X, τ) *is a fuzzy topological space.*

Example 1.23 *Let* $X = \{x, y\}$ *and* $\emptyset, X$ *and* A *be three fuzzy sets such that :*

$\emptyset = \{\langle x, 0\rangle, \langle y, 0\rangle\}; X = \{\langle x, 1\rangle, \langle y, 1\rangle\}.$

$A = \{\langle x, 0.8\rangle, \langle y, 0.3\rangle\}.$

$\emptyset \cap A = \{\langle x, 0\rangle, \langle y, 0\rangle\} = \emptyset; X \cap A = \{\langle x, 0.8\rangle, \langle y, 0.3\rangle\} = A$

$\emptyset \cup A = \{\langle x, 0.8\rangle, \langle y, 0.3\rangle\} = A; X \cup A = \{\langle x, 1\rangle, \langle y, 1\rangle\} = X$

So $\tau = \{\emptyset, X, A\}$ *is fuzzy topology and* (X, τ) *is fuzzy topologies space.*

Definition 1.24 *(fuzzy neighborhood)[3]*

A fuzzy set U in a fts (X, τ) is a neighborhood, or nbhd for short, of a fuzzy set A if and only if there exists an open fuzzy set 0 such that $A \subset 0 \subset U$.

Chapter 2

type-2 fuzzy sets,type-2 fuzzy relations and type-2 fuzzy topology

2.1 Type-2 fuzzy sets

[10] In this section, we define type-2 fuzzy sets and some important associated concepts. By doing this, we provide a simple collection of mathematically well-defined terms that will let us effectively communicate about type-2 fuzzy sets.

Definition 2.1 *[10] A type-2 fuzzy set, denoted $\tilde{\tilde{A}}$ is characterized by a type-2 membership function $\mu_{\tilde{\tilde{A}}}(x,u)$, where $x \in X$ and $u \in J_x \subseteq [0,1]$:*

$$\tilde{\tilde{A}} = \left\{((x,u), \mu_{\tilde{\tilde{A}}}(x,u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0,1]\right\}.$$

in which $0 \leq \mu_{\tilde{\tilde{A}}}(x,u) \leq 1$. $\tilde{\tilde{A}}$ can also be expressed as

$$\tilde{\tilde{A}} = \sum_{x\in X}\sum_{u\in J_x} \mu_{\tilde{\tilde{A}}}(x,u)/(x,u) J_x \subseteq [0,1].$$

Example 2.2 *Let $X = \{x_1, x_2, x_3\}$ and $\tilde{\tilde{A}}$ type-2 fuzzy set in universe X.*
$\tilde{\tilde{A}} = \{((x_1, 0.1), 0.2), ((x_2, 0.4), 0.6), ((x_2, 1), 0.5), ((x_3, 0.7), 0.3)\}$.

2.1.1 Operations on type-2 fuzzy sets

Consider the following two type-2 fuzzy sets $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ in the universe X. Let $\mu_{\tilde{\tilde{A}}(x)}$ and $\mu_{\tilde{\tilde{B}}(x)}$ be the these membership grades, which are indicated for each x $\in$ X,
$\mu_{\tilde{\tilde{A}}(x)} = \sum_{u\in J_x^u} f_x(u)/u$ and $\mu_{\tilde{\tilde{B}}(x)} = \sum_{w\in J_x^w} \mathrm{g}_x(w)/w$, where correspondingly $u \in J_x^u, w \in$

J_x^w indicate x and y primary membership $f_x(u)$, g $_x(w)$. The secondary membership (grades) of [0,1] are indicated x. The membership rating for the type-2 fuzzy sets union, intersection, and complement $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ have been defined as following way

Containment:[7] $\tilde{\tilde{A}}$ is a type-2 fuzzy set of $\tilde{\tilde{B}}$ denoted $\tilde{\tilde{A}} \subseteq \tilde{\tilde{B}}$ if $u < w$ and $f_x(u) \leq g_x(w)$ for every x$\in X$

Equality:[7] $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are equal type-2 fuzzy sets, denoted $\tilde{\tilde{A}} = \tilde{\tilde{B}}$ if $u = w$ and

$$f_x(u) = \mu_{\tilde{\tilde{A}}}(x,u) = g_x(w) = \mu_{\tilde{\tilde{B}}}(x,w) \ for\ every\ x \in X.$$

Join and meet

Define union and the intersection of two operations join and meet, where $\sqcup$ is join and $\sqcap$ is meet. The notation $\mu_{\tilde{\tilde{A}}}(x) \sqcup \mu_{\tilde{\tilde{B}}}(x)$ indicates the join between two membership functions $\mu_{\tilde{\tilde{A}}}(x)$ and $\mu_{\tilde{\tilde{B}}}(x)$, and $\mu_{\tilde{\tilde{A}}}(x) \sqcap \mu_{\tilde{\tilde{B}}}(x)$ indicates the meet between two membership functions $\mu_{\tilde{\tilde{A}}}(x)$ and $\mu_{\tilde{\tilde{B}}}(x)$. Explained as follows:

Union:[6] The union of two type-2 fuzzy sets

$$\begin{aligned}\tilde{\tilde{A}} \cup \tilde{\tilde{B}} \Leftrightarrow \mu_{\tilde{\tilde{A}} \cup \tilde{\tilde{B}}}(x) &= \mu_{\tilde{\tilde{A}}}(x) \sqcup \mu_{\tilde{\tilde{B}}}(x) \\ &= \sum_{u \in J_x^u} \sum_{w \in J_x^w} (f_x(u) \wedge g_x(w))/(u \vee w).\end{aligned} \tag{2.1}$$

Intersection:[6] The intersection of two type-2 fuzzy sets

$$\begin{aligned}\tilde{\tilde{A}} \cap \tilde{\tilde{B}} \Leftrightarrow \mu_{\tilde{\tilde{A}} \cap \tilde{\tilde{B}}}(x) &= \mu_{\tilde{\tilde{A}}}(x) \sqcap \mu_{\tilde{\tilde{B}}}(x) \\ &= \sum_{u \in J_x^u} \sum_{w \in J_x^w} (f_x(u) \wedge g_x(w))/(u \wedge w).\end{aligned} \tag{2.2}$$

where $\vee$ represents the max t-conorm and $\wedge$ represents a t-norm.

complement:[6]

$$\begin{aligned}\overline{\tilde{\tilde{A}}} \Leftrightarrow \mu_{\overline{\tilde{\tilde{A}}}}(x) &= \neg \mu_{\tilde{\tilde{A}}}(x) \\ &= \sum_{u \in J_x^u} f_x(u)/(1-u).\end{aligned} \tag{2.3}$$

Where,$\neg$ represent negation operation. The notation $\neg\mu_{\tilde{\tilde{A}}}(x)$ indicates the negation of the secondary membership function $\mu_{\tilde{\tilde{A}}}(x)$.

Example 2.3 *Let $X = \{x_1, x_2, x_3\}$ be a nonempty set, and let $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ two type-2 fuzzy sets in universe X.*

$\tilde{\tilde{A}} = \{((x_1, 0.2), 0.1), ((x_1, 0.4), 1), ((x_2, 0.3), 0.1), ((x_3, 0.5), 1), ((x_3, 0.6), 0.3)\}$.

$\tilde{\tilde{B}} = \{((x_1, 0.3), 0.1), ((x_2, 0.5), 0.3), ((x_2, 0.1), 1), ((x_3, 0.2), 1), ((x_3, 0.3), 1)\}$.

$\tilde{\tilde{A}} \cup \tilde{\tilde{B}}$ *for $x = x_1$:*

$$\begin{aligned}
\mu_{\tilde{\tilde{A}} \cup \tilde{\tilde{B}}}(x_1) &= \frac{0.1 \wedge 0.1}{0.2 \vee 0.3} + \frac{1 \wedge 0.1}{0.4 \vee 0.3} \\
&= \frac{0.1}{0.3} + \frac{0.1}{0.4} \\
&= \{(0.3, 0.1), (0.4, 0.1)\}
\end{aligned}$$

For $x = x_1$: $\tilde{\tilde{A}} \cup \tilde{\tilde{B}} = \{((x_1, 0.3), 0.1), ((x_1, 0.4), 0.1)\}$.

$\tilde{\tilde{A}} \cup \tilde{\tilde{B}}$ *for $x = x_2$:*

$$\begin{aligned}
\mu_{\tilde{\tilde{A}} \cup \tilde{\tilde{B}}}(x_2) &= \frac{0.1 \wedge 0.3}{0.3 \vee 0.5} + \frac{0.1 \wedge 1}{0.3 \vee 0.1} \\
&= \frac{0.1}{0.5} + \frac{0.1}{0.3} \\
&= \{(0.5, 0.1), (0.3, 0.1)\}
\end{aligned}$$

For $x = x_2$: $\tilde{\tilde{A}} \cup \tilde{\tilde{B}} = \{((x_2, 0.5), 0.1), ((x_2, 0.3), 0.1)\}$.

$\tilde{\tilde{A}} \cup \tilde{\tilde{B}}$ *for $x = x_3$:*

$$\begin{aligned}
\mu_{\tilde{\tilde{A}} \cup \tilde{\tilde{B}}}(x_3) &= \frac{1 \wedge 1}{0.5 \vee 0.2} + \frac{1 \wedge 1}{0.5 \vee 0.3} + \frac{0.3 \wedge 1}{0.6 \vee 0.2} + \frac{0.3 \wedge 1}{0.6 \vee 0.3} \\
&= \frac{1}{0.5} + \frac{1}{0.5} + \frac{0.3}{0.6} + \frac{0.3}{0.6} \\
&= \{(0.5, 1), (0.6, 0.3)\}
\end{aligned}$$

For $x = x_3$: $\tilde{\tilde{A}} \cup \tilde{\tilde{B}} = \{((x_3, 0.5), 1), ((x_3, 0.6), 0.3)\}$.

Hence,

$$\begin{aligned}
\tilde{\tilde{A}} \cup \tilde{\tilde{B}} = \; & \{((x_1, 0.3), 0.1), ((x_1, 0.4), 0.1), ((x_2, 0.5), 0.1), \\
& ((x_2, 0.3), 0.1), ((x_3, 0.5), 1), ((x_3, 0.6), 0.3)\}.
\end{aligned}$$

Example 2.4 *with the same values as the previous example(example 2.1), we study the intersection.*

$\tilde{\tilde{A}} \cap \tilde{\tilde{B}}$ *for* $x = x_1$:

$$\begin{aligned}
\mu_{\tilde{\tilde{A}}\cap\tilde{\tilde{B}}}(x_1) &= \frac{0.1 \wedge 0.1}{0.2 \wedge 0.3} + \frac{1 \wedge 1}{0.4 \wedge 0.5} \\
&= \frac{0.1}{0.2} + \frac{1}{0.4} \\
&= \{(0.2, 0.1), (0.4, 1)\}
\end{aligned}$$

$\tilde{\tilde{A}} \cap \tilde{\tilde{B}} = \{((x_1, 0.2), 0.1), ((x_1, 0.4), 1)\}$.

