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Theme

On the fuzzy numbers

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Contents

Introduction

Fuzzy sets were introduced in 1965 by Lotfi Zadeh with a view to reconcile mathematical modeling and human knowledge in the engineering sciences[19] .

Introduce by Alberto Nunes Cosenza and Mora-Camin [10] a particular class of fuzzy numbers, the fuzzy dual numbers, which present comparatively to classical fuzzy numbers additional attractive properties either from the point of view of theory as from the point for view of applications in the field of engineering and decision theory.

Someapproximation methods were introduced. For instance, Dubois and Prade [16] extended usual algebraic operations on real numbers to fuzzy numbers, and suggested a standard approximation to fuzzy arithmetic with efficient computation.

Fuzzy numbers allow us to make the mathematical model of linguistic expressions such as "many", "at least" or "about", fuzzy quantifiers or fuzzy cardinality. We can also build the quantified and probability statements [25]. And we present fuzzy numbers application in data analysis, artificial intelligence and decision making. We have counted on [1, 2, 4, 5, 17, 24].

In this memoir, we will study the fuzzy numbers.

The memoir is divided into three chapters :

The first chapter fuzzy sets and its operations, fuzzy relations and its operations, convex and concave fuzzy sets, characteristics of fuzzy set, as we explain cardinality of a fuzzy set, triangular norms and conorms.

The second chapter we present the concept of a fuzzy number and its operations, triangular fuzzy number and some kinds of fuzzy number of which triangular and Trapezoidal fuzzy number.

Finally, the final chapter is some properties of fuzzy numbers on fuzzy function, integration of fuzzy function and differentiation of fuzzy function.

Chapter 1

Preliminary on fuzzy sets, fuzzy relations, α-cuts and t-norms

This chapter reviews the concepts and notations of sets (crisp sets), and then introduces the concepts of fuzzy sets. The concept of fuzzy sets is a generalization of the crisp sets. Operations of fuzzy sets, fuzzy relations, convex and concave fuzzy set, characteristics of fuzzy set, cardinality of a fuzzy set, triangular norms and conorms. As we relied on [27, 22, 21]

1.1 Generalities on classic sets

Definition 1.1. *In classic set, we find that the element either belongs or does not belong to the set, for example, X in a qualitative set, and A is a subset of which the function μ_A is what gives each element $x \in X$ the degree of belonging to set A.*
if $x \in A \Longrightarrow \mu_A(x) = 1$
if $x \notin A \Longrightarrow \mu_A(x) = 0$
i.e., the function $\mu_A : X \longrightarrow \{0, 1\}$.

1.1.1 Operations of classic sets

In this section, we recall definition for some operations of classic sets, union, intersection, and negation.

Definition 1.2 (Union, Disjunction). *Let A and B be classes, the union of A and B is defined to be the class of all the elements which belong either to A, or B, or to both A and B. In symbols,*

$$A \cup B = \{x \in X / x \in A \quad or \quad x \in B\}.$$

Definition 1.3 (Intersection, Conjunction). *Let A and B the intersection of A and B is defined to be the class of all the elements which belong to both A and B. In symbols,*

$$A \cap B = \{x \in X / x \in A \quad and \quad x \in B\}.$$

Definition 1.4 (Negation, Complement). *The complement of a class A is the class of all the elements which do not belong to A. In symbols,*

$$\bar{A} = \{x \in X / x \notin A\}.$$

1.2 Fuzzy sets

A fuzzy set has been defined by lotfi Aliasker Zadeh in 1965 as a class of sets with a continuous characteristic function called its membership function. Let U be a reference set and consider its subsets.

Definition 1.5. [23]

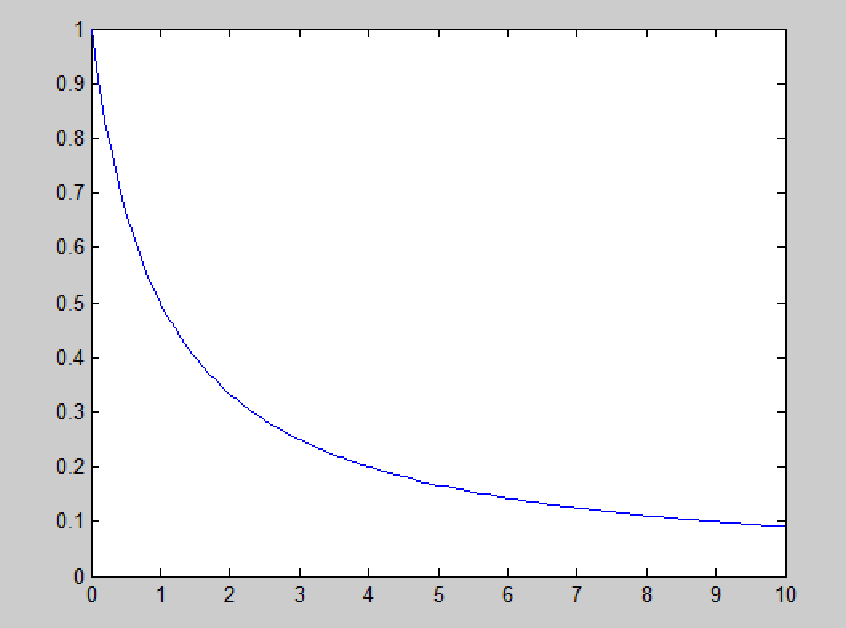
Let X be a nonempty set. A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.

Example 1.1. *(1) Let $X = \{0, 3\}$*

a fuzzy set $A = \{(0, 0.3), (1, 1), (2, 0.7), (3, 0.9)\}$.

(2) Let $X = [0, 10]$*, and* A *fuzzy subset in* X*, defined by :*

$$\mu_A(x) = \frac{1}{1+x}$$



Graph of μ_A

1.2.1 Operations of fuzzy sets

In this section, we recall definition for some operations of fuzzy sets, union, intersection, and complement.

Definition 1.6 (Union)**.** [18] *Membership value of member* x *in the union takes the greater value of membership between* A *and* B*.*

$\mu_{A\cup B}(x) = Max \quad [\mu_A(x), \mu_B(x)], x \in X.$

Definition 1.7 (Intersection)**.** [18] *We can find intersection of fuzzy sets* A *and* B *takes smaller value of membership function between* A *and* B*.*

$\mu_{A\cap B}(x) = Min \quad [\mu_A(x), \mu_B(x)], x \in X.$

Definition 1.8 (Complement)**.** [18] *The complement set of fuzzy set A likewise in crisp set. We denote the complement set of A as $\bar{A}$. Membership degree can be calculated as following.*

$\mu_{\bar{A}}(x) = 1 - \mu_A(x), x \in X.$

Definition 1.9 (Equality)**.** [18] *The equality of fuzzy sets A and B takes smaller value of membership function between A and B. We say that $A = B$, if and only if*

$\mu_A(x) = \mu_B(x) \quad \forall x \in X.$

Definition 1.10 (Inclusion)**.** [18] *The inclusion of fuzzy sets A and B takes smaller value of membership function between A and B. We say that $A \subseteq B$, if and only if*

$\mu_A(x) \leqslant \mu_B(x) \quad \forall x \in X.$

Definition 1.11 (Sum)**.** [18] *The sum of fuzzy sets A and B takes smaller value of membership function between A and B. The sum defined by $\forall x \in X$*

$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x).$

Definition 1.12 (Product)**.** [18] *The product of fuzzy sets A and B takes smaller value of membership function between A and B. The product defined by $\forall x \in X$*

$\mu_{A\times B}(x) = \mu_A(x)\mu_B(x).$

Example 1.2. *Let $X = \{a, b, c\}$, and let $A = \{(a, 0.2), (b, 0.9), (c, 0.5)\}$ a fuzzy set $\subset X$ and $B = \{(a, 0.7), (b, 0.1), (c, 1)\}$ a fuzzy set $\subset X$ we have :*

$A \cap B = \{(a, 0.2), (b, 0.1), (c, 0.5)\}.$

$A \cup B = \{(a, 0.7), (b, 0.9), (c, 1)\}.$

$A \times B = \{(a, 0.14), (b, 0.09), (c, 0.5)\}.$

$A + B = \{(a, 0.76), (b, 0.94), (c, 1)\}.$

$C(A) = \{(a, 0.8), (b, 0.1), (c, 0.5)\}.$

Proposition 1.1 (Fundamental properties of fuzzy sets operations)**.** *Let $A, B, C \in F(X)$, we have the following properties:*

Involution	$\bar{\bar{A}} = A$
Commutativity	$A \cup B = B \cup A, A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C) \quad and \quad (A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$\begin{cases} A \cap (B \cup C)) = (A \cap B) \cup (A \cap C) \quad and \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{cases}$
Absorption	$A \cup (A \cap B) = A, A \cap (A \cup B)) = A$
Idempotence	$A \cup A = A, A \cap A = A$
Absorption by X and $\emptyset$	$A \times X = X, A \cap \emptyset = \emptyset$
Identity	$A \cup \emptyset = A$
Law of contradiction	$A \cap \bar{A} = \emptyset$
Law of excluded middle	$A \cup \bar{A} = X$
De Morgan's laws	$\overline{A \cap B} = \bar{A} \cup \bar{B}$ *and* $\overline{A \cup B} = \bar{A} \cap \bar{B}$

1.3 Cartesian product of fuzzy sets

Let X and Y be any two sets. The Cartesian product set,
$X \times Y = \{(x, y)/x \in X, y \in Y\}$, fuzzy subset R is called fuzzy relation of X and Y, its membership function is denoted by $\mu_R(x, y) \in [0, 1]$.

