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The ideal of $\sigma(p)$-nuclear operators

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Table of contents

Notations

$\mathcal{L}(X,Y)$	The space of all bounded linear operators from X to$Y$.
$\mathbb{R}_+$	The field of non negative real numbers
E^*	The topological dual of E
p^*	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p^*} = 1$
$\mathbb{K}$	The field of real or complex numbers.
$\mathcal{K}(X,Y)$	The space of all compact operators from X to Y.
$\mathcal{L}_f(X,Y)$	The space o fall finite-rank operators from X to Y.
$\ell_\infty(B_{X^*})$	The Banach spaces of all bounded scalar families (λ_{x^*}) where $x^* \in B_{X^*}$.
I_X	The natural isometry $J : X \longrightarrow \ell_\infty(B_{X^*})$ is defined as $J(x) = (x^*(x))_{x^* \in B_{X^*}}$.
ℓ_∞	The Banach space of bounded scalar sequences.
c_0	The subspace of ℓ_∞ consisting of the scalar sequences $(x_n)_n$ such that $\lim x_n = 0$.
ℓ_p	The banach space of p-summabl scalar sequences.
T^*	The adjoint linear operator of T.
$\mathcal{N}_\sigma$	The set of all σ-nuclear linear operators.
$\mathcal{N}_{\sigma(p)}$	The set of all $\sigma(p)$-nuclear linear operators $(1 \leq p < \infty)$
$\ell_1(B_X)$	The Banach spaces of all absolutely summable scalar families (λ_x) where $x \in B_X$.
Q_X	The natural surjection $Q_X : \ell_1(B_X) \to X$ is defined as $Q_X(\lambda_x)_{x \in B_X} = \sum\limits_{x \in B_X} \lambda_x x$.

Introduction

The notion of ideal of linear operators between Banach spaces is due to Pietsch [17]. Nowadays, ideal operators are explored by several authors in many and diverse directions (see, for instance, [2, 3, 4, 6, 7], [18] and [16]). Among such ideals one can find the ideal of σ-nuclear operators (see, [17, Chapter 23]). This notion has been recently revisited in [13]. Kim and Yung defined a new tensor norm (σ-tensor norm) and show that it is associated with the ideal of σ-nuclear operators. It was shown [13], that the surjective hull and the injective hull of σ-nuclear linear operators coincide with the ideal of compact operators. In [8] the concept of σ-nuclear linear operators was generalized to $\sigma(p)$-nuclear linear for $p \geq 1$. Moreover, the concept of Lipschitz $\sigma(p)$-nuclear defined between a pointed metric space and a Banach space was introduced in [5], extending the $\sigma(p)$-nuclear linear operators. In this memory, we are going to do a detail study on the paper " Spaces of (p)-nuclear linear and multilinear operators and their duals " by Geraldo. Botelho and Ximena Mujica.

In the first Chapter is an overview of notions and basic concepts and results needed in the following chapters. These include vector-valued sequence spaces, also operator ideals and we describe the ideal of compact operators.

In the second Chapter we define the set $\mathcal{N}_\sigma$ of all σ-nuclear operators. We give some characterizations of this concept in term of factorization and we prove that $\mathcal{N}_\sigma^{sur}$ surjective hull and $\mathcal{N}_\sigma^{inj}$ injective hull of σ-nuclear equal to $\mathcal{K}$ (ideal of compact operators).

In the last Chapter we study the $\sigma(p)$-nuclear operators between Banach spaces. We prove that the class of $\sigma(p)$-nuclear linear operators is a Banach ideal. We give, under usual conditions on the underlying spaces, a simpler formula for the $\sigma(p)$-nuclear norm of a finite type operator. Also, we obtain a factorization of operators belonging to $\overline{\mathcal{L}_f}$. At the end of this chapter, we present an (incomplete) contribution of some properties of the injective and surjective hulls of the Banach operator ideal of $\sigma(p)$-nuclear operators. In [6], Ain, Lillemets and Oja they defined (p,r)-compact operators in an obvious way: a linear operator $T : E \to F$ is (p,r)-compact if $T(B_E)$ is a relatively (p,r)-compact subset of F. We show, that $\mathcal{K}_{(p,r)}$ ideal of (p,r)-compact operators is included in the $\mathcal{N}^{sur}_{\sigma(q^*)}$, where $\frac{1}{q^*} = \frac{1}{p} + \frac{1}{r}$.

The memory is based on papers [17], [14] and [8].

Chapter 1

Preliminaries

In this chapter we present a collection of some definitions, properties and basic formulas that will benefit us during this work (see [1], [10] and [17]). For example absolutely p-summable sequence, unconditionally p-summable sequences and operator ideals with some examples. We will write $\mathbb{K}$ for the real numbers field $\mathbb{R}$ or the complex numbers field $\mathbb{C}$. The set of all natural numbers $\{0, 1, \dots\}$ is denoted by $\mathbb{N}$. Along this work the letters X, Y, E and F denotes Banach spaces with the norm $\|.\|$. The closed unit ball of X is denoted by B_X that is the set $\{x \in X : \|x\| \leq 1\}$. The set of all *functionals* of a normed space X (that is the continuous linear mapping from X into the scalars) is a Banach space denoted by X^* and called the *topological dual* of X. For $x \in X$ we shall write $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$ for the action of the functional x^* on x $(x^*(x))$. The norm of $x^* \in X^*$ is

$$\|x^*\| = \sup\{|\langle x, x^* \rangle| : x \in B_X\}.$$

We denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators between X and Y with the norm

$$\|T\| = \sup_{x \in B_X} \|T(x)\|.$$

We write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. If $T \in \mathcal{L}(X, Y)$, the continuous linear operator $T^* : Y^* \longrightarrow X^*$ defined as

$$T^*(y^*)(x) = y^*(T(x)),$$

for every $y^* \in Y^*$ and $x \in X$ is called the *adjoint operator of* T with $\|T\| = \|T^*\|$.

1.1 Absolutely p-summable sequence

Let $1 \leq p \leq \infty$. The classical Banach sequence spaces ℓ_p, ℓ_∞ and c_0 are define by

- $\ell_p = \left\{(x_n)_n \subset \mathbb{K}\colon \| (x_n)_n \|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty\right\}, \, 0 \leq p < \infty.$
- $\ell_\infty = \{(x_n)_n \subset \mathbb{K}\colon \| (x_n)_n \|_\infty = \sup_{n\in\mathbb{N}} |x_n| < \infty\}, \, p = \infty.$
- $c_0 = \{(x_n)_n \subset \mathbb{K}\colon \lim_{n\to\infty} |x_n| = 0\}.$

For $p > 1$, we note p^* the conjugate of p defined by the formula $\frac{1}{p} + \frac{1}{p^*} = 1$. We pose $p^* = \infty$ si $p = 1$.

The following fact is well known, which is discussed in [10, 1].

Definition 1.1.1 *1. A sequence $(x_n)_n$, of elements of X is said to be absolutely p-summable if*

$$\|(x_n)_n\|_p = \begin{cases} (\sum_n \|x_n\|^p)^{\frac{1}{p}} < \infty & , \text{ if } 1 \leq p < \infty \\ \sup_n \|x_n\| < \infty & , \text{ if } p = \infty \end{cases}.$$

When $p = 1$, it is said that $(x_n)_n$ is absolutely summable. We denote by $\ell_p(X)$ the vector space of all absolutely p-summable sequences of elements of X. $(\ell_p(X), \|\cdot\|_p)$ is a Banach space.

2. The spaces $c_0(X\ )$ of norm null sequences in X is Branch spaces with the norm given by

$$\|(x_n)_n\|_\infty = \sup_n \|x_n\|.$$

3. A sequence $(x_n)_n$ in X is said to be weakly p-summable if

$$\sum_{n=1}^{\infty} |x^*(x_n)|^p < \infty,$$

for every $x^* \in B_{X^*}$. We denoted by $\ell_p^w(X)$ the Banach spaces of weakly p- summable sequence in X becomes a Banach spaces when equipped with the norm given by

$$\|(x_n)_n\|_p^w = \sup\left\{\left(\sum_{n=1}^{\infty} |x^*(x_n)|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\}.$$

Remark 1.1.1 *In the case $p = \infty$, then the spaces $\ell_\infty^w(X)$ of weakly bounded sequences coincide with the spaces $\ell_\infty(X)$,*

$$\|(x_n)_n\|_\infty^w = \|(x_n)_n\|_\infty .$$

1.2 Unconditionally p-summable sequences

We recall the notion of unconditionally p-summabl sequences.

Definition 1.2.1 *Let $1 \leqslant p \leqslant \infty$. A sequence $(x_n)_n \in X$ is said to be unconditionally p-summable if $(x_n)_n \in \ell_p^w(X)$ and*

$$\lim_{k \longrightarrow \infty} \|(x_n)_{n=k}^\infty\|_{p,w} = 0$$

and we denoted by

$$\ell_p^u(X) = \left\{ (x_n)_n \in \ell_p^u(X) : \lim_{k \longrightarrow \infty} \|(x_n)_{n=k}^\infty\|_{p,w} = 0 \right\}.$$

Definition 1.2.2 *A sequence $(x_i)_{i=n}^\infty$ in X is said unconditionally summable if, for all bijection $\eta : \mathbb{N} \longrightarrow \mathbb{N}$, the series $\sum_{i=1}^\infty x_{\eta(j)}$ is convergent in X.*

Proposition 1.2.1 *[9, Proposition 8.3]A sequence $(x_i)_{i=n}^\infty$ in X is said unconditionally summable if and only if, belongs to $\ell_1^u(X)$.*

Proposition 1.2.2 *$\ell_p^u(X)$ is a closed subset of $\ell_p^w(X)$ and therefore it is a Banach space, with the norm $\|\cdot\|_{p,w}$.*

Proposition 1.2.3 *Let $1 \leq p < \infty$. Then $\ell_p(X) \subseteq \ell_p^u(X) \subseteq c_0(X)$.*

1.3 Operator ideals

In this section, we recall some basic facts and properties about operator ideals. We also recall some of the classical examples.

Recall that a linear operator $T \in \mathcal{L}(X, Y)$ is said to have finite rank if $T(X)$ is a finite dimensional subspace of Y. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(X,Y)$. One can readily see that an operator $T \in \mathcal{L}(X,Y)$ has finite rank if, and only if, there exist $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$ such that

$$T(x) = \sum_{i=1}^{n} x_i^*(x) y_i,$$

for every $x \in X$.

Let us recall the definition of a Banach operator ideal, from [17] (see also [10]).

Definition 1.3.1 *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces X and Y its components $\mathcal{I}(X,Y) := \mathcal{L}(X,Y) \cap \mathcal{I}$ satisfy*

(i) $\mathcal{I}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ which contains the finite rank operators.

(ii) The ideal property: if $v \in \mathcal{L}(G,X)$, $u \in \mathcal{I}(X,Y)$ and $w \in \mathcal{L}(Y,H)$, then the composition $w \circ u \circ v$ is in $\mathcal{I}(G,H)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies

(i') $(\mathcal{I}(X,Y), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces X and Y.

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1, id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$, $id_{\mathbb{K}}(\lambda) = \lambda$.

(iii') $\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \|v\|_{\mathcal{I}} \|u\|$.

