PEOPLE's DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

جامعة محمد بوضياف - المسيلة
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Mohamed Boudiaf university of Msila
Faculty of Mathematics and computer sciences
Department of Mathematics

Master memory

Field : Mathematics and computer sciences
Branch : Mathematics
Option : Mathematical and numerical analysis

Theme

ABEL'S INTEGRAL EQUATIONS BY LAPLACE TRANSFORM

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University year 2020/2021

Thanks

First of all, I thank Allah, the Almighty, for giving me strength and patience.

I would like to particularly thank Dr.lakehali blkacem for his scientific support, as well as for his competence and good orientations, his valuable advice, but also his encouragement. I express my gratitude to him.

I would like to thank the members of the jury: Pro.selt Omar, and Prof. Bachir GAGUI,for the honor they do me by agreeing to chair and review this work.

Finally, I thank my family and all the people who participated directly or indirectly in carrying out this work.

DEDICATION

Thank you Allah for giving me the ability to overcome difficulties in my career study.

I dedicate this work to the symbol of tenderness, which sacrificed itself for my happiness, for my mother, she always wanted to achieve and succeed.

To my father, who was the cause of my courage and who was my shadow during all the years of my studies.

To my sisters, To my brothers.

To my friends: Hind, Hafsa, Marieme.

To all those I love

I dedicate this work.

Contents

Notations

Γ	is the gamma function
$*$	convolution
$\langle\rangle$	inner-product
$C(K)$	the space of continuous function
$\mathcal{L}$	Laplace transform
$\mathcal{L}^{-1}$	Inverse Laplace Transform
$\perp$	orthogonal
λ	scalar
$d(x,y)$	metric spaces with distance
$\int$	integrale
$A^{\perp}$	The orthogonal space
$\|\|$	normed

Introduction

In mathematics, an integral equation is an equation in which the unknown; generally a function of one or more variables, appears under the integral sign. this general definition takes into account many different specific forms and in practice several distinct types arise. for this reason, and in order to cover the main axes of our theme without getting involved in particularly inadequate situations, we are going to focus much more on linear integral equations of the form.

$$\int_a^b K(x,t)\varphi(t)dt = f(x) \quad a \leq x \leq b$$

and

$$\varphi(x) - \int_a^b K(x,t)\varphi(t)dt = f(x), \quad a \leq x \leq b$$

which are typical examples. in these equations the function φ is the unknown, the function $K(x,t)$ which is called kernel with the free term f are given. in another simple form in terms of operators, the preceding equations are written successively

$$A\varphi = f \qquad \varphi - A\varphi = f$$

Integral equations are one of the most important branches of mathematics, among which are Abel's integral equations.

Abel's integral equation occurs in many branches of scientific fields, such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation. Abel's integral equation is the earliest example of an integral

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt,$$

or of the second Kind

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt,$$

This dissertation is organized into fourths chapters:

In the first chapter, we will give reminders on functional spaces.

The second chapter in this chapter, we studied the integral Equation of Abel's.

The third chapter laplase transform.

The fourth chapter Application.

Chapter 1

Functional spaces

1.1 Banach spaces

A normed linear space is a metric space with respect to the metric d derived from its norm, where $d(x, y) = \|x - y\|$.

Definition 1 *A Banach space is a normed linear spece that is a complete metric space with respect to the metric derived from its norm. The following examples illustrate the definition. we will study many of these examples in greater detail later on, so we do not present proofs here.*

Example 2 *For $1 \leq p \leq \infty$, we define the p-norm on $\mathbb{R}^n$(or $\mathbb{C}^n$) by*

$$\|(x_1,x_2, x_{3...}x_n)\|_p = (|x_1|^p + |x_2|^p + ... |x_n|^p)^{1/p}.$$

For $p = \infty$, we define the ∞, or maximum, norm by

$$\|(x_1, x_2, x_3...x_n)\|_\infty = \max\{|x_1|, |x_2|, ... |x_n|\}.$$

then $\mathbb{R}^n$ equipped with the p-norm is a finite-dimensional Banach spece for $1 \leq p \leq \infty$.

Example 3 *The space $C([a, b])$ of continuous, real-valued (or complex-valued) function on $[a, b]$ with the sup-norm is Banach space.More generally, the space $C(K)$ of continuous function on a compact metri space K equipped with the sup-norm is a Banach spaces*

1.2 Hilbert Spaces

Inner Production Spaces

Definition 4 *[1] An* ***inner-product*** *on a vector space X is a map $\langle , \rangle : X \times X \to \mathbb{C}$*

such that

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle ,$$

$$\langle x, \lambda y \rangle = \lambda \langle x, y \rangle ,$$

$$\langle y, x \rangle = \overline{\langle x, y \rangle},$$

$$\langle x, x \rangle \geq 0; \quad \langle x, x \rangle = 0 \Leftrightarrow x = 0.$$

In words, it is said to be a positive-definite sesquilinear form. The simplest examples are $\mathbb{R}^N$ and $\mathbb{C}^N$ with $\langle x, y \rangle = \sum_{n=1}^{N} \overline{x}_n y_n$; the square matrices of size $N \times N$ also have an inner-product given by $\langle A, B \rangle = \sum_{i,j=1}^{N} \overline{A}_{ij} B_{ij}$,

1. $\langle x, x \rangle$ is real (and positive) and is denoted by $\|x\|^2$
2. $\langle x, y \rangle = 0 \ \forall x \Rightarrow y = 0$ (put $x = y$)
3. $\langle \lambda x, y \rangle = \overline{\lambda} \langle x, y \rangle$ (**anti-linear**); $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;
4. $\|\lambda x\| = |\lambda| \|x\|$.

