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وزارة التعليم العالي و البحث العلمي
جامعة محمد بوضياف بالمسيلة
كلية التكنولوجيا
قسم الهندسة الكهربائية

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
MOHAMED BOUDIAF UNIVERSITY OF M'SILA
FACULTY OF TECHNOLOGY
DEPARTMENT OF ELECTRICAL ENGINEERING

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شهادة إدارية

بخصوص مطبوعة الدروس الخاصة بالأستاذة

كحالي سليمة

بناءً على محضر اللجنة العلمية لقسم الهندسة الكهربائية تحت رقم: 131/ق.ه.ك/2025 المنعقد بتاريخ 25 ماي 2025 والمتضمن تعين الخبراء: الأستاذ بوقرة عبد الرحمان أستاذ بجامعة المسيلة الأستاذة لوكال كلثوم أستاذة محاضرة –أ بجامعة المسيلة، والأستاذ خميس رابح أستاذ بجامعة سكيكدة وذلك لتقييم مطبوعة الدروس الخاصة بالأستاذة كحالي سليمة أستاذة محاضرة –ب- بقسم الهندسة الكهربائية لجامعة المسيلة تحت عنوان :

" Méthode Numérique "

مطبوعة دروس مكتوبة باللغة الإنجليزية تحت عنوان:

" Numerical Methods"

وبعد إطلاع رئيس اللجنة العلمية ورئيس القسم على التقارير الواردة والتي كانت كلها ايجابية، وعليه فإن اللجنة لا ترى مانعا أن تتخذه سندا في تدريس طلبة السنة الثانية ليسانس كهروتقني وليسانس كهروميكانيك، شعبة كهروتقني وشعبة كهروميكانيك على التوالي، ميدان علوم وتكنولوجيا وأن تعتمد في أي تقييم للمسار العلمي للأستاذة المعنية.

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ملاحظة: سلمت هذه الشهادة للمعني(ة) لاستعمالها في حدود ما يسمح به القانون.

Algerian Democratic and People's Republic
الجمهورية الجزائرية الديمقراطية الشعبية
Ministry of Higher Education and Scientific Research
وزارة التعليم العالي و البحث العلمي

Mohamed Boudiaf University-M’sila
Faculty of Sciences and Technology
Electrical and Mechanical Engineering
2nd year Licence

جامعة محمد بوضياف - المسيلة
كلية التكنولوجيا
قسم : الهندسة الكهربائية

Course notes

Numerical Methods

Prepared by:

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Department of Electical and Mechanical Engineering

Academic Year: 2024-2025

National Program
Field: Science and Technology
Specialties: Electrical Engineering
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2nd year Licence

Semester : 4
Teaching Unit : UEF 2 .2 .2
Course Title : Numerical Methods
45h00 (Lecture: 1.5hourse, Tutorial: 1.5hourse)
Credits : 4
Coefficient : 2

Teaching Objectives: To introduce students to numerical methods and their applications in the field of mathematical computations.

Recommended Prerequisites:
Mathematics 1, Mathematics 2, Computer Science 1 and Computer Science 2.

Chapter 1: Solving Nonlinear Equations f(x) = 0 (3 Weeks)

1.Introduction to computational errors and approximations,2. Introduction to methods for solving nonlinear equations, 3. Bisection method, 4. Fixed-point iteration method (successive approximations), 5) Newton-Raphson method.

Chapter 2: Polynomial Interpolation (2 Weeks)

1.General Introduction, 2.Lagrange polynomial, 3. Newton polynomials.

Chapter 3: Numerical Integration (2 Weeks)

1.General introduction, 2. Trapezoidal Method, 3.Simpson's Method , 4. Quadrature formulas

Chapter 4: Solving Ordinary Differential Equations (Initial Value or Cauchy Problem) (2 Weeks).

1.General Introduction, 2. Euler's Method, 3. Improved Euler's Method, 4. Runge-Kutta Method.

Chapter 5: Direct Methods for Solving Systems of Linear Equations (2 Weeks)

1.Introduction and definitions, Cramer's Rule 3. Gaussian Elimination Method, Gaussian Elimination with Pivoting.

Chapter 6: Iterative Methods for Solving Systems of Linear Equations (2 Weeks)

1. Introduction and definitions, 2. Jacobi Method, 3. Gauss-Seidel Method.

Assessment Method: Continuous assessment: 40%; Final exam: 60%

Bibliographic References:

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Preface

Preface

This booklet serves as a teaching aid for students in the License 2 st years in the Department of Electrical and Mechanical Engineering. It provides a clear and structured summary of the methods covered throughout the chapters of the Numerical Methods course. The content is aligned with the official curriculum of the harmonized Licence programs, as established by the National Pedagogical Committees (CPND).

To fully benefit from this material, students are expected to have successfully completed the following prerequisite courses: Mathematics 1, Mathematics 2, Computer Science 1, Computer Science 2, and Computer Science 3.

Table of Contents

Table of Contents

Chapter I : Methods for Solving Nonlinear Equations

Chapter II : Solving Systems of Linear Equations (Direct Methods)

Chapter III : Solving Systems of Linear Equations (Iterative Methods)

Chapter IV : Numerical Integration

Chapter V : Differential Equations

Chapter VI : Interpolation

Introduction

Introduction

Numerical methods are essential techniques for approximating complex mathematical procedures. These approximations become necessary when exact analytical solutions are challenging or impossible to derive, particularly in fields like science and engineering where mathematical problems are often highly intricate. With the rapid advancement of computer technology, numerical methods have emerged as modern, indispensable tools for scientists and engineers, enabling the simplification and resolution of difficult problems

Numerical analysis pursues two primary objectives: first, to numerically solve practical problemswhether or not an analytical solution is knownand second, to evaluate the behavior of the methods employed, assessing their convergence speed toward an approximate solution and their accuracy in the presence of inherent computational errors. This course notes is designed for second-year university students pursuing a Bachelors degree in Science and Technology. It serves as a comprehensive manual, combining theoretical lessons with corrected exercises, covering the entire numerical analysis curriculum. The content is organized into six chapters as follows:

Chapter 1 examines methods for solving nonlinear equations, highlighting techniques such as the bisection method, the fixed-point method, and the Newton-Raphson method. It focuses on graphical interpretation, algorithms, convergence, and error estimation for each approach. Chapter 2 addresses the solution of linear equation systems using direct methods. It covers approaches like Cramers method (review), the method using the inverse matrix (Gauss), and the Gauss method with pivoting, emphasizing their algorithms, principles, and applications. Chapter 3 focuses on solving linear equation systems with iterative methods, including the Jacobi method and the Gauss-Seidel method. It also explores matrix A decomposition, stopping criteria, and convergence concepts. Chapter 4 delves into numerical integration, detailing methods such as the trapezoidal and Simpsons rules in both simple and generalized forms. It analyzes the errors associated with these methods and their impact on calculation accuracy. Chapter 5 tackles the numerical solution of ordinary differential equations. It presents classical methods like Eulers method and the Runge-Kutta method, explaining their general form, algorithms, and convergence properties. Finally, Chapter 6 is dedicated to numerical interpolation. It explores techniques such as Lagrange polynomial interpolation, Newtons method, and the Runge-Kutta method for interpolation. It also provides a detailed analysis of error and the necessary relations for coefficient calculations. At the end of each chapter, guided exercises (TD) with solutions are provided to help students apply the concepts learned. A list of bibliographic references is included at the end of this manual for further study.

Chapter I :Methods for Solving Nonlinear Equations

Introduction

Analytical methods for solving algebraic polynomial equations are limited to certain forms of low-degree equations, such as quadratic, cubic, and quartic equations. For polynomials of degree higher than four, it is generally not possible to find the roots using exact methods, and one must resort to numerical methods to find the roots.

Moreover, analytical methods limited to simple polynomial forms are not useful for transcendental equations such as:

$$xe^{x} - cos\, x + x^{2} + 1 = 0$$

It is therefore necessary to use numerical methods that allow for the calculation of roots with a certain precision.

In this chapter, we present several techniques for solving nonlinear equations. The proposed methods are: the bisection method, the Newton-Raphson method, and the fixed-point method.

A value of x that is a solution to $f(x) = 0$ is called a root or a zero of the function $f(x)$.

1.2. Bisection Method

This method is also called "dichotomy." It is based on an important theorem, which is the intermediate value theorem, as well as other methods.

Theorem If f is a continuous function on the interval $[a, b]$ and if $f(a).f(b) < 0$, then the equation $f(x) = 0$ has at least one root in the interval $[a, b]$.

1.2.1 Development of the Method

We ensure that if $f(a).f(b) < 0$ on the interval [a,b], the root is unique, i.e., f is monotone on the interval [a,b]. We divide the interval $[a, b]$ into two intervals $[a, c]$ and [c,b] such that $c = ((a + b)/2)$. We divide the interval $[a, b]$ into two intervals $[a, c]$ and [c,b] such that $c = ((a + b)/2)$. If $f(a).f(c) < 0$, this implies that the root is in $[a, c]$, and if $f(c) . f(b) < 0$, this implies that the root is in $[c, b]$. In either case, we select a single interval to continue the process. This ensures that after each iteration, the interval containing the root becomes smaller, providing a more precise approximation of the equation $f(x) = 0$.

Example 1: Calculate the root of the equation $f(x) = x^{5} - x - 1 = 0$ in the interval [1,2] with a precision .

Solution: The function $f(x)$ is a polynomial, hence continuous on [1,2].

The function f(x) is a polynomial, hence continuous on [1,2]. $f(1) = -1 , f(2) = 61 \Rightarrow f(1).f(2) < 0$, , so by the intermediate value theorem, there exists at least one root in $c \in [1,2]$ such that $f(c) = 0$. The function f(x) is a polynomial, hence continuous on [1,2].

$f(1) = -1$, $f(2) = 61 \;\Rightarrow\; f(1).f(2) \; < 0$, so by the intermediate value theorem, there exists at least one root in $c \in \; [1,2]$ such that $f(c) = 0$.

Moreover, since $f' \,(x) = 5x^4 - 1$, for all $x \in [1,2], f'(x) > 0 \;\Rightarrow f \;\nearrow$ (is monotone), so we deduce that the root of f in [1,2] is unique. We proceed to iteratively calculate an approximation of this root.

n	a	b	c	$b - c$	$f(c)$
1	1.0000	2.000	1.5000	0.50000	8.8906
2	1.0000	1.2500	1.1250	0.2500	1.5647
3	1.0000	1.2500	1.1250	0.1250	-0.0977
4	1.1250	1.2500	1.1875	0.0625	0.6167
.	.	.	.	.	.
10	1.1300	1.1348	1.1338	0.001	0.0096

The approximate root is $c = 1.1338$, obtained after 10 iterations.

Example 2: Solve the equation e^{x}= \cosx in the interval $[-0.5,0.5]$ with $eps = 0.0003$.

Solution: c=((a+b)/2)=0. We notice that f(c)=e^{0}-cos0=0

Thus, $c = 0$ is the exact root of the function in the interval $[-0.5,0.5]$.

1.2.2 Algorithm of the Method

Step 1: $c \leftarrow \; (a + b)/2$

Step 2: $If \; b - c \leq \; eps,$ c is considered the approximate root (stop).

Step 3: $If \;\; f(c) = 0$, c is considered the exact root (stop).

Step 4: $If \; f(a) \, . \, f(c) < 0$, then $c \leftarrow a$.

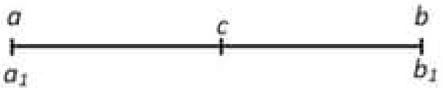
Otherwise, $c \leftarrow a$.

Step 5: Return to Step 1.

1.2.3 Convergence and Error Estimation

- We demonstrate that the bisection method converges to the unique root of the equation $f(x) = 0$ in the interval $[a, b]$.

- We seek to determine the maximum error committed when using the bisection method in the interval [a,b].



For i=1 : the length of the first interval $[a_1, b_1] \ = [a, b] \ is \ b_1 - a_1 = b - a$.

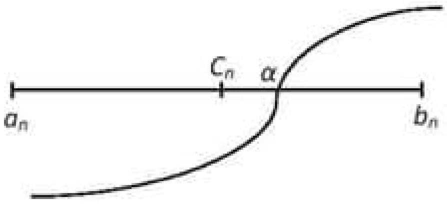
For $i = 2$: the length of the second interval $[a_2, b_2] \ is \ b_2 - a_2 \ = ((b - a)/2)$.

For $i = 3$: the length of the third interval $[a_3, b_3] \ is \ b_3 - a_3 \ = ((b - a)/(2^2))$.

For $i = 4$: the length of the fourth interval $[a_4, b_4] \ is \ b_4 - a_4 \ = ((b - a)/(2^3))$.

…

For $i = n$: the length of the fourth interval $[a_n, b_n] \ is \ b_n - a_n \ = ((b - a)/(2^{n-1}))$.



The approximate root is C_n =($(a_n + b_n)$/2). If we denote by α the exact solution, then we have:

$$|C_n - \alpha| \leq b_n - C_n = ((b_n - a_n)/2) = ((b - a)/(2^n)) \ \leq \ \varepsilon$$

Thus, the error

$$|\ C_n - \alpha| \leq \ ((b-a)/(2^n)) \qquad (1).$$

- The relation (1) also allows us to calculate in advance the maximum number of iterations required.

$$n \geq \frac{ln\left(\frac{b - a}{eps}\right)}{ln(2} \quad (2) \quad (n \ depends \ on \ f)$$

As an example: If the interval is $[1,2]$ and $\varepsilon \ = 10^{-3}$, then $n \geq 9.93$:

We take n=10. It is important to note that the number of iterations given by the formula above is,

, in many cases, an overestimation of the actual number of iterations required of iterations required (for the calculation, we can settle for n=9 while still achieving the desired precision).

1.3. Newton-Raphson Method

Let α be the exact root of the equation $f(x) = 0$. If f is continuous and differentiable in the vicinity of α , then the Taylor series expansion around x_0(where x_0 is the initial value) can be written as:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{\frac{f''(x_0)}{2!}(x - x^0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^2}{R}$$

Thus, we can write:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R$$

Assume x=α (where α is a root):

$$f(\alpha) = f(x_0) + (\alpha - x_0)f'(x_0) + R = 0$$

$\Rightarrow \alpha = x_0 - ((f(x_0))/(f'(x_0))) + R_1$

By ignoring R_1, we obtain a new approximation x_1 $(better\ than\ x_0\ )$:

$$x_1 = x_0 - ((f(x_0))/(f'(x_0)))$$

We proceed in the same way as before to find a new value x_2 using the Taylor series expansion around x_1:

$$x_2 = x_1 - ((f(x_1))/(f'(x_1)))$$

Thus, we obtain the following recursive relation:

$$\begin{cases} \quad x_0 \qquad\qquad given \\ x_k = x_{k-1} - ((f(x_{k-1}))/(f'(x_{k-1}))) \end{cases}$$

1.3.1 Graphical interpretation of the method

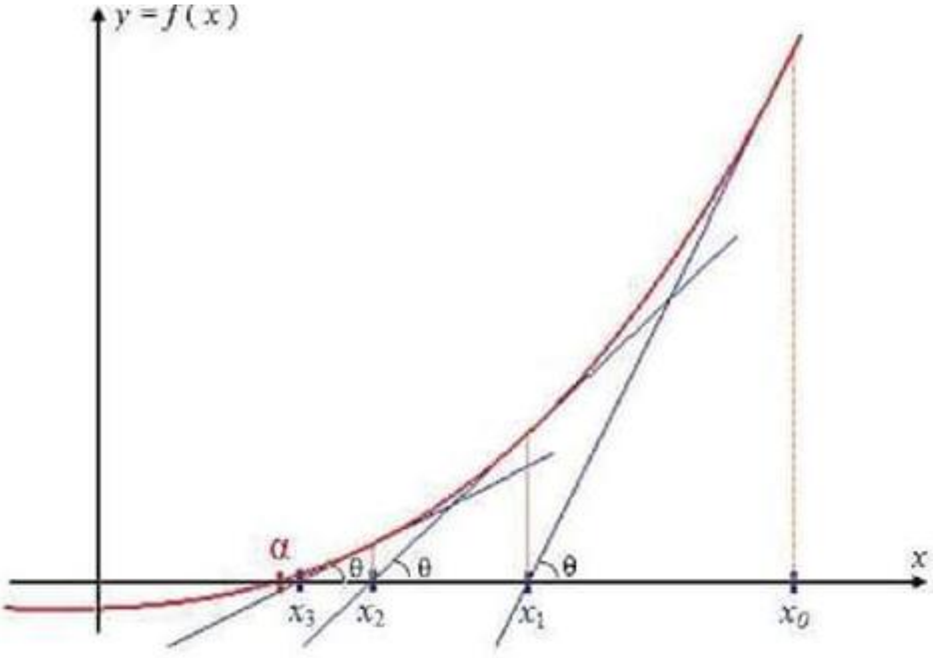


Figure1:Newton-Raphson method

$tg\varphi$ represents the derivative:

$$f(x_k) = \frac{f(x_{k-1}) - 0}{x_{k-1} - x_k}$$

Example : Find, using the Newton-Raphson method, the approximate value of the root of $x^5 - x + 2 = 0$ starting from $x_0 = -1$ with a precision $\varepsilon = 10^{-6}$.

Solution:

$f(x) = x^5 - x + 2 = 0 \;\Rightarrow\; f'(x) = 5x^4 - 1$

Applying the Newton-Raphson formula, we obtain:

$$x_k = x_{k-1} - \frac{{x_{k-1}}^5 - x_{k-1} + 2}{5{x_{k-1}}^4 - 1}$$

Thus, starting from $x_0 = -1$, we obtain the iterations illustrated in the following table:

k	x_k
0	-1.000000
1	-1.500000
2	-1.331620
3	-1.273516
4	-1.267237
5	-1.267168
6	-1.267168

$|x_k - x_{k-1}| = |x^6 - x^5| \leq \varepsilon$ thus the approximate root of the equation $f(x) = 0$,

$c = -1.267168.$

1.3.2 Algorithm of the Method

Step 1: $k \leftarrow 1$

Step 2: $x_k \leftarrow x_{k-1} - \frac{f(x_{k-1})}{f\prime(x_k)}$

Step 3: $If \ \ |x_k - x_{k-1}| \leq \varepsilon \, .So\ we\ stop\ (approximate\ root\ x).$

Otherwise, **Step 4** $k \leftarrow k + 1$.