Then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$ for the sup norm.

Theorem 1.3.1 *Let $\mathcal{I}$ be a subclass of $\mathcal{L}(X,Y)$ endowed with a non-negative function $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$. For Banach spaces X, Y, define*

$$\mathcal{I}(E,F) : = \mathcal{I} \cap \mathcal{L}(X,Y).$$

Then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a Banach ideal of linear operators if and only if the following conditions hold :

(i) The linear operator $id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$, $id_{\mathbb{K}}(\lambda) = \lambda$, belongs to $\mathcal{I}$ and $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$.

(ii) If $S_1, S_2, ... \in \mathcal{I}(E,F)$ and $\sum_{k=1}^{\infty} \|S_k\|_{\mathcal{I}} < \infty$, then $S = \sum_{k=1}^{\infty} \|S_k\| \in \mathcal{I}(X,Y)$ and $\|S\|_{\mathcal{I}} \leq \sum_{k=1}^{\infty} \|S_k\|$.

(iii) If $T \in \mathcal{L}(G,X)$, $S \in \mathcal{I}(X,Y)$ and $\in R \in \mathcal{L}(Y,H)$, then $R \circ S \circ T \in \mathcal{I}(G,H)$ and $\|R \circ S \circ T\|_{\mathcal{I}} \leq \|R\| \|S\|_{\mathcal{I}} \|T\|$.

Every Banach ideal $\mathcal{I}$ of linear operators contains the finite type linear operators

$$\|x(\cdot)b\|_{\mathcal{I}} = \|x\| \|b\|.$$

Corollary 1.3.1 *If* $S \in \mathcal{I}(X,Y)$, *so* $\|S\| \leq \|S\|_{\mathcal{I}}$.

Proof. We have for $x \in X, y^* \in Y^*$ and $S \in \mathcal{I}(X,Y)$

$$\begin{aligned} |(y^* \circ S)(x)| &= |(y^* \circ S)(x)| \|id_{\mathbb{K}}\|_{\mathcal{I}} = \|(y^* \circ T)(x) \cdot id_{\mathbb{K}}\|_{\mathcal{I}} \\ &= \|y^* \circ S \circ R_x\|_{\mathcal{I}} \leq \|y^*\| \|S\|_{\mathcal{I}} \|R_x\| \end{aligned}$$

where $R_x : \mathbb{K} \to X : R_x(\lambda) = \lambda x$ and $\|R\| = \|x\|$.

So,

$$\begin{aligned} \|S\| &= \sup_{\|x_i\| \leq 1} \|S(x)\|_Y \\ &= \sup_{\|x\| \leq 1, \|y^*\| \leq 1} |y^*(S(x))| \\ &\leq \sup_{\|x\| \leq 1, \|y^*\| \leq 1} \|y^*\| \|S\|_{\mathcal{I}} \|R_x\| \\ &= \|S\|_{\mathcal{I}}. \end{aligned}$$

■

1.3.1 Examples

We now give a list of examples.

$\mathcal{L}$: Ideal of continuous operators;

$\mathcal{L}_f$: Ideal of finite rank operators;

$\overline{\mathcal{I}}$: The closure (with the usual operator norm) of an operator ideal $\mathcal{I}$;

Approximable operators. An operator $T \in \mathcal{L}(X,Y)$ is called approximable operators if there are $T_n \in \mathcal{L}_f(X,Y)$, with

$$\lim_n \|T - T_n\| = 0.$$

We denote by $\overline{\mathcal{L}_f(X,Y)}$ the ideal space of all approximable operators from X to Y.

Compact linear operators. A linear operator $T \in \mathcal{L}(X,Y)$ is compact if

$$T\left(B_X\right) \text{ is relatively compact in } Y.$$

We denote by $\mathcal{K}\left(X,Y\right)$ the space of all compact operators between X and Y. If we provide $\mathcal{K}\left(X,Y\right)$ of the induced norm of $\mathcal{L}\left(X,Y\right)$, it becomes a Banach ideal.

1.3.2 The convex hull subset

Definition 1.3.2 *The convex hull of a sequence $(x_n)_n \in c_0(X)$ is defined as*

$$conv\{(x_n)_n\} = \left\{\sum_{n=1}^{\infty}\alpha_n x_n : (\alpha_n)_n \in B_{\ell_1}\right\}$$

Definition 1.3.3 *Let $1 \leq p \leq \infty$. The p-convex hull of a sequence $(x_n)_n \in \ell_p(X)$ is defined as*

$$p\text{-}conv\{(x_n)_n\} = \left\{\sum_{n=1}^{\infty}\alpha_n x_n : (\alpha_n)_n \in B_{\ell_{p^*}}\right\}.$$

Lemma 1.3.1 *[11] A subset K of a Banach space X is relatively compact if and only if for every $\epsilon > 0$,there exists $(x_n)_n \in c_0(X)$ with $\sup_{j\in\mathbb{N}} \|x_j\| \leq (1+\epsilon)\sup_{x\in K} \|x\|$ such that*

$$K \subset conv\{(x_n)_n\}.$$

Proposition 1.3.1 *Let $K \subset \ell_p$. The set A is relatively compact iff there exists $\lambda = \left(\lambda_n\right)_n \in c_0$ such that*

$$K \subset \left\{\sum_n \lambda_n d_n e_n : d \in B_{\ell_p}\right\}.$$

Chapter 2

The ideal of σ-nuclear linear operators

The theory of σ-nuclear linear operators on Banach spaces was introduced and developed by Pietsch [17, Chapter 23]. This notion has been recently revisited in [13]. Kim and Yung defined a new tensor norm (σ-tensor norm) and show that it is associated with the ideal of σ-nuclear operators. It was shown in [13], that the surjective hull and the injective hull of σ-nuclear linear operators coincide with the ideal of compact operators.

2.1 σ-nuclear linear operators

By [17], we have the following definition

Definition 2.1.1 *Let E, F be a Banach space and $T : E \to F$ be a bounded liniear operator we say that T is σ-nuclear operator if $T = \sum_{j=1}^{\infty} x_j^* \underline{\otimes} y_j$ with $(x_j^*)_{j=1}^{\infty} \subset E^*$ and $(y_j)_{i=1}^{\infty} \subset F$ such that the family $\left(x_j^* \underline{\otimes} y_j\right)$ is unconditionnally summable in the operator norm. Where $x_j^* \underline{\otimes} y_j$ is an operator from E to F defined by $x_j^* \underline{\otimes} y_j(x) = x_j^*(x) y_j$.*

The set of all σ-nuclear operators is denoted by $\mathcal{N}_\sigma(E, F)$ and the σ-nuclear norm is denoted by $\eta_\sigma(T)$ and it is definited by

$$\eta_\sigma(T) = \inf \sup \left\{ \sum_{j=1}^{\infty} \left| x_j^*(x) \, y^*(y_j) \right| , \ x \in B_E, y^* \in B_{F^*} \right\},$$

where the infimum is taken over all σ-nuclear representations.

Theorem 2.1.1 *By [17] $(\mathcal{N}_\sigma(E,F), \eta_\sigma(\cdot))$ is a Banach operator ideal.*

For extension σ-nuclear to $\sigma(p)$-nuclear we need the following characterization [8] of σ-nuclear operators.

Theorem 2.1.2 *An operator $T \in \mathcal{L}(E,F)$ is σ-nuclear if and only if there are sequences $(x_i^*)_{i=1}^\infty \subset E^*$ and $(y_i)_{i=1}^\infty \subset F$ such that $T = \sum\limits_{i=1}^{\infty} x_i^* \otimes y_i$,*

$$\sup_{x \in B_E,, y^* \in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x) y^*(y_i)| < \infty$$

and

$$\lim_{m \longrightarrow \infty} \sup_{x \in B_E,, y^* \in B_{F^*}} \sum_{i=m}^{\infty} |x_i^*(x) y^*(y_i)| = 0.$$

Proof. By [9, Propositions 8.3 and 8.1], a sequence $(z_i)_{i=1}^\infty$ is unconditionally summable in Banach space Z if and only if

$$\sup_{\varphi \in M} \sum_{i=1}^{\infty} |\varphi(z_i)| < \infty$$

and

$$\lim_{m \to \infty} \sup_{\varphi \in M} \sum_{i=m}^{\infty} |\varphi(z_i)| = 0,$$

for some norming set $M \subseteq B_{Z^*}$.

Let $x \in B_E$ and $y^* \in B_{F^*}$, define the map

$$\varphi_{x,y^*} : \mathcal{L}(E,F) \to \mathbb{K}, \qquad \varphi_{x,y^*}(T) := y^*(T(x)),$$

We can easily to prove that φ_{x,y^*} is a continuous linear form and $\|\varphi_{x,y^*}\| \leq 1$. Then, the set

$$M = \left\{ \varphi_{x,y^*}, \quad x \in B_E, y^* \in B_{F^*} \right\}$$

is subset of $B_{\mathcal{L}(E,F)^*}$.

Recall that for Banach space Z, a subset M of the dual unit ball B_{Z^*} is said to be norming, if $\|z\| = \sup_{z^* \in M} |z^*(z)|$ for all $z \in Z$.

Let us check that M is norming. Give $\varphi \in M$, we have

$$\begin{aligned} \sup_{\varphi \in M} |\varphi(T)| &= \sup_{\varphi_{x,y^*} \in M} |\varphi_{x,y^*}(T)| \\ &= \sup_{\substack{x \in B_E \\ y^* \in B_{F^*}}} |y^*(T(x))| \\ &= \sup_{x \in B_E} \|T(x)\| \\ &= \|T\|. \end{aligned}$$

For $Z = \mathcal{L}(E,F)$ and $(z_i)_{i=1}^{\infty} = (x_i^* \otimes y_i)_{i=1}^{\infty} \subset Z$. We take $\varphi \in M$

$$\begin{aligned} |\varphi(x_i^* \underline{\otimes} y_i)| &= \left|\varphi_{x,y^*}(x_i^* \underline{\otimes} y_i)\right| \\ &= |y^*(x_i^*(x) y_i)| \\ &= |x_i^*(x) y^*(y_i)| \end{aligned}$$

for $x \in B_E$, $y^* \in B_{F^*}$.

Since $(x_i^* \underline{\otimes} y_i)_{i=1}^{\infty}$ is unconditionally summable sequence in $\mathcal{L}(E,F)$, that equivalently with

$$\sup_{\varphi \in M} \sum_{i=1}^{\infty} |\varphi(x_i^* \underline{\otimes} y_i)| = \sup_{x \in B_E,, y^* \in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x) y^*(y_i)| < \infty$$

and

$$\lim_{m \to \infty} \sup_{\varphi \in M} \sum_{i=m}^{\infty} |\varphi(x_i^* \otimes y_i)| = \lim_{m \to \infty} \sup_{x \in B_E,, y^* \in B_{F^*}} \sum_{i=m}^{\infty} |x_i^*(x) y^*(y_i)| = 0.$$

This completes the proof. ■

Motivated by Theorem 2.1.2, we characterize the Definition 2.1.1 as follows.