Definition 5 *Two vectors x, y are* ***orthogonal*** *when $\langle x, y \rangle = 0$, written as $x \perp y$.the angle between two vectors is given by $\cos\theta = \langle x, y \rangle / \|x\| \|y\|$.*

1. $\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\,\langle x, y \rangle + \|y\|^2$.
2. (**Pythagoras**) If $z = x + y$ and $\langle x, y \rangle = 0$ then $\|z\|^2 = \|x\|^2 + \|y\|^2$.
3. For any orthogonal vectors x, y, $\|x\| \leq \|x+y\|$.
4. For any non-zero vectors x, y there is a unique vector z and a unique scalar λ such that $x = z + \lambda y$ and $z \perp y$.

Proof. The first three statements follow immediately from the axioms and the first proposition. For the last statement, let $z = x - \lambda y$ as required; ■

then $z \perp y$ holds $\Leftrightarrow \lambda = \langle y, x \rangle \, / \, \langle y, y \rangle$ (check).

Proposition 6 ***Cauchy-Schwarz***

$|\langle x, y \rangle| \leq \|x\| \, \|y\|$

Proof. Decompose x into orthogonal parts $x = (x - \lambda y) + \lambda y$ wehere $\lambda = \frac{\langle y,x \rangle}{\langle y,y \rangle}$(assuming $y \neq 0$); now use Pythagoras' theorem, and deduce that $|\lambda y| \leq \|x\|$ Note that Pythagoras' theorem and Cauchy-Schwarz's inequality are still ■

valid even if the 'inner -product' is not positive definite but just semi -definite,as long as $\|y\| \neq 0$.

$$\|x + y\| \leq \|x\| + \|y\|$$

The proof is simply an application of the Cauchy-Schwarz inequality to the expansion of $\|x + y\|^2$.

Hence $\|x\|$ is a norm,and all the facts about normed vector spaces apply to inner-product spaces. In particular they are metric spaces with distance $d(x, y) = \|x - y\|$, convergence of sequences makes sense as $x_n \to x \Leftrightarrow \|x_n - x\| \to 0$, continuity and dual spaces also make sence. Inner-product spaces are special normed spaces which not only have a concept o of length but also of angle.

Definition 7 *A Hilbert space is an inner-product space which is complete as a metric space.*

Orthogonality

Definition 8 *The* ***orthogonal*** *space of a set A is the set*

$$A^\perp = \{x : \langle y, x \rangle = 0 \ \forall y \in A\}.$$

Chapter 2

Integral Equation

2.1 Integral Equation

As stated in the previous chapter, an integral equation is the equation in which the unknown function $u(x)$ appears inside an integral .The most standard type of integral equation in $u(x)$ is of the form [2]

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt,$$

2.2 Classification of Integral Equation

Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation. In this text we will be concerned on the following types of integral equations.

2.2.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral equation in the form:

$$f(x) = \int_a^b K(x,t)u(t)dt. \tag{2.1}$$

This is called Fredholm integral equation of the first kind. However, for Fredholm integral equations of the second kind, the unknown function $u(x)$

appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \tag{2.2}$$

Examples of the two kinds are given by

$$\frac{\sin x - x\cos x}{x^2} = \int_0^1 \sin(xt)u(t)dt,$$

and

$$u(x) = x + \frac{1}{2}\int_{-1}^1 (x-t)u(t)dt, \tag{2.3}$$

respectively.

2.2.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x K(x,t)u(t)dt. \tag{2.4}$$

However, Volterra integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \tag{2.5}$$

Examples of the Volterra integral equations of the first kind are

$$xe^{-x} = \int_0^x e^{t-x}u(t)dt, \tag{2.6}$$

and

$$5x^2 + x^3 = \int_0^x (5+3x-3t)u(t)dt. \tag{2.7}$$

However, examples of the Volterra integral equations of the second kind are

$$u(x) = 1 - \int_0^x u(t)dt, \tag{2.8}$$

and

$$u(x) = x + \int_0^x (x-t)u(t)dt. \tag{2.9}$$

the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind, but will not be examined in this text. Examples of the two types are given by

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xu(t)dt - \int_0^1 tu(t)dt,$$

and

$$u(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau.$$

2.2.3 Abel's Integral Equation

Abel's Integral Equation and its Generalisations

Abel in 1823 investigated the motion of a particle that slides down along a smooth unknown curve, in a vertical plane, under the influence of the gravity. The particle takes the time $f(x)$ to move from the highest point of vertical height x to the lowest point 0 on the curve. The Abel's problem is derived to find the equation of that curve.

Abel derived the equation of motion of the sliding particle along a smooth curve by the singular integral equation

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \tag{2.10}$$

where $f(x)$ is a predetermined data function, and $u(x)$ is the solution that will be determined. It is to be noted that Abel's integral equation 2.10

is also called Volterra integral equation of the first kind. Besides the kernel $K(x,t)$ in Abel's integral equation 2.10 is

$$K(x,t) = \frac{1}{\sqrt{x-t}}, \tag{2.11}$$

where

$$K(x,t) \to \infty, \quad as \quad t \to x. \tag{2.12}$$

It is interesting to point out that although Abel's integral equation is a Volterra integral equation of the first kind, but two of the methods used before namely the series solution method and the conversion to a second kind Volterra equation, are not applicable here. The series solution cannot be used in this case especially if $u(x)$ is not analytic. Moreover, converting Abel's integral equation to a second kind Volterra equation is not obtainable because we cannot use Leibnitz rule due to the singularity. behavior of the kernel in 2.10

Exampels from Applications

The following problem is due to Huygens (Horologium oscillatorium;1673).

Let $y = \varphi(x)$ be a function starting from the origin (i.e.,$\varphi(0) = 0$) and monotonically increasing up to the ordinate $H \succ 0$. The graph of the function φ has to describe a trajectory along which a point mass is "sliding down".