Step 5: Return to Step 2

1.3.3 Convergence and Error Estimation

We demonstrate that if f is defined on the interval $[a, b]$ such that:

1. $(f(c), f(b) < 0$
2. $\forall x \in [a, b] \, , f'(x) \neq 0$
3. $\forall x \in [a, b] \, , f''(x) \neq 0$

Then the Newton-Raphson method generates a sequence that converges to the unique solution of $f(x) = 0$, starting from the approximation x_0, verifying:

$$f''(x_0).f(x_0) > 0$$

We also demonstrate that the error committed when using the Newton-Raphson method as a solution tool is written as:

$$|\alpha - x_k| \leq \frac{(x_k - x_{k-1})^2}{2m} avec \begin{cases} M = max\{|f''(x)|\}, & x \in [a,b] \\ m = min\{|f''(x)|\}, & x \in [a,b] \end{cases}$$

4.1. Fixed-Point Method

Before addressing the fixed-point method, it is important to define what is meant by a fixed point of a function.

Definition Let $g(x)$ be a function defined on the interval $[a,b]$. Any point $c \in [a,b]$ such that $c = g(c)$ is called a fixed point of the function $g(x)$.

The equation $f(x) = 0$, with f continuous on $[a,b]$, can be rewritten in the form $x = g(x), f(x) = x - g(x)$.

The choice of an initial root value x_0 allows us to obtain a first approximation x_1 such that $x_1 = g(x_0)$, then a better approximation x_2 such that $x_2 = g(x_1)$, and so on, leading to the following recursive relation:

$$\begin{cases} x_0 \\ x_k = g(x_{k-1}) \end{cases}$$

We aim to solve the equation $x^3 - 2 = 0$ starting from the initial value $x_0 = 1.2$.

Solution:

It is possible to transform the previous equation into several forms $x = g(x)$, for example:

$a)$ $x = g_1(x) = x^3 + x - 2$

$x_0 = 1.200$

$x_1 = 0.928$

$x_2 = -0.273$

$x_3 = -2.293$

$x_4 = -16.349$

We notice that we are moving away from the exact root $(2^{1/3})$ ⇒ Divergence.

b) $x = g_2(x) = (2 + 5x - x^3)/5$, we obtain:

$x_0 = 1.2000$

$x_1 = 1.2544$

$x_2 = 1.2596$

$x_3 = 1.2599$

$x_4 = 1.2599$

We notice that we are rapidly approaching the exact root $\left(2^{1/3}\right) \Rightarrow$ Convergence.

Thus, we conclude that the choice of the form $x = g(x)$ is crucial in determining the convergence or divergence of the fixed-point method.

4.1.1.Algorithm of the Method

Step 1: $k \leftarrow 1$

Step 2: $x_k \leftarrow g(x_{k-1})$

Step 3: If $|x_{k-1}$ -$x_k| \leq \varepsilon$, then stopx_k, approximate root)

Otherwise : Step 4: k←k+1

: Step 5: Return to Step 2.

4.1.2 Study of the Convergence of the Method

We demonstrate that if $g: [a, b] \rightarrow [a, b]$ has a fixed point in the interval $[a, b]$ and if $|g'(x)| \leq k < 1$, then that point is unique.

exampl We show that the function $g(x) = \frac{x^3-1}{3}$ has a unique fixed point in the interval [-1,1].

Solution: Divide the interval into two parts: $[-1,0]$ *and* $[0,1]$

$g(-1) = g(1) = 0 \in [-1,1]$

$$g(0) = -\frac{1}{3} \in [-1,1]$$

And we have:

$\forall x \in [-1,0]: g(x) \searrow \Rightarrow g(0) = -\frac{1}{3} \leq g(x) \leq g(-1) = 0$

$\forall x \in [-1,0]: g(x) \nearrow \Rightarrow g(0) = -\frac{1}{3} \leq g(x) \leq g(1) = 0$

Thus $g: [-1,1] \rightarrow [-1,1]$

Moreover, $|g'(x)| = |2x/3| \leq 2/3 < 1$

We conclude that the fixed point is unique in $[-1,1]$.

Theorem We demonstrate that if the function $g:[a,b] \to [a,b]$ satisfies $|g'(x)| \leq k < 1$ for all $x \in [a,b]$, then the sequence defined by the following recursive relation:

$$\begin{cases} x_0 \\ x_k = g(x_{k-1}) \end{cases}$$

is convergent and converges to the unique fixed point of g in $[a,b]$.

4.3.1.Error Estimation

We demonstrate that the error committed when using the fixed-point method as a solution tool is verified by the relation:

$$|x_k - \alpha| \leq \frac{k}{1-k}|x_n - x_{n-1}|, \qquad \begin{cases} \alpha : \text{ the exact solution} \\ x_n : \text{the approche solution} \\ \mathrm{k} = \dfrac{|x_n - x_{n-1}|}{|x_n - x_{n-2}|} \end{cases}$$

Chapter II:

Solving Systems of Linear Equations (Direct Methods)

Introduction

In practice, engineers often face problems whose solution involves solving a system of equations that models the various elements considered. For example, determining currents and voltages in electrical networks requires solving a system of linear or nonlinear equations.

In this chapter, we will discuss two main methods for solving linear systems, namely Gaussian elimination and Gaussian elimination with pivoting. Generally, solving a system of linear equations consists of finding a vector: $X = [x_1 , x_2, \ldots\ldots, x_n]^T\}$ ((where T denotes the transpose, making it a column vector), that satisfies:

$$\begin{cases} \mathrm{a_{11}x_1 + a_{12}x_2 + a_{13}x_3} + \ldots\ldots + a_{1n}x_n = \mathrm{b_1} \\ \mathrm{a_{21}x_1 + a_{22}x_2 + a_{23}x_3} + \ldots\ldots + a_{2n}x_n = \mathrm{b_2} \\ . \\ a_{1n}x_1 + a_{2n}x_2 + a_{3n}x_3 + + \ldots\ldots + a_{nn}x_n = b_n \end{cases}$$

We can use matrix notation, which is much more practical and compact. The system above can be rewritten as:

$$A\vec{X} = \vec{b}$$

where A is the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{12} & a_{22} & a_{23} & a_{2n} \\ & & . & \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{pmatrix}$$

and b is the vector: $\vec{b} = [b_1, b_2, \ldots\ldots, b_n]^T$ Of course, the matrix A and the vector b are known. The goal is to determine the vector X.

2.1 Cramer's Method

This is certainly the most well-known method for solving linear systems. However, we will see that it is the least recommended.

Solution Using Determinants

The system $\vec{A}X = \vec{b}$has a unique solution if and only if the determinant of matrix A is nonzero: det A=|A|≠0, In this case, the solution is given by:

$$x_1 = \frac{|A_1|}{|A|}\,, \qquad x_2 = \frac{|A_2|}{|A|}, \ldots\ldots, x_n = \frac{|A_n|}{|A|}$$

where A_i is obtained by replacing the i th column of A with the vector b.

Note: If $detA = 0$, the system has either infinitely many solutions or no solution.

Example Solve the following system using Cramer's Method:

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 50 \\ 3x_1 + 5x_2 - 4x_3 = 2 \\ 4x_1 + 7x_2 - 2x_3 = 31 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 3 & 4 \\ 3 & 5 & -4 \\ 4 & 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 50 \\ 2 \\ 31 \end{pmatrix}$$

det A=-8≠0⇒A is inversible, In this case, the solution is given by:

$x_1 = \frac{|A_1|}{|A|} = \frac{|A_1|}{-8}$, $A_1 = \begin{pmatrix} 50 & 3 & 4 \\ 2 & 5 & -4 \\ 31 & 7 & -2 \end{pmatrix}, and \;\; detA_1 = -21 \Rightarrow x_1 = \frac{|A_1|}{-8} = \frac{-21}{-8} = 3$

$x_2 = \frac{|A_2|}{|A|} = \frac{|A_2|}{-8}\;\; , A_2 = \begin{pmatrix} 1 & 50 & 4 \\ 3 & 2 & -4 \\ 4 & 31 & -2 \end{pmatrix}, and \;\; detA_1 = -40 \Rightarrow x_1 = \frac{|A_1|}{-8} = \frac{-40}{-8} = 5$

$x_3 = \frac{|A_3|}{|A|} = , A_2 = \begin{pmatrix} 1 & 3 & 50 \\ 3 & 5 & 2 \\ 4 & 7 & 31 \end{pmatrix}, and \;\; detA_1 = -64 \Rightarrow x_1 = \frac{|A_1|}{-8} = \frac{-64}{-8} = 8$

$$\Leftrightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$$

2.1.1 Solution Using the inverse matrix A^{-1}

If $det\ A = |A| \neq 0 \Rightarrow A^{-1}$ exists.

$$A\vec{X} = \vec{b}$$

$$A^{-1}A\vec{X} = \vec{b},\ \Rightarrow\ \vec{X} = A^{-1}\vec{b} \quad (\quad A^{-1} \text{ is calculated using the method of cofactors}).$$

Remark: Cramer's method requires a large number of computational operations $(n^2 + n)\,n!$ - 1). If n is large, the number of operations increases and consequently the computation time increases. Moreover, for medium and large systems, the cumulative rounding error increases with the number of operations and affects the accuracy of the results. We will address other methods that require a limited number of calculations, therefore faster and more accurate.

2.3Gaussian Elimination

Introduction (Principle)

This method transforms the system $\vec{A}X = \vec{b}$ into an equivalent system $\overrightarrow{A'}X = \overrightarrow{b'}$where A′ is an upper triangular matrix.

The transformation of the matrix A into A' and the vector b into b' goes through several which we present through the following example:

For better practicality, we form the matrix $\widetilde{A}$ such that $\tilde{A}$=[A⁝$\vec{b}$] which is called the augmented matrix of A.

Example : Solve the following system using Gaussian elimination:

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 50 \\ 3x_1 + 5x_2 - 4x_3 = 2 \ , \\ 4x_1 + 7x_2 - 2x_3 = 31 \end{cases} \quad \tilde{A} = \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 2 & 2 & 0 & \vdots & 2 \\ 3 & 2 & 6 & \vdots & 11 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

Step 1: Eliminate x_{1 }from rows E_2 and E_3

$$E_2 \leftarrow E_2 - \frac{2}{1} E_1, \ E_3 \leftarrow E_3 - \frac{3}{1} E_1$$

Resulting matrix:

$$\widetilde{A^{(1)}} = \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 0 & -4 & -6 & \vdots & 2 \\ 0 & -7 & -3 & \vdots & 11 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

Step 2: Eliminate $x_2$from E_3: $E_3 \leftarrow E_3 - \left(-\frac{7}{4}\right) E_2$

$$\widetilde{A^{(2)}} = \begin{bmatrix} 1 & 3 & 3 & \vdots & 0 \\ 0 & -4 & -6 & \vdots & 2 \\ 0 & 0 & 15/2 & \vdots & 11 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

Using back-substitution, we find:

15/12 x_3 =15/12 ⇒ x_3 =1

-4x_2}-6x_3 =2 ⇒x_2=-2

x_1+3x_2+3x_3=0⇒x_1 }=3

Thus, the solution is:

$$X = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

2.3.1 Algorithm of the Method

$$\begin{cases} \textbf{Triangularization}: \\ \omega = \dfrac{a_{ik}}{a_{kk}} \\ a_{ij} = a_{ij} - \omega a_{kj}; \quad \mathrm{j} = \mathrm{k}+1, \mathrm{n}+1;\ \mathrm{i} = \mathrm{k}+1, \mathrm{n};\ \mathrm{k} = 1, \mathrm{n}-1 \\ \textbf{Back} - \textbf{substitution}: \\ x_i = \left(b_i - \displaystyle\sum_{j=i+1}^{n} a_{ij} x_i \right) / a_{ii} \end{cases}$$

2.4 Gaussian Elimination with Pivoting

In Gaussian elimination, we assume at each step that $a_{kk} \neq 0$. However, if a_{kk}=0, we swap rows to ensure a_{kk}≠0

Additionally, dividing by small values can cause errors. To reduce this, we select a_{kk}as the largest absolute value in its column:

$$|a_{kk}| = Max\{|a_{ik}|\}\,;\ \ k \leq i \leq n$$

Chapter III :
Solving Systems of Linear Equations (Iterative Methods)

III.1. Introduction

We are interested in solving the linear system:

$$A\vec{X} = \vec{b}$$

A is a matrix of order n. The methods we present below are generalizations to the n- dimensional case of the resolution methods of $f(x) = 0$ studied in the one-dimensional case in Chapter 1. These methods involve using an initial vector $\vec{X}^{(0)}$ and generating a sequence of vectors: $\vec{X}^{(0)}$, $\vec{X}^{(1)}$......., $\vec{X}^{(n)}$ that converge as quickly as possible to the solution vector.

3.2 Principle We want to solve the linear system:

$$A\vec{X} = \vec{b}$$

Writing A in the form $A = M - N$ where A and M must be regular, the system can be

rewritten as: (M-N) $\vec{X}$=$\vec{b}$ or M $\vec{X}$=N $\vec{X}$+$\vec{b}$

Thus, $\vec{X}$=M^{-1}N$\vec{X}$+ $M^{-1}\vec{b}$

Starting from the initial vector $\vec{X}^{(0)}$we obtain the iterations $\vec{X}^{(k)}$as follows:

$$\vec{X}=M^{-1}N\vec{X}+ M^{-1}\vec{b} \quad ,k=0,1,2,\ldots\ldots$$

3.3 Decomposition of Matrix

We define the matrices $D, L,$ and U as follows:

- D Diagonal matrix where / d_{ii}=a_{ii}, $\forall\ i$.

- L : Lower triangular matrix where : $\begin{cases} l_{ii} = -a_{ii} & if\ \ i > j \\ l_{ii} = 0 & if\ \ i \leq j \end{cases}$

-U : Upper triangular matrix where:L : $\begin{cases} u_{ij} = -a_{ij} & if\ \ j > i \\ u_{ii} = 0 & if\ \ j \leq i \end{cases}$

From this, we obtain the relation:

$$A = D - L - U$$

3.4 Jacobi Method

Since matrix A is decomposed as:

$$A = M - N$$

Taking $M = D$ and $N = L + U$, relation (1) becomes:

$$\vec{X}^{(k+1)} = D^{-1}(L + U)\vec{X}^{(k)} + D^{-1}\vec{b} \quad (2)$$

which can be expanded as follows:

$$\begin{cases} \vec{X}_1^{(k+1)} = (b_1 - a_{12}\vec{X}_2^{(k)} - a_{13}\vec{X}_3^{(k)} - \cdots - a_{1n}\vec{X}_n^{(k)})/a_{11} \\ \vec{X}_2^{(k+1)} = (b_2 - a_{21}\vec{X}_2^{(k)} - a_{23}\vec{X}_3^{(k)} - \cdots - a_{2n}\vec{X}_n^{(k)})/a_{22} \\ \quad\quad\quad\quad . \\ \vec{X}_n^{(k+1)} = (b_n - a_{n1}\vec{X}_2^{(k)} - a_{n2}\vec{X}_3^{(k)} - \cdots - a_{nn}\vec{X}_{n-1}^{(k)})/a_{nn} \end{cases}$$

3.5 .Gauss-Seidel Method

Since matrix A is decomposed as: $A = M - N$

Taking $M = D - L$ and $N = U$, relation (1) becomes:

$$\vec{X}^{(k+1)} = (D - L)^{-1}U\vec{X}^{(k)} + (D - L)^{-1}\vec{b}$$

Since computing the inverse of $(D - L)$ can be complex, we prefer to write the system as follows:

$$\vec{X}^{(k+1)} = D^{-1}L\vec{X}^{(k+1)} + D^{-1}U\,\vec{X}^{(k)} + D^{-1}\vec{b} \quad (3)$$

By expanding this vector recurrence, we obtain:

$$\begin{cases} \vec{X}_1^{(k+1)} = (b_1 - a_{12}\vec{X}_2^{(k)} - a_{13}\vec{X}_3^{(k)} - \cdots - a_{1n}\vec{X}_n^{(k)})/a_{11} \\ \vec{X}_2^{(k+1)} = (b_2 - a_{21}\vec{X}_2^{(k+1)} - a_{23}\vec{X}_3^{(k)} - \cdots - a_{2n}\vec{X}_n^{(k)})/a_{22} \\ \vdots \\ \vec{X}_n^{(k+1)} = (b_n - a_{n1}\vec{X}_2^{(k+1)} - a_{n2}\vec{X}_3^{(k+1)} - \cdots - a_{nn}\vec{X}_{n-1}^{(k+1)})/a_{nn} \end{cases}$$

The Gauss-Seidel method is thus an improved variant of the Jacobi method. In fact, at iteration $k+1$, when computing$\vec{X}_2^{(k+1)}$, we already have a better approximation of $\vec{X}_1^{(k)}$than $\vec{X}_1^{(k+1)}$, as we have just computed . Similarly, when computing$\vec{X}_3^{(k+1)}$, we can use the newly computed $\vec{X}_1^{(k+1)}$and$\vec{X}_2^{(k+1)}$. More generally, for computing$\vec{X}_i^{(k+1)}$, we can use $\vec{X}_1^{(k+1)}$},$\vec{X}_2^{(k+1)}$,..., $\vec{X}_i^{(k+1)}$ from the previous iteration.

stopping test:

$$|\vec{X}_i^{(n)} - \vec{X}_i^{(n-1)}| \leq \varepsilon \ ,\forall\, i = \overline{1,n}$$

convergence condition

It can be shown that if A is a strictly diagonally dominant matrix (a sufficient condition), the Jacobi and Gauss-Seidel methods are convergent

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \leq |a_{ii}|, \quad \forall\, i = \overline{1,n}$$

Example 1 : Solve the system

$$\begin{cases} 5\mathrm{x}_1 + 2\mathrm{x}_2 - \mathrm{x}_3 = 6 \\ \mathrm{x}_1 + 6\mathrm{x}_2 - 3\mathrm{x}_3 = 4 \\ 2\mathrm{x}_1 + \mathrm{x}_2 + 4\mathrm{x}_3 = 7 \end{cases}$$

of equations using the Jacobi and Gausse Seidal methods from $X^{(0)} = (0,0,0)^t$^do only the first 3 iterations).

we have A= $\begin{pmatrix} 5 & 2 & -1 \\ 1 & 6 & -3 \\ 2 & 1 & 4 \end{pmatrix}$,

It can be shown that if A is a strictly diagonally dominant matrix : $\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \leq |a_{ii}|$, ∀i=1,n

$(|2| + |1| \leq |5|,\ \ |1| + |-3| \leq |6|,\ |2| + |1| \leq |4|)$, The condition is satisfied, and therefore the matrix A is diagonally dominant.

