MOHAMED BOUDIAF UNIVERSITY OF M'SILA
Faculty of Mathematics and Informatics
Department of Mathematics

Submitted for the degree OF **MASTER**

Field: MATHEMATICS AND computer sciences

Speciality: Mathematics

Option: Partial differential equations and applications

Presented by

Selma Nasri

Subject

Fractional Sobolev Spaces With Applications To Boundary Values Problems

Presented publicly in jun 2025

In view of the composed jury :

Dr. Sengouga Abdelmohouhcene	M.C.A. Univ of M'sila	Chairperson
Dr. Boughrara Brahim	M.C.A. Univ of M'sila	Supervisor
Dr. Mokhtari Abdelhake	M.C.A. Univ of M'sila	Examiner

Promotion: 2024/2025

Acknowledgement

First and foremost, I would like to thank **Allah** for granting me the strength and patience to complete this work.

I wish to express my sincere gratitude to my supervisor, **Dr. Boughrara Brahim**, for his guidance, support, and encouragement throughout this research. I truly appreciate the time and effort he devoted to supervising my work.

I am also deeply thankful to **Professor Saadi Abderrachid** and **Dr. Hammou Amouria** for their valuable advice and continuous support. Their contributions made a significant difference and greatly motivated me during this journey.

I would like to extend my thanks to the members of the jury for taking the time to read and evaluate my thesis, and for their insightful comments and suggestions.

Special thanks go to all my teachers in the Department of Mathematics, particularly those who taught partial differential equations, at the University of M'sila. I have acquired invaluable knowledge from them during my years of study.

Finally, I am profoundly grateful to my beloved family, especially my parents, for their unconditional love, support, and constant encouragement. This achievement would not have been possible without them.

Thank you all.

Dedication

To those who planted the seeds of ambition within me,
To those who were my greatest support after God,
To my dear parents,
Thank you for your prayers, your unwavering support, and countless sacrifices…
To you, I dedicate this achievement.

To my brothers and sisters, the source of constant encouragement and smiles,
Thank you for always being there.

To my friends and colleagues, partners in effort, fatigue, and success,
This accomplishment is not mine alone — it is the result of our shared journey.

To everyone who believed in me and stood by my side,
I dedicate this project as a token of gratitude and love.

Nasri Selma

Contents

Notation

$\mathbb{R}$	Set of real numbers.
$\mathbb{N}^*$	Set of positive integer numbers.
$L^P(a,b)$	Space of power function $p \in [1,\infty[$ integrable on (a,b).
$\|.\|_{L^p(a,b)}$	Norm in L^p.
$L^\infty(a,b)$	Space of functions essentially bounded on (a,b).
$\|.\|_{L^\infty}$	Norm in L^∞.
a.e.	almost everywhere.
$(.,.)$	Scalar product.
p'	conjugate of holder of p $\left(p' = \frac{p-1}{p}\right)$.
$AC(a,b)$	Space of absolute continuous function on (a,b).
$AC^n(a,b)$	Space of n absolute continuous function of order n on (a,b).
$W^{1,p}(a,b)$	The usual Sobolev space on (a,b).
$\|.\|_{W^{1.p}}$	Norm in $W^{1,p}$.
$\mathscr{D}(a,b)$	Space of infinitely differentiable functions with compact support.
$\mathcal{L}(E)$	The set of all linear and continuous maps from E to E.
$\Gamma(.)$	The Gamma function.
$B(.)$	The Beta function.
$I^\alpha_{a^+}$	The fractional integral on the left of order α in the sense of Riemann-Liouville.
$I^\alpha_{b^-}$	The fractional integral on the right of order α in the sense of Riemann-Liouville.
p^*_α	The fractional critical exponent $\left(p^*_\alpha = \frac{p}{1-\alpha p}\right)$.
$D^\alpha_{a^+}$	The fractional derivative on the left of order α in the sense of Riemann-Liouville.
$D^\alpha_{b^-}$	The fractional derivative on the right of order α in the sense of Riemann-Liouville.
${}^cD^\alpha_{a^+}$	The fractional derivative on the left of order α in the sense of Caputo.

${}^{c}D^{\alpha}_{b^-}$	The fractional derivative on the right of order α in the sense of Caputo.
$\mathcal{D}^{\alpha}_{a^+}$	The weak left fractional derivative .
$\mathcal{D}^{\alpha}_{b^-}$	The weak right fractional derivative .
$\frac{d}{dx}$	The usual derivative

Introduction

Mathematics is one of the most important tools in science. It helps us understand and describe many natural and physical phenomena. One of its key fields is differential equations, which are used to study things that change, like speed, temperature, or population growth.

As science progressed, researchers found that classical derivatives are sometimes not enough to model certain systems, especially those that depend on previous states or have memory. To solve this, a new type of derivative called fractional derivatives was introduced. This led to the development of fractional differential equations, which use derivatives of non-integer order. Important references in this area include the book by [6], which is a modern and detailed source on fractional equations, and the book by [9], which lays the theoretical foundations of fractional integrals and derivatives. Another helpful reference is the work by [8], which combines theory and applications.

These equations can be either linear or nonlinear. Nonlinear equations are harder to analyze and often require advanced mathematical tools. In this thesis, I study both linear and nonlinear fractional differential equations. The main goal is to prove the existence of weak solutions using methods from functional analysis. I also relied on other references such as [2] for nonlinear functional analysis and Sobolev spaces.

This thesis is divided into four chapters:

- In **Chapter 1**, I present the main mathematical tools and theorems needed in the rest of the work, especially those related to variational methods.
- In **Chapter 2**, I explain fractional integrals and the Riemann–Liouville derivative, with simple examples to help the reader understand the concepts.

- **Chapter 3** is about fractional Sobolev spaces and absolutely continuous functions. I also present embedding results and versions of the fractional integration by parts formula.
- In **Chapter 4**, I apply the theory to a nonlinear fractional differential equation using three methods: variational minimization, the Nehari manifold approach, and the eigenvalue method.

This work shows that fractional calculus is a powerful and useful tool for solving complex problems in mathematics and applied sciences.

CHAPTER 1

PRELIMINARIES

In this chapter, we present the essential mathematical tools and foundational results that will be used throughout the thesis.These concepts are essential for understanding and studying differential equations., particularly those involving fractional derivatives and variational methods.

1.1 Functional Spaces

1.1.1 The Lebegue Space

Definition 1.1. *[2] Let $(a, b) \subset \mathbb{R}$ open set and let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, the lebesgue space is defined by :*

$$L^p(a, b) = \left\{ f :]a, b[\longrightarrow \mathbb{R} \quad \textit{is measurable and} \int_a^b |f(x)|^p dx < \infty \right\}$$

equipped with the norm

$$||f||_{L^p(a,b)} = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$$

Definition 1.2. *The lebegue space $L^\infty(a, b)$ is defined by :*

$$L^\infty(a, b) = \Big\{ \ f :]a, b[\to \mathbb{R} \textit{ is measurable, } \exists \textit{ constant } C : |f(x)| \leq C \ a.e \ x \ on \ (a, b) \ \Big\},$$

equipped with the norm

$$\|f\|_{L^\infty(a,b)} = \inf \{C, |f(x)| \leq C \ a.e \ x \ on \ (a, b)\} = ess \sup_{x \in (a,b)} |f(x)| \ a.e \ x \ on \ (a, b)$$

Remark 1.1. *For $p = 2$,the space $L^2(a, b)$ is a Hilbert space with the inner product*

$$(u, v) = \int_a^b uv dx$$

Theorem 1.1. *[2] The spaces* $(L^p, ||.||_{L^p})$ *is*

- *Banach space for* $1 \leq p \leq \infty$
- *a reflexive space for* $1 < p < \infty$
- *a separable space for* $1 \leq p < \infty$

Remark 1.2. *The space* L^1 *is separable but not reflexive , while the space* L^∞ *is neither reflexive nor separable .*

Notation 1.1. *Let* $1 \leq p \leq \infty$ *we denote by* p' *the conjugate of* p *defined as follows* $p' = \frac{p}{p-1}$

Theorem 1.2. ***(The Hölder Inquality)***:*[2]*
Let $f \in L^p$ *and* $g \in L^{p'}$ *with* $1 \leq p \leq \infty$ *then* $f, g \in L^1$

$$\int_a^b |f(x)g(x)|dx \leq \|f\|_{L^p(a,b)} \|g\|_{L^{p'}(a,b)} \tag{1.1}$$

We have also the generalized Hölder inquality :

$$\int_a^b |f_1(x) \ldots f_m(x)|dx \leq \|f_1\|_{L^{p_1}(a,b)} \cdots \|f_m\|_{L^{p_m}(a,b)} \tag{1.2}$$

where $f_k(x) \in L^{p_k}(a,b), k = 1, 2, \ldots, m$, $\sum_{k=1}^{m} \frac{1}{p_k} = 1$

Remark 1.3. *whene* $p = q = 2$*, Hölder's inequality becomes:*

$$\left| \int_a^b f(x)g(x)\, dx \right| \leq \left(\int_a^b |f(x)|^2\, dx \right)^{1/2} \left(\int_a^b |g(x)|^2\, dx \right)^{1/2}, \tag{1.3}$$

which corresponds to the classical Cauchy-Schwarz inequality in the Hilbert space $L^2(a,b)$.

Theorem 1.3. ***(Young Inequality)***:*[2]*
for all $a, b \in \mathbb{R}$

$$ab \leq \frac{1}{p}|a|^p + \frac{1}{p'}|b|^{p'} \qquad (\frac{1}{p} + \frac{1}{p'} = 1)$$

Moreover : $\forall \epsilon > 0, \exists C > 0$ *such that :*

$$ab \leq \epsilon a^p + C b^{p'} \text{ where} \qquad C = \epsilon^{-\frac{1}{p-1}}$$

Theorem 1.4. ***(Interpolation inquality)****[2] If* $f \in L^p(a,b) \cap L^q(a,b)$ *with* $1 \leq p \leq q \leq \infty$ *, then* $f \in L^r(a,b) : p \leq r \leq q$ *:*

$$\|f\|_{L^r} \leq \|f\|^{\theta}_{L^p} \|f\|^{1-\theta}_{L^q} \ , \ \text{where} \ \ \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q} \ , \ 0 \leq \theta \leq 1$$

Theorem 1.5. ***(Fubini's theorem)****:[2][9]*
Let $\Omega_1 = [a,b], \Omega_2 = [c,d], -\infty \leq a < b \leq \infty$ *and let* $f(x,y)$ *be a measurable function defined on* $\Omega_1\Omega_2$ *if at least one of the integrals*

$$\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)dy\right)dx, \int_{\Omega_2}\left(\int_{\Omega_1} f(x,y)dx\right)dy, \iint_{\Omega_1\Omega 2} f(x,y)dxdy$$

is finit and absolutely convergent then

$$\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)dy\right)dx = \int_{\Omega_2}\left(\int_{\Omega_1} f(x,y)dx\right)dy = \iint_{\Omega_1\Omega 2} f(x,y)dxdy$$

the following particular case of Fubini's theorem holds,namely :

$$\int_a^b\left(\int_a^x f(x,y)dy\right)dx = \int_a^b\left(\int_y^b f(x,y)dx\right)dy$$

Theorem 1.6. ***(Lebesgue's dominated convergence theorem)****:[2]*
Let (f_n) *be a sequence of functions of* $L^1(a,b)$ *, we assume that*

a) $f_n(x) \longrightarrow f(x)$ *a.e* x *on* (a,b)

b)There exists a function $g \in L^1(a,b)$ *such that for each* n *,*$|f_n(x)| \leq g(x)$ *a.e* x *on* (a,b)

Then $f \in L^1(a,b)$ *and* $||f_n - f||_{L^1(a,b)} \longrightarrow 0$

Theorem 1.7. ***(Inverse of Lebesgue's Dominated Convergence Theorem)****:[2]*
Let $(f_n) \subset L^p(a,b)$*and* $f \in L^p(a,b)$ *such that* $\|f_n - f\|_{L^p(a,b)} \longrightarrow 0$*. Then, there exists a subsequence* (f_{n_k}) *such that:*

a) $f_{n_k}(x) \longrightarrow f(x)$ *a.e.* x *on* (a,b)

b) $|f_{n_k}(x)| \leq h(x) \quad \forall k$ *and a.e.* x *on* (a,b)*, with* $h \in L^p(a,b)$

1.1.2 Space of absolutely continuous functions

Definition 1.3. *Let* $[a,b], -\infty < a < b < +\infty$ *a finite interval of* $\mathbb{R}$ *and* $AC[a,b]$ *is the space of primitive function* f *, ie:*

$$AC([a,b]) = \left\{f : \exists\varphi \in L^1(a,b) : f(x) = C + \int_a^x \varphi(t)dt\right\}$$

and we call $AC[a,b]$ *the space of absolutely continuous functions on* (a,b)

Definition 1.4. *for* $n \in \mathbb{N}^*$ *, we denote* $AC^n([a,b])$ *the space of function* f *which have continuous derivatives up to order* $(n-1)$ *and such that* $f^{(n-1)} \in AC([a,b])$ *ie :*

$$AC^n([a,b]) = \left\{ f \in \mathcal{C}^{n-1}(a,b) / f^{(n-1)} \in AC([a,b]) \right\}$$

In particular ,if $n = 1$ *, we have* $AC^1([a,b]) = AC([a,b])$

Definition 1.5. *Let* $0 < \alpha < 1$ *the holder space* $\mathscr{C}^\alpha([a,b])$ *is define as follows*

$$\mathscr{C}^\alpha([a,b]) = \left\{ u \in \mathscr{C}([a,b]) \,\middle|\, \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C,\ \forall x,y \in [a,b],\ x \neq y \right\}$$

Remark 1.4. *The space* $\mathscr{C}^\alpha([a,b])$ *is a banach space with the norm*

$$\|u\|_{\mathscr{C}^\alpha} = \|u\|_{L^\infty(a,b)} + \sup_{\substack{x \neq y \\ x,y \in [a,b]}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

1.2 Some useful result

We use the following lemma to prove some next theorem.

Lemma 1.1. *Let* $x, y > 0$ *such that* $y \leq x$ *we have :*

$$\text{If } \mu \geq 1 \text{ then} \qquad (x-y)^\mu \leq x^\mu - y^\mu \tag{1.4}$$

$$\text{If } 0 < \mu \leq 1 \text{ then} \qquad y^\mu - x^\mu \leq (x-y)^\mu \tag{1.5}$$

Lemma 1.2. *For all* $x, y \geq 0$ *we have:*

$$\text{If } p \geq 1 \text{ then} \qquad (x+y)^p \leq 2^{p-1}(x^p + y^p), \tag{1.6}$$

$$\text{If } p < 1 \text{ then} \qquad (x+y)^p \leq x^p + y^p, \tag{1.7}$$

Lemma 1.3. *The following function is strictly increasing :*

$$\begin{array}{rcl} f : [a, a+h] & \longrightarrow & \mathbb{R} \\ s & \longmapsto & f(s) = (b-h-s)^\alpha - (b-s)^\alpha + (a+2h-s)^\alpha - (a+h-s)^\alpha \end{array}$$

Proof. We define:

$$f_1(s) = (b-h-s)^\alpha - (b-s)^\alpha, \quad f_2(s) = (a+2h-s)^\alpha - (a+h-s)^\alpha.$$

Then:

$$f_1'(s) = -\alpha \left[(b-h-s)^{\alpha-1} - (b-s)^{\alpha-1} \right],$$

$$f_1''(s) = \alpha(\alpha-1)\left[(b-h-s)^{\alpha-2} - (b-s)^{\alpha-2}\right].$$

Since $b-h-s < b-s$ for $s < b-h$, we have $f_1''(s) < 0$ for all $s \in (a, b-h)$. Thus, $f_1'(s)$ is decreasing on $(a, b-h)$, and we obtain:

$$f_1'(s) \geq f_1'(b-h) = \alpha h^{\alpha-1}.$$

Similarly, for f_2, we compute:

$$f_2'(s) = -\alpha\left[(a+2h-s)^{\alpha-1} - (a+h-s)^{\alpha-1}\right],$$

$$f_2''(s) = \alpha(\alpha-1)\left[(a+2h-s)^{\alpha-2} - (a+h-s)^{\alpha-2}\right].$$

Since $a+2h-s > a+h-s$, we have $f_2''(s) > 0$ for all $s < a+h$. Hence, $f_2'(s)$ is increasing on $[a, a+h]$, and:

$$f_2'(s) > f_2'(a) = -\alpha\left[(a+2h-a)^{\alpha-1} - (a+h-a)^{\alpha-1}\right] = -\alpha\left[(2h)^{\alpha-1} - h^{\alpha-1}\right].$$

Putting everything together, we find:

$$\begin{aligned} f'(s) &= f_1'(s) + f_2'(s) \\ &\geq \alpha h^{\alpha-1} - \alpha\left[(2h)^{\alpha-1} - h^{\alpha-1}\right] \\ &= \alpha\left[h^{\alpha-1} - (2h)^{\alpha-1} + h^{\alpha-1}\right] \\ &= \alpha h^{\alpha-1}\left[2 - 2^{\alpha-1}\right]. \end{aligned}$$

Since $\alpha \in (0,1)$, we have $2^{\alpha-1} < 1$, so $2 - 2^{\alpha-1} > 0$. Therefore, $f'(s) > 0$, and we conclude that f is strictly increasing on the considered interval. □

1.2.1 The Lax-Milgram theorem

Definition 1.6. *1. A bilinear form* $a : H \star H \to \mathbb{R}$ *is said to be*

(i) Continuous if there is a constant C *such that*

$$|a(u,v)| \leq C\|u\|\|v\| \quad \forall u, v \in H$$

(ii) Coercive if there is a constant $\alpha > 0$ *such that*

$$a(v, v \geq \alpha\|v\|^2 \quad \forall v \in H$$

- *A linear form* $\ell : H \to \mathbb{R}$ *is continuous or bounded if there is constant* $C > 0$ *such that*

$$\ell(v) \leq c\|v\| \qquad \forall v \in H$$

Theorem 1.8. ***(Lax-Milgram)****[2] Let* H *be a Hilbert space,* H' *its dual . Let* $a : H \star H \to \mathbb{R}$ *be a continuous and bilinear from , and* $\varphi \in H'$*. Then there exists a unique element* $u \in H$ *such that*

$$a(u, v) = \langle \varphi, v \rangle \quad \forall v \in H. \tag{1.8}$$

1.2.2 Some compactness result

Definition 1.7. ***(compact set)*** *Let* A *be a subset of a normed space* E*. Then* A *is called compact if every sequence* $(x_n) \subset A$ *has a convergent subsequence.*

Definition 1.8. ***(Precompact set)*** *A subset* A *of a normed space* E *is said to be* precompact *in* E *if, for every* $\varepsilon > 0$*, the set* A *can be covered by a finite number of subsets of* E*, each having diameter less than or equal to* ε*.*

Theorem 1.9. *[7] A normed space is compact if and only if it is precompact and complete.*

Definition 1.9. ***(Uniformly equicontinuous)*** *Let* E *be a subset of* $C([a, b])$ *(the space of continuous functions) we say that* E *is uniformly equicontinuous if:*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon\ \forall f \in E$$

Theorem 1.10. ***(Ascoli–Arzelà)****:[2]*
Let E *be a bounded set of* $C([a, b])$*. Assume that* E *is uniformly equicontinuous, Then the closure of* E *in* $C([a, b])$ *is compact*

Theorem 1.11. *[3] Let* $\Omega \subset \mathbb{R}^N$ *be an open set and* $1 \leq p < \infty$*. A bounded set* $K \subset L^p(\Omega)$ *is precompact in* $L^p(\Omega)$ *if and only if:*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \exists G \subset \Omega \text{ open such that } \forall u \in K,\ \forall h \in \mathbb{R}^N,\ |h| < \delta,\ |h| < d(G, \partial\Omega)$$

$$\int_G |u(x + h) - u(x)|^p \, dx < \varepsilon^p \quad \text{and} \quad \int_{\Omega \backslash G} |u(x)|^p \, dx < \varepsilon^p$$

1.2.3 Spectral decomposition

Definition 1.10. *A bounded operator $T \in \mathcal{L}(E)$ is said to be self-adjoint if $T^\star = T$, i.e.,*

$$(Tu, v) = (u, Tv) \qquad \forall u, v \in E$$

Definition 1.11. *Let E and F be normed spaces, and let $T : E \to F$ be a bounded linear operator. We say that T is* compact *if for every bounded sequence $(u_n)_{n\in\mathbb{N}}$ in E, there exists a subsequence (u_{k_n}) such that the sequence (Tu_{k_n}) converges in F.*

Theorem 1.12. *Let H be a separable Hilbert space and let T be a compact self-adjoint operator. Then there exists a Hilbert basis composed of eigenvectors of T*

1.3 Special Functions

Definition 1.12. *[9] the gamma function Γ is defined for a complex number z with $Re(z) > 0$ as follows :*

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$$

Proposition 1.1. *[9] We have following properties:*

1. *for all $z > 0 \in \mathbb{C}$ we have* $\quad \Gamma(z+1) = z\Gamma(z)$
2. $\Gamma(z) = (z-1)!$
3. *if $n \in \mathbb{N}^*$ then $\Gamma(n+1) = n!$*

Definition 1.13. *[9] the Beta function β is a type of euler integral defined by :*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} \qquad (p, q \in \mathbb{C}, Re(p) > 0, Re(q) > 0)$$

Proposition 1.2. *[9]*

1. *For all $p, q \in \mathbb{C}, Re(p) > 0, Re(q) > 0$, we have*
$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{1.9}$$
2. *If $n, m \in \mathbb{N}^*$then ,*
$$\beta(n,m) = \frac{(n-1)!(m-1)!}{(n+m+1)!}$$

CHAPTER 2

RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS AND DERIVATIVES

In this chapter, we will first learn about the idea of fractional integrals, followed by an explanation of the Riemann–Liouville derivative. Simple examples will be given to help make these concepts easier to understand. We will also look at some important properties.