Starting from the height $y \in [0, H]$ (i.e.,from a curve point (x, y) with $\varphi(x) = y$) with zero initial speed, the point mass needs a certain time denoted by $t = t(y)$ to arrive at the origin $(0, 0)$. Let a function $\tau : [0, H] \to \mathbb{R}$ be given. How must the function φ from above look like so that the time $t(y)$ coincides with the given function $\tau(y)$?

Chapter 3

LAPLACE TRANSFORM

3.1 Introduction

In this chapter, we present the formal definition of the Laplace transform and calculate the Laplace transforms of some elementary functions directly from the definition. The existence conditions for the Laplace transform are stated.

The basic operational properties of the Laplace transforms including, and the laplace transforms Methode.[3]

3.2 Definition of the Laplace Transform and Examples

We start with the fourier Integral, which expresses the representation of a function $f_1(x)$ defined on $-\infty < x < \infty$ in the form

$$f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} e^{-ikt} f_1(t) dt. \tag{3.1}$$

we next set $f_1(x) \equiv 0$ in $-\infty < x < 0$ and write

$$f_1(x) = e^{-cx} f(x) H(x) = e^{-cx} f(x), \quad x > 0, \tag{3.2}$$

where c is a positive fixed number, so that 3.1 becomes

$$f(x) = \frac{e^{cx}}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{0}^{\infty} \exp\{-t(c+ik)\} f(t) dt. \tag{3.3}$$

with a change of variable, $c + ik = s$, $i\ dk = ds$ we rewrite 3.3 as

$$f(x) = \frac{e^{cx}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp\{(s-c)x\} ds \int_0^\infty e^{-st} f(t) dt. \tag{3.4}$$

Thus, the Laplace transform of $f(t)$ is formally defined by

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t) dt, \quad \operatorname{Re} s > 0, \tag{3.5}$$

where e^{-st} is the kernel of the transform and s is the transform variable which is a complex number.Under broad conditions on $f(t)$, its transform $\overline{f}(s)$ is analytic in s in the half-plane, where $\operatorname{Re} s > a$

Rsult 3.4 then gives the formal definition of the inverse Laplace transform

$$\mathcal{L}^{-1}\{\overline{f}(s)\} = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \overline{f}(s) ds, c > 0. \tag{3.6}$$

obviously, $\mathcal{L}$ and $\mathcal{L}^{-1}$ are linear integral operators.

Table: [4] laplace Transform for some Elementary function

S.No.	$f(t)$	$L\{(t);s)$ or $F(s)$
1	1	$1/s, s > 0$
2	t^n, n is positive integer	$n!/s^{n+1}, s > 0$
3	$t^n, n > -1$	$\Gamma(n+1)/s^{n+1}, s > 0$
4	e^{at}	$1/(s-a), s > 0$
5	$\sin at$	$a/(s^2+a^2), s > 0$
6	$\cos at$	$s/(s^2+a^2), s > 0$
7	$\sinh at$	$a/(s^2-a^2), \lvert s\rvert > 0$
8	$\cosh at$	$s/(s^2-a^2), \lvert s\rvert > 0$
9	$J_0(at)$	$1/\sqrt{s^2+a^2}$
10	$J_n(at)$	$\frac{\{\sqrt{s^2-a^2}-s\}^n}{a^n\sqrt{s^2+a^2}}$
11	$\delta(t-a)$	e^{-as}
12	$\operatorname{erf}(\sqrt{t})$	$1/\{s\sqrt{s+1}\}$
13	t^{-a}	$\Gamma(1-a)s^{a-1}$

Example 9 *if* $f(t) = 1$ *for* $t > 0$,*then*

$$\overline{f}(s) = \mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \frac{1}{s} \tag{3.7}$$

Example
if $f(t) = e^{at}$, where a is a constant, then

$$\mathcal{L}\{e^{at}\} = \overline{f}(s) = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}, s > a. \tag{3.8}$$

Table:[4] Inverse Laplace Transform of some Elementary function.

S.No.	$F(s)$	$L^{-1}\{F(s);t\}$ or $f(t)$
1.	$1/s$	1
2.	$1/s^{n+1}(n$ is a positive integer $)$	$t^n/n!$
3.	$1/s^{\alpha+1}\{\text{Re}(\alpha)\rangle - 1\}$	$t^\alpha/\Gamma(\alpha+1)$
4.	$1/(s-a)$	e^{at}
5.	$1/(s^2+a^2)$	$\sin at/a$
6.	$s/(s^2+s^2)$	$\cos at$
7.	$1/(s^2-s^2)$	$(\sin hat)/a$
8.	$s/(s^2-s^2)$	$\cos hat$
9.	$1/\sqrt{(s^2+s^2)}$	$J_0(at)$
10.	$1/\{s\sqrt{(s+1)}\}$	$\text{erf}(\sqrt{t})$

3.3 Existence Conditions for the Laplace Transform

A function $f(t)$ is said to be of exponentail order $a(> 0)$ on $0 \leq t < \infty$ if there exists a positive constant K such that for all $t > T$,

$$|f(t)| \leq Ke^{at}, \tag{3.9}$$

and we write this symbolically as

$$f(t) = O(e^{at}) \quad \text{as} \quad t \to \infty. \tag{3.10}$$

Or, equivalently,

$$\lim_{t\to\infty} e^{-bt}\,|f(t)| \leq \; k \lim_{t\to\infty} e^{-(b-a)t} = 0, b > a. \tag{3.11}$$

such a function $f(t)$ is simple called an exponential order as $t \to \infty$, and clearly, it does not grow faster than Ke^{at} as $t \to \infty$.