1.method Jacobi

$$\begin{cases} 5\mathrm{x}_1 + 2\mathrm{x}_2 - \mathrm{x}_3 = 6 \\ \mathrm{x}_1 + 6\mathrm{x}_2 - 3\mathrm{x}_3 = 4 \\ 2\mathrm{x}_1 + \mathrm{x}_2 + 4\mathrm{x}_3 = 7 \end{cases} \Rightarrow \begin{cases} \vec{x}_1{}^{(k+1)} = \frac{1}{5}(6 - 2\vec{x}_2{}^{(k)} + \vec{x}_3{}^{(k)}) \\ \vec{x}_2{}^{(k+1)} = \frac{1}{6}(4 - \vec{x}_1{}^{(k)} + 3\vec{x}_3{}^{(k)}) \\ \vec{x}_3{}^{(k+1)} = \frac{1}{4}(7 - 2\vec{x}_1{}^{(k)} - \vec{x}_1{}^{(k)}) \end{cases}$$ Tapez une équation ici.

When $k = 0$, $X^{(0)}$=(0,0,0) ^{t}

$$\Leftrightarrow \begin{cases} \vec{x}_1{}^{(1)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(0)} + \vec{x}_3{}^{(0)}\right) = \frac{6}{5} = 1.200 \\ \vec{x}_2{}^{(1)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(0)} + 3\vec{x}_3{}^{(0)}\right) = \frac{4}{6} = 0.667 \\ \vec{x}_3{}^{(1)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(0)} - \vec{x}_1{}^{(0)}\right) = \frac{7}{4} = 1.750 \end{cases}$$

When $k = 1$,

$$\Leftrightarrow\begin{cases}\vec{x}_1{}^{(2)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(1)} + \vec{x}_3{}^{(1)}\right) = \frac{1}{5}(6 - 2(0.667) + 1.750) = 1.283\\ \vec{x}_2{}^{(2)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(1)} + 3\vec{x}_3{}^{(1)}\right) = \frac{1}{6}(4 - 1.200 + 3(1.750)) = 1.342\\ \vec{x}_3{}^{(2)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(1)} - \vec{x}_1{}^{(1)}\right) = \frac{1}{4}(7 - 2(1.200) - 0.667) = 0.983\end{cases}$$

When $k = 2$,

$$\Leftrightarrow\begin{cases}\vec{x}_1{}^{(3)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(2)} + \vec{x}_3{}^{(2)}\right) = \frac{1}{5}(6 - 2(1.342) + 0.983) = 0.860\\ \vec{x}_2{}^{(3)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{2} + 3\vec{x}_3{}^{(2)}\right) = \frac{1}{6}(4 - 1.283 + 3(0.983)) = 0.994\\ \vec{x}_3{}^{(3)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(2)} - \vec{x}_1{}^{(2)}\right) = \frac{1}{4}(7 - 2(1.283) - 1.342) = 0.773\end{cases}$$

$$\mathbf{X}=\begin{pmatrix}0.860\\0.994\\0.773\end{pmatrix}$$

2. Gausse Seidal method

$$\begin{cases}5\mathrm{x}_1 + 2\mathrm{x}_2 - \mathrm{x}_3 = 6\\ \mathrm{x}_1 + 6\mathrm{x}_2 - 3\mathrm{x}_3 = 4\\ 2\mathrm{x}_1 + \mathrm{x}_2 + 4\mathrm{x}_3 = 7\end{cases} \Rightarrow \begin{cases}\vec{x}_1{}^{(k+1)} = \frac{1}{5}(6 - 2\vec{x}_2{}^{(k)} + \vec{x}_3{}^{(k)})\\ \vec{x}_2{}^{(k+1)} = \frac{1}{6}(4 - \vec{x}_1{}^{(k+1)} + 3\vec{x}_3{}^{(k)}) \quad \text{Tapez une équation ici.}\\ \vec{x}_3{}^{(k+1)} = \frac{1}{4}(7 - 2\vec{x}_1{}^{(k+1)} - \vec{x}_2{}^{(k+1)})\end{cases}$$

When $k = 0$, $X^{(0)}$=(0,0,0) ^{t}

$$\Leftrightarrow\begin{cases}\vec{x}_1{}^{(1)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(0)} + \vec{x}_3{}^{(0)}\right) = \frac{6}{5} = 1.200\\ \vec{x}_2{}^{(1)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(1)} + 3\vec{x}_3{}^{(0)}\right) = \frac{1}{6}(4 - 1.200 + 3(0)) = 0.467 \quad ,\\ \vec{x}_3{}^{(1)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(1)} - \vec{x}_1{}^{(1)}\right) = \frac{1}{4}(7 - 2(1.200) - 0.467) = 1.033\end{cases}$$

$$\boldsymbol{X}^{(1)}=\begin{pmatrix}0.860\\0.994\\0.773\end{pmatrix}$$

When $k = 1$,

$$\Leftrightarrow\begin{cases} \vec{x}_1{}^{(2)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(1)} + \vec{x}_3{}^{(1)}\right) = \frac{1}{5}(6 - 2(0.467) + 1.033) = 1.220 \\ \vec{x}_2{}^{(2)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(2)} + 3\vec{x}_3{}^{(1)}\right) = \frac{1}{6}(4 - 1.220 + 3(1.033)) = 0.980, \\ \vec{x}_3{}^{(1)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(2)} - \vec{x}_1{}^{(2)}\right) = \frac{1}{4}(7 - 2(1.220) - 0.980) = 0.895 \end{cases}$$

$$\boldsymbol{X}^{(\mathbf{2})} = \begin{pmatrix} 1.220 \\ 0.980 \\ 0.895 \end{pmatrix}$$

When $k = 2$,

$$\Leftrightarrow\begin{cases} \vec{x}_1{}^{(3)} = \frac{1}{5}\left(6 - 2\vec{x}_2{}^{(2)} + \vec{x}_3{}^{(2)}\right) = \frac{1}{5}(6 - 2(0.980) + 0.895) = 0.987 \\ \vec{x}_2{}^{(3)} = \frac{1}{6}\left(4 - \vec{x}_1{}^{(3)} + 3\vec{x}_3{}^{(2)}\right) = \frac{1}{6}(4 - 0.987 + 3(0.895)) = 0.950, \\ \vec{x}_3{}^{(3)} = \frac{1}{4}\left(7 - 2\vec{x}_1{}^{(3)} - \vec{x}_1{}^{(3)}\right) = \frac{1}{4}(7 - 2(0.987) - 0.950) = 1.019 \end{cases}$$

$$\boldsymbol{X}^{(\mathbf{3})} = \begin{pmatrix} 0.987 \\ 0.950 \\ 1.019 \end{pmatrix}$$

Example 2 : Solve the system

$$\begin{cases} 6\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3 = 12 \\ 2\mathrm{x}_1 + 4\mathrm{x}_2 = 0 \\ \mathrm{x}_1 + 2\mathrm{x}_2 + 6\mathrm{x}_3 = 6 \end{cases}$$

of equations using the Jacobi and Gausse Seidal methods from $X^{(0)} = (2,2,2)^t$^do only the first 3 iterations).

we have A= $\begin{pmatrix} 6 & 1 & 1 \\ 2 & 4 & 0 \\ 1 & 2 & 6 \end{pmatrix}$, It can be shown that if A is a strictly diagonally dominant matrix : $\sum_{\substack{j=1 \\ j\neq i}}^{n} \left|a_{ij}\right| \leq |a_{ii}|$, ∀i=1,n

$(|2| + |1| \leq |5|,\ \ |1| + |-3| \leq |6|,\ |2| + |1| \leq |4|)$, The condition is satisfied, and therefore the matrix A is diagonally dominant.

1.method Jacobi

$$A = D - E - F$$

A= $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, E= $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & -2 & 0 \end{pmatrix}$, F= $\begin{pmatrix} 0 & -1 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$$\begin{cases} 6\mathrm{x}^1 + 2\mathrm{x}^2 + \mathrm{x}_3 = 12 \\ 2\mathrm{x}_1 + 4\mathrm{x}_2 = 0 \\ \mathrm{x}_1 + \mathrm{x}_2 + 6\mathrm{x}_3 = 6 \end{cases} \Rightarrow \begin{cases} \vec{x}_1^{\,(k+1)} = \dfrac{1}{6}(12 - \vec{x}_2^{\,(k)} - \vec{x}_3^{\,(k)}) \\ \vec{x}_2^{\,(k+1)} = \dfrac{1}{4}(2\vec{x}_1^{\,(k)}) \\ \vec{x}_3^{\,(k+1)} = \dfrac{1}{6}(6 - \vec{x}_1^{\,(k+1)} - 2\vec{x}_2^{\,(k+1)}) \end{cases}$$

When $k = 0$, $X^{(0)}$=$(2,2,2)^{\,t}$

$$\Leftrightarrow \begin{cases} \vec{x}_1^{\,(1)} = \frac{1}{6}(12 - \vec{x}_2^{\,(0)} - \vec{x}_3^{\,(0)}) = \frac{4}{3} \\ \vec{x}_2^{\,(1)} = \frac{1}{2}\left(\vec{x}_1^{\,(0)}\right) = -1 \\ \vec{x}_3^{\,(1)} = \frac{1}{6}\left(6 - \vec{x}_1^{\,(0)} - 2\vec{x}_1^{\,(0)}\right) = 0 \end{cases}$$

$\boldsymbol{X}^{(\mathbf{1})}$= $\begin{pmatrix} \frac{4}{3} \\ -1 \\ 0 \end{pmatrix}$, $\boldsymbol{X}^{(\mathbf{2})}$= $\begin{pmatrix} \frac{13}{6} \\ -\frac{2}{3} \\ \frac{10}{9} \end{pmatrix}$, $\boldsymbol{X}^{(\mathbf{3})}$= $\begin{pmatrix} \frac{52}{27} \\ -\frac{13}{12} \\ \frac{31}{36} \end{pmatrix}$

Example 3 : Solve the system

$$\begin{cases} 2x_1 - \mathrm{x}_2 = 1 \\ -x_2 + 2\mathrm{x}_3 = 1 \\ -x_1 + 2x_2 - \mathrm{x}_3 = 0 \end{cases} , \varepsilon = 10^{-1} = 0.1 \ , X^{(0)} = (0,0,0)^t$$

we have A= $\begin{pmatrix} 2 & -1 & 0 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$, It can be shown that if A is a strictly diagonally dominant matrix $\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \leq |a_{ii}|$, ∀i=1,n.

(|-1|+|0|≤|2|, |2|+|0|≤|-1|, |2|+|-1|≤|-1|), The condition is not satisfied, and therefore the matrix A is diagonally dominant.

1.method Jacobi

$$\begin{cases} 2x_1 - \mathrm{x}_2 = 1 \\ -x_1 + 2x_2 - \mathrm{x}_3 = 0 \\ -x_2 + 2\mathrm{x}_3 = 1 \end{cases} \Rightarrow \begin{cases} \vec{x}_1^{\,(k+1)} = \frac{1}{2}(1 + \vec{x}_2^{\,(k)}) \\ \vec{x}_2^{\,(k+1)} = \frac{1}{2}(\vec{x}_1^{\,(k)} + \vec{x}_3^{\,(k)}) \\ \vec{x}_3^{\,(k+1)} = \frac{1}{2}(1 + 2\vec{x}_2^{\,(k)}) \end{cases}$$ Tapez une équation ici.

When $k = 0$, $X^{(0)}$=(0,0,0) ^{t}

$$\Leftrightarrow \begin{cases} \vec{x}_1^{\,(1)} = \frac{1}{2}(1+0) = \frac{1}{2} = 0.500 \\ \vec{x}_2^{\,(1)} = \frac{1}{2}(0+0) = 0 \\ \vec{x}_3^{\,(1)} = \frac{1}{2}(1+0) = \frac{1}{2} = 0.500 \end{cases}$$

$$\boldsymbol{X}^{(\mathbf{1})} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \quad \boldsymbol{X}^{(\mathbf{2})} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \boldsymbol{X}^{(\mathbf{3})} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix}, \boldsymbol{X}^{(\mathbf{4})} = \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \end{pmatrix}, \quad \boldsymbol{X}^{(\mathbf{5})} = \begin{pmatrix} \frac{7}{8} \\ \frac{3}{4} \\ \frac{7}{8} \end{pmatrix}, \quad \boldsymbol{X}^{(\mathbf{6})} = \begin{pmatrix} \frac{7}{8} \\ \frac{7}{8} \\ \frac{7}{8} \end{pmatrix},$$

$$\boldsymbol{X}^{(\mathbf{7})} = \begin{pmatrix} \frac{15}{16} \\ \frac{7}{8} \\ \frac{15}{16} \end{pmatrix}, \quad \boldsymbol{X}^{(\mathbf{8})} = \begin{pmatrix} \frac{15}{16} \\ \frac{15}{16} \\ \frac{15}{16} \end{pmatrix}.$$

$|\vec{x}_i{}^{(k+1)}$ -$\vec{x}_i{}^{(k)}|\leq$ ε , i= $\overline{1,n}$

$|\vec{x}_i{}^{(8)}$ -$\vec{x}_i{}^{(7)}|\leq$ ε , i= $\overline{1,n}$: $\begin{cases} |\vec{x}_1{}^{(8)} - \vec{x}_1{}^{(7)} = |\frac{15}{16} - \frac{15}{16}| = 0 < \varepsilon \\ |\vec{x}_2{}^{(8)} - \vec{x}_2{}^{(7)} = |\frac{15}{16} - \frac{7}{8}| = 0{,}05 < \varepsilon\, 15 \\ |\vec{x}_3{}^{(8)} - \vec{x}_3{}^{(7)} = |\frac{15}{16} - \frac{15}{16}| = 0 < \varepsilon \end{cases}$

Example 4 Solve the system with Gausse Seidal method

$$\begin{cases} 4x_1 + \mathrm{x}_2 - x_3 = 3 \\ 2x_2 + 7x_2 + \mathrm{x}_3 = 19 \\ x_1 - 3x_2 + 12\mathrm{x}_3 = 31 \end{cases} , \varepsilon = 10^{-1} = 0.1 \ , X^{(0)} = (0,0,0)^t$$

we have A= $\begin{pmatrix} 4 & -1 & -1 \\ 2 & 7 & 1 \\ 1 & -3 & 12 \end{pmatrix}$, It can be shown that if A is a strictly diagonally dominant matrix $\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \leq |a_{ii}|$, ∀i=1,n.

(|1|+|-1|≤|4|, |2|+|1|≤|7|, |1|+|-3|≤|12|), The condition is satisfied, and therefore the matrix A is diagonally dominant.

Gausse Seidal method

$$\begin{cases} 4x_1 + \mathrm{x}_2 - x_3 = 3 \\ 2x_2 + 7x_2 + \mathrm{x}_3 = 19 \\ x_1 - 3x_2 + 12\mathrm{x}_3 = 31 \end{cases} \Rightarrow \begin{cases} \vec{x}_1{}^{(k+1)} = \frac{1}{4}\left(3 - \vec{x}_2{}^{(k)} + \vec{x}_3{}^{(k)}\right) \\ \vec{x}_2{}^{(k+1)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}^{(k+1)} - 3\vec{x}_3{}^{(k)}\right) \\ \vec{x}_3{}^{(k+1)} = \frac{1}{12}\left(31 - \vec{x}_1{}^{(k+1)} + 3\,\vec{x}_1{}^{(k+1)}\right) \end{cases}$$

When $k = 1$, $X^{(0)}$=(0,0,0) ^{t}

$$\Leftrightarrow \begin{cases} \vec{x}_1{}^{(1)} = \frac{1}{4}\left(3 - \vec{x}_2{}^{(0)} + \vec{x}_3{}^{(0)}\right) = \frac{1}{4}(3 - 0 + 0) = \frac{3}{4} = 0.75 \\ \vec{x}_2{}^{(1)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}^{(1)} - 3\vec{x}_3{}^{(0)}\right) = \frac{1}{7}(19 - 2(0.75) - 0) = 2.5, \\ \vec{x}_3{}^{(1)} = \frac{1}{12}\left(31 - \vec{x}_1{}^{(1)} + 3\,\vec{x}_1{}^{(1)}\right) = \frac{1}{12}(31 - 0.75 + 3(2.5)) = 3.15 \end{cases} , \boldsymbol{X^{(1)}} = \begin{pmatrix} \frac{3}{4} \\ \mathbf{2.5} \\ \mathbf{3.1} \end{pmatrix}$$

$$\boldsymbol{X}^{(\mathbf{1})}=\begin{pmatrix}0.75\\ \mathbf{2.5}\\ \mathbf{3.15}\end{pmatrix}$$

When $k = 1$,

$$\Leftrightarrow\begin{cases} \vec{x}_1{}^{(2)} = \frac{1}{4}\left(3 - \vec{x}_2{}^{(1)} + \vec{x}_3{}^{(1)}\right) = \frac{1}{4}(3 - 2.5 + 3.15) = 0.91 \\ \vec{x}_2{}^{(2)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}^{(2)} - 3\vec{x}_3{}^{(1)}\right) = \frac{1}{7}(19 - 2(0.91) - 3.15) = 2.00 \\ \vec{x}_3{}^{(2)} = \frac{1}{12}\left(31 - \vec{x}_1{}^{(2)} + 3\,\vec{x}_1{}^{(2)}\right) = \frac{1}{12}(31 - 0.91 + 3(2.00)) = 3.01 \end{cases}$$

$$\boldsymbol{X}^{(\mathbf{2})}=\begin{pmatrix}0.91\\ \mathbf{2.00}\\ \mathbf{3.01}\end{pmatrix}$$

When $k = 2$,

$$\Leftrightarrow\begin{cases} \vec{x}_1{}^{(3)} = \frac{1}{4}\left(3 - \vec{x}_2{}^{(2)} + \vec{x}_3{}^{(2)}\right) = \frac{1}{4}(3 - 2.00 + (3.01)) = 1{,}00 \\ \vec{x}_2{}^{(3)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}^{(3)} - 3\vec{x}_3{}^{(2)}\right) = \frac{1}{7}(19 - 2(1.00) - 3.01) = 2.00 \\ \vec{x}_3{}^{(3)} = \frac{1}{12}\left(31 - \vec{x}_1{}^{(3)} + 3\,\vec{x}_1{}^{(3)}\right) = \frac{1}{12}(31 - 1.00 + 3(2.00)) = 3.00 \end{cases}$$

$$X^{(3)}=\begin{pmatrix}1.00\\ 2.00\\ 3.00\end{pmatrix}$$

When $\boldsymbol{k = 3}$,

$$\begin{cases} \vec{x}_1{}^{(4)} = \frac{1}{4}\left(3 - \vec{x}_2{}^{(3)} + \vec{x}_3{}^{(3)}\right) = \frac{1}{4}(3 - 2.00 + (3.01)) = 1{,}00 \\ \vec{x}_2{}^{(4)} = \frac{1}{7}\left(19 - \overrightarrow{2x_1}^{(4)} - 3\vec{x}_3{}^{(3)}\right) = \frac{1}{7}(19 - 2(1.00) - 3.00) = 2.00 \\ \vec{x}_3{}^{(4)} = \frac{1}{12}\left(31 - \vec{x}_1{}^{(4)} + 3\,\vec{x}_1{}^{(4)}\right) = \frac{1}{12}(31 - 1.00 + 3(2.00)) = 3.00 \end{cases}$$

$$X^{(4)}=\begin{pmatrix}1.00\\ 2.00\\ 3.00\end{pmatrix}$$

so the approximate solution $X^{(4)}=\begin{pmatrix}1.00\\ 2.00\\ 3.00\end{pmatrix}$

Chapter IV:
Numerical Integration

IV1. Introduction

In this chapter, we will attempt to cover some numerical methods for calculating integrals: the trapezoidal rule, Simpson's rule, the generalized trapezoidal rule, and the generalized Simpson's rule. In practice, we often resort to these methods because the function to be integrated is generally known only at a finite number of points and is difficult to integrate analytically. These integration techniques aim to approximate the value of the integral using several values of the function to be integrated.