2.1 Riemann-Liouville Fractional Integrals

Definition 2.1. *Let $n-1<\alpha<n$ and $u \in L^1(a,b)$. The left Riemann–Liouville fractional integral $I^{\alpha}_{a^+}$ and right Riemann–Liouville fractional integral $I^{\alpha}_{b^-}$ of u are defined as follow:*

$$(I^{\alpha}_{a^+}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt \quad (a<x\leq b). \tag{2.1}$$

$$(I^{\alpha}_{b^-}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt \quad (a\leq x<b). \tag{2.2}$$

Example 2.1. *for $\alpha>0$,$\beta>0$,we put $f(x)=(x-a)^{\beta-1}$*

$$\begin{aligned}\left(I^{\alpha}_{a^+}(x-a)^{\beta-1}\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}(\tau-a)^{\beta-1}d\tau, \ \textit{we put}\ \tau-a=s(x-a)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 ((x-a)-s(x-a))^{\alpha-1}(s(x-a))^{\beta-1}(x-a)ds\\ &= \frac{1}{\Gamma(\alpha)}(x-a)^{\alpha+\beta-1}\int_0^1 s^{\beta-1}(1-s)^{\alpha-1}ds\\ &= \frac{1}{\Gamma(\alpha)}(x-a)^{\alpha+\beta-1}B(\alpha,\beta)\end{aligned}$$

By (1.9), $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$$\left(I_{a^+}^{\alpha}(x-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1} \tag{2.3}$$

same idea for the right-sided fractional integral (the change of variable is $b-\tau=s(b-x)$*) we obtien :*

$$\left(I_{b^-}^{\alpha}(b-x)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-x)^{\alpha+\beta-1}$$

Example 2.2. *Let* $0<\alpha<1$*, and let* $f(x)=c$*, where* c *is a constant.*

$$\begin{aligned}\left(I_{a^+}^{\alpha}c\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}c\ dt\\ &= \frac{c}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}dt=\left[\frac{-c}{\alpha\Gamma(\alpha)}(x-t)^{\alpha}\right]_a^x\end{aligned}$$

Hence,

$$\left(I_{a^+}^{\alpha}c\right)(x)=\frac{c}{\Gamma(\alpha+1)}(x-a)^{\alpha} \tag{2.4}$$

Theorem 2.1. *Let* $0<\alpha<1$ *and let* f*,* φ *such that*

$$I_{a^+}^{\alpha}\varphi(x)=f(x) \tag{2.5}$$

Then

$$\varphi(x)\in L^1(a,b)\Leftrightarrow \left(I_{a^+}^{1-\alpha}f\right)(x)\in AC([a,b]) \quad \textit{and} \quad \left(I_{a^+}^{1-\alpha}f\right)(x)(a)=0$$

Where

$$\left(I_{a^+}^{1-\alpha}f\right)(x)=\frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}f(t)dt \tag{2.6}$$

Proof. $(\Rightarrow)$:

The equation (2.5) may be solved in the following:

Changing x to t and t to s respectively ,and multiplying both sides of the equation by $(x-t)^{-\alpha}$ and integrating we have

$$\int_a^x (x-t)^{-\alpha}dt\int_a^t (t-s)^{\alpha-1}\varphi(s)ds=\Gamma(\alpha)\int_a^x (x-t)^{-\alpha}f(t)dt$$

We interchanging the order of integration in the left-hand side we arrive at :

$$\int_a^x \varphi(s)ds\int_s^x (x-t)^{\alpha-1}dt=\Gamma(\alpha)\int_a^x (x-t)^{-\alpha}f(t)dt \tag{2.7}$$

We put $t=s+\tau(x-s)$ we have :

$$\int_s^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt=\int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha}d\tau$$

$$= B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha)$$

We by substituting in (2.7) we have :

$$\int_a^x \varphi(s)ds = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} f(t)dt$$

Hence after differentiation we have :

$$\varphi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)dt \tag{2.8}$$

So, if (2.5) has a solution ,this solution is necessarily given by (2.8) and therefore it is unique.

We assum that the equation (2.5) has a solution $\varphi(x) \in L^1(a,b)$.

We define the function

$$f_{1-\alpha}(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} f(t)dt$$

By differentiating both sides with respect to x , we obtain :

$$\frac{d}{dx} f_{1-\alpha}(x) = \varphi(x)$$

Furthermore, evaluting at $x = a$, we get

$$I^{1-\alpha} f(a) = \frac{1}{\Gamma(1-\alpha)} \int_a^a (a-t)^{-\alpha} f(t)dt = 0$$

Thus, the necessity of the condition is establish

$\underline{(\Leftarrow)}$:

We have $I_{a^+}^{1-\alpha} f(x) \in AC(a,b)$ and satisfies $I_{a^+}^{1-\alpha} f(a) = 0$. We want to show that $\varphi(x) \in L^1(a,b)$

Since $I_{a^+}^{1-\alpha} f(x) \in AC([a,b])$, we have

$$\left(I_{a^+}^{1-\alpha} f\right)'(x) = \frac{d}{dx} I_{a^+}^{1-\alpha} f(x) = \varphi(x) \in L^1(a,b) \tag{2.9}$$

So, $\varphi(x)$ exists olmost everywhere and belongs to $L^1(a,b)$

We substitute (2.9) in (2.5) we get :

$$\frac{1}{\Gamma(\alpha)} \frac{d}{dx} \int_a^x (x-t)^{\alpha-1} \left(I_{a^+}^{1-\alpha} f\right)'(t)dt = g(x)$$

So,

$$g(x) = I_{a^+}^{\alpha} \left(I_{a^+}^{1-\alpha} f\right)'(x) \Leftrightarrow \left(I_{a^+}^{1-\alpha} f\right)'(x) = D_{a^+}^{\alpha} g(x)$$

Hence ,

$$\left(I_{a^+}^{1-\alpha}f\right)'(x)\in L^1(a,b)\Rightarrow\left(I_{a^+}^{1-\alpha}f\right)'(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x(x-t)^{-\alpha}g(t)dt=\left(I_{a^+}^{1-\alpha}g\right)'(x)$$

i.e.

$$\left(I_{a^+}^{1-\alpha}f\right)'(x)=\left(I_{a^+}^{1-\alpha}g\right)'(x)$$

Thus,

$$\left(I_{a^+}^{1-\alpha}f\right)(x)=\left(I_{a^+}^{1-\alpha}g\right)(x)+C \tag{2.10}$$

and we have as $x=a$

$$\left(I_{a^+}^{1-\alpha}f\right)(a)=0 \qquad \left(I_{a^+}^{1-\alpha}g\right)(a)=0 \qquad \text{because } \left(I_{a^+}^{1-\alpha}f\right)'(x)\in L^1(a,b)$$

Which implies that $C=0$ Hence by (2.10)

$$\left(I_{a^+}^{1-\alpha}f-I_{a^+}^{1-\alpha}g\right)=0\Rightarrow I_{a^+}^{1-\alpha}(f-g)=0\Rightarrow f-g=0$$

So,

$$f=g$$

This implies the choice:

$$\varphi(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x(x-t)^{-\alpha}f(t)dt$$

We must check that $\varphi(x)\in L^1(a,b)$. Since $I^{1-\alpha}f(x)$ is absolutely continuous, its derivative $\varphi(x)$ exists almost everywhere and belongs to $L^1(a,b)$. Thus, the solution $\varphi(x)$ exists in $L^1(a,b)$, proving that the equation is solvable. □

Lemma 2.1. *If $f(x)\in AC([a,b])$ then $I_{a^+}^{1-\alpha}f(x)\in AC([a,b])$ and*

$$I_{a^+}^{1-\alpha}f(x)=\frac{1}{\Gamma(2-\alpha)}\left(f(a)(x-a)^{1-\alpha}+\int_a^x f'(t)(x-t)^{1-\alpha}dt\right)$$

Proof. We have :

$$f\in AC(a,b) \text{ i.e } f(t)=f(a)+\int_a^t f'(s)ds \tag{2.11}$$

Substituting (2.11) into (2.6), we obtain:

$$I_{a^+}^{1-\alpha}f(x)=\frac{1}{\Gamma(1-\alpha)}\int_a^x(x-t)^{-\alpha}\left[f(a)+\int_a^t f'(s)ds\right]dt$$

Now we split the integral :

$$I_{a^+}^{1-\alpha}f(x)=\frac{f(a)}{\Gamma(1-\alpha)}\int_a^x(x-t)^{-\alpha}dt+\frac{1}{\Gamma(1-\alpha)}\int_a^x(x-t)^{-\alpha}\left(\int_a^t f'(s)ds\right)dt$$

So, we compute the first integral directly, and for the second integral, we apply fubini's Theorem(1.5):

$$I_{a^+}^{1-\alpha} f(x) = \frac{f(a)}{\Gamma(2-\alpha)}(x-a)^{1-\alpha} + \frac{1}{\Gamma(2-\alpha)}\int_a^x f'(s)(x-s)^{1-\alpha} ds$$

So:

$$I_{a^+}^{1-\alpha} f(x) = \frac{1}{\Gamma(2-\alpha)}\left(f(a)(x-a)^{1-\alpha} + \int_a^x f'(s)(x-t)^{1-\alpha} dt\right)$$

This function is absolutely continuous because: $f' \in L^1(a,b)$ and $(x-t)^{1-\alpha} \in L^1(a,b)$ for $0 < \alpha < 1$ and $(x-a)^{1-\alpha}$ is continuous.

Hence

$$I_{a^+}^{1-\alpha} f(x) \in AC(a,b)$$

□

Proposition 2.1. *Let* $\alpha > 0$ *,*$\beta > 0$ *et* $f \in L^1[a,b]$ *, Then*

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f = I_{a^+}^{\beta} I_{a^+}^{\alpha} f = I_{a^+}^{\alpha+\beta} f \tag{2.12}$$

Proof.

$$\begin{aligned} I_{a^+}^{\alpha} I_{a^+}^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_a^\tau (\tau-s)^{\beta-1} f(s) ds\right) d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \int_a^\tau (x-\tau)^{\alpha-1}(\tau-s)^{\beta-1} f(s) ds d\tau \end{aligned}$$

We change of integration order we get :

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(s)\int_s^x (x-\tau)^{\alpha-1}(\tau-s)^{\beta-1} d\tau ds$$

We use this change $u = \frac{\tau-s}{x-s}, du = \frac{d\tau}{x-s}, \tau = (x-s)u + s, u : 0 \longrightarrow 1$

$$\begin{aligned} I_{a^+}^{\alpha} I_{a^+}^{\beta} f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(s)(x-s)^{\alpha+\beta-1}\int_s^x (1-u)^{\alpha-1} u^{\beta-1} du ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(s)(x-s)^{\alpha+\beta-1} B(\alpha,\beta) du ds, \qquad (by 1.9) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_a^x f(s)(x-s)^{\alpha+\beta-1} ds \\ &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^x (x-s)^{\alpha+\beta-1} f(s) ds \\ &= I_{a^+}^{\alpha+\beta} f(x). \end{aligned}$$

□

2.1.1 Embedding of the Riemann-Liouville Integral

Theorem 2.2. *The Rieman-liouville integral* $I^{\alpha}_{a^+}f, I^{\alpha}_{b^-}f \;:\; L^p(a,b) \longrightarrow L^p(a,b)$ *is well defined and continous for* $1 \leq p \leq \infty$ *and we have :*

$$\|I^{\alpha}_{a^+}f\|_{L^p(a,b)} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p(a,b)} \tag{2.13}$$

$$\|I^{\alpha}_{b^-}f\|_{L^p(a,b)} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p(a,b)}. \tag{2.14}$$

Proof. We have :

$$|I^{\alpha}_{a^+}f(x)| = \left|\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt\right|$$

By Hölder's inequality(1.1), we obtain:

$$\begin{aligned}
|I^{\alpha}_{a^+}f(x)| &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x |f(t)|(x-t)^{\frac{\alpha-1}{p}}dt\right)\left(\int_a^x (x-t)^{\alpha-1}dt\right)^{\frac{1}{p'}} \\
&\leq \frac{(b-a)^{\alpha\left(\frac{p-1}{p}\right)}}{\Gamma(\alpha)\alpha^{\frac{p-1}{p}}}\left(\int_a^x |f(t)|^p(x-t)^{\alpha-1}dt\right)^{\frac{1}{p}}
\end{aligned}$$

Then ,

$$\|I^{\alpha}_{a^+}f\|^p_{L^p(a,b)} \leq \frac{(b-a)^{\alpha(p-1)}}{\Gamma^p(\alpha)\alpha^{p-1}}\int_a^b\int_a^x (x-t)^{\alpha-1}|f(t)|^p dtdx$$

Hence, by Fubini's Theorem (1.5), we obtain:

$$\begin{aligned}
\|I^{\alpha}_{a^+}f\|^p_{L^p(a,b)} &\leq \frac{(b-a)^{\alpha(p-1)}}{\Gamma^p(\alpha)\alpha^{p-1}}\int_a^b |f(t)|^p\int_t^b (x-t)^{\alpha-1}dxdt \\
&\leq \frac{(b-a)^{\alpha(p-1)}}{\Gamma^p(\alpha)\alpha^{p-1}}\frac{(b-a)^{\alpha}}{\alpha}\|f\|^p_{L^p(a,b)} \\
&= \frac{(b-a)^{\alpha p}}{\Gamma^p(\alpha+1)}\|f\|^p_{L^p(a,b)}
\end{aligned}$$

Hence ,

$$\|I^{\alpha}_{a^+}f\|_{L^p(a,b)} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p(a,b)}$$

- Same idea for (2.14) □

Theorem 2.3.
1. *If* $\alpha \in \left]0,\frac{1}{p}\right[$ *the operator* $I^{\alpha}_{a^+}, I^{\alpha}_{b^-} : L^p(a,b) \to L^q(a,b)$ *are continuous for every* $q \in [1, p^*_{\alpha}]$ *where* $p^*_{\alpha} = \frac{p}{1-\alpha p}$ *the fractional critical exponent.*
2. *If* $\alpha \in \left]\frac{1}{p},1\right[$ *the operator* $I^{\alpha}_{a^+}, I^{\alpha}_{b^-} : L^p(a,b) \to L^{\infty}(a,b)$ *are continuous*

Proof. 1. Let $\alpha \in \left]0, \frac{1}{p}\right[\subset (0,1)$ we show that :

$$\|I^{\alpha}_{a^+}u\|_{L^q(a,b)} \leq C\|u\|_{L^p(a,b)} \qquad \text{where } q \in [1, p^*_\alpha]$$

namely $I^{\alpha}_{a^+}$ is bounded from L^p , $1 \leq p \leq \frac{1}{\alpha}$ into L^r where $1 \leq r < p^*_\alpha$
We set $\epsilon = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{q}\right)$ then we have :

$$\Gamma(\alpha) I^{\alpha}_{a^+}u(x) = \int_a^x (x-t)^{\alpha-1}u(t)dt$$

By Hölder's inequality, we obtain:

$$|I^{\alpha}_{a^+}u(x)| \leq \frac{1}{\Gamma(\alpha)}\int_a^x \left(|u(t)|^{\frac{p}{r}}(x-t)^{\epsilon-\frac{1}{r}}\right)|u(t)|^{1-\frac{p}{r}}(x-t)^{\epsilon-\frac{1}{p'}}dt$$

Then, applying the generalized Hölder's inequality (1.2) ,where $p_1 = r$, $p_2 = \frac{rp}{r-p}$, $p_3 = p'$ we obtain:

$$\begin{aligned}
|I^{\alpha}_{a^+}u(x)| &\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x |u(t)|^p(x-t)^{r\epsilon-1}dt\right)^{\frac{1}{r}}\left(\int_a^x |u(t)|^p dt\right)^{\frac{1}{p}-\frac{1}{r}}\left(\int_a^x (x-t)^{p'\epsilon-1}dt\right)^{\frac{1}{p'}} \\
&\leq \frac{(b-a)^{\epsilon}}{\Gamma(\alpha)(\epsilon p')^{\frac{1}{p'}}}\|u(x)\|^{1-\frac{p}{r}}_{L^p(a,b)}\left(\int_a^x |u(t)|^p(x-t)^{r\epsilon-1}dt\right)^{\frac{1}{r}}
\end{aligned}$$

Hence ,

$$\begin{aligned}
\|I^{\alpha}_{a^+}u(x)\|_{L^r(a,b)} &= \left(\int_a^b |I^{\alpha}_{a^+}u(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq \frac{(b-a)^{\epsilon}}{\Gamma(\alpha)(\epsilon p')^{\frac{1}{p'}}}\|u(x)\|^{1-\frac{p}{r}}_{L^p(a,b)}\left(\int_a^b\int_a^x |u(t)|^p(x-t)^{r\epsilon-1}dtdx\right)^{\frac{1}{r}}
\end{aligned}$$

By Fubini's Theorem (1.5), we obtain:

$$\begin{aligned}
\|I^{\alpha}_{a^+}u(x)\|_{L^r(a,b)} &\leq \frac{(b-a)^{\epsilon}}{\Gamma(\alpha)(\epsilon p')^{\frac{1}{p'}}}\|u(x)\|^{1-\frac{p}{r}}_{L^p(a,b)}\left(\int_a^b |u(t)|^p dt\int_t^b (x-t)^{r\epsilon-1}dx\right)^{\frac{1}{r}} \\
&\leq \frac{(b-a)^{2\epsilon}}{\Gamma(\alpha)(\epsilon p')^{\frac{1}{p'}} r\epsilon^{\frac{1}{r}}}\|u(x)\|^{1-\frac{p}{r}}_{L^p(a,b)}\|u(x)\|^{\frac{p}{r}}_{L^p(a,b)} \\
&\leq C\|u(x)\|_{L^p(a,b)} \qquad \text{where } C = \frac{(b-a)^{2\epsilon}}{\Gamma(\alpha)(\epsilon p')^{\frac{1}{p'}} r\epsilon^{\frac{1}{r}}}
\end{aligned}$$

2. Let $\frac{1}{p} < \alpha < 1$ we show that :

$$\|I^{\alpha}_{a^+}u(x)\|_{L^\infty(a,b)} \leq \|u\|_{L^p(a,b)}$$

$$|I_{a^+}^\alpha u(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}|u(t)|dt$$

By Hölder's inequality (1.2), we obtain:

$$\begin{aligned}
|I_{a^+}^\alpha u(x)| &\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x (x-t)^{(\alpha-1)\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}} \left(\int_a^x |u(t)|^p dt\right)^{\frac{1}{p}} \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}} (x-a)^{\alpha-\frac{1}{p}} \|u\|_{L^p(a,b)} \\
&\leq \frac{c(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)} \|u\|_{L^p(a,b)}\ , \qquad \text{where} \quad c = \left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}
\end{aligned}$$

- Same idea for $I_{b^-}^\alpha u(x)$ □

Theorem 2.4. *Let $0 < \alpha < 1$ and $p \geq 1$. Then the operators: $I_{a^+}^\alpha$,$I_{b^-}^\alpha$:$L^p \longrightarrow L^p$ are compact.*

Proof. Since $L^p(a,b)$ is complete, it suffices to show that $I^\alpha(B_1)$ is precompact (see Theorem 1.9). For this purpose, we will apply Theorem 1.11.

