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Existence of a solution for a Linear Integral Equation via the Fixed point Theorem

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Dedications

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0.1 INTRODUCTION

Linear integral equations constitute an essential tool in applied mathematics, physics, engineering, and mathematical biology.

The importance of this topic lies in its exceptional capability to model various complex dynamic systems, such as heat transfer, population growth, and mechanical vibrations, where differential equations alone might not provide a complete description. Among the various mathematical frameworks used to study these equations, the Fixed Point Theorem stands out as a powerful and indispensable tool for proving the existence and uniqueness of solutions, transforming differential and integral problems into simpler operator equations.

Historically, previous studies regarding the concept of a fixed point date back to the beginnings of mathematical analysis in the 19th century, which were formally established by the Polish mathematician Stefan Banach in 1922 through his famous contraction principle. Following Banach's foundational work, numerous researchers and mathematicians, such as Henri Poincaré introduced important ideas related to dynamical systems that later influenced fixed-point theory.

Over the decades, extensive literature has been dedicated to extending these theorems from classic metric spaces to more general topological and Banach spaces, establishing a solid mathematical basis for solving linear and non-linear integral equations.

The main objectives of this study are to thoroughly investigate the existence and uniqueness of solutions for linear integral equations by employing modern fixed-point techniques. Specifically, this work aims to clarify how abstract fixed-point theorems can be practically applied to ensure the stability and solvability of integral equations in Banach spaces, and to bridge the gap between theoretical abstract functional analysis and practical numerical simulations.

To achieve these goals, the methodology adopted in this work follows a rigorous analytical and deductive approach. We begin by constructing the mathematical foundation through functional preliminaries, followed by classifying linear integral equations and studying their kernels.

We analytically apply Banach's fixed-point theorem to establish the core existence and uniqueness proofs.

The theoretical framework is validated through practical computation, where we present and analyze three structured, simplified numerical illustrative examples.

This work is organized into four chapters:

The First Chapter is devoted to presenting the basic concepts and necessary preliminaries, including metric spaces, Banach spaces, and linear and continuous operators.

The Second Chapter deals with linear integral equations, their types, general formulas, and the characteristics of integral kernels.

The Third Chapter is dedicated to the study of fixed point theorems, focusing specifically on Banach's Fixed Point Theorem and establishing the main existence and uniqueness results.

The Fourth Chapter is devoted to numerical applications, where we present three worked illustrative examples.

Chapter 1

Basic Concepts and Preliminaries

1.1 Metric Spaces

A metric space is one of the most fundamental structures in mathematical analysis. It provides a rigorous framework for measuring distances between objects, enabling the precise formulation of continuity, convergence, and completeness.

definition: let X be a non-empty set. A function d:$X \times X \longrightarrow \mathbb{R}$ is called a metric on X if for all elements $x, y, z \in X$,the following four axioms are satisfied:

1 Non negativity: $\mathbf{d}(x, y) \geq 0$
2 Identity of Indiscernibles: $d(x, y) = 0 \Leftrightarrow x = y$
3 Symmetry: $d(x, y) = d(y, x)$
4 Triangle Inequality: $d(x; z) \leq d(x, y) + d(y, z)$

these conditions are called the properties of a metric

Example: let $X = \mathbb{R}$ and define

$$d(x, y) = |x - y|$$

This metric satisfies the following conditions:
1-Non-negativity: $|x - y| \geq 0$,and $d(x, y) = 0$ if and only if $x = y$
2-Symmetry: $|x - y| = |y - x|$, which means $d(x, y) = d(y, x)$
3-Triangle Inequality: For any $x, y, z \in \mathbb{R}$, we have :
$|x - y| \leq \; |x - z| + |z - y| \Rightarrow d(x, y) \leq d(x, z) + d(z, y)$

1.1.1 Convergent Sequence

let $(x_n)_{n\geq 1}$ be a sequence in metric space (X,d) and let $x_0 \in X$
A sequence $(x_n)_{n\geq 1}$ is said to converge to an element x_0 denoted by:

$$x_n \longrightarrow x_0 \ or \ \lim_{n\longrightarrow\infty} x_n = x_0$$

$$\text{if } \forall \varepsilon > 0, \exists n_\varepsilon \geq 1 \text{ such that}$$

$$\forall n \geq n_\varepsilon \Rightarrow d(x_n, x_0) < \varepsilon$$

1.1.2 Cauchy Sequences

A sequence $(x_n)_{n\geq 1}$ is called a cauchy sequence if
$\forall \varepsilon > 0$, $\exists n_\varepsilon \geq 1$, such that $\forall p, q \geq n_\varepsilon \Rightarrow d(x_p, x_q) < \varepsilon$
Proposition 1.1: Every convergent sequence in a metric space is a Cauchy sequence.

1.1.3 Completeness

definition: A metric space (X,d) is said to be complete if every Cauchy sequence $(x_n)_{n\geq 1}$ in X converges to an element $x_0 \in X$:
$\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}, \forall p, q \geq N_\varepsilon, \Rightarrow d(x_p, x_q) \leq \varepsilon$

Note: in a complete space, the concepts of Cauchy sequence and convergent sequence are equivalent.

1.2 Banach space

Banach spaces provide the natural setting for studying linear integral equations, as they combine the algebraic structure of a vector space with a complete metric structure.

1.2.1 Normed Vector Space

definition: let E be a vector space over the field $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$
A norm on the space E is a function denoted by $\|.\|$ defined from E into $\mathbb{R}$ $\|.\| : E \longrightarrow \mathbb{R}_+$ such that:

1- Positive definiteness : $\forall x \in E$ $\|x\| = 0 \Leftrightarrow x = 0$
2- Absolute Homogeneity: $\forall x \in E$,$\forall \lambda \in \mathbb{k}$, $\|\lambda x\| = |\lambda| \|x\|$
3- Triangle Inequality: $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$

Remark: in general manner if $\begin{aligned} \|.\| : E &\longrightarrow \mathbb{R}_+ \\ x &\longmapsto \|x\| \end{aligned}$
is a norm on E , we denote the norm of x as $\|x\| = \|x\|_E$.