We are therefore interested in estimating the value of an integral of the form:

$$\boldsymbol{J = \int_a^b f(x)dx}$$

$\boldsymbol{f}$ is a continuous function defined on the interval $[\boldsymbol{a}, \boldsymbol{b}]$. The approximate value of the integral corresponds to the area $\boldsymbol{A}$ under the curve of $\boldsymbol{f(x)}$.

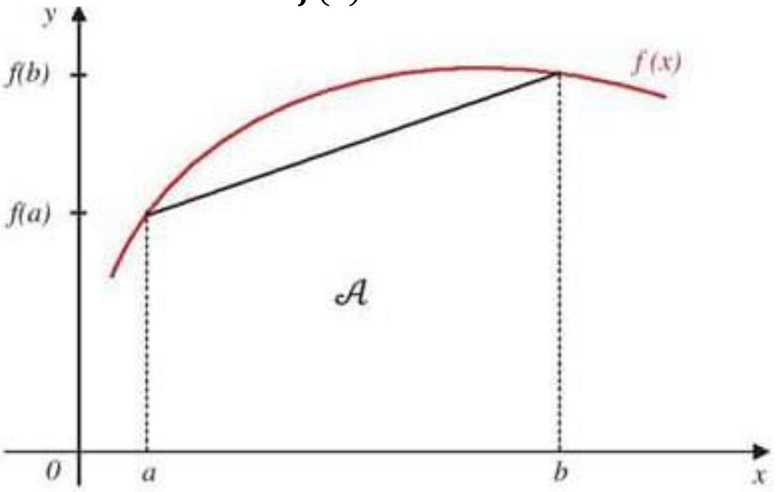
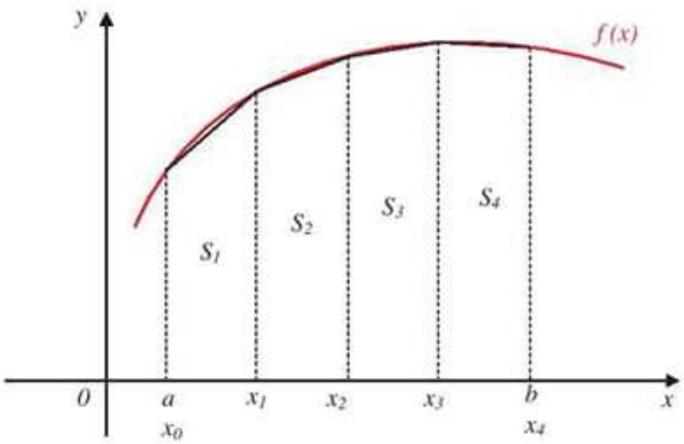


Figure.approximation of the integral by a trapezoid surface

The area corresponds to the surface of the trapezoid given by:

$$\mathrm{S} = \frac{\boldsymbol{b-a}}{\boldsymbol{2}}(\boldsymbol{f(a)} + \boldsymbol{f(b)})$$

We can divide the interval $[\boldsymbol{a}, \boldsymbol{b}]$ into n parts and consider the areas of the trapezoids.
Let's take n=4 small intervals:



Approximation of the integral by fourt trapezoid surface.

$h = x_{i+1} - x_i \; with \; i = \overline{0,3}$

We sum the areas of the 4 trapezoids:

$$S = \frac{h}{2}(f(x_i) - \; f(x_{i+1})$$

$$A = \sum_{i=1}^{4} S_i = \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$

In general, the integral $\int_a^b f(x)dx$can be written as:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

where $a_i$are the coefficients to be determined.

4.2. Trapezoidal and Simpson's Method

For $n = 1$: $f(x) = 1 \Rightarrow \int_a^b f(x)dx \; = b - a = a_0 * 1 + a_1 * 1$

$$f(x) = x \Rightarrow \int_a^b xdx \; = ((b^2)/2) - ((a^2)/2) = a_0 * a + a_1 * b$$

Solving this system of 2 equations gives: $a_0 = a_1 = ((b - a)/2)$

Thus, we obtain the formula of the trapezoidal rule:

$$\int_a^b f(x)dx \approx (b - a)[(1/2)f(a) + f(b)]$$

The trapezoidal rule can be obtained by approximating f(x) with the straight line segment joining the two points $(a, f(a))$ $and(b, f(b))$.

For $\boldsymbol{n=2}$: $\boldsymbol{f(x)=1 \Rightarrow \int_a^b 1dx = b-a = a_0 * 1 + a_1 * 1 + a_2 * 1}$

$$\boldsymbol{f(x)=x \Rightarrow \int_a^b x dx \; = \left(\frac{b^2}{2}\right) - \left(\frac{a^2}{2}\right) = a^0 * a + a^1 * \left(\left(\frac{a+b}{2}\right)\right) + a^2 * b}$$

$$\boldsymbol{f(x)=x^2 \Rightarrow \int_a^b x^2 dx \;\; = ((b^3)/3) - ((a^3)/3) = a_0 * a^2 + a_1 * (((a+b)/2))^2 + a_2 * b^2}$$

Solving this system of 3 equations gives: $\boldsymbol{a_0 = a_2 = ((b-a)/6)}$ et $\boldsymbol{a_1 = 4((b-a)/6)}$

Thus, we obtain Simpson's formula:

$$\boldsymbol{\int_a^b f(x)dx \approx (b-a)\left[\frac{1}{6}f(a) + \frac{4}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right]}$$

Simpson's formula can also be obtained by approximating f(x) with the parabola passing through the points $\boldsymbol{(a,f(a)), f(((a+b)/2)))\ et\ (b,f(b)}$

3. Generalized Trapezoidal Rule

We divide the interval $\boldsymbol{[a,b]}$ into n equal parts of length $\boldsymbol{h = ((b-a)/n)}$. The resulting subintervals are: $\boldsymbol{[x_0,x_1],[x_1,x_2],\ldots\ldots,[x_{n-1,}x_n]}$

By applying the trapezoidal rule to each subinterval, we get:

$$\boldsymbol{\int_a^b f(x)dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} f(x)dx = \sum_{i=0}^{n-1}\frac{h}{2}[f(x_i)+f(x_{i+1})]}$$

We observe that all the terms $\boldsymbol{f(x_i)}$are repeated twice, except the first and the last.

We conclude that:

$$\boldsymbol{\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + f(x_n) + 2\left(f(x_1) + f(x_3) + \cdots .. + f(x_n)\right)\right]}$$

This formula is called the generalized trapezoidal rule.

4. Generalized Simpson's Rule

We can improve the accuracy of Simpson's rule by applying it in a composite way.Since Simpson's rule requires an even number of intervals, we divide the integration interval $\boldsymbol{[a,b]}$ into $\boldsymbol{n = 2m}$ subintervals of length $\boldsymbol{h=(b-a)/n}$, and we use the simple Simpson's method in each pair of subintervals $\boldsymbol{[x_{2i}, x_{2i+2} x\_\{2i+2\}], i = 0,2,\ldots,m-1}$. Then:

$$\boldsymbol{\int_a^b f(x)dx = \sum_{i=0}^{m-1}\int_{x_{2i}}^{x_{2i+2}} f(x)dx = \sum_{i=0}^{m-1}\frac{h}{3}[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})]}$$

Note that the odd-indexed terms are multiplied by 4, while the even-indexed terms, except the first and the last, are multiplied by 2. Therefore, we conclude that:

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + f(x_n) + 4\big(f(x_1) + f(x_3) + \cdots..+f(x_{n-1})\big) + 2\big(f(x_2) + f(x_4) + \cdots..+f(x_{n-2})\big)\right]$$

This formula is called the .generalized Simpson's formula

5. Error Formulas
It is shown that:

Trapezoidal Error

$|R_T| \leq E_{max} = \frac{h^3}{12} M$

where

$$h = b - a \text{ and } M = Max\{|f''(\zeta)|\}, \qquad \zeta \in [a, b]$$

Generalized Trapezoidal Error

$|R_{TG}| \leq E_{max} = \frac{nh^3}{12} M$

where

$$h = \frac{b-a}{n}, \text{and } M = Max\{|f''(\zeta)|\}, \qquad \zeta \in [a, b]$$

Simpson's Error

$|R_S| \leq E_{max} = \frac{h^5}{90} M$

where

$$h = \frac{b-a}{2}, \text{and } M = Max\{|f^{(4)}(\zeta)|\}, \qquad \zeta \in [a, b]$$

Generalized Simpson's Error

$|R_{SG}| \leq E_{max} = \frac{nh^5}{180} M$

where

$$h = \frac{b-a}{2m},, \qquad m = \frac{n}{2}$$

$M = Max\{|f^{(4)}(\zeta)|\}, \ \zeta \in [a, b].$

Example: Consider the following integral:

$$J = \int_0^1 \ln(x+1)\,dx, \quad h = 0.25$$

1. Calculate the integral using Simpson's method.
2. What is the value of h such that the error$\leq 10^{-6}$

$$n = \frac{b-a}{h} = \frac{1-0}{4} = 4$$

$$\int_0^1 \mathbf{ln}(\boldsymbol{x}+\mathbf{1})\,\boldsymbol{dx} = \frac{\boldsymbol{h}}{\mathbf{3}}\left[\boldsymbol{f}(\mathbf{0}) + \boldsymbol{f}(\mathbf{1}) + \mathbf{2}\boldsymbol{f}(\mathbf{0.5}) + \mathbf{4}\big(\boldsymbol{f}(\mathbf{0.75}) + \boldsymbol{f}(\mathbf{0.25})\big) \approx \mathbf{0.3862}\right.$$

2. $\boldsymbol{E_{max}} = \frac{(\boldsymbol{b}-\boldsymbol{a})\boldsymbol{h}^{\mathbf{4}}}{\mathbf{180}}\boldsymbol{M}$ **,** where $\boldsymbol{M} = \boldsymbol{Max}\{|\boldsymbol{f}^{(\mathbf{4})}(\boldsymbol{\zeta})|\},\ \boldsymbol{\zeta} \in [\boldsymbol{a},\boldsymbol{b}]$.

where

$$\boldsymbol{h} = \frac{\boldsymbol{b}-\boldsymbol{a}}{\mathbf{2}\boldsymbol{m}},, \qquad \boldsymbol{m} = \frac{\boldsymbol{n}}{\mathbf{2}}$$

$f(x)$= ln(x+1)⇒ $\left|f^{(4)}(x)\right|$=$\left|\frac{-6}{(x+1)^4}\right|$=$\frac{6}{(x+1)^4}$, $max\left|f^{(4)}(x)\right| = 6$

E$\leq 10^{-6} \Rightarrow \frac{\boldsymbol{h}^{\mathbf{4}}}{\mathbf{180}}.\mathbf{6} \leq 10^{-6} \Rightarrow \boldsymbol{h}^{\mathbf{4}} \leq 30.10^{-3}$

$$h \leq 0.4161$$

Chapter V:
Differtial Equations

Introduction V1

The study of differential equations is a fundamental area in mathematics with widespread applications. Whether in electronics, mechanics, or heat transfer, solving differential equations is often essential.

While many differential equations can be solved analytically, most require numerical methods for resolution. In this chapter, we explore various numerical techniques for solving differential equations. These methods aim to be as simple as possible while maintaining progressive accuracy. For instance, we introduce basic methods like Euler's method, as well as more advanced techniques such as the fourth-order Runge-Kutta method, which offers significantly higher precision.

A first-order differential equation can be expressed as:

$$\frac{dy}{dx} = f(x,y) \quad or\; y' = f(x, y(x))$$

This equation has infinitely many solutions unless a specific initial value y_0 =y(x_0) (the initial condition) is provided. With this condition defined on the interval $[a, b]$, the problem becomes:

$$\begin{cases} y' = f(x, y(x)) \\ y_0 \;= \mathrm{y}(x_0) \end{cases}$$

This is known as a Cauchy problem, where the differential equation is paired with an initial condition y_0 =y(x_0). We can prove that a unique solution exists if the Lipschitz condition is satisfied: for some constants $L > 0$ and any x, y_1, $y_2$in $[a, b]$, the function f must satisfy:

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|$$

Additionally, f must be sufficiently regular to ensure that the solution y is differentiable and satisfies the boundary condition.

V . 2. Euler's Method

Euler's method is the simplest and most straightforward numerical approach for solving differential equations, despite its limited precision due to low accuracy.

Let $y' = f(x, y)$ with the initial condition y_n=y(x_n). Euler's method allows us to compute an approximate value y_{n+1}=y(x_{n+1}), where x_{n+1}= x_n + h and h is the step size for integration. The method is derived from the Taylor expansion of $y(x)$ around x_n:

$$y(x) = y(x_n) + (x - x_n)y'(x_n) + \frac{(x - x_n)^2}{2!}\; y''(x_n) + \frac{(x - x_n)^3}{3!}\; y^{\prime\prime\prime(x_n)} + \cdots$$

Substituting $x = x_{n+1}$ into the equation above, we obtain:

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}\; y''(x_n) + \frac{h^3}{3!}\; y'''(x_n) + \cdots \ldots ..$$

Considering only the first two terms of the Taylor expansion (for simplicity), we derive:

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + h^2E(h)$$

Here, $h^2E(h)$represents the truncation error introduced by neglecting higher-order terms. The previous equation simplifies, ignoring the error term, to Euler's formula:

$$y(x_{n+1}) \approx y(x_n) + hy'(x_n)$$

or equivalently:

$$x_{n+1} \approx y_n + hy'(x_n, y_n)$$

Example : Consider the differential equation:

$$\begin{cases} y' = y \\ \mathrm{y}(0) = 1 \end{cases}$$

Compute y_1 with $h = 1$, $h = 0.2$,and $h = 0.1$.

Solution:

The interval of interest is [0, 1].
Since $f(x, y) = y$, we can apply Euler's method as follows:

$$y(x_{n+1}) \approx y(x_n) + hf(x_n, y(x_n)$$

a. $\boldsymbol{h = 1}$ **:**

$$y(1) = y(0) + h\,f(0,1) = 1 + 1\,(1) = 2$$

Using Euler's method with $h = 1$, the approximation for $y(1)$ is 2.
The exact solution is y_{exact}(x) = e^x, so y_{exact}(1) = e^1 ≈ 2.718 . The absolute error is:

$$|y(1) - y_{exact}(1)|=|2 - 2.718|=0.718$$

b. $\boldsymbol{h = 2}$ **:**

With $h = 0.2$, the step size corresponds to 5 steps $\left(\frac{1-0}{0.2}\right) = 5$ over [0, 1]. Compute successive approximations for $y(0.2), y(0.4), y(0.6), y(0.8),$ and $y(1.0)$:

$y(0.2) = y(0) + h\,f(0,1) = 1 + 0.2 * 1 = 1.2$

y(0.4) = y(0.2) + h f(0.2, 1.2) = 1.2 + 0.2 ∗1.2 = 1.44

y(0.6) = y(0.4) + h f(0.4, 1.44) = 1.44 + 0.2 ∗ 1.44 = 1.728

y(0.8) = y(0.6) + h f(0.6, 1.728) = 1.728 + 0.2 ∗1.728 = 2.0736 y(1.0) = y(0.8) + h f(0.8, 2.0736) = 2.0736 + 0.2 ∗2.0736 = 2.48832

Thus, y(1) ≈2.488 . The exact value is $y_{exact}(1)$ =e^1 ≈ 2.718 , so the error is:

$$|y(1) - y_{exact}(1)| = |2.488 - 2.718| = 0.23$$

c. $\boldsymbol{h = 3}$ **:**

With $h = 0.1$, there are 10 steps$\left(\frac{1-0}{0.1}\right) = 10$. Compute successive approximations for \($y(0.1), y(0.2)$, ,……., $y(1.0)$:

$y(0.1) = y(0) + hf(0,1) = 1 + 0.1 * 1 = 1.1$

$y(0.2) = y(0.1) + hf(0.1, 1.1) = 1.1 + 0.1 * 1.1 = 1.21$

$y(0.3) = y(0.2) + hf(0.2, 1.21) = 1.21 + 0.1 * 1.21 = 1.331$
.
.
.

$y(1.0) = y(0.9) + hf(0.9, y(0.9)) = 2.593$

The exact value is $y_{exact}(1)$ = e^1 ≈ 2.718 , so the error is:

$$|y(1) - y_{exact}(1)\ | = |2.593 - 2.718| = 0.125$$

Remark: The exact solution is y_{exact} (x) = e^x, so $y_{exact}(1)$= e^1 ≈ 2.718. As the step size h decreases, the approximation improves, reducing the error.

. Single-Step Methods

V.3.1 General Form

Single-step methods, also known as step-by-step methods, take the general form:

$$y_{n+1} = y_n + h\,\emptyset(x_n y_n, h)$$

Here, $\emptyset$ is a function specific to the method. To obtain a numerical solution for $x_n = x_{n+1}$, single-step methods only require the solution at the previous step x_n . While many single-step methods exist, Euler's method is a well-known example where:

$$\emptyset(x, y) = f(x, y)$$

V.3.2.Runge-Kutta Methods

To improve the accuracy of Euler's method, higher-order methods like Runge-Kutta are preferred. While the first-order Euler method provides a basic approximation, Runge-Kutta methods offer multiple levels of precision. Their general form is:

$$y_{n+1} = y_n + R_1 k_1 + R_2 k_2 + R_3 k_3 + \cdots$$

where:

$$k_1 = h\, f(x_n, y_n)$$

$$k_2 = h\, f(x_n + ah, y_n + ak_1)$$

$$k_3 = h\, f(x_n + bh, y_n + aB)$$

$$\vdots$$

These methods are easy to implement and can be programmed efficiently.