We distinguish two cases:

- **Case** $p = 1$

We define

$$\Omega_h = (a+h, b-h), \qquad h > 0$$

and we consider the unit ball

$$B_1 = \{u \in L^1(a,b) \ : \ \|u\|_{L^1(a,b)} \leq 1\}$$

Let $u \in B_1$ we have

$$\begin{aligned}
\int_{a+h}^{b-h} |I_{a^+}^\alpha u(x+h) - I_{a^+}^\alpha u(x)|dx &= \frac{1}{\Gamma(\alpha)} \int_{a+h}^{b-h} \left|\left(\int_a^{x+h} (x+h-t)^{\alpha-1}u(t)dt - \int_a^x (x-t)^{\alpha-1}u(t)dt\ dx\right.\right. \\
&\quad \left.\left. - \int_a^x (x-t)^{\alpha-1}u(t)dt\right)\right| dx \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_{a+h}^{b-h} \int_a^x |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}||u(t)|dtdx\right. \\
&\quad \left. + \int_{a+h}^{b-h} \int_x^{x+h} |(x+h-t)^{\alpha-1}||u(t)|dtdx\right)
\end{aligned}$$

We denoted :

$$\int_{a+h}^{b-h} |I_{a^+}^\alpha u(x+h) - I_{a^+}^\alpha u(x)|dx \leq \frac{1}{\Gamma(\alpha)} (A_1 + A_2) \tag{2.15}$$

Let

$$\begin{aligned} A_1 &= \int_{a+h}^{b-h}\int_{a}^{a+h}|(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}||u(t)|dtdx \\ &\quad+\int_{a+h}^{b-h}\int_{a+h}^{x}|(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}||u(t)|dtdx \end{aligned}$$

By Fubini's Theorem (1.5), we obtain:

$$\begin{aligned} A_1 = \int_{a}^{a+h}|u(t)|\int_{a+h}^{b-h}\Big((x-t)^{\alpha-1}-(x+h-t)^{\alpha-1}\Big)\,dxdt \\ +\int_{a+h}^{b-h}|u(t)\int_{t}^{b-h}\Big((x-t)^{\alpha-1}-(x+h-t)^{\alpha-1}\Big)\,dxdt \end{aligned}$$

After computing the inner integral in both terms, we obtain:

$$\begin{aligned} A_1 = \frac{1}{\alpha}\int_{a}^{a+h}|u(t)|(b-h-t)^{\alpha}-(b-t)^{\alpha}+(a+2h-t)^{\alpha}-(a+h-t)^{\alpha}dt \\ +\frac{1}{\alpha}\int_{a+h}^{b-h}|u(t)|(b-h-t)^{\alpha}-(b-t)^{\alpha}+h^{\alpha}dt \end{aligned}$$

Since the function

$$f(t)=(b-h-t)^{\alpha}-(b-t)^{\alpha}+(a+2h-t)^{\alpha}-(a+h-t)^{\alpha}$$

By lemma 1.3 the function f is increasing , we bound it from above by evaluating it at $t=a+h$. In the second term, the function

$$f(t)=(b-h-t)^{\alpha}-(b-t)^{\alpha}+h^{\alpha}$$

is decreasing, so we bound it from above by evaluating it at $t=a+h$. We then obtain:

$$A_1 \leq \frac{(b-a-2h)^{\alpha}-(b-a-h)^{\alpha}+h^{\alpha}}{\alpha}\|u\|_{L^1(a,b)}+\frac{(b-a-2h)^{\alpha}-(b-a-h)^{\alpha}+h^{\alpha}}{\alpha}\|u\|_{L^1(a,b)}$$

$$A_1 \leq 2\frac{(b-a-2h)^{\alpha}-(b-a)^{\alpha}+h^{\alpha}}{\alpha} \tag{2.16}$$

Let

$$A_2=\int_{a+h}^{b-h}\int_{x}^{x+h}|(x+h-t)^{\alpha-1}||u(t)|dtdx$$

We perform the change of variable $s=t-x$, and by adjusting the bounds of the integral accordingly from x to 0 and from $x+h$ to h we obtain

$$A_2=\int_{a+h}^{b-h}\int_{0}^{h}(h-s)^{\alpha-1}|u(s+x)|dsdx$$

By fubini theorem (1.5) we obtain :

$$\begin{aligned} A_2 &= \int_{a+h}^{b-h} |u(s+x)| \int_0^h (h-s)^{\alpha-1} ds dx \\ &= \frac{h^\alpha}{\alpha} \int_{a+h}^{b-h} |u(s+x)| dx \end{aligned}$$

Hence ,

$$A_2 \leq \frac{h^\alpha}{\alpha} \|u\|_{L^1(a,b)} \tag{2.17}$$

Then by (2.16) and (2.17) we obtain :

$$\int_{a+h}^{b-h} |I_{a^+}^\alpha u(x+h) - I_{a^+}^\alpha u(x)| dx \leq \frac{1}{\Gamma(\alpha+1)} \left(2(b-a-2h)^\alpha - 2(b-a)^\alpha + 3h^\alpha\right)$$

So ,

$$\lim_{h \longrightarrow 0} \int_{a+h}^{b-h} |I_{a^+}^\alpha u(x+h) - I_{a^+}^\alpha u(x)| dx = 0$$

By the Dominated Convergence Theorem (1.6), we obtain:

$$\int_a^b |I_{a^+}^\alpha u(x)| dx = \lim_{h \longrightarrow 0} \int_{a+h}^{b-h} |I_{a^+}^\alpha u(x)| dx$$

So , given $\varepsilon > 0$, there is $\delta > 0$ such that :

$$\int_{(a,b)\setminus[a+h,b+h]} |I_{a^+}^\alpha u(x)| dx < \varepsilon \qquad 0 < h < \delta$$

Therefore ,by theorem (1.11) the $I_{a^+}^\alpha(B_1)$ is precompact in $L^p(a,b)$

Hence by theorem (1.9) the $I_{a^+}^\alpha(B_1)$ is compact

- **Case** $p > 1$

Let

$$u \in B_p = \{u \in L^p(a,b) : \quad \|u\|_{L^p(a,b)} \leq 1\}$$

We have :

$$\int_{a+h}^{b-h} |I_{a^+}^\alpha u(x+h) - I_{a^+}^\alpha u(x)|^p dx =$$

$$\int_{a+h}^{b-h} \left| \frac{1}{\Gamma(\alpha)} \left(\int_a^{x+h} (x+h-t)^{\alpha-1} u(t) dt - \int_a^x (x-t)^{\alpha-1} u(t) dt \right) \right|^p dx$$

By (1.6) ,we obtain :

$$\begin{aligned} \int_{a+h}^{b-h} |I_{a^+}^\alpha u(x+h) - I_{a^+}^\alpha u(x)|^p dx &\leq \frac{2^{p-1}}{\Gamma^p(\alpha)} \int_{a+h}^{b-h} \left(\int_a^x |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}||u(t)| dt \right)^p dx \\ &+ \frac{2^{p-1}}{\Gamma^p(\alpha)} \int_{a+h}^{b-h} \left(\int_x^{x+h} |(x+h-t)^{\alpha-1}||u(t)| dt \right)^p dx \end{aligned}$$

Then,

$$\int_{a+h}^{b-h} |I_{a^+}^{\alpha} u(x+h) - I_{a^+}^{\alpha} u(x)|^p \, dx = \frac{2^{p-1}}{\Gamma^p(\alpha)}(A_1 + A_2)$$

We have:

$$\begin{aligned} A_1 &= \int_{a+h}^{b-h} \left(\int_a^x \left| (x+h-t)^{\alpha-1} - (x-t)^{\alpha-1} \right| |u(t)| \, dt \right)^p dx \\ &= \int_{a+h}^{b-h} \left(\int_a^x \left[(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1} \right] |u(t)| \, dt \right)^p dx \end{aligned}$$

Furthermore, we split A_1as:

$$\begin{aligned} A_1 = &\int_{a+h}^{b-h} \left(\int_a^{a+h} \left[(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1} \right] |u(t)| \, dt \right)^p dx \\ &+ \int_{a+h}^{b-h} \left(\int_{a+h}^{x} \left[(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1} \right] |u(t)| \, dt \right)^p dx \end{aligned} \tag{2.18}$$

We apply Hölder's inequality (1.1), and we obtain:

$$\begin{aligned} A_1 \leq &\int_{a+h}^{b-h} \Bigg[\left(\int_a^{a+h} [(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}] |u(t)|^p \, dt \right) \\ &\left(\int_a^{a+h} [(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}] dt \right)^{\frac{P}{p'}} \Bigg] dx \\ &+ \int_{a+h}^{b-h} \Bigg[\left(\int_{a+h}^{x} [(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}] |u(t)|^p \, dt \right) \\ &\left(\int_{a+h}^{x} [(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}] dt \right)^{\frac{P}{p'}} \Bigg] dx \end{aligned}$$

Since

$$\begin{aligned} \left(\int_a^{a+h} [(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}] dt \right)^{\frac{P}{p'}} &= \left(\frac{2(x-a)^{\alpha} - (x-a-h)^{\alpha} - (x+h-a)^{\alpha}}{\alpha} \right)^{p-1} \\ \left(\int_{a+h}^{x} [(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}] dt \right)^{\frac{P}{p'}} &= \left(\frac{(x-a-h)^{\alpha} - (x-a)^{\alpha} + h^{\alpha}}{\alpha} \right)^{p-1} \end{aligned}$$

Hence,

$A_1 \leq$

$$\left(\frac{2(x-a)^{\alpha} - (x-a-h)^{\alpha} - (x+h-a)^{\alpha}}{\alpha} \right)^{p-1} \int_{a+h}^{b-h} \int_a^{a+h} [(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}] |u(t)|^p dt dx$$

$$+ \left(\frac{(x-a-h)^{\alpha} - (x-a)^{\alpha} + h^{\alpha}}{\alpha} \right)^{p-1} \int_{a+h}^{b-h} \int_{a+h}^{x} [(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}] |u(t)|^p dt dx$$

By Fubini's Theorem, we obtain:

$A_1 \leq$

$$\left(\frac{2(x-a)^\alpha-(x-a-h)^\alpha-(x+h-a)^\alpha}{\alpha}\right)^{p-1}\int_a^{a+h}|u(t)|^p\int_{a+h}^{b-h}[(x-t)^{\alpha-1}-(x+h-t)^{\alpha-1}]dxdt$$

$$+\left(\frac{(x-a-h)^\alpha-(x-a)^\alpha+h^\alpha}{\alpha}\right)^{p-1}\int_{a+h}^{b-h}|u(t)|^p\int_t^{b-h}[(x-t)^{\alpha-1}-(x+h-t)^{\alpha-1}]dxdt$$

Since,

$$\int_a^{a+h}|u(t)|^p\int_{a+h}^{b-h}[(x-t)^{\alpha-1}-(x+h-t)^{\alpha-1}]dxdt=\frac{1}{\alpha}\int_a^{a+h}(b-h-t)^\alpha-(a+h-t)^\alpha-(b-t)^\alpha$$
$$+(a+2h-t)^\alpha|u(t)|^pdt$$

$$\int_{a+h}^{b-h}|u(t)|^p\int_t^{b-h}[(x-t)^{\alpha-1}-(x+h-t)^{\alpha-1}]dxdt=\frac{1}{\alpha}\int_{a+h}^{b-h}(b-h-t)^\alpha-(b-t)^\alpha+h^\alpha|u(t)|^pdt$$

It has been previously shown that the function

$$f(t)=(b-h-t)^\alpha-(a+h-t)^\alpha-(b-t)^\alpha+(a+2h-t)^\alpha$$

by lemma 1.3 the function f is increasing,we bound it from above by $t=a+h$ and in the second term has been previously shown that the function

$$f(t)=(b-h-t)^\alpha-(b-t)^\alpha+h^\alpha$$

is decreasing,we bound it from above by $t=a+h$.We then obtain :

$$A_1\leq\left(\frac{2(b-a)^\alpha-(b-a-h)^\alpha-(b+h-a)^\alpha}{\alpha}\right)^{p-1}\cdot\frac{(b-a-2h)^\alpha-(b-a-h)^\alpha+h^\alpha}{\alpha}\|u\|^p_{L^p(a,b)}$$
$$+\left(\frac{(b-a-h)^\alpha-(b-a)^\alpha+h^\alpha}{\alpha}\right)^{p-1}\cdot\frac{(b-a-2h)^\alpha-(b-a-h)^\alpha+h^\alpha}{\alpha}\|u\|^p_{L^p(a,b)} \tag{2.19}$$

Now we estimate A_2. We apply Hölder's inequality and obtain:

$$\int_{a+h}^{b-h}\left(\int_x^{x+h}(x+h-t)^{\alpha-1}|u(t)|dt\right)^p dx\leq\int_{a+h}^{b-h}\left[\left(\int_x^{x+h}(x+h-t)^{\alpha-1}|u(t)|^pdt\right)^{\frac{1}{p}}\right.$$
$$\left.\left(\int_x^{x+h}(x+h-t)^{\alpha-1}dt\right)^{\frac{1}{p'}}\right]^p dx$$

Hence,

$$A_2 \leq \frac{h^{\frac{\alpha p}{p'}}}{\alpha^{\frac{p}{p'}}} \int_{a+h}^{b-h} \int_{x}^{x+h} (x+h-t)^{\alpha-1} |u(t)|^p dt dx$$

We perform the change of variable $s = t - x$. Then, when $t \to x$, we have $s \to 0$, and when $t \to x + h$, we have $s \to h$ we obtain :

$$A_2 \leq \frac{h^{\frac{\alpha p}{p'}}}{\alpha^{\frac{p}{p'}}} \int_{0}^{h} (h-t)^{\alpha-1} \int_{a+h}^{b-h} |u(x+s)|^p dx dt$$

Hence,

$$A_2 \leq \frac{h^{\alpha p}}{\alpha^p} \|u\|^p_{L^p(a,b)} \tag{2.20}$$

So,by (2.19) and (2.20) we obtain :

$$\lim_{h \longrightarrow 0} \int_{a+h}^{b-h} |I^\alpha_{a^+} u(x+h) - I^\alpha_{a^+} u(x)|^p dx = 0$$

So, By the Dominated Convergence Theorem (1.6), we obtain::

$$\int_a^b |{}_a I^\alpha_x u(x)|^p \, dx = \lim_{h \to 0} \int_{a+h}^{b-h} |{}_a I^\alpha_x u(x)|^p \, dx$$

next, given $\epsilon > 0$, there is $\delta > 0$ such that:

$$\int_{(a,b)\setminus(a+h,b-h)} |{}_a I^\alpha_x u(x)|^p \, dx < \epsilon \quad 0 < h < \delta$$

Therefore, by theorem (1.11) $I^\alpha_{a^+}(B_p)$ is precompact in $L^p(a,b)$, hence, by theorem (1.9) $I^\alpha_{a^+}(B_p)$ is compact In the same way , we can show that$I^\alpha_{b^-} : L^p(a,b) \longrightarrow L^p(a,b)$ is compact □

Theorem 2.5. *If* $\alpha \in \left(0, \frac{1}{p}\right)$ *the operator* $I^\alpha_{a^+}, I^\alpha_{b^-} : L^p(a,b) \to L^q(a,b)$ *are compact where* $q \in [1, p^*_\alpha[$

Proof. Let us consider the unit ball of $L^p(a,b)$:

$$B_p = \left\{ u \in L^p(a,b) : \|u\|_{L^p(a,b)} \leq 1 \right\}$$

We define $\Omega_h = (a+h, b-h)$ for $h > 0$. Let $1 \leq q < p^*_\alpha$. By the interpolation inequality(1.4), for any $0 \leq \theta \leq 1$, we obtain:

$$\|I^\alpha_{a^+} u(x+h) - I^\alpha_{a^+} u(x)\|_{L^q(\Omega_h)} \leq \|I^\alpha_{a^+} u(x+h) - I^\alpha_{a^+} u(x)\|^\theta_{L^1(\Omega_h)} \|I^\alpha_{a^+} u(x+h) - I^\alpha_{a^+} u(x)\|^{1-\theta}_{L^{p^*_\alpha}(\Omega_h)}$$

Next, we estimate the second term as follows:

$$\begin{aligned} \|I^\alpha_{a^+} u(x+h) - I^\alpha_{a^+} u(x)\|_{L^{p^*_\alpha}(\Omega_h)} &\leq \|I^\alpha_{a^+} u(x+h)\|_{L^{p^*_\alpha}(\Omega_h)} + \|I^\alpha_{a^+} u(x)\|_{L^{p^*_\alpha}(\Omega_h)} \\ &\leq 2\|I^\alpha_{a^+} u\|_{L^{p^*_\alpha}(a,b)} \end{aligned}$$

Thus,

$$\|I_{a^+}^{\alpha}u(x+h)-I_{a^+}^{\alpha}u(x)\|_{L^q(\Omega_h)} \leq \|I_{a^+}^{\alpha}u(x+h)-I_{a^+}^{\alpha}u(x)\|_{L^1(\Omega_h)}^{\theta}(2\|I_{a^+}^{\alpha}u\|_{L^{p_\alpha^*}(a,b)})^{1-\theta}$$

By Corollary(2.4), since $u \in B_p$, we have

$$\|I_{a^+}^{\alpha}u(x+h)-I_{a^+}^{\alpha}u(x)\|_{L^1(\Omega_h)} \to 0 \quad \text{as } h \to 0$$

Hence,

$$\|I_{a^+}^{\alpha}u(x+h)-I_{a^+}^{\alpha}u(x)\|_{L^q(\Omega_h)} \to 0 \quad \text{as } h \to 0$$

By Hölder's inequality (1.2), we obtain:

$$\begin{aligned}\|I_{a^+}^{\alpha}u\|_{L^q(E)} &\leq \left(\int_E |1|^{\frac{p_\alpha^*-q}{p_\alpha^*}}dx\right)^{1-\frac{q}{p_\alpha^*}}\left(\int_E |I_{a^+}^{\alpha}u(x)|^{\frac{p_\alpha^*}{q}}dx\right)^{\frac{q}{p_\alpha^*}} \qquad \text{where } E=(a,b)\backslash(a+h,b-h)\\ &\leq |2h|^{1-\frac{q}{p_\alpha^*}}\|I_{a^+}^{\alpha}u\|_{L^{p_\alpha^*}((a,b)\backslash(a+h,b-h))}\end{aligned}$$

By theorem (2.3)(1)

$$\|I_{a^+}^{\alpha}u\|_{L^q(E)} \leq C|2h|^{1-\frac{q}{p_\alpha^*}} \longrightarrow 0 \qquad \text{as } \ h \longrightarrow 0$$

Hence , by theorem (1.11) the $I_{a^+}^{\alpha}(B_p)$ is precompact in $L^q(a,b)$

Therefore,by theorem (1.9) the operator $I_{a^+}^{\alpha}: L^p(a,b) \to L^q(a,b)$ is compact.

- The same result holds for $I_{b^-}^{\alpha}$ by symmetry. □

Theorem 2.6. ***(Integration by part)****: Let $f \in L^p(a,b)$ and $g \in L^{p'}(a,b)$ such that $\frac{1}{p}+\frac{1}{p'} \leq 1+\alpha$, then we have*

$$\int_a^b f(x)I_{b^-}^{\alpha}g(x)dx = \int_a^b g(x)I_{a^+}^{\alpha}f(x)dx \tag{2.21}$$

Proof. By Theorem (2.3), the term $f(x)I_{b^-}^{\alpha}g(x)$ is well-defined. So

$$\begin{aligned}\int_a^b f(x)I_{b^-}^{\alpha}g(x)dx &= \frac{1}{\Gamma(\alpha)}\int_a^b f(x)\int_x^b (t-x)^{\alpha-1}g(t)dtdx\\ &= \frac{1}{\Gamma(\alpha)}\int_a^b\int_x^b (t-x)^{\alpha-1}f(x)g(t)dtdx\end{aligned}$$

using the Fubini's theorem we obtain:

$$\begin{aligned}\int_a^b f(x)I_{b^-}^{\alpha}g(x)dx &= \frac{1}{\Gamma(\alpha)}\int_a^b\int_a^t (t-x)^{\alpha-1}f(x)g(t)dtdx\\ &= \frac{1}{\Gamma(\alpha)}\int_a^b g(t)\int_a^t (t-x)^{\alpha-1}f(x)dxdt\\ &= \int_a^b g(t)\left(\frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}f(x)dx\right)dt\\ &= \int_a^b g(t)I_{a^+}^{\alpha}f(t)dt\end{aligned}$$

Hence

$$\int_a^b f(x) I_{b^-}^{\alpha} g(x) dx = \int_a^b g(t) I_{a^+}^{\alpha} f(t) dt$$

- By the same way for the right fractional integral □

2.2 Rieman Liouvill Fractional Derivatives

Definition 2.2. *Let* $0 < \alpha < 1$. *We define the left Riemann-Liouville fractional derivative* $D_{a^+}^{\alpha} f$ *of the function* f *as follows:*

$$(D_{a^+}^{\alpha} f)(x) = \frac{d}{dx}(I_{a^+}^{1-\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t) dt \tag{2.22}$$

And the right fractional derivative

$$(D_{b^-}^{\alpha} f)(x) = \frac{-d}{dx}(I_{b^-}^{1-\alpha} f)(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t) dt \tag{2.23}$$

Example 2.3. *for* $0 < \alpha < 1$,$\beta > 0$, *we put* $f(x) = (x-a)^{\beta-1}$, *according to the equation(2.22) we have*

$$(D_{a^+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right) \circ I_{a^+}^{1-\alpha} f(x) = \left(\frac{d}{dx}\right)\left(\frac{\Gamma(\beta)}{\Gamma(1-\alpha+\beta)}(x-a)^{\beta-\alpha}\right)$$

and

$$\left(\frac{d}{dx}\right)(x-a)^{\beta-\alpha} = (\beta-\alpha)(x-a)^{\beta-\alpha-1}$$

and on the other hand

$$\Gamma(1-\alpha+\beta) \quad = \quad (\beta-\alpha)\Gamma(\beta-\alpha), \qquad \Gamma(x+1) = x\Gamma(x)$$

therefore

$$\begin{aligned} (D_{a^+}^{\alpha} f)(x) &= \frac{\Gamma(\beta)}{\Gamma(1-\alpha+\beta)}(\beta-\alpha)(x-a)^{\beta-\alpha-1} \\ &= \frac{(\beta-\alpha)\Gamma(\beta)}{(\beta-\alpha)\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} \end{aligned}$$

Hence

$$(D_{a^+}^{\alpha} f)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} \tag{2.24}$$

same idea for the right-sided fractional derivative for $f(x) = (b-x)^{\beta-1}$ *we obtient :*

$$\left(D_{b^-}^{\alpha}(b-x)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1}$$

Example 2.4. *Now, we will have the derivative of order* α *of a constant function. According to the previous example, by taking* $\beta = 0$ *we obtain*

$$D_{a^+}^{\alpha}(x-a)^0 = D_{a^+}^{\alpha}1 = \frac{\Gamma(0+1)}{\Gamma(0-\alpha+1)}(x-a)^{0-\alpha}$$

$$D_{a^+}^{\alpha}1 = \frac{1}{\Gamma(\alpha+1)}(x-a)^{-\alpha}$$

The derivative of a constant in the sense of Riemann-Liouville is not zero
On the other hand, for $j = 1, 2, , [\alpha]+1,$

$$D_{a^+}^{\alpha}(x-a)^{\alpha-j}(x) = 0$$

- Same idea for the right-sided fractional derivative

2.2.1 Some properties of fractional derivatives

We present below some useful properties of the left and right fractional derivatives.

