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On The Spectral Properties for Drazin Invertible Operator

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Notations

$B(X,Y)$	The algebra of all bounded linear operators from X into itself
$N(T)$	The null space of T
$R(T)$	the range space of T,
$\mathcal{K}(X)$	the ideal of all compact operators on X
$\sigma(T)$	the spectrum of T,
$\rho(T)$	the resolvent set of T
$\alpha(T)$	The nullity of T is defined as the dimension of $N(T)$
$\beta(T)$	the deficiency of T is defined as the codimension of $R(T)$
I	operator of identity
$\Phi(X,Y)$	The set of fredholm operators
$\Phi_+(X,Y)$	The set of upper semi-Fredholm operators
$\Phi_-(X,Y)$	The set of lower semi-Fredholm operators
$L(X,Y)$	The algebra of all bounded linear operators from X into Y
$\sigma_p(T)$	The points spectrum of T
$\sigma_r(T)$	The residuel spectrum of T
$\sigma_c(T)$	The continous spectrum of T
$\Phi_\pm(X,Y)$	The set of semi-fredholm operator

Introduction

Many fundamental problems in operator theory and its various applications such as abstract algebra, numerical analysis, optimal control, and spectral analys, are inherently linked to the concept of invertibility. For instance, a linear operator equation of the form $Ax = b$ possesses a unique solution if and only if the operator A is invertible. However, in practical applications, mathematical models frequently involve non-invertible operators in both finite and infinite-dimensional spaces. To overcome this limitation, the theory of generalized inverses was developed, attracting the attention of numerous mathematicians throughout the twentieth century, including J. J. Koliha. The journey of generalized inverses began in 1936 when Jokn von Neumann introduced the notion of a generalized inverse for elements in a ring, which was later extended to algebras by l. kaplansky in 1948. Independently, the concept of the pseudo-inverse (or Moore-Penrose inverse) was established by E. H. Moore in 1922 and R. Penrose in 1955. In 1958, M. P. Drazin introduced a different yet remarkably powerful extension known as the Drazin inverse. The concept of Drazin invertibility was introduced in the more abstract setting of Banach algebras [13]. Drazin [17] proved a number of interesting properties, while some other basic properties of group invertible elements are investigated in [22, 24, 11]. In [15], a useful tool for finding the Drazin inverse of a sum of two elements and that of a block matrix is presented (see also [21]). Numerous mathematicians have studied the analogue of Cline's formula for other kinds of invertibility; see Barnes [10] and Corach et al. [16]. A section concerning Drazin invertible operators is patterned after Caradus [13], Caradus et al. [14], and [9]. Drazin invertibility of an operator has also been characterized in several other ways: by means of ascent and descent in [20]; by the condition that 0 is a pole of the resolvent in [18] and [12]; and through the existence of a closed invariant subspace in [20]. Characterizations of left Drazin invertible and right Drazin invertible operators among those having topological uniform descent were given by Aiena and Triolo [8]. Finally, the local spectrum properties of the Drazin inverse were investigated by Aiena in [4]. Unlike traditional pseudo-inverses, the Drazin inverse preserves essential spectral relationships, making it a vital tool in abstract Cauchy problems, countable Markov chains, systems of singular differential equations, and the study of isolated points in the spectrum. The primary objective of this dissertation is to investigate the algebraic, spectral, and local spectral properties of Drazin invertible operators on complex Banach spaces. By relying fundamentally on localized geometric structures specifically the analytic core $K(T)$, the quasi-nilpotent part $H_0(T)$, and the framework of local spectral theory this work explores some techniques and establishes rigorous characterizations of operator behavior under the Drazin transformation. Regarding the thesis structure, this work is systematically organized into four interconnected chapters. Chapter 1 establishes the necessary mathematical background by recalling the classical properties of bounded linear operators on Banach spaces, including the dual operator, the inverse operator, and the decomposition of the spectrum into point, continuous, and residual parts, while also providing the residual parts, whil also providing the essentials of Fredholm theory, discussing ascent and descent, the Fredholm index, Weyl spectra, Riesz operators, and algebraic operators. Chapter 2 is devoted to the presentation of the notion of generalized inverses, transitioning from relatively regular operators and the limitations of pseudo-inverses to the core-nilpotent decomposition and spectral properties of the Drazin inverse, alongside characterizations of the analytic core and the quasi-nilpotent part in describing isolated spectral points. Chapter 3 provides the foundations of local spectral theory by introducing the localized versions of the resolvent set and the spectrum, leading into a detailed study of the Single-Valued Extension Property (SVEP). Within this local framework, the chapter examines the structural characterizations of local spectral subspaces and explores key topological properties, namely Dunford's property (C), Bishop's property (β), property (Q), and the class of decomposable operators. Finally, chapter 4 addresses the transmission of these local spectral properties between a Drazin invertible operator and its Drazin inverse. After illustrating the failure of property transmission in general pseudo-inverses, it is demonstrated that SVEP, property (C), property (Q), polaroid properties, Bishop's property (β), property(δ), and decomposability are faithfully preserved under the Drazin inverse. Furthermore, a clean, reciprocal relationship for the non-zero local spectrum is verified, and these transmission results are successfully extended to the related operators R^2S and S^2R, before concluding with a summary of open problems and directions for future research.

Chapter 1

Preliminaries and Foundational Concepts

1.1 Bounded Linear Operators on Banach Space

1.1.1 Linear operator

Definition 1.1.1 *Let X and Y be any two normed spaces over a field $\Bbbk$, T is a function from X to Y, then we say T is linear operator if :*

1 $T(x+y) = T(x) + T(y)$; for all x , $y \in X$,

2 $T(\alpha x) = \alpha T(x)$; for each $\alpha \in K$, $x \in X$.

Example 1.1.1 *Let $T : X \longrightarrow Y$ such that $X = C([0,1])$ and $Y = C([0,1])$ be defined by $T(f)(s) = \int_0^1 k(s,t)f(t)dt$; where $f \in C([0,1])$, $k(s,t)$ is a real valued continous function over $[0,1] \times [0,1]$, that is $F(s) = T(f)(s)$, where $F \in C[0,1]$ $s,t \in [0,1]$, cleary T is a linear operator. $L(X,Y)$ denote the set of linear operators from X into Y.*

In this case X is called the domain of T denoted by $D(T)$, the image of X under T is called the range of T denoted by ($R(T)$ or $Im(T)$) is the set of all $y \in Y$ such that $Tx = y$ for some $x \in X$.

in other word

$$R(T) = \{y \in Y, y = Tx\}.$$

The null space (or kernel) of linear operator T denotted by $(N(T)$ or $ker(T))$ is the set of all $x \in X$ such that $T(x) = 0$.

$$N(T) = \{x \in X, T(x) = 0\}.$$

The graph of T is the set $G(T)$ such that

$$G(T) = \{(x, Tx)/x \in D(T)\} \subset X \times Y.$$

$X \times Y$ is the norm $\|(x,y)\|^2_{X\times Y} = \|x\|^2_X + \|y\|^2_Y$wich makes $X \times Y$ is a Banach space

1.1.2 Bounded linear operator

Definition 1.1.2 *Let X, Y be normed vector spaces and $T : X \longrightarrow Y$ a linear operator we say that T is bounded if there exists a number $C > 0$ such that*

$$\|T(x)\| \leq C\,\|x\| \ \textit{for each } x \in X$$

the lower bound of all $C > 0$ such that $\|T(x)\| \leq C\,\|x\|$ for each $x \in X$ is called norm of the operator T and is denoted by $\|T\|$

Theorem 1.1.1 *T is bounded defined as*

$$\begin{aligned} \|T\| &= \sup_{x\neq 0} \tfrac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| \\ &= \sup\{\|Tx\|\,;\|x\| \leq 1\} \end{aligned}$$

If $\|Tx\| \leq C$, if for every $x \in X$ with $\|x\| = 1$, then $\|T\| \leq C$

We denote the set of bounded linear operators from X into Y by $B(X,Y)$ and by $B(X)$ if $X = Y$

1.1.3 The Dual Operator

Definition 1.1.3 *Let X and Y be two Banach spaces. We denote by $X^* = L(X, \mathbb{C})$ the dual space of X, that is, the space of all continuous linear functionals on X. Let $T \in L(X, Y)$ be a bounded linear operator.*

The dual operator (or adjoint operator) of T, denoted by T^* is defined as the operator

$$T^* : Y^* \to X^*$$

given by

$$(T^* f)(x) = f(Tx), \text{ forall } x \in X \text{ and } f \in Y^* .$$

In other words, the adjoint operator T^* associates to each functional $f \in Y^*$ the functional $T^* f \in X^*$ obtained by composing f with T

1.1.4 Inverse Operator

Definition 1.1.4 *let X and Y be two Banach spaces and let $T : X \to Y$.*

We say that T is invertible if there exists a bounded linear operator

$$T^{-1} : Y \to X$$

$$T^{-1}T = I_X \text{ and } TT^{-1} = I_Y, \text{ where } I_Y \text{ is the identity operator on } Y$$

In this case, T^{-1} is called the **inverse operator** of T

By the **Bounded Inverse Theorem**, if T is a bounded linear operator between Banach spaces and is bijective, then its inverse T^{-1} is automatically bounded.