. V.3.2.1.Second-Order Runge-Kutta Formulas

The second-order Runge-Kutta method takes the form:

$$y_{n+1} = y_n + R_1 k_1 + R_2\, k_2$$

where:

$k_1 = h\, f(x_n, y_n)$

$k_2 = h\, f(x_n + a\, h, y_n + \alpha k_1)$

The calculation of the values $a, \alpha, R_1, R_2, k_1,$ requires determining $y', y'', and\ y''' starting$ from $y' = f(x,y)$, followed by a Taylor expansion. This results in a system of equations involving fewer unknowns than equations, so there is no unique solution. This provides several variants of the second-order Runge-Kutta method. The two most commonly used are:

- **Improved Tangent Formula (Midpoint Method:**

$$y_n = y_n + hf\left(x_n + \frac{h}{2}, y_{n+\frac{1}{2}}\right) with\ y_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(x_n, y_n)$$

This formula demonstrates why the midpoint method is often called the improved tangent method: the function $f(x, y)$ is evaluated at the midpoint of the interval $[x_n, x_{n+1}]$.

- Modified Euler-Cauchy Formula:

$$y(x_{n+1}) \approx y(x_n) + \frac{h}{2}\left[f\left(x_n, y(x_n)\right) + f(x_{n+1}, y_{n1})\right] \text{ with } y_{n+1} = y(x_n) + h\, f(x_n, y(x_n))$$

To simplify calculations, y_{n+1} is computed in two steps. First, a provisional value \(y_{n+1} \) is calculated using Euler’s method. Then, this value is refined in the second step to improve accuracy, resembling a predictor-corrector approach.

To simplify the calculations, the evaluation of y_{n+1} is divided into two steps. The variable y_{n1} corresponds directly to one iteration of Euler’s method, providing a predicted value y_{n1} of the solution at x_{n+1} , which is then corrected and improved in the second step. This approach is thus a predictor-corrector method.

V.3.2.2. Fourth-Order Runge-Kutta Formulas

The fourth-order Runge-Kutta method is the most widely used, requiring four evaluations of the function f. While it demands more computational effort than second-order methods, it provides greater precision for complex functions. The fourth-order Runge-Kutta formula is:

$$y(x_{n+1}) = y(x_n) + 1\frac{1}{6}(k_1 + k_2 + 2k_3 + k_4 +)$$

where:

$$k_1 = hf(x_n, y(x_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2},\ y(x_n) + \frac{k_2}{2}\right)$$
$$k_3 = hf\left(x_n + \frac{h}{2},\ y(x_n) + \frac{k_2}{2}\right)$$
$$k_4 = h(x_n + h,\ y(x_n) + k_3)$$

Chapter VI:

Interpolation

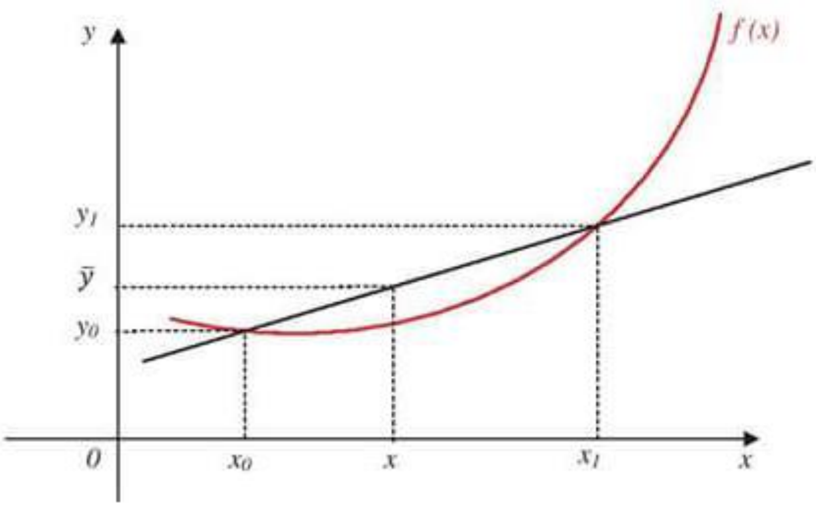
V.1. Introduction

Given a function $f(x)$ known at $(n+1)$ points of the form $(x_i, f(x_i))$ for $i = 0, 1, 2, \dots, n$, is it possible to construct an approximation of $f(x)$? Furthermore, for any x, can we derive an approximation using the interpolation points $(x_i(fx_i))$ \)? These points may be obtained from experimental data, a table, or other sources. Even when the function $f(x)$ is not explicitly defined, interpolation enables us to estimate $f(x)$ for values of x distinct from the x_i.

The interpolation problem is relatively straightforward. The solution involves constructing a polynomial of degree at most n that passes through the $n+1$ interpolation points. This polynomial is referred to as the interpolation polynomial.

VI.2. Lagrange Polynomial

Determine $y(x)$ for $x = 1.5$, given the values $y(1) = 2.7128\, and\, y(2) = 7.3890$.



Interpolation of a function by a straight line](image_placeholder)

The properties of the straight line within the interval $[x_0, x_1]$ allow us to establish the following relationship:

$$\frac{y_1 - \bar{y}}{x_1 - x} = \frac{y_1 - y_0}{x_1 - x_0}$$

Solving for $\bar{y}$, we obtain:

$$y = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

This expression can then be used to compute $\bar{y}$ (1.5).

This formula is known as the Lagrange polynomial. In this case, it represents the equation of a straight line, which is a polynomial of degree 1 derived from two measurement points.

VI.2.1 General Formula

It can be demonstrated that the Lagrange polynomial $P_n(x)$ of degree at most n for $n+1$ measurement points is expressed as:

$$\bar{y}=P_n(x)=\frac{(x-x_1)(x-x_2)\ldots\ldots.(x_0-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_1)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_1+$$

$$+\cdots\ldots+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_1)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}y_n$$

By defining the Lagrange basis polynomial:

$$L_K(x)=(x-x_0)(x-x_1)\ldots\ldots\ldots(x-x_{K-1})(x-x_{K+1})\ldots\ldots(x-x_n)$$

the simplified form for $(n+1)$ points becomes:

$$P_n(x)=\sum_{K=0}^{n}\frac{L_K(x)}{L_K(x_K)}y_K$$

VI.2.2 Theorem

It can be proven that the Lagrange interpolation polynomial is unique, satisfying the property:

$$\sum_{K=0}^{n}\frac{L_K(x)}{L_K(x_K)}=1$$

Example: Compute the Lagrange polynomial for the following measurement points:

x_i	x_0=0	x_1=1	x_2=2	x_3=3
y_i	y_0=-4	y_1=-2	y_3=3	y_3=14

With 4 points, the degree of the polynomial is at most 3.

$$P_3(x)=\sum_{K=0}^{3}\frac{L_K(x)}{L_K(x_K)}y_K+\frac{L_0(x)}{L_0(x_0)}y_0+\frac{L_1(x)}{L_1(x_1)}y_1+\frac{L_2(x)}{L_2(x_2)}y_2+\frac{L_3(x)}{L_3(x_3)}y_3$$

For k = 0

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_1)(x_0-x_3)} = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}(x-1)(x-2)(x-3)$$

For k = 1

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}x(x-2)(x-3)$$

For k = 2

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = -\frac{1}{2}x(x-1)(x-3)$$

For k = 3

$$\frac{L_3(x)}{L_3(x_3)} = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-2)(x-3)}{(2-0)(2-1)(2-3)} = \frac{1}{6}x(x-1)(x-2)(2)$$

Thus:

$$P_n(x) = -\frac{1}{6}(x-1)(x-2)(x-3) + \frac{1}{2}x(x-2)(x-3) - \frac{1}{2}x(x-1)(x-3)(2) + \frac{1}{6}x(x-1)(x-2)$$

2.3. Error Estimation

It can be shown that the error incurred when approximating the original function $f(x)$ with the Lagrange interpolation polynomial P_n(x) over $n+1$ points is bounded by:

$$|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}\prod_{i=0}^{n}|x-x_i|$$

where : $M = Max\left[\left(f^{(n+1)}(\xi)\right)\right],\ \xi \in [x_0,\ x_n]$

VI.3 Newton Method

VI.3 1. General Form

When considering the general representation of a polynomial, the Newton interpolation form is a natural choice, distinct from the standard power basis:

$$P_n(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots \ldots \ldots . + C_n x^n$$

However, for interpolation purposes, a more suitable form is:

$$P_n(x) = C_0 + C_1(x - x_0) + C_2(x - x_0)(x - x_1) + \cdots \ldots . + C_n(x - x_0)(x - x_{n-1})$$

In this form, the coefficient C_n is associated with a product of terms $(x - x_i)$, ensuring that the polynomial is of degree at most n . The key advantage of this form lies in its structure, which facilitates the determination of the $(n + 1)$ coefficients C_i such that the polynomial $P_n(x)$ passes through the $(n + 1)$ interpolation points $(x_i, f(x_i))$. Specifically, we require:

$$P_n(x) = f(x_i) = y_i$$

This condition ensures that the polynomial exactly matches the given data points.

3.2. Relationships Necessary for Calculating the Coefficients C_i

To calculate the coefficients C_i, we use the following relationships:

$$\Delta y_i = y_{i+1} - y_i$$

$$\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(y_{i+1}) - \Delta y_i = y_{i+2} - y_{i+1} + y_i = y_{i+2} - 2y_{i+1} + y_i$$

$$\Delta^3 y_i = \Delta(\Delta^2 y_i) = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

More generally, for the k -th differences:

$$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^k y_{i+1} \qquad (1)$$

$$\Delta^k y_0 = y_k - \binom{k}{1} y_{k-1} + \binom{k}{2} y_{k-2}\Delta + \cdots \ldots . + (-1)^k y_0 \qquad (2)$$

$$\binom{i}{1} = C_k^i = \frac{k!}{i!\,(k-i)!}$$

Using the relationship (1), we can construct the following table:

i	x_i	y_i	Δy_i $\Delta y_{i+1} - \Delta y_i$	$\Delta^2 y_i$ $\Delta y_{i+1} - \Delta y_i$	$\Delta^3 y_i$ $\Delta^2 y_{i+1} - \Delta^2 y_i$	$\Delta^4 y_i$ $\Delta^3 y_i - \Delta^3 y_i$
0	x_0	y_0				
			Δy_0			
1	x_1	y_1		$\Delta^2 y_0$		
			Δy_1		$\Delta^3 y_i$	
2	x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_i$
			Δy_2		$\Delta^3 y_i$	
3	x_3	y_3		$\Delta^2 y_2$		
			Δy_3			
4	x_4	y_4				
.	.					
.	.					
.	.					

VI.3 2. Calculation of the Coefficients C_i

We take a polynomial of degree 3 in general. Suppose the points \\$x_i$) are equidistant ($x_{i+1} - x_i \ = \ h$, i = $\forall\ i = \overline{0, n-1}$.

$$P_3(x) = C_0 + C_1(x - x_0) + C_2(x - x_0)(x - x_1) + C_3(x - x_0)(x - x_1)(x - x_2)$$

$$\begin{cases} P_3(x_0) = C_0 \\ P_3(x_1) = C_0 + hC_1 \\ P_3(x_2) = C_0 + 2hC_1 + 2h^2C_2 \\ P_3(x_3) = C_0 + hC_1 + 6h^2C_2 + 6h^3C_3 \end{cases}$$

Solving the system gives (using the formula (2)):

$$C_0 = y_0$$
$$C_1 = \Delta^2 y_0 / 2h^2$$
$$C_2 = \Delta^3 y_0 / 6h^3$$

$$P_3(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{6h^3}(x - x_0)(x - x_1)(x - x_2)$$

For the general case of a Newton interpolation polynomial of degree n , we obtain the following expression:

$$P_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + .. + \frac{\Delta^n y_0}{n!\,h^n}(x - x_0)(x - x_1) \dots (x - x_2)$$

Remarks:
- Unlike the Lagrange method, the Newton method requires recalculating the coefficients with each change in the data.
- If the points x_i are not equidistant, h and the successive differences can be replaced by the respective differences between the various x_i values.

Example: Find the Newton polynomial that passes through the following equidistant points:

x_i	x_0=0	x_1=1	x_2=2	x_3=3
y_i	y_0=1	y_1=4	y_3=8	y_3=14

On 4 points where the degree of the polynomial is $n = 3$, the Newton polynomial passing through these 4 points is:

$$P_3(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{6h^3}(x - x_0)(x - x_1)(x - x_2)$$

where $h = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = 1$.

x_0=0 → y_0=1

$\Delta y_0 = y_1 - y_1 = 3$

$\Delta^2 y_0 = \Delta y_0 - \Delta y_1 = 1$

x_1=1→ y_1=4

$\Delta^2 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = 1$

$\Delta y_1 = y_1 - y_1 = 4$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = 2$

x_2=2 → y_2=8

$\Delta y_2 = y_1 - y_1 = 6$

x_3=3 → y_2=8

The polynomial is:

$$P_n(x) = 1 + \frac{3}{1}(x - 0) + \frac{1}{2}(x - 0)(x - 1) + \frac{1}{6}(x - 0)(x - 1)(x - 2)$$

$$= \frac{1}{6}x^3 + \frac{17}{6}x + 1$$

SERIES OF EXERCISES
Numerical Methods

Tutorial N.o1

Exercise 1: Consider the following equation: $x^3 - x - 1 = 0$

1. Show that this equation has a solution in the interval $[1,2]$.

2. Is this solution unique?

3. Compute an approximation of this solution using the bisection method with a precision of 10^{-2}.

Exercise 2: Use the bisection method to approximate the solution of the equation:

$1 - xe^{x}=0$ in the interval [0,1] with a precision $\varepsilon = 10^{-3}$.

Exercise 3: Consider the following equation: $x - 0.8 - 0.2\,sin\,x = 0$

Using the Newton-Raphson method, solve the equation in the interval with a precision

$\varepsilon = 10^{-5}$and initial value and x_0

Exercise 4: We want to evaluate $\sqrt{a}$ using the Newton-Raphson method.

1. Write the recurrence equation.

2. Now, consider $a = 7$ and the interval [1,4].

   a. Verify that the Newton-Raphson method converges to a unique solution.

   b. Compute the first four iterations for both cases: $x_0 = 1$ and $x_0 = 3$.

Exercise 5: Consider the following equation: $cos\,x - x = 0$

1. Show that this equation has a solution in the interval $[0,1]$.

2. Find the function $g(x)$ that ensures the convergence of the fixed-point method.

3. Compute an approximate solution with a precision $\varepsilon = 10^{-2}$ and $x_0 = 0.5$.

Exercise 6: We want to solve the equation: $x^3 - x - 1 = 0$ using the fixed-point method in the interval $[1,2]$.

1. Show that the function $x = g(x) = \sqrt[3]{x+1}$ satisfies the conditions for convergence.

2. Compute the approximate solution with a precision $\varepsilon = 10^{-2}$ and $x_0 = 0.5$

Compare the number of iterations obtained with those in Exercise 1. What can be concluded

Solution of Tutorial N.o1

Exercice 1 :

Consider the equation : $f(x) = x^3 - x - 1 = 0 , x \in [1,2]$

1. ***Prove that the equation has a solution in the interval [1, 2] :***

Let $f(\boldsymbol{x}) = \boldsymbol{x^3 - x - 1}$. Since f is a polynomial, it is continuous on [1 , 2] ,Evaluate at the endpoints:

$f(1) = -1, f(2) = 5$; $f(1). f(2) < 0$, by the Intermediate Value Theorem, there exists $c \in [\ 1\ , 2\ ]$ such that $f(c) = 0$.

2. ***L'unicité de la solution :***

$f'(x) = 3x^2 - 1 > 0 \ \forall\ x \ \in [1,2] \Rightarrow f \nearrow$, so f is strictly increasing.

Since f is monotone, the solution$c$ c is unique.

3. ***La bissection :*** *We apply the bisection method with $\varepsilon = 10^{-2}$.. We use 3 decimal places, and the number of iterations* $\boldsymbol{n}$*can be calculated in advance using the formula:* $\boldsymbol{n} \geq \frac{\frac{\boldsymbol{b-a}}{\boldsymbol{\varepsilon}}}{\boldsymbol{ln2}} = \frac{\boldsymbol{ln10^2}}{\boldsymbol{ln2}} \Rightarrow \boldsymbol{n = 7\ iterarion}$

a	b		$b - c$	$f(c)$
1.000	2.000	1.500	0.500	+0.875
1.000	1.500	1.250	0.250	−0.297
1.250	1.500	1.375	0.125	+0.225
1.250	1.375	1.312	0.063	−0.054
1.312	1.375	1.343	0.032	+0.079
1.312	1.343	1.327	0.016	+0.010
1.312	1.327	1.319	0.008	−0.024

$b - c = 1.327 - 1.319 = 0.008 \leq \varepsilon = 0.010$ donc on arrête les calculs et la solution est $c \simeq 1.319$ et on peut écrire $c = 1.319 \pm 0.010$.

Exercice 2 :

Using the bisection method, we compute an approximate solution to the equation:

$f(x) = 1 - xe^{x} = 0,\ x \in [0,1],\ \varepsilon = 10^{-3}$

f is is continuous on [0,1].

$f(0) = 1, f(1) = -1.718;\ \ f(0).f(1) < 0$

Moreover, , $f'(x) = -e^{x}(1+x) < 0\ \ \forall\ x\ \in [1,2]$, so f is strictly decreasing.

Since f is monotone, the solution c is unique.

The number of iterations n is:

On prendra 4 chiffres après la virgule et le nombre d'itérations n est :

$$n \geq \frac{ln\frac{b-a}{\varepsilon}}{ln2} = \frac{ln10^{3}}{ln2} = 9.97 \Rightarrow n = 10 \text{ iterations}$$

n	a	b	$C = \frac{a+b}{2}$	$b - c$	$f(x)$
1	0.0000	1.0000	0.5000	0.5000	+0.1756
2	0.5000	1.0000	0.7500	0.2500	−0.5877
3	0.5000	0.7500	0.6250	0.1250	−0.1676
4	0.5000	0.6250	0.5625	0.0625	+0.0128
5	0.5625	0.6250	0.5937	0.0313	−0.0750
6	0.5625	0.5937	0.5781	0.0156	−0.0305
7	0.5625	0.5781	0.5703	0.0078	−0.0087
8	0.5625	0.5703	0.5664	0.0039	+0.0020
9	0.5664	0.5703	0.5683	0.0020	−0.0032
10	0.5664	0.5683	0.5673	0.0010	+0.0004

$b - c = 0.5683 - 0.5673 = 0.0010 \leq \varepsilon = 0.0010$ so we stop the calculations.

The solution is $c \simeq 0.5673$ and we can write $c = 0.5673 \pm 0.0010$.