Definition 2.1.2 *We say that an operator T between two Banach spaces E and F is σ-nuclear if there are sequences $(\lambda_i)_{i=1}^{\infty} \in \ell_{\infty}, (x_i^*)_{i=1}^{\infty} \subset E^*$ and $(y_i)_{i=1}^{\infty} \subset F$ such that $T = \sum_{i=1}^{\infty} \lambda_i x_i^* \underline{\otimes} y_i$,*

$$\sup_{x \in B_E,, y^* \in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x) y^*(y_i)| < \infty$$

and

$$\lim_{m \longrightarrow \infty} \sup_{x \in B_E,, y^* \in B_{F^*}} \sum_{i=m}^{\infty} |x_i^*(x) y^*(y_i)| = 0.$$

In this case, the σ-nuclear norm becomes defined as

$$\eta_{\sigma(1)}(T) = \inf\left\{ \|(\lambda_i)_{i=1}^{\infty}\|_{\infty} \sup_{x\in B_{E},,y^*\in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x)y^*(y_i)| \right\},$$

where the infimum is taken over all σ-nuclear representations of T.

Proposition 2.1.1 *For all σ-nuclear operator T, we have* $\eta_{\sigma}(T) = \eta_{\sigma(1)}(T)$

Proof. *Now, consider* $T = \sum_{i=1}^{\infty} x_i^* \underline{\otimes} y_i = \sum_{i=1}^{\infty} \lambda_i x_i^* \underline{\otimes} y_i$ *with* $\lambda_i = 1$*, we have*

$$\begin{aligned} \eta_{\sigma(1)}(T) &= \inf\left\{ \|(\lambda_i)_{i=1}^{\infty}\|_{\infty} \sup_{x\in B_{E},,y^*\in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x)y^*(y_i)| \right\} \\ &\leq \inf\left\{ \sup_{x\in B_{E},,y^*\in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x)y^*(y_i)| \right\} = \eta_{\sigma}(T). \end{aligned}$$

On the other hand, let $T = \sum_{i=1}^{\infty} \lambda_i x_i^* \underline{\otimes} y_i = \sum_{i=1}^{\infty} x_i^* \underline{\otimes} z_i$ *where* $z_i = \lambda_i y_i$*, we have*

$$\begin{aligned} \eta_{\sigma}(T) &= \inf\left\{ \sup_{x\in B_{E},,y^*\in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x)y^*(z_i)| \right\} \\ &= \inf\left\{ \sup_{x\in B_{E},,y^*\in B_{F^*}} \sum_{i=1}^{\infty} |\lambda_i|\,|x_i^*(x)y^*(y_i)| \right\} \\ &\leq \inf\left\{ \|(\lambda_i)_{i=1}^{\infty}\|_{\infty} \sup_{x\in B_{E},,y^*\in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x)y^*(y_i)| \right\} = \eta_{\sigma(1)}(T), \end{aligned}$$

which completes the proof. ■

Example 2.1.1 *Let* $(x_i)_{i=1}^{\infty} \in c_0(E)$*, we define the map*

$$S : \ell_1 \longrightarrow E : \alpha \longmapsto S(\alpha) = \sum_{i=1}^{\infty} e_i^*(\alpha) x_i,$$

where $(e_i^*)_{i=1}^{\infty}$ *is the standard unit vector in* ℓ_1^* *(i.e.* c_0*). Then S is σ-nuclear operator. Indeed, obvious that S is linear continues operator and* $S = \sum_{i=1}^{\infty} e_i^* \otimes x_i$*. We have*

$$\begin{aligned} \sup_{\alpha\in B_{\ell_1},x^*\in B_{E^*}} \sum_{i=1}^{\infty} |e_i^*(\alpha)x^*(x_i)| &= \sup_{\alpha\in B_{\ell_1},x^*\in B_{E^*}} \sum_{i=1}^{\infty} |\alpha_i x^*(x_i)| \\ &\leq \sup_{\alpha\in B_{\ell_1}} \sum_{i=1}^{\infty} |\alpha_i|\,\|x_i\| \\ &\leq \sup_{1\leq i\leq\infty} \|x_i\| \sup_{\alpha\in B_{\ell_1}} \|\alpha\|_{\ell_1} < \infty \end{aligned}$$

and

$$\begin{aligned}\lim_{m\to\infty}\sup_{\alpha\in B_{\ell_1},x^*\in B_{E^*}}\sum_{i=m}^{\infty}\left|e_i^*(\alpha)x^*\left(x_i\right)\right| &= \lim_{m\to\infty}\sup_{\alpha\in B_{\ell_1},x^*\in B_{E^*}}\sum_{i=m}^{\infty}\left|\alpha_i x^*\left(x_i\right)\right| \\ &\leq \lim_{m\to\infty}\sup_{\alpha\in B_{\ell_1}}\sum_{i=m}^{\infty}\left|\alpha_i\right|\left\|x_i\right\| \\ &\leq \lim_{m\to\infty}\sup_{1\leq i\leq\infty}\left\|x_i\right\|\sup_{\alpha\in B_{\ell_1}}\sum_{i=m}^{\infty}\left|\alpha_i\right| = 0.\end{aligned}$$

So, S is σ-nuclear operator and

$$\begin{aligned}\eta_\sigma\left(S\right) &\leq \sup\left\{\sum_{j=1}^{\infty}\left|\alpha_j x^*(x_j)\right| : (\alpha_j)_j\in B_{\ell_1}, x^*\in B_{E^*}\right\} \\ &\leq \sup_{j\in\mathbb{N}}\left\|x_j\right\| = \left\|x\right\|_\infty .\end{aligned}$$

2.2 Factorization theorem

A sequence (e_j) with $e_j \neq 0$ is called a hyperorthogonal basis of the Banach space U if its linear span is dense and if $\left|\xi_j\right| \leq \left|\eta_j\right|$ implies $\left\|\sum_{j=1}^{n}\xi_j e_j\right\| \leq \left\|\sum_{j=1}^{n}\eta_j e_j\right\|$ for $n = 1, 2, ...$

Remark 2.2.1 *Every hyperorthogonal basis is unconditional conversely, a Banach space with an unconditional basis can be renormed such that basis becomes hyperorthogonal.*

Example 2.2.1 *The unit sequence basis of ℓ_p with $1 \leq p < \infty$ and c_0 is hyperorthogonal basis.*

The σ-nuclear operator can be factorized as the following theorem (see [17, Theorem 23.2.5] and [14]). Here we give the proof in more detail .

Theorem 2.2.1 *An operator $T \in \mathcal{N}_\sigma\left(E, F\right)$ if and only if, the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ A\downarrow & \nearrow B & \\ U & & \end{array}$$

where, $A \in \overline{\mathcal{L}_f(E,G)}, B \in \overline{\mathcal{L}_f(G,F)}$ and G is a Banach space having a hyperorthogonal basis $(e_i)_{i=1}^{\infty}$. And we have $\eta_\sigma\left(T\right) = \inf\left\|A\right\|\left\|B\right\|$ where the infimum is taken over all possible factorizations.

Proof. $\Rightarrow$) Consider $\varepsilon > 0$, $T \in \mathcal{N}_\sigma(E,F)$ and let $\sum_{i=1}^{\infty} \lambda_i x_i^* \underline{\otimes} y_i$ be a representation of T, such that

$$\sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x) y^*(y_i)| \leq (1+\varepsilon) \|(\lambda_i)_{i=1}^{\infty}\|_\infty^{-1} \eta_\sigma(T).$$

We have for all $x \in E$ and $y^* \in F^*$,

$$\begin{aligned} \sum_{i=1}^{\infty} \left| \frac{x_i^*(x)}{\|x\|} \frac{y^*(y_i)}{\|y^*\|} \right| &\leq \sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=1}^{\infty} |x_i^*(x) y^*(y_i)| \\ &\leq (1+\varepsilon) \|(\lambda_i)_{i=1}^{\infty}\|_\infty^{-1} \eta_\sigma(T), \end{aligned}$$

it follows that

$$\sum_{i=1}^{\infty} |x_i^*(x) y^*(y_i)| \leq (1+\varepsilon) \|(\lambda_i)_{i=1}^{\infty}\|_\infty^{-1} \eta_\sigma(T) \|x\| \|y^*\|.$$

We put $|x_i^*(x) y^*(y_i)| = |\alpha_i|$, obvious that $(\alpha_i)_{i=1}^{\infty} \in \ell_1$. By Lemma 8.6.4 in [17], there exists a decreasing sequence $(\varrho_i)_{i=1}^{\infty} \in c_0$ with $0 \leq \varrho_i \leq 1$ such that

$$\sum_{i=1}^{\infty} \varrho_i^{-1} |\alpha_i| \leq (1+\varepsilon) \sum_{i=1}^{\infty} |\alpha_i|.$$

We conclude that

$$\begin{aligned} \sum_{i=1}^{\infty} \varrho_i^{-1} |x_i^*(x) y^*(y_i)| &\leq (1+\varepsilon) \sum_{i=1}^{\infty} |x_i^*(x) y^*(y_i)| \\ &\leq (1+\varepsilon)^2 \|(\lambda_i)_{i=1}^{\infty}\|_\infty^{-1} \eta_\sigma(T) \|x\| \|y^*\|. \end{aligned}$$

We can consider $\varrho_i = \beta_i^2$, where $(\beta_i) \in c_0$ such that $1 \geq \beta_1 \geq \beta_2 \geq ... \geq 0$. Thus,

$$\sum_{i=1}^{\infty} \beta_i^{-2} |x_i^*(x) y^*(y_i)| \leq (1+\varepsilon)^2 \|(\lambda_i)_{i=1}^{\infty}\|_\infty^{-1} \eta_\sigma(T) \|x\| \|y^*\|.$$

Let

$$G := \left\{ z = (\tau_i)_{i=1}^{\infty} \in \mathbb{C} : (\tau_i \beta_i^{-1} y_i)_{i=1}^{\infty} \text{ is unconditionally summable} \right\}$$

and

$$\|z\|_G := \sup_{y^*\in B_{F^*}} \sum_{i=1}^{\infty} |\beta_i^{-1} \tau_i y^*(y_i)|.$$

Then $(G, \|\cdot\|_G)$ is a Banach space and the sequence $(e_i)_{i=1}^{\infty}$ of standard unit vectors forms a hyperorthogonal basis in G. Indeed, let $|\gamma_i| \leq |\theta_i|$, we have

$$\begin{aligned}
\left\|\sum_{i=1}^{n}\gamma_i e_i\right\|_G &= \|(\gamma_i)_{i=1}^{n}\|_G \\
&= \sup_{y^*\in B_{F^*}}\sum_{i=1}^{n}\left|\gamma_i\beta_i^{-1}y^*(y_i)\right| \\
&\leq \sup_{y^*\in B_{F^*}}\sum_{i=1}^{n}\left|\theta_i\beta_i^{-1}y^*(y_i)\right| \\
&= \|(\theta_i)_{i=1}^{n}\|_G = \left\|\sum_{i=1}^{n}\theta_i e_i\right\|_G,
\end{aligned}$$

for all $n \in \mathbb{N}$.

Define $A \in \mathcal{L}(E, G)$ and $B \in \mathcal{L}(G, F)$ by

$$A(x) = x_i^*(x), \qquad \text{for } x \in E, x_i^* \in E^*$$

and

$$B(z) = B\left((\tau_i)_{i=1}^{\infty}\right) = B(\sum_{i=1}^{\infty}\tau_i e_i) = \sum_{i=1}^{\infty}\tau_i B(e_i) = \sum_{i=1}^{\infty}\lambda_i\tau_i y_i,$$

for $z \in G$ and $(\lambda_i)_{i=1}^{\infty} \in \ell_\infty$.