Example 1 *let $a, b \in \mathbb{R}$ with $a < b$ and consider the space* $E = C^0[a,b]$ *defined as*

$$E = C^0[a,b] = \{f : \ [a,b] \longrightarrow \mathbb{R} \backslash f \text{ is continuous on } [a;b]\}$$

the application $\|.\| : C^0[a,b] \longrightarrow [0;+\infty[$ *defined by:*

$f \mapsto \int\limits_a^b |f(x)|\, dx$

The application $\int\limits_a^b |f(x)|\, dx$ is a norm on $E = C^0[a,b]$

denoted $\|f\|_1$

1.2.2 Banach spaces

definition: let $(E, \|.\|_E)$ be a normed vector space we say that E is a Banach space if it is complete , meaning every Cauchy sequence $\{x_n\} \subset E$ converges to a limit $x \in E$

Example : we define the space L^p by:

$L^p([a,b]) = \left\{ f : \ [a,b] \longrightarrow \mathbb{R} \backslash \text{ f is measurable and } \int\limits_a^b |f(x)|^p\, dx < \infty \right\}$

we define the norm on this space by: $\|f\|_p = \left(\int\limits_a^b |f(x)|^p\, dx \right)^{\frac{1}{p}}$

for any p such that $1 \leq p < \infty$ the space $\left(L^p, \|.\|_p\right)$ is a Banach space

1.3 The Completeness of the Space $C([a,b])$

The space $C([a,b])$ of all continuous real-valued functions defined on the closed interval $[a,b]$, equipped with the supremum norm defined by:

$$\|f\|_\infty = \sup_{t \in [a,b]} |f(t)| = \max_{t \in [a,b]} |f(t)|$$

is a Banach Space.

proof:

Let (f_n) be a Cauchy sequence in $C([a,b])$. Thus, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$\forall n, m \geq N \ \ \|f_n - f_m\|_\infty < \epsilon$$

1-For each $t \in [a,b]$, the sequence $(f_n(t))$ is Cauchy in $\mathbb{R}$.since $\mathbb{R}$ is complete we define $f(t) = \lim_{n \longrightarrow \infty} f_n(t)$

2-taking the limit as $m \longrightarrow \infty$ in the inequality $|f_n(t) - f_m(t)| < \epsilon$ we get $|f_n(t) - f(t)| \leq \epsilon \qquad \forall\ n \geq N$

Since N is independent of t, taking the supremum over $[a,b]$yields

$$\|f_n - f\|_\infty \leq \epsilon \quad \forall n \geq N$$

thus $f_n \longrightarrow f$

3- since f_n are continuous and the convergence is uniform, the limit function f is continuous $(f \in C([a,b]))$

so: Every Cauchy sequence in $C([a,b])$ converges within the space, making it a Banach space.

1.4 Linear and continuous operators

definition (linear operator) : let E and F two normed vector spaces over the same field $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$ $\{(E, \|.\|_E)$ and $(F, \|.\|_F)\}$an application $T : E \longrightarrow F$ is called
a linear operator if it satisfies the following two conditions for all $x, y \in E$, $\lambda \in \mathbb{k}$:
$1-$ $T(x+y) = T(x) + T(y)$
2- $T(\lambda x) = \lambda T(x)$

1.4.1 Continuity of linear operators

definition: if E and F two normed spaces , A linear operator A is defined on $G \subset E$ into F is said to be continuous at a point $x_0 \in G$ if for every sequence (x_n) in G
such that : $x_n \longrightarrow x_0$ the sequence $A(x_n)$ converge to $A(x_0)$

if and only if T is continuous on all of E, which is equivalent to T being a bounded operator.

$$\lim_{n \longrightarrow \infty} (Ax_n) = A\left(\lim_{n \longrightarrow \infty} x_n\right) = A(x_0)$$

Remark: A linear operator is said to be continuous on E if it is continuous at one point.

Theorem: let E and F be two normed spaces , A linear operator T defined on a subset $G \subset E$ into F is said to be continuous at a single point $x_0 \in G$

definition(*bounded operators*)**:** A linear operator T:$E \longrightarrow F$ is said to be bounded if there exists a positive constant $C > 0$ such that:

$$\|T(x)\|_F \leq C \|x\|_E \quad \forall x \in E$$

Theorem (Continuity and Boundedness)**:**

let $T : E \longrightarrow F$ be a linear operator. The following statements are equivalent:

- T is continuous at a single point $x_0 \in E$.
- T is continuous on E.
- T is a bounded operator.

Example:

Consider the integral operator $T : C([a,b]) \longrightarrow C([a,b])$ defined by:

t represents global or cumulative properties of the unknown function over a specific interval or domain.

integral equations are primarily classified into three main types:

$$(T\phi)(x) = \int_a^b K(x,t)\,\phi(t)\,dt$$

where $K(x,t)$ is continuous kernel on $[a,b] \times [a,b]$

$1-$It is easy to verify that $T(\alpha\phi + \beta\psi) = \alpha T(\phi) + \beta T(\psi)\,, T$ is a linear operator.

2- Since K is continuous on a closed domain, it is bounded by some constant M. This ensures that $\|T_\phi\|_\infty \leq M(b-a)\|\phi\|_\infty$ proving that T is a bounded (and thus continuous) operator.

1.4.2 the operator norm

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two normed spaces and let $T : X \longrightarrow Y$ be a bounded linear operator ,the operator norm of T is defined as:

$$\|T\| = \sup_{x \neq 0 \ x \in X} \frac{\|T(x)\|_Y}{\|x\|_X}$$

1.4.3 Norm estimate of the operator

1- upper bound: finding a constant C such that $\|T(x)\|_F \leq C \|x\|_E$ for all x this shows that $\|A\| \leq C$

2-Attaining the supermum: finding a specific $x \in E$ or a sequence where the equality is approached proving that $\|A\| = C$

Example:
let T:$(C\left[0,1\right], \left\|.\right\|_{\infty}) \longrightarrow \mathbb{R}$ be defined by

$$T\left(f\right) = \int\limits_{0}^{1} f\left(t\right)dt$$

1-upper bound:
$\left|T\left(f\right)\right| = \left|\int\limits_{0}^{1} f\left(t\right)dt\right| \leq \int\limits_{0}^{1} \left|f\left(t\right)dt\right| \leq \int\limits_{0}^{1} \left\|f\right\|_{\infty} dt = \left\|f\right\|_{\infty}$
Thus $\left\|T\right\| \leq 1$

2-Attainment:
$T\left(f_0\right) = \int\limits_{0}^{1} 1dt = 1$ Take $f_0\left(t\right) = 1$, then $\left\|f_0\right\|_{\infty} = 1$
since $\left|T\left(f_0\right)\right| = \left\|f_0\right\|_{\infty}$
the norm of the operator is $\left\|T\right\| = 1$

Chapter 2

Linear integral equations

2.1 definition of integral equation

An integral equation is an equation in which the unknown function appears under one or more integral signs, in these equations , the objective is to determine a function that satisfies given relation involving an integral transform
it represents global or cumulative properties of the unknown function over a specific interval or domain.

integral equations are primarily classified into three main types:

2.1.1 Fredholm integral equation

definition: a Fredholm integral equation of the first kind is an equation of the form:

$$\int_a^b l(x,\mu)\,\varphi(\mu)\,d\mu = f(x)$$

where φ is the unknown function, while f and l are known functions
in this type of equation the limits of integration are constant which is the
primary characteristic of Fredholm equations.

definition: A Fredholm integral equation of the second kind is an equation of the form:

$$\varphi(x) = f(x) + \lambda_2 \int_a^b l_2(x,\mu)\,\varphi(\mu)\,d\mu \qquad x \in [a,b]$$

where λ is a scalar parameter.