If a function $f(t)$ is continuous or piecewise continuous in every finite interval $(0, T)$, and of exponential order e^{at} , then the Laplace transform of $f(t)$ exists for all s provided $Res > a$.

Proof. we have

$$\begin{aligned} \left|\overline{f}(s)\right| &= \left|\int_0^\infty e^{-st} f(t) dt\right| \le \int_0^\infty e^{-st} \left|f(t)\right| dt & (3.12) \\ &\le K \int_0^\infty e^{-t(s-a)} dt = \frac{k}{s-a}, \quad \text{for} \quad \operatorname{Re} s > a. & (3.13) \end{aligned}$$

■

Thus, the proof is complete. It is noted that the conditions as stated in are sufficient rether than necessary conditions, It also follows from 3.12 that $\lim_{s \to \infty} \left|\overline{f}(s)\right| = 0$, that is, $\lim_{s \to \infty} \overline{f}(s) = 0$.this result can be regarded as the limiting property of the Laplace transform.

However, $\bar{f}(s) = s$ or s^2 is not the Laplace transform of any continuous (or piecewise continuous) function because $\overline{f}(s)$ does not tend to zero as $s \to \infty$.

Further, a function $f(t) = exp(at^2)$, $a > 0$ cannot have a Laplace transform even though it is continuous but is not of the exponential order because

$$\lim_{t \to \infty} exp(at^2 - st) = \infty.$$

3.4 Basic Properties of Laplace Transforms

Theorem 10 *(Heaviside's first Shifting Theorem).*

If $\mathcal{L}\{f(t)\} = \overline{f}(s)$, then

$$\mathcal{L}\{e^{-at} f(t)\} = \overline{f}(s+a), \tag{3.14}$$

where a is a real constant.

Proof. we have, by definition, ■

$$\mathcal{L}\left\{e^{-at} f(t)\right\} = \int_0^\infty e^{-(s+a)t} f(t) dt = \overline{f}(s+a).$$

Example 11 *The following results readily follow from 3.14*

$$\mathcal{L}\{t^n e^{-at}\} = \frac{n!}{(s+a)^{n+1}}, \tag{3.15}$$

$$\mathcal{L}\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}, \tag{3.16}$$

$$\mathcal{L}\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}. \tag{3.17}$$

If $\mathcal{L}\{f(t)\} = \overline{f}(s)$, them the second shifting property holds:

$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s) = e^{-as}\mathcal{L}\{f(t)\}, \quad a > 0. \tag{3.18}$$

Or, equivalenty,

$$\mathcal{L}\left\{f(t)H(t-a)\right\} = e^{-as}\mathcal{L}\left\{f(t+a)\right\}. \tag{3.19}$$

where $H(t-a)$ is the Heaviside unit step function defined by It follows from the definition that

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st}f(t-a)H(t-a)dt$$

$$= \int_a^\infty e^{-st}f(t-a)dt,$$

which is, by putting $t-a=\tau$,

$$= e^{-sa}\int_0^\infty e^{-s\tau}f(\tau)d\tau = e^{-sa}\overline{f}(s).$$

we leave it to the reader to prove 3.19
In particular, if $f(t)=1$, them

$$\mathcal{L}\{H(t-a)\} = \frac{1}{s}\exp(-sa). \tag{3.20}$$

Use the shifting propety 3.18 or 3.19 to find the laplace tranform of
(a)

$$f(t) = \left\{\begin{array}{cc} 1, & 0<t<1 \\ -1, & 1<t<2 \\ 0, & t>2 \end{array}\right\},$$

(b)

$$g(t) = \sin t \; H(t-\pi).$$

To find $\mathcal{L}\{f(t)\}$, we write $f(t)$ as

$$f(t) = 1 - 2H(t-1) + H(t-2).$$

Hence,

$$\begin{aligned} \overline{f}(s) &= \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - 2\mathcal{L}\{H(t-1)\} + \mathcal{L}\{H(t-2)\} \\ &= \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}. \end{aligned}$$

To obtain $\mathcal{L}\{g(t)\}$, we use 3.19 so that

$$\overline{g}(s) = \mathcal{L}\{\sin t H(t-\pi)\} = -e^{-\pi s}\mathcal{L}\{\cos t\} = -\frac{se^{-\pi s}}{s^2+1}.$$

scaling property:

$$\mathcal{L}\{f(at)\} = \frac{1}{|a|}\overline{f}(\frac{s}{a}), a \neq 0. \tag{3.21}$$

show that the Laplace transform of the square wave function $f(t)$ defined by

$$f(t) = H(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + ... \tag{3.22}$$

$$\bar{f}(s) = \frac{1}{s}\tanh(\frac{as}{2}). \tag{3.23}$$

$$f(t) = H(t) - 2H(t-a) = 1 - 2 \cdot 0 = 1, 0 < t < a$$

$$f(t) = H(t) - 2H(t-a) + 2H(t-2a)$$

$$= 1 - 2 \cdot 1 + 2 \cdot 0 = -1, \quad 0 < a < t < 2a.$$

Thus,

$$\overline{f}(s) = \frac{1}{s} - 2\cdot\frac{e^{-as}}{s} + 2\cdot\frac{e^{-2as}}{s} - 2\cdot\frac{e^{-3as}}{s} + ...$$

$$= \frac{1}{s}[1 - 2r(1 - r + r^2 - ...)] \quad \text{where} \quad r = e^{-as}$$

$$= \frac{1}{s}\left[1 - \frac{2r}{1+r}\right] = \frac{1}{s}\left[1 - \frac{2e^{-as}}{1+e^{-as}}\right]$$

$$= \frac{1}{s}\left(\frac{1-e^{-as}}{1+e^{-as}}\right) = \frac{1}{s}\left(\frac{e^{\frac{sa}{2}} - e^{\frac{-as}{2}}}{e^{\frac{as}{2}} + e^{\frac{-as}{2}}}\right) = \frac{1}{s}\tanh\left(\frac{as}{2}\right).$$