Hence, it follows that

$$B \circ A(x) = B((x_i^*(x))_{i=1}^{\infty}) = \sum_{i=1}^{\infty}\lambda_i x_i^*(x)y_i = Tx.$$

Usually, we take $P_m : G \to G$ such that $P_m((\xi_i)_{i=1}^{\infty}) := (\xi_i)_{i=1}^{m}$. Then

$$\begin{aligned}
\|(A - P_mA)x\|_G &= \left\|(x_i^*(x))_{i=m+1}^{\infty}\right\|_G \\
&= \sup_{y^*\in B_{F^*}}\sum_{i=m+1}^{\infty}\beta_i^{-1}|x_i^*(x)y^*(y_i)| \\
&= \sup_{y^*\in B_{F^*}}\sum_{i=m+1}^{\infty}\beta_i\beta_i^{-2}|x_i^*(x)y^*(y_i)| \\
&\leq \beta_{m+1}\sup_{y^*\in B_{F^*}}\sum_{i=m+1}^{\infty}\beta_i^{-2}|x_i^*(x)y^*(y_i)| \\
&\leq \beta_{m+1}(1+\varepsilon)^2\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\eta_\sigma(T)\|x\|
\end{aligned}$$

and

$$\begin{aligned}
\|(B - BP_m)z\| &= \left\| \sum_{i=m+1}^{\infty} \lambda_i \tau_i y_i \right\| \\
&= \sup_{y^* \in B_{F^*}} \sum_{i=m+1}^{\infty} |\lambda_i \tau_i y^*(y_i)| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_\infty \sup_{y^* \in B_{F^*}} \sum_{i=m+1}^{\infty} |\tau_i y^*(y_i)| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_\infty \sup_{y^* \in B_{F^*}} \sum_{i=m+1}^{\infty} \beta_i \beta_i^{-1} |\tau_i y^*(y_i)| \\
&\leq \beta_{m+1} \|(\lambda_i)_{i=1}^{\infty}\|_\infty \sup_{y^* \in B_{F^*}} \sum_{i=m+1}^{\infty} |\tau_i \beta_i^{-1} y^*(y_i)| \\
&\leq \beta_{m+1} \|(\lambda_i)_{i=1}^{\infty}\|_\infty \|t\|_G.
\end{aligned}$$

Hence $\|A - P_m A\| \leq \beta_{m+1}(1+\varepsilon)^2 \|(\lambda_i)_{i=1}^{\infty}\|_\infty^{-1} \eta_\sigma(T)$ and $\|B - BP_m\| \leq \beta_{m+1} \|(\lambda_i)_{i=1}^{\infty}\|_\infty$.

So, $A \in \overline{\mathcal{L}_f(E,G)}$ and $B \in \overline{\mathcal{L}_f(G,F)}$ with

$$\|A\| \leq (1+\varepsilon)^2 \|(\lambda_i)_{i=1}^{\infty}\|_\infty^{-1} \eta_\sigma(T) \text{ and } \|B\| \leq \|(\lambda_i)_{i=1}^{\infty}\|_\infty.$$

It follows that $\|A\|\|B\| \leq (1+\varepsilon)^2 \eta_\sigma(T)$. Then, $\eta_\sigma(T) = \inf \|A\|\|B\|$ where the infimum is taken over all possible factorizations.

$\Leftarrow$) Assume that T have the precedent factorization. Consider $A \in \overline{\mathcal{L}_f(E,G)}, B \in \overline{\mathcal{L}_f(G,F)}$ and G is Banach space having a hyperorthogonal basis.

Let $(e_i)_{i=1}^{\infty}$ be a hyperorthogonal basis of G and $(v_i) \in G^*$ be a sequence of corresponding coordinate functionals, we have

$$A(x) = \sum_{i=1}^{\infty} v_i(A(x)) e_i = \sum_{i=1}^{\infty} A^* v_i(x) e_i.$$

By composing with B, we get

$$T(x) = B \circ A(x) = B(\sum_{i=1}^{\infty} A^* v_i(x) e_i) = \sum_{i=1}^{\infty} A^* v_i(x) B(e_i).$$

Therefore, considering $x_i^* = A^* v_i \in E^*$, as there are $(\lambda_i)_{i=1}^{\infty} \in \ell_\infty \setminus \{0\}$ and $(y_i)_{i=1}^{\infty} \in F$ such that $B(e_i) = \lambda_i y_i$. Then T has the representation

$$T = \sum_{i=1}^{\infty} \lambda_i x_i^* \underline{\otimes} y_i.$$

In order to show that T is σ-nuclear operator. We have

$$\begin{aligned}
\sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=1}^{m} |x_i^*(x)y^*(y_i)| &= \sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=1}^{m} \left| A^* v_i(x) y^*(\frac{B(e_i)}{\lambda_i}) \right| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=1}^{m} |v_i(Ax)y^*(B(e_i))| \\
&= \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \sup_{\substack{u\in Im(A)\\ y^*\in B_{F^*}}} \sum_{i=1}^{m} |v_i(u)\, y^*(B(e_i))| \,.
\end{aligned}$$

Let $\varepsilon_i = \pm 1$, we have to estimate the latter expression. Observe that

$$\begin{aligned}
\sup_{\substack{u\in Im(A)\\ y^*\in B_{F^*}}} \sum_{i=1}^{m} |v_i(u)\, y^*(B(e_i))| &= \sup_{\substack{u\in Im(A)\\ y^*\in B_{F^*}}} \left| \sum_{i=1}^{m} \varepsilon_i v_i(u)\, y^*(B(e_i)) \right| \\
&= \sup_{\substack{u\in Im(A)\\ y^*\in B_{F^*}}} \left| y^* \left(B \left(\sum_{i=1}^{m} \varepsilon_i v_i(u)\, e_i \right) \right) \right| \\
\text{(by Hahn-Banach)} \quad &= \sup_{u\in Im(A)} \left\| B \left(\sum_{i=1}^{m} \varepsilon_i v_i(u)\, e_i \right) \right\| \\
&\leq \|B\| \sup_{u\in Im(A)} \left\| \sum_{i=1}^{m} \varepsilon_i v_i(u)\, e_i \right\| .
\end{aligned}$$

Since $(e_i)_i$ is hyperorthogonal basis and $|\varepsilon_i| = 1$, for all $i \geq 1$, then

$$\sup_{u\in Im(A)} \left\| \sum_{i=1}^{m} \varepsilon_i v_i(u)\, e_i \right\| \leq \sup_{u\in Im(A)} \left\| \sum_{i=1}^{m} v_i(u)\, e_i \right\| .$$

Therefore

$$\begin{aligned}
\sup_{\substack{x\in B_E\\ y^*\in B_{F^*}}} \sum_{i=1}^{m} |x_i^*(x)y^*(y_i)| &\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \|B\| \sup_{u\in Im(A)} \left\| \sum_{i=1}^{m} v_i(u)\, e_i \right\| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \|B\| \sup_{u\in Im(A)} \|P_m(u)\| ,
\end{aligned}$$

where $(P_m)_m$ the sequence of canonical projections, $P_m(u) = \sum_{i=1}^{m} v_i(u)e_i$.

Thanks to [15, Corollary 26.3 (a)], there is a constant $C \geq 1$ such that $\|P_m(u)\| \leq C\|u\|$. It follows that

$$\begin{aligned}
\sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=1}^{m} |x_i^*(x)y^*(y_i)| &\leq C\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\|B\| \sup_{u\in Im(A)} \|u\| \\
&= C\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\|B\| \sup_{x\in B_E} \|A(x)\| \\
&= C\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\|B\|\|A\| < \infty.
\end{aligned}$$

Now we will prove that

$$\lim_{m\longrightarrow\infty} \sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=m}^{\infty} |x_i^*(x)y^*(y_i)| = 0.$$

We have

$$\begin{aligned}
\sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=m+1}^{\infty} |x_i^*(x)y^*(y_i)| &= \sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=m+1}^{\infty} \left| A^*v_i(x)y^*\left(\frac{B(e_i)}{\lambda_i}\right)\right| \\
&\leq \|(\lambda_i)_{i=m+1}^{\infty}\|_{\infty}^{-1} \sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{i=m+1}^{\infty} |v_i(Ax)y^*(B(e_i))| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \sup_{\substack{u\in Im(A)\\ y^*\in B_{F^*}}} \sum_{i=m+1}^{\infty} |v_i(u)\,y^*(B(e_i))| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \sup_{\substack{u\in \overline{Im(A)}\\ y^*\in B_{F^*}}} \sum_{i=m+1}^{\infty} |v_i(u)\,y^*(B(e_i))|
\end{aligned}$$

Let $\varepsilon_i = \pm 1$, we have to estimate the latter expression. Observe that

$$\begin{aligned}
\sup_{\substack{u\in \overline{Im(A)}\\ y^*\in B_{F^*}}} \sum_{i=m+1}^{\infty} |v_i(u)\,y^*(B(e_i))| &= \sup_{\substack{u\in \overline{Im(A)}\\ y^*\in B_{F^*}}} \left|\sum_{i=m+1}^{\infty} \varepsilon_i v_i(u)\,y^*(B(e_i))\right| \\
&= \sup_{\substack{u\in \overline{Im(A)}\\ y^*\in B_{F^*}}} \left| y^*\left(B\left(\sum_{i=m+1}^{\infty} \varepsilon_i v_i(u)\,e_i\right)\right)\right| \\
&= \sup_{u\in \overline{Im(A)}} \left\| B\left(\sum_{i=m+1}^{\infty} \varepsilon_i v_i(u)\,e_i\right)\right\| \\
&\leq \|B\| \sup_{u\in \overline{Im(A)}} \left\|\sum_{i=m+1}^{\infty} \varepsilon_i v_i(u)\,e_i\right\|.
\end{aligned}$$

Since $(e_i)_i$ is hyperorthogonal basis and $|\varepsilon_i| = 1$, for all $i \geq 1$, then

$$\sup_{u\in \overline{Im(A)}} \left\|\sum_{i=m+1}^{\infty} \varepsilon_i v_i(u)\,e_i\right\| \leq \sup_{u\in \overline{Im(A)}} \left\|\sum_{i=m+1}^{\infty} v_i(u)\,e_i\right\|.$$

Therefore

$$\begin{aligned}\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{i=m+1}^{\infty}|x_i^*(x)y^*(y_i)| &\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\|B\|\sup_{u\in\overline{Im(A)}}\left\|\sum_{i=m+1}^{\infty}v_i(u)\,e_i\right\| \\ &\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\|B\|\sup_{u\in\overline{Im(A)}}\|u-P_m(u)\|,\end{aligned}$$

We know that every approximable operators are compact operators, it follows that $\overline{Im(A)}$ is compact in G. We have by [15, Corollary 26.3 (b)] the sequence (P_m) converges to the identity uniformly on $\overline{Im(A)}$, i.e.

$$\lim_{m\to\infty}\sup_{u\in\overline{Im(A)}}\|u-P_m(u)\|=0.$$

Consequently,

$$\lim_{m\longrightarrow\infty}\sup_{\substack{x\in B_E\\ y^*\in B_{F^*}}}\sum_{i=m}^{\infty}|x_i^*(x)y^*(y_i)|=0.$$

This proves that T is σ-nuclear operator. ■

2.3 The surjective and injective hull of σ-nuclear linear operators

Let $\mathcal{A}$ be an operator ideal. The surjective hull $[\mathcal{A},\|\cdot\|_{\mathcal{A}}]^{sur}$ of an operator ideal $[\mathcal{A},\|\cdot\|_{\mathcal{A}}]$ is defined as follows;

$$\mathcal{A}^{sur}(E,F):=\{T\in\mathcal{L}(E,F):Q_E\in\mathcal{A}(\ell_1(B_E),F)\},$$

where $Q_E:\ell_1(B_E)\to E$ is the natural surjection operator defined as $Q_X(\lambda_x)_{x\in B_E}=\sum_{x\in B_E}\lambda_x x$, and $\|T\|_{\mathcal{A}^{sur}}:=\|TQ_E\|_{\mathcal{A}}$ for $T\in\mathcal{A}^{sur}(E,F)$.