1.2 The Spectrum and Resolvent

Definition 1.2.1 *Let X be a complex Banach space and let $T : X \to X$ be a bounded linear operator. The **spectrum** of T, denoted by $\sigma(T)$,is defined as. The spectrum $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$.*

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B(X)\}.$$

In other words, a complex number λ belongs to the spectrum of T if the operator $T - \lambda I$ does not admit a bounded inverse. The complement of the spectrum in $\mathbb{C}$ is called the **resolvent set** of T, and is denoted by

$$\rho(T) = \mathbb{C} \backslash \sigma(T)$$

For every $\lambda \in \rho(T)$,the operator

$$R(\lambda, T) = (T - \lambda I)^{-1}$$

Moreover, the spectrum can be decomposed into three disjoint parts:

1. **The point spectrum**

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective}\}$$

2. **The continuous spectrum**

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{ is injective, has dense range, but is not surjective}\}$$

3. **The residual spectrum**

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{ is injective but its rang is not dense}\}$$

Thus

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$ [(1)]

[(1)] P.Aiena, Fredholm and local Spectral Theory, with Applications to Multipliers, kluwer Academic Publishers, 2004 (p.509)

Spectrum of the Inverse Operator

If the operator$T \in L(X)$ is invertible, then the spectrum of its inverse operator T^{-1} consists of the reciprocals of the spectral values of the original operator T:

$$\sigma(T^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(T) \right\}$$

Since the operator is invertible, zero does not belong to its spectrum $(0 \notin \sigma(T))$. This allows the application of the spectral Mapping Theorem using the analytic reciprocal function:

$$f(\lambda) = \frac{1}{\lambda}$$

This reciprocal relation $(1/\lambda)$ remains valid and hold true when studying the spectrum with respect to any spectral regularity R within the Banach space.[(1)]

1.3 Fredholm Theory Essentials

1.3.1 Ascent and Descent

Definition 1.3.1 ***(Ascent)***

The ascent of T, denoted by $a(T)$, is defined as the smallest non-negative integer p such that

$$\ker(T^p) = \ker(T^{p+1})$$

If no such integer exists, we say that the ascent of T is infinite, and we write

$$a(T) = \infty$$

Thus, the ascent measures the stabilization of the increasing sequence of null spaces

$$\ker(T) \subseteq \ker(T^2) \subseteq \ker(T^3) \subseteq ...$$

(1) P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, klwer Academic Publishers, 2004 (p.288)

In contrast to the ascent, the descent measures the stabilization of the decreasing sequence of ranges

Definition 1.3.2 ***(Descent)***

The descent of T , denoted by $d(T)$, is defined as the smallest non-negative integer q such that

$$\mathrm{Ran}(T^q) = \mathrm{Ran}(T^{q+1})$$

If no such integer exists, we say that the descent of T is infinte, and we write

$$d(T) = \infty$$

Thus, the descent measures the stabilization of the decreasing sequence of ranges

$$\mathrm{Ran}(T) \supseteq \mathrm{Ran}(T^2) \supseteq \mathrm{Ran}(T^3) \supseteq \ldots$$

Fundamental Result

If both ascent and descent are finite, then

$$a(T) = d(T)$$

Moreover, in this case, the space admits the direct sum decomposition

$$X = \ker(T^p) \oplus \mathrm{Ran}(T^p), \text{ where } p = a(T) = d(T).$$

These concepts are closely related to Fredholm operators, which we introduce next along with the Fredholm index.[(1)]

[(1)] P. Aiena, Fredholm and Local Spectral Theory, with applications to Multipliers, kluwer Academic Publishers, 2004 (p.08)

1.3.2 Fredholm Theory and Fredholm Operators

The Fredholm theory, named after the Swedish mathematician Erik lvar Fredholm, is one of the fundamental pillars of modern functional analysis. It playsntral role in the study of linear equation on Banach and Hilbert spaces, particularly in connection with integral equations and compact operators. Fredholm theory is mainly concerned with the solvability of linear equations of the form $Tx = y$

Definition 1.3.3 ***(Fredholm Operator)***

Let $T \in B(X, Y)$. The operator T is said to be a **Fredholm operator** if it satisfies the folowing conditions:

1. $\dim(\ker T) < \infty$
2. $\operatorname{codim}(\operatorname{Im} T) < \infty$
3. The range $\operatorname{Im} T$ is closed in Y

Here:

- $\ker T$ denotes the kernel (null space) of T
- $\operatorname{Im} T$ denotes the range of T
- $\operatorname{codim}(\operatorname{Im} T)$ is the dimension of the quotient space $Y/\operatorname{Im} T$.[1]

The Fredholm Index

Assoiated with every Fredholm operator is an integer called the **Fredholm index**,defined by

$$\operatorname{ind}(T) = \dim(\ker T) - \operatorname{codim}(\operatorname{Im} T).$$

The index is a fundamental invariant in operator theory .

One of its most important properties is its stability under compact perturbations: if K is a compact operator, then

$$\operatorname{ind}(T + K) = \operatorname{ind}(T)$$

(1) P. Aiena [1], p. 02

1.3.3 Weyl Spectra

Definition 1.3.4 *Let X be a Banach space, and let $T \in L(X)$ be a bounded linear operator. The **Weyl spectrum** of T, denoted by $\sigma_w(T)$, is defined as the subset of the spectrum $\sigma(T)$ obtained by omitting the isolated eigenvalues of finite multiplicity.*

Mathematically, this is expressed by the following identity: $\sigma_w(T) = \sigma(T) \backslash \pi_{00}(T)$

- $\pi_{00}(T)$: The set of all isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. It is characterized by the following conditions:

1. λ is an isolated point of $\sigma(T)$.

2. The dimension of the kernel (nullity) is finite and positive, i.e.,

$$0 < \alpha(\lambda I - T) < \infty$$

(where $\alpha(\lambda I - T) = \dim \ker(\lambda I - T)$).[1]

1.3.4 Riesz Operator

Definition 1.3.5 *Let X be a complex Banach space, and let $L(X)$ be the space of all bounded linear operators defined from X into itself. An operator $T \in L(X)$ is said to be a **Riesz operator** if and only if the operator: $\lambda I - T$ is a **Fredholm operator** for every non-zero complex number $\lambda \in \mathbb{C} \backslash \{0\}$.*

Basic Concepts Related to Riesz Operator

In order to be a Fredholm operator, the operator $\lambda I - T$ must satisfy the following three conditions simultaneously for all $\lambda \neq 0$:

1. **Finite-dimensional kernel:**

$$\dim(\ker(\lambda I - T)) < \infty$$

2. **Closed range**: The range $R(\lambda I - T)$ is a closed set in X.

[1] P. Aiena [1], p.214

3. **Finite codimension:**

$$\dim(X/R(\lambda I - T)) < \infty$$

[(1)]

1.3.5 Algebraic Operators

Definition 1.3.6 *Let X be a Banach space, and let $L(X)$ denote the space of all bounded (continuous) linear operators on X into itself. An operator $T \in L(X)$ is said to be an* ***algebraic operator*** *if there exists a nontrivail polynomial $h(t)$ such that it annihilates the operator, i.e.,*

$$h(T) = 0$$

In other words, if the polynomial $h(t)$ of degree n is given by:

$$h(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$$

then substituting the operator yields the zero operator:

$$a_n T^n + a_{n-1} T^{n-1} + \ldots + a_1 T + a_0 I = 0$$

(where I denotes the identity operator on $L(X)$).[(2)]

Weyl-Type Spectra for Invertible operators

Theorem 1.3.1 *If $T \in L(X)$ is invertible, then:*

$$\sigma_w(T^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_w(T) \right\}$$

Proof. Since T is invertible, we have $0 \notin \sigma(T)$. Consider the analytic function $f(\lambda) = \frac{1}{\lambda}$ defined on an open neighborhood of $\sigma(T)$ not containing 0. By the spectral Mapping Theorem for the Weyl spectrum, we obtain

$$\sigma_w(f(T)) = f(\sigma_w(T))$$

Since $f(T) = T^{-1}$, it follows that $\sigma_w(T^{-1}) = f(\sigma_w(T)) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma_w(T) \right\}$. ■

[(1)] P. Aiena [1], p.214
[(2)] P. Aiena [1], p.253

Chapter 2

Generalized Inverses: From Pseudo-Inverses to Drazin Inverses

2.1 Relatively Regular Operator

Definition 2.1.1 *An operator $T \in L(X)$ is said to be relatively regular (or to admit a pseudo-inverse) if there exists an operator $S \in L(X)$ such that*

$$TST = T.$$

Pseudo-inverses are not unique, and, as shown by the unilateral shift example, they do not preserve spectral relationships in a simple way.