Proposition 2.2. *If* $0 < \alpha < 1$ *and* $f(x) \in AC(a,b)$ *then :*

$$\left(D_{a^+}^{\alpha}f\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(f(a)(x-a)^{-\alpha} + \int_a^x (x-t)^{-\alpha}f'(t)dt\right) \tag{2.25}$$

And

$$\left(D_{b^-}^{\alpha}f\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(f(b)(b-x)^{-\alpha} + \int_x^b (t-x)^{-\alpha}f'(t)dt\right) \tag{2.26}$$

Proof. •By (2.22) and (2.11) we obtain :

$$\begin{aligned} D_{a^+}^{\alpha}f(x) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \left(f(a) + \int_a^t f'(s)ds\right)(x-t)^{-\alpha}dt \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(f(a)\int_a^x (x-t)^{-\alpha}dt + \int_a^x\int_a^t f'(s)(x-t)^{-\alpha}dsdt\right) \\ &= \frac{1}{\Gamma(1-\alpha)}\left(f(a)(x-a)^{-\alpha} + \frac{d}{dx}\int_a^x\int_a^t f'(s)(x-t)^{-\alpha}dsdt\right) \end{aligned}$$

By fubini theorem (1.5) we obtian :

$$\begin{aligned} D_{a^+}^{\alpha}f(x) &= \frac{1}{\Gamma(1-\alpha)}\left(f(a)(x-a)^{-\alpha} + \frac{d}{dx}\int_a^x f'(s)\int_s^x (x-t)^{-\alpha}dtds\right) \\ &= \frac{1}{\Gamma(1-\alpha)}\left(f(a)(x-a)^{-\alpha} + \int_a^x f'(s)(x-s)^{-\alpha}ds\right) \end{aligned}$$

- By the same way for (2.26) □

Proposition 2.3. *Let* $0 < \alpha < 1, 0 < \beta < 1$*, we have the following properties:*

1. *if* $f, g \in L^1(a,b)$ *and* $\forall \lambda, \mu \in \mathbb{R}$ *Then the operator* $D^{\alpha}_{a^+}, D^{\alpha}_{b^-}$ *is linear*
$$D^{\alpha}_{a^+}(\lambda f + \mu g)(x) = \lambda D^{\alpha}_{a^+} f + \mu D^{\alpha}_{a^+} g$$
2. *if* $f(x) \in L^1([a,b])$*, then*
$$\left(D^{\alpha}_{a^+} I^{\alpha}_{a^+} f\right)(x) = f(x), \qquad \text{and} \qquad \left(D^{\alpha}_{b^-} I^{\alpha}_{b^-} f\right)(x) = f(x) \tag{2.27}$$
3. *if* $f(x) \in L^1([a,b])$*, where* $0 < \alpha < 1$*, and* $I^{1-\alpha}_{a^+} f \in AC([a,b])$ *we have*
$$\left(I^{\alpha}_{a^+} D^{\alpha}_{a^+} f\right)(x) = f(x) - \frac{I^{1-\alpha}_{a^+} f(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} \tag{2.28}$$
$$\left(I^{\alpha}_{b^-} D^{\alpha}_{b^-} f\right)(x) = f(x) - \frac{I^{1-\alpha} f(b)}{\Gamma(\alpha)}(b-x)^{\alpha-1} \tag{2.29}$$

Proof. Let $0 < \alpha < 1$

1. The proof is obvious.
2. - By (2.22) and the proposition (2.1) we get :
$$D^{\alpha}_{a^+} I^{\alpha}_{a^+} f(x) = \frac{d}{dx} I^{1-\alpha}_{a^+} I^{\alpha}_{a^+} f(x) = f(x)$$
   - By the same way for the right fractional derivative
3. We have
$$\begin{aligned} \left(I^{\alpha}_{a^+} D^{\alpha}_{a^+} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \left(\int_a^x (x-t)^{\alpha-1} D^{\alpha}_{a^+} f(t) dt \right) \\ &= \frac{d}{dx} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^x (x-t)^{\alpha} D^{\alpha}_{a^+} f(t) dt \right) \end{aligned}$$
   We put :
$$\begin{aligned} G(u) &= \frac{1}{\Gamma(\alpha+1)} \int_a^x (x-t)^{\alpha} D^{\alpha}_{a^+} f(t) dt \\ &= \frac{1}{\Gamma(\alpha+1)} \int_a^x (x-t)^{\alpha} \frac{d}{dt} I^{\alpha-1}_{a^+} f(t) dt \end{aligned}$$

Then, we apply integration by parts we obtain:

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}\int_a^x (x-t)^\alpha \frac{d}{dt} I_{a^+}^{\alpha-1} f(t)dt &= \frac{1}{\Gamma(\alpha+1)}\left[(x-t)^\alpha I_{a^+}^{1-\alpha}\right]_a^x + \alpha\int_a^x (x-t)^{\alpha-1} I_{a^+}^{1-\alpha} f(t)dt \\ &= -\frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha+1)}(x-a)^\alpha + I_{a^+}^{\alpha}(I_{a^+}^{1-\alpha} f(x))\end{aligned}$$

By (2.1) we obtain :

$$\frac{1}{\Gamma(\alpha+1)}\int_a^x (x-t)^\alpha \frac{d}{dt} I_{a^+}^{1-\alpha} f(t)dt = -\frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha+1)}(x-a)^\alpha + I^1 f(x)$$

So ,

$$\begin{aligned}\frac{d}{dx}\left(-\frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha+1)}(x-a)^\alpha + I_{a^+}^{1} f(x)\right) &= \frac{d}{dx} I_{a^+}^{1} f(x) - \frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha+1)}\frac{d}{dx}(x-a)^\alpha \\ &= f(x) - \frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}\end{aligned}$$

- By the same way for (2.29)

□

2.3 Caputo Fractional Derivatives

Definition 2.3. *Where $x \in [a,b]$, moreover ,the left and right caputo fractional derivative are defined as :*

$$^{c}D_{a^+}^{\alpha} u(x) = I_{a^+}^{1-\alpha} u'(x) \tag{2.30}$$

and

$$^{c}D_{b^-}^{\alpha} u(x) = -I_{b^-}^{1-\alpha} u'(x) \tag{2.31}$$

Remark 2.1. *If $u(x) \in \mathscr{D}(a,b)$ we have*

$$^{c}D_{a^+}^{\alpha} = D_{a^+}^{\alpha} \ , \ ^{c}D_{b^-}^{\alpha} = D_{b^-}^{\alpha}$$

CHAPTER 3

FRACTIONAL SOBOLEV SPACES

In this chapter, we introduce the concepts of fractional absolutely continuous functions and fractional Sobolev spaces. We also present some fundamental properties of these spaces, including embedding results, and establish several versions of the fractional integration by parts theorem.

3.1 Weak fractional derivative

Definition 3.1. • *Let* $u \in L^1_{Loc}(a,b)$ *and* $0 < \alpha < 1$, *if there is* $v \in L^1_{Loc}$ *such that :*

$$\int_a^b u(x) D^{\alpha}_{b^-}\varphi(x)dx = \int_a^b v(x)\varphi(x)dx, \qquad \forall \varphi \in \mathscr{D}(a,b) \tag{3.1}$$

Then v *is called weak left fractional derivative of* u *we denote by :*

$$\mathcal{D}^{\alpha}_{a^+}u(x) = v(x)$$

• *In the same way, if there is* $w \in L^1_{Loc}(a,b)$ *such that :*

$$\int_a^b u(x) D^{\alpha}_{a^+}\varphi(x)dx = \int_a^b w(x)\varphi(x)dx, \qquad \forall \varphi \in \mathscr{D}(a,b) \tag{3.2}$$

Then w *is called weak right fractional derivative of* u *we denote by :*

$$\mathcal{D}^{\alpha}_{b^-}u(x) = w(x)$$

Proposition 3.1. *If* $u \in C^1([a,b])$ *then the weak left and right fractional derivative* $\mathcal{D}^{\alpha}_{a^+}$,$\mathcal{D}^{\alpha}_{b^-}$ *of* u *coincides with the classical fractional derivative* $D^{\alpha}_{a^+}$,$D^{\alpha}_{b^-}$ *of* u.

Proof. We have

$$\int_a^b u(x)(D^\alpha_{b^-}\varphi)(x)dx \;=\; \int_a^b u(x)^c D^\alpha_{b^-}\varphi(x)dx \qquad \forall\varphi\in\mathscr{D}(a,b)$$

By the remark (2.1) and since ${}^cD^\alpha_{b^-}\varphi = -I^{1-\alpha}_{b^-}\frac{d}{dx}\varphi(x)$

$$\begin{aligned}\int_a^b u(x)(D^\alpha_{b^-}\varphi)(x)dx &= \int_a^b u(x)^c D^\alpha_{b^-}\varphi(x)dx = -\int_a^b u(x)I^{1-\alpha}_{b^-}\varphi'(x)dx\\ &= -\int_a^b I^{1-\alpha}_{a^+}u(x)\varphi'(x)dx \qquad (by(2.6))\\ &= \int_a^b v(x)\varphi(x)dx \qquad \forall\varphi\in\mathscr{D}(a,b)\end{aligned}$$

Therefore: $\mathcal{D}^\alpha_{a^+}u(x)=v(x)=\frac{d}{dx}I^{1-\alpha}_{a^+}u(x)$
where $\frac{d}{dx}$ is understood in the weak sense
Hence,

$$D^\alpha_{a^+}=\mathcal{D}^\alpha_{a^+} \qquad D^\alpha_{b^-}=\mathcal{D}^\alpha_{b^-}$$

□

Remark 3.1. *The weak fractional deravitive $\mathcal{D}^\alpha_{a^+}$,$\mathcal{D}^\alpha_{b^-}$ are lineair*

3.2 Spaces$AC^{\alpha,p}_{a^+}(a,b)$ and $AC^{\alpha,p}_{b^-}(a,b)$

Definition 3.2. *Let $0<\alpha<1$ and $1\le p<\infty$ we define the space $AC^{\alpha,p}_{a^+}(a,b)$,$AC^{\alpha,p}_{b^-}(a,b)$ as follow*

$$AC^{\alpha,p}_{a^+}(a,b)=\left\{f/\exists\varphi\in L^1(a,b): f(x)=\frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I^\alpha_{a^+}\varphi(x)\right\} \quad where\ c=I^{1-\alpha}_{a^+}f(a) \tag{3.3}$$

and

$$AC^{\alpha,p}_{b^-}(a,b)=\left\{g/\exists\psi\in L^1(a,b): g(x)=\frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1}+I^\alpha_{b^-}\psi(x)\right\} \quad where\ d=I^{1-\alpha}_{b^-}g(b) \tag{3.4}$$

Proposition 3.2. *Let $0<\alpha<1$, $f\in L^1(a,b)$ then $D^\alpha_{a^+}f\in L^1(a,b)$ if and only if*

$$f(x)=\frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I^\alpha_{a^+}\varphi(x) \qquad x\in[a,b]\ a.e \tag{3.5}$$

In this case , $c=\left(I^{1-\alpha}_{a^+}f\right)(a)$ and $\varphi(x)=\left(D^\alpha_{a^+}f\right)(x)$

Proposition 3.3. *Let $0<\alpha<1$, $g\in L^1(a,b)$ then $D^\alpha_{b^-}g\in L^1(a,b)$ if and only if*

$$g(x)=\frac{d}{\Gamma(\alpha)}(b-x)^{\alpha-1}+I^\alpha_{b^-}\psi(x) \qquad x\in[a,b]\ a.e \tag{3.6}$$

In this case , $d=\left(I^{1-\alpha}_{b^-}g\right)(b)$ and $\psi(x)=\left(D^\alpha_{b^-}g\right)(x)$

3.3 Spaces $E_L^{\alpha,p}(a,b)$ and $E_R^{\alpha,p}(a,b)$

• Let $\alpha \in (0,1)$. We define the fractional Sobolev-type spaces associated with the left and right Riemann–Liouville derivatives as follows:

Definition 3.3. *The spaces $E_L^{\alpha,p}(a,b)$ and $E_R^{\alpha,p}(a,b)$ are defined by:*

$$E_L^{\alpha,p}(a,b) := \left\{ u \in L^p(a,b) \mid \mathcal{D}_{a^+}^{\alpha} u \in L^p(a,b) \right\}, \tag{3.7}$$

endowed with the norm:

$$\|u\|_{E_L^{\alpha,p}(a,b)} := \left(\int_a^b |u(x)|^p \, dx + \int_a^b |\mathcal{D}_{a^+}^{\alpha} u(x)|^p \, dx \right)^{1/p}.$$

Similarly,

$$E_R^{\alpha,p}(a,b) := \left\{ u \in L^p(a,b) \mid \mathcal{D}_{b^-}^{\alpha} u \in L^p(a,b) \right\}. \tag{3.8}$$

endowed with the norm:

$$\|u\|_{E_R^{\alpha,p}(a,b)} = \left(\int_a^b |u(x)|^p \, dx + \int_a^b |\mathcal{D}_{b^-}^{\alpha} u(x)|^p \, dx \right)^{1/p}.$$

• In particular, when $p = 2$, $E_L^{\alpha,2}(a,b)$ we denote $E_L^{\alpha}(a,b)$ it is Hilbert space with inner product

$$\|u\|_{E_L^{\alpha}(a,b)} = \left(\int_a^b |u(x)|^2 \, dx + \int_a^b |{}_aD_x^{\alpha} u(x)|^2 \, dx \right)^{1/2}$$

Similarly,

$$\|u\|_{E_R^{\alpha}(a,b)} = \left(\int_a^b |u(x)|^2 \, dx + \int_a^b |{}_xD_b^{\alpha} u(x)|^2 \, dx \right)^{1/2}$$

Theorem 3.1. *[4] For any $0 < \alpha < 1$, $1 \leq p < \infty$, f has the left Riemann Liouville derivative $D_{a^+}^{\alpha} f \in L^p(a,b)$ if and only if $f \in AC_{a^+}^{\alpha,p}(a,b)$ that is-*

$$f(x) = \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a^+}^{\alpha}\varphi(x)$$

With $c = I_{a^+}^{\alpha-1} f(a)$ and $\varphi(x) = D_{a^+}^{\alpha} f(x)$

Proof. • **(⇒)** We show that if $D_{a^+}^{\alpha} f(x) \in L^p(a,b)$, then $f \in AC_{a^+}^{\alpha,p}(a,b)$.

Let $\varphi(x) = D_{a^+}^{\alpha} f(x) \in L^p(a,b)$. Define:

$$g(x) = I_{a^+}^{\alpha}\varphi(x) = I_{a^+}^{\alpha} D_{a^+}^{\alpha} f(x) \in L^p(a,b) \qquad \text{(by (2.2))}$$

Then

$$g \in AC_{a^+}^{\alpha,p}(a,b).$$

Since

$$D_{a^+}^{\alpha} f(x) = D_{a^+}^{\alpha} g(x),$$

it follows that

$$D_{a^+}^{\alpha}(f(x) - g(x)) = 0.$$

Using the definition of the Riemann–Liouville fractional derivative, we obtain:

$$D_{a^+}^{\alpha}(f-g)(x) = \frac{d}{dx}\left[I_{a^+}^{1-\alpha}(f(x) - g(x))\right] = 0.$$

This implies that the function $I_{a^+}^{1-\alpha}(f(x) - g(x))$ is constant:

$$I_{a^+}^{1-\alpha}(f(x) - g(x)) = c.$$

Applying the operator $D_{a^+}^{1-\alpha}$ to both sides gives:

$$f(x) - g(x) = D_{a^+}^{1-\alpha} c = \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1},$$

Therefore,

$$f(x) = g(x) + \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1}, \quad \text{with } g(x) = I_{a^+}^{\alpha}\varphi(x).$$

Hence $f \in AC_{a^+}^{\alpha,p}(a,b)$, which completes the proof.

- ($\Leftarrow$) We show that if $f \in AC_{a^+}^{\alpha,p}(a,b)$, then $D_{a^+}^{\alpha} f \in L^p(a,b)$.

By the definition of $AC_{a^+}^{\alpha,p}(a,b)$ (see 2.11), we have:

$$f(x) = \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I_{a^+}^{\alpha}\varphi(x), \quad \text{with } \varphi \in L^p(a,b).$$

We apply the left Riemann–Liouville fractional derivative of order α to both sides of the equation:

$$D_{a^+}^{\alpha} f(x) = D_{a^+}^{\alpha}\left(\frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1}\right) + D_{a^+}^{\alpha}\left(I_{a^+}^{\alpha}\varphi(x)\right).$$

By the definition of the Riemann–Liouville derivative (see Definition 2.22), and the semigroup property $D_{a^+}^{\alpha} I_{a^+}^{\alpha}\varphi = \varphi$ (see Equation(2.27)), we obtain:

$$\begin{aligned} D_{a^+}^{\alpha} f(x) &= \frac{d}{dx}\left(I_{a^+}^{1-\alpha}\left(\frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1}\right)\right) + \varphi(x) \\ &= \frac{d}{dx}(c) + \varphi(x) \qquad \text{(see Equation 2.3).} \end{aligned}$$

Since c is constant, we have $\frac{d}{dx}(c) = 0$, and thus:

$$D^{\alpha}_{a+} f(x) = \varphi(x) \in L^p(a,b).$$

□

Proposition 3.4. ***(A relation between*** $AC^{\alpha,p}_{a^+}(a,b)$ ***and*** $E_L^{\alpha,p}(a,b)$***)*** *Let* $1 \leq p < \infty$. *Then we have:*

$$E_L^{\alpha,p}(a,b) = AC^{\alpha,p}_{a^+}(a,b) \cap L^p(a,b).$$

Proof.

- **First, we show that**

$$E_L^{\alpha,p}(a,b) \subset AC^{\alpha,p}_{a^+}(a,b) \cap L^p(a,b).$$

Let $u \in E_L^{\alpha,p}(a,b)$. by definition 3.3 $u \in L^p(a,b)$ and $\mathcal{D}^{\alpha}_{a+} u \in L^p(a,b)$.

By Theorem 3.1, it follows that $u \in AC^{\alpha,p}_{a^+}(a,b)$, i.e.,

$$u(x) = \frac{c}{\Gamma(\alpha)}(x-a)^{1-\alpha} + I^{\alpha}_{a+}\varphi(x),$$

where $\varphi(x) = \mathcal{D}^{\alpha}_{a+} u(x) \in L^p(a,b)$, and $c = \left(I^{1-\alpha}_{a^+} u\right)(a)$.

Thus, we obtain:

$$E_L^{\alpha,p}(a,b) \subset AC^{\alpha,p}_{a^+}(a,b) \cap L^p(a,b). \tag{3.9}$$

- **Next, we show that**

$$AC^{\alpha,p}_{a^+}(a,b) \cap L^p(a,b) \subset E_L^{\alpha,p}(a,b).$$

Let $u \in AC^{\alpha,p}_{a^+}(a,b) \cap L^p(a,b)$. Then $u \in AC^{\alpha,p}_{a^+}(a,b)$, so there exists $\varphi \in L^p(a,b)$ such that:

$$u(x) = \frac{c}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^{\alpha}_{a+}\varphi(x),$$

with $\varphi(x) = D^{\alpha}_{a+} u(x) \in L^p(a,b)$ and $u \in L^p(a,b)$ by assumption.

Therefore, by (3.7), we have $u \in E_L^{\alpha,p}(a,b)$, and hence:

$$AC^{\alpha,p}_{a^+}(a,b) \cap L^p(a,b) \subset E_L^{\alpha,p}(a,b). \tag{3.10}$$

Hence, by (3.9) and (3.10), we conclude that:

$$E_L^{\alpha,p}(a,b) = AC^{\alpha,p}_{a^+}(a,b) \cap L^p(a,b).$$

□

Theorem 3.2. *The fractional spaces* $(E_L^\alpha(a,b), \|.\|_L)$ *and* $(E_R^\alpha(a,b), \|.\|_R)$ *are separable and reflexive banach spaces.*

Proof. We equipped the product space $E = L^2(a,b) \star L^2(a,b)$ with the Hilbert space norm:

$$\|(u_1,u_2)\|_E = \left(\|u_1\|^2_{L^2(a,b)} + \|u_2\|^2_{L^2(a,b)}\right)^{\frac{1}{2}}$$

Next, we consider the fractional space $E_L^\alpha(a,b)$ with its norm:

$$\|u\|_{E_L^\alpha(a,b)} = \left(\|u\|^2_{L^2(a,b)} + \|\mathcal{D}^\alpha_{a^+}u\|^2_{L^2(a,b)}\right)^{\frac{1}{2}}$$

Now, define the operator: :

$$\begin{array}{rcc} T : E_L^\alpha(a,b) & \rightarrow & E \\ u & \mapsto & (u, \mathcal{D}^\alpha_{a^+}u) \end{array}$$

For any $u \in E_L^\alpha(a,b)$, we have:

$$\|Tu\|_E = \left(\|u\|^2_{L^2(a,b)} + \|\mathcal{D}^\alpha_{a^+}\|^2_{L^2(a,b)}\right)^{\frac{1}{2}} = \|u\|_{E_L^\alpha(a,b)}$$

Therefore, T is an isometry, i.e it is injective and preserves the norm.
Let $(u_n) \subset E_L^\alpha(a,b)$ be a sequence such that $T(u_n) \to w \in E$. Then:

$$T(u_n) = (u_n, \mathcal{D}^\alpha_{a^+}u_n) \to (u,v) \in E$$

Therefore,

$$\begin{cases} u_n & \longrightarrow & u & \text{in } L^2(a,b) \\ D^\alpha_{a^+}u_n & \longrightarrow & v & \text{in } L^2(a,b) \end{cases}$$

From the convergence $u_n \to u$ in $L^2(a,b)$, we have $u_n \to u$ in $\mathscr{D}'(a,b)$, hence:

$$D^\alpha_{a^+}u_n \to D^\alpha_{a^+}u \quad \text{in } \mathscr{D}'(a,b)$$

Hence , $v = D^\alpha_{a^+}u$,i.e.

$$w = (u, D^\alpha_{a^+}) = T(u)$$

Consequently, the image $T(E_L^\alpha(a,b))$ it is a closed subspace of E.Since $E = L^2(a,b)L^2(a,b)$ is separable and reflexive (being a Hilbert space), every closed subspace of E is also separable and reflexive. Hence, $T(E_L^\alpha(a,b))$ is also separable and reflexive
Thus $E_L^\alpha(a,b)$s separable and reflexive.

- Similarly for $E_R^\alpha(a,b)$,completing the proof. □

Now, let us explore some properties of $I^\alpha_{a^+}$ and $\mathcal{D}^\alpha_{a^+}$

Lemma 3.1.