Definition 1.13 (Cartesian product)**.** [18] *The cartesian product applied to multiple fuzzy sets can be defined as follows.*
Denoting $\mu_{A_1}(x), \mu_{A_2}(x), ...\mu_{A_n}(x)$, *as membership functions of* $A_1, A_2, ..., A_n$ *for* $\forall x_1 \in A_1, x_2 \in A_2, ...x_n \in A_n$. *Then, the probability for n-tuple* $(x_1, x_2, ..., x_n)$ *to be involved in fuzzy set* $A_1 \times A_2 \times ... \times A_n$ *is ,*

$$\mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) = min\left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\right\}.$$

Example 1.3. *Lets* $X_1 = \{a, b, c\}$, $X_2 = \{\alpha, \beta\}$ *and let* A_1, A_2 *be two fuzzy sets, respectively defined on* X_1 *and* X_2 *by:*
$A_1 = \{(a, 0.1), (b, 0.4), (c, 0.8)\}$ *; it can be written like this* $A_1 = 0.1/a + 0.4/b + 0.8/c$

$A_2 = \{(\alpha, 0.2), (\beta, 0.6)\}$ *; it can be written like this* $A_2 = 0.2/\alpha + 0.6/\beta$.
Therefore,
$A_1 \times A_2 = \{((a,\alpha), 0.1), ((a,\beta), 0.1), ((b,\alpha), 0.2), ((b,\beta), 0.4), ((c,\alpha), 0.2), ((c,\beta), 0.6)\}$.

1.4 Fuzzy relations

[3] A fuzzy relation is an extension of an ordinary relation, and it playing an important role in fuzzy mathematics. In an ordinary set theory, relation R describes the relation between affirmation and negation with "yes" and "no". It has a range of 0, 1. 1 is yes, 0 is no. Besides the "absolutely" relation with "yes" and "no", the objective things also have many vague concepts, such as "some relation","close relation" and so on.
Fuzzy relations are a subsets of the cartesian product $X \times Y$, where X and Y are an universes of discourse.

Definition 1.14 (Fuzzy relations)**.** [18] *Fuzzy relation has degree of membership whose value lies in* $[0, 1]$.

$$\mu_R : A \times B \longrightarrow [0, 1]$$

$R = \{((x, y), \mu_R(x, y))/\mu_R(x, y) \geq 0, x \in A, y \in B\}$.

1.4.1 α-cut of fuzzy relations

We have learned about α-cut for fuzzy sets, and we know a fuzzy relation is one kind of fuzzy sets. Therefore, we can apply the α-cut to the fuzzy relation.

Definition 1.15 (α-cut)**.** [18] *We can obtain α-cut relation from a fuzzy relation by taking the pairs which have membership degrees no less than α. Assume $R \subseteq A \times B$, and R_α is a α-cut relation. Then*

$$R_\alpha = \{(x, y)|\mu_R(x, y) \geq \alpha, x \in A, y \in B\}.$$

Note that R_α is a crisp relation.

Example 1.4. *Let* $A = \{a, b\}$ *and we have a fuzzy relation* R *on* $A \times A$ *such that,*
$M_R = \begin{bmatrix} 0.6 & 0.1 \\ 0.9 & 0.5 \end{bmatrix}$
then we can have some α*-cut relations in the following.*

If $\alpha = 0.5$

$$M_{R_{0.5}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

If $\alpha = 0.6$

$$M_{R_{0.6}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

If $\alpha = 0.9$

$$M_{R_{0.9}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

1.4.2 Fuzzy relation and representations with matrix

We recall the matrix as fuzzy matrix, whose elements contain values within the closed interval $[0, 1]$.

When X and Y are finite domains, fuzzy relation R of X to Y can be represented by fuzzy matrix.

In general, $X = \{x_1, x_2 ... x_m\}$ and $Y = \{y_1, y_2 ... y_n\}$ are finite sets, the fuzzy relation of $X \times Y$ can be represented by the following $m \times n$ matrix:

$$\begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) & ... & \mu_R(x_1, y_n) \\ \mu_R(x_2, y_1) & \mu_R(x_2, y_2) & ... & \mu_R(x_2, y_n) \\ ... & ... & ... & ... \\ \mu_R(x_m, y_1) & \mu_R(x_m, y_2) & ... & \mu_R(x_m, y_n) \end{bmatrix}$$

The general form of fuzzy matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

Where $0 \leq a_{ij} \leq 1, 1 \leq i \leq m, 1 \leq j \leq n$, it is recorded as $A = [a_{ij}]_{m\times n}$.

1.4.3 Operations of fuzzy relations

A fuzzy relation is a fuzzy subset on $X \times Y$, being a special kind of fuzzy sets. The following are five common operations: Suppose R_1 and R_2 are fuzzy relations on $X \times Y$ *let* $(x, y) \in X \times Y$, then there are:

Definition 1.16 (Equality)**.** [12] $R_1 = R_2 \Longleftrightarrow \mu_{R_1}(x, y) = \mu_{R_2}(x, y)$.

Definition 1.17 (Containing)**.** [12] $R_1 \subset R_2 \Longleftrightarrow \mu_{R_1}(x, y) \leqslant \mu_{R_2}(x, y)$.

Definition 1.18 (Union)**.** [12] $R_1 \cup R_2 \Longleftrightarrow \mu_{(R_1\cup R_2)}(x, y) = Max\{\mu_{R_1}(x, y), \mu_{R_2}(x, y)\} = \mu_{R_1}(x, y) \vee \mu_{R_2}(x, y)$.

Definition 1.19 (intersection)**.** [12] $R_1 \cap R_2 \Longleftrightarrow \mu_{(R_1\cap R_2)}(x, y) = Min\{\mu_{R_1}(x, y), \mu_{R_2}(x, y)\} = \mu_{R_1}(x, y) \wedge \mu_{R_2}(x, y)$.

Definition 1.20 (complement)**.** [12] $\bar{R}_1 \Longleftrightarrow \mu_{\bar{R}_1}(x, y) = 1 - \mu_{R_1}(x, y), (x, y) \in X \times Y$.

Example 1.5. *Suppose R_1, R_2 and R_3 are fuzzy relations on $X \times Y, (x, y) \in X \times Y$ and,*

$$R_1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.9 & 0.6 \end{bmatrix}, R_2 = \begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.5 \end{bmatrix}, R_3 = \begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.5 \end{bmatrix}$$

(1) Equality $R_2 = R_3 \Leftrightarrow \mu_{R_2}(x, y) = \mu_{R_3}(x, y)$.

(2) Containing $R_2 \subset R_1 \Leftrightarrow \mu_{R_2}(x, y) \leqslant \mu_{R_1}(x, y)$.

(3) Union $R_1 \cup R_2 \Leftrightarrow \mu_{(R_1\cup R_2)}(x, y) = Max\{\mu_{R_1}(x, y), \mu_{R_2}(x, y)\} = \mu_{R_1}(x, y) \vee \mu_{R_2}(x, y) = \begin{bmatrix} 0.4 & 0.2 \\ 0.9 & 0.6 \end{bmatrix} \vee \begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.4 \vee 0.3 & 0.2 \vee 0.1 \\ 0.9 \vee 0.7 & 0.6 \vee 0.5 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 \\ 0.9 & 0.6 \end{bmatrix}$.

(4) Intersection $R_1 \cap R_2 \iff \mu_{(R_1\cap R_2)}(x,y) = Min\{\mu_{R_1}(x,y), \mu_{R_2}(x,y)\} = \mu_{R_1}(x,y) \wedge \mu_{R_2}(x,y) = \begin{bmatrix} 0.4 & 0.2 \\ 0.9 & 0.6 \end{bmatrix} \wedge \begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.4\wedge 0.3 & 0.2 \wedge 0.1 \\ 0.9 \wedge 0.7 & 0.6 \wedge 0.5 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.5 \end{bmatrix}$
$= R_2$.

(5) Complement $\mu_{\bar{R}_1}(x,y) = 1 - \mu_{R_1}(x,y)$.

$$\bar{R}_1 = \begin{bmatrix} 1-0.4 & 1-0.2 \\ 1-0.9 & 1-0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0.1 & 0.4 \end{bmatrix}.$$

1.5 Convex and Concave fuzzy sets

The notion of convexity and concavity can be generalized to fuzzy sets of a universe X, which we shall assume to be a real Euclidean N-dimensional space (Zadeh, 1965).

1.5.1 Convex fuzzy sets

Definition 1.21. [15] *A fuzzy set* $A \subset X$ *is convex if and only if its* α*-cuts are convex. An equivalent definition of convexity is: A is convex if and only if*

$\forall x_1 \in X, \forall x_2 \in X, \lambda \in [0,1],$
$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_A(x_1), \mu_A(x_2))$.

Example 1.6. *Let a fuzzy set* $A = \{(6,0.5),(8,0.9),(7,1)\}$ *and* $\lambda = \frac{1}{2}$
$x_1 = 6, x_2 = 8$.
We have $\mu_A(\lambda x_1 + (1-\lambda)x_2) = \mu_A(3+4) = \mu_A(7) = 1 \geq min(\mu_A(x_1), \mu_A(x_2)) = min(\mu_A(6), \mu_A(8)) = min(0.5, 0.9) = 0.5$. *then A is convex.*

1.5.2 Concave fuzzy sets

Definition 1.22. [6]

A fuzzy set $A \in X$ *is a concave fuzzy set if and only if*
$\forall x_1 \in X, \forall x_2 \in X, \lambda \in [0,1]$, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \leq max(\mu_A(x_1), \mu_A(x_2))$.

Example 1.7. *Let a fuzzy set* $A = \{(1,0.1),(-1,1),(0,0.5)\}$ *and* $\lambda = \frac{1}{2}, x_1 = 1$,
$x_2 = -1$.