Basic properties

There is no loss of generality in requiring only the condition $T = TST$ Indeed, if this holds, then the operator $S^{'} = STS$ also satisfies

$$T = TS^{'}TandS^{'} = S^{'}TS^{'}$$

Furthermore, if S is a pseudo-inverse of T, then all operators of the form

$$STS + U - STUTS, U \in L(X)$$

This formula gives all pseudo-inverses of T .

Theorem 2.1.1 *A bunded operator $T \in L(X, Y)$ is relatively regular if and only if* $\ker(T)$ *and $T(X)$ are complemented*[(1)]

2.2 Quasi-nilpotent part of an operator and analytic core

Let $T \in B(X)$, the quasi-nilpotent part $H_0(T)$ of T and the analytic core of T, $K(T)$, are defined respectively by

$$H_0(T) = \left\{ x \in X : \lim_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}$$

and $K(T) = \{x \in X :$ there exists a sequence (x_n) in X and a constant $\delta > 0$ such that $Tx_1 = x$, Tx_n

These two subspaces were extensively studied by M.Mbekhta in [49]; they play an important role in local spectral theory (see [1,48]) as well as in classical spectral theory. The following two lemmas gather some properties of the spaces $H_0(T)$ and $K(T)$; they will often be used later (see [49,52])

Lemma 2.2.1 *Let $T \in B(X)$, then*

1. $H_0(T)$ is a subspace not necessarily closed in X.
2. $T(H_0(T)) \subseteq H_0(T)$.
3. For all $j \geq 0, N(T^j) \subseteq H_0(T)$.
4. $x \in H_0(T) \Leftrightarrow Tx \in H_0(T)$.
5. $H_0(T) = X \Leftrightarrow T$ is quasi-nilpotent.
6. If T is bounded below, then $H_0(T) = \{0\}$

Lemma 2.2.2 *Let $T \in B(X)$,then*

(1) P. Aiena [1], p88

1. $K(T)$ is a subspace not necessarily closed in X.

2. $T(K(T)) = K(T)$.

3. If X_0 is closed subspace of X with $T(X_0) = X_0$, then $X_0 \subseteq K(T)$.

4. If T is quasi-nilpotent, then $K(T) = \{0\}$.

5. If T is surjective, then $K(T) = X$.

Examples

- Let $X := \ell^2 \oplus \ell^2$...endowed with the norm $\|x\| := (\sum_{n=1}^{\infty} \|x_n\|^2)^{\frac{1}{2}}$ for all $x = (x_n) \in X$. We define

$$T_n e_i := \begin{cases} e_{i+1}, \text{ if } i=1,...,n, \\ \frac{e_{i+1}}{i-n}, \text{ if } i>n. \end{cases}$$

It is easy to verify that $\left\|T_n^{n+k}\right\| = \frac{1}{k!}$ and $(\frac{1}{k!})^{\frac{1}{n+k}} \to 0 \ (k \to \infty)$.
We now define the operator T by .

$$T := T_1 \oplus ... \oplus T_n \oplus ...$$

we see that $|T_n| = 1$, hence we obtain $|T| = 1$.Since $\sigma_p(T_n) = \phi$,then $\sigma_p(T) = \phi$. Consider the sequence $x = (x_n) \subset X$ defined by $x_n = \frac{e_1}{n}$ for all $n \in \mathbb{N}$. We have

$$\|x\| := (\sum_{n=1}^{\infty} \frac{1}{n^2})^{\frac{1}{2}} < \infty$$

which implies that $x \in X$. Moreover

$$\|T_n x\|^{\frac{1}{n}} \geq \left\|T_n^n \frac{e_1}{n}\right\|^{\frac{1}{n}} = (\frac{1}{n})^{\frac{1}{n}}$$

and this last term does not converge to 0. Hence $x \notin H_0(T)$.On the other hand,

$$\ell^2 \oplus \ell^2 \oplus ... \oplus \ell^2 \oplus \{0\} ... \subset H_0(T),$$

where the nonzero terms are finite in number, thus $H_0(T)$ is dense in X.Since $H_0(T) \neq X$, it follows that $H_0(T)$ is not closed. In the following example, we give an operator T which has a closed quasi-nilpotent part.

- Let $T : \ell^2 \to \ell^2$ be defined by

$$Tx = (\tfrac{x_2}{2}, ..., \tfrac{x_n}{n}, ...), \text{ with } x = (x_1, ..., x_n...).$$

We see that $\left|T^k\right| = \frac{1}{(k+1)!}$, thus T is quasi-nilpotent, hence $H_0(T) = \ell^2$.

- The left shift $Tx = (x_2, x_3, ...)$ is a surjective operator on ℓ^2 hence $K(T) = \ell^2$.The following theorem gives an important characterization of isolated points of the spectrum of an operator $T \in B(X)$ in terms of the quasi-nilpotent part and the analytic core.

Theorem 2.2.1 *Let $T \in B(X)$. Then the following conditions are equivalent:*

1. λ_0 is an isolated point in the spectrum $\sigma(T)$ of T;
2. $X = K(T - \lambda_0 I) \oplus H_0(T - \lambda_0 I)$ where $H_0(T - \lambda_0 I) \neq 0$ and $\oplus$ denotes the topological direct sum. Moreover, λ_0 is a pole of order k of the resolvent $R(\lambda, T)$, if and only if

$$K(T - \lambda_0 I) = R((T - \lambda_0 I)^k) \text{ and } H_0(T - \lambda_0 I) = N((T - \lambda_0 I)^k).$$

Theorem 2.2.2 *Let$T \in B(X)$. If λ_0 is an isolated point in $\sigma(T)$, then for all $\lambda \neq \lambda_0$*

$$\{0\} \subset H_0(T - \lambda_0 I) \subseteq K(T - \lambda I)$$

Corollary 2.2.1 *Let $T \in B(X)$. If $\lambda_0 \neq 0$ is an isolated point in $\sigma(T)$, then*

$$T(H_0(T - \lambda_0 I)) = H_0(T - \lambda_0 I)$$

Corollary 2.2.2 *Let $T \in B(X)$. If $\lambda_0 \in \sigma(T)$ satisfies, $K(T - \lambda_0 I) = 0$ then λ_0 is the only isolated point in $\sigma(T)$. If 0 is an isolated point in $\sigma_{su}(T)$, we have the following result:*

Theorem 2.2.3 *Let $T \in B(X)$ such that $0 \in \sigma_{su}(T)$. then 0 is an isolated point in $\sigma_{su}(T)$ if and only if $X = K(T) + H_0(T)$.*

Corollary 2.2.3 *Assume that $0 \in \sigma_{ap}(T)$. Then 0 is an isolated point in $\sigma_{ap}(T)$ if and only if $X^* = K(T^*) + H_0(T^*)$.*

Proposition 2.2.1 *Let $T \in B(X)$, such that $0 \in \sigma_{ap}(T)$. Then:*

1. $K(T)$ and $H_0(T)$ are closed;

2. $K(T) \cap H_0(T) = \{0\}$;

3. $K(T) \oplus H_0(T)$ is closed and there exists $\lambda_0 \neq 0$ such that

$$K(T) \oplus H_0(T) = K(T - \lambda_0 I) = \bigcap_{n=0}^{\infty} R((T - \lambda_0 I)^n)$$

[(1)]

2.3 The Drazin Inverse: Definition, Uniqueness, and Core-Nilpotent Decomposition

Definition 2.3.1 ***(Drazin Invertibility)***

Let $R \in L(X)$. The operator R is said to be **Drazin invertible** if there exist an operator $S \in L(X)$ and an integer $n \in \mathbb{N}$ such that:

$$RS = SR, SRS = S, R^n SR = R^n.$$

The smallest such integer n is called the **index** of R, denoted by $i(R)$. The operator S is called the **Drazin inverse** of R and is denoted by R^D.

Theorem 2.3.1 ***(Characterization of Drazin Invertibility)***

Let $R \in L(X)$. Then R is Drazin invertible if and only if

$$a(R) = d(R) < \infty$$

[(2)]

[(1)] P. Aiena [1], pp.118-122

[(2)] P. Aiena [1], p.85

Theorem 2.3.2 ***(Core-Nilpotent Decomposition)***

Let $R \in L(X)$ be a Drazin invertible operator with index $\nu = i\ (R)$. Then there exist two closed R-invariant subspaces Y and Z such that:

$$X = Y \oplus Z,\ R = R_1 \oplus R_2,$$

- $R_1 = R|_Y$ is nilpotent of order ν
- $R_2 = R|_Z$ is invertible

Moreover,the Drazin inverse $S = R^D$ is given by: $S = 0 \oplus R_2^{-1}$. This decomposition plays a central role in the study of Drazin invertible operators. The subspace Y is often called the **cor** (or algebraic spectral subspace corresponding to 0), while Z is its complementary subspace.