(The Laplace Transform of a Periodic Function) If $f(t)$ is a periodic function of period a, and if $\mathcal{L}\{f(t)\}$ exists, show that

$$\mathcal{L}\{f(t)\} = [1 - exp(-as)]^{-1}\int_0^a e^{-st}f(t)dt. \tag{3.24}$$

we have, by definition,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = \int_0^a e^{-st}f(t)dt + \int_a^\infty e^{-st}f(t)dt.$$

letting $t = \tau + a$ in the second integral gives

$$\overline{f}(s) = \int_0^a e^{-st}f(t)dt + \exp(-sa)\int_0^\infty e^{-st}f(\tau + a)d\tau,$$

which is, due to $f(\tau + a) = f(\tau)$ and replacing the dummy variable τ by t in the second integral,

$$= \int_0^a e^{-st}f(t)dt + \exp(-sa)\int_0^\infty e^{-st}f(t)dt.$$

Finally, combining the second term with the left hand side, we obtain3.24

In particular, we calculate the Laplace transform of a rectified sine wave, that is, $f(t) = |\sin at|$ .this is a periodic function with period $\frac{\pi}{a}$.we have

$$\int_0^{\frac{\pi}{a}} e^{-st}\sin atdt = \left[\frac{e^{-st}(-a\cos at - s\sin at)}{(s^2+a^2)}\right]_0^{\frac{\pi}{a}} = \frac{a\left\{1 + \exp(-\frac{s\pi}{a})\right\}}{(s^2+a^2)}.$$

Clearly, the property 3.24 gives

$$\mathcal{L}\{(f(t)\} = \frac{a}{(s^2+a^2)} \cdot \frac{1+\exp\left(-\frac{s\pi}{a}\right)}{1-\exp\left(-\frac{s\pi}{a}\right)}$$

$$= \frac{a}{(s^2+a^2)} \left[\frac{\exp\left(\frac{s\pi}{2a}\right)+\exp\left(-\frac{s\pi}{2a}\right)}{\exp\left(\frac{2\pi}{2a}\right)-\exp\left(-\frac{s\pi}{2a}\right)}\right]$$

$$= \frac{a}{s^2+a^2}\coth(\frac{s\pi}{2a}).$$

3.5 The Laplace Transform Method

Although the Laplace transform method was presented before, but a brief summary will be helpful. In the convolution theorem for the Laplace transform method, it was stated that if the kernel $K(x,t)$ of the integral equation [2]

$$f(x) = \int_0^x K(x,t)u(t)dt, \tag{3.25}$$

Depends on the difference $x-t$, then it is called a difference kernel.The Abel's integral equation can thus be expressed as

$$f(x) = \int_0^x K(x-t)u(t)dt. \tag{3.26}$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each.Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by,

$$\begin{aligned} \mathcal{L}\{f_1(x)\} &= F_1(s), \\ \mathcal{L}\{f_2(x)\} &= F_2(s), \end{aligned} \tag{3.27}$$

The Laplace convolution product of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt, \tag{3.28}$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t) f_1(t) dt. \tag{3.29}$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x). \tag{3.30}$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{f_1 * f_2)(x)\} = F_1(s)F_2(s). \tag{3.31}$$

Based on this summary, we will examine Abel's integral equation where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform Taking Laplace transforms of both sides of 2.10 leads to

$$\mathcal{L}\{f(x)\} = \mathcal{L}\{u(x)\}\mathcal{L}\{x^{-\frac{1}{2}}\}, \tag{3.32}$$

or equivalently

$$F(s) = U(s)\frac{\Gamma(1/2)}{s^{1/2}} = U(s)\frac{\sqrt{\pi}}{s^{1/2}}, \tag{3.33}$$

that gives

$$U(s) = \frac{s^{1/2}}{\sqrt{\pi}} F(s), \tag{3.34}$$

where Γ is the gamma function, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ In Appendix D, the definition of the gamma function and some of the relations related to it are given.

The last equation 3.34 can be rewritten as

$$U(s) = \frac{s}{\pi}(\sqrt{\pi} s^{-\frac{1}{2}} F(s)), \tag{3.35}$$

which can be rewritten by

$$\mathcal{L}\{u(x)\} = \frac{s}{\pi}\mathcal{L}\{y(x)\}, \tag{3.36}$$

where

$$y(x) = \int_0^x (x-t)^{-\frac{1}{2}} f(t)dt. \tag{3.37}$$

Using the fact

$$\mathcal{L}\{y'(x)\} = s\mathcal{L}\{y(x)\} - y(0), \tag{3.38}$$

into 3.36 we obtain

$$\mathcal{L}\{u(x)\} = \frac{1}{\pi}\mathcal{L}\{y'(x)\}. \tag{3.39}$$

Applying L^{-1} to both sides of 3.39 gives the formula

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt, \tag{3.40}$$

Chapter 4

APPLICATIONS

4.1 Introduction

In this section, we present some numerical applications to demonstrate the effectiveness of Laplace transform for the solution of Abel's Integral Equation.