Exercice 3 :

$f(x) = x - 0.8 - 0.2\sin x = 0$, x= 10^{-5} $x \in [\pi/4, \pi/2]$ –

ε=10 −5 , $x_0 = \pi/4$

Convergence study:

The function f is defined on the interval $[\pi/4, \pi/2]$such that:

a) $f(\pi/4) = -0.16$,$f(\pi/2) = 0.57$

b) $f'(x) = 1 - 0.2cos.$
If $f'(x) = 0$, then $1 - 0.2cos\,x = 0 \implies cos\,x = 5$, which is impossible since $cos\,x \in [-1,1]$.
Thus, $f'(x) > 0\ \forall x \in [\pi/4, \pi/2]$.

c) $f''(x) = 0.2sin\,x > 0\ \forall x \in [\pi/4, \pi/2]$

The function f satisfies the three conditions, so the Newton-Raphson method converges to a unique solution.

Recursive formula:

$$\boldsymbol{x_k = x_{k-1} - \frac{f(x_{k-1})}{x_k} = x_{k-1} - \frac{x_{k-1} - 0.8 - 0.2sin(x_{k-1})}{1 - 0.2cos(x_{k-1})}}$$

k	x_k	$\lvert x_k - x_{k-1} \rvert$
0	$\frac{\pi}{4}$	—
1	0.967120	0.181722
2	0.964335	0.002785
3	0.964334	0.000001

$|x_k - x_{k-1}|$ ≤|=|0.964334−0.964335|=0.000001≤ ε=10 −5 .

The approximate solution is $c \approx 0.964334$.

Exercise 4:

1. Recursive equation: To find $\sqrt{a}$ solve $x^2 - a = 0$.

Let f(x)= $x^2 - a$.

The Newton-Raphson iteration is:

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{x_k} \Rightarrow x_k = x_{k-1} - \frac{{x_{k-1}}^2 - a}{2x_{k-1}} \Rightarrow x_k = \frac{{x_{k-1}}^2 + a}{2x_{k-1}}$$

2. Case: $\boldsymbol{a = 7}$, interval $[\mathbf{1}, \mathbf{4}]$

a.Convergence of the Newton-Raphson method:

For a=7, we have $f(x) = x2 - 7 = 0$.

The function f is defined on [1 ,4] such that:

- $f(1) = -6$, $f(4) = 9$, $f(1) \cdot f(4) < 0$
- f′(x)=2x>0 ∀ $x \in [1, 4]$.
- $f''(x) = 2 > 0$.

The convergence conditions are satisfied, so the Newton-Raphson method converges to a unique solution] $c \in [1,4]$

4. First four iterations: The recursive equation is:

$$x_k = \frac{{x_{k-1}}^2 + a}{2x_{k-1}}$$

For $x_0 = 1$

k	x_k
0	1.0000
1	4.0000
2	2.8750
3	2.6549
4	2.6458

Forr $x_0 = 3$

k	x_k
0	3.0000
1	2.6667
2	2.6458
3	2.6457
4	2.6457

Exercise 5:

$f(x) = cosx - x = 0$

1. Existence of a solution in the interval [0,1]:

f is continuous on [0,1].
$f(0) = 1\ , f(1) = -0.46\ ;\ f(0) \cdot f(1) < 0$.
Moreover, $f'(x) = -sin\,x - 1 < 0\ \forall x \in [0{,}1]$, so f is strictly decreasing.

Since f f is monotone, the solution c is unique.

2. Function g(x) ensuring convergence of the fixed-point method:

$f(x) = cos\,x - x = 0 \Longrightarrow cos\,x = x$

Let $g(x) = cos\,x$

We verify the conditions for convergence of the fixed-point method:

a) $\boldsymbol{Is\ g}([\mathbf{0},\mathbf{1}]) \subseteq [\mathbf{0},\mathbf{1}]\ \boldsymbol{g}([\mathbf{0},\mathbf{1}])\ \backslash \boldsymbol{subseteq}\ [\mathbf{0},\mathbf{1}]\ \boldsymbol{g}([\mathbf{0},\mathbf{1}]) \subseteq [\mathbf{0},\mathbf{1}]$?

The function $g(x)$ is continuous on [0,1].
For $x \in [0, \pi/2]$, $0 \leq cosx \leq 1$.
Thus, g: [0,1] → [0,1]], and since $[0{,}1] \subseteq \left[0, \frac{\pi}{2}\right]$,g([0,1])⊆[0,1]

b) $Is\ \mid g'(x) \mid \leq k < 1\ \forall x \in [0{,}1] \mid$

$g'(x) = -sin\,x \Longrightarrow \mid g'(x) \mid = sin$

Since $(sin\,x)' = cos\,x > 0\ \forall x \in [0{,}1]$, $sinx$ is increasing, and its maximum is at $x = 1$.
$k = sin(1) \approx 0.84 \Longrightarrow \mid g'(x) \mid \leq k = 0.84 < 1$ .Both conditions are satisfied, so the fixed-point method converges. The recursive formula is:

$$x_k = g(x_{k-1}) = cos(x_{k-1})$$

3. Approximate solution: With 10^{-2}, x_0 =0.5

n	x_k	$\lvert x_k - x_{k-1} \rvert$
0	0.500	—
1	0.878	0.378
2	0.639	0.239
3	0.803	0.164

4	x_k	$\lvert x_k - x_{k-1} \rvert$
5	00.769	0.075
6	0.719	0.050
7	0.752	0.033
8	0.730	0.022
9	0.745	0.015
10	0.735	0.010

The approximate solution is $c \approx 0.735$.

Exercise 6:

$f(x) = x^3 - x - 1 = 0,\ x \in [1,2]$

1. Verification of convergence conditions:

g(x)=$\sqrt[3]{(x+1)}$

We verify two conditions:

a) Is g([1,2])⊆[1,2] ?

The function $g(x)$ is continuous on [1,2].
g′(x)= $\frac{1}{3\sqrt[3]{(x+1)}} > 0\ \forall x \in [1,2], so\ g$ is increasing.
$g(1) = 23 \approx 1.$, $g(2) = 33 \approx 1.44$

Thus, $\forall x \in [1,2],\ 1.26 \leq g(x) \leq 1.44 \implies g([1,2]) \subseteq [1,2]$

b) Is $\mid g'(x) \mid \leq k < 1\ \forall x \in [1,2]$?

$$g'(x) = \frac{1}{3\sqrt[3]{(x+1)}} \Rightarrow |g'(x)| = \frac{1}{3\sqrt[3]{(x+1)^2}}$$

$\forall x_1 \in [1,2]$, $\forall x_2 \in [1,2]$: $x_1 < x_2 \Rightarrow \frac{1}{3\sqrt[3]{(x_1+1)^2}} > \frac{1}{3\sqrt[3]{(x_2+1)^2}}$

$$\Rightarrow |g'(x_1)| > |g'(x_2)|$$

Thus, $\mid g'(x) \mid$ is decreasing, and its maximum is at $x = 1$

$k = \mid g'(1) \mid = 0.21 < 1$

Both conditions are satisfied, so the fixed-point method converges. The recursive formula is:

$$x_k = g(x_{k-1}) = 3\sqrt[3]{x_{k-1}+1}$$

3. Approximate solution:

n	x_k	$\lvert x_\mathrm{k} - x_\mathrm{k-1}\rvert$
0	1.500	—
1	1.357	0.143
2	1.331	0.019
3	1.326	0.005

The approximate solution is $c \approx 1.326$.

4. Comparison and conclusion:

The number of iterations for the fixed-point method $(n = 3)$ is less than that for the bisection method $(n = 7)$.
We conclude that the fixed-point method converges faster than the bisection method.

Tutorial N.o2

Exercise 1: Solve the following system of equations using the Cramer's method.

$$\begin{cases} 2x_1 + x_2 - 4x_3 = 1 \\ 3x_1 + 3x_2 - 5x_3 = 1 \\ 4x_1 + 5x_2 - 2x_3 = 8 \end{cases}$$

Exercise 2:

Solve the following system of equations using the Gauss method.

$$\begin{cases} 4x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - x_3 = -5 \\ x_1 + x_2 - 3x_3 = -9 \end{cases}$$

Exercise 3: Consider the following system of equations:

$$\begin{cases} x_1 + 3x_2 + 3x_3 = 0 \\ x_1 + x_2 = 1 \\ 3x_1 + 2x_2 + 6x_3 = 11 \end{cases}$$

1. Solve the system using the Gauss method.
2. Solve the system using the Gauss method with pivoting.

Exercise 4:

Consider the following two matrices:

$$A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad , B=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{bmatrix}$$

1. Compute the determinant of A and B using the Gauss method.
2. Deduce the determinant of A^{-1}, B^{-1} and$(AB)^{-1}$.

Solution of Tutorial N.o2

Exercise 1: Solving the System of Equations Using the Gaussian Elimination Method

The system can be written in the following matrix form:

$A.\vec{X}$=$\vec{b}$ avec $A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 4 & -1 \\ 1 & 1 & -3 \end{bmatrix}$, $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\overrightarrow{b}$=$\begin{bmatrix} 9 \\ -5 \\ -9 \end{bmatrix}$,

The augmented matrix $\tilde{A}$ is defined as:

$$\tilde{A} = \left[\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right] \begin{array}{l} E_1 \\ E_2, \\ E_3 \end{array}$$

Step 1: Eliminate the Elements Below the Diagonal in the First Column Using Row E_1

$E_2 \rightarrow E_2 - \frac{1}{2} E_1$

$E_3 \rightarrow E_3 - \frac{1}{4} E_1$

$$\tilde{A}^{(1)} = \left[\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 0 & 7/2 & -1 & -19/2 \\ 0 & 3/4 & -3 & -45/4 \end{array}\right] \begin{array}{l} E_1 \\ E_2, \\ E_3 \end{array}$$

Step 2: Eliminate the Elements Below the Diagonal in the Second Column Using Row E_2

$E_3 \rightarrow E_3 - \frac{3}{14} E_2$

$$\tilde{A}^{(2)} = \left[\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 0 & 7/2 & -2 & -19/2 \\ 0 & 0 & -43/14 & -129/14 \end{array}\right] \begin{array}{l} E_1 \\ E_2, \\ E_3 \end{array}$$

This results in the following upper triangular matrix:

$-\frac{43}{14} x_3 = -\frac{129}{14} \Rightarrow x_3 = 3$

$\frac{7}{2} x_2 - 2x_3 = \frac{19}{2} \Rightarrow x_3 = -1$

$4x_2 + x_2 + 2x_3 = 9 \Rightarrow x_3 = 1$

Therefore, the solution is:

$$\vec{X} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix},$$

Exercise 2: Solving the system of equations using the Gaussian elimination method

The system can be written in the following matrix form:

$$A.\vec{X}=\vec{b} \text{ avec } A=\begin{bmatrix}1&3&3\\1&1&0\\3&2&6\end{bmatrix},\quad \vec{X}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\ \text{ and } \overrightarrow{b}=\begin{bmatrix}0\\1\\11\end{bmatrix},$$

$K=1 \rightarrow$ Pivot$^{(1)}$= max(1,1,3)=3, so we swap rows 1 and 3 to obtain:

$$\begin{bmatrix}3&2&6\vdots&11\\1&1&0\vdots&1\\1&3&3\vdots&0\end{bmatrix}\rightarrow\begin{bmatrix}3&2&6&\vdots\, 11\\0&1/3&-2&\vdots\, -8/3\\0&7/3&1&\vdots\, -11/3\end{bmatrix}$$

$K=2 \rightarrow$ Pivot$^{(2)}$= max$(1/3,\ 7/3)$= $7/3$, so we swap rows 2 and 3 to obtain:

$$\begin{bmatrix}3&2&6&\vdots&11\\0&7/3&0&\vdots&-11/3\\0&1/3&-2&\vdots&-8/3\end{bmatrix}\rightarrow\begin{bmatrix}3&2&6&\vdots\, 11\\0&7/3&1&\vdots\, -11/3\\0&0&-15/7&\vdots\, -15/7\end{bmatrix}$$

This results in the following upper triangular matrix:

$$A.\vec{X}=\vec{b}\Leftrightarrow\begin{cases}x_3=1\\x_2=\frac{3}{7}\left(-\frac{11}{3}-x_3\right)\\x_3=\frac{1}{3}(11-2x_2-6x_3)\end{cases}\Leftrightarrow\begin{cases}x_3=1\\x_2=-2\\x_2=3\end{cases}$$

Exercise 3:

1. Calculation of the determinant of matrices A and B using the Gauss method:

$$\tilde{A}=\begin{bmatrix}2&1&1\\1&2&1\\1&1&3\end{bmatrix}\begin{array}{l}E_1\\E_2,\\E_3\end{array}$$

Step 1: Eliminate the elements below the diagonal in the first column using Row E_1

$$E_2\rightarrow E_2-\frac{1}{2}E_1$$

$$E_3\rightarrow E_3-\frac{1}{2}E_1$$

$$\tilde{A}^{(1)}=\begin{bmatrix}2&1&1\\0&3/2&1/2\\0&1/2&5/2\end{bmatrix}\begin{array}{l}E_1\\E_2,\\E_3\end{array}$$

Step 2: Eliminate the elements below the diagonal in the second column using row E_2

$$E_3\rightarrow E_3-\frac{1}{3}E_2$$

$$\tilde{A}^{(2)} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 7/3 \end{bmatrix} \begin{matrix} E_1 \\ E_2, \\ E_3 \end{matrix}$$

The determinant of a triangular matrix is the product of its diagonal elements, so:

$$\det(A) = \det\left(\tilde{A}^{(1)}\right) = \det\left(\tilde{A}^{(2)}\right) = 2 * \frac{3}{2} * \frac{7}{3} = 7$$

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

Step 1: Elimination of the elements below the diagonal in the first column using row E_1

$E_2 \rightarrow E_2 - 2E_1$
$E_3 \rightarrow E_3 - 3E_1$

$$\tilde{B}^{(1)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -7 & -4 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

Step 2: Elimination of the elements below the diagonal in the second column using row E_2

$$E_3 \rightarrow E_3 - \frac{7}{3} E_2$$

$$\tilde{B}^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}$$

The determinant of a triangular matrix is the product of its diagonal elements, so:

$$\det(B) = \det\left(B^{(1)}\right) = \det\left(\tilde{B}^{(2)}\right) = 1 * (-3) * 3 = -9$$

2. Determinants of $\boldsymbol{A^{-1}}$, $\boldsymbol{B^{-1}}$ $\boldsymbol{and}$ $\boldsymbol{(A.B)^{-1}}$

Det(A^{-1})=?

We know that: $A.A^{-1} = I \Rightarrow det(A.A^{-1}) = det(I) \Rightarrow det(A) * det(A^{-1}) = det(I) \Rightarrow det(A^{-1}) = det(I) / det(A)$

Since $det(I) = 1$, $we\ have\ det(A^{-1}) = 1 / det(A) = 1 / 7$

$det(B^{-1})$ = ? ,det(B^{-1}) $= \frac{1}{det(B)} = -1/9$

$det(A.B)^{-1}$ = ? , $(A.B^{-1})$=$B^{-1}A^{-1}$, det$(A.B)^{-1}$=det($B^{-1}A^{-1}$)=det (B^{-1})$.det(A^{-1}) = \left(-\frac{1}{9}\right) * \frac{1}{7}$

=-1/63

Tutorial N.o3

Exercise 1:

1.Rewrite the linear system so that it becomes diagonally dominant:

$$\begin{cases} -2x_1 + 10x_3 = 7 \\ 10x_1 - x_2 = 9 \\ -x_1 + 10x_2 - 2x_3 = 10 \end{cases}$$

2.Using the **Jacobi method**, then the **Gauss-Seidel method**, compute the first 3 iterations by taking

X^0=(0,0,0)^T

Exercise 2:

1.Consider the following linear system:

$$\begin{cases} 3x_1 + x_2 - x_3 = 2 \\ x_1 + 5x_2 + 2x_3 = 17 \\ 2x_1 - x_2 - 6x_3 = -18 \end{cases}$$

Starting from X^0=(0,0,0)^T,compute the first 5 iterations of both the **Jacobi** and **Gauss-Seidel** methods.

2.Knowing that the exact solution is X^0=(1,2,3)^T, what can you conclude?

Exercise 3 (Supplementary): Using the **Jacobi method**, then the **Gauss-Seidel method**, compute the first 3 iterations by taking X^0=(0,0,0)^T

$$\begin{cases} \mathbf{2}x_1 - x_2 + x_3 = -1 \\ 3x_1 - 3x_2 + 9x_3 = 0 \\ 3x_1 + 2x_2 + 5x_3 = 10 \end{cases}$$

2. Why this divergence and which method diverges the fastest.

Solution of Tutorial N.o3

Exercise 1:

1. Rewriting the system to be diagonally dominant:

For the system to be diagonally dominant, the condition must be satisfied:

$$\sum_{\substack{j=1\\ j\neq i}}^{3} |a_{ij}| < |a_{ii}| \ \forall\, i = \overline{1,n}$$

The diagonally dominant system is:

$$\begin{cases} 10x_1 - x_2 = 9 \\ x_1 + 10x_2 - 2x_3 = 10 \\ -2x_1 + 10x_3 = 7 \end{cases}$$

Verification:

$$\begin{cases} |-1| + |0| < |10| \\ |-1| + |-2| < |10| \\ |-1| + |-2| < |10| \end{cases}$$

2. Calculation of the first three iterations:

a. Jacobi Method:

Recursive system:

$$\begin{cases} {x_1}^{k+1} = \dfrac{{x_2}^{k} + 9}{10} \\ {x_2}^{k+1} = \dfrac{{x_1}^{k} + 2{x_3}^{k} + 10}{10} \\ {x_3}^{k+1} = \dfrac{2{x_1}^{k} + 10}{10} \end{cases}$$

Starting from $X^0 = [0, 0, 0]^T$:

k	*0*	*1*	*2*	*3*
x_1	*0.0000*	0.9000	*1.0000*	1.0230
x_2	*0.0000*	1.0000	1.2300	1.2760
x_3	0.0000	0.7000	0.8800	0.9000

Approximate solution: $X = [1.0230, 1.2760, 0.9000]^T$

b. Gauss-Seidel Method:

Recursive system:

$$\begin{cases} x_1^{k+1} = \dfrac{x_2^{k} + 9}{10} \\ x_2^{k+1} = \dfrac{x_1^{k+1} + 2x_3^{k} + 10}{10} \\ x_3^{k+1} = \dfrac{2x_1^{k+1} + 10}{10} \end{cases}$$

Starting from $X^0 = [0,0,0]^T$:

k	*0*	*1*	*2*	*3*
x_1	*0.0000*	0.9000	*1.0090*	1.0277
x_2	*0.0000*	1.0900	1.2769	1.2831
x_3	0.0000	0.8800	0.9018	0.9055

Approximate solution: $X = [1.0277, 1.2831, 0.9055]^T$

Comparison with Exact Solution:

Exact solution: X = $[507/493 \approx 1.0284,\ 633/493 \approx 1.2840,\ 893/986 \approx 0.9057]^T$

Observation:

For the same number of iterations, the Gauss-Seidel method provides a more accurate solution and converges faster than the Jacobi method.