Proof. Since R is Drazin invertible, by Theorem there exists an integer

$$p = a(R) = d(R)$$

such that

$$X = \ker(R^p) \oplus R^p(X).$$

Set

$$Y := \ker(R^p), \qquad Z := R^p(X)$$

Both Y and Z are closed and R-invariant subspaces. Define

$$R_1 := R|_Y, \qquad R_2 := R|_Z.$$

For every $x \in Y = \ker(R^p)$, $T^p x = 0$, hence $R_1^p = 0$. Therefore, R_1 is nilpotent. On the other hand, since $Z = R^p(X) = R^{p+1}(X)$, the restriction R_2 is invertible on Z. Consequently, $R = R_1 \oplus R_2$, with R_1 nilpotent and R_2invertible. Moreover, the Drazin inverse $S = R^D$ can be represented with respect to the decomposition $X = Y \oplus Z$ as $S = 0 \oplus R_2^{-1}$. ■

2.4 Spectral Properties of the Drazin Inverse

Theorem 2.4.1 *Suppose that $R \in L(X)$ is Drazin invertible with Drazin inverse S. Then $\sigma(S)\setminus\{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma(R)\setminus\{0\}\right\}$ [1]*

Proof. If R is invertible, then $S = R^{-1}$,

$$\sigma(S)\setminus\{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma(R)\setminus\{0\}\right\}$$

Now suppose that

$$0 \in \sigma(R). \quad R = R_1 \oplus R_2, \quad S = 0 \oplus S_2, \quad S_2 = R_2^{-1}$$

Clearly,

$$\sigma(R) = \sigma(R_1) \cup \sigma(R_2) = \{0\} \cup \sigma(R_2), \ \sigma(S) = \{0\} \cup \sigma(S_2), \ 0 \notin \sigma(R_2) \text{ and } 0 \notin \sigma(S_2).$$

Since S_2 is the inverse of R_2, the nonzero part of the spectrum of S is given by the reciprocals of the nonzero points of the spectrum of R. Hence,

$$\sigma(S)\setminus\{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma(R)\setminus\{0\}\right\}$$

[(1)] ■

Spectral Relation for the Drazin Inverse

Theorem 2.4.2 *Let $R \in L(X)$ be a Drazin invertible operator with Drazin inverse S. If R is not nilpotent, then:*

$$\sigma_R(S)\setminus\{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_R(R)\setminus\{0\}\right\}$$

This shows that the nonzero spectrum of S is the reciprocal of the nonzero spectrum of R.[(2)]

Browder-Type Spectra

Theorem 2.4.3 *Let $R \in L(X)$ be Drazin invertible with inverse S. Then:*

(1) P. Aiena [1], p.288

(2) P. Aiena [1], Theorem 3.125, p.288

1. R is a Browder operator if and only if S is a Browder operator.

2. $\sigma_b(S)\setminus\{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_b(R)\setminus\{0\}\}$

3. $\sigma_{ub}(S)\setminus\{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{ub}(R)\setminus\{0\}\}$

4. $\sigma_{lb}(S)\setminus\{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{lb}(R)\setminus\{0\}\}$ [(1)]

Riesz Operators

Corollary 2.4.1 *If a Drazin invertible operator $R \in L(X)$ is a Riesz operator, then its Drazin inverse S is also a Riesz operator.* [(2)]

Relations Involving Powers and Kernels

Theorem 2.4.4 *(for $\lambda \neq 0$)*

1. If T is invertible:
$$(\lambda I - T)^k(X) = (\frac{1}{\lambda}I - T^{-1})^k(X), \forall k \in \mathbb{N}.$$

2. If R is Drazin invertible:
$$\ker(\lambda I - S)^k = \ker(\frac{1}{\lambda}I - R)^k$$

3. $(\lambda I - S)^k(X) = (\frac{1}{\lambda}I - R)^k(X)$

4. λ is a pole of the resolvent of $R \Leftrightarrow \frac{1}{\lambda}$ is a pole of the resolvent of S

5. $H_0(\lambda I - S) = H_0(\frac{1}{\lambda}I - R)$ [(3)]

Drazin Invertibility of Algebraic Operators

Corollary 2.4.2 *If $T \in L(X)$ is algebraic, then its Drazin inverse is also algebraic.* [(4)]

Weyl-Type Spectra for the Drazin Inverse

(1) P. Aiena [1], p.289
(2) P. Aiena[1],p.289
(3) P. Aiena[1], p. 290 (Theorem 3.128)
(4) P. Aiena [1], pp. 255-256 (Theorem 3.68)

Theorem 2.4.5 *Let $R \in L(X)$ be Drazin invertible with inverse S. Then*

1. $\sigma_w(S) \setminus \{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_w(R) \setminus \{0\}\right\}$
2. $\sigma_{uw}(S) \setminus \{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{uw}(R) \setminus \{0\}\right\}$
3. $\sigma_{lw}(S) \setminus \{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{lw}(R) \setminus \{0\}\right\}$

The main principle established in this section is:

- The nonzero spectrum of the Drazin inverse is the reciprocal of the nonzero spectrum of the original operator.
- This reciprocal relationship extends to several important spectra, including:
  - Regular spectrum
  - Browder spectrum
  - Weyl spectrum
  - Upper and lower semi-spectra
- Moreover:
  - Spectral properties such as **Browder** and Riesz are preserved under the Drazin inverse.
  - Poles of the resolvent satisfy the transformation: $\lambda \leftrightarrow \frac{1}{\lambda}$ [(1)]

[(1)] P. Aiena ,S.Triolo [2], pp.435-439

Chapter 3

Foundations of Local Spectral Theory

3.1 The Local Resolvent Set and Local Spectrum

Let $T \in L(X)$ and $x \in X$. The notion of the **local resolvent** and the **local spectrum** plays a central role in local spectral theory.

Definition 3.1.1 *The* ***local resolvent set*** *of T at the point x, denoted by $\rho_T(x)$, is defined as the union of all open sets $U \subset \mathbb{C}$ for which there exists an analytic function $f : U \to X$ satisfing*

$$(\lambda I - T) f(\lambda)\,(x) = x \quad \text{forall } \lambda \in U$$

The **local spectrum** of T at x, denoted by $\sigma_T(x)$, is defined by $\sigma_T(x) = \mathbb{C} \backslash \rho_T(x)$[(1)]

- These notions are among the fundamental tools introduced in local spectral theory. In fact, as presented in this chapter, the study of the **local spectrum** and the associated **local spectral subspaces** $X_T(F)$, where $F \subset \mathbb{C}$, provides a deeper understanding of the behavior of operators at each vector $x \in X$. Moreover, the local spectral framework allws us to characterize important subspaces such as the **analytic core** $k(T)$, which appears as a particular case of a local spectral subspace. This connection shows howl local spectral theory refines the classical (global) spectral theory. In addition, these

(1) P. Aiena [1], pp. 95-97

concepts are closely related to the **single-valued extension property (SVEP)**, which is a key property studied in this chapter. The validity of SVEP ensures good analytic behavior of the local resolvent functions and plays a crucial role in many spectral decomposition problems.[(1)]

Remark 3.1.1 ***Remark 3.1.2*** *The local spectrum is also connected with other important subspaces, such as the* ***glocal subspaces****, which generalize analytic local subspaces, the* ***quasi-nilpotent part*** *of an operator, corresponding to the spectral value* 0*, and other subspaces related to the kernel and range of operators of the form* $\lambda I - T$*.*[(2)]

These relationships highlight the importance of the local resolvent and local spectrum in understanding both local and global properties of bounded linear operators.

3.2 Single-Valued Extension Property– SVEP

The study of local spectral theory begins with the analytic solutions of the equation

$$(\lambda I - T)f(\lambda) = x$$

For a given $x \in X$, the **local resolvent set** $\rho_T(x)$ consists of all $\lambda \in \mathbb{C}$ for which there exists an analytic function f defined in a neighborhood of λ satisfying

$$(\mu I - T)f(\mu) = x$$

$$\sigma_T(x) = \mathbb{C} \backslash \rho_T(x)$$

A central concept in this theory is the **Single-Valued Extension Property(SVEP)**

Definition 3.2.1 *Let* $T \in L(X)$ *and* $\lambda_0 \in \mathbb{C}$*. The operator* T *is said to have the* ***single-valued extension property (SVEP)*** *at* λ_0 *if, for every*

(1) P. Aiena [1], pp.95-104

(2) P. Aiena [1], pp.95-96

open neighborhood U of λ_0, the only analytic function $f : U \to X$

$$(\lambda I - T)f(\lambda) = 0, \forall \lambda \in D$$

The operator T is said to have **SVEP** if it has SVEP at every point $\lambda \in \mathbb{C}$.[(1)]

3.2.1 Basic Properties of SVEP

The SVEP ensures both **existence and uniqueness** of analytic solutions of the local resolvent equation. If T has the SVEP at $\lambda_0 \in \sigma_T(x)$, then there exists a **unique analytic function** f such that

$$(\lambda I - T)f(\lambda) = x$$

- It guarantees the existence of a **maximal analytic extension** of the resolvent function:
$$\widetilde{f}(\mu) = (\mu I - T)^{-1}x \text{ for } \mu \in \sigma(T)$$

- The SVEP is **inherited by restictions**: If M is a closed T-invariant subspace, then $T|_M$ has the SVEP.