$\tilde{\tilde{A}} \cap \tilde{\tilde{B}}$ *for* $x = x_2$:

$$\begin{aligned}
\mu_{\tilde{\tilde{A}}\cap\tilde{\tilde{B}}}(x_2) &= \frac{0.1 \wedge 0.3}{0.3 \wedge 0.5} + \frac{0.1 \wedge 1}{0.3 \wedge 0.1} \\
&= \frac{0.1}{0.3} + \frac{0.1}{0.1} \\
&= \{(0.3, 0.1), (0.1, 0.1)\}
\end{aligned}$$

$\tilde{\tilde{A}} \cap \tilde{\tilde{B}} = \{((x_2, 0.3), 0.1), ((x_2, 0.1), 0.1)\}$.

$\tilde{\tilde{A}} \cap \tilde{\tilde{B}}$ *for* $x = x_3$:

$$\begin{aligned}
\mu_{\tilde{\tilde{A}}\cap\tilde{\tilde{B}}}(x_3) &= \frac{1 \wedge 1}{0.5 \wedge 0.2} + \frac{1 \wedge 1}{0.5 \wedge 0.3} + \frac{0.3 \wedge 1}{0.6 \wedge 0.2} + \frac{0.3 \wedge 1}{0.6 \wedge 0.3} \\
&= \frac{1}{0.2} + \frac{1}{0.3} + \frac{0.3}{0.2} + \frac{0.3}{0.3} \\
&= \{(0.2, 1), (0.3, 1), (0.2, 0.3), (0.3, 0.3)\}
\end{aligned}$$

$\tilde{\tilde{A}} \cap \tilde{\tilde{B}} = \{((x_3, 0.2), 1), ((x_3, 0.3), 1), ((x_3, 0.2), 0.3), ((x_3, 0.3), 0.3)\}$.

Hence,

$$\begin{aligned}
\tilde{\tilde{A}} \cap \tilde{\tilde{B}} &= \{((x_1, 0.2), 0.1), ((x_1, 0.4), 1), ((x_2, 0.3), 0.1), \\
&\quad ((x_2, 0.1), 0.1), ((x_3, 0.2), 1), ((x_3, 0.2), 0.3)\}.
\end{aligned}$$

Example 2.5 *(complement)*

Let $X = \{x_1, x_2, x_3\}$ *be a nonempty set, and* $\tilde{\tilde{A}}$ *type-2 fuzzy set in universe* X.

$\tilde{\tilde{A}} = \{((x_1, 0.4), 1), ((x_1, 0.7), 0.2), ((x_2, 0.3), 0.1), ((x_3, 0.6), 1), ((x_3, 0.2), 0.3)\}$.

For $x = x_1$:

$$\begin{aligned}\overline{\tilde{\tilde{A}}}(x_1) &= \frac{1}{1-0.4} + \frac{1}{1-0.7} \\ &= \frac{1}{0.6} + \frac{1}{0.3} \\ &= \{((x_1, 0.6), 1), ((x_1, 0.3), 1)\}\end{aligned}$$

For $x = x_2$:

$$\begin{aligned}\overline{\tilde{\tilde{A}}} &= \frac{0.1}{1-0.3} = \frac{0.1}{0.7} \\ &= \{((x_2, 0.7), 0.1)\}\end{aligned}$$

For $x = x_3$:

$$\begin{aligned}\overline{\tilde{\tilde{A}}} &= \frac{1}{1-0.6} + \frac{0.3}{1-0.2} \\ &= \frac{1}{0.4} + \frac{0.3}{0.8} \\ &= \{((x_3, 0.4), 1), ((x_3, 0.8), 0.3)\}\end{aligned}$$

so $\overline{\tilde{\tilde{A}}} = \{((x_1, 0.6), 1), ((x_1, 0.3), 0.2), ((x_2, 0.7), 0.1), ((x_3, 0.4), 1), ((x_3, 0.8), 0.3)\}$.

Theorem 2.6 *[13] Assume that* $\tilde{\tilde{A}}, \tilde{\tilde{B}}$ *and* $\tilde{\tilde{C}}$ *are type-2 fuzzy sets on X. For any* $x \in X$*, If secondary membership functions* $\mu_{\tilde{\tilde{A}}}(x), \mu_{\tilde{\tilde{B}}}(x)$ *and* $\mu_{\tilde{\tilde{C}}}(x)$ *are convex fuzzy sets, we obtain*

$$(\tilde{\tilde{A}} \cup \tilde{\tilde{B}}) \cap \tilde{\tilde{C}} = (\tilde{\tilde{A}} \cap \tilde{\tilde{C}}) \cup (\tilde{\tilde{B}} \cap \tilde{\tilde{C}}) \tag{2.4}$$

and

$$(\tilde{\tilde{A}} \cap \tilde{\tilde{B}}) \cup \tilde{\tilde{C}} = (\tilde{\tilde{A}} \cup \tilde{\tilde{C}}) \cap (\tilde{\tilde{B}} \cup \tilde{\tilde{C}}) \tag{2.5}$$

Proof. [13] For any x $\in$ X, since secondary membership functions $\mu_{\tilde{\tilde{A}}}(x), \mu_{\tilde{\tilde{B}}}(x)$ and $\mu_{\tilde{\tilde{C}}}(x)$ are convex T1 fuzzy sets in [0,1], from [1], they satisfy distributive laws under $\sqcup$ and $\sqcap$.

As a result, we obtain

$$\left[\mu_{\tilde{\tilde{A}}}(x) \sqcup \mu_{\tilde{\tilde{B}}}(x)\right] \sqcap \mu_{\tilde{\tilde{C}}}(x) = \left[\mu_{\tilde{\tilde{A}}}(x) \sqcap \mu_{\tilde{\tilde{C}}}(x)\right] \sqcup \left[\mu_{\tilde{\tilde{B}}}(x) \sqcap \mu_{\tilde{\tilde{C}}}(x)\right]$$

and

$$\left[\mu_{\tilde{\tilde{A}}}(x) \sqcap \mu_{\tilde{\tilde{B}}}(x)\right] \sqcup \mu_{\tilde{\tilde{C}}}$$
$$= \left[\mu_{\tilde{\tilde{A}}}(x) \sqcup \mu_{\tilde{\tilde{C}}}(x)\right] \sqcap \left[\mu_{\tilde{\tilde{B}}}(x) \sqcup \mu_{\tilde{\tilde{C}}}(x)\right]$$

□

Operations under collection of type-2 fuzzy sets [1]

Let $\left\{\tilde{\tilde{A}}_i : i \in \mathbb{N}\right\}$ be an arbitrary collection of type-2 fuzzy sets subset of X such that $\mathbb{N}$ is a countable set, operation are possible under an arbitrary collection of type-2 fuzzy sets.

1. The union $\bigcup_{i\in\mathbb{N}} \tilde{\tilde{A}}$ is defined as

$$\left[\bigcup_{i\in\mathbb{N}} \tilde{\tilde{A}}_i\right](x) = \sum_{x\in X}\sum_{u\in J_x^u} \frac{\wedge_{i\in\mathbb{N}}(f_x(u))_i}{\vee_{i\in\mathbb{N}}(u)_i} \tag{2.6}$$

2. The intersection $\bigcap_{i\in\mathbb{N}} \tilde{\tilde{A}}_i$ is defined as

$$\left[\bigcap_{i\in\mathbb{N}} \tilde{\tilde{A}}_i\right](x) = \sum_{x\in X}\sum_{u\in J_x^u} \frac{\wedge_{i\in\mathbb{N}}(f_x(u))_i}{\wedge_{i\in\mathbb{N}}(u)_i} \tag{2.7}$$

Proposition 2.7 *[1] Let $\left\{\tilde{\tilde{A}}_i : i \in \mathbb{N}\right\}$ be an arbitrary collection of type-2 fuzzy sets subset of X such that $\mathbb{N}$ is a countable set and $\tilde{\tilde{B}}$ be another type-2 fuzzy set of X, then*

1. $\tilde{\tilde{B}} \bigcap \left[\bigcup_{i\in\mathbb{N}} \tilde{\tilde{A}}_i\right] = \bigcup_{i\in\mathbb{N}} \left(\tilde{\tilde{B}} \bigcap \tilde{\tilde{A}}_i\right)$
2. $\tilde{\tilde{B}} \bigcup \left[\bigcap_{i\in\mathbb{N}} \tilde{\tilde{A}}_i\right] = \bigcap_{i\in\mathbb{N}} \left(\tilde{\tilde{B}} \bigcup \tilde{\tilde{A}}_i\right)$
3. $1 - \left[\bigcup_{i\in\mathbb{N}} \tilde{\tilde{A}}_i\right] = \bigcap_{i\in\mathbb{N}} \left(1 - \tilde{\tilde{A}}_i\right)$
4. $1 - \left[\bigcap_{i\in\mathbb{N}} \tilde{\tilde{A}}_i\right] = \bigcup_{i\in\mathbb{N}} \left(1 - \tilde{\tilde{A}}_i\right)$

2.2 Type-2 fuzzy relations

Definition 2.8 *[2] A type-2 fuzzy relation is a type-2 fuzzy set defined on the Cartesian product of the crisp sets $X_1, X_2, ..., X_n$; that is, where the tuples $(x_1, x_2, ..., x_3)$ have different degrees of membership and are type-1 fuzzy sets. In other words, the type-2 fuzzy relation denotes a degree of membership that is a type-1 fuzzy set and of itself, rather than a number in the range [0,1]. It's easy to see why this is called a fuzzy-valued fuzzy relation [1].*

The type-2 fuzzy relation $\tilde{\mathcal{R}}$ on $X \times Y$ is considered, where $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_m\}$. $\tilde{\mathcal{R}}$ The type-2 fuzzy relation that may be written follows :

$$\mu_{\tilde{\mathcal{R}}}(X, Y) = \begin{pmatrix} \mu_{\tilde{\mathcal{R}}}(x_1, y_1) & \cdots & \mu_{\tilde{\mathcal{R}}}(x_1, y_m) \\ \mu_{\tilde{\mathcal{R}}}(x_2, y_1) & \cdots & \mu_{\tilde{\mathcal{R}}}(x_2, y_m) \\ \vdots & \ddots & \vdots \\ \mu_{\tilde{\mathcal{R}}}(x_n, y_1) & \cdots & \mu_{\tilde{\mathcal{R}}}(x_n, y_m) \end{pmatrix}$$

where each $\mu_{\tilde{\mathcal{R}}}(x_i, y_j)$ is a type-1 fuzzy set or a secondary membership grade of $\tilde{\mathcal{R}}$.