Lemma 2.3.1 *[17] Let $[\mathcal{A},\|\cdot\|_{\mathcal{A}}]$ be a Banach operator ideal and let E and F be a Banach spaces.A linear map $T\in\mathcal{A}^{sur}(E,F)$ if and only if there exists a Banach space Z and an $S\in(Z,F)$ such that $T(B_E)\subset S(B_Z)$.*

In this case,

$$\|T\|_{\mathcal{A}^{sur}}=\inf\|S\|_{\mathcal{A}}.$$

where the infimum is taken over all the above inclusions.

Theorem 2.3.1 *[13, Theorem 2] The surjective hull* $[\mathcal{N}_\sigma, \eta_\sigma(\cdot)]^{sur}$ *of the ideal of* σ*-nuclear operators can be identified with the ideal* $[\mathcal{K}, \|\cdot\|]$ *of compact operators.*

Proof. Since $[\mathcal{N}_\sigma, \eta_\sigma(\cdot)] \subset [\overline{\mathcal{L}_f}, \|\cdot\|]$ and $[\overline{\mathcal{L}_f}, \|\cdot\|]^{sur} = [\mathcal{K}, \|\cdot\|]$(see [9, Ex. 9.12] , then

$$[\mathcal{N}_\sigma, \eta_\sigma(\cdot)]^{sur} \subset [\mathcal{K}, \|\cdot\|].$$

To show the opposite inclusion, let E and F be Banach spaces. Let $T \in \mathcal{K}(E, F)$ and let $\epsilon > 0$ Then by Lemma 1.3.1, there exists a null sequence$(x_j)_j$ in E with

$$\sup_{j\in\mathbb{N}} \|x_j\| \leq (1+\epsilon)\|T\|$$

such that

$$T(B_F) \subset \left\{ \sum_{j=1}^{\infty} a_j x_j : (a_j)_j \in B_{\ell_1} \right\}.$$

Let us consider the map $S : \ell_1 \to E$, $A = \sum_{j=1}^{\infty} e_j \underline{\otimes} x_j$, where each e_j is the standard unit vector in c_0. By Example 2.1.1, $S \in \mathcal{N}_\sigma(\ell_1, E)$ and

$$\eta_\sigma(S) \leq \sup_{j\in\mathbb{N}} \|x_j\| \leq (1+\epsilon)\|T\|.$$

Since $T(B_F) \subset A(B_{\ell_1})$, by Lemma 2.3.1, $T \in \mathcal{N}_\sigma^{sur}(E, F)$ and $\eta_\sigma(T) \leq \eta_\sigma(S) \leq (1+\epsilon)\|T\|$.

■

Following [17], for a Banach operator ideal $\mathcal{A}$, the injective hull $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{inj}$ of an operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is defined as follows;

$$\mathcal{A}^{inj}(E, F) := \{T \in \mathcal{L}(E, F) : i_F T \in \mathcal{A}(E, \ell_\infty(B_{F^*})\},$$

where $i_F : F \to \ell_\infty(B_{F^*})$ is the natural isometry defined as $i_F(z) = (z^*(z))_{z^* \in B_{F^*}}$, and $\|T\|_{\mathcal{A}^{inj}} := \|i_F T\|_{\mathcal{A}}$ for all $T \in \mathcal{A}^{inj}(E, F)$. If $\mathcal{A}$ is an operator ideal, then $\mathcal{A}^{inj}$ is also an operator ideal. The dual operator ideal $\mathcal{A}^{dual}$ consists of all operators T such that $T^* \in \mathcal{A}$. An operator ideal $\mathcal{A}$ is called symmetric if $\mathcal{A} \subset \mathcal{A}^{dual}$. in case $\mathcal{A} = \mathcal{A}^{dual}$ the operator ideal is said to be completely symmetric [17, 4.4.5].

Lemma 2.3.2 *[13, Lemma 4]If* $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ *is a symmetric Banach operator ideal, then* $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{inj} = [[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{sur}]^{dual}$

Using the principle of local reflexivity, Hutton [12](see also, [17, Proposition 4.4.7]), proved that a continuous linear operator $S \in \mathcal{L}(E,F)$ between Banach spaces can be approximated, in the usual operator norm, by finite rank operators if and only if its adjoint $S^* \in \mathcal{L}(F^*, E^*)$ can be approximated, in the usual operator norm, by finite rank operators. (i.e., $\overline{\mathcal{L}_f(E,F)}$ is completely symetric).

Theorem 2.3.2 $[\mathcal{N}_\sigma(E,F), \eta_\sigma(\cdot)]$ *is symmetric.*

Proof. By Theorem 2.2.1 $T \in \mathcal{N}_\sigma(E,F)$ if and only if, there exist two operators $A \in \overline{\mathcal{L}_f(E,G)}, B \in \overline{\mathcal{L}_f(G,F)}$ and G is a Banach space having a hyperorthogonal basis $(e_i)_{i=1}^\infty$ such that $T = BA$. Hutton's theorem assures that B (respectively A) can be approximated by finite rank operators if and only if B^*(respectively A^*) can be approximated by finite rank operators. This proves that $T^* = A^*B^* \in \mathcal{N}_\sigma(F^*, E^*)$. ■

The injective hull of the ideal of nuclear operators is identified in [17, Proposition 8.5.5] (see also [13, Theorem 2]). The following theorem is a consequence of the fact that the ideal $[\mathcal{N}_\sigma(E,F), \eta_\sigma(\cdot)]$ is symmetric.

Theorem 2.3.3 $[\mathcal{N}_\sigma, \eta_\sigma(\cdot)]^{inj} = [\mathcal{K}, \|\cdot\|]$.

Proof. Since $[\mathcal{N}_\sigma, \eta_\sigma(\cdot)]$ is symmetric, by Theorem 3.3.1 and Lema 2.3.2,

$$[\mathcal{N}_\sigma, \eta_\sigma(\cdot)]^{inj} = ([\mathcal{N}_\sigma, \eta_\sigma(\cdot)]^{sur})^{dual} = [\mathcal{K}, \|\cdot\|].$$

■

Chapter 3

The ideal of $\sigma(p)$-nuclear linear operators

In [14] and [8] the Banach operator ideal of $\sigma(p)$-nuclear linear operators for $p \geq 1$ is introduced and studied. In [8] the concept of σ-nuclear linear operators was generalized to $\sigma(p)$-nuclear linear operators for $p \geq 1$. In this chapter we prove that the class of $\sigma(p)$-nuclear linear operators is a Banach ideal, in particular the class of σ-nuclear linear operators is a Banach operator ideal. Also prove, under usual conditions on the underlying spaces, a simpler formula for the $\sigma(p)$-nuclear norm of a finite type operator. At the end of this section, we present an (incomplete) contribution of some properties of the surjective hull of the Banach operator ideal of $\sigma(p)$-nuclear operators.

3.1 $\sigma(p)$-nuclear linear operators

Now we present the classes of $\sigma(p)$-nuclear linear operators for $p \geq 1$ introduced by Botelho and Mujica in [8] and [14].

Definition 3.1.1 *For $1 \leq p < \infty$, we say that a linear operator $T : E \to F$ is $\sigma(p)$-nuclear if there are sequences $(\lambda_j)_{j=1}^{\infty} \in \ell_{p^*}, (x_j^*)_{j=1}^{\infty}$ in E^* and $(y_j)_{j=1}^{\infty}$ in F, such that*

$$T = \sum_{j=1}^{\infty} \lambda_j x_j^* \underline{\otimes} y_j,$$

$$\sup_{x \in B_E,, y^* \in B_{F^*}} (\sum_{j=1}^{\infty} \left|x_j^*(x) y^*(y_j)\right|^p)^{\frac{1}{p}} < \infty$$

and

$$\lim_{m\to\infty}\sup_{x\in B_E,,y^*\in B_{F^*}}(\sum_{j=m}^{\infty}\left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}}=0.$$

In this case we say that $T=\sum_{j=1}^{\infty}\lambda_j x_j^*\underline{\otimes}y_j$, is $\sigma(p)$-nuclear representation of T and define

$$\|T\|_{\sigma(p)}=\inf\{\left\|(\lambda_j)_{j=1}^{\infty}\right\|_{p^*}\cdot\sup_{x\in B_E,,y^*\in B_{F^*}}(\sum_{j=m}^{\infty}\left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}}\},$$

where the infimum runs over all $\sigma(p)$-nuclear representation of T. The set of all such linear operators is denoted by $\mathcal{N}_{\sigma(p)}(E;F)$.