Example 12 *Solve the following Abel's integral equation*

$$2\pi\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{4.1}$$

Substituting $f(x) = 2\pi\sqrt{x}$ in 3.40 gives

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{2\pi\sqrt{t}}{\sqrt{x-t}}dt = \frac{d}{dx}(\pi x) = \pi, \tag{4.2}$$

Example 13 *Solve the following Abel's integral equation*

$$\frac{4}{3}x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{4.3}$$

Substituting $f(x) = \frac{4}{3}x^{\frac{3}{2}}$ in 3.40 gives

$$u(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{4}{3}t^{\frac{3}{2}}}{\sqrt{x-t}}dt = \frac{1}{2}\frac{d}{dx}(x^2) = x \tag{4.4}$$

Example 14 *Solve the following Abel's integral equation*

$$\frac{8}{3}x^{\frac{3}{2}} + \frac{16}{5}x^{\frac{5}{2}} = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt. \tag{4.5}$$

Using 3.40 gives

$$\begin{aligned} u(x) &= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\frac{8}{3}t^{\frac{3}{2}} + \frac{16}{5}t^{\frac{5}{2}}}{\sqrt{x-t}}dt, \\ &= \frac{1}{\pi}\frac{d}{dx}(\pi x^2 + \pi x^3) = 2x + 3x^2. \end{aligned} \tag{4.6}$$

Example 15 *Find two-terms approximation of the solution of the following Abel's integral equation*

$$\sin x = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt, \quad x \in [0,1]. \tag{4.7}$$

Using 3.40 gives

$$\begin{aligned} u(x) &= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\sin t}{\sqrt{x-t}}dt = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{t - \frac{1}{3!}t^3}{\sqrt{x-t}}dt, \\ &= \frac{1}{\pi}\frac{d}{dx}\left(\frac{4}{3}x^{\frac{3}{2}} - \frac{16}{105}x^{\frac{7}{2}}\right) = \frac{1}{\pi}\left(\frac{1}{2}\sqrt{x} - \frac{8}{15}x^{\frac{5}{2}}\right). \end{aligned} \tag{4.8}$$

$$x = \int_0^x \frac{1}{\sqrt{x-t}}u(t)dt, \tag{4.9}$$

Taking Laplace transform of both sides of 4.9, we have:

$$\mathcal{L}\left\{x\right\} = \mathcal{L}\left\{\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt\right\}$$

$$\Rightarrow \frac{1}{v^2} = \mathcal{L}\left\{x^{-1/2} * u(x)\right\} \tag{4.10}$$

Applying convolution theorem of Laplace transform in 4.10, we have:

$$\frac{1}{v^2} = \mathcal{L}\left\{x^{-1/2}\right\}\mathcal{L}\left\{u(x)\right\}$$

$$\Rightarrow \frac{1}{v^2} = \left(\frac{\sqrt{\pi}}{v^{1/2}}\right) \mathcal{L}\left\{u(x)\right\}$$

$$\Rightarrow \mathcal{L}\left\{u(x)\right\} = \frac{v^{-3/2}}{\sqrt{\pi}} \tag{4.11}$$

Applying inverse Laplace transform on both sides of 4.11, we get:

$$u(x) = \frac{1}{\sqrt{\pi}} \mathcal{L}^{-1}\left\{v^{-3/2}\right\}$$

$$\Rightarrow u(x) = \frac{2}{\pi} x^{1/2} \tag{4.12}$$

which is the required solution of 4.9

$$1 + x + x^2 = \int_0^x \frac{1}{\sqrt{x-1}} u(t)dt \tag{4.13}$$

Taking Laplace transform of both sides of 4.13, we have:

$$\mathcal{L}\left\{1\right\} + \mathcal{L}\left\{x\right\} + \mathcal{L}\left\{x^2\right\} = \mathcal{L}\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t)dt\right\}$$

$$\Rightarrow \frac{1}{v} + \frac{1}{v^2} + \frac{2}{v^3} = \mathcal{L}\left\{x^{-1/2} * u(x)\right\} \tag{4.14}$$

Applying convolution theorem of Laplace transform in 4.14, we have:

$$\frac{1}{v} + \frac{1}{v^2} + \frac{2}{v^3} = \mathcal{L}\{x^{-1/2}\}\mathcal{L}\left\{u(x)\right\}$$

$$\Rightarrow \frac{1}{v} + \frac{1}{v^2} + \frac{2}{v^3} = \left(\frac{\sqrt{\pi}}{v^{1/2}}\right) \mathcal{L}\left\{u(x)\right\}$$

$$\Rightarrow \mathcal{L}\left\{u(x)\right\} = \frac{1}{\sqrt{\pi}}\left[\frac{1}{v^{1/2}} + \frac{1}{v^{3/2}} + \frac{2}{v^{5/2}}\right] \tag{4.15}$$

Applying inverse Laplace transform on both sides of 4.15 we get:

$$u(x) = \frac{1}{\sqrt{\pi}} \mathcal{L}^{-1}\left\{\frac{1}{v^{1/2}} + \frac{1}{v^{3/2}} + \frac{2}{v^{5/2}}\right\}$$

$$\Rightarrow u(x) = \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\mathcal{L}^{-1}\left\{\frac{1}{v^{1/2}}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{v^{3/2}}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{v^{5/2}}\right\}\right\}$$

$$\Rightarrow u(x) = \frac{1}{\pi}\left[x^{-1/2} + 2x^{1/2} + \frac{8}{3}x^{3/2}\right] \tag{4.16}$$

which is the required solution of 4.13

4.2 Numerical Examples

Several test examples are presented to illustrate the accuracy and stability of the method described in this paper To examine the accuracy of the results, absolute errors are employed to assess the efficiency of this method. As for the stability of this method, we take the right-hand side prescribed function $f(s)$ with small errors. To model this character, in the following examples we take the right-hand side prescribed function as $f^{\varepsilon}(s) = f(s)[1+\varepsilon\theta(s)]$, where $f(s)$ denotes a true function corresponding to the exact solution, ε is a constant and (s) denotes a uniform random variable with values in $[-1,1]$ such that the maximum relative error $max|f^{\varepsilon}(s) - f(s)|/|f(s)| \leq \varepsilon$.[5] Standard Abel integral equations refer to