2.2.1 Composition of type-2 fuzzy relations

[6] If $\mathcal{R}(\tilde{\mathcal{R}})$ and $\mathcal{S}(\tilde{\mathcal{S}})$ are two type-1 (type-2) fuzzy relations on U $\times$ V and V $\times$ W , respectively, then the membership for any pair (u,w), u $\in$ U and w $\in$ W, is non-zero if and only if there exists at least one v $\times$ V such that $\mu_{\mathcal{R}}$(u, v) $\neq 0$ ($\mu_{\tilde{\mathcal{R}}}$(u,v) $\neq 0$) and $\mu_{\mathcal{S}}$(v,w) $\neq 1/0$ ($\mu_{\tilde{\mathcal{S}}}$(v,w) $\neq 1/0$).

$$\mu_{\tilde{\mathcal{R}} \circ \tilde{\mathcal{S}}}(u, w) = \bigsqcup_{v \in V} \left[\mu_{\tilde{\mathcal{R}}}(u, v) \sqcap \mu_{\tilde{\mathcal{S}}}(v, w) \right].$$

Example 2.9 *Let $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{S}}$ two type-2 fuzzy relations, $\tilde{\mathcal{R}} \in (X \times Y)$, $\tilde{\mathcal{S}} \in (Y \times Z)$, where $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2, z_3\}$.*

$$\tilde{\mathcal{R}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} \qquad\qquad\qquad y_1 & y_2 \qquad\qquad\qquad \end{array} \\ \begin{bmatrix} \frac{0.2}{0.3} + \frac{1}{0.5} + \frac{0.4}{0.7} & \frac{0.6}{0.6} + \frac{0.8}{0.4} + \frac{0.3}{0.1} \\ \frac{0.9}{0.1} + \frac{0.6}{0.8} + \frac{0.4}{0.4} & \frac{1}{0.7} + \frac{0.4}{0.9} + \frac{0.8}{0.8} \\ \frac{1}{0.8} + \frac{0.5}{0} + \frac{0.2}{0.6} & \frac{0.7}{0.6} + \frac{1}{0.2} + \frac{0.3}{0.6} \end{bmatrix} \end{array}$$

$$\tilde{\mathcal{S}} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \left[\begin{array}{ccc} \frac{0.2}{1} + \frac{05}{0} + \frac{0.3}{0.4} & \frac{0.7}{0.7} + \frac{0.3}{0.5} + \frac{0.8}{0.4} & \frac{0.2}{0.6} + \frac{1}{0.7} + \frac{0.9}{0.5} \\ \frac{0.8}{1} + \frac{0.6}{0.2} + \frac{0}{0.9} & \frac{0.4}{0.7} + \frac{0.2}{0.1} + \frac{0.1}{0.3} & \frac{0.6}{0} + \frac{1}{0.7} + \frac{0.8}{0.8} \end{array} \right] \end{array}$$

$\mu_{\tilde{\mathcal{R}}\circ\tilde{\mathcal{S}}}(x_i, z_j) = \bigsqcup\left[\mu_{\tilde{\mathcal{R}}}(x_i, y_1) \sqcap \mu_{\tilde{\mathcal{S}}}(y1, z_j)\right]$. *where* $i = \{1, 2, 3\}$, $j = \{1, 2, 3\}$.

$$\tilde{\mathcal{R}} \circ \tilde{\mathcal{S}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \left[\begin{array}{ccc} \frac{0.2}{0.3} + \frac{0.5}{0.2} + \frac{0}{0.7} & \frac{0.2}{0.6} + \frac{0.2}{0.5} + \frac{0.1}{0.4} & \frac{0.2}{0.3} + \frac{0.8}{0.5} + \frac{0.3}{0.5} \\ \frac{0.2}{0.7} + \frac{0.4}{0.2} + \frac{0}{0.4} & \frac{0.4}{0.7} + \frac{0.2}{0.5} + \frac{0.1}{0.4} & \frac{0.2}{0.1} + \frac{0.4}{0.7} + \frac{0.4}{0.8} \\ \frac{0.2}{0.6} + \frac{0.5}{0.2} + \frac{0.2}{0.5} & \frac{0.4}{0.7} + \frac{0.2}{0.1} + \frac{0.1}{0.4} & \frac{0.2}{0.6} + \frac{0.5}{0.2} + \frac{0.2}{0.6} \end{array} \right] \end{array}$$

Theorem 2.10 *[13] Suppose that $\tilde{\mathcal{R}}, \tilde{\mathcal{S}}$ and $\tilde{\mathcal{L}}$ are Type-2 fuzzy relations on product spaces $U\times V$, $V\times W$ and $W\times Q$, respectively, and $\mu_{\tilde{\mathcal{R}}}(u,v)$, $\mu_{\tilde{\mathcal{S}}}(v,w)$ and $\mu_{\tilde{\mathcal{L}}}(w,q)$ are convex T1 fuzzy sets. If spaces V and W are finite, then*

$$(\tilde{\mathcal{R}} \circ \tilde{\mathcal{S}}) \circ \tilde{\mathcal{L}} = \tilde{\mathcal{R}} \circ (\tilde{\mathcal{S}} \circ \tilde{\mathcal{L}})$$

Proof. [7] From Definition 1, for any (u,q) $\in$ U×Q, we have

$$\begin{aligned} \mu_{(\tilde{\mathcal{R}}\circ\tilde{\mathcal{S}})\circ\tilde{\mathcal{L}}}(u, q) &= \sqcup_{w\in W}\left[\mu_{\tilde{\mathcal{R}}\circ\tilde{\mathcal{S}}}(u, w) \sqcap \mu_{\tilde{\mathcal{L}}}(w, q)\right] \\ &= \sqcup_{w\in W}\left[\sqcup_{v\in V}\left[\mu_{\tilde{\mathcal{R}}}(u, v) \sqcap \mu_{\tilde{\mathcal{S}}}(v, w)\right] \sqcap \mu_{\tilde{\mathcal{L}}}(w, q)\right] \end{aligned}$$

$\mu_{\tilde{\mathcal{R}}}(u, v) \sqcap \mu_{\tilde{\mathcal{S}}}(v, w)$ is a convex type-1 fuzzy set in [0,1]

$$\begin{aligned} \mu_{\tilde{\mathcal{R}}\circ(\tilde{\mathcal{S}}\circ\tilde{\mathcal{L}})} &= \sqcup_{v\in V}\left[\mu_{\tilde{\mathcal{R}}}(u, v) \sqcap \mu_{\tilde{\mathcal{S}}\circ\tilde{\mathcal{L}}}(v, q)\right] \\ &= \sqcup_{v\in V}\left[\mu_{\tilde{\mathcal{R}}}(u, v) \sqcup_{w\in W}\left[\mu_{\tilde{\mathcal{S}}}(v, w) \sqcap \mu_{\tilde{\mathcal{L}}}(w, q)\right]\right] \\ &= \sqcup_{w\in W}\left[\sqcup_{v\in V}\left[\mu_{\tilde{\mathcal{R}}}(u, v) \sqcap \mu_{\tilde{\mathcal{S}}}(v, w) \sqcap \mu_{\tilde{\mathcal{L}}}(w, q)\right]\right] \end{aligned}$$

implies $\mu_{(\tilde{\mathcal{R}}\circ\tilde{\mathcal{S}})\circ\tilde{\mathcal{L}}}$ (u,q)= $\mu_{\tilde{\mathcal{R}}\circ(\tilde{\mathcal{S}}\circ\tilde{\mathcal{L}})}$ (u,q) for any (u,q) $\in U \times Q$ $\square$

Theorem 2.11 *[13] Let $\tilde{\mathcal{R}}, \tilde{\mathcal{S}}$ be two T2 fuzzy relations defined on $U\times V$, and $\tilde{\mathcal{L}}$ be a T2 fuzzy relation on $V\times W$. If $\mu_{\tilde{\mathcal{R}}}(u,v)$, $\mu_{\tilde{\mathcal{S}}}(u,v)$ and $\mu_{\tilde{\mathcal{L}}}(v,w)$ are convex fuzzy sets for any (u,v) $\in U\times V$ and (v,w) $\in V\times W$, then we have*

$$(\tilde{\mathcal{R}} \cup \tilde{\mathcal{S}}) \circ \tilde{\mathcal{L}} = (\tilde{\mathcal{R}} \circ \tilde{\mathcal{L}}) \cup (\tilde{\mathcal{S}} \circ \tilde{\mathcal{L}})$$

Proof. [7]

$$\begin{aligned}\mu_{(\tilde{\mathcal{R}}\cup\tilde{\mathcal{S}})\circ\tilde{\mathcal{L}}}(u,w) &= \bigsqcup_{v\in V}[\mu_{\tilde{\mathcal{R}}\cup\tilde{\mathcal{S}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]\\ &= \bigsqcup_{v\in V}[[\mu_{\tilde{\mathcal{R}}}(u,v)\sqcup\mu_{\tilde{\mathcal{S}}}(u,v)]\sqcap\mu_{\tilde{\mathcal{L}}}(u,w)]\end{aligned}$$

for any $(u,w)\in U\times W$, can be written

$$\begin{aligned}\mu_{(\tilde{\mathcal{R}}\circ\tilde{\mathcal{L}})\sqcup(\tilde{\mathcal{S}}\circ\tilde{\mathcal{L}})}(u,w) &= \mu_{\tilde{\mathcal{R}}\circ\tilde{\mathcal{L}}}(u,w)\sqcup\mu_{\tilde{\mathcal{S}}\circ\tilde{\mathcal{L}}}(u,w)\\ &= \bigsqcup_{v\in w}[\mu_{\tilde{\mathcal{R}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]\sqcup\bigsqcup_{v\in V}[\mu_{\tilde{\mathcal{S}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]\end{aligned}$$

since arbitrary fuzzy sets in [0,1] satisfy associative and commutative laws under $\sqcup$ [1], we have

$$\begin{aligned}&\bigsqcup_{v\in V}[\mu_{\tilde{\mathcal{R}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]\sqcup\bigsqcup_{v\in V}[\mu_{\tilde{\mathcal{S}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]\\ &=\bigsqcup_{v\in V}[[\mu_{\tilde{\mathcal{R}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]\sqcup[\mu_{\tilde{\mathcal{S}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]]\end{aligned}$$