We have $\mu_A(\lambda x_1 + (1-\lambda)x_2) = \mu_A(\frac{1}{2} + \frac{1}{2}(-1)) = \mu_A(0) = 0.5 \leq max(\mu_A(x_1),$ $\mu_A(x_2)) = max(\mu_A(1), \mu_A(-1)) = max(0.1, 1) = 1.$ *then* A *is concave.*

Proposition 1.2. [9]

If A *is concave, then its complement* $\bar{A} = 1 - A$ *is convex and vice versa.*

Proof. If A is concave, then for any two points p and q and any point r on the line segment $\overline{pq}$

$$A(r) \leq max[A(p), A(q)],$$

i.e., $\bar{A}(r) \geqslant 1 - max[1-\bar{A}(p), 1-\bar{A}(q)]$. If $1-\bar{A}(p) \geqslant 1-\bar{A}(q)$, then $\bar{A}(r) \geqslant \bar{A}(p)$, i.e.,

$$if \quad \bar{A}(q) \geqslant \bar{A}(p), \quad then \quad \bar{A}(r) \geqslant \bar{A}(p). \tag{1.1}$$

Similarly, if $1 - \bar{A}(q) \geqslant 1 - \bar{A}(p)$ then $\bar{A}(r) \geqslant \bar{A}(q)$,
i.e.,

$$if \quad \bar{A}(p) \geqslant \bar{A}(q), then \quad \bar{A}(r) \geqslant \bar{A}(q). \tag{1.2}$$

Relations 1.1 and 1.2 can be combined into

$$\bar{A}(r) \geq min[\bar{A}(p), \bar{A}(q)],$$

which is the relation to state that t is a convex fuzzy set. ☐

1.6 Characteristics of fuzzy sets

1.6.1 α-cuts

Let A be a fuzzy set on X and A_α its α-cut. A_α can be written $\mu_A^{-1}([\alpha, 1])$, i.e., the inverse image of the $[\alpha, 1]$. Let $\mu_{[\alpha,1]}$ be the characteristic function of the interval [α, 1] in the universe [0, 1]. We get $\mu_{A_{\alpha(x)}} = \mu_{[\alpha,1]}(\mu_A(x)), x \in X$.

Definition 1.23 (α-cuts)**.** [18] *The α-cut set A_α is made up of members whose membership is not less than α. $A_\alpha = \{x \in X / \mu_A(x) \geqslant \alpha\}$ note that α is arbitrary. This α-cut set is a crisp set.*

Definition 1.24 (Height)**.** [20] *The height of a fuzzy set A is the largest membership grade of any element in A.*

$$H(A) = Max \mu_A(x).$$

Definition 1.25 (Support)**.** [20] *The support of a fuzzy set A, denoted by $S(A)$, we mean all elements of X that belong to a non zero degree. That is $S(A)$ is a calssical set defined by*

$$S(A) = \{x \in X | \mu_A(x) > 0\}.$$

Definition 1.26 (Kernel)**.** [20] *The ker of a fuzzy set A, denoted by $ker(A)$, we mean all elements of X that belong to a equal one. That is $ker(A)$is a calssical set defined by*

$$ker(A) = \{x \in X | \mu_A(x) = 1\}.$$

Example 1.8. *Let $X = \{x_1, x_2, x_3, x_4\}$ be a universal set, and let $A = \{(x_1, 0, 3), (x_2, 0, 5), (x_3, 0, 8), ($ an fuzzy subset in X.*
If $\alpha = 0, 5$, $A_{0,5} = \{x_2, x_3\}$.
If $\alpha = 0, 3$, $A_{0,3} = \{x_1, x_2, x_3\}$.
$H(A) = 1$.
$S(A) = \{x_1, x_2, x_3, x_4\}$.
$ker(A) = \{x_4\}$.

1.7 Cardinality of fuzzy sets

1.7.1 Scalar Cardinality

When X is a finite set, the cardinality $|A|$ of a fuzzy set A on X is defined as

$$|A| = \sum_{x \in X} \mu_A(x).$$

$|A|$ is sometimes called the power of A.

$\|A\| = \frac{|A|}{|X|}$ is the relative cardinality. It can be interpreted as the proportion of elements of X that are in A.
When X is not finite, $|A|$ does not always exist. However, if A has a finite support,$|A| = \sum_{x \in supA} \mu_A(x)$.

Example 1.9. *Let* $X = [1, 2, 3]$
A fuzzy set $A = \{(1, 0.5), (2, 1), (3, 0.9)\}$
The cardinality
$|A| = \sum_{x \in X} \mu_A(x) = \mu_A(1) + \mu_A(2) + \mu_A(3) = 0.5 + 1 + 0.9 = 2.4$
The relative cardinality
$\|A\| = \frac{|A|}{|X|} = \frac{2.4}{3}$

1.8 T-norms and T-conorms

There are two types of operators in fuzzy sets: t-norms and t-conorms. They are often called as triangular-norm and triangular-conorm respectively

1.8.1 Triangular norms

Definition 1.27 (Triangular norms). [18] *A triangular norm (T-norm) is a binary operation* $t : [0, 1] \times [0, 1] \longrightarrow [0, 1]$*, satisfying the following conditions.* $\forall x, y, z \in [0, 1]$

1. *Commutativity* $T(x, y) = T(y, x)$
2. *Associativity* $T(x, T(y, z)) = T(T(x, y), z)$
3. *Monotonicity if* $y \leq z$ *then* $T(x, y) \leq T(x, z)$
4. *Identity* $T(1, x) = x$

Notice that for all T-norm $T(0, x) = 0$.

Example 1.10. $T_M(x,y) = min(x,y) = x \wedge y$

$T_M : [0,1] \times [0,1] \longrightarrow [0,1]$

Let $x \in [0,1] \Longrightarrow x \leqslant 1 \Longrightarrow T_M(x,1) = min(x,1) = x.$

Let $x, y \in [0,1] \Longrightarrow T_M(x,y) = min(x,y) = min(y,x) = T_M(y,x).$

$T_P(x,y) = x \cdot y$ *(product t-norm).*

$T_L(x,y) = max(x+y-1,0).$

$T_L : [0,1] \times [0,1] \longrightarrow [0,1]$

Let $x \in [0,1] \Longrightarrow x \geqslant 0 \Longrightarrow T_L(x,1) = max(x+1-1,0) = max(x,0) = x.$

Let $x, y \in [0,1] \Longrightarrow T_L(x,y) = max(x+y-1,0) = max(0,y+x-1) = T_L(y,x).$

1.8.2 Triangular conorms

Definition 1.28 (Triangular conorms)**.** [18] *A triangular conorm (T-conorm) is a binary operation* $s : [0,1] \times [0,1] \longrightarrow [0,1]$*, satisfying the following conditions.* $\forall x, y, z \in [0,1]$

1 . *Commutativity* $T(x,y) = T(y,x)$

2 . *Associativity* $T(x,T(y,z)) = T(T(x,y),z)$

3 . *Monotonicity if* $y \leq z$ *then* $T(x,y) \leq T(x,z)$

4 . *Identity* $T(0,x) = x$

Notice that for all T-conorm $T(1,x) = 1.$

Example 1.11. $T_M(x,y) = max(x,y)$

$T_M : [0,1] \times [0,1] \longrightarrow [0,1]$

Let $x \in [0,1] \Longrightarrow x \geqslant 0 \Longrightarrow T_M(x,0) = max(x,0) = x.$

Chapter 2

Definition of a fuzzy number and types of a fuzzy numbers, arithmetic operations on fuzzy numbers

This chapter describes fuzzy numbers. First we'll look into interval, the fundamental concept of fuzzy number, then operations of fuzzy numbers. In addition, we'll introduce special kind of fuzzy number such as triangular fuzzy number and trapezoidal fuzzy number.

2.1 Concept of fuzzy numbers

2.1.1 Fuzzy number

Fuzzy number is expressed as a fuzzy set defining a fuzzy interval in the real number $\mathbb{R}$.The α-cut operation can be also applied to the fuzzy number. If we denote α-cut interval for fuzzy number A as A_α, the obtained interval A_α is defined as

$$A_\alpha = [a_1^\alpha, a_3^\alpha]$$

Definition 2.1. [14] *A fuzzy number $\bar{A}$ is a fuzzy set constrained by a membership function,*

$$\mu_{\bar{A}} : \mathbb{R} \longrightarrow [0,1],$$

that satisfies:

1. *A is normal, i.e., there exists a real number m, such that $\mu_A(m) = 1$.*
2. *A is fuzzy convex, i.e., for any pair x_1, x_2 belonging to support (A),*

   $$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_A(x_1), \mu_A(x_2)) \quad \forall \lambda \in [0,1],$$

   where support (A) is the support of A and $S(A) = \{x \in \mathbb{R}/\mu_A(x) > 0\}$.
3. *A is upper semicontinuous, i.e, for each $\alpha \in (0,1)$, the α-level set $[A]_\alpha = \{x \in \mathbb{R}/\mu_A(x) \geq \alpha\}$is closed.*

Definition 2.2. [8] *Consider a fuzzy subset of the real line $A : R \longrightarrow [0,1]$. Then A is a fuzzy number if it satisfies the following properties:*

1. *A is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $A(x_0) = 1$*
2. *A is fuzzy convex, i.e.,*

   $$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_A(x_1), \mu_A(x_2)) \quad \forall \lambda \in [0,1], x_1, x_2 \in \mathbb{R}$$

3. *A is upper semicontinuous on $\mathbb{R}$ i.e., $\forall \epsilon > 0, \exists \delta > 0$ such that $A(x) - A(x_0) < \epsilon, |x - x_0| < \delta$*
4. *A is compactly supported i.e.,$cl\{x \in \mathbb{R}, A(x) > 0\}$ is compact, where $cl(B)$ denotes the closure of the set B*