1. *The operator* $I_{a^+}^{\alpha} : L^2(a,b) \longrightarrow E_L^{\alpha}(a,b)$ *is linear and bounded*

2. *The operator* $\mathcal{D}_{a^+}^{\alpha} : E_L^{\alpha}(a,b) \longrightarrow L^2(a,b)$ *is linear and bounded*

Proof. 1. From the definition of fractional integral (2.1) , we see that $I_{a^+}^{\alpha}$ is linear operator , let $u \in L^2(a,b)$, $v = I_{a^+}^{\alpha} u$ then by lemma 3.1 , shows $v \in E_L^{\alpha}(a,b)$

We have :

$$I_{a^+}^{1-\alpha} v = I_{a^+}^{1-\alpha} I_{a^+}^{\alpha} u = I_{a^+}^{1} u \qquad \text{(By proposition 2.1)}$$

So $D_{a^+}^{\alpha} v = u$ exist , which means $\mathcal{D}_{a^+}^{\alpha} v = D_{a^+}^{\alpha} v$, thus we obtain :

$$\|\mathcal{D}_{a^+}^{\alpha} v\|_{L^2(a,b)} = \|u\|_{L^2(a,b)} \tag{3.11}$$

and

$$I_{a^+}^{\alpha} u = v \in E_L^{\alpha}(a,b)$$

Together with (3.7) and (3.11) we have :

$$\|I_{a^+}^{\alpha} u\|_{E_L^{\alpha}(a,b)} = \left(\|I_{a^+}^{\alpha} u\|_{L^2(a,b)}^2 + \|\mathcal{D}_{a^+}^{\alpha} I_{a^+}^{\alpha} u\|_{L^2(a,b)}^2\right)^{\frac{1}{2}}$$

Then by (2.13) we get :

$$\begin{aligned} &\le \left(\frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha+1)}\|u\|_{L^2(a,b)}^2 + \|u\|_{L^2(a,b)}^2\right)^{\frac{1}{2}} \\ &\le \left(\frac{(b-a)^{2\alpha}}{\Gamma^2(\alpha+1)} + 1\right)^{\frac{1}{2}} \|u\|_{L^2(a,b)} \end{aligned}$$

which means that $I_{a^+}^{\alpha} : L^2(a,b) \longrightarrow E_L^{\alpha}(a,b)$ is bounded .

2. We have previously proven in lemma 3.1 that the operator $\mathcal{D}_{a^+}^{\alpha}$ is linear
Moreover , it is easy to see

$$\|\mathcal{D}_{a^+}^{\alpha} u\|_{L^2(a,b)} \le \left(\|u\|_{L^2(a,b)}^2 + \|\mathcal{D}_{L^2(a,b)}^{\alpha} u\|^2\right)^{\frac{1}{2}} = \|u\|_{E_L^{\alpha}(a,b)}$$

Which means :$\mathcal{D}_{a^+}^{\alpha} : E_L^{\alpha}(a,b) \longrightarrow L^2(a,b)$ is bounded operator

□

3.4 The space $E^{\alpha}_{L,0}(a,b)$, $E^{\alpha}_{R,0}(a,b)$

Our analysis will focus on the space $E^{\alpha}_{L,0}(a,b)$ with analogous results holding for $E^{\alpha}_{R,0}(a,b)$

Definition 3.4. *Let $\alpha \in (0,1)$,we denote by $E^{\alpha}_{L,0}(a,b)$ as the closure of $\mathscr{D}(a,b)$ in $E^{\alpha}_{L}(a,b)$ endowed with the norm of $E^{\alpha}_{L}(a,b)$, that is :*

$$E^{\alpha}_{L,0}(a,b) = \overline{\mathscr{D}(a,b)}^{\|\cdot\|_{E^{\alpha}_{L}(a,b)}} \tag{3.12}$$

Lemma 3.2. *[10] Let $u \in E^{\alpha}_{L,0}(a,b)$ then :*

$$I^{\alpha}_{a^+} D^{\alpha}_{a^+} u(x) = u(x) \qquad a.e \;\; x \in (a,b) \tag{3.13}$$

Proof. From the definition of space $E^{\alpha}_{L,0}(a,b)$ there exists a sequence $(\varphi_n) \subset \mathscr{D}(a,b)$ such that :

$$\|\varphi_n - u\|_{E^{\alpha}_{L}(a,b)} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

For $\varphi_n \in \mathscr{D}(a,b)$, on has $\mathcal{D}^{\alpha}_{a^+}\varphi_n = D^{\alpha}_{a^+}\varphi_n$
which together with (2.28) we get

$$I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}\varphi_n = I^{\alpha}_{a^+}D^{\alpha}_{a^+}\varphi_n = \varphi_n - \frac{I^{1-\alpha}\varphi_n(a)}{\Gamma(\alpha)}(x-t)^{\alpha-1}$$

Since $I^{1-\alpha}\varphi_n(a) = 0$, we have

$$I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}\varphi_n = \varphi_n$$

By lemma 3.1 we add and subtract the term$I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}\varphi_n$:

$$\begin{aligned}
\|I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u - u\|_{E^{\alpha}_{L}} &= \|(I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u - I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}\varphi_n) + (I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}\varphi_n - u)\|_{E^{\alpha}_{L}(a,b)} \\
&\leq \|(I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u - I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}\varphi_n)\|_{E^{\alpha}_{L}(a,b)} + \|(I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}\varphi_n - u)\|_{E^{\alpha}_{L}(a,b)} \\
&\leq \|(I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}(u - \varphi_n)\|_{E^{\alpha}_{L}(a,b)} + \|I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}\varphi_n - u\|_{E^{\alpha}_{L}(a,b)}
\end{aligned}$$

By the continuity of $I^{\alpha}_{a^+}$ and $\mathcal{D}^{\alpha}_{a^+}$, it follows that:

$$\begin{aligned}
\|I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u - u\|_{E^{\alpha}_{L}} &\leq c\|\varphi_n - u\|_{E^{\alpha}_{L}(a,b)} + \|u - \varphi_n\|_{E^{\alpha}_{L}(a,b)} \\
&= (c+1)\|\varphi_n - u\| \longrightarrow 0, \text{ as } n \longrightarrow \infty
\end{aligned}$$

Where $c > 0$ is constant

Hence :

$$I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u(x) = u(x) \qquad \text{a.e } x \in [a,b]$$

□

Corollary 3.1. ***(Fractional Poincare inequality)***: *For evry $u \in E^{\alpha}_{L,0}(a,b)$ we have :*

$$\|u\|_{L^2(a,b)} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|\mathcal{D}^{\alpha}_{a} u\|_{L^2(a,b)} \tag{3.14}$$

Proof. By combining lemma 3.2 with (2.13) we get :

$$\begin{aligned} \|u\|_{L^2(a,b)} &= \|I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)} \\ &\leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)} \end{aligned}$$

□

Remark 3.2. *By Corollary 3.1 , we can endow the space $E^{\alpha}_{L,0}(a,b)$ with the below norm, which is equivalent to the norm of $E^{\alpha}_{L}(a,b)$:*

$$\|u\|_{E^{\alpha}_{L,0}(a,b)} = \left(\int_a^b |\mathcal{D}^{\alpha}_{a}u|^2 \, dx\right)^{\frac{1}{2}}.$$

Theorem 3.3. *1. If $\alpha \in]0, \frac{1}{2}[$,then*

$$E^{\alpha}_{L,0}(a,b) \subset \left\{ u \in L^2(a,b), \mathcal{D}^{\alpha}_{a^+}u \in L^2(a,b) \text{ and } \lim_{x \longrightarrow a^+} I^{1-\alpha}_{a^+}u(x) = 0 \right\}$$

2. And , if $\alpha \in]\frac{1}{2}, 1[$ then :

$$E^{\alpha}_{L,0}(a,b) \subset \left\{ u \in L^2(a,b), \mathcal{D}^{\alpha}_{a^+}u \in L^2(a,b) \text{ and } u(a) = u(b) = 0 \right\}$$

Proof.

- **For** $\alpha \in \left]0, \frac{1}{2}\right[$ by definition 3.4 we have :

$$E^{\alpha}_{L,0}(a,b) = \overline{\mathscr{D}(a,b)}^{\|\cdot\|_{E^{\alpha}_{L}(a,b)}}$$

So, for every $u \in E^{\alpha}_{L,0}(a,b)$, there is $(\varphi_n) \subset \mathscr{D}(a,b)$ such that :

$$\lim_{n\to\infty} \|\varphi_n - u\|_{E^{\alpha}_{L}(a,b)} = 0$$

Then by theorem 2.3 we have :

$$\lim_{x\to a^+} I^{1-\alpha}_{a^+}u(x) = I^{1-\alpha}_{a^+}u(a) = \lim_{n\to\infty} I^{1-\alpha}_{a^+}\varphi_n(a) = 0 \qquad \forall \varphi \in \mathscr{D}(a,b)$$

Hence,

$$\lim_{x\to a^+} I^{1-\alpha}u(x) = 0 \qquad \forall x > a$$

because $1-\alpha > \frac{1}{2}$ So,

$$E^{\alpha}_{L,0}(a,b) \subset \left\{ u \in L^2(a,b), \mathcal{D}^{\alpha}_{a^+} \in L^2(a,b) \text{ and } \lim_{x \longrightarrow a^+} I^{1-\alpha}_{a^+}u(x) = 0 \right\}$$

- **Now, for** $\alpha \in \left]\frac{1}{2}, 1\right[$ we recall from the definition 3.4 that: For any $u \in E^{\alpha}_{L,0}(a,b)$, there exists a sequence $(u_n) \subset \mathscr{D}(a,b)$ such that

  $$u_n \longrightarrow u \quad \text{in } E^{\alpha}_{L,0}(a,b)$$

  By the compact embedding theorem 3.7 , we have:

  $$\|u\|_{L^{\infty}(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}\|u\|_{E^{\alpha}_{L,0}(a,b)}$$

  That is,

  $$\max_{x\in[a,b]} |u(x)-u_n(x)| \leq C\|u-u_n\|_{E^{\alpha}_{L,0}(a,b)} \longrightarrow 0$$

  where

  $$C = \frac{(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}$$

  Therefore, we obtain:

  $$0 \leq |u(a)| = |u(a)-u_n(a)| \leq \|u-u_n\|_{L^{\infty}} \longrightarrow 0 \quad \text{as } n \to \infty$$

  Which implies that $u(a) = 0$. In the same way, we also have $u(b) = 0$ So,

  $$E^{\alpha}_{L,0}(a,b) \subset \left\{u \in L^2(a,b), \mathcal{D}^{\alpha}_{a^+} \in L^2(a,b) \textit{ and } u(a) = u(b) = 0\right\}$$

□

3.5 Embedding results for Sobolev spaces $E^{\alpha}_{L,0}(a,b), E^{\alpha}_{R,0}(a,b)$

3.5.1 Continuous embedding

Theorem 3.4. *[10] If $\alpha \in \left]0, \frac{1}{2}\right[$, then embedding $E^{\alpha}_{L,0}[a,b] \hookrightarrow L^q[a,b]$ is continous for every $q \in [1, 2^*_{\alpha}]$, where $2^*_{\alpha} = \frac{2}{1-2\alpha}$ is the fractional critical exponent*

Proof. By lemma 3.2 we have

$$\|u(x)\|_{L^{2^*}(a,b)} \quad = \quad \|I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u(x)\|_{L^{2^*}(a,b)}$$

By theorem 2.3

$$\|u(x)\|_{L^{2^*}(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a^+}u(x)\|_{L^2(a,b)} \tag{3.15}$$

Since

$$\|u(x)\|_{L^q(a,b)} \leq C\|u(x)\|_{L^{2^*}(a,b)} \qquad \forall q \in [1, 2^*] \tag{3.16}$$

Hence, by combining (3.15) and (3.16), we obtain:

$$\|u(x)\|_{L^q(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{2}}C}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a^+}u(x)\|_{L^2(a,b)}$$

Then ,

$$\|u(x)\|_{L^q(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{2}}C}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|u(x)\|_{E^{\alpha}_{L,0}(a,b)}$$

□

Theorem 3.5. *[10] If $\alpha \in]\frac{1}{2},1[$, then the embedding $E^{\alpha}_{L,0} \hookrightarrow \mathscr{C}^{\alpha-\frac{1}{2}}[a,b]$ is continuous Moreover :*

$$\|u\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)} \tag{3.17}$$

Proof. Let $u \in E^{\alpha}_{L,0}(a,b)$. Then, by lemma (3.2) and Hölder's inequality (1.2), we obtain:

$$\begin{aligned}
|u(x)| = |I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u(x)| &= \frac{1}{\Gamma(\alpha)}\left|\int_a^x (x-t)^{\alpha-1}\mathcal{D}^{\alpha}_{a^+}u(x)dt\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^x (x-t)^{2(\alpha-1)}dt\right)^{\frac{1}{2}}\left(\int_a^x |\mathcal{D}^{\alpha}_{a^+}u(t)|^2dt\right)^{\frac{1}{2}} \\
&= \frac{\|\mathcal{D}^{\alpha}_{a^+}u(t)\|_{L^2(a,b)}}{(2\alpha-1)^{\frac{1}{2}}\Gamma(\alpha)}(x-a)^{\alpha-\frac{1}{2}}
\end{aligned}$$

On other hande , let $a < x_1 < x_2 < b$ then by lemma 3.2 and theorem 2.3 we obtain :

$$\begin{aligned}
|u(x_2)-u(x_1)| &= |I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u(x_2) - I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u(x_1)| \\
&= \left|\frac{1}{\Gamma(\alpha)}\left(\int_a^{x_2}(x_2-t)^{\alpha-1}\mathcal{D}^{\alpha}_{a^+}u(t)dt - \int_a^{x_1}(x_1-t)^{\alpha-1}\mathcal{D}^{\alpha}_{a^+}u(t)dt\right)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^{x_1}\left|(x_2-t)^{\alpha-1}(x_1-t)^{\alpha-1}\right||\mathcal{D}^{\alpha}_{a^+}u(t)|dt + \int_{x_1}^{x_2}|(x_2-t)^{\alpha-1}||\mathcal{D}^{\alpha}_{a^+}u(t)|dt\right)
\end{aligned}$$

By the Cauchy–Schwarz inequality (1.3) , we obtain:

$$|u(x_2)-u(x_1)| \leq \frac{1}{\Gamma(\alpha)}\left(\int_a^{x_1}|(x_2-t)^{\alpha-1}-(x_1-t)^{\alpha-1}|^2\right)^{\frac{1}{2}}\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)} + \frac{(x_2-x_1)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a^+}\|_{L^2(a,b)}$$

Using (1.4), we obtain the following estimate for the first term:

$$\begin{aligned}
|u(x_2)-u(x_1)| &\leq \frac{\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)}}{\Gamma(\alpha)}\left(\int_a^{x_1}((x_1-t)^{2\alpha-1}-(x_2-t)^{2\alpha-1})\right)^{\frac{1}{2}} + \frac{(x_2-x_1)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)} \\
&\leq \frac{\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)}}{(2\alpha-1)^{\frac{1}{2}}\Gamma(\alpha)}\left((x_2-x_1)^{\alpha-\frac{1}{2}}+(x_1-a)^{\alpha-\frac{1}{2}}-(x_2-a)^{\alpha-\frac{1}{2}}\right) \\
&\quad + \frac{(x_2-x_1)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)}
\end{aligned}$$

Using (1.5) we obtain :

$$|u(x_2)-u(x_1)| \leq \frac{\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)}}{(2\alpha-1)^{\frac{1}{2}}\Gamma(\alpha)}\left((x_2-x_1)^{\alpha-\frac{1}{2}}+(x_2-x_1)^{\alpha-\frac{1}{2}}+(x_2-x_1)^{\alpha-\frac{1}{2}}\right)$$

Hence ,

$$|u(x_2)-u(x_1)| \leq \frac{3(x_2-x_1)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)} \tag{3.18}$$

Therefore,

$$\begin{aligned}
\|u\|_{\mathscr{C}^{\alpha-\frac{1}{2}}(a,b)} &= \|u\|_{L^\infty(a,b)} + \sup_{(x_1,x_2)\in(a,b)} \frac{|u(x_2)-u(x_1)|}{|x_2-x_1|^{\alpha-\frac{1}{2}}} \\
&\leq \frac{(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a^+}u\|_{L^2(a,b)} + \frac{3}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|\mathcal{D}^{\alpha}_{a}u\|_{L^2(a,b)} \\
&= \left(\frac{(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}+3\right)\|\mathcal{D}^{\alpha}_{a}u\|_{L^2(a,b)}
\end{aligned}$$

Hence ,

$$\|u\|_{\mathscr{C}^{\alpha-\frac{1}{2}}(a,b)} \leq \left(\frac{(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}+3\right)\|u\|_{E^{\alpha}_{L,0}(a,b)}$$

□

3.5.2 Compact embedding

Theorem 3.6. *[10] For $\alpha \in]0,\frac{1}{2}[$ the embedding $E^{\alpha}_{L,0}(a,b) \hookrightarrow L^p(a,b)$ is compact for every $p \in [1, 2^*_\alpha[$*

Proof. By Theorem 3.4 , the operator

$$i : E^{\alpha}_{L,0}(a,b) \;\;\rightarrow\;\; L^q(a,b)$$

is continuous for $q \in [1, 2^*]$ which means there exist a constante C such that :

$$\|u_n\|_{L^q(a,b)} \leq C\|u_n\|_{E^{\alpha}_{L,0}(a,b)} \qquad \text{for all } q \in [1,2^*]$$

Therefore, if $u_n \rightharpoonup u$ in $E^{\alpha}_{L,0}(a,b)$ and the sequence is bounded, then, by the inequality above, it is also bounded in $L^p(a,b)$.

Since $\mathcal{D}^{\alpha}_{a^+}u_n$ is bounded in $L^2(a,b)$, it follows from Theorem 2.5 that $I^{\alpha}_{a^+}(\mathcal{D}^{\alpha}_{a^+}u_n)$ is precompact in $L^q(a,b)$, for every $q \in [1, 2^*[$,That means there is a subsequence (u_{k_n}) we denoted $(u_n)_{n\in\mathbb{N}}$ and $u \in L^q(a,b)$ such that

$$\|I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u_n - u\|_{L^q(a,b)} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

Since by lemma 3.2 we have

$$u_n(x) = I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u_n(x) \Rightarrow i(u_n(x)) = i(I^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}u_n(x))$$

So, by the continuity of i, the composition is compact.
Therefore,

$$i : E^{\alpha}_{L,0}(a,b) \longrightarrow L^q(a,b)$$

is a compact operator for every $q \in [1, 2^*[$.

□

Theorem 3.7. *[10] if* $\alpha \in]\frac{1}{2}, 1[$ *the embedding* $E^{\alpha}_{L,0}(a,b) \hookrightarrow C([a,b])$ *is compact*

Proof. Let $u_n \rightharpoonup u$ in $E^{\alpha}_{L,0}(a,b)$, we know that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $E^{\alpha}_{L,0}(a,b)$. Then, by Theorem 3.5 , it follows that $(u_n)_{n\in\mathbb{N}}$ is also bounded in $C([a,b])$ i.e :

$$\|u_n\|_{E^{\alpha}_{L,0}(a,b)} \leq K$$

Hence by (3.17) we obtain :

$$\|u_n\|_{L^{\infty}(a,b)} \leq \frac{K(b-a)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}$$

Thus, $\|u_n\|_{L^{\infty}(a,b)}$ is uniformly bounded, meaning that the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([a,b])$.
Moreover , by (3.18) we have

$$|u_n(x_2) - u_n(x_1)| \leq \frac{K(x_2-x_1)^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}$$

Let

$$\forall \varepsilon > 0, |x_2 - x_1| < \delta \Rightarrow |u_n(x_1) - u_n(x_2)| < \varepsilon$$

We choose δ such that

$$\frac{k\,\delta^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\,(2\alpha-1)^{\frac{1}{2}}} = \varepsilon$$

which gives

$$\delta = \left(\frac{\Gamma(\alpha)\,(2\alpha-1)^{\frac{1}{2}}\,\varepsilon}{k}\right)^{\frac{1}{\alpha-\frac{1}{2}}} > 0$$

Hence,

$$|x_2 - x_1| < \delta \Rightarrow |u_n(x_1) - u_n(x_2)| \leq \varepsilon$$

Therefore, the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous.

Since the sequence (u_n) is uniformly bounded and equicontinuous on the compact interval $[a,b]$, by the Arzelà–Ascoli Theorem 1.10 , it is precompact in $C([a,b])$.