Furthermore, noting that $\mu_{\tilde{\mathcal{R}}}$(u,v), $\mu_{\tilde{\mathcal{S}}}$(u,v) and $\mu_{\tilde{\mathcal{L}}}$(v,w) are convex fuzzy sets in [0,1], form [1], they satisfy distributive laws under $\sqcup$ and $\sqcap$. Therefore, we get

$$\begin{aligned}&\bigsqcup_{v\in V}[[\mu_{\tilde{\mathcal{R}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]\sqcup[\mu_{\tilde{\mathcal{S}}}(u,v)\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]]\\ &=\bigsqcup_{v\in V}[[\mu_{\tilde{\mathcal{R}}}(u,v)\sqcup\mu_{\tilde{\mathcal{S}}}(u,v)]\sqcap\mu_{\tilde{\mathcal{L}}}(v,w)]\\ &\mu_{(\tilde{\mathcal{R}}\cup\tilde{\mathcal{S}})\circ\tilde{\mathcal{L}}}(u,w)=\mu_{(\tilde{\mathcal{R}}\circ\tilde{\mathcal{L}})\cup(\tilde{\mathcal{S}}\circ\tilde{\mathcal{L}})}(u,w)\end{aligned}$$

for any (u,w) $\in U\times W$. Thus,

$$(\tilde{\mathcal{R}}\cup\tilde{\mathcal{S}})\circ\tilde{\mathcal{L}}=(\tilde{\mathcal{R}}\circ\tilde{\mathcal{L}})\cup(\tilde{\mathcal{S}}\circ\tilde{\mathcal{L}})$$

□

Theorem 2.12 *[13] Let $\tilde{\mathcal{R}},\tilde{\mathcal{S}}$ be two T2 fuzzy relations defined on $U\times V$, and $\tilde{\mathcal{L}}$ be a T2 fuzzy relation on $W\times U$. If $\mu_{\tilde{\mathcal{R}}}$(u,v), $\mu_{\tilde{\mathcal{S}}}(u,v)$ and $\mu_{\tilde{\mathcal{L}}}$(w,u) are convex T1 fuzzy sets for any (u,v) $\in U\times V$ and (w, u) $\in W\times U$, then we have*

$$\tilde{\mathcal{L}}\circ(\tilde{\mathcal{R}}\cup\tilde{\mathcal{S}})=(\tilde{\mathcal{L}}\circ\tilde{\mathcal{R}})\cup(\tilde{\mathcal{L}}\circ\tilde{\mathcal{S}})$$

Remark 2.13 $(\tilde{\mathcal{R}}\cap\tilde{\mathcal{S}})\circ\tilde{\mathcal{L}}\neq(\tilde{\mathcal{R}}\circ\tilde{\mathcal{L}})\cap(\tilde{\mathcal{S}}\circ\tilde{\mathcal{L}})$

2.2.2 The reflexive, symmetric, transitive type-2 fuzzy relation

Definition 2.14 *[5] Let $\tilde{\mathcal{R}}$ be a Type-2 fuzzy relation on X. Then $\tilde{\mathcal{R}}$ is said to be*

(1) *reflexive if* $\tilde{\mathcal{R}}(x,x) = \overline{1}$, *for all* $x \in X$.

(2) *antireflexive if* $\tilde{\mathcal{R}}(x,x) = \overline{0}$, *for all* $x \in X$.

(3) *symmetric if* $\tilde{\mathcal{R}}(x,y) = \tilde{\mathcal{R}}(y,x)$, *for all* $x, y \in X$.

(4) *antisymmetric if* $\tilde{\mathcal{R}}$ *satisfies* $\tilde{\mathcal{R}}(x,y) = \overline{0}$ *or* $\tilde{\mathcal{R}}(y,x) = \overline{0}$, *for all* $x, y \in X(x \neq y)$.

(5) *transitive if* $\tilde{\mathcal{R}} \circ \tilde{\mathcal{R}} \sqsubseteq \tilde{\mathcal{R}}$, *where* $\tilde{\mathcal{R}} \circ \tilde{\mathcal{R}}$ *is defined by*

$$\tilde{\mathcal{R}} \circ \tilde{\mathcal{R}}(x,y) = \tilde{sup}_{z \in X} \left\{ \tilde{\mathcal{R}}(x,z) \tilde{\wedge} \tilde{\mathcal{R}}(z,y) \right\}.$$

$\overline{0}$ and $\overline{1}$ memberships are denoted as 1/0 and 1/1 in type-2 fuzzy sets respectively. $\overline{0}$ membership in a type-2 fuzzy set means that it has a secondary membership equal to 1 corresponding to the primary membership of 0, and if it has all other secondary memberships equal to 0. Similarly, the meaning of $\overline{1}$ is same as $\overline{0}$[4].

Example 2.15 *Let $\tilde{\tilde{\mathcal{R}}}$ type-2 fuzzy relation on X, where $X = \{x, y\}$, given by*

$$\tilde{\mathcal{R}} = \begin{bmatrix} \frac{1}{1} + \frac{0.9}{0.9} & \frac{0.3}{0.3} + \frac{0.4}{0.4} \\ \frac{0.3}{0.3} + \frac{0.4}{0.4} & \frac{0.5}{1} + \frac{0.6}{1} \end{bmatrix}$$

(i) $\tilde{\mathcal{R}}(x,x) = \overline{1}$, *for all* $x \in X$.

Therefore, $\tilde{\mathcal{R}}$ is a type-2 fuzzy reflexive relation.

(ii) $\tilde{\mathcal{R}}(x,y) = \tilde{\mathcal{R}}(y,x)$, *for all* $x, y \in X$.

Therefore, $\tilde{\mathcal{R}}$ is a type-2 fuzzy symmetric relation.

(iii) $\tilde{\mathcal{R}} \circ \tilde{\mathcal{R}} \sqsubseteq \tilde{\mathcal{R}}$

$$\tilde{\mathcal{R}} \circ \tilde{\mathcal{R}} = \begin{bmatrix} \frac{1}{1} + \frac{0.9}{0.9} & \frac{0.3}{0.3} + \frac{0.4}{0.4} \\ \frac{0.3}{0.3} + \frac{0.4}{0.4} & \frac{0.5}{1} + \frac{0.6}{1} \end{bmatrix} \circ \begin{bmatrix} \frac{1}{1} + \frac{0.9}{0.9} & \frac{0.3}{0.3} + \frac{0.4}{0.4} \\ \frac{0.3}{0.3} + \frac{0.4}{0.4} & \frac{0.5}{1} + \frac{0.6}{1} \end{bmatrix}$$

$$\tilde{\mathcal{R}} \circ \tilde{\mathcal{R}} = \begin{bmatrix} \frac{0.3}{1} + \frac{0.4}{0.9} & \frac{0.3}{0.3} + \frac{0.4}{0.4} \\ \frac{0.3}{0.3} + \frac{0.4}{0.4} & \frac{0.5}{1} + \frac{0.6}{1} \end{bmatrix}$$

Thus, $\tilde{\mathcal{R}} \circ \tilde{\mathcal{R}} \sqsubseteq \tilde{\mathcal{R}}$.

Hence, $\tilde{\mathcal{R}}$is a type-2 fuzzy transitive relation.

2.3 Type-2 fuzzy topology

Definition 2.16 *[1] Let $\tilde{\tilde{\tau}}$ be the collection of type-2 fuzzy sets over X. $\tilde{\tilde{\tau}}$ is said to be a type-2 fuzzy topology if*

1. $\tilde{\tilde{\emptyset}}, \tilde{\tilde{X}} \in \tilde{\tilde{\tau}}$.

2. $\tilde{\tilde{A}} \cap \tilde{\tilde{B}} \in \tilde{\tilde{\tau}}$, *for any* $\tilde{\tilde{A}}, \tilde{\tilde{B}} \in \tilde{\tilde{\tau}}$.

3. $\underset{i\in\mathbb{N}}{\cup} \tilde{\tilde{A}}_i \in \tilde{\tilde{\tau}}$, *for any* $\tilde{\tilde{A}}_i \in \tilde{\tilde{\tau}}$, $\mathbb{N}$ *countable set.*

The pair $(X, \tilde{\tilde{\tau}})$ is called type-2 fuzzy topological space over X.

Remark 2.17 *$\tilde{\tilde{A}}$ is a type-2 fuzzy close set in X.*

Proposition 2.18 *[1] Let $(X, \tilde{\tilde{\tau}})$ be general type-2 fuzzy topological space over X then the following conditions hold:*

1. *$\tilde{\tilde{\emptyset}}, \tilde{\tilde{X}}$ are type-2 fuzzy closed sets.*

2. *Arbitrary intersection of type-2 fuzzy closed sets is closed sets.*

3. *Finite union of type-2 fuzzy closed sets is closed sets.*

Proof. [1]

1. $\tilde{\tilde{\emptyset}}, \tilde{\tilde{X}}$ are type-2 fuzzy closed sets because they are the complements of the type-2 fuzzy open sets $\tilde{\tilde{\emptyset}}, \tilde{\tilde{X}}$ is respectively.

2. Let $\left\{\tilde{\tilde{A}}_i : i \in \mathbb{N}\right\}$ be an arbitrary collection of type-2 fuzzy closed sets, then

$$\begin{aligned}
\left[\bigcap_{i\in\mathbb{N}} \tilde{\tilde{A}}_i\right](x) &= \sum_{x\in X}\sum_{u\in J_x^u} \frac{\wedge_{i\in\mathbb{N}}\left(f_x(u)\right)_i}{\wedge_{i\in\mathbb{N}}(u)_i} \\
&= \sum_{x\in X}\sum_{u\in J_x^u} \frac{\wedge_{i\in\mathbb{N}}\left(f_x(u)\right)_i}{1-\left(\vee_{i\in\mathbb{N}}(1-u)\right)_i} \\
&= \left[\bigcup_{i\in\mathbb{N}} \overline{\tilde{\tilde{A}}_i}\right](x)
\end{aligned}$$

since the arbitrary union of type-2 fuzzy open sets are open $\left[\bigcup_{i\in\mathbb{N}}\overline{\tilde{\tilde{A}}_i}\right](x)$ is an open and $\left[\bigcap_{i\in\mathbb{N}}\tilde{\tilde{A}}_i\right](x)$ is a type-2 fuzzy closed set.

3. If $\tilde{\tilde{A}}_i(i \in \mathbb{N})$ is type-2 fuzzy closed sets, then $\bigcup_{i\in\mathbb{N}}\tilde{\tilde{A}}_i$ is a type-2 fuzzy closed set, (finite intersection of type-2 fuzzy open sets are open).