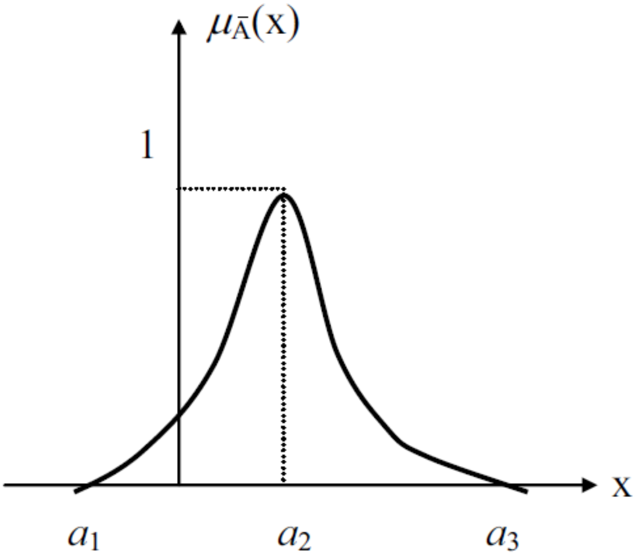


Figure 2.1: Fuzzy Number $\bar{A} = (a_1, a_2, a_3)$

Example 2.1. *Between whether the function* $A : \mathbb{R} \longrightarrow [0.1]$, *defined as a form*

$$A(x) = \begin{cases} 0 \quad if \quad x < -1 \\ \frac{(x+1)}{2} \quad if \quad -1 \leqslant x < 1 \\ \frac{(4-x)}{3} \quad if \quad 1 \leqslant x < 4 \\ 0 \quad if \quad x \geqslant 4 \end{cases}$$

is a fuzzy number because

A is a fuzzy set

We choose $x_0 = 1 \Longrightarrow A(x_0) = A(1) = 1$, *then A is normal*

And $[A]_\alpha = \{x \in \mathbb{R}/A(x) \geq \alpha\}$

If $-1 \leqslant x < 1$ *the* $\alpha = \frac{(x+1)}{2} \Longrightarrow x = 2\alpha - 1$

If $1 \leqslant x < 4$ *the* $\alpha = \frac{(4-x)}{3} \Longrightarrow x = 4 - 3\alpha$, *then* $[A]_\alpha = [2\alpha - 1, 4 - 3\alpha]$ *is closed*

For each $\alpha \in (0, 1)$ *the* $S(A) = \{x \in \mathbb{R}/\mu_A(x) > 0\}$, *then A is fuzzy number.*

2.2 Operations of fuzzy numbers

2.2.1 Operation of α-cut interval

We referred to α-cut interval of fuzzy number $\bar{A} = [a_1, a_3]$ as crisp set

$$A_\alpha = [a_1^\alpha, a_3^\alpha], \forall \alpha \in [0,1], a_1, a_3, a_1^\alpha, a_3^\alpha \in \mathbb{R}$$

so A_α is a crisp interval. As a result, the operations of interval reviewed in the previous section can be applied to the α-cut interval A_α.
If α-cut interval B_α of fuzzy number $\bar{B}$ is given

$$\bar{B} = [b_1, b_3], b_1, b_3 \in \mathbb{R}$$

$$B_\alpha = [b_1^\alpha, b_3^\alpha], \forall \alpha \in [0,1], b_1^\alpha, b_3^\alpha \in \mathbb{R},$$

operations between A_α and B_α can be described as follows :

$$A_\alpha(+)B_\alpha = [a_1^\alpha, a_3^\alpha](+)[b_1^\alpha, b_3^\alpha] = [a_1^\alpha + b_1^\alpha, a_3^\alpha + b_3^\alpha].$$

$$A_\alpha(-)B_\alpha = [a_1^\alpha, a_3^\alpha](-)[b_1^\alpha, b_3^\alpha] = [a_1^\alpha - b_3^\alpha, a_3^\alpha - b_1^\alpha].$$

$$A_\alpha(\times)B_\alpha = [a_1^\alpha, a_3^\alpha](\times)[b_1^\alpha, b_3^\alpha] = [a_1^\alpha \times b_1^\alpha, a_3^\alpha \times b_3^\alpha].$$

$$A_\alpha(/)B_\alpha = [a_1^\alpha, a_3^\alpha](/)[b_1^\alpha, b_3^\alpha] = [\frac{a_1^\alpha}{b_3^\alpha}, \frac{a_3^\alpha}{b_1^\alpha}], b_3^\alpha \neq 0, b_1^\alpha \neq 0.$$

The multiplication of a fuzzy number A in $\mathbb{R}$ by a real number $k > 0$.

$$(k \times A)_\alpha = k \times A_\alpha = [k \times a_1^\alpha, k \times a_3^\alpha]$$

Example 2.2. *Let* $\bar{A_\alpha} = [1+\alpha, 4-2\alpha]$ *and*
$\bar{B_\alpha} = [2+2\alpha, 6-2\alpha]$

$$\bar{A}_\alpha(+)\bar{B}_\alpha = [(1+\alpha)+(2+2\alpha), (4-2\alpha)+(6-2\alpha)] = [3+3\alpha, 10-4\alpha]$$

For $\alpha = 0$

$\bar{A}_0(+)\bar{B}_0 = [3, 10]$

For $\alpha = 1$

$\bar{A}_1(+)\bar{B}_1 = [6, 6]$

$\bar{A}(+)\bar{B} \cong [3, 6, 10]$

$$\bar{A}_\alpha(-)\bar{B}_\alpha = [(1+\alpha)-(6-2\alpha), (4-2\alpha)-(2+2\alpha)] = [-5+3\alpha, 2-4\alpha]$$

For $\alpha = 0$

$\bar{A}_0(-)\bar{B}_0 = [-5, 2]$

For $\alpha = 1$

$\bar{A}_1(-)\bar{B}_1 = [-2, -2]$

$\bar{A}(-)\bar{B} \cong [-5, -2, 2]$

$$\bar{A}_\alpha(\times)\bar{B}_\alpha = [(1+\alpha)\times(2+2\alpha), (4-2\alpha)\times(6-2\alpha)]$$

For $\alpha = 0$

$\bar{A}_0(\times)\bar{B}_0 = [2, 24]$

For $\alpha = 1$

$\bar{A}_1(\times)\bar{B}_1 = [8, 8]$

$\bar{A}(\times)\bar{B} \cong [2, 8, 24]$

$$\bar{A}_\alpha(/)\bar{B}_\alpha = [(1+\alpha)/(6-2\alpha), (4-2\alpha)/(2+2\alpha)]$$

For $\alpha = 0$

$\bar{A}_0(/)\bar{B}_0 = [1/6, 2]$

For $\alpha = 1$

$\bar{A}_1(/)\bar{B}_1 = [1/2, 1/2]$

$\bar{A}(/)\bar{B} \cong [1/6, 1/2, 2]$

$(k \times \bar{A})_\alpha$ *for* $k = 1$

$1 \times \bar{A}_\alpha = [1(1+\alpha), 1(4-2\alpha)] = [1+\alpha, 4-2\alpha]$

2.2.2 Operation of fuzzy numbers

(1) Addition: $\bar{A} \oplus \bar{B}$

$$\mu_{\bar{A}\oplus\bar{B}}(z) = \bigvee_{z=x+y}(\mu_{\bar{A}}(x) \wedge \mu_{\bar{B}}(y))$$

(2) Subtraction: $\bar{A} \ominus \bar{B}$

$$\mu_{\bar{A}\ominus\bar{B}}(z) = \bigvee_{z=x-y}(\mu_{\bar{A}}(x) \wedge \mu_{\bar{B}}(y))$$

(3) Multiplication: $\bar{A} \otimes \bar{B}$

$$\mu_{\bar{A}\otimes\bar{B}}(z) = \bigvee_{z=x\times y}(\mu_{\bar{A}}(x) \wedge \mu_{\bar{B}}(y))$$

(4) Division: $\bar{A} \oslash \bar{B}$

$$\mu_{\bar{A}\oslash\bar{B}}(z) = \bigvee_{z=x/y}(\mu_{\bar{A}}(x) \wedge \mu_{\bar{B}}(y))$$

(5) Minimum: $\bar{A}(\wedge)\bar{B}$

$$\mu_{\bar{A}(\wedge)\bar{B}}(z) = \bigvee_{z=x\wedge y}(\mu_{\bar{A}}(x) \wedge \mu_{\bar{B}}(y))$$

(6) Maximum: $\bar{A}(\vee)\bar{B}$

$$\mu_{\bar{A}(\vee)\bar{B}}(z) = \bigvee_{z=x\vee y}(\mu_{\bar{A}}(x) \wedge \mu_{\bar{B}}(y))$$

Example 2.3. *Let two fuzzy numbers* $\bar{A} = \{(1,1),(4,0.5)\}$ *and*
$\bar{B} = \{(2,1),(1,0.5)\}$
$\forall x \in \bar{A}, y \in \bar{B}.$
$\bullet z \in \bar{A} \oplus \bar{B}$

1. $z < 2 \quad \mu_{\bar{A}\oplus\bar{B}}(z) = 0.$
2. $z = 2 \quad x + y = 1 + 1$
   $\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(1) = 1 \wedge 0.5 = 0.5$
   $\mu_{\bar{A}\oplus\bar{B}}(2) = \vee(0.5) = 0.5.$
3. $z = 3 \quad x + y = 1 + 2$
   $\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(2) = 1 \wedge 1 = 1$
   $\mu_{\bar{A}\oplus\bar{B}}(3) = \vee(1) = 1.$
4. $z = 5 \quad x + y = 4 + 1$
   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(1) = 0.5 \wedge 0.5 = 0.5$
   $\mu_{\bar{A}\oplus\bar{B}}(5) = \vee(0.5) = 0.5.$
5. $z = 6 \quad x + y = 4 + 2$
   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(2) = 0.5 \wedge 1 = 0.5$
   $\mu_{\bar{A}\oplus\bar{B}}(6) = \vee(0.5) = 0.5.$
6. $z > 6 \quad \mu_{A\oplus\bar{B}}(z) = 0.$

so $\bar{A} \oplus \bar{B}$ *can be written as*
$\bar{A} \oplus \bar{B} = \{(2,0.5),(3,1),(5,0.5),(6,0.5)\}.$
$\bullet z \in \bar{A} \ominus \bar{B}$