- Every operator has the SVEP on the **resolvent set** $\rho(T)$, and also on the **boundary of the spectrum** $\partial\sigma(T)$, in particular at isolated spectral points.

- The SVEP is stable under **translations**: T has the SVEP if and only if $\lambda I - T$ has the SVEP.

3.2.2 Characterization via Local Spectral Subspaces

Define the **local spectral subspace** associated with a set $F \subset \mathbb{C}$ by

$$X_T(F) = \{x \in X : \sigma_T(x) \subset F\}$$

These subspaces satisfy:

(1) P. Aiena [1], pp.96-112

- $X_T(F)$ is a **linear, T-hyperinvariant subspace**
- If $\lambda \notin F$, then
  - $(\lambda I - T)(X_T(F)) = X_T(F)$
  - $X_T\left(\cap_j F_j\right) = \cap_j X_T(F_j)$

3.2.3 Important Results on SVEP

- **Equivalence**: The following are equivalent:
  - T has the SVEP,
  - $X_T(\varnothing) = \{0\}$,
  - $X_T(\varnothing)$ is closed
- **Direct sum:** $T_1 \oplus T_2$ has the SVEP if and only if both T_1 and T_2 have the SVEP, and

$$\sigma_{T_1 \oplus T_2}(x_1 \oplus x_2) = \sigma_{T_1}(x_1) \cup \sigma_{T_2}(x_2)$$

- **Decomposition result**: If $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$

3.2.4 Local Decomposition Property

If T has the SVEP and F_1,F_2 are disjoint closed subsets of $\mathbb{C}$, then

$$X_T(F_1 \cup F_2) = X_T(F_1) \oplus X_T(F_2)$$

This decomposition is obtained using contour integrals and analytic functional calculus.

3.2.5 Analytic Core Characterization

An important consequence is the description of the **anlytic core**:

$$K(T) = X_T(\mathbb{C}\setminus\{0\}) = \{x \in X : 0 \notin \sigma_T(x)\}$$

Conclution

The SVEP is a fundamental property in local spectral theory. It ensures the uniqueness of analytic solutions, provides a structure for local spectral subspaces, and allows decomposition of the space according to spectral behavoir. It also plays a crucial role in linking analytic methods with operator theory, particularly through resolvent functions and contour integration techniques.[1]

3.3 Local Spectral Subspaces

In this section, we introduce the notion of local spectral subspaces associated with a bounded linear operator $T \in L(X)$. These subspaces constitute a variant of the analytic subspaces $X_T(F)$. They are particularly useful in local spectral theory, especially in the case where the operator T does not possess the single-valued extension property (SVEP). Let $F \subseteq \mathbb{C}$ be a closed subset. The local spectral subspace associated with F is defined by

$$\chi_T(F) = \{x \in X : \text{there is an ananalytic function } f : \mathbb{C}\setminus F \to X,\ (\lambda I - T)f(\lambda) = x, \forall\lambda \in \mathbb{C}\setminus F\}$$

It is easy to verify that $\chi_T(F)$ is a linear subspace of X. Moreover, $\chi_T(F) \subseteq X_T(F)$ for every closed subset $F \subseteq \mathbb{C}$. The following theorem presents some basic properties of local spectral subspaces.[2]

Theorem 3.3.1 *let $T \in L(X)$. Then :*

1. $\chi_T(\phi) = \{0\}$ and $\chi_T(\sigma(T)) = X$.
2. If F and G are closed subsets of $\mathbb{C}$, then

$$\chi_T(F) = \chi_T(F \cap \sigma(T)), (\lambda I - T)\chi_T(F) = \chi_T(F)$$

[1] P. Aiena [1], pp.96-118
[2] P. Aiena [1], p.112

3. If $(\lambda I - T)x \in \chi_T(F)$ for some $\lambda \in F$, then $x \in \chi_T(F)$

4. For every pair of disjoint closed subsets $F_1,\, F_2 \subseteq \mathbb{C}$,

$$\chi_T(F_1 \cup F_2) = \chi_T(F_1) + \chi_T(F_2)$$

5. The operator T has the SVEP if and only if

   $\chi_T(F) = X_T(F)$

   $\chi_T(F) \cap \chi_T(G) = \{0\}$

   Conversely, if $X_T(F) = \chi_T(F)$

   If $T \in L(X)$ is onto, then

$$X = K(T) = \chi_T(\mathbb{C}\setminus \{0\})$$

   Furhermore, there exists $\delta > 0$ such that

$$X = \chi_T(\mathbb{C}\setminus\overline{D(0,\delta)}) = \chi_T(\mathbb{C}\setminus D(0,\delta))$$

   If T is of kato-type, then exists $\delta > 0$ such that

$$K(T) = \chi_T(\mathbb{C}\setminus\overline{D(0,\delta)}) = \chi_T(\mathbb{C}\setminus D(0,\delta))$$

   Another important result concerns the decomposition of the space when the spectrum is disconnected.[1]

Theorem 3.3.2 *suppose that* $\sigma(T) = F_1 \cup F_2$

$$X = \chi_T(F_1) \oplus \chi_T(F_2)$$

We also have the following characterization.

Theorem 3.3.3 *If* $T \in L(X)$ *and* $F \subseteq \mathbb{C}$ *is closed, then:*

1. $\chi_T(F) = X$ if and only if $\sigma_S(T) \subseteq F$

(1) P. Aiena [1], Theorem 2.23, p.113

2. $\chi_T(F) = \{0\}$ if and only if $\sigma_{ap}(T) \cap F = \phi$

Let U be an open neighborhood of $\sigma(T)$ and let $f : U \to \mathbb{C}$ be analytic. Define

$$g(\mu, \lambda) = \begin{cases} \frac{f(\mu)-f(\lambda)}{\mu-\lambda}, \ \mu \neq \lambda \\ f'(\lambda) \ , \ \mu = \lambda \end{cases}$$

Then g is analytic in both variables and satisfies

$$f(\mu) - f(\lambda) = (\mu - \lambda) g(\mu, \lambda)$$

From the functional calculus one obtains

$$f(T) - f(\lambda) = (T - \lambda I) g(T, \lambda)$$

This leads to the following canonical behavior of local spectral subspace under analytic functional calculus.(1)

Theorem 3.3.4 *If $T \in L(X)$ and $f : U \to \mathbb{C}$ is analytic on an open neighborhood U of $\sigma(T)$, then*

$$\chi_{f(T)}(F) = \chi_T(f^{-1}(F))$$

for every closed subset $F \subseteq \mathbb{C}$. Finally, the local spectral subspace associated with the closed disc $\overline{D(0, \varepsilon)}$ admits the characterization

$$\chi_T(\overline{D(0, \varepsilon)}) = \left\{ x \in X : \limsup_{n \to \infty} \|T^n x\|^{\frac{1}{n}} \leq \varepsilon \right\}.$$

This characterization connects local spectral subspace with the asymptotic behavior of the sequence $(T^n x)$.(2)

3.3.1 The Quasi-nilpotent Part

For a single point $\{\lambda\}$, the glocal spectral subspace coincides with the quasi-nilpotent part:

$$\chi_T(\{\lambda\}) = H_0(\lambda I - T) = \left\{ x \in X : \lim_{n \to \infty} \|(\lambda I - T)^n x\|^{\frac{1}{n}} = 0 \right\}.$$

(1) P. Aiena [1], Theorem 2.27, p.113
(2) P. Aiena [1], Theorem 2.29, p.116

3.4 Key Properties

Let X be a Banach space and $T \in L(x)$. In local spectral theory, several important properties play a central role in the study of spectral operators and decomposable operators. Among the most important are Dunford's property (C), Bishop's property(β), the decomposition property (δ), and decomposability.

3.4.1 Dunford's Property (C)

A bounded operator $T \in L(X)$ is said to have **Dunford's property (C)** if, if for every closed subset $F \subseteq \mathbb{C}$, the analytic spectral subspace $X_T(F)$ is closed. This property was introduced by Dunford and plays an important role in local spectral theory and in the characterization of spectral operators. Moreover, property (C) implies the single-valued extension property (SVEP).

Indeed, if T has property (C), then T has the SVEP. Furthermore, quasi-nilpotent operators provide examples of operators having property (C). In fact, if T is quasi-nilpotent on a Banach space X, then T has property (C). Another important fact is that property (C) is inherited by restrictions to closed invariant subspaces. More precisely, if T has property (C) and Y is a T-invariant closed subspace of X, then the restriction $T|_Y$ also has property (C).