Here the integration is over the fixed interval $[a,b]$, making this a global equation the value of φ at any point x depends on its values everywhere in $[a,b]$. This corresponds structurally to Boundary Value Problems.

Physical interpretation
Fredholm equations appear in steady-state and global interaction models:
• Potential theory and electrostatics
• Scattering theory in quantum mechanics
• Image reconstruction and inverse problems
• Epidemiological models with spatial diffusion

2.1.2 Volterra integral equation

Volterra equations may be viewed as integral equations with variable upper limit.

definition: A linear Volterra integral equation of the first kind is an equation with one unknown function expressed in the following form:

$$\int_a^x l(x,\mu)\,\varphi(\mu)\,d\mu = f(x)$$

definition: A linear Volterra integral equation of the Second Kind is an equation with an unknown function $\varphi(x)$ expressed in the following form:

$$\varphi(x) = f(x) + \lambda_1 \int_a^x l_1(x,\mu)\,\varphi(\mu)\,d\mu \qquad x \in [a,b]$$

The key feature is that the upper limit of integration is the variable x itself, making the equation causal in nature the value of φ at x depends only on its values for $\mu \leq x$. This structure naturally corresponds to Initial Value Problems in ordinary differential equations (ODEs).

Physical interpretation
Volterra equations model memory effects and hereditary phenomena:
• Population dynamics with age structure
• Heat conduction with time-dependent boundary conditions
• Viscoelastic materials (stress depends on strain history)
• Renewal processes in probability theory

2.1.3 Mixed Volterra-Fredholm equations

A Volterra-Fredholm integral equation is a combination of distinct Volterra and Fredholm integrals appearing within a single integral equation.

definition: Volterra and Fredholm integral equations is defined by the following form:

$$\varphi(x) = f(x) + \lambda_1 \int_a^x l_1(x,\mu)\,\varphi(\mu)\,d\mu + \lambda_2 \int_a^b l_2(x,\mu)\,\varphi(\mu)\,d\mu \quad , x \in [a,b]$$

Example:

$$\varphi(x) = \sin(x) + x^2 + \int_0^x (x+\mu)\,\varphi(\mu)\,d\mu + \int_0^1 \exp^{(x-\mu)}\,\varphi(\mu)\,d\mu$$

The fundamental difference between Fredholm and Volterra integral equations lies in the domain of influence and the nature of the integration limits:

- Fredholm Equations: The fixed limits of integration $[a,b]$ reflect a global behavior, where the solution at any point depends on the boundaries. This mirrors Boundary Value Problems (BVPs).

- Volterra Equations: The variable upper limit x reflects a local or causal behavior, where the solution depends only on past history . This mirrors Initial Value Problems (IVPs).

2.2 General Form of Linear Integral Equations

the general form of a linear integral equation of the second kind is expressed as follows :

$$\varphi(x) - \lambda_1 \int_a^x l_1(x,\mu)\varphi(\mu)d\mu - \lambda_2 \int_a^b l_2(x,\mu)\varphi(\mu)d\mu = f(x) \qquad x \in [a,b]$$

this formula consists of the following mathematical elements:

$1-the$ unknown function $\varphi(x)$

2- the source term $f(x)$: A known continuous function, if $f(x)$=0 the equation is called homogeneous , otherwise it is non-homogeneous.

3- the kernel $l(x,\mu)$: A known function of two variables x and μ.

4- the parameter λ : A scalar real or complex number.

5- the limits of integration $[a,b]$.

2.3 Properties of the integral kernels and consequences

The behavior of solutions of integral equations is strongly influenced by the regularity and structure of the kernel $l(x,\mu)$. the main cases are summarized below:

Table 2.1

Kernel Property	**Consequence for the operator T**
$l \in C\ ([a,b] \times [a,b])$	T is bounded on $C\ ([a,b])$,with $\|T_\varphi\|_\infty \leq M\ (b-a)\ \|\varphi\|_\infty$
$l \in L^2([a,b] \times [a,b])$	T is a Hilbert-Schmidt operator, hence compact on $L^2\ ([a,b])$
$\lvert l(x,\mu)\rvert \leq L$ (bounded kernel)	T is a contraction if $\lvert\lambda\rvert L(b-a) < 1$
$l(x,\mu) = l(\mu,x)$(symmetric kernel)	T is self-adjoint in $L^2\ ([a,b])$
$l(x,\mu) = 0$ for $\mu > x$	**volterra operator:** triangular structure ⇒existence and uniqueness without spectral conditions

The properties of the kernel $l\ (x, \mu)$ directly determine the mathematical nature of the integral operator T and the behavior of its solutions:
1- Continuity ($l \in C$): Ensures the boundedness of the operator on $C\ ([a,b])$, satisfying :

$$\|T_\phi\|_\infty = M\ (b-a)\ \|\phi\|_\infty$$

2- Square-Integrability $\left(l \in L^2\right)$: Defines a Hilbert-Schmidt operator, implying that T is compact in $L^2([a,b] \times [a,b])$

3- Boundedness: Guarantees that the operator is a strict contraction under the spectral condition:

$$|\lambda|\ L\ (b-a) < 1$$

4- Symmetry: Makes the operator self-adjoint, which ensures that all its eigenvalues are real numbers.

5- Triangular Structure: Characterizes the Volterra operator, guaranteeing existence and uniqueness for any scalar λ without additional constraints.

2.3.1 Influence on the existence of the solution

The topological properties of the kernel, particularly continuity and boundedness, directly influence the existence and uniqueness of the
solution for both Fredholm and Volterra integral equations by dictating the behavior of the corresponding integral operator within the functional space. Specifically, when the kernel $l\ (x, \mu)$ is continuous on the closed domain
$[a,b] \times [a,b]$ it ensures that the linear integral operator T maps the space of continuous functions $C\ ([a,b])$ into itself. This self-mapping property is a fundamental prerequisite for establishing existence results, as it guarantees that for any continuous input function, the resulting output remains free of singularities or discontinuities. Consequently, given that the non-homogeneous term $f\ (x)$
is also continuous, the entire mapping defined by the integral equation operates strictly within the boundaries of $C\ ([a,b])$. Without this continuity

and the associated boundedness, the operator could map successive approximations
outside of our complete metric space, which would break the underlying structure required to guarantee the convergence of analytical or numerical solutions.

Chapter 3

Fixed Point Theorems and their Applications to Existence and Uniqueness

3.1 Historical Background

The concept of a fixed point a point x such that $T(x) = x$ has deep roots in mathematical analysis.

Henri Poincaré implicitly used fixed point ideas in his celebrated study of the three-body problem around 1890.

The first general fixed point theorem for contractions was formulated by the Polish mathematician Stefan Banach in 1922, providing the modern foundation for the entire field.