1. $z < -1 \quad \mu_{\bar{A}\ominus\bar{B}}(z) = 0.$

2. $z=-1 \quad x-y=1-2$

   $\mu_{\bar{A}}(1) \wedge \mu_{B}(2) = 1 \wedge 1 = 1$

   $\mu_{\bar{A}\ominus\bar{B}}(-1) = \vee(1) = 1.$

3. $z=0 \quad x-y=1-1$

   $\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(1) = 1 \wedge 0.5 = 0.5$

   $\mu_{\bar{A}\ominus\bar{B}}(0) = \vee(0.5) = 0.5.$

4. $z=2 \quad x-y=4-2$

   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(2) = 0.5 \wedge 1 = 0.5$

   $\mu_{\bar{A}\ominus\bar{B}}(2) = \vee(0.5) = 0.5.$

5. $z=3 \quad x-y=4-1$

   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(1) = 0.5 \wedge 0.5 = 0.5$

   $\mu_{\bar{A}\ominus\bar{B}}(3) = \vee(0.5) = 0.5.$

6. $z \geqslant 4 \quad \mu_{\bar{A}\ominus\bar{B}}(z) = 0.$

so $\bar{A}\ominus\bar{B}$ *can be written as*

$\bar{A}\ominus\bar{B} = \{(-1,1),(0,0.5),(2,0.5),(3,0.5)\}.$

$\bullet z \in \bar{A}\otimes\bar{B}$

1. $z<1$

   $\mu_{\bar{A}\otimes\bar{B}}(z) = 0$

2. $z=1 \quad x\times y = 1\times 1$

   $\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(1) = 1 \wedge 0.5 = 0.5$

   $\mu_{\bar{A}\otimes\bar{B}}(1) = \vee(0.5) = 0.5$

3. $z=2 \quad x\times y = 1\times 2$

   $\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(2) = 1 \wedge 1 = 1$

   $\mu_{\bar{A}\otimes\bar{B}}(2) = \vee(1) = 1$

4. $z = 4 \quad x \times y = 4 \times 1$

   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(1) = 0.5 \wedge 0.5 = 0.5$

   $\mu_{\bar{A}\otimes\bar{B}}(4) = \vee(0.5) = 0.5$

5. $z = 8 \quad x \times y = 4 \times 2$

   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(2) = 0.5 \wedge 1 = 0.5$

   $\mu_{\bar{A}\otimes\bar{B}}(8) = \vee(0.5) = 0.5$

6. $z > 8$

   $\mu_{\bar{A}\otimes\bar{B}}(z) = 0$

so $\bar{A} \otimes \bar{B}$ *can be written as*

$\bar{A} \otimes \bar{B} = \{(1, 0.5), (2, 1), (4, 0.5), (8, 0.5)\}.$

$\bullet z \in \bar{A} \oslash \bar{B}$

1. $z < 1/2$

   $\mu_{\bar{A}\oslash\bar{B}}(z) = 0$

2. $z = 1/2 \quad x/y = 1/2$

   $\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(2) = 1 \wedge 1 = 1$

   $\mu_{\bar{A}\oslash\bar{B}}(1/2) = \vee(1) = 1$

3. $z = 1 \quad x/y = 1/1$

   $\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(1) = 1 \wedge 0.5 = 0.5$

   $\mu_{\bar{A}\oslash\bar{B}}(1) = \vee(0.5) = 0.5$

4. $z = 2 \quad x/y = 4/2$

   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(2) = 0.5 \wedge 1 = 0.5$

   $\mu_{\bar{A}\oslash\bar{B}}(2) = \vee(0.5) = 0.5$

5. $z = 4 \quad x/y = 4/1$

   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(1) = 0.5 \wedge 0.5 = 0.5$

   $\mu_{\bar{A}\oslash\bar{B}}(4) = \vee(0.5) = 0.5$

6. $z > 4$

$\mu_{\bar{A}\oslash\bar{B}}(z) = 0$

so $\bar{A} \oslash \bar{B}$ *can be written as*

$\bar{A} \oslash \bar{B} = \{(1/2, 1), (1, 0.5), (2, 0.5), (4, 0.5)\}.$

$\bullet z \in \bar{A}(\vee)\bar{B}$

1. $z < 1 \quad \mu_{\bar{A}(\vee)\bar{B}}(z) = 0.$

2. $z = 1 \quad x \vee y = 1 \vee 1$

$\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(1) = 1 \wedge 0.5 = 0.5$

$\mu_{\bar{A}(\vee)\bar{B}}(1) = \vee(0.5) = 0.5.$

3. $z = 2 \quad x \vee y = 1 \vee 2$

$\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(2) = 1 \wedge 1 = 1$

$\mu_{\bar{A}(\vee)\bar{B}}(2) = \vee(1) = 1.$

4. $z = 4 \quad x \vee y = 4 \vee 2$ *and* $x \vee y = 4 \vee 1$

$\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(2) = 0.5 \wedge 1 = 0.5$

$\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(1) = 0.5 \wedge 0.5 = 0.5$

$\mu_{\bar{A}(\vee)\bar{B}}(4) = \vee(0.5, 0.5) = 0.5.$

5. $z > 4 \quad \mu_{\bar{A}(\vee)\bar{B}}(z) = 0.$

so $\bar{A} \vee \bar{B}$ *can be written as*

$\bar{A} \vee \bar{B} = \{(1, 0.5), (2, 1), (4, 0.5)\}.$

$\bullet z \in \bar{A}(\wedge)\bar{B}$

1. $z < 1 \quad \mu_{\bar{A}(\wedge)\bar{B}}(z) = 0.$

2. $z = 1 \quad x \wedge y = 1 \wedge 1$ *and* $x \wedge y = 1 \wedge 2$ *and* $x \wedge y = 4 \wedge 1$

$\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(1) = 1 \wedge 0.5 = 0.5$

$\mu_{\bar{A}}(1) \wedge \mu_{\bar{B}}(2) = 1 \wedge 1 = 1$

$\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(1) = 0.5 \wedge 0.5 = 0.5$

$\mu_{\bar{A}(\wedge)\bar{B}}(1) = \vee(0.5, 1, 0.5) = 1.$

3. $z = 2 \quad x \wedge y = 4 \wedge 2$

   $\mu_{\bar{A}}(4) \wedge \mu_{\bar{B}}(2) = 0.5 \wedge 1 = 0.5$

   $\mu_{\bar{A}(\wedge)\bar{B}}(4) = \vee(0.5) = 0.5.$

4. $z > 2 \quad \mu_{\bar{A}(\wedge)\bar{B}}(z) = 0.$

so $\bar{A} \wedge \bar{B}$ *can be written as*

$\bar{A} \wedge \bar{B} = \{(1, 1), (2, 0.5)\}.$

2.3 Triangular fuzzy numbers

2.3.1 Definition of triangular fuzzy number

Definition 2.3. [7] *It is a fuzzy number represented with three points as follows :*

$$\bar{A} = (a_1, a_2, a_3)$$

this representation is interpreted as membership functions and holds the following conditions

1. a_1 *to* a_2 *is increasing function*
2. a_2 *to* a_3 *decreasing function*
3. $a_1 \leq a_2 \leq a_3$.

$$\mu_{\bar{A}}(x) = \begin{cases} 0, \quad x < a_1 \\ x - a_1/a_2 - a_1, \quad a_1 \leqslant x \leqslant a_2 \\ a_3 - x/a_3 - a_2, \quad a_2 \leqslant x \leqslant a_3 \\ 0, \quad x > a_3 \end{cases}$$

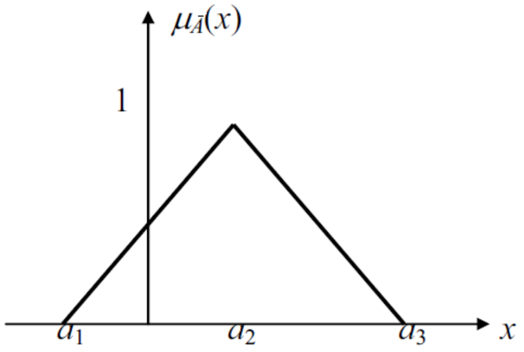


Figure 2.2: Triangular Fuzzy Number $\bar{A} = (a_1, a_2, a_3)$

Example 2.4. *The triangular fuzzy number* $\bar{A} = (-6, -2, 2)$, *the membership function value will be,*

$$\mu_{\bar{A}}(x) = \begin{cases} 0, \quad x < -6 \\ x + 6/4, \quad -6 \leqslant x \leqslant -2 \\ 2 - x/4, \quad -2 \leqslant x \leqslant 2 \\ 0, \quad x > 2 \end{cases}$$

2.3.2 Operations of triangular fuzzy number

First, consider addition and subtraction. Here we need not use membership function. Suppose triangular fuzzy numbers $\bar{A}$ and $\bar{B}$ are defined as,

$$\bar{A} = (a_1, a_2, a_3), \bar{B} = (b_1, b_2, b_3)$$

1. Addition $\bar{A} \oplus \bar{B} = (a_1, a_2, a_3)(+)(b_1, b_2, b_3)$
   $= (a_1 + b_1, a_2 + b_2, a_3 + b_3)$: triangular fuzzy number

2. Subtraction $\bar{A} \ominus \bar{B} = (a_1, a_2, a_3)(-)(b_1, b_2, b_3)$
   $= (a_1 - b_3, a_2 - b_2, a_3 - b_1)$: triangular fuzzy number

3. Multiplication $\bar{A} \otimes \bar{B} = (a_1, a_2, a_3) \times (b_1, b_2, b_3)$
   $= (\alpha, a_2 \times b_2, \beta)$

in which

$\alpha = min\{a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3\}$

$\beta = max\{a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3\}$

4. Division $\bar{A} \oslash \bar{B} = \bar{A} \otimes \bar{B}^{-1} = (a_1, a_2, a_3) \times (\frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1})$
   $= (\alpha, \frac{a_2}{b_2}, \beta)$ in which $b_1 \neq 0, b_2 \neq 0, b_3 \neq 0$.
5. Symmetric image $-(A) = (-a_3, -a_2, -a_1)$: triangular fuzzy number

Example 2.5. *Let* $\bar{A} = (2, 5, 3)$ *and* $\bar{B} = (1, 6, 7)$

$\bar{A} \oplus \bar{B} = (3, 11, 10)$.