This means that there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, and a function $u \in C[a,b]$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty(a,b)} = 0$$

□

3.6 Integration by parts

Theorem 3.8. *Let $f \in I^\alpha_{a^+}(L^p), g \in I^\alpha_{a^+}(L^p)$ such that $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ then ,*

$$\int_a^b f(x) D^\alpha_{b^-} g(x)dx = \int_a^b g(x) D^\alpha_{a^+} f(x)dx \tag{3.19}$$

Proof. We put $f = I^\alpha_{a^+}\varphi$ and $g = I^\alpha_{b^-}\psi$ we consider : $\varphi = D^\alpha_{a^+} f$ and $\psi = D^\alpha_{b^-} g$ we obtain :

$$\begin{aligned}\int_a^b f(x) D^\alpha_{b^-} g(x)dx &= \int_a^b I^\alpha_{a^+}\varphi(x) D^\alpha_{b^-} g(x)dx \\ &= \int_a^b I^\alpha_{a^+}\varphi(x)\psi(x)dx\end{aligned}$$

By (2.21) we obtain :

$$\begin{aligned}\int_a^b f(x) D^\alpha_{b^-} g(x)dx &= \int_a^b \varphi(x) I^\alpha_{b^-}\psi(x)dx \\ &= \int_a^b g(x) D^\alpha_{a^+} f(x)dx\end{aligned}$$

□

- By the same way for the right fractional derivative

$$\int_a^b f(x) D^\alpha_{a^+} g(x)dx = \int_a^b g(x) D^\alpha_{b^-} f(x)dx$$

Theorem 3.9. *Let $f \in E^\alpha_L(a,b)$ and $g \in AC([a,b])$ then :*

$$\int_a^b f(x) D^\alpha_{a^+} g(x)dx = \int_a^b g(x) D^\alpha_{b^-} f(x) + g(b) I^{1-\alpha}_{b^-} f(b) \tag{3.20}$$

$$\int_a^b f(x) D^\alpha_{b^-} g(x)dx = \int_a^b g(x) D^\alpha_{b^+} f(x) + g(a) I^{1-\alpha}_{a^+} f(a) \tag{3.21}$$

Proof. By (2.25) we obtain :

$$\begin{aligned}
\int_a^b f(x)D^\alpha_{a^+}g(x)dt &= \frac{1}{\Gamma(1-\alpha)}\int_a^b f(x)\left(g(a)(x-a)^{-\alpha}+\int_a^x g'(t)(x-t)^{-\alpha}dt\right)dx\\
&= \frac{g(a)}{\Gamma(1-\alpha)}\int_a^b f(x)(x-a)^{-\alpha}dx+\frac{1}{\Gamma(1-\alpha)}\int_a^b\int_a^x f(x)(x-t)^{-\alpha}g'(t)dtdx\\
&= g(a)I^{1-\alpha}_{b^-}f(a)+\int_a^b f(x)I^{1-\alpha}_{a^+}g'(x)dx
\end{aligned}$$

By theorem (2.6) we obtain :

$$\int_a^b f(x)D^\alpha_{a^+}g(x)dx = g(a)I^{1-\alpha}_{b^-}f(a)+\int_a^b I^{1-\alpha}_{b^-}f(x)g'(x)dx$$

We apply integration by parts to the second term we obtain:

$$\int_a^b f(x)D^\alpha_{a^+}g(x)dx = g(a)I^{1-\alpha}_{b^-}f(a)+I^{1-\alpha}_{b^-}f(b)g(b)-I^{1-\alpha}_{b^-}f(a)g(a)-\int_a^b\frac{d}{dx}I^{1-\alpha}_{b^-}f(x)g(x)dx$$

Hence ,

$$\int_a^b f(x)D^\alpha_{a^+}g(x)dx = \int_a^b D^\alpha_{b^-}f(t)g(x)+I^{1-\alpha}_{b^-}f(b)g(b)$$

□

- The same idea for (3.21)

Theorem 3.10. *If* $0\le\frac{1}{p}<\alpha<1$ *and* $0\le\frac{1}{q}<\alpha<1$, *then*

$$\int_a^b(D^\alpha_{a+}f)(x)g(x)\,dx=\int_a^b f(x)D^\alpha_{b-}g(x)\,dx+f(b)(I^{1-\alpha}_{b-}g)(b)-(I^{1-\alpha}_{a+}f)(a)g(a)$$

For any $f\in AC^{\alpha,p}_{a+},\ g\in AC^{\alpha,q}_{b-}$

Proof. From (3.3) and (3.4) there exist $\varphi\in L^p(a,b)$ and $\psi\in L^q(a,b)$ such that

$$f(x)=\frac{I^{1-\alpha}_{a+}f(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}+I^\alpha_{a+}\varphi(x)$$

$$g(x)=\frac{I^{1-\alpha}_{b-}g(b)}{\Gamma(\alpha)}(b-x)^{\alpha-1}+I^\alpha_{b-}\psi(x)$$

On the one hand, we have:

$$\begin{aligned}
\int_a^b(D^\alpha_{a+}f)(x)g(x)\,dx &= \int_a^b\varphi(x)g(x)\,dx\\
&= \int_a^b\varphi(x)\left(\frac{I^{1-\alpha}_{b-}g(b)}{\Gamma(\alpha)}(b-x)^{\alpha-1}+I^\alpha_{b-}\psi(x)\right)dx\\
&= \frac{I^{1-\alpha}_{b-}g(b)}{\Gamma(\alpha)}\int_a^b\varphi(x)(b-x)^{\alpha-1}dx+\int_a^b\varphi(x)I^\alpha_{b-}\psi(x)dx\\
&= I^{1-\alpha}_{b-}g(b)(I^\alpha_{a+}\varphi)(b)+\int_a^b I^\alpha_{a+}\varphi(x)\psi(x)dx \qquad \text{(By(2.21))}
\end{aligned}$$

$$\int_a^b (D^\alpha_{a+}f)(x)g(x)\,dx = I^{1-\alpha}_{b-}g(b)\cdot f(b) + \int_a^b I^\alpha_{a+}\varphi(x)\psi(x)dx \tag{3.22}$$

On auther hand ,we obtain

$$\begin{aligned}
\int_a^b f(x)(D^\alpha_{b-}g)(x)\,dx &= \int_a^b f(t)\psi(x)\,dx \\
&= \int_a^b \left(\frac{I^{1-\alpha}_{a+}f(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} + I^\alpha_{a+}\varphi(x)\right)\psi(x)\,dx \\
&= \frac{I^{1-\alpha}_{a+}f(a)}{\Gamma(\alpha)}\int_a^b (x-a)^{\alpha-1}\psi(x)\,dx + \int_a^b I^\alpha_{a+}\varphi(x)\psi(x)\,dx \\
&= I^{1-\alpha}_{a+}f(a)(I^\alpha_{b-}\psi)(a) + \int_a^b I^\alpha_{a+}\varphi(x)\psi(x)\,dx,
\end{aligned}$$

$$\int_a^b f(x)(D^\alpha_{b-}g)(x)\,dx = I^{1-\alpha}_{a+}f(a)g(a) + \int_a^b I^\alpha_{a+}\varphi(x)\psi(x)\,dx \tag{3.23}$$

By subtracting equation (3.22) from equation (3.23), we obtain:

$$\begin{aligned}
\int_a^b \Big(f(x)(D^\alpha_{b-}g)(x) - (D^\alpha_{a+}f)(x)g(x)\Big)\,dx &= I^{1-\alpha}_{a+}f(a)\cdot g(a) + \int_a^b I^\alpha_{a+}\varphi(x)\psi(x)\,dx \\
&\quad - f(b)I^{1-\alpha}_{b-}g(b) - \int_a^b I^\alpha_{a+}\varphi(x)\psi(x)\,dx \\
&= \left(I^{1-\alpha}_{a+}f\right)(a)g(a) - \left(I^{1-\alpha}_{b-}g\right)(b)f(b)
\end{aligned}$$

So,

$$\int_a^b (D^\alpha_{a+}f)(x)g(x)\,dx = \int_a^b f(x)D^\alpha_{b-}g(x)\,dx + f(b)(I^{1-\alpha}_{b-}g)(b) - (I^{1-\alpha}_{a+}f)(a)g(a)$$

□

Theorem 3.11. *Let $f \in E^\alpha_{L,0}(a,b)$ and $g \in E^\alpha_{R,0}(a,b)$ we have :*

$$\int_a^b f(x)D^\alpha_{b^-}g(x)dx = \int_a^b D^\alpha_{a^+}f(x)g(x)dx \tag{3.24}$$

$$\int_a^b D^\alpha_{b^-}f(x)g(x)dx = \int_a^b f(x)D^\alpha_{a^+}g(x)dx \tag{3.25}$$

Proof. By lemma 3.2 we obtain :

$$\begin{aligned}
\int_a^b f(x)D^\alpha_{b^-}g(x)dx &= \int_a^b I^\alpha_{a^+}D^\alpha_{a^+}f(x)D^\alpha_{b^-}g(x)dx \\
&= \int_a^b D^\alpha_{a^+}f(x)I^\alpha_{b^-}D^\alpha_{b^-}g(x)dx \qquad \text{(by (2.21))} \\
&= \int_a^b D^\alpha_{a^+}f(x)g(x)dx \qquad \text{(by lemma (3.2))}
\end{aligned}$$

Hence,

$$\int_a^b f(x)D^\alpha_{b^-}g(x)dx = \int_a^b D^\alpha_{a^+}f(x)g(x)dx$$

The same idea for (3.25) □

CHAPTER 4

APPLICATION TO THE BOUNDARY VALUE PROBLEMS

In this chapter, we investigate a nonlinear fractional differential equation involving Riemann–Liouville derivatives. We study three main cases:

- In the **sublinear case** $1 \leq q < 2$, we employ variational minimization methods;
- In the **superlinear case** $2 < q < 2^*$, we use the Nehari manifold technique;
- In the **linear case** $q = 2$, we distinguish between the cases based on the values of λ.

In each case, we establish the existence of non trivial weak solutions.

4.1 Position of the problem

Definition 4.1. ***(The problem*** (P)***)*** *We consider the following problem:*

$$(\mathrm{P})\left\{\begin{array}{rclcl} D_{b^-}^{\alpha} D_{a^+}^{\alpha} u(x) & = & \lambda |u|^{q-2}u & in &]a,b[\\ B_\alpha(u) & = & 0 & on &]a,b[\end{array}\right.$$

where $0 < \alpha < 1$ *and* $B_\alpha(u)$ *represent the boundary condition of the problem* (P)*which depends on the value of* $\alpha \in (0,1)$*, and is given by: :*

$$B_\alpha(u) = \left\{\begin{array}{lll} \lim\limits_{x \longrightarrow a^+} I^{1-\alpha}u(x) = 0 & if & \alpha \in \left]0, \frac{1}{2}\right[\\ u(a) = u(b) = 0 & if & \alpha \in \left]\frac{1}{2}, 1\right[\end{array}\right.$$

Definition 4.2. ***(The Weak Solution)*** *A function* $u \in E_{L,0}^{\alpha}(a,b)$ *is called a weak solution of problem* (P) *if it satisfies the following variational formulation.*

$$\int_a^b D_{a^+}^{\alpha} u(x) D_{a^+}^{\alpha} v(x)dx = \lambda \int_a^b |u(x)|^{q-2}u(x)v(x)dx \quad \forall v \in \mathscr{D}(a,b). \tag{4.1}$$

To find the variational formulation it is necessary to follow the following steps :

- We multiply the first equation in problem (P)by a test function v that is sufficiently smooth and compactly supported in (a, b), and integrate over (a, b):

$$\int_a^b D^\alpha_{b^-}(D^\alpha_{a^+}u(x))v(x)\,dx = \lambda\int_a^b |u(x)|^{q-2}u(x)v(x)\,dx, \quad \forall v \in \mathscr{D}(a,b).$$

- Using fractional integration by parts (3.19), the left-hand side transforms into:

$$\int_a^b D^\alpha_{a^+}u(x)D^\alpha_{a^+}v(x)dx = \lambda\int_a^b |u(x)|^{q-2}u(x)v(x)dx \quad \forall v \in \mathscr{D}(a,b). \tag{4.2}$$

Definition 4.3. ***(Energy Functional)*** *We define the functional associated with problem* (P) *as follows:*

$$\begin{array}{rcl} I : E^\alpha_{L,0}[a,b] & \to & \mathbb{R} \\ u & \mapsto & I(u) = \dfrac{1}{2}\displaystyle\int_a^b |D^\alpha_a u(x)|^2 dx - \int_a^b F(u)dx \end{array} \tag{4.3}$$

where :

$$F(u) = \frac{\lambda}{q}|u|^q$$

Proposition 4.1. *The functional I is well-defined and differentiable on $E^\alpha_{L,0}(a,b)$. And the differential is given by*

$$\langle I'(u), v\rangle = \int_a^b D^\alpha_{a^+}u(x)D^\alpha_{a^+}v(x)dx - \lambda\int_a^b |u(x)|^{q-2}u(x)v(x)dx \qquad \forall u, v \in E^\alpha_{L,0}(a,b) \tag{4.4}$$

Proof. • Firstly, the functional (4.3) is well-defined. Indeed, for any $u \in E^\alpha_{L,0}(a,b)$ we have :

$$|I_1(u)| = \frac{1}{2}\,\|D^\alpha_{a^+}u\|^2_{L^2(a,b)} < \infty \tag{4.5}$$

For the second term:

$$\begin{aligned} |I_2(u)| &= \frac{\lambda}{q}\left|\int_a^b u^q\,dx\right| \\ &\leq \frac{\lambda}{q}\int_a^b |u(x)^q|\,dx = \frac{\lambda}{q}\|u\|^q_{L^q(a,b)} < \infty \qquad (E^\alpha_{L,0}(a,b) \hookrightarrow L^q(a,b)) \end{aligned} \tag{4.6}$$

Hence, by (4.5) and (4.6), the functional (4.3) is well-defined.

• Let us now show that I is Gâteaux differentiable and $I'(u)$ is continuous i.e :

$$\lim_{t \longrightarrow 0}\frac{I(u+tv) - I(u)}{t} =< I'(u), v >$$

We set:

$$I_1(u) = \frac{1}{2}\int_a^b |D_{a^+}^{\alpha} u(x)|^2\, dx$$

and

$$I_1(u+tv) = \frac{1}{2}\int_a^b |D_{a^+}^{\alpha} (u+tv)|^2\, dx$$

by lemma(3.1) we obtain:

$$|D_{a^+}^{\alpha}(u+tv)(x)|^2 = |D_{a^+}^{\alpha}u(x) + tD_{a^+}^{\alpha}v(x)|^2$$

Integrating over (a,b) we get :

$$I_1(u+tv) = \frac{1}{2}\int_a^b |D_{a^+}^{\alpha}u(x)|^2 dx + t\int_a^b D_{a^+}^{\alpha}u(x)D_{a^+}^{\alpha}v(x)dx + \frac{t^2}{2}\int_a^b |D_{a^+}^{\alpha}v(x)|^2 dx$$

Thus

$$I_1(u+tv) - I_1(u) = t\int_a^b D_{a^+}^{\alpha}u(x)D_{a^+}^{\alpha}v(x)dx + \frac{t^2}{2}\int_a^b |D_{a^+}^{\alpha}v(x)|^2 dx$$

Dividing by t and taking the limite as $t \longrightarrow 0$, we obtain :

$$\lim_{t\longrightarrow 0}\frac{I_1(u+tv) - I_1(u)}{t} = \int_a^b D_{a^+}^{\alpha}u(x)D_{a^+}^{\alpha}v(x)dx$$

Thus,by Poincaré inequality (3.14) ,we can show easily that $I_1'(u)$ is continuous in $E_{L,0}^{\alpha}(a,b)$ and the gataux derivative is

$$< I_1'(u), v >= \int_a^b D_{a^+}^{\alpha}u(x)D_{a^+}^{\alpha}v(x)dx \tag{4.7}$$

Now for the nonlinear term:

$$I_2(u) = \int_a^b |u|^q dx$$

And

$$I_2(u+tv) = \int_a^b |u+tv|^q dx$$

Thus,

$$\lim_{t\longrightarrow 0}\frac{I_2(u+tv) - I_2(u)}{t} = \lim_{t\longrightarrow 0}\int_a^b \frac{|u+tv|^q - |u|^q}{t}dx$$

We apply the Dominated Convergence Theorem 1.6 on :

$$\frac{|u+tv|^q - |u|^q}{t}$$

1.

$$\begin{aligned}\lim_{t\longrightarrow 0}\frac{|u+tv|^q - |u|^q}{t} &= \lim_{t\longrightarrow 0} q|u+tv|^{q-2}(u+tv)v \\ &= q|u|^{q-2}uv\end{aligned}$$

2. we define

$$g(s) = |u + sv|^q$$

is continous and differentiable , then by means value theorem , $\exists\ \theta \in (0, t)$ such that

$$\begin{aligned} g(t) - g(0) = g'(\theta_t)t \quad &\Leftrightarrow \quad |u + tv|^q - |u|^q = q|u + \theta_t v|^{q-2}(u + \theta_t v)vt \\ &\Leftrightarrow \quad \frac{|u + tv|^q - |u|^q}{t} = q|u + \theta_t v|^{q-2}(u + \theta_t v)v \end{aligned}$$

Then

$$\left|\frac{|u + tv|^q - |u|^q}{t}\right| = q|u + \theta_t v|^{q-1}|v|$$

We applay the inquality (1.6)

when : $a = u$, $b = \theta_t v$

Hence

$$\begin{aligned} \left|\frac{|u + tv|^q - |u|^q}{t}\right| &\leq 2^{q-1}q(|u|^{q-1} + |v|^{q-1})|v| \\ &\leq 2^{q-1}q(|u|^{q-1}|v| + |v|^{q-1}|v|) \\ &\leq 2^{q-1}q(|u|^{q-1}|v| + |v|^q) \end{aligned}$$

Then by Hölder inquality $|u|^{q-1}|v| \in L^1(a, b)$, and $|v|^q \in L^q \subset L^1$

Hence we take $h = 2^{q-1}q(|u|^{q-1}|v| + |v|^q) \in L^1$

By the Dominated Convergence Theorem (D.C.T) 1.6 we conclude that

$$\lim_{t \longrightarrow 0} \int_a^b \frac{|u + tv|^q - |u|^q}{t} dx = q|u|^{q-2}uv$$

Hence, by applying holder inquality (1.2) and Poincaré inequality (3.14) , we show that $I_2'(u)$ is continous and the Gâteaux derivative of I_2 is

$$< I_2'(u), v >= \lambda \int_a^b |u(x)|^{q-2}u(x)v(x)dx \tag{4.8}$$

By (4.7) and (4.8), the functional I is differentiable on $E^{\alpha}_{L,0}(a, b)$ and :

$$< I'(u), v >= \int_a^b D^{\alpha}_{a^+}u(x)D^{\alpha}_{a^+}v(x)dx - \lambda \int_a^b |u(x)|^{q-2}u(x)v(x)dx$$

□

Now we try to prove that :

$$I' : E^{\alpha}_{L,0}(a, b) \longrightarrow (E^{\alpha}_{L,0}(a, b))'$$

is continous

We aim show that

$$u_n \to u \text{ in } E^{\alpha}_{L,0}(a,b) \Rightarrow \|I'(u_n) - I'(u)\|_{(E^{\alpha}_{L,0}(a,b))'} \longrightarrow 0$$

Where

$$\|I'(u_n) - I'(u)\|_{(E^{\alpha}_{L,0}(a,b))'} = \sup_{\|v\|=1} | < I'(u_n) - I'(u), v > |$$

Let $v \in E^{\alpha}_{L,0}(a,b)$ such that $\|v\| = 1$, we have

$$\|I'(u_n) - I'(u)\|_{(E^{\alpha}_{L,0}(a,b))'} \quad = \quad \sup_{\|v\|=1} \left| \int_a^b \left(D^{\alpha}_{a^+}u_n - D^{\alpha}_{a^+}u\right) D^{\alpha}_{a^+}v dx + \int_a^b \left(|u_n|^{q-2}u_n - |u|^{q-2}u\right) v \right|$$

By Holder inquality we obtain :

$$\begin{aligned}
\|I'(u_n) - I'(u)\|_{(E^{\alpha}_{L,0}(a,b))'} &\leq \sup_{\|v\|=1} \left(\|D^{\alpha}_{a^+}u_n - D^{\alpha}_{a^+}u\|_{L^2(a,b)} \|D^{\alpha}_{a^+}v\|_{L^2(a,b)} \right) \\
&\quad + \left(\int_a^b \left| |u_n|^{q-2}u_n - |u|^{q-2}u \right|^{\frac{q}{q-1}} \|v\|_{L^q(a,b)}\, dx \right)^{\frac{q-1}{q}} \\
&\leq \left(\|u_n - u\|_{E^{\alpha}_{L,0}(a,b)} + \left(\int_a^b \left| |u_n|^{q-2}u_n - |u|^{q-2}u \right|^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}} \right) \\
&\leq \|u_n - u\|_{E^{\alpha}_{L,0}(a,b)} + \left(\int_a^b \left| |u_n|^{q-2}u_n - |u|^{q-2}u \right|^{\frac{q}{q-1}} dx \right)^{\frac{q-1}{q}} \\
&= \|u_n - u\|_{E^{\alpha}_{L,0}(a,b)} + \| |u_n|^{q-2}u_n - |u|^{q-2}u \|_{L^{\frac{p}{p-1}}(a,b)}
\end{aligned}$$

We set $w_n = |u_n|^{q-2}u_n$, we prove that $\|w_n - w\|_{L^{\frac{q}{q-1}}(a,b)} \longrightarrow 0$. Where $w = |u|^{q-2}u$

Since $u_n \rightharpoonup u$ in $E^{\alpha}_{L,0}(a,b)$, by compact embedding $u_n \longrightarrow u$ in $L^q(a,b)$. Then by converse of dominated convergent theorem there exists a subsequence still denoted (u_n) such that

1. $u_n \longrightarrow u \quad a.e \ \Rightarrow |u_n|^{q-2}u_n \longrightarrow |u|^{q-2}u$ that is $w_n \longrightarrow w \qquad a.e$

2. $\exists g \in L^q(a,b) : |u_n| \leq g(x) \quad a.e \ \Rightarrow |u_n|^{q-1} \leq g^{q-1}$ Hence

$$\begin{aligned}
|w_n| &= \left| |u_n|^{q-2}u_n \right| \\
&= |u_n|^{q-1} \leq g^{q-1}
\end{aligned}$$

   Since $g \in L^q(a,b) \Rightarrow g^{q-1} \in L^{\frac{q}{q-1}}$

Hence by dominated convergent theorem $w_n \longrightarrow w \Rightarrow \| |u_n|^{q-2}u_n - |u|^{q-2}u \|_{L^{\frac{q}{q-1}}(a,b)} \longrightarrow 0$

Hence

$$\|I'(u_n) - I'(u)\|_{(E^{\alpha}_{L,0}(a,b))'} \longrightarrow 0$$

Remark 4.1. *The function $u \in E^{\alpha}_{L,0}(a,b)$ is weak solution of problem* (P) *if and only if u is a critical point of I*

4.2 Existence of the weak solution for the sublinear case $1 \leq q < 2$

In this case, we prove that the problem (P) admits a nontrivial weak solution. We use variational minimization methods by showing that the functional (4.3) is coercive, weakly lower semicontinuous, and bounded from below.