□

Example 2.19 *Let $X = \{x_1, x_2\}$ and let $\tilde{\tilde{\emptyset}}, \tilde{\tilde{X}}$ and $\tilde{\tilde{A}}$ be three type-2 fuzzy sets in X which are*

$$\tilde{\tilde{\emptyset}} = \{((x_1, 0), 1), ((x_2, 0), 1)\}\,;\ \tilde{\tilde{X}} = \{((x_1, 1), 0.9), ((x_2, 1), 0.9)\}$$

$$\tilde{\tilde{A}} = \{((x_1, 0.6), 0.2), ((x_1, 0.3), 0.7), ((x_1, 0.4), 0.5), ((x_2, 0.8), 1), ((x_2, 0.5), 0.2), ((x_2, 0.6), 0.7)\}$$

$\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}}$ *for $x = x_1$:*

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}}\cup\tilde{\tilde{X}}}(x_1) &= \frac{1 \wedge 0.9}{0 \vee 1} = (1, 0.9) \\ \tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}} &= \{((x_1, 1), 0.9)\} \end{aligned}$$

$\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}}$ *for $x = x_2$:*

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}}\cup\tilde{\tilde{X}}}(x_2) &= \frac{1 \wedge 0.9}{0 \vee 1} = (1, 0.9) \\ \tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}} &= \{((x_2, 1), 0.9)\} \end{aligned}$$

$\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}}$ *for $x = x_1$:*

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}}\cap\tilde{\tilde{X}}}(x_1) &= \frac{1 \wedge 0.9}{0 \wedge 1} = (0, 0.9) \\ \tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}} &= \{((x_2, 0), 0.9)\} \notin \tilde{\tilde{\tau}} \end{aligned}$$

then $\tilde{\tilde{\tau}} = \left\{\tilde{\tilde{\emptyset}}, \tilde{\tilde{X}}, \tilde{\tilde{A}}\right\}$ is not type-2 fuzzy topology.

Example 2.20 *Let $X = \{x_1, x_2\}$ and let $\tilde{\tilde{\emptyset}}, \tilde{\tilde{X}}$ and $\tilde{\tilde{A}}$ be three type-2 fuzzy sets in X which are*

$$\tilde{\tilde{\emptyset}} = \{((x_1, 0), 1), ((x_2, 0), 1)\}\,; \tilde{\tilde{X}} = \{((x_1, 1), 1), ((x_2, 1), 1)\}\,.$$

$\tilde{\tilde{A}} = \{((x_1, 0.6), 0.2), ((x_1, 0.3), 1), ((x_1, 0.4), 0.5), ((x_2, 0.8), 1), ((x_2, 0.5), 0.2), ((x_2, 0.6), 0.7)\}$.

$\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}}$ *for* $x = x_1$*:*

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}}}(x_1) &= \frac{1 \wedge 1}{0 \vee 1} = (1, 1) \\ \tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}} &= \{((x_1, 1), 1)\} \end{aligned}$$

$\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}}$ *for* $x = x_2$:

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}}}(x_2) &= \frac{1 \wedge 1}{0 \vee 1} = (1, 1) \\ \tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}} &= \{((x_2, 1), 1)\} \end{aligned}$$

$\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{X}} = \{((x_1, 1), 1), ((x_2, 1), 1)\} = \tilde{\tilde{X}}$

$\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}}$ *for* $x = x_1$:

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}}}(x_1) &= \frac{1 \wedge 1}{0 \wedge 1} = (0, 1) \\ \tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}} &= \{((x_1, 0), 1)\} \end{aligned}$$

$\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}}$ *for* $x = x_2$:

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}}}(x_2) &= \frac{1 \wedge 1}{0 \wedge 1} = (0, 1) \\ \tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}} &= \{((x_2, 0), 1)\} \end{aligned}$$

$\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{X}} = \{((x_1, 0), 1), ((x_2), 0), 1)\} = \tilde{\tilde{\emptyset}}$

- $\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{A}}$ *for* $x = x_1$:

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{A}}}(x_1) &= \frac{1 \wedge 0.2}{0 \vee 0.6} + \frac{1 \wedge 1}{0 \vee 0.3} + \frac{1 \wedge 0.5}{0 \vee 0.4} \\ &= \{(0.6, 0.2), (0.3, 1), (0.4, 0.5)\} \\ \tilde{\tilde{\emptyset}} \cup \tilde{\tilde{A}}(x_1) &= \{((x_1, 0.6), 0.2), ((x_1, 0.3), 1), ((x_1, 0.4), 0.5)\} \end{aligned}$$

- $\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{A}}$ *for* $x = x_2$:

$$\begin{aligned} \mu_{\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{A}}}(x_2) &= \frac{1 \wedge 1}{0 \vee 0.8} + \frac{1 \wedge 0.2}{0 \vee 0.5} + \frac{1 \wedge 0.7}{0 \vee 0.6} \\ &= \{(0.8, 1), (0.5, 0.2), (0.6, 0.7)\} \\ \tilde{\tilde{\emptyset}} \cup \tilde{\tilde{A}}(x_2) &= \{((x_2, 0.8), 1), ((x_2, 0.5), 0.2), ((x_2, 0.6), 0.7)\} \end{aligned}$$

$$\tilde{\tilde{\emptyset}} \cup \tilde{\tilde{A}} = \{((x_1, 0.6), 0.2), ((x_1, 0.3), 0.7), ((x_1, 0.4), 0.5), ((x_2, 0.8), 1), ((x_2, 0.5), 0.2), ((x_2, 0.6), 0.7)\} = \tilde{\tilde{A}}.$$

- $\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{A}}$ *for* $x = x_1$:

$$\begin{aligned}
\mu_{\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{A}}}(x_1) &= \frac{1 \wedge 0.2}{0 \wedge 0.6} + \frac{1 \wedge 1}{0 \wedge 0.3} + \frac{1 \wedge 0.5}{0 \wedge 0.4} \\
&= \frac{0.2}{0} + \frac{1}{0} + \frac{0.5}{0} \\
&= (0, max(0.2, 1, 0.5)) \\
&= (0, 1)
\end{aligned}$$

- $\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{A}}$ *for* $x = x_2$:

$$\begin{aligned}
\mu_{\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{A}}}(x_2) &= \frac{1 \wedge 1}{0 \wedge 0.8} + \frac{1 \wedge 0.2}{0 \wedge 0.5} + \frac{1 \wedge 0.7}{0 \wedge 0.6} \\
&= \frac{1}{0} + \frac{0.2}{0} + \frac{0.7}{0} \\
&= (0, max(1, 0.2, 0.7)) \\
&= (0, 1)
\end{aligned}$$

$\tilde{\tilde{\emptyset}} \cap \tilde{\tilde{A}} = \{((x_1, 0), 1), ((x_2, 0), 1)\} = \tilde{\tilde{\emptyset}}$

In the same way we find:

$-\tilde{\tilde{X}} \cup \tilde{\tilde{A}} = \{((x_1, 1), 1), ((x_2, 1), 1)\} = \tilde{\tilde{X}}$

$$-\tilde{\tilde{X}} \cap \tilde{\tilde{A}} = \{((x_1, 0.6), 0.2), ((x_1, 0.3), 1), ((x_1, 0.4), 0.5), ((x_2, 0.8), 1), ((x_2, 0.5), 0.2), ((x_2, 0.6), 0.7)\} = \tilde{\tilde{A}}.$$

Hence, $\tilde{\tilde{\tau}}$ *is the type-2 fuzzy topology.*

Chapter 3

Fuzzy topology generated by fuzzy relation

In this chapter, we discuss the presentation of the work of the scientist Mishra and Srivastava (2018) on the topic of fuzzy topology generated by a fuzzy relation.

3.1 Fuzzy topology generated by fuzzy relation

Definition 3.1 *[11] Let $\mathcal{R}$ be a fuzzy relation on a set X. Then for $x \in X$, the fuzzy sets L_x and R_x, which are defined as*

$$L_x(y) = \mathcal{R}(y, x), \ for\ all\ y \in X.$$

$$R_x(y) = \mathcal{R}(x, y), \ for\ all\ y \in X.$$

are called lower and upper contour, respectively, of the element $x \in X$.

The fuzzy topology generated by the collection $\mathcal{S}_1$ of all lower contours (i.e.,$\mathcal{S}_1 = \{L_x : x \in X\}$) will be denoted by τ_1, and the fuzzy topology generated by the collection $\mathcal{S}_2$ of all upper contours (i.e.,$\mathcal{S}_2 = \{R_x : x \in X\}$) will be denoted by τ_2.

Definition 3.2 *[11]*

The fuzzy topology which is generated by the set $\mathcal{S} = \{L_x\}_{x\in X} \cup \{R_x\}_{x\in X}$ is called the fuzzy topology generated by $\mathcal{R}$ and is denoted by $\tau_{\mathcal{R}}$.

Example 3.3 *Let $\mathcal{R}$ be a fuzzy relation on $X = \{x, y\}$, given by*

$\mathcal{R}$	x	y
x	0.6	0.8
y	0.4	0.5

Then, L_x, L_y, R_x, R_y *are the fuzzy sets in X given by:*

$$L_x = \{\langle x, 0.6\rangle, \langle y, 0.4\rangle\}; \ L_y = \{\langle x, 0.8\rangle, \langle y, 0.5\rangle\}.$$

$$R_x = \{\langle x, 0.6\rangle, \langle y, 0.8\rangle\}; \ R_y = \{\langle x, 0.4\rangle, \langle y, 0.5\rangle\}.$$

$\tau_1 = \{0_X, 1_X, L_x, L_y\}$ *fuzzy topology.*

$\tau_2 = \{0_X, 1_X, R_x, R_y\}$ *fuzzy topology.*

$L_x \cap R_y = \{\langle x, 0.4\rangle, \langle y, 0.4\rangle\}; L_y \cap R_x = \{\langle x, 0.6\rangle, \langle y, 0.5\rangle\}.$

$L_x \cup R_y = \{\langle x, 0.6\rangle, \langle y, 0.5\rangle\}; \ L_y \cup R_x = \{\langle x, 0.8\rangle, \langle y, 0.8\rangle\}.$

the fuzzy topology generated by fuzzy relation is

$$\tau_{\mathcal{R}} = \{0_X, 1_X, L_x, L_y, R_x, R_y, L_x \cap R_y, , L_y \cap R_x, L_x \cup R_y, L_y \cup R_x\}.$$

Example 3.4 *Let* $\mathcal{R}$ *be a fuzzy relation on* $X = \{x, y, z\}$, *given by*