$\bar{A} \ominus \bar{B} = (-5, -1, 2)$.

$\bar{A} \otimes \bar{B} = (\alpha, 5 \times 6, \beta)$

in which

$\alpha = min\{2, 14, 3, 21\} = 2$

$\beta = max\{2, 14, 3, 21\} = 21$

$\bar{A} \otimes \bar{B} = (2, 30, 21)$.

$\bar{A} \oslash \bar{B} = \bar{A} \otimes \bar{B}^{-1} = (2, 5, 3) \otimes (\frac{1}{7}, \frac{1}{6}, 1)$

$\alpha = min\{\frac{2}{7}, 2, \frac{3}{7}, 3\} =$

$\beta = max\{\frac{2}{7}, 2, \frac{3}{7}, 3\} = 3$

$\bar{A} \oslash \bar{B} = (\frac{2}{7}, \frac{5}{6}, 3)$.

$-(\bar{A}) = (-3, -5, -2)$.

2.4 Types of fuzzy numbers

2.4.1 Trapezoidal fuzzy number

[18] Another shape of fuzzy number is trapezoidal fuzzy number. This shape is originated from the fact that there are several points whose membership degree is maximum $\alpha = 1$.

Definition 2.4. *We can define trapezoidal fuzzy number* $\bar{A}$ *as*

$$\bar{A} = (a_1, a_2, a_3, a_4)$$

the membership function of this fuzzy number will be interpreted as follows

$$\mu_{\bar{A}}(x) = \begin{cases} 0, \quad x < a_1 \\ x - a_1/a_2 - a_1, \quad a_1 \leqslant x \leqslant a_2 \\ 1, \quad a_2 \leqslant x \leqslant a_3 \\ a_4 - x/a_4 - a_3, \quad a_3 \leqslant x \leqslant a_4 \\ 0, \quad x > a_4 \end{cases}$$

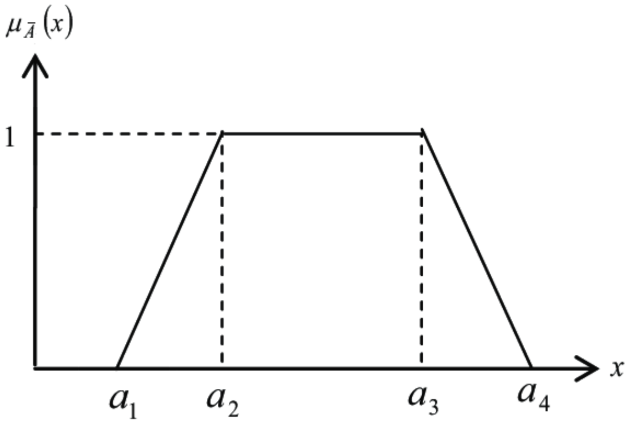


Figure 2.3: Trapezoidal-Fuzzy-Number

α-cut interval for this shape is written below.

$\forall \alpha \in [0, 1]$

$$A_\alpha = [(a_2 - a_1)\alpha + a_1, -\alpha(a_4 - a_3) + a_4]$$

when $a_2 = a_3$, the trapezoidal fuzzy number coincides with triangular one.

Example 2.6. $\bar{A} = (-2, 0, 1, 2)$ *is a trapezoidal fuzzy number*

$$\mu_{\bar{A}}(x) = \begin{cases} 0, & x < -2 \\ \frac{x+2}{2}, & -2 \leqslant x \leqslant 0 \\ 1, & 0 \leqslant x \leqslant 1 \\ 2 - x, & 1 \leqslant x \leqslant 2 \\ 0, & x > 2 \end{cases}$$

α-cut interval for this

$A_\alpha = [2\alpha - 2, 2 - \alpha]$.

2.4.2 Operations of trapezoidal fuzzy number

[18] Let's talk about the operations of trapezoidal fuzzy number as in the triangular fuzzy number,
$\bar{A} = (a_1, a_2, a_3, a_4)$ and $\bar{B} = (b_1, b_2, b_3, b_4)$

(1) Addition and subtraction between fuzzy numbers become trapezoidal fuzzy number
$\bar{A}(+)\bar{B} = [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4]$
$\bar{A}(-)\bar{B} = [a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1]$

(2) Multiplication, division, and inverse need not be trapezoidal fuzzy number.

(3) Max and Min of fuzzy number is not always in the form of trapezoidal fuzzy number.

Example 2.7. *Let $\bar{A} = (-1, 1, 2, 4)$ and $\bar{B} = (0, 1, 2, 3)$ two trapezoidal fuzzy numbers*

$$\bar{A}(+)\bar{B} = (-1, 1, 2, 4) + (0, 1, 2, 3) = [-1 + 0, 1 + 1, 2 + 2, 4 + 3] = [-1, 2, 4, 7]$$

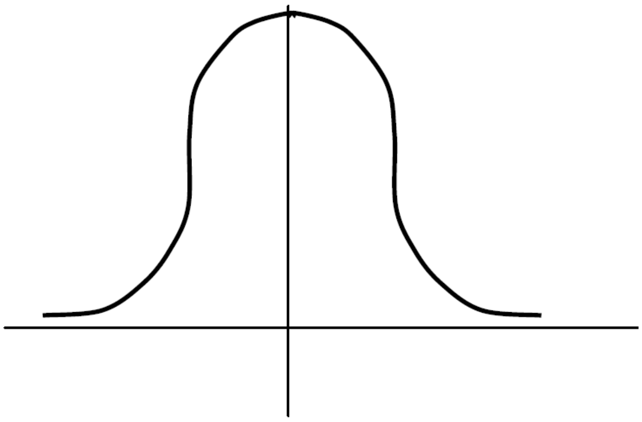
$$\bar{A}(-)\bar{B} = (-1, 1, 2, 4) - (0, 1, 2, 3) = [-1 - 3, 1 - 2, 2 - 1, 4 - 0] = [-4, -1, 1, 4]$$

2.4.3 Bell Shape fuzzy number

[18] Bell shape fuzzy number is often used in practical applications and its function is defined as follows

$$\mu_f(x) = \exp\{-(x - m_f)^2/2\delta_f^2\}$$

where μ_f is the mean of the function, δ_f is the standard deviation.



Bell shape fuzzy number

2.4.4 L-R fuzzy number

Definition 2.5. [11] *A fuzzy number* $\bar{A}$ *is called a L-R fuzzy number if its membership function* $\mu_{\bar{A}} : \mathbb{R} \longrightarrow [0, 1]$ *has the following form:*

$$\mu_{\bar{A}}(x) = \begin{cases} L(\dfrac{x-a}{\alpha}), (a - \alpha) \leq x < a, \alpha > 0, \\ 1, \qquad a \leq x \leq b, \\ R(\dfrac{x-b}{\beta}), b < (b + \beta), \beta > 0, \\ 0, \qquad otherwise, \end{cases}$$

where $L(.)$ *and* $R(.)$ *are piecewise continuous functions,* $L(.)$ *is increasing, R(.) is decreasing and* $L(0) = R(0) = 1$*. The L-R fuzzy number* A *as described above will* *be represented as* $A = (a, b, \alpha, \beta)_{LR}$*. Here* L *and* R *are called as the left and right reference functions,* a *and* b *are respectively called starting and end points of the flat interval,* α *is called the left spread and* β *is called the right spread. The general shape of a L-R fuzzy number* $A = (a, b, \alpha, \beta)_{LR}$ *will be as follows*

Remark 2.1. [11] *Let* $A = (a_1, b_1, \alpha, \beta)_{LR}$ *and* $B = (a_2, b_2, \gamma, \delta)_{LR}$ *be two L-R fuzzy numbers. Then it can be verified that*

$$A \oplus B = (a_1 + a_2, b_1 + b_2, \alpha + \gamma, \beta + \delta)_{LR}$$

and

$$-A = -(a_1, b_1, \alpha, \beta)_{LR} = (-b_1, -a_1, \beta, \alpha)_{RL}$$

For defining the difference $A \ominus B$*the original number* B *should be a R-L fuzzy number so that* $-B$ *becomes a L-R fuzzy number and one can compute* $A - B$ *as* $A \oplus (-B)$*. Therefore for* $A = (a_1, b_1, \alpha, \beta)_{LR}$ *and* $B = (a_2, b_2, \gamma, \delta)_{LR}$ *we get*

$$\begin{aligned} A \ominus B &= (a_1, b_1, \alpha, \beta)_{LR} \oplus (-(a_2, b_2, \gamma, \delta)_{RL}) \\ &= (a_1, b_1, \alpha, \beta)_{LR} \oplus (-b_2, -a_2, \delta, \gamma)_{LR} \\ &= (a_1 - b_2, b_1 - a_2, \alpha + \delta, \beta + \gamma)_{LR} \end{aligned}$$

Chapter 3

Some properties of fuzzy numbers

In this chapter, we introduce the concept of fuzzy function. Fuzzy functions consist of crisp function with fuzzy constraint and fuzzifying function. Integration of fuzzifying function in crisp interval and fuzzy interval. We also offer differentiation of crisp function on fuzzy points and differentiation of fuzzifying function the crisp interval.

3.1 Fuzzy function

[18] Fuzzy function can be classified into the following three groups according to which aspect of the crisp function the fuzzy concept was applied.