L. E. J. Brouwer (1912) proved that any continuous map from a closed ball in $\mathbb{R}^n$ to itself has at least one fixed point. Schauder (1930) extended this to compact operators in infinite dimensions (see [1], [5]). However, for our purposes existence and uniqueness with explicit approximation Banach's theorem is the central tool.

3.2 Banach fixed point theorem

3.2.1 The concept of contraction mapping

definition 3.1

let (X, d) be a complete metric space , the Mapping
$T{:}X \longrightarrow X$ is called a contraction if there exists a constant $k \in [0, 1[$ such that for
all $x, y \in X$:

$$d\left(T\left(x\right),T\left(y\right)\right)\leq k.d\left(x,y\right)$$

k is called the contraction constant ,that the distance between the images is strictly smaller than the distance between the original points.

The condition $k<1$ implies that the mapping reduces distances between points. Consequently, successive iterations become closer and closer, which is the key mechanism behind Banach's Fixed Point Theorem.
Example:
Let T: $\mathbb{R}\longrightarrow\mathbb{R}$ be an operator defined by

$$T\left(x\right)=\frac{1}{2}x$$

$$\left|T\left(x\right)-T\left(y\right)\right|=\left|\frac{1}{2}x-\frac{1}{2}y\right|=\left|\frac{1}{2}\left(x-y\right)\right|=\frac{1}{2}\left|x-y\right|$$

since there exists a constant $k=\frac{1}{2}$ such that $0\leq k<1$
T is contraction

3.2.2 Statement of Banach Fixed Point Theorem

Theorem 3.1
Let (X,d) be a complete metric space and $T:X\rightarrow X$ a contraction mapping with contraction constant $k\in[0,1[$. Then:

(i) EXISTENCE: T has at least one fixed point in X.

(ii) UNIQUENESS: T has exactly one fixed point $x^{*}\in X$.

(iii) APPROXIMATION: For any $x_0\in X$, the Picard iterates

$x_{n+1}=T(x_n)$, $n=0,1,2,...$

converge to x^{*} with a priori error estimate:

$d(x_n,x*)\leq k^n/(1-k)\cdot\ d(x_1,x_0).\ \ n\geq0$

(iv) RATE: The sequence (x_n) converges geometrically to x^{*} with ratio k.

proof:

let $x_0\in X$ be an arbitrary starting point. we define the sequence $\left(x_n\right)_{n\geq0}$ by the iterative process (Picard iteration):

$$x_{n+1}=T\left(x_n\right),\quad\forall n\in\mathbb{N}$$

$x_1 = T(x_0), x_2 = T(x_1) = T^2(x_0)$

$$x_n = T^n(x_0)$$

1) showing $\{x_n\}$ is cauchy sequence
using the contraction property of $T\,(d(T_x, T_y)) \leq k.d(x, y)$, we observe that:
$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq k.d(x_n, x_{n-1})$$

by induction, we find:

$$d(x_{n+1}, x_n) \leq k^n \; d(x_1, x_0)$$

for any $m > n$ using the triangle inequality:

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots + d(x_{n+m-1}, x_{n+m})$$

substituting our previous result :

$$d(x_n, x_{n+m}) \leq \left(k^n + k^{n+1} + \ldots. + k^{n+m-1}\right) d(x_1, x_0)$$

$$d(x_n, x_{n+m}) \leq k^n \left(1 + k + k^2 + \ldots + k^{m-1}\right) d(x_1, x_0)$$

by using the geometric series sum formula :

$$d(x_n, x_{n+m}) \leq k^n \; \frac{1 - k^m}{1 - k} d(x_1, x_0)$$

since $0 \leq k < 1$, as $n \longrightarrow \infty$, $k^n \longrightarrow 0, that\ d(x_n, x_{n+m}) \longrightarrow 0$

which proves that $\{x_n\}$ is a cauchy sequence

2) Existence

since X is a complete metric space, Thus

$$\lim_{n \longrightarrow \infty} x_n = x^* \in X$$

by the continuity of the contraction mapping T:

$$T(x^*) = T\left(\lim_{n \longrightarrow \infty} x_n\right) = \lim_{n \longrightarrow \infty} T(x_n) = \lim_{n \longrightarrow \infty} x_{n+1} = x^*$$

Thus x^* is a fixed point of T.

3) uniqueness:

suppose there exist two fixed points $x^*$and y^* such that $T(x^*) = x^*$,
$T(y^*) = y^*$ then :

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq kd(x^*, y^*)$$

this implies:

$$(1-k)\, d(x^*, y^*) \leq 0$$

since $1-k>0$ and $d(x^*, y^*) \geq 0, it$ means that $d(x^*, y^*) = 0$
Thus :

$$x^* = y^*$$

3.3 Application to Volterra Integral Equations

We rewrite the Volterra equation as a fixed point problem. Define the operator $T : C([a, b]) \longrightarrow C([a, b])$
by:

$$(T\varphi)(x) = f(x) + \lambda_1 \int_a^x l_1(x, \mu)\varphi(\mu)d\mu$$

Finding φ such that $\varphi = T\varphi$ is precisely solving the Volterra equation.

Proposition 3.1 : Contractivity of the Volterra Operator
Assume $l_1 \in C([a, b] \times [a, b])$ with $|l_1(x, \mu)| \leq M_1$ for all $x, \mu. \in [a, b]$ let

$k_1 = |\lambda_1| \, M_1(b-a)$. The contractivity of the Volterra integral operator T is
analyzed through two distinct cases:
1: Direct Contraction (T is a strict contraction)
If the parameters satisfy the condition:
$k_1 = |\lambda_1| \, M_1(b-a) < 1$
Then the operator T itself is a strict contraction on the complete metric space $C([a, b])$.
Case 2: Iterated Contraction (T^n is a strict contraction)
if $k_1 \geq 1$ the operator T is not a strict contraction. However, for the n-th iterated operator T^n, we have:

$$\left\|T_{\varphi}^{n}-T_{\psi}^{n}\right\|_{\infty} \leq \frac{\left(\left|\lambda_{1}\right| M_{1}(b-a)\right)^{n}}{n!}\|\varphi-\psi\|_{\infty}$$

$$\lim_{n \longrightarrow \infty} \frac{\left(\left|\lambda_{1}\right| M_{1}(b-a)\right)^{n}}{n!}=0$$

Therefore, there strictly exists a sufficiently large integer $n \in \mathbb{N}$ such that

$$K_{n_0}=\frac{\left(\left|\lambda_{1}\right| M_{1}(b-a)\right)^{n_0}}{n_0!}<1$$

According to the Generalized Banach Fixed-Point Theorem ([5]), since the iterated power T^{n_0} is a strict contraction, the original operator T is fully guaranteed to possess the exact same unique fixed point φ^*.

proof:

let $\phi, \psi \in C([a, b])$, we evaluate the successive iterations of the operator T