Standard Abel integral equations refer to

$$\int_0^s \frac{\varphi(t)}{(s-t)^v}dt = f(s), \quad s > 0, 0 < v < 1, \tag{4.17}$$

and

$$\int_s^a \frac{\varphi(t)}{(t-s)^v}dt = f(s), \quad s < a, 0 < v < 1, \tag{4.18}$$

Theorem 16 *Assume that $\varphi(s)$ is $(n+1)$ times continually differentiable, and $f(0) = 0$. Then the solution to Abel equation 4.17 can be approximated by*

$$\varphi_n(s) = \frac{1}{s^{1-v}C_n(1-v)}\begin{vmatrix} f(s) & \frac{1}{2-v} & \cdots & \frac{1}{n+1-v} \\ \frac{1-v}{s}\int_0^s f(t)dt & \frac{1}{3-v} & \cdots & \frac{1}{n+2-v} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(n+1-v)}{\Gamma(n)\Gamma(1-v)s^n}\int_0^s (s-t)^{n-1}f(t)dt & \frac{1}{n+2-v} & \cdots & \frac{1}{2n+1-v} \end{vmatrix}$$

which is called the nth-order approximation, where $\Gamma\ (\star)$ is the Gamma function. Moreover, this nth-order approximation is exact for a solution of polynomial of degree equal to or less than n.

For Abel equation 4.18, omitting a detailed procedure, we similarly can prove the following

Example 17 *We consider Abel equation*

$$\int_0^s \frac{\varphi(t)}{(s-t)^{1/3}} dt = f(s), \quad 0 < s \leq 1, \tag{4.19}$$

with $f(s) = s^{5/3}$, and its exact solution is $\varphi(s) = 10s/9$.

Based on **Theorem17**, we can obtain explicit forms of several lower-order approximate solutions. By direct computation, one can find that $\varphi_n(s)$ reduces to the desired exact solution only if $n \geq 1$, i.e., $\varphi_n(s) = 10s/9$, as $n \geq 1$. This can be explained as a result of the fact that the desired solution is just a polynomial of degree one. Therefore, if a desired solution is a polynomial of degree n, the nth-order approximation collapses to its exact solution.

Example 18 $f(s) = s^{7/6}$. *In this case, its exact solution can be found to be*

$$\varphi(s) = \frac{7\Gamma(1/6)}{18\sqrt{\pi}\Gamma(2/3)}\sqrt{s}. \tag{4.20}$$

Tabel 1 Approximate solutions and maximum absolute errors between $\varphi(s)$ and $\varphi_n(s)$ of **Exampel 19**

n	1	2	3
$\varphi_n(s)$	$\frac{110}{117}\sqrt{s}$	$\frac{5920}{6669}\sqrt{s}$	$\frac{54560}{60021}\sqrt{s}$
$\max\limits_{[0,1]} \lvert\varphi_n(s) - \varphi(s)\rvert$	0.0383	0.0142	7.11×10^{-3}

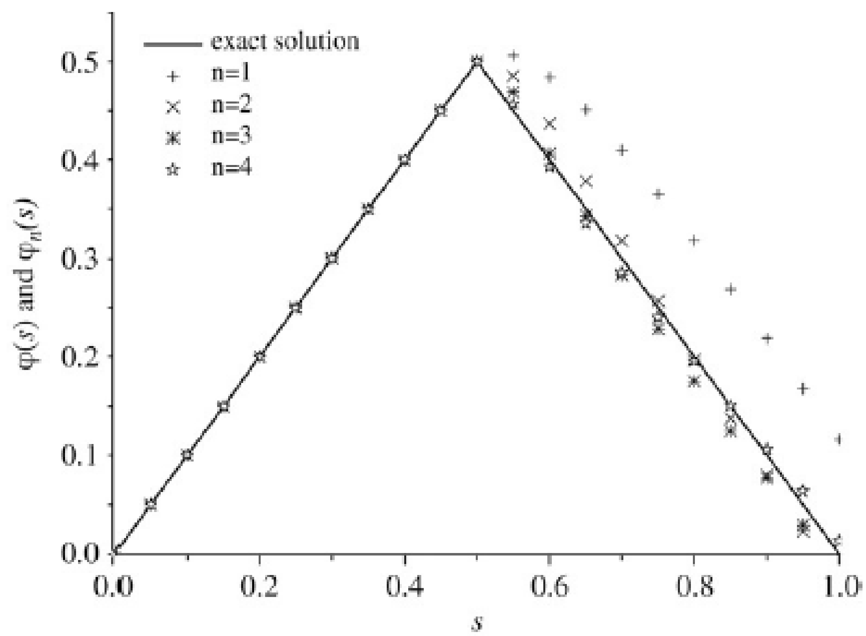

Fig.1.Comparison of the exact and approximate solutions of Example 20.

Example 19 *Consider Abel equation*

$$\int_0^s \frac{\varphi(t)}{(s-t)^{1/2}} dt = f(s) \tag{4.21}$$

where

$$f(s) = \begin{cases} \frac{4}{3}s^{3/2} & ,0 \leq s < \frac{1}{2} \\ \frac{4}{3}s^{3/2} - \frac{8}{3}\left(s-\frac{1}{2}\right)^{3/2} & ,\frac{1}{2} \leq s \leq 1, \end{cases} \tag{4.22}$$

Its exact solution is

$$\varphi(s) = \begin{cases} s, & 0 \leq s < \frac{1}{2}, \\ 1-s, & \frac{1}{2} \leq s \leq 1. \end{cases} \tag{4.23}$$

It is noted that the desired exact solution actually is continuous, but not differential at $s = 0.5$, which violates the assumption of **Theorem17**. However, according to the method described above, lower-order approximations except for $n = 1$ still give quite satisfactory accuracy. This can be observed from **Fig.1**, in which the exact and corresponding approximate solutions for $n = 1, 2, 3, 4$ are plotted. It is seen from **Fig.1** that when $n = 3, 4$, the corresponding approximations are very close to the exact solution, implying that the suggested methods are also applicable to a class of Abel equations having non-differential solutions.