1. Crisp function with fuzzy constraint.
2. Propagation of fuzziness by crisp function
3. Fuzzifying function of crisp variable

3.1.1 Function with fuzzy constraint

Definition 3.1. [18] *Let X and Y be crisp sets, and f be a crisp function. $\bar{A}$ and $\bar{B}$ are fuzzy sets defined on universal sets X and Y respectively. Then the function* *satisfying the condition* $\mu_{\bar{A}}(x) \leq \mu_{\bar{B}}(f(x))$ *is called a function with constraints on fuzzy domain A and fuzzy range B.*

$$f : \bar{X} \longrightarrow Y$$

Example 3.1. *Let tow fuzzy sets*
$\bar{A} = \{(2, 0.4), (3, 0.6), (4, 0)\}$
$\bar{B} = \{(4, 0.5), (6, 0.9)\}$
and a function $y = f(x) = 2x$
$\Longrightarrow \mu_{\bar{A}}(x) \leq \mu_{\bar{B}}(f(x))$.

3.1.2 Propagation of fuzziness by crisp function

Definition 3.2 (Fuzzy extension function)**.** [18] *The fuzzy extension function propagates the ambiguity of independent variables to dependent variables. When* f *is a crisp function from* X *to* Y*, the fuzzy extension function* f *defines the image* $f(\overline{X})$ *of fuzzy set* $\bar{X}$*. That is, the extension principle is applied.*

$$\mu_{\bar{B}}(y) = \begin{cases} \underset{x \in f^{-1}(y)}{max} \quad \mu_{\bar{A}}(x), \quad if \quad f^{-1}(y) \neq \emptyset \\ 0, \quad if \quad f^{-1}(y) = \emptyset \end{cases}$$

$$\mu_{f(\bar{x})}(y) = \begin{cases} \underset{x \in f^{-1}(y)}{max} \quad \mu_{\bar{x}}(x), \quad if \quad f^{-1}(y) \neq \emptyset \\ 0, \quad if \quad f^{-1}(y) = \emptyset \end{cases}$$

Example 3.2. *let crisp function* $f(x) = 2\bar{x} + 1$ *in which*
$f : A \longrightarrow B$
$\bar{x} \longrightarrow f(\bar{x})$
on suppose $A = \{(0, , 0.7), (1, 0.3), (2, 0.5), (3, 0.6), (4, 0.2)\}$
its range, $B = [0, 20]$
$\bar{B} = \{(1, 0.7), (3, 0.3), (5, 0.5), (7, 0.6), (9, 0.2)\}$

3.1.3 Fuzzifying function of crisp variable

Definition 3.3 (Single fuzzifying function)**.** [18] *Fuzzifying function from X to Y is the mapping of X in fuzzy power set $\bar{P}(y)$.*

$$\bar{f} : \bar{X} \longrightarrow \bar{P}(y)$$

That is to say, the fuzzifying function is a mapping from a domain to a fuzzy set of ranges. Fuzzifying function and the fuzzy relation coincides with each other in a mathematical manner. So to speak, fuzzifying function can be interpreted as fuzzy relation R defined as following :

$$\forall (x, y) \in X \times Y$$

$$\mu_{\bar{f}(x)}(y) = \mu_R(x, y)$$

Example 3.3. *Let two crisp sets $A = \{2, 3, 4\}$ and $B = \{2, 3, 4, 6, 8, 9, 12\}$*
$\bar{f}(2) = \bar{B}_1, \bar{f}(3) = \bar{B}_2, \bar{f}(4) = \bar{B}_3.$
$\bar{f} = [\bar{B}_1, \bar{B}_2, \bar{B}_3]$
$\bar{B}_1 = \{(2, 0.5), (4, 1), (6, 0.5)\}$
$\bar{B}_2 = \{(3, 0.5), (6, 1), (9, 0.5)\}$
$\bar{B}_3 = \{(4, 0.5), (8, 1), (12, 0.5)\}$

Definition 3.4 (Fuzzy bunch of functions)**.** [18] *A fuzzy bunch of crisp functions from X to Y is defined with a fuzzy set of crisp function $f_i (i = 1, ..., n)$ and it is denoted as*

$$\bar{f} = \{(f_i, \mu_{\bar{f}}(f_i))/f_i : X \longmapsto Y, i \in \mathbb{N}\}$$

$$f_i = f(x), \forall x \in X$$

This function produces a fuzzy set as its outcome.

Example 3.4. *Let* $X = \{1, 2, 3\}$, $\bar{f} = \{(f_1, 0.4), (f_2, 0.7), (f_3, 0.5)\}$

$f_1 = x, f_2 = x^2, f_3 = x + 1$

for $\bar{f}_1 = \{(1, 0.4), (2, 0.4), (3, 0.4)\}$.

for $\bar{f}_2 = \{(1, 0.7), (4, 0.7), (9, 0.7)\}$.

for $\bar{f}_3 = \{(2, 0.5), (3, 0.5), (4, 0.5)\}$.

$\bar{f}(1) = \{(1, 0.4), (1, 0.7), (2, 0.5)\} = \{(1, 0.7), (2, 0.5)\}$.

$\bar{f}(2) = \{(2, 0.4), (3, 0.5), (4, 0.7)\}$.

$\bar{f}(3) = \{(3, 0.4), (4, 0.5), (9, 0.7)\}$.

3.2 Integration of fuzzy function

3.2.1 Integration of fuzzifying function in a crisp interval

Definition 3.5. [15] *Let* f *be a fuzzifying function from* $[a, b] \subseteq \mathbb{R}$ *to* $\mathbb{R}$ *such that* $\forall x \in [a, b], \bar{f}(x)$ *is a fuzzy number, i.e., a piecewise continuous convex normalized fuzzy set on* $\mathbb{R}$. $\forall \alpha \in]0, 1]$, *the equation* $\mu_{\bar{f}(x)}(y) = \alpha$ *with* x *and* α *as parameters is assumed to have two and only two continuous solutions* $y = f_\alpha^+(x)$ *and* $y = f_\alpha^-(x)$ *for* $\alpha \neq 1$ *and only one,* $y = f(x)$, *for* $\alpha = 1$, *which is also continuous.* f_α^+ *and* f_α^- *are defined such that*

$f_{\alpha\prime}^+ \geqslant f_\alpha^+(x) \geqslant f(x) \geqslant f_\alpha^-(x) \geqslant f_{\alpha\prime}^-(x) \qquad \forall \alpha, \alpha\prime. \quad$ *with* $\quad \alpha\prime \leqslant \alpha$.

An intuitive way of defining the integral $\bar{I}(a, b)$ *of* $\bar{f}$ *over* $[a, b]$ *is to assign the membership value* α *to the integral of any* α*-level curve of* $\bar{f}$ *over* $[a, b]$. *Using Zadeh's notation,* $\bar{I}(a, b)$*is the fuzzy set on* $\mathbb{R}$

$$\bar{I}(a, b) = \{(\int_a^b f_\alpha^-(x)dx + \int_a^b f_\alpha^+(x)dx, \alpha)/\alpha \in [0, 1]\}$$

Definition 3.6. [18] *In crisp interval* $[a, b] \in \mathbb{R}$, *let the fuzzifying function have fuzzy value* $\bar{f}(x)$ *for* $x \in [a, b]$. *Integration* $\bar{I}(a, b)$ *of the fuzzifying function in* $[a, b]$ *is defined as follows:*

$$\bar{I}(a, b) = \{(\int_a^b f_\alpha^-(x)dx + \int_a^b f_\alpha^+(x)dx, \alpha)/\alpha \in [0, 1]\}$$

Were f^-_α *and* f^+_α *are* α*-cut functions of* $\bar{f}(x)$ *.*

Example 3.5. *There is a fuzzy bunch of functions and we want to get integration in* $[1, 2]$

$\bar{f} = \{(f_1, 0.2), (f_2, 0.2), (f_3, 0.6)\}$

$f_1 = x, f_2 = x^2, f_3 = x + 1$

$X = [1, 2]$*, find the integration of* $\bar{f}$ *in* $[1, 2]$

(i) $\alpha = 0.6,$

$f = f_3(x) = x + 1$

$$I_\alpha(1, 2) = \int_1^2 (x + 1)dx = \frac{1}{2}x^2 + x] = \frac{5}{2}$$

Therefore, $\bar{I}_{0.6}(1, 2) = \{(\frac{5}{2}, 0.6)\}$

(ii) $\alpha = 0.2$*, there are two functions*

$$f^+ = f_1(x) = x$$

$$f^- = f_2(x) = x^2$$

$$I^+_\alpha(1, 2) = \int_1^2 xdx = \frac{1}{2}x^2] = \frac{3}{2}$$

$$I^-_\alpha(1, 2) = \int_1^2 x^2dx = \frac{1}{3}x^3] = \frac{7}{3}$$

The integration results are $\frac{3}{2}$ *with possibility* 0.2 *and* $\frac{7}{3}$ *with* 0.2*.*

Then, $\bar{I}_{0.2}(1, 2) = \{(\frac{3}{2}, 0.2), (\frac{7}{3}, 0.2)\}$

Finally, we have the total integration.

$$\bar{I}(1, 2) = \{(\frac{5}{2}, 0.6), (\frac{3}{2}, 0.2), (\frac{7}{3}, 0.2)\}.$$

3.2.2 Integration crisp function in fuzzy interval

Definition 3.7. [15] *Let* A *and* B *be two fuzzy sets on* $\mathbb{R}$*. The extension principle allows defining the integral of a real-valued ordinary function* f *over the fuzzy interval* $(\bar{A}, \bar{B})$ *bounded by* $\bar{A}$ *and* $\bar{B}$*, say* $\bar{I}(\bar{A}, \bar{B})$*:*

$$\mu_{\bar{I}(\bar{A},\bar{B})}(z) = \underset{z=\int_x^y f(\mu)d\mu}{Sup} min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)].$$

Definition 3.8. [18] *Integration $\bar{I}(\bar{A},\bar{B})$ of crisp function f in fuzzy interval $[\bar{A},\bar{B}]$ is defined as,*

$$\mu_{\bar{I}(\bar{A},\bar{B})}(z) = \underset{z=\int_x^y f(\mu)d\mu}{Max} Min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)].$$