For $n=1$:

$$|(T \phi)(x)-(T \psi)(x)|=\left|\lambda_{1} \int_{a}^{x} l_{1}(x, \mu)(\phi(\mu)-\psi(\mu)) d \mu\right|$$

$$|(T \phi)(x)-(T \psi)(x)| \leq\left|\lambda_{1}\right| \int_{a}^{x}\left|l_{1}(x, \mu)\right| .|\phi(\mu)-\psi(\mu)| d \mu$$

by using the bounds $\left|l_{1}(x, \mu)\right| \leq M_{1}$ and $|\phi(\mu)-\psi(\mu)| \leq\|\phi-\psi\|_{\infty}$

For $n=2$

$$\left|\left(T^{2} \phi\right)(x)-\left(T^{2} \psi\right)(x)\right| \leq\left|\lambda_{1}\right| \int_{a}^{x}\left|l_{1}(x, \mu)\right| .|(T \phi)(\mu)-(T \psi)(\mu)| d \mu$$

substituting the result on $n=1$ into the integrand:

$$\left|\left(T^{2} \phi\right)(x)-\left(T^{2} \psi\right)(x)\right| \leq\left|\lambda_{1}\right| \int_{a}^{x} M_{1} .\left[\left|\lambda_{1}\right| M_{1}(\mu-a)\|\phi-\psi\|_{\infty} d \mu\right]$$

$$\left|\left(T^{2} \phi\right)(x)-\left(T^{2} \psi\right)(x)\right| \leq\left|\lambda_{1}\right|^{2} M_{1}^{2}\|\phi-\psi\|_{\infty} \int_{a}^{x}(\mu-a) d \mu$$

$$\left|\left(T^{2} \phi\right)(x)-\left(T^{2} \psi\right)(x)\right| \leq \frac{\left|\lambda_{1}\right|^{2} M_{1}^{2}(x-a)^{2}}{2}\|\phi-\psi\|_{\infty}$$

Proceeding by mathematical induction, one obtains
$\forall n \in \mathbb{N}$

$$|(T^n\phi)(x) - (T^n\psi)(x)| \leq \frac{|\lambda_1|^n M_1^n (x-a)^n}{n!} \|\phi - \psi\|_\infty$$

since $x \in [a,b]$, we have $(x-a) \leq (b-a)$ .therefore

$$|(T^n\phi)(x) - (T^n\psi)(x)| \leq \frac{(|\lambda_1| M_1 (b-a))^n}{n!} \|\phi - \psi\|_\infty$$

by taking the supermum norm on $[a,b]$

$$\|(T^n\phi)(x) - (T^n\psi)(x)\|_\infty \leq k_n \|\phi - \psi\|_\infty$$

where k_n is a constant :

$$k_n = \frac{(|\lambda_1| M_1 (b-a))^n}{n!}$$

as $n \longrightarrow \infty$

$$\lim_{n \longrightarrow \infty} k_n = \lim_{n \longrightarrow \infty} \frac{(|\lambda_1| M_1 (b-a))^n}{n!} = 0$$

$$\Rightarrow \exists n_0 \in \mathbb{N} \text{ such that } k_{n_0} < 1$$

Thus, T^{n_0} is a strict contraction mapping on $C([a,b])$.

Since T^{n_0} is a contraction on the complete metric space $C([a,b])$, Banach's theorem implies that T^{n_0} has a unique fixed point. This fixed point is also a fixed point of T, hence T admits a unique fixed point.

if T^n is a contraction for some integer n, then T has a unique fixed point (see [2], [4])

3.4 Application to Fredholm Integral Equations

Similarly, for the Fredholm equation, define:

$$(T\varphi)(x) = f(x) + \lambda_2 \int_a^b l_2(x,\mu)\varphi(\mu)d\mu$$

Proposition 3.2 : Contractivity of the Fredholm Operator
Assume $|l_2(x,\mu)| \leq M_2$. Then:

$\|T\varphi - T\psi\|_\infty \le |\lambda_2| M_2(b-a)\|\varphi - \psi\|_\infty$.

Contraction condition:$|\lambda_2| < \frac{1}{M_2(b-a)}$

proof:

let $\phi, \psi \in C([a,b])$ for any $x \in [a,b]$:

$$|(T\phi)(x) - (T\psi)(x)| = \left|(f(x) + \lambda_2 \int_a^b l_2(x,\mu)\,\phi(\mu)\,d\mu) - (f(x) + \lambda_2 \int_a^b l_2(x,\mu)\,\psi(\mu)\,d\mu\right|$$

$$= \left|\lambda_2 \int_a^b l_2(x,\mu)\,[\phi(\mu) - \psi(\mu)]d\mu\right|$$

$$|(T\phi)(x) - (T\psi)(x)| \le |\lambda_2| \int_a^b |l_2(x,\mu)| \,.\, |\phi(\mu) - \psi(\mu)|\,d\mu$$

using $|l_2(x,\mu)| \le M_2$

$$|(T\phi)(x) - (T\psi)(x)| \le |\lambda_2|\, M_2 \int_a^b |\phi(\mu) - \psi(\mu)|\,d\mu$$

using $|\phi(\mu) - \psi(\mu)| \le \|\phi - \psi\|_\infty$

$$|(T\phi)(x) - (T\psi)(x)| \le |\lambda_2|\, M_2\, \|\phi - \psi\|_\infty \int_a^b d\mu$$

$$\le |\lambda_2|\, M_2\,(b-a)\,\|\phi - \psi\|_\infty$$

$$\|T_\phi - T_\psi\|_\infty \le |\lambda_2|\, M_2\,(b-a)\,\|\phi - \psi\|_\infty$$

under the condition $|\lambda_2| < \frac{1}{M_2(b-a)}$, we have $k = |\lambda_2|\, M_2\,(b-a) < 1$ which proves that the Fredholm operator T is a strict contraction on the Banach space $C([a,b])$

3.5 Main Existence–Uniqueness Theorem

We now state and prove the main result of this work, unifying both Volterra and Fredholm cases.

Theorem 3.2: Existence and Uniqueness
Consider the mixed Volterra–Fredholm integral equation:

$$\varphi(x) - \lambda_1 \int_a^x l_1(x,\mu)\varphi(\mu)d\mu - \lambda_2 \int_a^b l_2(x,\mu)\varphi(\mu)d\mu = f(x), x \in [a,b].$$

Suppose:

(H1) $f \in C([a,b])$

(H2) $l_1, l_2 \in C([a,b] \times [a,b])$ with $|l_1(x,\mu)| \leq M_1$ and $|l_2(x,\mu)| \leq M_2$

(H3) $k := |\lambda_1| M_1 (b-a) + |\lambda_2| M_2 (b-a) < 1$

Then there exists a UNIQUE solution $\varphi^* \in C([a,b])$. The Picard iteration

$$\varphi_{n+1}(x) = f(x) + \lambda_1 \int_a^x l_1(x,\mu)\varphi_n(\mu)d\mu + \lambda_2 \int_a^b l_2(x,\mu)\varphi_n(\mu)d\mu$$

converges to φ^* with rate $\|\varphi_n - \varphi^*\|_\infty \leq \frac{k^n}{1-k} \cdot \|\varphi_1 - \varphi_0\|_\infty$.

proof

We apply Banach's Fixed Point Theorem (Theorem 3.1) to the operator $T : C([a,b]) \longrightarrow C([a,b])$ defined by:

$$(T_\varphi)(x) \quad = \quad f(x) + \lambda_1 \int_a^x l_1(x,\mu)\varphi(\mu)d\mu + \lambda_2 \int_a^b l_2(x,\mu)\varphi(\mu)d\mu$$

Step 1 : T maps $C([a,b])$ to itself: Since f, l_1, l_2 are continuous and $\varphi \in C([a,b])$, the function T_φ is continuous by standard results on continuity of integrals with continuous integrands.