$\mathcal{R}$	x	y	z
x	*0.3*	*0.5*	*0.7*
y	*0.2*	*0.6*	*0.9*
z	*0.8*	*0.1*	*0.4*

$L_x, L_y, L_z, R_x, R_y, R_z$ *are the fuzzy sets in X and* τ_1, τ_2 *where :*

$$L_x = \{\langle x, 0.3\rangle, \langle y, 0.2\rangle, \langle z, 0.8\rangle\}; L_y = \{\langle x, 0.5\rangle, \langle y, 0.6\rangle, \langle z, 0.1\rangle\};$$

$$L_z = \{\langle x, 0.7\rangle, \langle y, 0.9\rangle, \langle z, 0.4\rangle\}$$

$$R_x = \{\langle x, 0.3\rangle, \langle y, 0.5\rangle, \langle z, 0.7\rangle\}; R_y = \{\langle x, 0.2\rangle, \langle y, 0.6\rangle, \langle z, 0.9\rangle\};$$

$$R_z = \{\langle x, 0.8\rangle, \langle y, 0.1\rangle, \langle z, 0.4\rangle\}$$

$\tau_1 = \{0_x, 1_x, L_x, L_y, L_z\}; \tau_2 = \{0_x, 1_x, R_x, R_y, R_z\}$ *two fuzzy topology.*

$L_x \cup R_z = \{\langle x, 0.8\rangle, \langle y, 0.2\rangle, \langle z, 0.8\rangle\}$ *and* $L_z \cup R_y = \{\langle x, 0.7\rangle, \langle y, 0.9\rangle, \langle z, 0.9\rangle\}.$

$R_x \cup L_y = \{\langle x, 0.5\rangle, \langle y, 0.6\rangle, \langle z, 0.7\rangle\}.$

So $\tau_{\mathcal{R}} = \{0_x, 1_x, L_x, L_y, L_z, R_x, R_y, R_z, L_x \cup R_z, L_z \cup R_y, R_x \cup L_y\}.$

Proposition 3.5 *[11]*

If $\mathcal{R}$ *is a symmetric fuzzy relation, then* $\tau_1 = \tau_2$.

Proof. [11] Since $\mathcal{R}$ is a symmetric fuzzy relation, so $\mathcal{R}$(x, y) = $\mathcal{R}$(y, x), for each x, y ∈X. This implies that $R_x(y) = L_x(y)$, for each x, y ∈ X and hence $R_x = L_x$, for each x ∈X.

Thus the topologies τ_1 and τ_2, which are generated by $\{L_x : x \in X\}$ and $\{R_x : x \in X\}$, respectively, are same. $\square$

Example 3.6 *Let $\mathcal{R}$ be a symmetric fuzzy relation on $X = \{x, y\}$, given by*

$$\mathcal{R} = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.2 \end{bmatrix} \end{array}$$

then, L_x, L_y, R_x, R_y are the fuzzy sets in X and τ_1, τ_2, where:

$$L_x = \{\langle x, 0.4\rangle, \langle y, 0.3\rangle\}, \; L_y = \{\langle x, 0.3\rangle, \langle y, 0.2\rangle\}.$$
$$R_x = \{\langle x, 0.4\rangle, \langle y, 0.3\rangle\}, \; R_y = \{\langle x, 0.3\rangle, \langle y, 0.2\rangle\}.$$

$$L_x, L_y \in \tau_1, \text{ then } L_x \cap L_y = \{\langle x, 0.3\rangle, \langle y, 0.2\rangle\} = L_y \in \tau_1.$$
$$L_x \in \tau_1 \text{ for each } x \in X, \text{ then } \cup L_x \in \tau_1, L_y \in \tau_1 \text{ for each } y \in X, \text{ then } \cup L_y \in \tau_1.$$
$$L_x \cup L_y = \{\langle x, 0.4\rangle, \langle y, 0.3\rangle\} = L_x \in \tau_1.$$

hence, $\tau_1 = \{0, 1, L_x, L_y\}$ fuzzy topology.

$$R_x, R_y \in \tau_2, \text{ then } R_x \cap R_y = \{\langle x, 0.3\rangle, \langle y, 0.2\rangle\} = R_y \in \tau_2.$$
$$R_x \in \tau_2 \text{ for each } x \in X, \text{ then } \cup R_x \in \tau_2, R_y \in \tau_2 \text{ for each } y \in X, \text{ then } \cup R_y \in \tau_2.$$
$$R_x \cup R_y = \{\langle x, 0.4\rangle, \langle y, 0.3\rangle\} = R_x \in \tau_2.$$

hence, $\tau_2 = \{0, 1, R_x, R_y\}$ fuzzy topology.

we have, $L_x = R_x = \{\langle x, 0.2\rangle, \langle y, 0.3\rangle\}$ *and* $L_y = R_y = \{\langle x, 0.3\rangle, \langle y, 0.2\rangle\}$.

Hence, According to the previous property, the $\tau_1 = \tau_2$.

Proposition 3.7 *[11] If $\mathcal{R}$ is a fuzzy preorder relation, then*

1. *If $A \in \tau_1$, then $\bigcup\limits_{x:A(x)=1} R_x \subseteq A$.*
2. *If $A \in \tau_2$, then $\bigcup\limits_{x:A(x)=1} L_x \subseteq A$.*

Proof. [11] To show that $\bigcup\limits_{x:A(x)=1} L_x \subseteq A$. This indicates that there is some x for which $A(x) = 1$ is true and $y_r \in L_x$. So $r < \mathcal{R}(y, x)$. Now since A is open and $A(x) = 1$, so

$x_r \in A$ and there exists a basic fuzzy open set $\bigcap_{i=1}^{n} L_{x_i}$ such that

$$\begin{array}{rcl} x_r \in \bigcap_{i=1}^{n} L_{x_i} \subseteq A & \Rightarrow & r < \mathcal{R}(x, x_i),\ for\ each i = 1, 2, ..., n \\ & \Rightarrow & r < \min\{\mathcal{R}(y, x), \mathcal{R}(x, x_i)\} \\ & \leq & \mathcal{R}(y, x_i),\ for\ each\ i = 1, 2, ..., n \\ & \Rightarrow & y_r \in L_{x_i},\ for\ each\ i = 1, 2, ..n \\ & \Rightarrow & y_r \in \bigcap_{i=1}^{n} L_{x_i} \subseteq A \\ & \Rightarrow & y_r \in A \\ & \Rightarrow & \bigcup_{x:A(x)=1} L_x \subseteq A \end{array}$$

□

Theorem 3.8 *[11]*

Let (X, τ) be a fuzzy topological space where τ has a subbase $\{U_x, V_x : x \in X\}$ such that $U_y(x) = V_x(y)$, for each x, y $\in X$. Consider the fuzzy relation $\mathcal{R} : X \times X \longrightarrow I$ defined by $\mathcal{R}(x, y) = U_y(x) = V_x(y)$, for each (x, y) $\in X \times X$. Then the following properties hold good:

1. *$\mathcal{R}$ is reflexive if and only if $U_x(x) = 1$, for each x $\in X$.*

2. *$\mathcal{R}$ is irreflexive if and only if $U_x(x) \neq 1$, for some x $\in X$.*

3. *$\mathcal{R}$ is symmetric if and only if $U_x = V_x$, $\forall x \in X$.*

4. *$\mathcal{R}$ is antisymmetric if and only if $(Ux \cap Vx)(y) = 0$, $\forall x, y \in X$, such that $x \neq y$.*

5. *$\mathcal{R}$ is transitive if and only if $U_z(x) \geq (V_x \cap U_z)(y)$ holds for each x,y,z $\in X$.*

6. *$\mathcal{R}$ is total if and only if $U_x \cup V_x = 1_X$, for each $x \in X$.*

Chapter 4

Type-2 fuzzy topology generated by type-2 fuzzy relation

In this chapter, we try to create a type-2 fuzzy topology generated by a type-2 fuzzy relation by extending the concept introduced by Mishra with different definitions and examples.

Definition 4.1 *Let $\tilde{\tilde{\mathcal{R}}}$ be a type-2 fuzzy relation on a set X. The fuzzy sets $\tilde{\tilde{L}}_x$ and $\tilde{\tilde{R}}_x$, for $x \in X$, defined as :*

1. $\tilde{\tilde{L}}_x(y) = \tilde{\tilde{\mathcal{R}}}(y, x)$, *for all* $y \in X$.

2. $\tilde{\tilde{R}}_x(y) = \tilde{\tilde{\mathcal{R}}}(x, y)$, *for all* $y \in X$.

The type-2 fuzzy topology generated by the collection $\mathcal{Q}_1$, where $\mathcal{Q}_1 = \{L_x : x \in X\}$, will denote $\tilde{\tilde{\tau}}_1$, and The fuzzy topology type-2 generated by the collection $\mathcal{Q}_2$,
where $\mathcal{Q}_2 = \{R_x : x \in X\}$, will denote $\tilde{\tilde{\tau}}_2$.

Definition 4.2 *The fuzzy topology type-2 generated by the set $\mathcal{Q} = \left\{\tilde{\tilde{L}}_x\right\}_{x\in X} \sqcup \left\{\tilde{\tilde{R}}_x\right\}_{x\in X}$ is known as the $\tilde{\tilde{\mathcal{R}}}$ generated and is denoted by $\tilde{\tilde{\tau}}_{\tilde{\tilde{\mathcal{R}}}}$.*

Example 4.3 *Let $\tilde{\tilde{\mathcal{R}}}$ be a type-2 fuzzy relation on $X = \{x, y\}$, given by:*

$$\tilde{\tilde{\mathcal{R}}} = \begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \left[\begin{array}{cc} \frac{0.1}{1} & \frac{0.2}{0.4} \\ \frac{0.3}{0.5} & \frac{0.7}{0.3} \end{array}\right] \end{array}$$

then, $\tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y, \tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y$ are the type-2 fuzzy sets on X and $\tilde{\tilde{\tau}}_1, \tilde{\tilde{\tau}}_2$, given by :

$$\tilde{\tilde{L}}_x = \frac{0.1}{1} + \frac{0.3}{0.5}, \tilde{\tilde{L}}_y = \frac{0.2}{0.4} + \frac{0.7}{0.3}.$$
$$\tilde{\tilde{R}}_x = \frac{0.1}{1} + \frac{0.2}{0.4}, \tilde{\tilde{R}}_y = \frac{0.3}{0.5} + \frac{0.7}{0.3}.$$

$\tilde{\tilde{0}}, \tilde{\tilde{1}} \in \tilde{\tilde{\tau}}_1.$

$\tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y \in \tilde{\tilde{\tau}}_1$, *then* $\tilde{\tilde{L}}_x \sqcap \tilde{\tilde{L}}_y = \frac{0.1}{0.4} + \frac{0.3}{0.3}$.