4.2.1 Main Result

Theorem 4.1. *Let $\alpha \in (0,1)$ and $1 \leq q < 2$. Then, the problem given by equations* (P) *admits non trivial weak solution in $E^{\alpha}_{L,0}(a,b)$,*

We will applay the following theorem , wich gurranties the existence of a critical point of I

Theorem 4.2. *Let E be a reflexive Banach space, and let $I : E \to \mathbb{R}$ be a functional that is bounded from below, coercive, and weakly lower semi-continuous. Then I attains its minimum at some point $u \in E$.*

Proof. To prove Theorem 4.2, it suffices to establish the following lemmas

Lemma 4.1. *The functional I is coercives in $E^{\alpha}_{L,0}[a,b]$: i.e*

$$\lim_{\|u\| \longrightarrow +\infty} I(u) = +\infty$$

Proof. By (4.3)

$$\begin{aligned} I(u) &= \frac{1}{2}\int_a^b |D^{\alpha}_a u(x)|^2 dx - \int_a^b F(u)dx \\ &= \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{\lambda}{q}\int_a^b |u|^q \qquad \text{by Remark (3.2)} \end{aligned}$$

We apply Hölder's inequality to the second term and obtain:

$$I(u) \geq \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{\lambda}{q}\|u\|^q_{L^q}$$

By continuous embedding(see theorem 3.4) , we obtain.

$$I(u) \geq \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{\lambda}{q}\|u\|^q_{E^{\alpha}_{L,0}(a,b)}$$

Hence

$$\lim_{\|u\| \longrightarrow +\infty} I(u) > \lim_{\|u\| \longrightarrow +\infty} \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)} = +\infty \qquad (1 \leq q < 2)$$

□

Lemma 4.2. *The functional I is weakly lower semi-continuous(w.l.s.c) in $E^{\alpha}_{L,0}(a,b)$: i.e. For any sequence $(u_n) \subset E^{\alpha}_{L,0}(a,b)$ if*

$$u_n \rightharpoonup u \quad in \quad E^{\alpha}_{L,0}(a,b) \quad then \quad \liminf_{n\longrightarrow\infty} I(u_n) \geq I(u)$$

Proof. By the functional (4.3) we set:

i)

$$I_1(u) = \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)}$$

Since $\|u\|_{E^{\alpha}_{L,0}(a,b)}$ is a norm in Banach space(Hilbert space), we have

$$(u_n, v) \to (u, v)$$

We put $v = u$

$$\lim_{n\longrightarrow\infty}(u_n, u) = (u, u) = \|u\|^2$$

Thus

$$\liminf_{n\longleftrightarrow\infty} \|u_n\|_{E^{\alpha}_{L,0}(a,b)}\|u\|_{E^{\alpha}_{L,0}(a,b)} \geq \|u\|^2$$

Hence

$$\liminf_{n\longleftrightarrow\infty} \|u_n\|_{E^{\alpha}_{L,0}(a,b)} \geq \|u\|$$

$\|u\|_{E^{\alpha}_{L,0}(a,b)}$ is (w. l. s.c) .So I_1 is (w.l.s.c) i.e

$$\frac{1}{2}\liminf_{n\longrightarrow+\infty} \|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} \geq \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)}$$

Consequently:

$$\liminf_{n\longrightarrow+\infty} I_1(u_n) \geq I_1(u) \tag{4.9}$$

ii)

$$I_2(u) = \frac{\lambda}{q}\|u\|^q_{L^q}$$

By compact embedding we obtain :

$$\frac{\lambda}{q}\lim_{n\longrightarrow+\infty}\int_a^b u_n^q = \frac{\lambda}{q}\int_a^b u^q$$

Consequently :

$$\frac{\lambda}{q}\lim_{n\longrightarrow+\infty} I_2(u_n) = \frac{\lambda}{q} I_2(u) \tag{4.10}$$

Hence by (4.9) and (4.10)

$$\liminf_{n\longrightarrow+\infty} I(u_n) = \frac{1}{2}\liminf_{n\longrightarrow+\infty} \|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{\lambda}{q}\lim_{n\longrightarrow+\infty} I_2(u_n) \geq \frac{1}{2}I_1(u) - \frac{\lambda}{q}I_2(u) = I(u)$$

$\square$

Lemma 4.3. *The functional I is bounded from below in $E^{\alpha}_{L,0}(a,b)$: i.e*

$$\exists M \in \mathbb{R} : I(u) > M, \forall u \in E^{\alpha}_{L,0}(a,b)$$

Proof. By (4.3) we obtian:

$$I(u) = \frac{1}{2}||u||^2_{E^{\alpha}_{L,0}(a,b)} - \frac{\lambda}{q}\|u^q\|_{L^q}$$

By the continuous embedding, we obtain:

$$I(u) \geq \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{\lambda c}{q}\|u\|^q_{E^{\alpha}_{L,0}(a,b)}$$

We using Young inquality(1.3) we get :

$$\begin{aligned} I(u) &\leq \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{\lambda c}{q}\epsilon\|u\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{c_\epsilon \lambda c}{q} \\ &= \left(\frac{1}{2} - \frac{\lambda c}{q}\epsilon\right)\|u\|^2_{E^{\alpha}_{L,0}(a,b)} - C \qquad \left(C = \frac{c_\epsilon \lambda c}{q}\right) \end{aligned}$$

For $\epsilon < \frac{q+1}{2\lambda c}$, we have $\left(\frac{1}{2} - \frac{\lambda c}{q}\epsilon\right) \geq 0$
Hence

$$I(u) \geq -C$$

$\square$

$\square$

Proof of Theorem 4.1. According to Theorem 4.2, the functional I attains its minimum at some $u \in E^{\alpha}_{L,0}(a,b)$. Since I is differentiable (see Proposition 4.1), u is a critical point of I. which is the non-trivial weak solution of (P_λ) .
we have $I(u) = \inf I(u)$, let $w \in E^{\alpha}_{L,0}(a,b)$ with $\|w\|_{L^q(a,b)} = 1$

$$\begin{aligned} I(tw) &= t^2\|D^{\alpha}_{a^+}w\|_{L^2(a,b)} - t^q\|w\|^q_{L^q(a,b)} \\ &= t^q\left(t^{2-q}\|D^{\alpha}_{a^+}w\|_{L^2(a,b)} - \|w\|^q_{L^q(a,b)}\right) \\ &\leq t^q\left(t^{2-q}\|D^{\alpha}_{a^+}w\|_{L^2(a,b)} - 1\right) < 0 \end{aligned}$$

for t small emough

$$I(tw) < 0$$

Hence $u \neq 0$

4.3 Existance of the weak solution for the superlinear case $2 < q < 2^*$

In this case, we observe that the functional defined by (4.3) is not bounded from below on the whole space $E^{\alpha}_{L,0}(a,b)$. However, it may be bounded below on certain subsets of this space. One such subset is known as the Nehari manifold.

4.3.1 Preliminaries

Definition 4.4. ***(Nehari manifold)*** *We define the Nehari manifold as follows:*

$$M = \left\{ u \in E^{\alpha}_{L,0}(a,b) \;\middle|\; \langle I'(u), u\rangle = 0 \right\}, \tag{4.11}$$

which is equivalent to:

$$M = \left\{ u \in E^{\alpha}_{L,0}(a,b) \;\middle|\; u \neq 0,\ \|u\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda \|u\|^q_{L^q(a,b)} \right\}. \tag{4.12}$$

Remark 4.2. *If $u \in M$ then we have*

$$I(u) = \left(\frac{1}{2} - \frac{1}{q}\right) \|u\|^2_{E^{\alpha}_{L,0}(a,b)}. \tag{4.13}$$

That is, the functional I restricted to the Nehari manifold M takes the above simplified form.

Remark 4.3. *The functional I given by (4.13) is coercive on M. Indeed, since $q \in]2, 2^*[$, we have $q > 2$, and hence*

$$\frac{1}{2} - \frac{1}{q} > 0$$

Therefore,

$$\lim_{\|u\| \to +\infty,\, u \in M} I(u) = +\infty$$

This shows that I is coercive on M, allowing us to apply the direct method of the calculus of variations to obtain a minimizer on M.

4.3.2 Main Result

Theorem 4.3. *Let* $q \in]2, 2^*[$, *and* $\lambda > 0$. *Then the problem (4.1) admits a non trivial weak solution in* M.

The proof of this theorem is based on the proofs of the following lemmas

Lemma 4.4. *The Nehari manifold defined by (4.12) is not empty.*

Proof. Let $u \in E^{\alpha}_{L,0}(a,b)$ with $u \neq 0$. We aim to show that there exists a real $t > 0$ such that $tu \in M$, that is:

$$\|tu\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda \|tu\|^q_{L^q(a,b)}$$

Then

$$t^2 \|u\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda t^q \|u\|^q_{L^q(a,b)}$$

Dividing both sides by t^2 (since $t > 0$), we obtain:

$$\|u\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda t^{q-2} \|u\|^q_{L^q(a,b)}$$

which leads to:

$$t^{q-2} = \frac{\|u\|^2_{E^{\alpha}_{L,0}(a,b)}}{\lambda \|u\|^q_{L^q(a,b)}}$$

Thus, we define:

$$t = \left(\frac{\|u\|^2_{E^{\alpha}_{L,0}(a,b)}}{\lambda \|u\|^q_{L^q(a,b)}} \right)^{\frac{1}{q-2}} > 0$$

Therefore, there exists $t > 0$ such that $tu \in M$, which implies that the Nehari manifold M is nonempty. □

Lemma 4.5. *We define* $m = \inf\limits_{u \in M} I(u)$ *and we have* $m > 0$.

Proof. Let $u \in M$ and by embedding inquality we have :

$$\|u\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda \|u\|^q_{L^q} \leq \lambda C \|u\|^q_{E^{\alpha}_{L,0}(a,b)}$$

for some $C > 0$ we have

$$\|u\|_{E^{\alpha}_{L,0}(a,b)} \geq \left(\frac{1}{\lambda C} \right)^{\frac{1}{q-2}} \tag{4.14}$$

We have

$$\begin{aligned} m = \inf_{u \in M} I(u) &= \left(\frac{1}{2} - \frac{1}{q} \right) \inf_{u \in M} \|u\|^2_{E^{\alpha}_{L,0}(a,b)} \\ &\geq \left(\frac{1}{2} - \frac{1}{q} \right) \left(\frac{1}{\lambda C} \right)^{\frac{2}{q-2}} \end{aligned}$$

Hence

$$m = \inf_{u \in M} I(u) > 0$$

□

Lemma 4.6. *The energy functional I attains its minimum over the Nehari manifold M at some $u \neq 0$, that is , There exists $u \in M$, with $u(x) \geq 0$ almost everywhere in (a,b) , such that :*

$$m = \inf_{u \in M} I(u) = I(u)$$

Proof. Let $(u_n) \subset M$ be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} I(u_n) = \inf_{u \in M} I(u) = m$$

By the definition of the set M, we have:

$$(u_n)_{n\in\mathbb{N}} \in M \;\Leftrightarrow \begin{cases} u_n \in E^{\alpha}_{L,0}(a,b), \quad u_n \neq 0, \\ \|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda \|u_n\|^q_{L^q(a,b)} \end{cases} \Rightarrow \begin{cases} |u_n| \in E^{\alpha}_{L,0}(a,b), \quad |u_n| \neq 0, \\ \|\,|u_n|\,\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda \|\,|u_n|\,\|^q_{L^q(a,b)} \end{cases} \Rightarrow |u_n| \in M$$

Since $(|u_n|) \subset M$ and $I(u_n) = I(|u_n|)$, we may assume that

$$u_n(x) \geq 0$$

Therefore, we can consider a new minimizing sequence consisting of nonnegative functions. Hence,

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{q}\right) \|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} = m$$

Since the functional I is coercive, the sequence (u_n) is bounded in $E^{\alpha}_{L,0}(a,b)$. Moreover, we know that

$$E^{\alpha}_{L,0}(a,b) \hookrightarrow_c L^q(a,b)$$

i.e., the embedding is compact. Therefore, up to a subsequence, we have:

$$u_n \rightharpoonup u \quad \text{in } E^{\alpha}_{L,0}(a,b), \quad \text{and} \quad u_n \to u \quad \text{in } L^q(a,b)$$

which implies that

$$u_n(x) \to u(x) \quad \text{a.e. } x \in (a,b)$$

We have

$$I(u) = \frac{1}{2}\|u\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{1}{q}\|u\|^q_{L^q(a,b)}$$

Since $u_n \rightharpoonup u$ in $E^{\alpha}_{L,0}(a,b) \Rightarrow \|u\| \leq \liminf\limits_{n\longrightarrow+\infty} \|u_n\|$

$$\begin{aligned}
I(u) &\leq \frac{1}{2}\liminf_{n\longrightarrow+\infty} \|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{1}{q}\liminf_{n\longrightarrow+\infty} \|u_n\|^q_{L^q(a,b)} \\
&= \liminf_{n\longrightarrow+\infty} \left(\frac{1}{2}\|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} - \frac{1}{q}\|u_n\|^q_{L^q(a,b)}\right) \\
&= \liminf_{n\longrightarrow+\infty} I(u_n) = m
\end{aligned}$$

Hence

$$I(u) \leq m \tag{4.15}$$

- If $u \in M$, then

$$I(u) \geq m = \inf_{u\in M} I(u) \tag{4.16}$$

Since $(u_n) \in M$, we have

$$\|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda\|u_n\|^q_{L^q(a,b)}$$

Now, suppose by contradiction that

$$\|u_n\|_{E^{\alpha}_{L,0}(a,b)} \to 0 \quad \text{as } n \to \infty$$

Then, since $E^{\alpha}_{L,0}(a,b)$ is a reflexive Banach space, there exists a subsequence (still denoted u_n) such that

$$u_n \rightharpoonup u \quad \text{weakly in } E^{\alpha}_{L,0}(a,b)$$

Using the weak lower semi-continuity of the norm, we get

$$\|u\| \leq \liminf_{n\to\infty} \|u_n\| = 0 \quad \Rightarrow \quad u = 0$$

This contradicts inequality (4.14), which asserts the existence of a constant $C > 0$ such that for all $u \in M$,

$$\|u\| \geq \left(\frac{1}{\lambda C}\right)^{\frac{1}{p-2}} > 0$$

Hence, the assumption $\|u_n\| \to 0$ must be false,then

$$u \neq 0$$

- It remains to show that

$$\|u\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda\|u\|^q_{L^q(a,b)}.$$

By the weak lower semi-continuity of the norm, we have:

$$\|u\|^2_{E^{\alpha}_{L,0}(a,b)} \leq \liminf_{n\to\infty} \|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} = \lambda \liminf_{n\to\infty} \|u_n\|^q_{L^q(a,b)} = \lambda\|u\|^q_{L^q(a,b)}.$$

Assume, by contradiction, that:

$$\|u\|^2_{E^\alpha_{L,0}(a,b)} < \lambda \|u\|^q_{L^q(a,b)}$$

Define

$$t = \frac{\|u\|^2_{E^\alpha_{L,0}(a,b)}}{\lambda \|u\|^q_{L^q(a,b)}} < 1$$

Then, $tu \in M$ i.e:

$$\begin{aligned} I(tu) &= \left(\frac{1}{2} - \frac{1}{q}\right) \|tu\|^2_{E^\alpha_{L,0}(a,b)} \\ &= t^2 \left(\frac{1}{2} - \frac{1}{q}\right) \|u\|^2_{E^\alpha_{L,0}(a,b)} \\ &\leq t^2 \liminf_{n\to\infty} \left(\frac{1}{2} - \frac{1}{q}\right) \|u_n\|^2_{E^\alpha_{L,0}(a,b)} \\ &= t^2 \liminf_{n\to\infty} I(u_n) \\ &\leq t^2 m. \end{aligned}$$

Hence by (4.16) we get :

$$m < I(tu) \leq t^2 m \Rightarrow t \geq 1$$

This leads to a contradiction , thus our assumption was false, and we conclude that:

$$\|u\|^2_{E^\alpha_{L,0}(a,b)} = \lambda \|u\|^q_{L^q(a,b)} \Rightarrow u \in M$$

Finally, by (4.15) and (4.16), it follows that:

$$I(u) = m$$

□

Lemma 4.7. *Let $u \in M$ be a minimizer of the energy functional I over the Nehari manifold $M \subset E^\alpha_{L,0}(a,b)$. Then u is a critical point of I, that is,*

$$I'(u) = 0 \quad in \quad E^\alpha_{L,0}(a,b).$$

Proof. Let $v \in E^\alpha_{L,0}(a,b)$ and we define the function as follow :

$$\Phi(s) = \left(\frac{\|u + sv\|^2_{E^\alpha_{L,0}(a,b)}}{\|u + sv\|^q_{L^q(a,b)}}\right)^{\frac{1}{p-2}}$$

Where $s \in (-\epsilon, \epsilon)$ is small emough , and $\Phi(s) > 0$ such that $\Phi(s)(u + sv) \in M$
And we have :

$$\Phi(0) = \left(\frac{\|u\|^2_{E^\alpha_{L,0}(a,b)}}{\|u\|^q_{L^q(a,b)}} \right)^{\frac{1}{p-2}} = 1$$

Thus , we define the function γ as

$$\gamma(s) = I(\Phi(s)(u + sv))$$

Since u is a minimizer of I on M , the function $\gamma(s)$ has minimum at $s = 0$
Hence

$$\gamma(0) = I(u) \leq \gamma(s)$$

Therfore

$$\gamma'(s) = 0$$

Differentiating γ with respect to s, we obtain:

$$\gamma'(s) = \langle I'\left(\Phi(s)(u + sv)\right), \Phi'(s)(u + sv) + \Phi(s)v \rangle$$

Since $\Phi(0) = 1$ and $u \in M$, this yields:

$$\begin{aligned} \gamma'(0) &= \langle I'\left(\Phi(0)u\right), \Phi'(0)u + v \rangle \\ &= \langle I'(u), \Phi'(0)u + v \rangle \end{aligned}$$

Using the fact that $\langle I'(u), u \rangle = 0$, we conclude:

$$\gamma'(0) = \langle I'(u), v \rangle$$

□

Proof of Theorem 4.3 By above lemma the functional (I) admits a critical point whish is the non-trivial weak solution in M

4.4 Existence of the weak solution for the linear case $q = 2$

In this section, we study the linear problem associated with the Riemann-Liouville fractional operator, where we fix the exponent $q = 2$. Our goal is to determine the eigenvalues, in particular the first eigenvalue λ_1, using the weak formulation and variational characterization. We also show that the problem admits only the trivial solution when $\lambda < \lambda_1$, while a nontrivial solution exists when $\lambda \geq \lambda_1$.

4.4.1 Preliminaries

Definition 4.5. ***(Eigen value problem** (P_λ)**)***the eigen value* (P_λ)*) is defien as follow :*

$$(\mathrm{P}_\lambda)\begin{cases} D^\alpha_{b^-}D^\alpha_{a^+}u = \lambda u & \textit{in } x \in \quad]a,b[, \\ B_\alpha(u) = 0 & \textit{on } x \in \partial]a,b[\end{cases}$$

Where :

$$B_\alpha(u) = \begin{cases} \lim\limits_{x \longrightarrow a^+} I^{1-\alpha}u(x) = 0 & \textit{if} \quad \alpha \in \left]0, \frac{1}{2}\right[\\ u(a) = u(b) = 0 & \textit{if} \quad \alpha \in \left]\frac{1}{2}, 1\right[\end{cases}$$

λ is called an eigenvalue of the operator $D^\alpha_{b^-}D^\alpha_{a^+}$, and the non-trivial weak solution u is called the eigenfunction associated with λ.

Definition 4.6. *We define the bilinear form associated with* (P_λ) *as follows:*

$$a(u,v) = \int_a^b D^\alpha_{a^+}u(x)\, D^\alpha_{a^+}v(x)\, dx - \lambda \int_a^b u(x)v(x)\, dx \qquad \forall u, v \in E^\alpha_{L,0}(a,b). \tag{4.17}$$

Remark 4.4. *Let us note that the above problem admits* 0 *as a solution, which is called the trivial solution. Our intention is to investigate the existence of nontrivial weak solutions of* (P_λ) *.*

4.4.2 The case $\lambda < \lambda_1$

Theorem 4.4. *The problem* (P_λ) *admits a unique weak solution wich is the trivial solution in* $E^\alpha_{L,0}(a,b)$ *when* $-\infty < \lambda < \lambda_1$ *with*

$$\lambda_1 = \inf_{\substack{w \in E^\alpha_{L,0}(a,b) \\ w \neq 0}} \frac{\|D^\alpha_{a^+}w\|^2_{L^2(a,b)}}{\|w\|^2_{L^2(a,b)}}$$

Proof. We consider two distinct subcases

- **Case $\lambda \leq 0$**

To prove the theorem (4.4) in this case , we apply the Lax-Milgram Theorem.

$i)$ We define the bilinear form as follow

$$a(u,v) = \int_a^b D^\alpha_{a^+}u(x) D^\alpha_{a^+}v(x)\ dx - \lambda \int_a^b u(x)v(x)\ dx$$

We verify the conditions of the Lax-Milgram theorem:

1. **Continuity:** For all $u, v \in E^{\alpha}_{L,0}(a,b)$,

$$\begin{aligned} |a(u,v)| &= \left| \int_a^b D^{\alpha}_{a^+}u(x) D^{\alpha}_{a^+}v(x)\,dx - \lambda \int_a^b u(x)v(x)\ dx \right| \\ &\leq \left| \int_a^b D^{\alpha}_{a^+}u(x) D^{\alpha}_{a^+}v(x)\,dx \right| + |\lambda| \left| \int_a^b u(x)v(x)\,dx \right| \end{aligned}$$

   By Hölder's inequality, we obtain

$$\begin{aligned} |a(u,v)| &\leq \|D^{\alpha}_{a^+}u\|_{L^2(a,b)} \|D^{\alpha}_{a^+}v\|_{L^2(a,b)} + |\lambda| \|u\|_{L^2(a,b)} \|v\|_{L^2(a,b)} \\ &\leq C \|u\|_{E^{\alpha}_{L,0}(a,b)} \|v\|_{E^{\alpha}_{L,0}(a,b)}, \end{aligned}$$

   where the norm $\|u\|_{E^{\alpha}_{L,0}(a,b)}$ is defined appropriately for the space $E^{\alpha}_{L,0}(a,b)$.