Example 3.6. *On suppose that*
$\bar{A} = \{(3, 0.4), (4, 1), (5, 0.5)\}$
$\bar{B} = \{(6, 0.9), (7, 1), (8, 0.3)\}$
$f(x) = 2, x \in [3, 8], \int_{\bar{A}}^{\bar{B}} f(x)dx = \int_{\bar{A}}^{\bar{B}} 2dx.$

1. *for* $[3, 6]$ $2x]_3^6 = 12 - 6 = 6$ $min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(3), \mu_{\bar{A}}(6)) = min(0.4, 0.9) = 0.4$
2. *for* $[3, 7]$ $2x]_3^7 = 14 - 6 = 8$ $min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(3), \mu_{\bar{A}}(7)) = min(0.4, 1) = 0.4$
3. *for* $[3, 8]$ $2x]_3^8 = 16 - 6 = 10$ $min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(3), \mu_{\bar{A}}(8)) = min(0.4, 0.3) = 0.3$
4. *for* $[4, 6]$ $2x]_4^6 = 12 - 8 = 4$ $min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(4), \mu_{\bar{A}}(6)) = min(1, 0.9) = 0.9$
5. *for* $[4, 7]$ $2x]_4^7 = 14 - 8 = 6$ $min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(4), \mu_{\bar{A}}(7)) = min(1, 1) = 1$
6. *for* $[4, 8]$ $2x]_4^8 = 16 - 8 = 8$ $min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(4), \mu_{\bar{A}}(8)) = min(1, 0.3) = 0.3$
7. *for* $[5, 6]$ $2x]_5^6 = 12 - 10 = 2$ $min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(5), \mu_{\bar{A}}(6)) = min(0.5, 0.9) = 0.5$

8. *for* $[5,7]$ $\quad 2x]_5^7 = 14 - 10 = 4 \quad min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(5), \mu_{\bar{A}}(7)) = min(0.5, 1) = 0.5$

9. *for* $[5,8]$ $\quad 2x]_5^8 = 16 - 10 = 6 \quad min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(5), \mu_{\bar{A}}(8)) = min(0.5, 0.3) = 0.3$

finally
$\bar{I}[\bar{A}, \bar{B}] = \{(6,1), (8,0.4), (10,0.3), (4,0.9), (2,0.5)\}$

Example 3.7. *Let* $\bar{A} = \{(1,0.2), (2,1)\}$
and $\bar{B} = \{(3,0.7), (4,1)\}$
$f(x) = 2x, x \in [1,4], \int_{\bar{A}}^{\bar{B}} f(x)dx = \int_{\bar{A}}^{\bar{B}} 2Xdx.$

1. *for* $[1,3]$ $\quad x^2]_1^3 = 9 - 1 = 8 \quad min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(1), \mu_{\bar{A}}(3)) = min(0.2, 0.7) = 0.2$

2. *for* $[1,4]$ $\quad x^2]_1^4 = 16 - 1 = 15 \quad min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(1), \mu_{\bar{A}}(4)) = min(0.2, 1) = 0.2$

3. *for* $[2,3]$ $\quad x^2]_2^3 = 9 - 4 = 5 \quad min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(2), \mu_{\bar{A}}(3)) = min(1, 0.7) = 0.7$

4. *for* $[2,4]$ $\quad x^2]_2^4 = 16 - 4 = 12 \quad min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)] = min(\mu_{\bar{A}}(2), \mu_{\bar{A}}(4)) = min(1, 1) = 1$

finally
$\bar{I}[\bar{A}, \bar{B}] = \{(5,0.7), (8,0.2), (12,1), (15,0.2)\}$

3.2.3 Integration of fuzzifying function in fuzzy interval

Definition 3.9. [15] *Zadeh's extension principle gives now*

$$\mu_{\bar{I}(\bar{A},\bar{B})}(z) = \underset{\substack{z=\int_x^y \bar{I}(t)dt, \\ \bar{I}\in L,(x,y)\in\mathbb{R}^2, x\leqslant y}}{Sup} min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(y), \mu_{\bar{f}}(\bar{I})].$$

Where $\bar{f}$ is a fuzzifying function satisfying the assumptions of a and A, B are fuzzy numbers that delimit a fuzzy interval, $\bar{I}(A, B)$ is the fuzzy integral.

$$\mu_{\bar{I}(\bar{A},\bar{B})}(z) = \underset{x \leqslant y}{Sup} \quad min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(y), \underset{\substack{\bar{I} \in L \\ z=\int_x^y \bar{I}(t)dt}}{Sup} \quad \mu_{\bar{f}}(\bar{I})]$$

$$\mu_{\bar{I}(\bar{A},\bar{B})}(z) = \underset{x \leqslant y}{Sup} \quad min[\mu_{\bar{A}}(x), \mu_{\bar{B}}(y), \mu_{\bar{I}(x,y)}(z)]$$

3.3 Differentiation of fuzzy function

3.3.1 Differentiation of crisp function on fuzzy points

Definition 3.10. [18] *By the extension principle, differentiation $f\prime(A)$ of a function f at a fuzzy point or a fuzzy set A is defined as*

$$\mu_{f\prime(A)}(y) = \underset{f(x)=y}{Max} \mu_A(x)$$

Example 3.8. *When differentiating function $f(x) = 2x^3$ at fuzzy point A,*

$A = \{(-2, 0.3), (2, 0.7), (3, 1)\}$

From $f\prime(x) = 6x^2$*,*

$f\prime(x) = \{(24, 0.3), (24, 0.7), (54, 1)\}$

$= \{(24, 0.7), (54, 1)\}$

Example 3.9. *When differentiating function $f(x) = x$ at fuzzy point A,*

$A = \{(1, 0.1), (2, 0.6), (3, 0.9)\}$

From $f\prime(x) = 1$*,*

$f\prime(x) = \{(1, 0.1), (1, 0.6), (1, 0.9)\}$

$= \{(1, 0.9)\}$

3.3.2 Differentiation of fuzzifying function in crisp interval

Definition 3.11. [18] *For all elements x in domain D, we define the differentiation of fuzzifying function. Let any αcut of f be differentiable for arbitrary $x \in D$. At this time, we define differentiation $(\overline{df}/dx)(x_0)$ of at real point x_0:*

$$\mu_{(\overline{df}/dx)(x_0)}(p) = \max_{f_\alpha : (df_\alpha/dx)(x_0)) = p} \mu_{\bar{f}}(f_\alpha)$$

Example 3.10. *There is a fuzzifying function*

$\bar{f} = \{(f_1, 0.2), (f_2, 0.5), (f_3, 0.2)\}$

$f_1 = x + 1, f_2 = x, f_3 = x^2$

First, we have

$f\prime_1(x) = 1, f\prime_2(x) = 1, f\prime_3(x) = 2x$

differentiation at $x_0 = 0.3$ is obtained through

$f\prime_1(0.3) = 1, \quad when = 0.2$

$f\prime_2(0.3) = 1, \quad when = 0.5$

$f\prime_3(0.3) = 0.6, \quad when = 0.2$

$\frac{\overline{df}}{dx}(x_0) = \frac{\overline{df}}{dx}(0.3) = \{(1, 0.2), (1, 0.5), (0.6, 0.2)\}$

$= \{(1, 0.2), (1, 0.5)\}$

Example 3.11. *There is a fuzzifying function*

$\bar{f} = \{(f_1, 0.6), (f_2, 1), (f_3, 1)\}$

$f_1 = \frac{1}{2}x^2, f_2 = x^3, f_3 = 5x + 2$

First, we have

$f\prime_1(x) = x, f\prime_2(x) = 3x^2, f\prime_3(x) = 5$

differentiation at $x_0 = 1$ is obtained through

$f\prime_1(1) = 1, \quad when = 0.6$

$f\prime_2(1) = 3, \quad when = 1$

$f\prime_3(1) = 5, \quad when = 1$

$\frac{\overline{df}}{dx}(x_0) = \frac{\overline{df}}{dx}(1) = \{(1, 0.6), (3, 1), (5, 1)\}$

$= \{(1, 0.6), (5, 1)\}$

Conclusion

In this memoir, we have studied the fuzzy number, operations on fuzzy numbers, as well as the fuzzy sets and some fuzzy relations, and characteristics of the fuzzy set. We found among the fuzzy numbers is the triangular fuzzy number. Fuzzy numbers have properties, including the integrals of fuzzy functions at a fuzzy number and a crisp number, and crisp functions at a fuzzy number as well as for the differentiation of fuzzy functions at a crisp number and the differentiation of crisp functions at a fuzzy number.
We study fuzzy differential equations, and other characteristics of fuzzy numbers.

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ملخص

في هذه المذكرة، قمنا بدراسة مفهوم العدد الضبابي، كما تطرقنا الى العمليات الحسابية على الأعداد الضبابية، واستعمالها في حساب تكاملات محدودة لدوال ضبابية في حالتي الحدود الضبابية والحدود العادية، والدوال العادية عند حدود ضبابية، وكذلك في حساب التفاضل الضبابي.

كلمات مفتاحية

المجموعات الضبابية، الاعداد الضبابية، الدوال الضبابية، تكامل الدوال الضبابية، تفاضل الدوال الضبابية.

Abstract

In this memoir, we have studied the concept of fuzzy number. We also discussed arithmetic operations on fuzzy numbers, and its use in calculating finite integrals of fuzzy functions in the cases of fuzzy terms and crisp terms, and crisp functions at the fuzzy interval, and as well in computation fuzzy differential.

Key words

Fuzzy sets, fuzzy numbers, fuzzy functions, fuzzy functions integration, fuzzy functions differentiation.

Résumé

Dans ce mémoire, nous avons étudié le concept de nombre flou, nous avons également discuté des opérations arithmétiques sur les nombres flous, et son utilisation dans le calcul d'intégrales finies de fonctions floues dans les cas de bornes flous et nettes, et de fonctions nettes à intervalle flou, et aussi dans calcul différentiel flou.

Mot-clés

Ensembles flous, nombres flous, fonctions flous, intégration des fonctions flous, differentiation des fonctions flous.