Exercise 2:

Linear System:

$$\begin{cases} 3x_1 + x_2 - x_3 = 2 \\ x_1 + 5x_2 + 2x_3 = 17 \\ 2x_1 - x_2 - 6x_3 = -18 \end{cases}$$

1. Calculation of the first five iterations:

a. Jacobi Method:

Recursive system:

$$\begin{cases} x_1^{k+1} = \dfrac{-x_2^{k} + -x_3^{k} + 2}{3} \\ x_2^{k+1} = \dfrac{-x_1^{k} - 2x_3^{k} + 17}{5} \\ x_3^{k+1} = \dfrac{2x_1^{k} - x_3^{k} + 18}{6} \end{cases}$$

Starting from $X^0 = [0,0,0]^T$:

k	0	1	2	3	4	5
x_1	0.0000	0.6667	0.5333	0.8630	0.8674	0.9406
x_2	0.0000	3.4000	2.0667	2.2311	2.0941	2.0602
x_3	0.0000	3.0000	2.6556	2.8333	2.9158	2.9401

Approximate solution: $X = [0.9406, 2.0602, 2.9401]^T$

b. Gauss-Seidel Method:

Recursive system:

$$\begin{cases} {x_1}^{k+1} = \dfrac{-{x_2}^k + -{x_3}^k + 2}{3} \\ {x_2}^{k+1} = \dfrac{-{x_1}^{k+1} - 2{x_3}^k + 17}{5} \\ {x_3}^{k+1} = \dfrac{2{x_1}^{k+1} - {x_3}^{k+1} + 18}{6} \end{cases}$$

Starting from $X^0 = X = [0,0,0]^T$

k	0	1	2	3	4	5
x_1	0.0000	0.6667	0.4704	0.8498	0.9381	0.99775
x_2	0.0000	3.2667	2.2348	2.1163	2.0402	2.0154
x_3	0.0000	2.6778	2.7843	2.9305	2.9727	2.9899

Approximate solution: Approximate solution: $X = [0.9406, 2.0154, 2.9899]^T$

2. Conclusion:

The exact solution is: $X = [1, 2, 3]^T$

By comparing the approximate solutions to the exact solution, the Gauss-Seidel method provides a more accurate solution after five iterations. The Gauss-Seidel solution $[0.9775, 2.0154, 2.9899]^T$) is closer to the exact solution than the Jacobi solution ($[0.9406, 2.0602, 2.9401]^T$). The Gauss-Seidel method generally converges faster due to its use of the most recent values, though this is not always true for all systems.

Solution to Exercise 3

$$\begin{cases} 3x_1 - \ x_2 + x_3 = \ -1 \\ 3x_1 - \ 3x_2 + \ 9x_3 \ = \ 0 \\ 3x_1 + 2\,x_2 + 5x_3 = \ 10 \end{cases}$$

1. Calculation of the first five iterations:

a. Jacobi Method:

Recursive system:

$$\begin{cases} x_1{}^{k+1} \ = \dfrac{x_2{}^k + -x_3{}^k - 1}{2} \\ x_2{}^{k+1} \ = \ x_1{}^k + 3x_3{}^k \\ x_3{}^{k+1} \ = \dfrac{10 - 3x_1{}^k - 2x_3{}^k}{5} \end{cases}$$

Starting from $X^0 = X \ = [0,0,0]^T$

k	0	1	2	3
x_1	0.00	-0.50	-1.50	1.10
x_2	0.00	0.00	5.50	5.40
x_3	0.00	2.00	2.30	0.70

b. Gauss-Seidel Method: Recursive system:

$$\begin{cases} x_1{}^{k+1} \ = \dfrac{x_2{}^k + -x_3{}^k - 1}{2} \\ x_2{}^{k+1} \ = \ x_1{}^{k+1} + 3x_3{}^k \\ x_3{}^{k+1} \ = \dfrac{10 - 3x_1{}^{k+1} - 2x_3{}^{k+1}}{5} \end{cases}$$

Starting from $X^0 = X \ = [0,0,0]^T$

k	0	1	2	3
x_1	0.00	-0.50	-2.00	1.75
x_2	0.00	-0.50	5.50	4.75
x_3	0.00	2.50	1.00	-.095

Analysis and Conclusion: - The solution diverges because the linear system is not diagonally dominant (the sufficient condition for convergence is not satisfied).

- The table below calculates the relative error for each iteration:

k	$\frac{\left\Vert \vec{X}^{(k)} - \vec{X}^{(k-1)} \right\Vert}{\left\Vert \vec{X}^{(k)} \right\Vert}$	
	Jacobi	Gauuse.S
1	100%	100%
2	91.06%	107.07%
3	54.95%	83.30%

Tutorial No4.

Exercise 1: Consider the following integral:

$$J = \int_0^2 \sqrt{x}\ dx$$

1. Calculate the integral using the trapezoidal method.
2. Calculate the integral using Simpson's method.
3. Compare the two results obtained with the exact value.

Exercise 2:

Consider the following integral:

$$J = \int_0^{\pi} sinx^2\ dx$$

1. Calculate the integral using the generalized trapezoidal method with 5, then 10 intervals.
2. Knowing that the exact value is 0.7726, compare the results obtained with the exact value.

Exercise 3:

Consider the following integral:

$$J = \int_0^1 \frac{1}{x+1} dx$$

1. Calculate the integral using the generalized Simpson's method with 4, then 8 intervals.
2. Compare the results obtained with the exact value.
3. Calculate the maximum error committed in the two previous cases.

Exercise 4

Consider the function $f(x)$ defined by the following table:

x_i	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$f(x_i)$	0	0.382683	0.707107	0.923880	1

1. Calculate the integral $\int_0^{\frac{\pi}{2}} f(x)$ using the generalized trapezoidal method.
2. Recalculate this integral using the generalized Simpson's method.
3. Knowing that$f(x) = sinx$ compare the results obtained with the exact value.
4. Find the number of intervals nn necessary to achieve an error of 10^{-6}using the generalized Simpson's method.

Exercise 5 (Supplementary): Consider the following integral:

$$J = \int_0^1 e^x\, dx$$

1. Calculate the integral using the generalized trapezoidal method with 4, then 8 intervals.
2. Compare the results obtained with the exact value.
3. Calculate the maximum error committed in the two previous cases.

Solution of TD n° 4

Exercise 1:

1. Calculation of the integral using the trapezoidal method

$\int_a^b f(x)dx \cong J_T = (\text{b - a})\left[\frac{1}{2}f(a) + \frac{1}{2}f(b)\right]$

$\int_a^b \sqrt{x}\, dx \cong J_T = (2 - 0)\left[\frac{1}{2}\sqrt{x} + \frac{1}{2}\sqrt{2}\right] \approx 1.4142$

2. Calculation of the integral using the Simpson method

$\int_a^b f(x)dx \cong J_S = (\text{b - a})\left[\frac{1}{6}f(a) + \frac{1}{6}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right]$

$\int_a^b \sqrt{x}\, dx \cong J_S = (2 - 0)\left[\frac{1}{6}\sqrt{0} + \frac{4}{6}\sqrt{1} + \frac{1}{6}\sqrt{2}\right] \approx 1.8047$

3. Comparison

$J_{exacte} \approx \int_0^2 \sqrt{x}\, dx \cong J_T = \int_0^2 \sqrt{3}\, dx \cong \left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[\sqrt{x^3}\right]_0^2 = \frac{2}{3}\left[\sqrt{x^3}\right]_0^2 \approx 1.8856$

$|J_{exacte} - J_T|$=$|1.8856 - 1.4142|$=0.4714

$|J_{exacte} - J_S|$=$|1.8856 - 1.0809|$=0.0809

We therefore conclude that the approximation of the integral given by the Simpson method is better than that obtained by the trapezoidal method.

Exercise 2

1. Calculation of the integral using the generalized trapezoidal method

a. With 5 small intervals

$$J = \int_0^{\pi} sinx^2\, dx$$

$$h = \frac{b-a}{n} = \frac{\pi - 0}{5} = \frac{\pi}{5}$$

x_i	x_0=$\pi/5$	x_1=$\pi/5$	x_2=$2\pi/5$	x_3=$3\pi/5$	x_4=$4\pi/5$	x_5=π
$f(x_i)$	0.0000	0.3846	1.0000	-.03999	0.0333	-0.4303

With:

$\int_a^b f(x)dx \cong J_{T_{G_5}} = \frac{h}{2}\left[f(x_0) + f(x_5) + 2\big(f(x_1) + f(x_2) + f(x_3) + f(x_4)\big)\right]$

$J_{T_{G_5}} = \frac{\pi}{10}\left[0.0000 - 0.4303 + 2(0.3846 + 1.0000 - 0.3999 + 0.03333)\right] = 0.5044$

b. With 10 small intervals :

$$h = \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10}$$

x_i	x_0=$\pi/5$	x_1=$\pi/10$	x_2=$\pi/5$	x_3=$3\pi/10$	x_4=$2\pi/5$	x_5=$\pi/2$
$f(x_i)$	0.0000	0.0958	0.3846	0.7760	1.0000	0.6243

x_6=$3\pi/5$	x_7=$7\pi/10$	x_8=$4\pi/5$	x_9=$9\pi/10$	x_{10}=π
-.03999	-0.9924	0.0333	0.9902	-.0.4303

With:

$\int_a^b f(x)dx \cong J_{T_{G_{10}}} = \frac{h}{2}\left[f(x_0) + f(x_{10}) + 2\big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + \cdots + f(x_9)\big)\right]$

$J_{T_{G_{10}}} = \frac{\pi}{20}$ $[0.0000 - 0.4303 + 2(0.0958 + 0.3846 - 0.7760 + 1.0000 - 0.3999 - 0.9924 + 0.0333 +$ $0.9902)]$

$J_{T_{G_{10}}}$=0.7224

2. Comparison

$J_{exacte} = 0.7224$

$\left|J_{exacte} - J_{T_{G_{15}}}\right|$=$|0.7726 - 0.5044|$=0.2682

$\left|J_{exacte} - J_{T_{G_{10}}}\right|$=$|0.7726 - 0.7224|$=0.0502

For the case $n = 5$, the absolute error is 0.2682 compared to the exact solution. For the case $n = 10$, the absolute error has been reduced to 0.0502. This absolute error is about 5 times smaller than the error obtained with 5 intervals.

Exercise03

Calculate the integral using the generalized trapezoidal method.

1. With four small intervals

$$J = \int_0^1 \frac{1}{x+1}dx$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x_i	x_0=0	x_1=0.25	x_2=$2\pi/5$	x_3=$3\pi/5$	x_4=$4\pi/5$
$f(x_i)$	1	0.8	0.666667	0.571429	0.5

With:

$\int_a^b f(x)dx \cong J_{T_{G_4}} = \frac{h}{3}\left[f(x_0) + f(x_5) + 4\big(f(x_1) + f(x_3)\big) + f(x_3) + 2f(x_2)\right]$

$J_{S_{G_4}} = \frac{0.25}{3}\,[1 + 0.5 + 4(0.8 + 0.571429) + 2 * 0.666667)] = 0.693254$

2. With 8 small intervals

$$h = \frac{b-a}{n} = \frac{1-0}{8} = \frac{1}{8}$$

x_i	x_0=0	x_1=0.125	x_2=0.250	x_3=0.375	x_4=0.500	x_5=0.625	x_6=0.750	x_7=0.875	x_8=1.000
$f(x_i)$	1	0.888889	0.8	0727273	0.666667	0.615385	0.571429	0.533333	0.5

With:

$$\int_a^b f(x)dx \cong J_{SG_8} =$$

J_{SG_8}=$\frac{h}{3}\left[f(x_0) + f(x_8) + 4\big(f(x_1) + f(x_2) + f(x_3) + f(x_5) + f(x_7) + 2(f(x_2) + f(x_4) + f(x_6)\big)\right]$

$J_{SG_8} = \frac{0.125}{3}\,[1 + 0.5 + 4(0.888889 + 0727273 + 0.615385 + 0.533333 + 2(0.8 + 0.666667 + 0.571429)]$

J_{SG_8}=0.693155

3. Comparison

$J_{exacte} = \int_0^1 \frac{1}{x+1}dx = [\ln(x+1)]_0^1 = ln2$

$|J_{exacte} - J_{SG_4}|$=$|ln2 - 0.693254|$=$1.0682.10^{-4}$.

$\left|J_{exacte} - J_{SG_8}\right|$=$|ln2 - 0.693155|$=7.8194.$10^{-6}$.

For the case $n = 4$, the absolute error is $1.0682.10^{-4}$ relative to the exact solution. For the case n = 8 , the absolute error has been reduced to 7.8194.10^{-6}.This absolute error is approximately 14 times smaller than the error obtained with 4 intervals.

3. Maximal error

$$|\boldsymbol{R_{SG}}| \leq \boldsymbol{E_{max}} = \frac{\boldsymbol{nh^5}}{\boldsymbol{2*90}} \boldsymbol{M}$$

With $h = \frac{b-a}{n}$ $, M = Max\{|f^{(4)}(\xi)|\}$, $\boldsymbol{\xi} \in [\boldsymbol{a}, \boldsymbol{b}]$

So $\boldsymbol{E_{max}} = \frac{\boldsymbol{(b-a)^5}}{\boldsymbol{180n^4}} \boldsymbol{M}$

Calculus of M

$f(x) = \frac{1}{x+1}$, $f'(x) = -\frac{1}{(x+1)^2}$, , $f''(x) = 2\frac{1}{(x+1)^3}$, , $f'''(x) = -6\frac{1}{(x+1)^4}$, , $f''''(x) = 24\frac{1}{(x+1)^5}$,

$M = Max\left\{\left|24\frac{1}{(x+1)^5}\right|\right\} = Max\left\{\left|24\frac{1}{(x+1)^5}\right|\right\}$ $, x \in [0,1]$

$\forall x_1 \in [0,1]$, $\forall x_2 \in [0,1]$, $x_1 < x_2 \Rightarrow x_1 + 1 < x_2 + 1 \Rightarrow (x_1 + 1)^5 < (x_2 + 1)^5$

$\Rightarrow \frac{1}{(x_1+1)^5} > \frac{1}{(x_2+1)^5} \Rightarrow \frac{24}{(x_1+1)^5} > \frac{24}{(x_2+1)^5} \Rightarrow |f^{(4)}(x_1)| > |f^{(4)}(x_2)|$

$\Rightarrow |f^{(4)}(x_2)| \searrow$ $\Rightarrow$The maximum is obtained for $x = 0$

$M = 24\frac{1}{(0+1)^5}$=25.

For the case $n = 4,$, we obtain:

$$E_{max} = \frac{(1-0)^5}{180*4^4} 24 = 5.2083.10^{-4}$$

Exercise 4:

1. Calculation of the integral using the generalized trapezoidal method

$$J = \int_0^{\pi/2} f(x)dx$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x_i	x_0=0	x_1=$\pi/8$	x_2=$\pi/4$	x_3=$3\pi/8$	x_4=$\pi/2$
$f(x_i)$	1	0.382683	0.707107	0.923880	1

$$h = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = \pi/2$$

$\int_a^b f(x)dx \cong J_{T_{G_4}} = \frac{h}{2}\left[f(x_0) + f(x_4) + 2\big(f(x_1) + f(x_2) + f(x_3)\big)\right]$

$J_{T_{G_4}} = \frac{\pi}{16}\,[0 + 1 + 2(0.382683 + 0.707107 + 0.923880)]$=0.987116

2. Calculation of the integral using the generalized Simpson's method

$\int_a^b f(x)dx \cong J_{T_{G_4}} = \frac{h}{3}\left[f(x_0) + f(x_4) + 4\big(f(x_1) + f(x_3)\big) + 2f(x_2)\big)\right]$

$J_{SG_4} = \frac{\pi}{24}\,[0 + 1 + 4(0.382683 + 0.923880) + 2 * 0.707107]$=1.000135

3. Comparison

$$J_{exacte} = \int_0^{\pi/2} sinx\,dx = -[\cos x]_0^{\pi/2} = 1$$

$\left|J_{exacte} - J_{SG_4}\right|$=$|1 - 0.987116|$ =0.012884.

$\left|J_{exacte} - J_{SG_8}\right|$=$|1 - 1.000135|$=0.000135

It can be observed that the Simpson's method provides better precision than the trapezoidal method.

4. Number of intervals n

The theoretical error committed by the Simpson's method is verified by:

$$|\boldsymbol{R_{SG}}| \le \boldsymbol{E_{max}} = \frac{\boldsymbol{nh^5}}{\boldsymbol{2 * 90}}\,\boldsymbol{M}$$

Where $h = \frac{b-a}{n}$, $M = Max\{|f^{(4)}(\xi)|\}$, $\boldsymbol{\xi} \in [\boldsymbol{0}, \frac{\boldsymbol{\pi}}{\boldsymbol{2}}]$

Thus, for $|\boldsymbol{R_{SG}}| \le \varepsilon$ it is sufficient that n satisfies

$\frac{nh^5}{180}\,M = \frac{(b-a)^5}{180n^4}\,M \le \varepsilon \Rightarrow n^4 \ge \frac{(b-a)^5}{180\varepsilon}\,M$

$f(x) = sinx \;\Rightarrow f^{(4)}(x) = sinx$

$f^{(4)}(x) \sim [0, \frac{\pi}{2}] \Rightarrow M = f^{(4)}\left(\frac{\pi}{2}\right) = 1$

Thus $n^4 \ge \frac{\left(\frac{\pi}{2} - 0\right)^5}{180 * 10^{-6}} * 1 \Rightarrow n \ge 15.18$

We take $n = 16$.

Exercise 5 (Supplementary)

1. Calculation of the Integral Using the Generalized Trapezoidal Method

a. With four Small Intervals

$$J = \int_0^1 e^x dx$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x_i	x_0=0	x_1=0.25	x_2=0.50	x_3=0.75	x_4=1
$f(x_i)$	1	1.2840	1.6487	2.1170	2.7138

$\int_a^b f(x)dx \cong J_{T_{G_4}} = \frac{h}{2}\left[f(x_0) + f(x_4) + 2\big(f(x_1) + f(x_2) + f(x_3)\big)\right]$

$J_{TG_4} = \frac{0.25}{2}$ [1 + 2.7138 + 2(1.2840 + 1.6487 + 2.7138)]=01.7272

2. With eight small intervals

$$h = \frac{b-a}{n} = \frac{1-0}{8} = \frac{1}{8}$$

x_i	x_0=0	x_1=0.125	x_2=0.250	x_3=0.375	x_4=0.500	x_5=0.625	x_6=0.750	x_7=0.875	x_8=1.000
$f(x_i)$	1	1.1331	1.2840	1.4550	1.6487	1.8682	2.1170	2.3989	2.7183

With:

$$\int_a^b f(x)dx \cong J_{T_{G_8}}$$

$J_{T_{G_8}}$=$\frac{h}{2}\left[f(x_0) + f(x_8) + 2\big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7)\big)\right]$

$J_{S_{G_8}} = \frac{0.125}{2}$ [1 + 2.7183 + 4(1.1331 + 1.2840 + 1.4550 + 1.6487 + 1.8682 + 2.1170 + 2.3989)]

$J_{S_{G_8}}$=1.7205

3. Comparison

$J_{exacte} = \int_0^1 e^x \, dx = -[e^x]_0^1 = 2.7183$

$|J_{exacte} - J_{TG_4}| = |1.7183 - 0.987116| = 0.0089.$

$|J_{exacte} - J_{TG_8}| = |1.7183 - 1.000135| = 0.0022$

For the case $n = 4$, the absolute error is 0.0089 compared to the exact solution. For the case $n = 8$, the absolute error is reduced to 0.0022. This absolute error is approximately 4 times smaller than the error obtained with 4 intervals.