Proposition 3.1.1 *Let $T\in\mathcal{L}(E,F)$ and $1\leq p<\infty$. Then*

$$\mathcal{N}_{\sigma(p)}(E,F)\subset\mathcal{N}_{\sigma}(E,F).$$

Proof. Given $T\in\mathcal{N}_{\sigma(p)}(E,F)$. We have $T=\sum_{j=1}^{\infty}\lambda_j x_j^*\underline{\otimes}y_j=\sum_{j=1}^{\infty}x_j^*\underline{\otimes}b_j$, where $b_j=\lambda_j y_j\in F$, note that T have a σ-nuclear representation. Thanks to Hölder's inequality, we have

$$\begin{aligned}\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{j=1}^{\infty}\left|x_j^*(x)y^*(b_j)\right| &= \sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{j=1}^{\infty}|\lambda_j|\left|x_j^*(x)y^*(y_j)\right|\\ &\leq \|(\lambda_j)_{j=1}^{\infty}\|_{p^*}\sup_{x\in B_E,,y^*\in B_{F^*}}\left(\sum_{i=1}^{\infty}\left|x_j^*(x)y^*(y_j)\right|^p\right)^{\frac{1}{p}}<\infty\end{aligned}$$

and

$$\lim_{m\longrightarrow\infty}\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{i=m}^{\infty}\left|x_j^*(x)y^*(b_j)\right|\leq\|(\lambda_j)_{j=1}^{\infty}\|_{p^*}\lim_{m\longrightarrow\infty}\sup_{x\in B_E,,y^*\in B_{F^*}}\left(\sum_{j=m}^{\infty}\left|x_j^*(x)y^*(y_j)\right|^p\right)^{\frac{1}{p}}=0.$$

Hence, T is σ-nuclear, with

$$\begin{aligned}\eta_{\sigma}^{L}(T) &= \inf\left\{\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{j=1}^{\infty}\left|x_j^*(x)y^*(b_j)\right|\right\}\\ &\leq \inf\left\{\|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\sup_{x\in B_E,,y^*\in B_{F^*}}\left(\sum_{i=1}^{\infty}\left|x_j^*(x)y^*(y_j)\right|^p\right)^{\frac{1}{p}}\right\}=\eta_{\sigma(p)}^{L}(T).\end{aligned}$$

This completes the proof. ■

Example 3.1.1 *Let $1 \leq p, q, r < \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, $(x_i)_{i=1}^{\infty} \in \ell_q(E)$ and $(\lambda_j)_{j=1}^{\infty} \in \ell_{p^*}$, we define the map*

$$R : \ell_r \longrightarrow E : \alpha \longmapsto S(\alpha) = \sum_{i=1}^{\infty} \lambda_j e_i^*(\alpha) x_i,$$

where $(e_i^*)_{i=1}^{\infty}$ is the standard unit vector in ℓ_{p^*} (i.e. c_0). Then R is $\sigma(p)$-nuclear operator. Indeed, obvious that S is linear continues operator and have a $\sigma(p)$-nuclear representation $R = \sum_{i=1}^{\infty} \lambda_j e_i^* \underline{\otimes} x_i$. We have

$$\begin{aligned} \sup_{\alpha \in B_{\ell_r}, x^* \in B_{E^*}} \Big(\sum_{j=1}^{\infty} \left|e_j^*(\alpha) x^*(x_j)\right|^p\Big)^{\frac{1}{p}} &\leq \sup_{\alpha \in B_{\ell_r}, x^* \in B_{E^*}} \Big(\sum_{j=1}^{\infty} |\alpha_j x^*(x_j)|^p\Big)^{\frac{1}{p}} \\ &\leq \sup_{\alpha \in B_{\ell_r}} \Big(\sum_{j=1}^{\infty} (|\alpha_j| \, \|x_j\|)^p\Big)^{\frac{1}{p}} \\ &\leq \sup_{\alpha \in B_{\ell_r}} \Big(\sum_{j=1}^{\infty} |\alpha_j|^r\Big)^{\frac{1}{r}} \Big(\sum_{j=1}^{\infty} \|x_j\|^q\Big)^{\frac{1}{q}} \\ &\leq \|(x_j)_j\|_q < \infty \end{aligned}$$

and

$$\lim_{m \to \infty} \sup_{\alpha \in B_{\ell_r}, x^* \in B_{E^*}} \Big(\sum_{j=m}^{\infty} \left|e_j^*(\alpha) x^*(x_j)\right|^p\Big)^{\frac{1}{p}} \leq \lim_{m \to \infty} \Big(\sum_{j=m}^{\infty} \|x_j\|^q\Big)^{\frac{1}{q}} = 0$$

Then, R is $\sigma(p)$-nuclear operator and

$$\begin{aligned} \|R\|_{\sigma(p)} &\leq \|(\lambda_j)_{j=1}^{\infty}\|_{p^*} \sup_{\alpha \in B_{\ell_r}, x^* \in B_{E^*}} \Big(\sum_{j=1}^{\infty} \left|e_j^*(\alpha) x^*(x_j)\right|^p\Big)^{\frac{1}{p}} \\ &\leq \|(\lambda_j)_{j=1}^{\infty}\|_{p^*} \|(x_j)_j\|_q. \end{aligned}$$

Proposition 3.1.2 *For $1 \leq p < \infty$, $(\mathcal{N}_{\sigma(p)}, \| \,.\, \|_{\sigma(p)})$ is a Banach ideal of linear operator. In particular the class of σ-nuclear operator linear is a Banach operator ideal.*

Proof. Condition (i). It follows from $id_{\mathbb{K}} = 1 \cdot id_{\mathbb{K}} \underline{\otimes} 1$ ($id_{\mathbb{K}}(\alpha) = \sum\limits_{j=1}^{\infty} \delta_{n,j} \cdot id_{\mathbb{K}}(\alpha) \cdot 1 = \alpha$) and $\|id_{\mathbb{K}}\|_{\sigma(p)} = \sup\limits_{\|\alpha\| \leq 1, \|\gamma\| \leq 1} |1 \cdot id_{\mathbb{K}}(\alpha) \gamma(1)| \leq 1$. Moreover, let $id_{\mathbb{K}} = \sum\limits_{j=1}^{\infty} \lambda_{,j} \beta_j \underline{\otimes} \eta_j \in \mathcal{L}_{\sigma(p)}$ be any $\sigma(p)$-nuclear representation. Then $1 = id_{\mathbb{K}}(1) = \sum\limits_{j=1}^{\infty} \lambda_{,j} \beta_j \eta_j$. Consenquently,

$$1 = \|id_{\mathbb{K}}\| \leq \sup_{\|\gamma\| \leq 1} \sum_{j=1}^{\infty} \left|\lambda_{,j} \beta_j \gamma(\eta_j)\right| \leq \|(\lambda_j)_{j=1}^{\infty}\|_{p^*} \sup_{\|\gamma\| \leq 1} \Big(\sum_{j=1}^{\infty} \left|\beta_j \gamma(\eta_j)\right|^p\Big)^{\frac{1}{p}}.$$

and therefore $1 \leq \|id_{\mathbb{K}}\|_{\sigma(p)}$.

Let us prove condition (ii) of Theorem 1.3.1. First, for every $T \in \mathcal{N}_{\sigma(p)}(E;F)$ and ε given we remark that

$$\begin{aligned} \|T\| &= \sup_{x\in B_E} \|T(x)\| = \sup_{x\in B_E} \left\| \sum_{j=1}^{\infty} |\lambda_j| \left|x_j^*(x) y_j\right| \right\|_F \\ &= \sup_{x\in B_E,,y^*\in B_{F^*}} \sum_{j=1}^{\infty} |\lambda_j| \left|x_j^*(x) y^*(y_j)\right| \\ &\leq \|(\lambda_j)_{j=1}^{\infty}\|_{p^*} \sup_{x\in B_E,,y^*\in B_{F^*}} \left(\sum_{i=1}^{\infty} \left|x_j^*(x) y^*(y_j)\right|^p \right)^{\frac{1}{p}} \\ &< (1+\varepsilon) \|T\|_{\sigma(p)} \end{aligned}$$

Let $A_1, A_2, ... \in \mathcal{N}_{\sigma(p)}(E;F)$ be such that $\sum_{j=1}^{\infty} \|A_k\|_{\sigma(p)} < \infty$. Since$\|\cdot\| \leq \|\cdot\|_{\sigma(p)}$the series $\sum_{k=1}^{\infty} A_k$ is absolutely convergent in the Banach space $\mathcal{L}(E;F)$,therefore $A := \sum_{k=1}^{\infty} A_k$ converges in $\mathcal{L}(E;F)$. Now we shall see A is $\sigma(p)$-nuclear .Given $\epsilon > 0$ for each k take $\sigma(p)$-nuclear representation $A_K = \sum_{j=1}^{\infty} \lambda_{kj} x_{kj}^* \underline{\otimes} y_{kj}$ such that

$$\left\|(\lambda_{kj})_{j=1}^{\infty}\right\|_{p^*} \leq ((1+\epsilon) \|A_k\|_{\sigma(p)})^{\frac{1}{p^*}},$$

and

$$\sup_{x\in B_E,,y^*\in B_{F^*}} (\sum_{j=1}^{\infty} \left|x_{kj}^*(x) y^*(y_{kj})\right|^p)^{\frac{1}{p}} \leq ((1+\epsilon) \|A_k\|_{\sigma(p)})^{\frac{1}{p}}.$$

Let us see that $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{kj} x_{kj}^* \underline{\otimes} y_{kj}$ $\sigma(p)$-nuclear representation of A.

we have,

$$\begin{aligned} \| \quad (\lambda_{kj})_{k,j=1}^{\infty} \|_{p^*}^{p^*} &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_{kj}|^{p^*} \\ &\leq \sum_{k=1}^{\infty} ([(1+\epsilon) \|A_k\|_{\sigma(p)}]^{\frac{1}{p^*}})^{p^*} \\ &= (1+\epsilon). \sum_{k=1}^{\infty} \|A_k\|_{\sigma(p)} < \infty, \end{aligned}$$

and

$$\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left|x_{kj}^*(x)y^*(y_{kj})\right|^p \le \sum_{k=1}^{\infty}\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{j=1}^{\infty}\left|x_{kj}^*(x)y^*(y_{kj})\right|^p \le (1+\epsilon)\sum_{k=1}^{\infty}\|A_k\|_{\sigma(p)}<\infty.$$

To check the condition concerning the tail of the series, let $\delta>0$ be given. Observing that, for each $m\in\mathbb{N}$, such that

$$\begin{aligned} A &= \sum_{k=1}^{\infty}A_k=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_{kj}x_{kj}^*\underline{\otimes}y_{kj} \\ &= \underbrace{\sum_{k=1}^{m-1}\sum_{j=1}^{m-1}\lambda_{kj}x_{kj}^*\underline{\otimes}y_{kj}+\sum_{k=1}^{m-1}\sum_{j=m}^{\infty}\lambda_{kj}x_{kj}^*\underline{\otimes}y_{kj}+\sum_{k=m}^{\infty}\sum_{j=1}^{\infty}\lambda_{kj}x_{kj}^*\underline{\otimes}y_{kj}}_{\text{tail}}, \end{aligned}$$

We have to show that there is $M\in\mathbb{N}$ such that

$$\sup_{x\in B_E,,y^*\in B_{F^*}}\{(\sum_{k=1}^{m-1}\sum_{j=m}^{\infty}\left|x_{kj}^*(x)y^*(y_{kj})\right|^p+\sum_{k=m}^{\infty}\sum_{j=1}^{\infty}\left|x_{kj}^*(x)y^*(y_{kj})\right|^p)^{\frac{1}{p}}\}<\delta,$$

for every $m\ge M$. By $\sum_{j=1}^{\infty}\|A_k\|_{\sigma(p)}<\infty$, there exists $k_\delta\in\mathbb{N}$ such that $\sum_{k=k_\delta}^{\infty}\|A_k\|_{\sigma(p)}<\frac{\delta^p}{2(1+\epsilon)}$. Hence

$$\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{k=k_\delta}^{\infty}\sum_{j=1}^{\infty}\left|x_{kj}^*(x)y^*(y_{kj})\right|^p \le \sum_{k=k_\delta}^{\infty}\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{j=1}^{\infty}\left|x_{kj}^*(x)y^*(y_{kj})\right|^p \le (1+\epsilon).\sum_{k=k_\delta}^{\infty}\|A_k\|_{\sigma(p)}\le\frac{\delta^p}{2}.$$

For $k=1,...,k_{\delta-1}$, since $\sum_{j=1}^{\infty}\lambda_{kj}x_{kj}^*\underline{\otimes}y_{kj}$ is a $\sigma(p)-$ nuclear representation of A_k, there is $j_k\in\mathbb{N}$, such that

$$\sup_{x\in B_E,,y^*\in B_{F^*}}\sum_{j=j_k}^{\infty}\left|x_{kj}^*(x)y^*(y_{kj})\right|^p\le\frac{\delta^p}{2^{k+1}}.$$

Choosing $M = \max\{K_\delta, J_1, ..., J_{k_\delta - 1}\}$ we have for $m \geq M$: This completes the proof that A is $\sigma(p)$-nuclear. From

$$\begin{aligned}
\| \quad & (\lambda_{kj})_{j,k=1}^{\infty} \|_{p^*} \cdot \sup_{x \in B_E,, y^* \in B_{F^*}} (\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| x_{kj}^*(x) y^*(y_{kj}) \right|^p)^{\frac{1}{p}} \\
\leq \quad & (1+\epsilon)^{\frac{1}{p^*}} . (\sum_{k=1}^{\infty} \|A_k\|_{\sigma(p)})^{\frac{1}{p^*}} . (1+\epsilon)^{\frac{1}{p}} . (\sum_{k=1}^{\infty} \|A_k\|_{\sigma(p)})^{\frac{1}{p}} \\
= \quad & (1+\epsilon) . \sum_{k=1}^{\infty} \|A_k\|_{\sigma(p)} ,
\end{aligned}$$

letting $\epsilon \to 0$, we conclude that $\|A\|_{\sigma(p)} \leq \sum_{k=1}^{\infty} \|A_k\|_{\sigma(p)}$.