3.4.2 Bishop's Property (β)

Let $H(U, X)$ denote the space of all analytic functions from an open subset $U \subseteq \mathbb{C}$ into X, endowed with the topology of locally uniform convergence. An operator $T \in L(X)$ is said to have **Bishop's property (δ)** at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subseteq D(\lambda, r)$, and for every sequence of analytic functions $(f_n) \subset H(U, X)$ satisfying

$$(\mu I - T)f_n(\mu) \to 0 \text{ in } H(U, X), \text{we also have } f_n(\mu) \to 0 \text{ in } H(U, X).$$

The operator T is said to have property (β) if it has property (β) at every $\lambda \in \mathbb{C}$. The following implication is fundamental in local spectral theory:

$$\textbf{property } (\beta) \Rightarrow \textbf{property (C)} \Rightarrow \textbf{SVEP}$$

Thus, Bishop's property (β) is stronger than Dunford's property (C).

3.4.3 Decomposable Operators

An operator $T \in L(X)$ is called **decomposable** if for every open covering U_1 ,U_2 of the complex plane $\mathbb{C}$, there exist two closed T-invariant subspaces Y_1 and Y_2 such that

$$Y_1 + Y_2 = X \text{ and } \sigma(T|_{Y_i}) \subseteq U_i, \quad i = 1, 2$$

Decomposable operators constitute an important class of operators in local spectral theory and admit a rich spectral structure. A deep result in local spectral theory states that decomposability can be characterized by the combination of properties (β) and (δ). More precisely,

$$T \text{ is decomposable} \Leftrightarrow T \text{ has both properties } (\beta) \text{ and } (\delta)$$

Equivalently,

$$T \text{ is decomposable} \Leftrightarrow T \text{ has both properties } (C) \text{ and } (\delta)$$

These properties are therefore strongly interconnected and form one of the fundamental aspects of local spectral theory.(1)

3.5 Property (Q)

Let X be a Banach space and let $T \in L(X)$. An operator T is said to satisfy property (Q) if the quasi-nilpotent part $H_0(\lambda I - T)$ is closed for every $\lambda \in \mathbb{C}$. The quasi-nilpotent part associated with the operator $\lambda I - T$ is defined by

$$H_0(\lambda I - T) = \left\{ x \in X : \lim_{n\to\infty} \|(\lambda I - t)^n x\|^{\frac{1}{n}} = 0 \right\}$$

(1) P. Aiena [1],pp. 112-135

Property (Q) is one of the important notions in local spectral theory. It is closely related to the single-valued extension property (SVEP) and to property (C). In fact, property (Q) lies between these two properties in the sense that

$$\textbf{property(C)} \Rightarrow \textbf{property(Q)} \Rightarrow \textbf{SVEP}$$

Terefore, every operator havingproperty (C) also has property (Q), while every operator satisfying property (Q) possesses the SVEP. The importance of property (Q) comes from the fact that the closedness of the quasi-nilpotent part provides useful information about the local spectral behavior of the operator. Moreover, this property appears naturally in the study of decomposability, local spectral subspaces and spectral properties of bounded linear operators on Banach spaces. Property (Q) is also connected with several classes of operators studied in local spectral theory, especially operators the having the SVEP and operators satisfying local spectral conditions. Consequently, it plays a significant role in the investigation of spectral theory and Fredholm theory.[(1)]

[(1)] P. Aiena [1], p.118

Chapter 4

Transmission of SVEP an the Local Spectrum

Before proving the results for Drazin inverse, it is instructive to recall the counterexample for pseudo-inverses. The unilateral right shift T on $\ell^2(\mathbb{N})$ has SVEP. Its pseudo-inverse, the left shift L, does NOT have SVEP. Moreover, the local spectrum of T at any non-zero vector is the entire closed unit disc, so a reciprocal relationship cannot hold for any pseudo-inverse. The notion of pseudo-inverse appears naturally in the study of generalized inverses and spectral theory. However, several local spectral properties that hold for invertible operators are not necessarily preserved for pseudo-inverses. In particular, before studying the Drazin inverse, it is useful to recall a counterexample concerning pseudo-inverses. Consider the unilateral right shift operator T on $\ell^2(\mathbb{N})$.The operator T has the single-valued extension property (SVEP). Its pseudo-inverse, namely the left shift operator L, does not have the SVEP. Moreover, the local spectrum of T at every nonzero vector coincideswith the closed unit disc. Therefore, a reciprocal relationship between the local spectral properties of an operator and those of its pseudo-inverse cannot generally be expected. This example shows that pseudo-inverses do not preserve several important spectral properties. For this reason, the concept of Drazin invertibility plays an essential role in local spectral theory. Indeed, many spectral properties are transmitted between a Drazin inverse, unlike the case of pseudo-inverses.[(1)]

[(1)] P. Aiena, The Single-Valued Extension Property, pp.97-101

4.1 SVEP Transmition

In this section, we study the transmission of the single-valued extension property (SVEP) for Drazin invertible operators. Let $R \in L(X)$ be a Drazin invertible operator with Drazin inverse S. The aim is to show that the local spectral properties of R are closely related those of its Drazin inverse. Recall that if R is Drazin invertible, then there exist two closed invariant subspces Y and Z such that $X = Y \oplus Z$ with $R = R_1 \oplus R_{2,}$ where $R_1 = R|_Y$ is nilpotent and $R_2 = R|_Z$ is invertible. The Drazin inverse of R is then given by $S = 0 \oplus R_2^{-1}$. The following lemma establishes the transmission of SVEP between an operator and its Drazin inverse.

Lemma 4.1.1 *Let $R \in L(X)$, X a Banach sapce, be Drazin invertible with Drazin inverse S then R has SVEP if and only if S has SVEP.* [2]

Proof. Assume R has SVEP. If $0 \notin \sigma(R)$, then S is the inverse of R, and the result follows from the spectral mapping for SVEP . Assume $0 \in \sigma(R)$. Use the core-nilpotent decomposition $X = Y \oplus Z$, $R = R_1 \oplus R_2$ with R_1 nilpotent and R_2 invertible, and $S = 0 \oplus R_2^{-1}$.

- Nilpotent operators trivially have SVEP
- R_2, being invertible and a restriction of R to a closed invariant subspace, inherits SVEP from R.
- R_2^{-1} has SVEP.
- Therefore, $S = 0 \oplus R_2^{-1}$ is the direct sum of two operators with SVEP. The direct sum of operators with SVEP has SVEP. The converse is proved similarly

■

4.2 Reciprocal Relationship for the Non-zero Local Spectrum

The reciprocal relationship between the local spectrum of a Drazin invertible operator and its Drazin inverse plays an important role in local spectral theory. Under the SVEP condition, the non-zero local spectra of the operator and its Drazin inverse are connected through an inversion formula. The following theorem describes this relationship.

Theorem 4.2.1 *Suppose $R \in L(X\ )$ is Drazin invertible with Drazin inverse S. If R has SVEP, then for every $x \in X$ we have:*

$$\sigma_S(x)\backslash\{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_R(x)\backslash\{0\}\right\}$$

[1]

Proof. *If $0 \notin \sigma(R\ )$, then $S = R$ and the result follows from the spectral mapping theorem for the local spectrum. Assume $0 \in \sigma(R\ )$ and use the core-nilpotent decomposition as above. Write $x = y + z$ with $y \in Y$, $z \in Z$. From $\sigma_R(x) = \sigma_{R_1}(y) \cup \sigma_{R_2}(z)$. Since R_1 is nilpotent, $\sigma_{R_1}(y)$ is either $\{0\}$ (if $y \neq 0$) or $\emptyset$ (if $y = 0$). Similarly, $\sigma_S(x) = \sigma_0(y) \cup \sigma_{R_2^{-1}}(z)$, where $\sigma_0(y)$ is $\{0\}$ if $y \neq 0$ and $\emptyset$ if $y = 0$. If $y = 0$, then $\sigma_R(x) = \sigma_{R_2}(x)$ and $\sigma_S(x) = \sigma_{R_2^{-1}}(x)$. The result follows from the invertible case. If $y \neq 0$, then $\sigma_R(x)\backslash\{0\} = \sigma_{R_2}(z)$ and $\sigma_S(x)\backslash\{0\} = \sigma_{R_2^{-1}}(z)$. Applying the spectral mapping theorem to R_2 and R_2^{-1} gives the desired relationship. This theorem shows that, unlike for general pseudo-inverses, the Drazin inverse preseves the local spectral information in a clean, reciprocal manner.* ∎

4.3 Transmission of Property (C) and Property (Q)

Local Subspaces of a Quasi-nilpotent Operator

A key lemma is needed to handle the nilpotent part R_1.

Lemma 4.3.1 *Let $T \in L(X\ )$ be quasi-nilpotent (i;e., $\sigma(T\ ) = \{0\}$). For a closed set $F \subseteq \mathbb{C}$:*

- If $0 \in F$, then $X_T(F) = X$.