$\tilde{\tilde{L}}_x \in \tilde{\tilde{\tau}}_1$ *for each* $x \in X$, *then* $\sqcup \tilde{\tilde{L}}_x \in \tilde{\tilde{\tau}}_1, \tilde{\tilde{L}}_y \in \tilde{\tilde{\tau}}_1$ *for each* $y \in X$, *then* $\sqcup \tilde{\tilde{L}}_y \in \tilde{\tilde{\tau}}_1$.

$$\begin{aligned} \tilde{\tilde{L}}_x \sqcup \tilde{\tilde{L}}_y &= \frac{0.1 \wedge 0.2}{1 \vee 0.4} + \frac{0.3 \wedge 0.7}{0.5 \vee 0.3} \\ &= \frac{0.1}{1} + \frac{0.3}{0.5} \\ &= \tilde{\tilde{L}}_x \in \tilde{\tilde{\tau}}_1. \end{aligned}$$

hence, $\tilde{\tilde{\tau}}_1 = \left\{ \tilde{\tilde{0}}, \tilde{\tilde{1}}, \tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y \right\}$ *type-2 fuzzy topology.*

$\tilde{\tilde{0}}, \tilde{\tilde{1}} \in \tilde{\tilde{\tau}}_1.$

$\tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y \in \tilde{\tilde{\tau}}_2$, *then* $\tilde{\tilde{R}}_x \sqcap \tilde{\tilde{R}}_y = \frac{0.1}{0.5} + \frac{0.2}{0.3}$.

$\tilde{\tilde{R}}_x \in \tilde{\tilde{\tau}}_2$ *for each* $x \in X$, *then* $\sqcup \tilde{\tilde{R}}_x \in \tilde{\tilde{\tau}}_2$, $\tilde{\tilde{R}}_y \in \tilde{\tilde{\tau}}_2$ *for each* $y \in X$, *then* $\sqcup \tilde{\tilde{R}}_y \in \tilde{\tilde{\tau}}_2$

$$\begin{aligned} \tilde{\tilde{R}}_x \sqcup \tilde{\tilde{R}}_y &= \frac{0.1 \wedge 0.3}{1 \vee 0.5} + \frac{0.2 \wedge 0.7}{0.4 \vee 0.3} \\ &= \frac{0.1}{1} + \frac{0.2}{0.4} \\ &= \tilde{\tilde{R}}_x \in \tilde{\tilde{\tau}}_2. \end{aligned}$$

hence, $\tilde{\tilde{\tau}}_2 = \left\{ \tilde{\tilde{0}}, \tilde{\tilde{1}}, \tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y \right\}$ *type-2 fuzzy topology.*

$\tilde{\tilde{L}}_x \sqcup \tilde{\tilde{R}}_x = \frac{0.1}{1} + \frac{0.2}{0.5} = L_x$, $\tilde{\tilde{L}}_x \sqcup \tilde{\tilde{R}}_y = \frac{0.1}{1} + \frac{0.3}{0.5} = \tilde{\tilde{L}}_x$

$\tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_x = \frac{0.1}{1} + \frac{0.2}{0.4} = \tilde{\tilde{R}}_x$, $\tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_y = \frac{0.2}{0.5} + \frac{0.7}{0.3}, (\tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_y \sqsubseteq \tilde{\tilde{R}}_y)$.

the type-2 fuzzy topology generated by type-2 fuzzy relation is:

$$\tilde{\tilde{\tau}}_{\tilde{\tilde{\mathcal{R}}}} = \left\{ \tilde{\tilde{0}}, \tilde{\tilde{1}}, \tilde{\tilde{L}}_x, , \tilde{\tilde{L}}_y, , \tilde{\tilde{R}}x, , \tilde{\tilde{R}}_y, , \tilde{\tilde{L}}_y \sqcup, \tilde{\tilde{R}}_y \right\}.$$

Example 4.4

Let $\tilde{\tilde{\mathcal{R}}}$ *be a type-2 fuzzy relation on* $X = \{x, y, z\}$, *given by:*

$\tilde{\tilde{\mathcal{R}}}$	x	y	z
x	$\frac{0.3}{0.2}$	$\frac{0.8}{0.5}$	$\frac{0.3}{0.1}$
y	$\frac{0.4}{0.2}$	$\frac{0.5}{0.6}$	$\frac{0.4}{0.7}$
z	$\frac{0.7}{0.7}$	$\frac{0.4}{0.3}$	$\frac{0.9}{0.2}$

$\tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y, \tilde{\tilde{L}}_z, \tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y, \tilde{\tilde{R}}_z$ *are the fuzzy sets in X and* $\tilde{\tilde{\tau}}_1$, $\tilde{\tilde{\tau}}_2$, *given by:*

$\tilde{\tilde{L}}_x = \frac{0.3}{0.2} + \frac{0.4}{0.2} + \frac{0.7}{0.7}$, $\tilde{\tilde{L}}_y = \frac{0.8}{0.5} + \frac{0.5}{0.6} + \frac{0.4}{0.3}$, $\tilde{\tilde{L}}_z = \frac{0.3}{0.1} + \frac{0.4}{0.7} + \frac{0.9}{0.2}$.

$\tilde{\tilde{R}}_x = \frac{0.3}{0.2} + \frac{0.8}{0.5} + \frac{0.3}{0.1}$, $\tilde{\tilde{R}}_y = \frac{0.4}{0.2} + \frac{0.5}{0.6} + \frac{0.4}{0.7}$, $\tilde{\tilde{R}}_z = \frac{0.7}{0.7} + \frac{0.4}{0.3} + \frac{0.9}{0.2}$.

Therefor

$\tilde{\tilde{\tau}}_1 = \left\{0_x, 1_x, \tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y, \tilde{\tilde{L}}_z\right\}$.

$\tilde{\tilde{\tau}}_2 = \left\{0_x, 1_x, \tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y, \tilde{\tilde{R}}_z\right\}$.

$\tilde{\tilde{L}}_x \sqcap \tilde{\tilde{L}}_y = \frac{0.3}{0.2} + \frac{0.4}{0.2} + \frac{0.4}{0.3}$; $\tilde{\tilde{L}}_x \sqcap \tilde{\tilde{L}}_z = \frac{0.3}{0.1} + \frac{0.4}{0.2} + \frac{0.7}{0.2}$ *and* $\tilde{\tilde{L}}_y \sqcap \tilde{\tilde{L}}_z = \frac{0.3}{0.1} + \frac{0.4}{0.6} + \frac{0.4}{0.2}$.

$\tilde{\tilde{L}}_x \sqcap \tilde{\tilde{R}}_x = \frac{0.3}{0.2} + \frac{0.4}{0.2} + \frac{0.3}{0.1}$; $\tilde{\tilde{L}}_x \sqcap \tilde{\tilde{R}}_y = \frac{0.3}{0.2} + \frac{0.4}{0.2} + \frac{0.4}{0.7}$ *and* $\tilde{\tilde{L}}_x \sqcap \tilde{\tilde{R}}_z = \frac{0.3}{0.2} + \frac{0.4}{0.2} + \frac{0.7}{0.2}$.

$\tilde{\tilde{R}}_x \sqcap \tilde{\tilde{R}}_y = \frac{0.3}{0.2} + \frac{0.5}{0.5} + \frac{0.3}{0.1}$; $\tilde{\tilde{R}}_x \sqcap \tilde{\tilde{R}}_z = \frac{0.3}{0.2} + \frac{0.4}{0.3} + \frac{0.3}{0.1}$ *and* $\tilde{\tilde{R}}_y \sqcap \tilde{\tilde{R}}_z = \frac{0.4}{0.2} + \frac{0.4}{0.3} + \frac{0.4}{0.2}$.

$\tilde{\tilde{L}}_y \sqcap \tilde{\tilde{R}}_y = \frac{0.4}{0.2} + \frac{0.5}{0.6} + \frac{0.4}{0.3}$; $\tilde{\tilde{L}}_y \sqcap \tilde{\tilde{R}}_x = \frac{0.3}{0.2} + \frac{0.5}{0.5} + \frac{0.3}{0.1}$ *and* $\tilde{\tilde{L}}_y \sqcap \tilde{\tilde{R}}_z = \frac{0.7}{0.5} + \frac{0.4}{0.3} + \frac{0.4}{0.2}$.

$\tilde{\tilde{L}}_z \sqcap \tilde{\tilde{R}}_x = \frac{0.3}{0.1} + \frac{0.4}{0.7} + \frac{0.3}{0.1}$; $\tilde{\tilde{L}}_z \sqcap \tilde{\tilde{R}}_y = \frac{0.3}{0.2} + \frac{0.4}{0.6} + \frac{0.4}{0.2}$ *and* $\tilde{\tilde{L}}_z \sqcap \tilde{\tilde{R}}_z = \frac{0.3}{0.1} + \frac{0.4}{0.3} + \frac{0.9}{0.2}$.

$\tilde{\tilde{L}}_x \sqcup \tilde{\tilde{R}}_x = \frac{0.3}{0.2} + \frac{0.4}{0.5} + \frac{0.3}{0.7}$; $\tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_y = \frac{0.4}{0.5} + \frac{0.5}{0.6} + \frac{0.4}{0.7}$ *and* $\tilde{\tilde{L}}_z \sqcup \tilde{\tilde{R}}_z = \frac{0.3}{0.7} + \frac{0.4}{0.7} + \frac{0.9}{0.2}$.

$\tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_x = \frac{0.3}{0.5} + \frac{0.5}{0.6} + \frac{0.3}{0.3}$; $\tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_z = \frac{0.7}{0.7} + \frac{0.4}{0.6} + \frac{0.4}{0.3}$ *and* $\tilde{\tilde{L}}_z \sqcup \tilde{\tilde{R}}_x = \frac{0.3}{0.2} + \frac{0.4}{0.7} + \frac{0.3}{0.2}$.

$\tilde{\tilde{L}}_z \sqcup \tilde{\tilde{R}}_y = \frac{0.3}{0.2} + \frac{0.4}{0.7} + \frac{0.4}{0.2}$; $\tilde{\tilde{R}}_x \sqcup \tilde{\tilde{R}}_y = \frac{0.3}{0.2} + \frac{0.5}{0.6} + \frac{0.3}{0.7}$ *and* $\tilde{\tilde{R}}_x \sqcup \tilde{\tilde{R}}_z = \frac{0.3}{0.7} + \frac{0.4}{0.5} + \frac{0.3}{0.2}$.