(iii) Now consider the linear operators $R \in \mathcal{L}(F, G)$ and $S \in \mathcal{L}(H, E)$. Let $T \in \mathcal{L}_{\sigma(p)}(E; F)$ and $\varepsilon > 0$. We have $T = \sum_{j=1}^{\infty} \lambda_j x_j^* \underline{\otimes} y_j$ such that

$$\left\| (\lambda_j)_{j=1}^{\infty} \right\|_{p^*} \sup_{x \in B_E,, y^* \in B_{F^*}} (\sum_{j=1}^{\infty} \left| x_j^*(x) y^*(y_j) \right|^p)^{\frac{1}{p}} \leq (1+\varepsilon) \|T\|_{\sigma(p)}$$

and

$$\lim_{m \to \infty} \sup_{x \in B_E,, y^* \in B_{F^*}} (\sum_{j=m}^{\infty} \left| x_j^*(x) y^*(y_j) \right|^p)^{\frac{1}{p}} = 0.$$

For every $u \in H$,

$$\begin{aligned}
RTS(u) &= R(\sum_{j=1}^{\infty} \lambda_j x_j^*(S(u) y_j) = R(\sum_{j=1}^{\infty} \lambda_j S^*(x_j^*)(u) y_j) \\
&= \sum_{j=1}^{\infty} \lambda_j S^*(x_j^*)(u) R(y_j) = \sum_{j=1}^{\infty} \lambda_j h_j^*(u) g_j
\end{aligned}$$

and

$$\begin{aligned}
& \left\| (\lambda_j)_{j=1}^{\infty} \right\|_{p^*} \sup_{u \in B_H,, g^* \in B_{G^*}} (\sum_{j=1}^{\infty} \left| (S^*(x_j^*))(u) \cdot g^*(R(y_j)) \right|^p)^{\frac{1}{p}} \\
= \quad & \left\| (\lambda_j)_{j=1}^{\infty} \right\|_{p^*} \sup_{x \in B_E,, y^* \in B_{F^*}} (\sum_{j=1}^{\infty} \left| (x_j^*(S(u)) \cdot R^*(g^*)(y_j)) \right|^p)^{\frac{1}{p}} \\
= \quad & \|S\| \, \|R^*\| \left\| (\lambda_j)_{j=1}^{\infty} \right\|_{p^*} \sup_{x \in B_E,, y^* \in B_{F^*}} (\sum_{j=1}^{\infty} \left| (x_j^*(\frac{S(u)}{\|S\|}) \cdot \frac{R^*(g^*)}{\|R^*\|}(y_j)) \right|^p)^{\frac{1}{p}} \\
\leq \quad & \|S\| \, \|R\| \, (1+\varepsilon) \, \|T\|_{\sigma(p)} .
\end{aligned}$$

Moreover

$$\lim_{m\to\infty} \sup_{u\in B_H,,g^*\in B_{G^*}} (\sum_{j=m}^{\infty} \left|(S^*(x_j^*))(u)\cdot g^*(R(y_j))\right|^p)^{\frac{1}{p}} \leq$$

$$\|S\|\,\|R\| \lim_{m\to\infty} \sup_{x\in B_E,,y^*\in B_{F^*}} (\sum_{j=m}^{\infty} \left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}} = 0.$$

So $RTS \in \mathcal{L}_{\sigma(p)}(H,G)$ and $\|RTS\|_{\sigma(p)} \leq \|R\|\,\|T\|_{\sigma(p)}\,\|S\|$. ■

Remark 3.1.1 *By Proposition 3.1.2 and Proposition 2.1.1 the case $p = 1$ recovers the Banach ideal of σ-nuclear linear operators.*

3.2 Some properties

This section is mainly based on [8].

Once $(\mathcal{L}_{\sigma(p)}, \|\ .\ \|_{\sigma(p)})$ is a Banach ideal linear operators,we have $\mathcal{L}_f(E;F) \subseteq \mathcal{L}_{\sigma(p)}(E;F)$, and

$$\|\lambda x^* \underline{\otimes} y\|_{\sigma(p)} = |\lambda|\,\|x^*\|\,\|y\|,$$

for all $\lambda \in \mathbb{K}, x^* \in E^*, y \in F$.

Remark 3.2.1 *It follows from the definition that $\mathcal{L}_f(E;F)$ is $\|\ .\ \|_{\sigma(p)}$-dense in $\mathcal{L}_f(E;F)$. Therefore the operator $S = \sum_{j=1}^{\infty} \lambda_j x_j^* \underline{\otimes} y_j$ is approximable. since every $S \in \mathcal{L}_f(E;F)$ has a finite reprsentation of the form $S = \sum_{j=1}^{m} \lambda_j x_j^* \underline{\otimes} y_j$.*

For $S \in \mathcal{L}_f(E;F)$, define

$$\|S\|_{\sigma(p)f} = \inf\{\|\ (\lambda_j)_{j=1}^m\ \|_{p^*} \cdot \sup_{x\in B_E,,y^*\in B_{F^*}} (\sum_{j=1}^{m} \left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}}\}.$$

Where the infimum runs over all finite representations $S = \sum_{j=1}^{m} \lambda_j x_j^* \underline{\otimes} y_j$.

Remark 3.2.2 *It is easy to check that* $\|\cdot\|_{\sigma(p)f}$ *is a norm on* $\mathcal{L}_f(E;F)$ *such that* $\|\cdot\| \leq \|\cdot\|_{\sigma(p)f}$. *It follows that* $\|\cdot\|_{\sigma(p)f}$ *is a complete norm on* $\mathcal{L}_f(E;F)$. *It is clear that* $\|\cdot\|_{\sigma(p)} \leq \|\cdot\|_{\sigma(p)f}$

For further use, we shall establish conditions under which the equality holds

Lemma 3.2.1 *If the norms* $\|\cdot\|_{\sigma(p)}$ *and* $\|\cdot\|_{\sigma(p)f}$ *are equivalent on* $\mathcal{L}_f(E;F)$ *then they coincide on this space.*

Proof. By assumption there is a constant $C > 0$ such that $\|\cdot\|_{\sigma(p)f} \leq C\|\cdot\|_{\sigma(p)}$on $\mathcal{L}_f(E;F)$. Given $S \in \mathcal{L}_f(E;F)$ and $\varepsilon > 0$, take an infinite $\sigma(p)$-nuclear representations $S = \sum_{j=1}^{\infty} \lambda_j x_j^* \underline{\otimes} y_j$ such that

$$\left\|(\lambda_j)_{j=1}^{\infty}\right\|_{p^*} \cdot \sup_{x\in B_E,,y^*\in B_{F^*}} (\sum_{j=1}^{\infty} \left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}} \leq (1+\frac{\varepsilon}{2})\|S\|_{\sigma(p)}$$

In particular, for each $m \in \mathbb{N}$,

$$\| \sum_{j=1}^{m-1} \lambda_j x_j^* \underline{\otimes} y_j \|_{\sigma(p)f} \leq \| (\lambda_j)_{j=1}^{m-1} \|_{p^*} \cdot \sup_{x\in B_E,,y^*\in B_{F^*}} (\sum_{j=1}^{m} \left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}} \leq (1+\frac{\varepsilon}{2})\|S\|_{\sigma(p)}.$$

Since,

$$\lim_{m\to\infty} \sup_{x\in B_E,,y^*\in B_{F^*}} (\sum_{j=m}^{\infty} \left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}} = 0,$$

for a sufficiently large $m \in \mathbb{N}$ we get,

$$\begin{aligned}
\left\|\sum_{j=m}^{\infty} \lambda_j x_j^* \underline{\otimes} y_j\right\|_{\sigma(p)} &\leq \left\|(\lambda_j)_{j=m}^{\infty}\right\|_{p^*} \cdot \sup_{x\in B_E,,y^*\in B_{F^*}} (\sum_{j=m}^{\infty} \left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}} \\
&\leq \left\|(\lambda_j)_{j=1}^{\infty}\right\|_{p^*} \cdot \sup_{x\in B_E,,y^*\in B_{F^*}} (\sum_{j=1}^{\infty} \left|x_j^*(x)y^*(y_j)\right|^p)^{\frac{1}{p}} \\
&\leq \frac{\varepsilon}{2C}\|S\|_{\sigma(p)}
\end{aligned}$$

It follows that,

$$\begin{aligned}
\|S\|_{\sigma(p)f} &= \left\| \sum_{j=1}^{m-1} \lambda_j x_j^* \underline{\otimes} y_j + \sum_{j=m}^{\infty} \lambda_j x_j^* \underline{\otimes} y_j \right\|_{\sigma(p)f} \\
&\leq \left\| \sum_{j=1}^{m-1} \lambda_j x_j^* \underline{\otimes} y_j \right\|_{\sigma(p)f} + \left\| \sum_{j=m}^{\infty} \lambda_j x_j^* \underline{\otimes} y_j \right\|_{\sigma(p)f} \\
&\leq (1+\frac{\varepsilon}{2}) \|S\|_{\sigma(p)} + C \left\| \sum_{j=m}^{\infty} \lambda_j x_j^* \underline{\otimes} y_j \right\|_{\sigma(p)} \\
&\leq (1+\frac{\varepsilon}{2}) \|S\|_{\sigma(p)} + \frac{\varepsilon}{2} \|S\|_{\sigma(p)} \\
&= (1+\varepsilon) \|S\|_{\sigma(p)} .
\end{aligned}$$

And as this holds for every $\varepsilon > 0$, the result follows. ■

Corollary 3.2.1 *Suppose that E is a finite-dimensional normed space. Then $\|S\|_{\sigma(p)f} = \|S\|_{\sigma(p)}$ for every $S \in \mathcal{L}(E;F)$.*