2. **Coercivity:** by Poincaré inquality (3.14)

$$a(u,u) = \|D^{\alpha}_{a^+}u\|^2_{L^2} - \lambda \|u\|^2_{L^2} \geq \|D^{\alpha}_{a^+}u\|^2_{L^2} \geq \|u\|^2_{E^{\alpha}_{L,0}}.$$

ii) The linear form $\ell(v)$ is equal to zero.then ℓ is continuous

Hence, by the Lax-Milgram theorem, there exists a unique weak solution which is the trivial solution $u \in E^{\alpha}_{L,0}(a,b)$ to the problem.

- **Case** $0 < \lambda < \lambda_1$

In this case we applay Poincaré inequality. By the bilinear form (4.17), we have:

$$a(u,u) = \|D^{\alpha}_{a^+}u\|^2_{L^2} - \lambda \|u\|^2_{L^2}$$

Applying the Poincaré inequality (3.14), which asserts:

$$\|u\|^2_{L^2} \leq c \|D^{\alpha}_{a^+}u\|^2_{L^2}, \quad \forall u \in E^{\alpha}_{L,0}(a,b)$$

we deduce:

$$\begin{aligned} a(u,u) &\geq \|D^{\alpha}_{a^+}u\|^2_{L^2} - \lambda c \|D^{\alpha}_{a^+}u\|^2_{L^2} \\ &= (1 - \lambda c) \|D^{\alpha}_{a^+}u\|^2_{L^2}. \end{aligned}$$

Thus, $a(u,u)$ is coercive if $1 - \lambda c > 0$, that is:

$$\lambda < \frac{1}{c}$$

Moreover, Poincaré's inequality implies:

$$\frac{1}{c} \leq \inf_{u \in E^{\alpha}_{L,0}(a,b)} \frac{\|D^{\alpha}_{a^+}u\|^2_{L^2}}{\|u\|^2_{L^2}} = \lambda_1$$

Therefore, coercivity holds whenever $\lambda < \lambda_1$

□

4.4.3 The case $\lambda = \lambda_1$

Theorem 4.5. *For* $\lambda = \lambda_1$ *the problem* (P_{λ_1}) *has a non trivial weak solution* u *which called the first eigen function of* $D^{\alpha}_{b^-} D^{\alpha}_{a^+}$

Proof. we introduce its weak formulation, which consists in finding a function $u \in E^{\alpha}_{L,0}(a,b)$ such that :

$$\int_a^b D^{\alpha}_{a^+} u \, D^{\alpha}_{a^+} v \, dx = \lambda_1 \int_a^b uv \, dx \qquad \forall v \in \mathscr{D}(a,b)$$

We put $v = u$ we obtain :

$$\|D^{\alpha}_{a^+} u\|^2_{L^2(a,b)} = \lambda_1 \|u\|^2_{L^2(a,b)}$$

Let we put :

$$\phi(w_n) = \frac{\|D^{\alpha}_{a^+} w_n\|^2_{L^2}}{\|w_n\|^2_{L^2}}$$

The weak solution u satisfies :

$$\lambda_1 = \frac{\|D^{\alpha}_{a^+} u\|^2_{L^2(a,b)}}{\|u\|^2_{L^2(a,b)}} = \phi(u)$$

Then $\lambda_1 = \inf\limits_{w \in E^{\alpha}_{L,0}(a,b)} \phi(w)$ Let $(w_n) \subset E^{\alpha}_{L,0}(a,b)$ be a minimizing sequence, i.e.,

$$\lim_{n \to \infty} \phi(w_n) = \inf_{w \in E^{\alpha}_{L,0}(a,b)} \phi(w) = \lambda_1 > 0$$

Now define the normalized sequence:

$$u_n = \frac{v_n}{\|v_n\|_{L^2}(a,b)} \quad \Rightarrow \quad \|u_n\|_{L^2}(a,b) = 1 \quad \text{and} \quad D^{\alpha}_{a^+} u_n = \frac{D^{\alpha}_{a^+} v_n}{\|v_n\|_{L^2}(a,b)}. \tag{4.18}$$

Then:

$$\phi(v_n) = \|D^{\alpha}_{a^+} u_n\|^2_{L^2} = \phi(u_n) \to \lambda_1$$

Hence, the sequence (u_n) is bounded in $E^{\alpha}_{L,0}(a,b)$, and there exists a subsequence (still denoted (u_n)) and $u \in E^{\alpha}_{L,0}(a,b)$ such that:

$$u_n \rightharpoonup u \quad \text{in } E^{\alpha}_{L,0}(a,b)$$

By the compact embedding $E^{\alpha}_{L,0}(a,b) \hookrightarrow_c L^2(a,b)$, we obtain:

$$u_n \to u \quad \text{in } L^2(a,b), \quad \|u_n\|_{L^2} = 1 \to \|u\|_{L^2} = 1$$

Also, since the norm is weakly lower semicontinuous and $\phi(u_n) \to \lambda_1$, we have:

$$\begin{aligned}
\phi(u) = \|D^{\alpha}_{a^+}u\|^2_{L^2} &\leq \liminf_{n\to\infty} \|D^{\alpha}_{a^+}u_n\|^2_{L^2} \\
&= \liminf_{n\to\infty} \frac{\|D^{\alpha}_{a^+}u_n\|^2_{L^2}}{\|u_n\|^2_{L^2}} \qquad \text{(by dividing by } \|u_n\|^2_{L^2}) \\
&= \liminf_{n\to\infty} \phi(u_n) \\
&= \lambda_1
\end{aligned}$$

Therefore,

$$\phi(u) = \frac{\|D^{\alpha}_{a^+}u\|^2_{L^2(a,b)}}{\|u\|^2_{L^2(a,b)}} \leq \lambda_1$$

But by definition of λ_1, we have:

$$\phi(u) \geq \lambda_1$$

which implies:

$$\phi(u) = \lambda_1$$

Thus, if ϕ attains its minimum at u, we must show that u is a critical point of ϕ. Let

$$\phi(u) = \frac{\|D^{\alpha}_{a^+}u\|^2_{L^2}}{\|u\|^2_{L^2}}$$

We have $I : u \longmapsto \|D^{\alpha}_{a^+}u\|^2_{L^2(a,b)}$ is differentiable.

$$< I'(u), v >= 2\int_a^b D^{\alpha}_{a^+}u \ D^{\alpha}_{a^+}v \ dx \qquad \text{see the proof of (4.7)}$$

In the same way, we can show that $J : u \longmapsto \|u\|^2_{L^2(a,b)}$ is differentiable and we have ,

$$\langle J'(u), v\rangle = 2\langle u, v\rangle$$

Then by using the quotient rule for differentiation, we find that the derivative is given by:

$$\begin{aligned}
\langle \phi'(u), v\rangle &= \frac{2\langle D^{\alpha}_{a^+}u, D^{\alpha}_{a^+}v\rangle \|u\|^2_{L^2} - 2\langle u, v\rangle \|D^{\alpha}_{a^+}u\|^2_{L^2}}{\|u\|^4_{L^2}} \\
&= \frac{2\|u\|^2_{L^2}\left(\langle D^{\alpha}_{a^+}u, D^{\alpha}_{a^+}v\rangle - \langle u, v\rangle \frac{\|D^{\alpha}_{a^+}u\|^2_{L^2}}{\|u\|^2_{L^2}}\right)}{\|u\|^4_{L^2}}.
\end{aligned}$$

Thus,

$$\langle \phi'(u), v\rangle = \frac{2}{\|u\|^2_{L^2}}\left(\langle D^{\alpha}_{a^+}u, D^{\alpha}_{a^+}v\rangle - \lambda_1 \langle u, v\rangle\right)$$

Hence, $u \in E^{\alpha}_{L,0}(a,b) \setminus \{0\}$ and $\phi(u) = \lambda_1 = \inf\limits_{u \in E^{\alpha}_{L,0}(a,b)} \phi(w)$

Hence

$$\phi'(u) = 0 \Rightarrow \langle \phi'(u), v \rangle = 0 \ , \ \forall v$$

We have $\phi'(u) = 0$, that is,

$$\frac{2}{\|u\|^2_{L^2}} \left(\langle D^{\alpha}_{a^+} u, D^{\alpha}_{a^+} v \rangle - \lambda_1 \langle u, v \rangle \right) = 0$$

Hence,

$$\langle D^{\alpha}_{a^+} u, D^{\alpha}_{a^+} v \rangle - \lambda_1 \langle u, v \rangle = 0, \tag{4.19}$$

Hence, u is a weak solution of (P) . Since $\|u\|_{L^2(a,b)} = 1$, then u is non trivial solution. $\square$

4.4.4 The case $\lambda > \lambda_1$

In this section, we use the decomposition spectral theorem 1.12

Definition 4.7. *Let* $\alpha \in (0,1)$. *We define the following linear operator*

$$\begin{array}{rcl} T : L^2(a,b) & \to & L^2(a,b) \\ u & \mapsto & Tv = u \end{array} \tag{4.20}$$

where u *is the solution of the boundary value problem*

$$\left\{ \begin{array}{rcll} D^{\alpha}_{b^-} D^{\alpha}_{a^+} u & = & v & \text{in }]a,b[\\ B_{\alpha}(u) & = & 0 & \text{on } \partial]a,b[\end{array} \right.$$

Then we state the following theorem

Theorem 4.6. *There exists an eigenvalue sequence* (λ_n) *such that* (P_{λ_n}) *has non trivial solution*
Furthermore $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \ , \ \lambda_n \to +\infty$

To prove this theorem, we verify the conditions of the spectral decomposition theorem.

Proposition 4.2. *The operator* T *is well defined and linear*

Proof.
- To prove that the operator T is well defined, we apply the Lax-Milgram theorem. We consider the bilinear form

$$a(u,v) = \int_a^b D^{\alpha}_{a^+} u(x) \, D^{\alpha}_{a^+} v(x) \, dx$$

This bilinear form satisfies the following properties:

- **Continuity**: By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|a(u,v)| &\leq \|D^{\alpha}_{a^+}u\|_{L^2(a,b)}\, \|D^{\alpha}_{a^+}v\|_{L^2(a,b)} \\ &= \|u\|_{E^{\alpha}_{L,0}(a,b)}\, \|v\|_{E^{\alpha}_{L,0}(a,b)} \qquad \text{(by Remark 3.2).}\end{aligned}$$

- **Coercivity**:

$$a(u,u) = \int_a^b |D^{\alpha}_{a^+}u(x)|^2\, dx = \|u\|^2_{E^{\alpha}_{L,0}(a,b)}$$

Furthermore, the linear form is:

$$\ell(v) = \int_a^b u(x)v(x)\, dx$$

- **Continuous** : By the Cauchy–Schwarz inequality and the continuity embedding, we obtain:

$$\begin{aligned}\ell(v) &\leq \|u\|_{L^2(a,b)}\|v\|_{L^2(a,b)} \\ &\leq \|u\|_{E^{\alpha}_{L,0}(a,b)}\|v\|_{E^{\alpha}_{L,0}(a,b)}\end{aligned}$$

By the Lax-Milgram theorem, there exists a unique $u \in E^{\alpha}_{L,0}(a,b)$ such that

$$a(u,v) = \ell(v), \quad \forall v \in E^{\alpha}_{L,0}(a,b)$$

Hence, the operator $T\colon v \mapsto u$ is well defined.

- Now, we prove the linearity of the operator T

  Let $v_1, v_2 \in L^2(a,b)$ and $\forall\ \mu_1, \mu_2 \in \mathbb{R}$, we show that :

$$T(\mu_1 v_1 + \mu_2 v_2) \;=\; \mu_1 T(v_1) + \mu_2 T(v_2)$$

Since u_1, u_2 are solutions of the following boundary value problems, respectively:

$$u_1 = T(v_1), \quad \text{i.e.} \begin{cases} D^{\alpha}_{b^-}D^{\alpha}_{a^+}u_1 = v_1 & \text{in }]a,b[\\ B_{\alpha}(u_1) = 0 & \text{on } \partial]a,b[\end{cases} \tag{4.21}$$

$$u_2 = T(v_2), \quad \text{i.e.} \begin{cases} D^{\alpha}_{b^-}D^{\alpha}_{a^+}u_2 = v_2 & \text{in }]a,b[\\ B_{\alpha}(u_2) = 0 & \text{on } \partial]a,b[\end{cases} \tag{4.22}$$

We multiply equations (4.21) and (4.22) by $\mu_1$and μ_2 respectively, and add them term by term.We obtain:

$$D^{\alpha}_{b^-}D^{\alpha}_{a^+}\left(\mu_1 u_1 + \mu_2 u_2\right) = \mu_1 v_1 + \mu_2 v_2$$

Hence,

$$T(\mu_1 v_1 + \mu_2 v_2) = \mu_1 u_1 + \mu_2 u_2 = \mu_1 T(v_1) + \mu_2 T(v_2)$$

Now consider the boundary condition. Since B_α is a linear boundary operator:

$$B_\alpha(u) = B_\alpha(\lambda u_1 + \mu u_2) = \mu_1 B_\alpha(u_1) + \mu_2 B_\alpha(u_2) = \lambda \cdot 0 + \mu \cdot 0 = 0$$

So T is **linear**.

□

Proposition 4.3. *The operator T is self-adjoint in $L^2(a,b)$ that is :*

$$< Tv_1, v_2 >=< v_1, Tv_2 > \qquad \forall v_1, v_2 \in L^2(a,b)$$

Proof. Let $v_1, v_2 \in L^2(a,b)$ i.e $Tv_1 = u_1$ and $Tv_2 = u_2$
So

$$\int_a^b D^\alpha_{a^+} u_1(x) D^\alpha_{a^+} u_2(x)dx = \int_a^b v_1(x)u_2(x)dx \tag{4.23}$$

And

$$\int_a^b D^\alpha_{a^+} u_2(x) D^\alpha_{a^+} u_1(x)dx = \int_a^b v_2(x)u_1(x)dx \tag{4.24}$$

Then by (4.23) and (4.24) we obtain :

$$\int_a^b v_1(x)u_2(x)dx = \int_a^b v_2(x)u_1(x)dx$$

Hence ,

$$<Tv_1, v_2> = <v_1, Tv_2>$$

□

Proposition 4.4. *The operator T is compact.*

Proof. Let (v_n) be bounded sequence in $L^2(a,b)$ such that :

$$Tv_n = u_n \quad \Leftrightarrow \quad \left\{ \begin{array}{lcll} D^\alpha_{b^-} D^\alpha_{a^+} u_n & = & v_n & \text{in }]a,b[\\ B_\alpha(u_n) & = & 0 & \text{on } \partial]a,b[\end{array} \right.$$

$$\Rightarrow \quad \int_a^b D^\alpha_{a^+} u_n D^\alpha_{a^+} \varphi dx = \int_a^b v_n \varphi \qquad \forall \varphi \in \mathscr{D}(a,b)$$

We put $\varphi = u_n$ we obtain :

$$\|D^\alpha_{a^+} u_n\|^2_{L^2(a,b)} = \int_a^b v_n u_n dx$$

By the Cauchy–Schwarz theorem and the Poincaré inequality (3.1), we obtain:

$$\|u_n\|^2_{E^{\alpha}_{L,0}(a,b)} \leq C\|v_n\|_{L^2(a,b)}\|u_n\|_{E^{\alpha}_{L,0}(a,b)}$$

which implies that

$$\|u_n\|_{E^{\alpha}_{L,0}(a,b)} \leq C\|v_n\|_{L^2(a,b)} = C'$$

By compact embedding theorem (3.6) we have $E^{\alpha}_{L,0}(a,b) \hookrightarrow_c L^2(a,b)$ i.e:

$$\begin{array}{lccl} \exists\, u_{k_n}: & u_{k_n} & \longrightarrow & u \text{ in } L^2(a,b) \\ & Tv_{k_n} & \longrightarrow & u \end{array}$$

Hence, the operator T is compact ☐

Proof of Theorem 4.6 By the spectral decomposition theorem (1.12), we know that the space $L^2(a,b)$ admits a Hilbert basis $(u_n)_{n\in\mathbb{N}^*}$ consisting of eigenfunctions associated with eigenvalues $(\mu_n)_{n\in\mathbb{N}^*}$, where (μ_n) is a decreasing sequence converging to zero:

$$Tu_n = \mu_n u_n \quad \text{with} \quad \mu_n \to 0.$$

It follows that

$$D^{\alpha}_{b^-} D^{\alpha}_{a^+}(\mu_n u_n) = u_n,$$

and hence,

$$D^{\alpha}_{b^-} D^{\alpha}_{a^+} u_n = \frac{1}{\mu_n} u_n = \lambda_n u_n,$$

where (λ_n) is an increasing sequence such that $\lambda_n \to +\infty$.

Hence , the problem (P$_\lambda$) admit a non trivial weak solution u_n at $\lambda = \lambda_n = \frac{1}{\mu_n}$, for all n

Conclusion

In summary, this thesis explores the theoretical and practical aspects of fractional differential equations, with a particular focus on proving the existence of weak solutions for both linear and nonlinear problems. By combining classical tools from functional analysis with modern techniques in fractional calculus, this work contributes to the understanding of how fractional derivatives can be effectively applied in mathematical modeling. The structure of the thesis is designed to guide the reader progressively—from foundational concepts to advanced applications—highlighting both the richness and the challenges of this rapidly growing field.

Bibliography

[1] **L.Bourdin, D. Idczak.**, A fractional fundamental lemma and a fractional integration by parts formula – Applications to critical points of Bolza functionals and to linear boundary value problems.Advances in Differential Equations, 2015, 20 (3-4), pp.213-232.

[2] **H. Brezis**, Functional Analysis: Sobolev Spaces and Partial Differential Equations. Springer, New York ,2011

[3] **F.Demengel,G.Demengel** : Functional Spaces for the Theory of Elliptic Partial Differential Equations,Springer, London, 2012.

[4] **D. Idczak , S. Walczak**, Fractionnal Sobolev spaces via Riemann-Liouville derivatives, Journal of Function Spaces and Applications, 2013, 1-15, (2013).

[5] **H.Jin , W.Liu** , Eigenvalue problem for fractional differential operator containing left and right fractional derivative. Adv. Differ. Equ, 246 ,2016.

[6] **A.A. Kilbas, H.H. Srivastava, J.J. Trujillo**: Theory and Applications of Fractional Differential Equations, Elsevier Science B.V., Amsterdam, 2006.

[7] **Nawfal Elhage Hassan**, Topologie générale et espaces normés. Dunod, Paris, 2011.

[8] **I. Podlubny**, Fractional Differential Equations, Academic Press,United States of America, 1999.

[9] **S.G. Samko, A.A. Kilbas and O.I. Marichev**: Fractional Integrals and Derivatives-Theory and Applications, Gordonand Breach Science Publishers, Amsterdam, The Netherlands, 1993.

[10] **E. Torres Ledesma1, and C. Montalvo Bonilla:** Fractional Sobolev space with Riemann–Liouville fractional derivative and application to a fractional concave–convex problem.Tusi Mathematical Research Group, 1–38, 2021.

Abstract

This thesis aims to study fractional differential equations of both linear and nonlinear types, based on Riemann–Liouville derivatives of non-integer order.

We begins by presenting the necessary mathematical background, including Lebesgue and fractional Sobolev spaces, and introduces the fundamental concepts of fractional integration and differentiation. These tools are then applied to analyze three main cases of the studied equation: the sublinear case $1 \leq q < 2$, the superlinear case $2 < q < 2^*$, and the linear case $q = 2$. In each case, we prove the existence of weak solutions within suitable functional spaces, taking into account the variation of conditions related to the spectral parameter λ.

Keywords: Riemann–Liouville integral, Riemann–Liouville derivative, fractional Sobolev spaces, fractional boundary value problems.

Résumé

Ce mémoire a pour objectif l'étude des équations différentielles fractionnaires de type linéaire et non linéaire, en s'appuyant sur les dérivées de Riemann–Liouville d'ordre non entier.

Le travail commence par la présentation des bases mathématiques nécessaires, incluant les espaces de Lebesgue et de Sobolev fractionnaires, ainsi que les notions fondamentales d'intégration et de dérivation fractionnaires.

Ces outils sont ensuite appliqués à l'analyse de trois cas principaux de l'équation étudiée : le cas sous-linéaire $1 \leq q < 2$, le cas sur-linéaire $2 < q < 2^*$, et le cas linéaire $q = 2$. Dans chaque situation, nous démontrons l'existence de solutions faibles dans des espaces fonctionnels appropriés, en tenant compte des différentes conditions imposées sur le paramètre spectral λ.

Mots-clés : Intégrale de Riemann–Liouville, dérivée de Riemann–Liouville, espaces de Sobolev fractionnaires, problèmes aux limites fractionnaires.

ملخص

تهدف هذه المذكرة إلى دراسة المعادلات التفاضلية الكسرية من النوع الخطي واللاخطي بالاعتماد على مشتقات ريمان-ليوفيل ذات الرتبة غير الصحيحة. نبدأ بتقديم الخلفية الرياضية اللازمة، بما في ذلك فضاءات ليباغ وسوبوليف الكسرية، ونعرض المفاهيم الأساسية للتكامل والمشتقة الكسرية. ثم نطبق هذه الأدوات على دراسة ثلاث حالات رئيسية من المعادلة المدروسة: الحالة تحت-خطية $1 \leq q < 2$، الحالة فوق-خطية $2 < q < 2^*$، والحالة الخطية $q = 2$. في كل حالة، نثبت وجود حلول ضعيفة ضمن فضاءات دالية مناسبة، مع مراعاة اختلاف الشروط على المعامل الطيفي λ.

الكلمات المفتاحية: التكامل ريمان–ليوفيل، المشتقة ريمان–ليوفيل، الفضاءات الكسرية من نوع سوبوليف، مسائل القيم الحدية الكسرية.