Step 2 : T is a contraction: For $\varphi, \psi \in C([a,b])$:

$$|(T_\varphi)(x) - (T_\psi)(x)| \leq |\lambda_1| \int_a^x |l_1(x,\mu)||\varphi(\mu) - \psi(\mu)|d\mu| + \\ |\lambda_2| \int_a^b |l_2(x,\mu)||\varphi(\mu) - \psi(\mu)|d\mu$$

$$\leq |\lambda_1|\, M_1 \int_a^x |\varphi - \psi|\, d\mu + |\lambda_2|\, M_2 \int_a^b |\varphi - \psi|\, d\mu \\ \leq (|\lambda_1| M_1 (b-a) + |\lambda_2| M_2 (b-a)) \|\varphi - \psi\|_\infty$$

$$= k\|\varphi - \psi\|_\infty$$

Taking the supremum:$\|T\varphi - T\psi\|_\infty \leq k\|\varphi - \psi\|_\infty$. Since $k < 1$ by assumption (H3), T is a contraction.

Step 3 : Completeness: $(C([a,b]), \|\cdot\|_\infty)$ is a complete metric space (Banach space).

Step 4 : Conclusion: All hypotheses of Theorem 3.1 are satisfied. Therefore T has a unique fixed point $\varphi^* \in C([a,b])$, which is the unique solution of the integral equation.

3.6 A Priori Error Estimate

Corollary 3.1 : Quantitative Error Bound

Under the hypotheses of Theorem 3.1, starting from $\varphi_0 \in C([a,b])$:

$$\|\varphi_n - \varphi^*\|_\infty \leq \frac{k^n}{1-k} \cdot \|\varphi_1 - \varphi_0\|_\infty$$

This allows us to determine in advance how many iterations are needed to achieve a prescribed accuracy $\varepsilon > 0$:

it suffices that :

$$n \geq \frac{log\left(\frac{\varepsilon(1-k)}{\|\varphi_1 - \varphi_0\|_\infty}\right)}{log(k)}$$

.

Chapter 4

Numerical Applications

4.1 Example 1 : Volterra Equation with Exact Solution

Consider the Volterra integral equation on $[0, 1]$:

$$\varphi(x) = x + \int_0^x (x-\mu)\varphi(\mu)d\mu, \; x \in [0,1]$$

Here, the components are defined as:
$f(x) = x$
$\lambda_1 = 1$
$l_1(x, \mu) = x - \mu$
$M_1 = \max\left\{[0,1]^2\right\} |x - \mu| = 1$, $b - a = 1$
so
$k_1 = |\lambda_1| M_1(b-a) = 1.1.1 = 1$

Since the initial condition $k = 1$ is not strictly less than 1, the operator T is not a strict contraction on $C([0,1])$, and the classical Banach Fixed-Point Theorem cannot be applied directly.

However, we can investigate the iterated operator T^2 . For any $\varphi, \psi \in C([0,1])$, we evaluate the first iteration

$$|(T_\varphi)(x) - (T_\psi)(x)| \leq \int_0^x |x-\mu| \, . \, |\varphi(\mu) - \psi(\mu)| \, d\mu \leq M_1 \|\varphi - \psi\|_\infty . x$$

Proceeding to the second iteration T^2, we obtain:

$$|(T^2_\varphi)(x) - (T^2_\psi)(x)| \leq \int_0^x |x-\mu| \, . \, |(T_\varphi)(\mu) - (T_\psi)(\mu)| \, d\mu$$

$$|(T_{\varphi}^2)(x) - (T_{\psi}^2)(x)| \leq \int_0^x 1.\,(M_1\|\varphi - \psi\|_{\infty}.\mu)\,d\mu = M_1\|\varphi - \psi\|_{\infty}\frac{x^2}{2}$$

$$\left\|T_{\varphi}^2 - T_{\psi}^2\right\|_{\infty} \leq \frac{1}{2}\|\varphi - \psi\|_{\infty}$$

According to the Generalized Banach Fixed-Point Theorem, if any iterated power T^n is a strict contraction on a complete space, then the original operator T is fully guaranteed to have the exact same unique fixed point, and the Picard iterations will strictly converge to it

Since the new contraction constant is $k_2 = \frac{1}{2} < 1$, the iterated operator T^2 is indeed a strict contraction. Therefore, the convergence of the Picard successive approximations is fully guaranteed by the modified Banach theorem.The exact solution to this equation is $\varphi^*(x) = \sinh(x)$. We apply the Picard iteration with
$\varphi_0(x) = 0$ as summarized in Table 4.1 below:

Table 4.1: Successive Approximations for Example 1

Iteration	$\varphi_n(x)$
0	$\varphi_0(x) = 0$
1	$\varphi_1(x) = x$
2	$\varphi_2(x) = x + \frac{x^3}{6}$
3	$\varphi_3(x) = x + \frac{x^3}{6} + \frac{x^5}{120}$
$n \longrightarrow \infty$	$\varphi^*(x) = \sinh(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + ...$

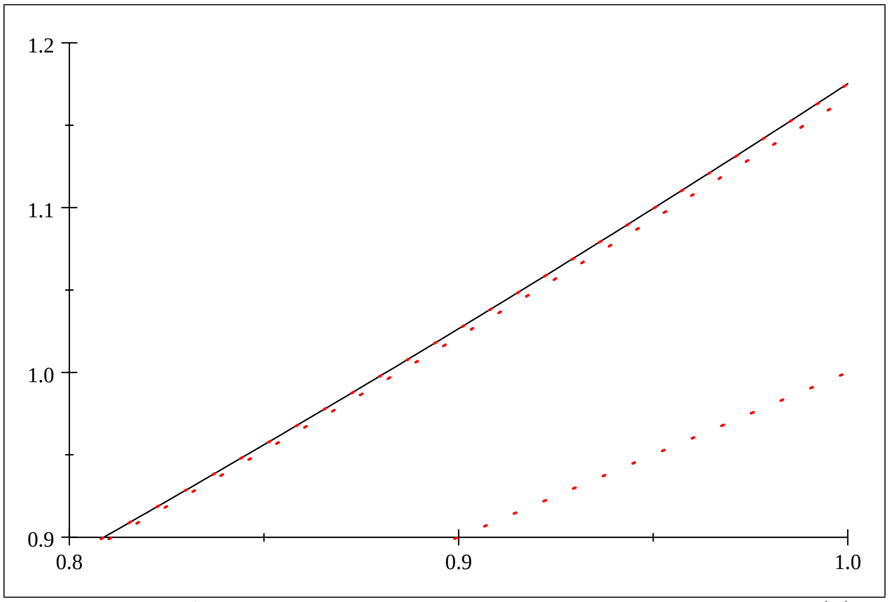
Each iterate adds the next term of the Taylor series of $\sinh(x)$.
To demonstrate the accuracy of the applied method, we evaluated the absolute error at the point $x = 1$, it was observed that the approximate results significantly converge to the exact value $\sinh(x) \approx 1.175201$