Proof. In this case $\mathcal{L}(E;F) = \mathcal{L}_f(E;F)$ and this is a complete space for both norms $\|\cdot\|_{\sigma(p)f}$ and $\|\cdot\|_{\sigma(p)}$. By the isomorphisme theorem these norms are equivalent. The result follows from Lemma 3.2.1. ■

Lemma 3.2.2 *If $A \in \mathcal{L}_{\sigma(p)}(E;F)$ and $T \in \mathcal{L}_f(D;E)$ then*

$$\|A \circ T\|_{\sigma(p)f} \leq \|A\|_{\sigma(p)} \|T\| .$$

Proof. Letting $J : T(D) \to E$ be the formal inclusions and $\widetilde{T} : D \to T(D)$ be defined by $\widetilde{T}(u) = T(u)$, we can write $T = J \circ \widetilde{T}$. Since each $T(D)$ is finite dimensional,

we have

$$\mathcal{L}_f(T(D);F) = \mathcal{L}_f(T(D);F) = \mathcal{L}_{\sigma(p)}(T(D);F).$$

So,

$\mathcal{L}_f(T(D);F)$ is complete with both norms $\| \;.\; \|_{\sigma(p)}$and $\| \;.\; \|_{\sigma(p)f}$. By the inequality $\| \;.\; \|_{\sigma(p)} \leq \| \;.\; \|_{\sigma(p)f}$and the open mapping theorem we conclude that these norms are equivalent on $\mathcal{L}_f(T(D);F)$. By Lemma 3.2.1 we get

$$\begin{aligned} \| \quad & A \circ J \|_{\sigma(p)f} = \| A \circ J \|_{\sigma(p)} \\ \leq \quad & \| A \|_{\sigma(p)} \| J \| \\ = \quad & \| A \|_{\sigma(p)} \end{aligned}$$

from which it follows that

$$\begin{aligned} \| \quad & A \circ T \|_{\sigma(p)f} = \| A \circ J \circ \widetilde{T} \|_{\sigma(p)f} \\ \leq \quad & \| A \circ J \|_{\sigma(p)f} \, . \, \| \widetilde{T} \| \\ = \quad & \| A \|_{\sigma(p)} \| T \| \, . \end{aligned}$$

This completes the proof. ■

Now, we recall Banach spaces, which have the approximation property.
Let E be a Banach space and let $1 \leq \gamma < \infty$. We say that E has the γ-*bounded approximation property* if, for every compact subset $K \subset E$ and every $\varepsilon > 0$, there is a finite rank operator $T \in \mathcal{L}_f(E,E)$ such that

$$\|x - Tx\| \leq \varepsilon,$$

for all $x \in K$ and $\|T\| \leq \gamma$.

A Banach space is said to have the *bounded approximation property* if it has γ-*bounded approximation property* for some γ. A Banach space is said to have the *metric approximation property* if it has 1-*bounded approximation property.*

Proposition 3.2.1 *If E^* have the bounded approximation property, then $\|\cdot\|_{\sigma(p)f} = \|\cdot\|_{\sigma(p)}$ on $\mathcal{L}_f(E,F)$ regardless of the Banach space F.*

Proof. Let $\gamma \geq 1$ be such that E^* has the γ-bounded approximation property. Given $A \in \mathcal{L}_f(E,F)$,Given $\epsilon > 0$,by [17, Lemma 10.2.6] there exists $T \in \mathcal{L}_f(E,F)$, such that

$$\|T\| \leq (1+\epsilon)\gamma$$

and

$$A \circ T = A$$

By 3.2.2 we have

$$\begin{aligned} \|A \circ T\|_{\sigma(p)f} &= \|A \circ T\|_{\sigma(p)f} \\ &\leq \|A\|_{\sigma(p)} \| T \| \\ &\leq (1+\epsilon)\gamma \|A\|_{\sigma(p)f} \end{aligned}$$

Letting $\epsilon \to 0$ we get

$$\|A\|_{\sigma(p)f} \leq \gamma \|A\|_{\sigma(p)}$$

The result follows from by 3.2.1. ■

3.3 Factorization theorem, surjective hull of $\sigma(p)$-nuclear linear operators

We give the factorization theorem, for its proof we use the same technique in Theorem 2.2.1 (see also, [14, 1.6.2 Teorema] for $n = 1$).

Theorem 3.3.1 *Let $1 < p < \infty$ and $T \in \mathcal{L}(E, F)$. An operator T is $\sigma(p)$-nuclear if and only if, the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ A \downarrow & \nearrow B & \\ U & & \end{array}$$

where, $A \in \overline{\mathcal{L}_f(E, U)}, B$ diagonal and U is a Banach space having a hyperorthogonal basis $(e_i)_{i=1}^{\infty}$, with $B(e_i) = \lambda_i y_i$, $(\lambda_i)_{i=1}^{\infty} \in \ell_{p^*}$ and $y_i \in F$ such that $(\tau_i y_i)_{i=1}^{\infty} \in \ell_p^u(F)$ for $\tau = \sum_{i=1}^{\infty} \tau_i e_i \in U$. And we have $\eta_{\sigma(p)}^L(T) = \inf \|B\| . \|B\|$ where the infimum is taken over all possible factorizations.

Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. The (p, r)-convex hull of a sequence $(x_n)_n \in \ell_p(E)$ is defined as

$$(p, r)\text{-conv}\{(x_n)_n\} = \left\{ \sum_{n=1}^{+\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_r} \right\}.$$

A subset K of a Banach space E is relatively (p,r)-compact (see [6]) if and only if for every $\epsilon > 0$,there exists a sequence $(x_j)_j \in \ell_p(E)$ such that

$$K \subset \left\{ \sum_{n=1}^{+\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_r} \right\}.$$

for some $(x_n)_n \in \ell_p(E)((x_n)_n \in c_0(E)$ when $p = \infty)$.

Definition 3.3.1 *[6]Let $1 \leq p \leq \infty$, $1 \leq r \leq p^*$. A linear operator $T : X \to Y$ is (p,r)-compact if $T(B_E)$ is a relatively (p,r)-compact subset of Y.*

Denote the class of all (p,r)-compact operators acting between arbitrary Banach spaces by $\mathcal{K}_{(p,r)}$.

Remark 3.3.1 *It is clear that $\mathcal{K}_{(\infty,1)} = \mathcal{K}$, $\mathcal{K}_{(p,p^*)} = \mathcal{K}_p$.*

Proposition 3.3.1 *Let $1 \leq p,q,r < \infty$ such that $\frac{1}{q^*} = \frac{1}{p} + \frac{1}{r}$, $r \leq p^*$ and $p \leq q$. Then $\left[\mathcal{K}_{(p,r)}(E,F), \|\cdot\|_{\mathcal{K}_{(p,r)}}\right] \subset \left[\mathcal{N}_{\sigma(q^*)}(E,F), \|\cdot\|_{\sigma(q^*)}\right]^{sur}$.*

Proof. Let $T \in \mathcal{K}_{(p,r)}(E,F)$ and let $\epsilon > 0$. Then by Lemma 1.3.1, there exists a null sequence $(y_j)_j$ in $\ell_p(F)$ with

$$\|(y_j)_j\|_p \leq (1+\epsilon)\|T\|_{\mathcal{K}_{(p,r)}}$$

such that

$$T(B_E) \subset \left\{ \sum_{j=1}^{\infty} a_j y_j : (a_j)_j \in B_{\ell_r} \right\}.$$

Let us consider the map $R : \ell_r \longrightarrow F : \alpha \longmapsto R(\alpha) = \sum_{i=1}^{\infty} e_i^*(\alpha) y_i = \sum_{i=1}^{\infty} \|(y_j)_j\|_p e_i^*(\alpha) \frac{y_i}{\|(y_j)_j\|_p} = \sum_{i=1}^{\infty} \lambda_j e_i^*(\alpha) \frac{y_i}{\|(y_j)_j\|_p}$, where each e_j is the standard unit vector in c_0. As $\ell_p \subset \ell_q$

Then R has a $\sigma(q^*)$-reprsentation,

$$\begin{aligned} \sup_{\alpha \in B_{\ell_r}, y^* \in B_{F^*}} \left(\sum_{j=1}^{\infty} \left|e_j^*(\alpha) y^*(y_j)\right|^{q^*}\right)^{\frac{1}{q^*}} &\leq \sup_{\alpha \in B_{\ell_r}} \left(\sum_{j=1}^{\infty} |\alpha_j|^r\right)^{\frac{1}{r}} \left(\sum_{j=1}^{\infty} \|y_j\|^p\right)^{\frac{1}{p}} \\ &= \|(y_j)_j\|_p \end{aligned}$$

and

$$\|R\|_{\sigma(q^*)} \leq \|(y_j)_j\|_p \leq (1+\epsilon)\|T\|_{\mathcal{K}_{(p,r)}}$$

Since $T(B_E) \subset R(B_{\ell_r}) = \left\{ \sum_{j=1}^{\infty} a_j y_j : (a_j)_j \in B_{\ell_r} \right\}$, by Lemma 2.3.1, $T \in \mathcal{N}_{\sigma(q^*)}^{sur}(E,F)$ and $\|T\|_{\mathcal{N}_{\sigma(q^*)}^{sur}} \leq \|R\|_{\mathcal{N}\sigma(q^*)} \leq (1+\epsilon)\|T\|$. ■

Conclusion

The concept of σ-nuclear linear operators was generalized to $\sigma(p)$-nuclear linear operators for $p \geq 1$. The class of $\sigma(p)$-nuclear linear operators is a Banach ideal, in particular the class of σ-nuclear linear operators is a Banach operator ideal. Also, under usual conditions on the underlying spaces, a simpler formula for the $\sigma(p)$-nuclear norm of a finite type operator. Also, we obtain a factorization of operators belonging to $\overline{\mathcal{L}_f}$. The surjective hull and the injective hull of $\sigma(1)$-nuclear linear operators coincide with the ideal of compact operators. But, in the case $p > 1$ until now, we don't know the answer, we have the following " $\mathcal{K}_{(p,r)}$ the ideal of (p,r)-compact operators is included in $\mathcal{N}^{sur}_{\sigma(q^*)}$ ".

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الملخص .

الهدف من هذا العمل هو دراسة تفصيلية لمثاليات باناخ للمؤثرات الخطيه المستمره σ(p) نيكليار المعرفةبين فضاءات باناخ. نقدم مناقشة لبعض خصائص هذا المثالي وتفكيك هذه المثل بفئات المؤثرين المقربة .

الكلمات المفتاحية: مثالي المؤثرات الخطية، مؤثر متراص، متتاليات بناخ p جمعية غير المشروطة، مؤثرات مقربة، مؤثرات σ(p) نيكليار.

Abstract.

The aim of this work is a detail study of the Banach ideal of sigma(p)-nuclear linear operators, between Banach spaces. We present a discussion of some properties and factorized this ideal by approximable operators.

Keywords: Linear operator ideals, compact operator, unconditionally p-summable sequences, approximable operators, sigma nuclear.

Résumé.

Le but de ce travail est d'étudier en détail les idéaux d'opérateurs sigma(p) nucléaires entre espaces de Banach. Nous présentons une discussion de quelques propriétés, et le théorème de factorisation de cet idéal par des opérateurs approximables.

Mots clés: L'idéal linéaire, Opérateur compact, Suite inconditionnelle p-sommable, Opérateur approximable, sigma nucléaire