- If $0 \notin F$, then $X_T(F) = \{0\}$.

Proof. Since $\sigma(T) = \{0\}$, we have $X_T(F) = X_T(F \cap \sigma(T)) = X_T(F \cap \{0\})$. if $0 \notin F$, then $F \cap \{0\} = \emptyset$ and $X_T(\emptyset) = \{0\}$.If $0 \in F$, then

$F \cap \{0\} = \{0\}$. since T has SVEP (quasi-nilpotent operators do), $X_T(\{0\}) = H_0(T)$. For a quasi-nilpotent operator, $H_0(T) = X$.[(1)] ■

4.4 Transmission of Property (C)

Theorem 4.4.1 *Let $R \in L(X)$ be Drazin invertible with Drazin inverse S. Then R has prperty (C) if and only if S has property (C).*

Proof. Assume R has property (C) Again, use the decomposition $X = Y \oplus Z$ with $R = R_1 \oplus R_2$ and $S = 0 \oplus R_2^{-1}$. By [1, Corollary 2.10] , for any closed $F \subseteq \mathbb{C}$,

$$X_S(F) = Y_0(F) \oplus Z_{R_2^{-1}}(F),$$

where $Y_0(F\ )$ is the local spectral subspace of the zero operator on Y. Since R has property (C), its restriction R_2 also has property (C). As R_2 is invertible,[2, Corollary 2.4] implies that R_2^{-1} has property (C), so $Z_{R_2^{-1}}(F)$ is closed. From [2, Lemma 2.7] , $Y_0(F\ )$ is either Y (if $0 \in F$) or $\{0\}$ (if $0 \notin F$). In both cases, $X_S(F)$ is the direct sum of closed subspaces, hence closed. Therefore, S has property (C). The converse is proved similarly[(2)]

■

4.5 Transmission of Property (Q)

Theorem 4.5.1 *Let $R \in L(X)$ be Drazin invertible with Drazin inverse S.Then R has property (Q) if and only if S has property (Q).*

Proof. the proof follows the same structure as[2, Theorem 2.8] , but using the definition of property (Q) and the fact that

[(1)] P. Aiena, Fredholm and Local Spectral Theory, p.145

[(2)] P. Aiena, Fredholm and Local Spectral Theory, Section 2.12, p.204

$$H_0(\lambda I - T) = X_T(\{\lambda\})$$

when T has SVEP. Since property (Q) implies SVEP, we can use the direct sum decomposition for H_0 from [[1, $Theorem 2.9$]]. The closedness of H_0 for S follows from the closedness for its parts, using [2, Lemma 2.7] for the zero part and property (Q) for R_2^{-1}[(1)]

■

4.6 Transmission of Polaroid and a-Polaroid Properties

Definition 4.6.1 *An operator $T \in L(X)$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the reslvent (equivalently, $\lambda I - T$ is Drazin invertible for every $\lambda \in$ iso $\sigma(T\ )$. It is a-polaroid if every isolated point of $\sigma_a(T)$ is a pole.*

The study of transmission properties of polaroid and a-polaroid operators is an important topic in local spectral theory. These properties describe how the polaroid condition behaves under perturbations, quasi-affinities, intertwining operators, and algebraic transformations. Several results show that the polaroid and a-polaroid properties be preserved under suitable assumptions. In particular, if two operators are intertwined by an injective operator, some spectral properties may pass from one operator to another. This phenomenon is called the transmission of polaroid properties. Moreover, hereditary polaroid operators play an important role in this theory because of their relationship with the SVEP property and Weyl-type theorems. These operators are also closely connected with Drazin invertibility and local spectral behavior. Furthermore, the transmission of polaroid and a-polaroid properies has several applications in Weyl-type theorems, Browder theory, and spectral decomposition of operators.[(2)]

(1) P. Aiena [1], Theorem 2.187, pp.206-207

(2) P. Aiena [1], p.352

4.7 Relationship Between Kernels

A useful lemma relates the kernels of powers of R and S.

Lemma 4.7.1 *For a Drazin invertible operator R with Drazin inverse S, and for $\lambda \neq 0$, we have*

$$\ker(\lambda I - R)^k = \ker(\frac{1}{\lambda} I - S)^k \text{ for all } k \in \mathbb{N}.$$

Proof. The proof uses the core- nilpotent decomposition. For the invertible part R_2 and its inverse S_2, it follows from a direct calculation.

The nilpotent part contributes nothing for $\lambda \neq 0$[(1)] ■

Theorem 4.7.1 *Let $R \in L(X)$ be Drazin invertible with Drazin inverse S.*

- If R is polaroid, then S is polaroid.
- If R is a-polaroid, then S is a-polaroid.

Proof. We outline the proof for the polaroid case. Let $\mu \in iso\ \sigma(S)$. If $\mu = 0$, then since S is Drazin invertible, 0 is a pole.If $\mu \neq 0$, then by the reciprocal spectral relationship (4), $\lambda = 1/\mu \in iso\ \sigma(R)$. Since R is polaroid, $\lambda I - R$ is Drazin invertible and $H_0(\lambda I - R) = \ker(\lambda I - R)^p$ for some p. Using the decomposition and [2, Lemma 2.11] , one shows that $H_0(\mu I - S) = H_0(\lambda I - R)$ and $\ker(\mu I - S)^p = \ker(\lambda I - R)^p$. Hence $H_0(\mu I - S) = \ker(\mu I - S)^p$, so μ is a pole. This result has implications for the validity of Weyl-type theorems for S when they hold for R[(2)] ■

(1) P. Aiena, S. Triolo, Local spectral theory for Drazin invertible operators

(2) P. Aiena, S. Triolo, Local spectral theory for Drazin invertible operators

4.8 Transmission of Bishop's Property (β), property (δ), and Decomposability

4.8.1 Characterization of Property (β)

Bishop's property (β) plays a fundamental role in local spectral theory, acting as a powerful tool to bridge the gap between global and local spectral properties of bounded linear operators. An equivalent and very useful characterization of property (β) is defined through the stability of convergence within the space of analytic functions. Specifically,

Proposition 4.8.1 *An operator $T \in L(X)$ satisfies property (β) if and only if for every open set $U \subseteq \mathbb{C}$, the induced operator:*

$$H(U,X) \to H(U,X), (T_U f)(\lambda) = (\lambda I - T)f(\lambda)$$

has a **closed range**. Here, $H(U,X)$ denotes the Fréchet space of X-valued analytic functions on U. This characterization defines T as an operator that preserves the "injectivity" of the limit process in analytic function spaces, ensuring that spectral subspaces remain topologically well-behaved.[(1)]

Commutativity and Product Symmetry

One of the most significant attributes of property (β) is its symmetry with respect to the order of multiplication. According to established theorems in spectral theory, for two operators $S \in L(X,Y)$ and $R \in L(Y,X)$:

Proposition 4.8.2 *The operator $SR \in L(Y)$ possesses property (β) if and only if the operator $RS \in L(X)$ possesses property (β).*

This implies that property (β) is a shared spectral attribute within the algebra of operators, remaining invariant regardless of the product sequenc

has closed range. Here, $H(U,X$)isthe Fréchet space of X-valued analytic functions on U.[(2)]

[(1)] P. Aiena [1], p. 208

[(2)] P. Aiena [1], p. 211

4.8.2 Transmission of Property (β)

Theorem 4.8.1 *Let $R \in L(X)$ be Drazin invertible with Drazin inverse S. Then R has property (β) if and only if S has property (β).*

Proof. Assume R has property (). Use the decomposition $X = Y \oplus Z$, which induces a topological direct sum decomposition of the Fréchet space $H(U, X) \cong H(U, Y) \oplus H(U, Z)$ (Gleason's theorem). For the Drazin inverse $S = 0 \oplus R_2^{-1}$, the operator S_U acts as

$$S_U = 0_U \oplus (R_2^{-1})_U.$$

- Since R_2 has property (β), and is invertible, R_2^{-1} also has property (β) ([2, Corollary 2.4]). Therefore, $(R_2^{-1})_U$ has closed range.
- The zero operator 0_U on $H(U, Y)$ trivially has closed range (it is the zero operator). The direct sum of operators with closed range has closed range, so S_U has closed range. Hence, S has property (β). The converse is analogous..[1]

■

Remark 4.8.1 *Property (β) is situated within a clear hierarchy of local spectral conditions:*

$$\textit{Property } (\beta) \Rightarrow \textit{property } (C) \Rightarrow \textit{SVEP}$$

Where:

- **Property (C):** Refers to Dunford's property, ensuring that spectral subspaces are closed.
- **SVEP:** The Single Valued Extension Property, which only ensures uniqueness.

Furthermore, property (β) serves as one of the two essential pillars of **decomposability** in Banach spaces. An operator T is decomposable if and only if it possesses both property (β) and its dual, property ()[2]

(1) P. Aiena [1], Theorem. 2.190, pp. 208-209
(2) P. Aiena [1], p. 210

4.8.3 Transmission of Property (δ) and Decomposability

Corollary 4.8.1 *Let $R \in L(X)$ be Drazin invertible with Drazin inverse S.*

- If R has property (δ), then S has property (δ).
- If R is decomposable, then S is decomposable.