Example 20 *Finally, we consider an example of the stability of approximations of Abel inversion, i.e., consider*

$$\int_0^s \frac{\varphi(t)}{(s-t)^{1/2}} dt = \left(\frac{4}{3} s^{3/2} - \frac{32}{35} s^{7/2} \right) [1 + \varepsilon \theta(s)], \tag{4.24}$$

where ε is a small parameter and $\theta(s)$ denotes a uniform random variable with values in $[-1, 1]$.

It is easily found that when $\varepsilon = 0$ the exact solution is

$$\varphi(s) = s - s^3. \tag{4.25}$$

We adopt **Theorem 17** and get the second- and third-order approximations as

$$\varphi_2(s) = s - \frac{226}{231} s^3, \tag{4.26}$$

$$\varphi_3(s) = s - s^3, \tag{4.27}$$

Respectively. In the following, we set $\varepsilon = 0.01$, and reconstruct $\varphi(s)$ in the case of random errors. For comparison, **Fig2** and **3** show the second- and third-order reconstructions from the right-hand side non-homogeneous term with random noise

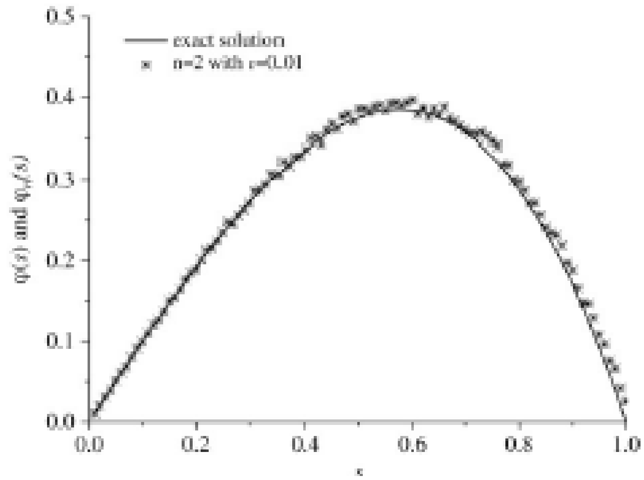

Fig.2 $n = 2$ from noisydata with $\varepsilon = 0.01$.

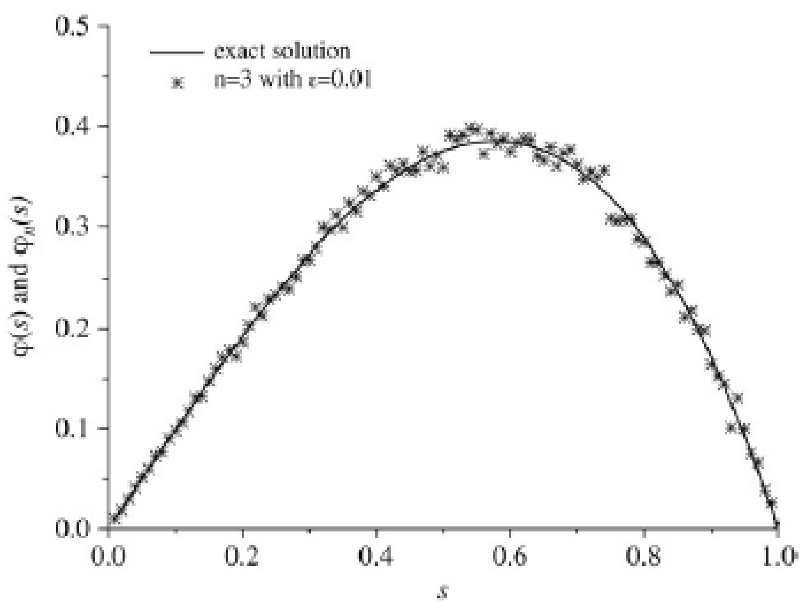

Fig.3 $n = 3$ from noisydata with $\varepsilon = 0.01$.

Chapter 5

Conclusion and Perspective

5.1 Conclusion

In this memory, we have successfully discussed Laplace transform for the solution of Abel's Integral Equation. The given numerical applications in the application section explain the complete procedure for the solution of Abel's Integral Equation using Laplace transform. The results show that Laplace transform is a powerful integral transform method for the solution of Abel's Integral Equation. In future, Laplace transform can be used for solving other singular integral equations.

5.2 Perspective

In the speces of the 2 or 3 dimension , this subject is very complicated whis we holpe studied in my dectorat thesis.

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Résumé

Dans ce mémoire, On a utiisé la transformation de Laplace pour résoudre l'équation intégrale d'Abel. Quelques exemples numériques sont donnés pour montrer l'efficacité de la transformation de Laplace pour la résolution de l'équation intégrale d'Abel.

Mots-clés : équation intégrale d'Abel, transformations de Laplace, transformation de Laplace inverse,

Abstract

In this paper, we have use the Laplace transform for the solution of Abel's Integral Equation and some numerical example are given to demonstrate the effectiveness of Laplace transform for the solution of Abel's Integral Equation.

Keywords: Abel's Integral Equation, Laplace Transforms, Inverse Laplace Transform,

الملخص

في هذه المذكرة استعملنا تحويلات لابلاس لحل معدلات أبيل التكاملية

مع إعطاء بعض الامثلة العددية لحل هذه المعادلات

الكلمات المفتاحية

معادلة أبيل المتكاملة ، تحويلات لابلاس ، تحويل لابلاس العكسي