$\tilde{\tilde{R}}_y \sqcup \tilde{\tilde{R}}_z = \frac{0.4}{0.7} + \frac{0.4}{0.6} + \frac{0.4}{0.9}$.

so

$$\begin{aligned}
\tilde{\tilde{\tau}}_{\tilde{\tilde{\mathcal{R}}}} = \Big\{ & 0_x, 1_x, \tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y, \tilde{\tilde{L}}_z, \tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y, \tilde{\tilde{R}}_z, \tilde{\tilde{L}}_x \sqcap \tilde{\tilde{L}}_y, \tilde{\tilde{L}}_x \sqcap \tilde{\tilde{L}}_z, \tilde{\tilde{L}}_y \sqcap \tilde{\tilde{L}}_z, \tilde{\tilde{L}}_x \sqcap \tilde{\tilde{R}}_x, \tilde{\tilde{L}}_x \sqcap \tilde{\tilde{R}}_y, \\
& \tilde{\tilde{L}}_x \sqcap \tilde{\tilde{R}}_z, \tilde{\tilde{L}}_y \sqcap \tilde{\tilde{R}}_y, \tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_x, \tilde{\tilde{L}}_y \sqcap \tilde{\tilde{R}}_z, \tilde{\tilde{L}}_z \sqcup \tilde{\tilde{R}}_x, \tilde{\tilde{L}}_z \sqcap \tilde{\tilde{R}}_y, \tilde{\tilde{L}}_z \sqcap \tilde{\tilde{R}}_z, \tilde{\tilde{R}}_x \sqcap \tilde{\tilde{R}}_y, \\
& \tilde{\tilde{R}}_x \sqcap \tilde{\tilde{R}}_z, \tilde{\tilde{R}}_y \sqcap \tilde{\tilde{R}}_z, \tilde{\tilde{L}}_x \sqcup \tilde{\tilde{L}}_y, \tilde{\tilde{L}}_x \sqcup \tilde{\tilde{L}}_z, \tilde{\tilde{L}}_y \sqcup \tilde{\tilde{L}}_z, \tilde{\tilde{L}}_x \sqcup \tilde{\tilde{R}}_x, \tilde{\tilde{L}}_x \sqcup \tilde{\tilde{R}}_y, \tilde{\tilde{L}}_x \sqcup \tilde{\tilde{R}}_z, \\
& \tilde{\tilde{R}}_x \sqcup \tilde{\tilde{R}}_x, \tilde{\tilde{R}}_x \sqcup \tilde{\tilde{R}}_z, \tilde{\tilde{R}}_y \sqcup \tilde{\tilde{R}}_z, \tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_x, \tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_y, \tilde{\tilde{L}}_y \sqcup \tilde{\tilde{R}}_z, \tilde{\tilde{L}}_z \sqcup \tilde{\tilde{R}}_x, \tilde{\tilde{L}}_z \sqcup \tilde{\tilde{R}}_y, \tilde{\tilde{L}}_z \sqcup \tilde{\tilde{R}}_z \Big\}.
\end{aligned}$$

Proposition 4.5

If $\tilde{\tilde{\mathcal{R}}}$ *is a symmetric type-2 fuzzy relation, then* $\tilde{\tilde{\tau}}_1 = \tilde{\tilde{\tau}}_2$.

Proof.

Since $\tilde{\tilde{\mathcal{R}}}$ is a symmetric type-2 fuzzy relation, so $\tilde{\tilde{\mathcal{R}}}(x, y) = \tilde{\tilde{\mathcal{R}}}(y, x)$, for each x, y $\in$ X.

This implies that $\tilde{\tilde{R}}_x(y) = \tilde{\tilde{L}}_x(y)$, for each x, y $\in$ X and hence $\tilde{\tilde{R}}_x = \tilde{\tilde{L}}_x$, for each x $\in$ X. Thus the topologies $\tilde{\tilde{\tau}}_1$ and $\tilde{\tilde{\tau}}_2$, are generated by $\left\{\tilde{\tilde{L}}_x : x \in X\right\}$ and $\left\{\tilde{\tilde{R}}_x : x \in X\right\}$. $\square$

Example 4.6 *Let $\tilde{\tilde{\mathcal{R}}}$ be a symmetric type-2 fuzzy relation on $X = \{x, y\}$, given by*

$$\tilde{\tilde{\mathcal{R}}} = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc} \frac{0.1}{0.2} & \frac{0.4}{0.5} \\ \frac{0.4}{0.5} & \frac{0.2}{0.6} \end{array}\right] \end{array}$$

then, $\tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y, \tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y$ are type-2 fuzzy sets and $\tilde{\tilde{\tau}}_1, \tilde{\tilde{\tau}}_2$, given by

$$\tilde{\tilde{L}}_x = \frac{0.1}{0.2} + \frac{0.4}{0.5}, \tilde{\tilde{L}}_y = \frac{0.4}{0.5} + \frac{0.2}{0.6}.$$

$$\tilde{\tilde{R}}_x = \frac{0.1}{0.2} + \frac{0.4}{0.5}, \tilde{\tilde{R}}_y = \frac{0.4}{0.5} + \frac{0.2}{0.6}.$$

we have,

$$\tilde{\tilde{0}}_X, \tilde{\tilde{1}}_X \in \tilde{\tilde{\tau}}_1.$$

$$\tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y \in \tilde{\tilde{\tau}}_1, \text{ then } \tilde{\tilde{L}}_x \sqcap \tilde{\tilde{L}}_x = \tfrac{0.1}{0.2} + \tfrac{0.2}{5}, (\tilde{\tilde{L}}_x \sqcap \tilde{\tilde{L}}_x \sqsubseteq \tilde{\tilde{L}}_y).$$

$$\tilde{\tilde{L}}_x \in \tilde{\tilde{\tau}}_1 \text{ for each } x \in X, \ \sqcup\tilde{\tilde{L}}_x \in \tilde{\tilde{\tau}}_1; \ \tilde{\tilde{L}}_y \in \tilde{\tilde{\tau}}_1 \text{ for each } y \in X, \ \sqcup\tilde{\tilde{L}}_y \in \tilde{\tilde{\tau}}_1.$$

$$\tilde{\tilde{L}}_x \sqcup \tilde{\tilde{L}}_y = \tfrac{0.1}{0.5} + \tfrac{0.2}{0.6}, \ (\tilde{\tilde{L}}_x \sqcup \tilde{\tilde{L}}_y \sqsubseteq \tilde{\tilde{L}}_y).$$

hence, $\tilde{\tilde{\tau}}_1 = \left\{\tilde{\tilde{0}}, \tilde{\tilde{1}}, \tilde{\tilde{L}}_x, \tilde{\tilde{L}}_y\right\}$ *type-2 fuzzy topology.*

$$\tilde{\tilde{0}}_X, \tilde{\tilde{1}}_X \in \tilde{\tilde{\tau}}_2.$$

$$\tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y \in \tilde{\tilde{\tau}}_2, \text{ then } \tilde{\tilde{R}}_x \sqcap \tilde{\tilde{R}}_y = \tfrac{0.1}{0.2} + \tfrac{0.2}{0.5}, (\tilde{\tilde{R}}_x \sqcap \tilde{\tilde{R}}_y \sqsubseteq \tilde{\tilde{R}}_x).$$

$$\tilde{\tilde{R}}_x \in \tilde{\tilde{\tau}}_2 \text{ for each } x \in X, \ \sqcup\tilde{\tilde{R}}_x \in \tilde{\tilde{\tau}}_2; \tilde{\tilde{R}}_y \in \tilde{\tilde{\tau}}_2 \text{ for each } y \in X, \ \sqcup\tilde{\tilde{R}}_y \in \tilde{\tilde{\tau}}_2.$$

$$\tilde{\tilde{R}}_x \sqcup \tilde{\tilde{R}}_y = \tfrac{0.1}{0.5} + \tfrac{0.2}{0.6}, \ (\tilde{\tilde{R}}_x \sqcup \tilde{\tilde{R}}_y \sqsubseteq \tilde{\tilde{R}}_y).$$

hence, $\tilde{\tilde{\tau}}_2 = \left\{\tilde{\tilde{0}}, \tilde{\tilde{1}}, \tilde{\tilde{R}}_x, \tilde{\tilde{R}}_y\right\}$ *type-2 fuzzy topology.*

we have $\tilde{\tilde{L}}_x = \tilde{\tilde{R}}_x = \frac{0.1}{0.2} + \frac{0.4}{0.5}$ *and* $\tilde{\tilde{L}}_y = \tilde{\tilde{R}}_y = \frac{0.4}{0.5} + \frac{0.2}{0.6}$.

hence, According to a previous property, the $\tilde{\tilde{\tau}}_1 = \tilde{\tilde{\tau}}_2$.

Conclusion

In this memory, we have extended the definition given by prof.Mishra (fuzzy topology generated by fuzzy relation) to the type-2 fuzzy topology. We created two type-2 fuzzy sets $\tilde{\tilde{L}}_x$, $\tilde{\tilde{R}}_x$ and we did the union and intersection between them, so we got a type-2 fuzzy topology generated by type-2 fuzzy relation and is denoted by $\tilde{\tilde{\tau}}_{\tilde{\tilde{\mathcal{R}}}}$, giving examples and properties that indicate this.

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ملخص

الطوبولوجيا هي أحد فروع الرياضيات التي اكتسبت أهمية كبيرة . في هذه المذكرة درسنا النمط الثاني للطبولوجيا الضبابية المستخلصة من النمط الثاني من العلاقات الضبابية بتوسعة الطبولوجيا الضبابية نمط واحد المستخلصة من النمط واحد للعلاقات الضبابية .

كلمات مفتاحية : النمط الثاني للمجموعات الضبابية ، النمط الثاني للعلاقات الضبابية،النمط الثاني للطبولوجيا الضبابية، الطبولوجيا الضبابية المستخلصة من العلاقة الضبابية.

Abstract

Topology is one the branches of mathematics that has great received importance . In this memory, we studied the type-2 of fuzzy topology generated by type-2 fuzzy relations by expanding type-1 fuzzy topology generated by type-1 fuzzy relations .

Key words : Type-2 fuzzy sets, type-2 fuzzy relations, type-2 fuzzy topology, fuzzy topology generated by fuzzy relation.

Résumé

La topologie est l'une des branches des mathématiques qui a une grand importance. Dans ce mémoire, nous avons étudié la topologie floue de type-2 générée par des relations floues type-2 en développant la topologie floue type-1 générée par les relations floues de type-1.

Mot-clés : ensembles flou de type-2, relations flou, topology flou de type-2, topologie floue générée par une relation floue .