Table 4.2: Error Estimation at $x = 1$

n	Numerical Solution $\varphi_n(1)$	Exact Solution $\varphi^*(1)$	Absolute Error $\lvert\varphi^* - \varphi_n\rvert$
1	1.000000	1.175201	0.175201
2	1.166667	1.175201	0.008534
3	1.175000	1.175201	0.000201

The results in Table 4.2 show a significant decrease in the absolute error as the number of iterations n increases. With only 3 iterations, the relative error reached less than 10^{-4}

Figure 1:comparison of successive approximations and the exact solution for example 1



comparison of successive approximations and the exact solution $\varphi^*(x)$

——	exact sol
......	approximate sol

4.2 Example 2: Fredholm Equation (Small Parameter)

Consider the Fredholm equation on $[0, 1]$:

$$\varphi(x) = 1 + \lambda_2 \int_0^1 x \ \mu \ \varphi(\mu) d\mu, \ x \in [0, 1] \ , \ |\lambda_2| < 1$$

The kernel $l_2(x, \mu) = x\mu$ satisfies $M_2 = 1$. The contraction condition gives $|\lambda_2| < 1$.

Since $T_\varphi = 1 + \lambda_2 x \int_0^1 \mu\varphi(\mu) d\mu$

Since the kernel $l_2(x, \mu) = x\mu$ is separable, the variable x can be factored out of the integral operator. The equation becomes:

$$\varphi(x) = 1 + \lambda_2 x \int_0^1 \mu\varphi(\mu) d\mu$$

Since the integration limits are fixed, the definite integral represents a constant. Let

$$\beta = \int_0^1 \mu\varphi(\mu)d\mu$$

$$\varphi\,(x) = 1 + \lambda_2 \beta x$$

Comparing constant terms on both sides gives $\alpha = 1$,Comparing coefficients of x gives $\beta = \lambda_2 \left(\frac{\alpha}{2} + \frac{\beta}{3}\right)$, hence $\beta = \frac{3\lambda_2}{6-2\lambda_2}$

$$\varphi\,(x) = \alpha + \beta x$$

$$\alpha + \beta x = 1 + \lambda_2\ x \int_0^1 \mu\ (\alpha + \beta\mu)d\mu = 1 + \lambda_2\ x\ (\frac{\alpha}{2} + \frac{\beta}{3})$$

$$\Longrightarrow \alpha = 1 \quad , \quad \beta = \lambda_2(\frac{\alpha}{2} + \frac{\beta}{3}) \quad \Longrightarrow \quad \beta = \frac{3\lambda_2}{6-2\lambda_2}$$

$$Exact\ solution : \varphi^*(x) = 1 + \frac{3\lambda_2\ x}{6-2\lambda_2}$$

For $\lambda = 0.5$: $\varphi^*(x) = 1 + 0.3x$.

Table 4.3: Numerical results of Picard iterations for Example 2

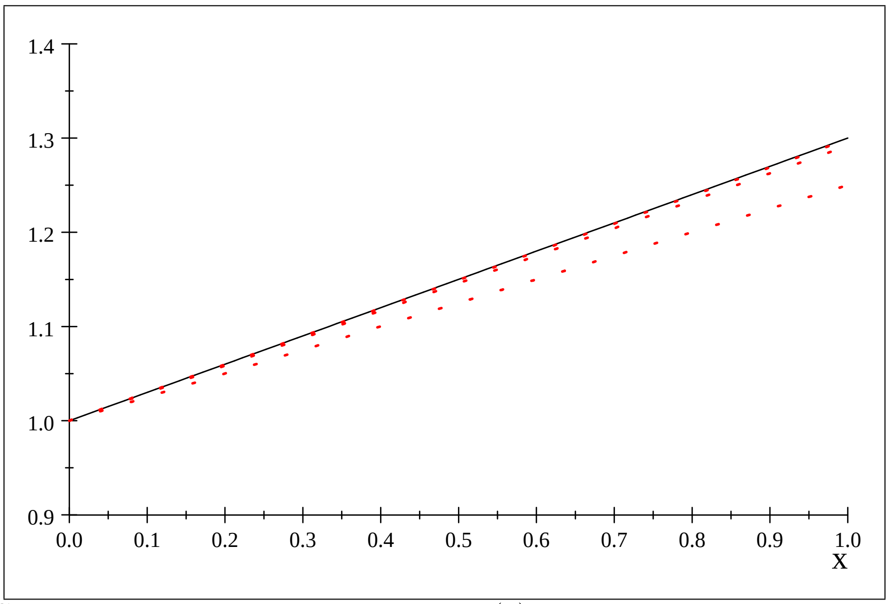
x	$\varphi_1\,(x)$	$\varphi_2\,(x)$	$\varphi_3\,(x)$
0.0	1.0000	1.0000	1.0000
0.2	1.0000	1.0500	1.0583
0.4	1.0000	1.1000	1.1167
0.6	1.0000	1.1500	1.1750
0.8	1.0000	1.2000	1.2333
1.0	1.0000	1.2500	1.2917

Table 4.4: Absolute error between the exact solution and approximations

x	$\mathrm{E}_1 = \lvert\varphi^* - \varphi_1\rvert$	$\mathrm{E}_2 = \lvert\varphi^* - \varphi_2\rvert$	$\mathrm{E}_3 = \lvert\varphi^* - \varphi_3\rvert$
0.0	0.0000	0.0000	0.0000
0.2	0.0600	0.0100	0.0017
0.4	0.1200	0.0200	0.0033
0.6	0.1800	0.0300	0.0050
0.8	0.2400	0.0400	0.0067
1.0	0.3000	0.0500	0.0083

The Picard iteration converges in 4 steps with $\|\varphi_4 - \varphi^*\|_\infty < 10^{-5}$.