Proof. This follows from the duality between property (β) and property (δ). R has (β) iff R^* has (δ). If R is Drazin invertible, then R^* also Drazin inverse S^*.By [2, Theorem 2.13], R^* has (δ) implies S^* has (δ), which in turn implies S has (β). Decomposability follws from combining this with[2, Theorem 2.13] [(1)] ■

4.9 Properties of Related Operators $\mathbf{R^2S}$ and $\mathbf{S^2R}$

In the study of local spectral theory, the relationships between an operator R and its Drazin inverse S provide deep insights into the stability of spectral properties. From the fundamental defining equations of the Drazin inverse, we observe that:

$$U := R^2S = RSR.$$

this operator U is of particular interest in operator theory because it serves as the **Drazin inverse of S with index** 1, and it possesses the property of being always relatively regular. This section explores how local spectral properties, such as the Single Valued Extension Property (SVEP) and properties (C) and (Q), are transmitted to these related operators.[(2)]

Theorem 4.9.1 *Suppose $R \in L(X)$ is Drazin invertible with Drazin inverse S. If R has SVEP, then for all $x \in X$,*

$$\sigma_{R^2S}(x)\setminus\{0\} = \sigma_R(x)\setminus\{0\}.$$

(1) P. Aiena [1],Cor. 2.191, p. 209

(2) P.Aiena, Fredholm and Local Spectral Theory, with Applicatios to Multipliers,Kluwer Academic Publishers, 2004,Chapter 2, Section 2.11, pp.197-203

Moreover, if R has SVEP (respectively, property (C), (Q), (β), (δ), or is decomposable), then both R^2S and S^2R have the same property.

Proof. The proof is structured based on the transmission of spectral identities through the Drazin inverse relationships:

1. **Equality of Local Spectra:** The core of this identity relies on the spectral mapping theorem for local spectra. By applying the results established in [2, Theorem 2.6] twice, we see that the local spectrum $\sigma_{R^2S}(x)$ is directly related to $\sigma_S(x)$, which in turn is inherently tied to $\sigma_R(x)$. Since R^2S acts as a "bridge" operator, the non-zero ps of their locaartl spectra must coincide

2. **Transmission of Properties:** The transmission of properties like (C), (Q), and (β) follows from the fact that R^2S is itself a Drazin inverse of S. As demonstrated in the previous chapters (referencing [1, Theorem 2.187 ,2.189, 2.190]), these properties are invariant under the Drazin transformation. Specifically:

   - If R has **Property (Q),** then S must have it, which subsequently ensures R^2S (as a function of S) also satisfies the condition of having closed spectral subspaces $H_0(\lambda I - T)$.
   - If R is Decomposable, it satisfies both (β) and (δ); these are inherited by S and consequently by R^2S.

3. **The case of** S^2R : The analysis for the operator S^2R is even more direct. By utilizing the defining properties of the Drazin inverse (where $SRS = S$), we find that:

   $$S^2R = S(SR) = (SS)R = S$$

   Since S^2R is simply the operator S itself, the result that it shares the same properties as R (and S) becomes trivial[(1)]

■

[(1)] P.Aiena, Fredholm and Local Spectral Theory, with Applicatios to Multipliers,Kluwer Academic Publishers, 2004,Chapter 2, Section 2.12, pp. 204-209

Conclusion

In this dissertation, we have conducted a detailed investigation into the local spectral theory of Drazin invertible operators, following the work of Aiena and Triolo. We have shown that the core-nilpotent decomposition is a powerful tool that allows us to understand the relationship between an operator R and its Drazin inverse S. Our main findings can be summarized as follows:

- The single-valued extension property (SVEP) is transmitted between R and S.
- For operators with SVEP, the non-zero parts of the local spectra at any vector x are reciprocals of each other.
- The deeper local spectral properties, namely Dunford's property (C) , property (Q) , Bishop's property (β), property (δ), and decomposability, are all transmitted between R and S.
- The polaroid and a-polaroid properties are also transmitted
- These results extend naturally to the operators R^2S and S^2R

These results highlight that the Drazin inverse, unlike more general pseudo-inverses, faithfully preserves the local spectral structure of the original operator. the work presented here opens several avenues for future investigation:

1. **Perturbation Theory:** How do the local spectral properties of a Drazin invertible operator behave under small perturbations? If R is Drazin invertible and has property (C), does a small perturbation $R+K$ (with some conditions on K) also have property (C)?

2. **Analytic Functional Calculus:** For an operator T with property (), we know that $f(T)$ also has property () for functions f analytic on a neighbourhood of $\sigma(T\ )$. Can this be extended to functions that are analytic only on a neighbourhood of the non-zero part of the spectrum, using the Drazin inverse?

3. **Applications to Differential Operators:** Many differential operators on bounded domains with certain boundary conditions are Fredholm and have a discrete spectrum. Investigating the Drazin invertibility and local spectral properties of such operators could have practical implications.

4. **Operators with a Generalised Drazin Inverse:** The concept of the Drazin inverse has been extended to the "generalised Drazin inve- res" for operators where 0 is not necessarily a pole, but a point of the spectrum where the quasi-nilpotent part is isolated. It would be interesting to see which of the results presented here carry over to this more general setting.

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Abstract

This dissertation presents a comprehensive study of the local spectral properties of Drazin invertible operators on complex Banach spaces, building upon the foundational work of Aiena and Triolo. We systematically investigate the transmission of key spectral properties including the single-valued extension property (SVEP), Dunford's property (C), Bishop's property (β), and property (Q) between an operator R and its Drazin inverse S. The analytical framework relies heavily on the core-nilpotent decomposition, which enables an independent treatment of the invertible and nilpotent parts. We establish a reciprocal relationship for the non-zero points of the local spectrum and prove that the aforementioned properties are indeed transmission of the polaroid and a-polaroid properties, and discuss the algebraic and spectral implications for related operators such as RS and SR. This work aims to provide a self-contained, detailed exposition of these results, supplemented by the necessary background material and illustrative examples.

Key Words : Drazin invertible ; Local spectrum ; SVEP Property ; Weyl essential spectrum

Résumé

cette dissertation présente une étude approfondie des propriétés spectrales locales des opérateurs inversibles au sens de Drazin sur des espaces de Banach complexes, en se basant sur les travaux fondateurs d'Aiena et Triolo. Nous étudions de maniére systématique la transmission des propriétés spectrales clés notamment la propriété de l'extension analytique unique (SVEP), la propriété (C) de Dunford, la propriété (β) de Bishop, et la propriété (Q) entre un opérateur R et son inverse de Drazin S. Le cadre analytique repose fortement sur la décomposition coeur- nilpotent (core-nilpotent decomposition), qui permet un traitement indépendanr des parties inversibles et nilpotentes. Nous établissons une relation réciproquee pour les points non nuls du spectre local et prouvons que les propriétés mentionnées ci- dessus caractérisent en effet la transmission des propriétés polaroide et a-polaroide, tout en discutant les implications algébriques et spectrales pour les opérateurs associés tels que RS et SR. Ce travail vise à fournir un exposé détaillé et autonome dz ces résultats, complété par le matériel de base nécessaire et des exemples illustratifs.

Mots clès : Inverse de Drazin ; Spectre local ; Properiété du PEVS ; Spectre de Weyl

الملخص

تقدم هذه الاطروحة دراسة شاملة للخواص الطيفية المحلية للمؤثرات القابلة للعكس بمفهوم "درازين" على فضاءات "باناخ" بناء على العمل التأسيسي لكل من "ايينا" و "تريولو" نحن نتقصى بشكل منهجي انتقال الخواص الطيفية الرئيسية, بما في ذلك: خاصية التمديد وحيد القيمة , خاصية "دنفورد", خاصية "بيشوف" و الخاصية (Q) بين المؤثر ومعكوسه درازين الخاص به S.

يعتمد الاطار التحليلي بشكل كبير على تفكيك النواة المعدومة القوة و الذي يتيح معالجة مستقلة للأجزاء القابلة للعكس و الاجزاء المعدومة القوة. نثبت وجود علاقة تبادلية للنقاط غير الصفرية للطيف المحلي, ونبرهن على ان الخواص المذكورة اعلاه هي بالفعل انتقال لخواص "بولارويد" و "ا-بولارويد" كما نناقش التضمينات الجبرية و الطيفية للمؤثرات ذات الصلة، يهدف هذا العمل الى تقديم عرض مفصل و مكثف ذاتيا لهذه النتائج, مدعوما بالمادة الاساسية اللازمة و الامثلة التوضيحية.

الكلمات المفتاحية: معاكس ذرازن، الطيف المحلي، خاصية تمديد القيمة الواحدة، طيف ويل.