Figure 2: Comparison between the exact solution $\varphi^*(x)$ and the Picard approximations for example 2



Comparison between the exact solution $\varphi^*(x)$ and the Picard approximations $\varphi_n(x)$

——	exact sol
......	approximate sol

4.3 Example 3 : Mixed Volterra–Fredholm Equation

Consider on $[0, 1]$:

$$\varphi(x) - 0.2\int_0^x (x-\mu)\varphi(\mu)d\mu \; - \; 0.3\int_0^1 x\mu\varphi(\mu)d\mu = f(x)$$

where:

$$f(x) = \exp(x) - 0.2(\exp(x) - x - 1) \; - \; 0.3x\, \frac{\exp(1) - 2}{2}$$

is chosen so that $\varphi^*(x) = \exp(x)$ is the exact solution.

Checking the contraction condition:

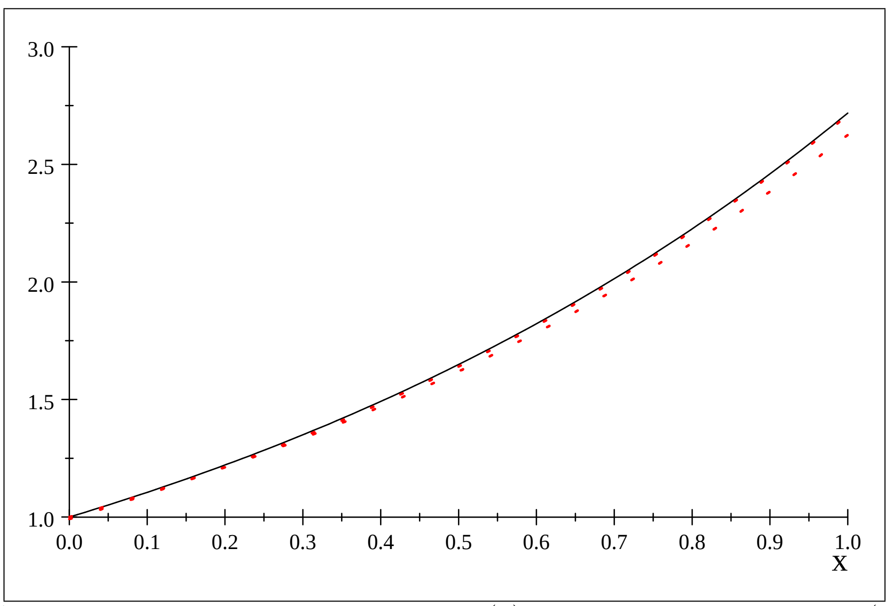
$k = 0.2 \cdot M_1 (b - a) + 0.3 \cdot M_2 = 0.2 \cdot 1 \cdot 1 + 0.3 \cdot 1 \cdot 1 = 0.5 < 1.$

Table 4.5: Numerical results for Example 3 (Mixed Equation)

Iteration n	$\Vert\varphi_n - \varphi^*\Vert_\infty$
1	2.14×10^{-1}
3	2.67×10^{-2}
5	3.34×10^{-3}
8	1.04×10^{-4}
12	$\approx 10^{-7}$ (very small numerical error)

this table shows that the absolute error decreases rapidly across the iterations. Starting from 2.14×10^{-1} at $n = 1$, it reaches very small numerical error $\left(\approx 10^{-7}\right)$ at $n = 12$. The geometric decay rate matches the theoretical prediction:$\Vert\varphi_n - \varphi^*\Vert_\infty \approx (0.5)^n \cdot \Vert\varphi_1 - \varphi_0\Vert_\infty$,confirming the sharpness of the a priori estimate.

Figure 3:Comparison between the exact solution $\varphi^*(x)$ and the Picard iterations for example 3



Comparison between the exact solution $\varphi^*(x)$ and the Picard iterations $\varphi_n(x)$

——	exact sol
......	approximate sol

4.4 Comparative Summary

Table 4.6

Equation Type	*Convergence Condition*
Volterra (pure)	$\lvert\lambda_1\rvert\, M_1(b-a) < 1$ (Contraction condition in L^∞ or convergence by Neumann iteration)
Fredholm (pure)	$\lvert\lambda_2\rvert\, M_2(b-a) < 1\,(or\ \ \lVert T\rVert < 1)$,Contraction condition of the integral operator
Mixed V–F	$\lvert\lambda_1\rvert M_1(b-a) + \lvert\lambda_2\rvert M_2(b-a) < 1$ (Global contraction condition)
General Banach space	T compact + $(I-T)^{-1}$ $\Leftrightarrow$Fredholm Alternative (Unique solution or non-trivial kernel)

(see Proposition 3.1 , Proposition 3.2)

This comparative summary highlights the distinct convergence behaviors of integral equations in Banach spaces. Volterra equations converge globally due to the quasi-nilpotent nature of their operator, making them less dependent on the domain's size. Conversely, Fredholm equations are highly sensitive to the interval length, strictly requiring the contraction condition $\lvert\lambda_2\rvert < \frac{1}{M_2(b-a)}$ to guarantee a unique solution via the Banach Fixed Point Theorem. Mixed Volterra-Fredholm equations combine both behaviors, demanding a strict balance between the kernel bounds M_1 and M_2 to maintain a global contraction. Furthermore, understanding these specific bounds is essential for verifying the stability of the system.

Finally, when dealing with compact operators in general Banach spaces, the analysis transcends simple contractions and utilizes the celebrated Fredholm Alternative. This theorem establishes a fundamental duality, ensuring either a unique solution for every free term or the existence of non-trivial eigenfunctions. This comprehensive theoretical distinction is crucial for developing robust numerical methods to approximate the exact solutions, thereby bridging the gap between abstract functional analysis and practical applications.

Conclusion

This research has established a classical framework for proving the existence and uniqueness of solutions to linear integral equations of Volterra, Fredholm, and mixed types, using Banach's Fixed Point Theorem as the central analytical tool.

The main achievements of this work are:

• A precise formulation of the contraction condition $k < 1$ for the integral operator, expressed in terms of the kernel bound and the parameter λ.

• A constructive proof via Picard iteration, providing not only existence and uniqueness, but also an explicit a priori error estimate.

• Three worked numerical examples confirming the theoretical results and illustrating the

exponential convergence of Picard iterates.

• A unified treatment of the mixed Volterra–Fredholm case with a single, verifiable condition on the parameters λ_1 and λ_2.

These results open natural directions for future investigation:

1. Extension to nonlinear integral equations of the form $\varphi = f + \lambda K(\varphi)$

where K is a nonlinear operator, using Schauder's or Krasnoselskii's fixed point theorems.

2. Analysis in weighted Sobolev spaces H^s to handle kernels with reduced regularity.

3. Adaptive a posteriori error estimates for practical computation without prior knowledge of M_1 and M_2.

4. Application to systems of integral equations arising in multi-species population models and coupled thermoelastic systems.

The numerical results confirm the theoretical convergence established by Banach's Fixed Point Theorem and demonstrate the effectiveness of Picard iterations for solving linear integral equations.

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