4. Maximum Error

$|\boldsymbol{R_{TG}}| \leq \boldsymbol{E_{max}} = \frac{\boldsymbol{nh^3}}{\boldsymbol{12}} \boldsymbol{M}$

Where $h = \frac{b-a}{n}$ $, M = Max\{|f''(\xi)|\}$, $\boldsymbol{\xi} \in [\boldsymbol{a}, \boldsymbol{b}]$

So $\boldsymbol{E_{max}} = \frac{(\boldsymbol{b}-\boldsymbol{a})^3}{\boldsymbol{12n^2}} \boldsymbol{M}$

Calculus of M

$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x,$

$M = Max\{|e^x|\} = Max\{e^x\} \quad , x \in [0,1]$

$(e^x)' = e^x > 0 \Rightarrow f''(x) \nearrow \Rightarrow M = e^1 = e$

For the case $n = 4,$, we obtain:

$$E_{max} = \frac{(1-0)^3}{12 * 4^2} e = 0.0142$$

For the case $n = 8,$, we obtain:

$$E_{max} = \frac{(1-0)^3}{12 * 8^2} e = 0.0035.$$

Tutorial No.5

Exercise 1: Consider the following differential equation:

$$\begin{cases} y' = y + x \\ y(0) = 1 \end{cases}$$

Compute the approximate solution of this equation at $x = 1$ using Euler's method, dividing the interval into 10 equal parts.

Knowing that the exact solution is $y_{exact} = -1 - x + e^{x}$, compare the result obtained with $y_{exact}(1)$.

Exercise 2: Consider the differential equation:

Consider the differential equation:

$$\begin{cases} y' = y + e^{2x} \\ y(0) = 2 \end{cases}$$

1. Using a step size h=0.1, perform one iteration of the Euler-Cauchy (modified Euler) method and calculate the error by comparing the result with the exact solution $y_{\text{exact}}(0.1)$
2. Using a step size h=0.05 h = 0.05 h=0.05, perform two iterations of the Euler-Cauchy method and calculate the error at $y(0.1)$ by comparing the result with the exact solution$y_{exact}(1)$).
3. Compute the ratio of the errors from parts 1) and 2), and provide a comment on the result obtained.

Exercise 3: Consider the following differential equation:

$$\begin{cases} y' = y + x + 1 \\ y(0) = 1 \end{cases}$$

1. Compute the approximate solution $y(0.2)$ using Euler's method and the 4th-order Runge-Kutta method with a step size $h = 0.1$.
2. For each method, calculate the error by comparing the result with the exact solution $y_{exact}(0.2)$1.0187370780. Comment on the results obtained.

Note: Use 9 significant digits after the decimal point.

Solution of Tutorial N.o5

Exercise 1

$$\begin{cases} y' = y + x \\ y(0) = 1 \end{cases}$$

1. Calculation of the approximate solution of the differential equation at $x = 1$ using Euler's method

The working interval is $[0, 1]$.

The step size h is: $\frac{1-0}{10} = 0.1$.

Moreover, we have: $f(x, y) = y + x$.

Thus, we can use Euler's method:

$$y(x_{n+1}) \approx y(x_n) + h\, f(x_n, y(x_n))$$

and successively obtain approximations of

$y(0.1), y(0.2), y(0.3), y(0.4), y(0.5), y(0.6), y(0.7), y(0.8), y(0.9),)\ and\ (\ y(1.0)\ )$. The first iteration yields:

$$y(0.1) \approx y(0) + h\, f(0, 1) = 1 + 0.1(1 + 0) = 1.1.$$

Similarly, the second iteration gives:

$$y(0.2) \approx y(0.1) + h\, f(0.1, 1.1) = 1.1 + 0.1(1.1 + 0.1) = 1.22.$$

And the third iteration gives:

$$y(0.3) \approx y(0.2) + h\, f(0.2, 1.22) = 1.22 + 0.1(1.22 + 0.2) = 1.362.$$

The following table summarizes the results of the first ten iterations:

x_n	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$f(x_n)$	1.0000	1.1000	1.2200	1.3620	1.5282	1.72120	1.943122	2.1974342	2.48717762	2.815895382	3.1874849202

Thus, the approximate solution of the differential equation at $x = 1$ is:

$$y(1) = 3.1874849202$$

2. Comparison of the obtained result with the exact solution

The exact solution is: $y_{exact}(x) = 1 - x - e^{x}$.Thus, the exact solution at $x = 1$ gives:

$$y_{execte}(1) = 1 - 1 - e^{1} = 0.2490787367$$

The error committed is:

$E_e = |\, y_{exact}(x) - y(1)| = |: y_{exact}(1) - y(1)| = |0.2490787367 - 3.1874849202|$=0.2490787367

Exercise 2

$$\begin{cases} y' = y + e^{2x} \\ \quad y(0) = 2 \end{cases}$$

1. Calculation of the first iteration of the Euler-Cauchy method

The step size is $h = 0.1$.

Moreover, we have $f(x, y) = y + e^{x}$.

Thus, we can use the Euler-Cauchy method:

$$y(x_{n+1}) \approx \; y(x_n) + \frac{h}{2}\,[f(x_n, y(x_n) + f(x_{n+1}, y(x_{n1})]$$

with

$$y_{n1} = y(x_n) + h\, f(x_n, y(x_n))$$

and obtain the approximation of $y(0.1)$. The first iteration yields:

$$y_{n1} = y(0) + h\, f(0, 2) = 2 + 0.1(2 + e^{2*0}\}) = 2.3$$

which is the result obtained using Euler's method. The second step gives:

$$y(0.1) = y(0) + \frac{h}{2}[\, f(0, 2) + f(0.1,2,3)] = 2 + \frac{0.1}{2}[2 + e^{2*0} + 2.3 + e^{2*0.1}) = 2.326070138$$

<u>Error committed</u>

The exact solution is: $y_{execte}(x) = e^{x} + e^{2x}$.

The exact solution at 0.1 gives , $y_{execte}(0.1) = e^{0.1} + e^{2*0.1}$.=2.326573676, so the error comitted is

$E_{ec(h=0.1)} = |\, y_{execte}(0.1) - y(0.1)| = |2.326573676 - 2.326070138|$=0.000503538

2. Calculation of the first two iterations of the Euler-Cauchy method

The step size is $h = 0.05$.

Moreover, we have $f(x, y) = y + e^{2x}$.

Thus, we can use the Euler-Cauchy method and successively obtain the approximations of $y(0.05)$ and

$y(0.1)$. The first iteration yields:

$$y_{n1} = y(0) + h\,f(0.2) = 2 + 0.05(2 + e^{2*0}) = 2.15$$

which is the result obtained using Euler's method. The second step gives:

$$y_{n1=}y(0.05) + h\,[f(00.05,2.156379273) = 2.156379273 + 0.05(2.156379273 + e^{2*0.05})\,]\,]$$
$$= 2.319456782$$

Similarly, the second iteration gives:

$$y(0.05) \approx y(0) + \frac{h}{2}\,[f(0,2) + f(0.05, 2.15)] = 2 + \frac{0.05}{2}[2 + e^{2*0} + 2.15 + e^{2*0.05}]$$
$$= 2.156379273$$

The correction then leads to:

$$y(0.1) \approx y(0.05) + \frac{h}{2}\,[f(0.05,2.156379273) + f(0.1,2.319456782)]$$
$$= 2.156379273 + \frac{0.05}{2}[2.156379273 + e^{2*0.05} + 2.319456782 + e^{2*0.1}]$$
$$= 2.326439516$$

Error committed

The error committed is:

$E_{ec(h=0.1)} = |\,y_{exect}(0.1) - y(0.1)| = |2.326573676 - 2.326439516| = 0.00013416$

3. Error ratio

The error ratio is:

R = $\frac{E_{ec(h=0.1)}}{E_{ec(h=0.05)}}$=3.75$\approx 2^2$

It is clear that the difference between the exact solution and the approximate solution decreases by a factor of $3.75 \approx 2^2$ when the step size h is halved, which confirms that the Euler-Cauchy method is of order 2.

Exercise 3

1. Calculation of the approximation of $y(0.2)$:

The working interval is $[0, 0.2]$.

The step size h is: $h = 0.1$.

Moreover, we have: $f(x,y) = -y + x + 1$.

a. Euler's method

Thus, we can use Euler's method:

$$y(x_{n+1}) \approx y(x_n) + h\,f(x_n, y(x_n)),$$

and successively obtain approximations of $y(0.1)$ and $y(0.2)$. The first iteration yields:

$$y(0.1) \approx y(0) + h\,f(0,1) = 1 + 0.1(-1 + 0 + 1) = 1.$$

Similarly, the second iteration gives:

$$y(0.2) \approx y(0.1) + h\,f(0.1,1) = 1 + 0.1(-1 + 0.1 + 1) = 1.01.$$

Thus, the approximation of $y(0.2)$ using Euler's method is $y(0.2) \approx 1.01$.

b. Runge-Kutta method of order 4

Thus, we can use the Runge-Kutta method of order 4:

$k_1 = h\,f(x_n, y(x_n))$

$k_2 = h\,f(x_n + \frac{h}{2}, y(x_n) + \frac{k_1}{2})$

$k_3 = h\,f(x_n + \frac{h}{2}, y(x_n) + \frac{k_2}{2})$

$k_4 = h\,f(x_n + h, y(x_n) + k_3)$

$$y(x_{n+1}) \approx y(x_n) + \frac{1}{6}(\,k_1 + 2k_2 + 2k_3 + k_4)$$

and successively obtain approximations of $y(0.1)$ and $y(0.2)$. The first iteration yields:

$k_1 = h\,f(0.1) = 0.1(-1 + 0 + 1) = 0$

$k_2 = h\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}\right) = 0.1f(0.05{,}1) = 0.1(-1 + 0.05 + 1) = 0.005$

$k_3 = h\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}\right) = 0.1f(0.05{,}1.0025) = 0.1(-1.0025 + 0.05 + 1) = 0.00475$

$k_4 = h\,f(0 + 0.1{,}1 + 0.00475) = 0.1f(0.1{,}1.00475) = 0.1(-1.00475 + 0.1 + 1)$

which leads to:

$$y(0.1) \approx y(0) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0 + 2 * 0.005 + 2 * 0.00475 + 0.009525) = 1.0048375$$

Similarly, the second iteration gives:

$k_1 = h\,f(0.1, 1.0048375) = 0.1(-1.0048375 + 0.1 + 1) = 0.00951625$

$$k_2 = h\,f(0.1 + \frac{0.1}{2}, 1.0048375 + \frac{0.00951625}{2}) = 0.1\,f(0.15, 1.009595625)$$

$$= 0.1(-1.009595625 + 0.15 + 1) = 0.014040437$$

$$k_3 = h\,f(0.1 + \frac{0.1}{2}, 1.0048375 + \frac{0.014040437}{2}) = 0.1\,f(0.15, 1.011857718)$$

$$= 0.1(-1.011857718 + 0.15 + 1) = 0.013814228$$

$$k_4 = h\,f(0.1 + 0.1, 1.0048375 + 0.013814228) = 0.1\,f(0.2, 1.018651728)$$

$$= 0.1(-1.018651728 + 0.2 + 1) = 0.018134827$$

which leads to:

$$y(0.2) \approx y(0.1) + \frac{1}{6}(\,k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.0048375 + \frac{1}{6}(0.00951625 + 2 * 0.014040437 + 2 * 0.013814228 + 0.018134827)$$

$$= 1.018730901$$

Thus, the approximation of $y(0.2)$) using the Runge-Kutta method of order 4 is $y(0.2)$ ≈1.01873090.

2. Error committed

The exact solution is: $y_{exact}(0.2) = 1.018730780$

a. Error committed by Euler's method:

$E_e = |\, y_{exect}(0.2) - y(0.2)| = |1.018730780 - 1.01| = 0.00870750 = 0.873078 * 10^{-2}$

b. Error committed by the Runge-Kutta method of order 4:

E_{RK4}= $|\, y_{exect}(0.2) - y(0.2)|$ = |1.018730780 - 1.018730901| = 0.000000121 = $0.121 * 10^{-6}$

It is clear that Euler's method is much less accurate than the Runge-Kutta method of order 4. Therefore, it is preferable to use methods of as high an order as possible.

Tutorial N.o6

Exercise 1 Given the three points $(0, 1), (1, 0.5), and\ (3, 0.25)$ for the function $f(x)$:

1. Determine the Lagrange polynomial that passes through these points.
2. Find an approximation of $f(1.5)$.
3. Knowing that $f(x) = 1 - (x + 1)$, calculate the maximum error when approximating $f(x)$ with the Lagrange polynomial $P(x)$, and compare it with the exact error.

Exercise 2:

Given the polynomials $P(x)\ and\ Q(x)$ defined as follows:

0	0	1	2
$P(x)$	-6	3	21
$Q(x)$	10	15	40

Find the points of intersection using the Lagrange interpolation method.

Exercise 3: Using the following data, find an approximation of $f(4.5)$ with a Newton polynomial of degree 2:

x	1.0	3.0	5.0	7.0
$f(x)$	0.0000	1.2528	1.6094	1.9456

Exercise 4:

1. Using Newton's interpolation method, calculate an approximation of $\sqrt{1.6}$. Take x0=1 x_1 =1, x_2 =2, and x_3=3.
2. Calculate the error committed

Exercise 5 (Supplementary):

Determine the Lagrange polynomial for the function $f(x) = \mid x \mid$ that passes through the points with x-coordinates $-1, -0.5, 0, 0.5, and\ 1$ 1.

Solution of Tutorial N.o6

Exercise 1:

x_i	$x_0 = 0$	x_1=1	x_2=2
$f(x_i)$= y_i	y_0=1	$y_1 = 0.5$	y_2=0.25

1. Lagrange Polynomial Passing Through the 3 Points:

We have 3 points, so the degree of the polynomial is n ≤ 2.

$$P_2(x) = \sum_{k=0}^{2} \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

For $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(0 - x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3)$$

For $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x(x-3)$$

For $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_2)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x(x-1)$$

Thus,

$P_2(x)$=$\frac{1}{3}(x-1)(x-3) - \frac{1}{2}x(x-3) + \frac{1}{6}x(x-1) = 0.125x^2 - 0.625x + 1.$

2. Approximation of $f(1.5)$

Using the Lagrange polynomial$P_2(x)$, we find:

$y(x) = P_2(1.5) = 0.125(1.5^2) - 0.625(1.5) + 1$

$y(1.5) \approx 0.344$

3. Calculation of the Maximum Error:

$$|f(x)\ -\ P_2(x)| \leq E_{Max}(x)$$

Where:

$$E_{Max}(x) = \frac{M}{(n+1)!}\prod_{i=0}^{2}|x - x_i|$$

$M = Max\left\{\left|f^{(3)}(\xi)\right|\right\}, \quad \xi \in [0{,}3]$

$f(x) = -\frac{1}{(x+1)^2}\ , \quad f'' = \frac{2}{(x+1)^3}, \qquad\qquad f''' = -\frac{6}{(x+1)^4}$

$M = Max\left\{\left|-\frac{6}{(\xi+1)^4}\right|\right\} = Max\left\{\left|\frac{6}{(\xi+1)^4}\right|\right\}, \quad \xi \in [0{,}3],$

$\forall\, x_1 \in [0{,}3], \qquad\qquad \forall\, x_2 \in [0{,}3],$

$x_1 < x_2 \Rightarrow x_1 + 1 < x_2 + 1 \Rightarrow (x_1+1)^4 < (x_2+1)^4 \Rightarrow \frac{1}{(x_1+1)^4} > \frac{1}{(x_2+1)^4} \Rightarrow \frac{6}{(x_1+1)^4} > \frac{6}{(x_2+1)^4}$

$\Rightarrow |f'''(x_1)| > |f'''(x_2)| \Rightarrow |f'''(x)| \searrow$

The maximum is obtained for $x\ =\ 0$. Thus,

$M = \frac{6}{(0+1)^4} = 6,$

$E_{Max}(x) = \frac{6}{3*2}|x||x-1||x-3|, \quad x \in [0{,}3]$

For $x\ =\ 1.5,\ E_{Max}(1.5) = |1.5||1.5-1||1.5-3|$

$E_{Max}(1.5) = 1.125$

$E_{exact}\ = |f(1.5) - P_2(1.5)||0.4 - 0.344|$

$E_{exact}\ = 0.056$

Comparison:

We observe that $E_{exact} < E_{Max}$.

Exercise 2:

Lagrange Polynomial

a. $P(x)$

x_i	$x_0 = 0$	x_1=1	x_2=2
$f(x_i)$=y_i	y_0=-6	$y_1 = 3$	y_2=21

We have 3 points, so the degree of the polynomial is n ≤ 2.

$$P_2(x) = \sum_{k=0}^{2} \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

For $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2)$$

For $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2)$$

For $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_2)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{2}x(x-1)$$

Thus,

$P_2(x)$=$\frac{1}{2}(x-1)(x-2)(-6) - x(x-2)(3) + \frac{1}{2}x(x-1)(21)$

$P_2(x)$=$\frac{1}{2} = \frac{9}{2}x^2 + \frac{9}{2}x - 6.$

b. Q(x)

x_i	$x_0 = 0$	x_1=1	x_2=2
$f(x_i)$=y_i	y_0=10	$y_1 = 15$	y_2=40

We have 3 points, so the degree of the polynomial is n ≤ 2.

$$Q_2(x) = \sum_{k=0}^{2} \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

For $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2)$$

For $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -x(x-2)$$

For $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_2)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{2}x(x-1)$$

Thus,

$Q_2(x)$=$\frac{1}{2}(x-1)(x-2)(10) - x(x-2)(15) + \frac{1}{2}x(x-1)(40)$

$Q_2(x) = 10x^2 - 5x + 10.$

<u>Exercise 3:</u>

<u>Approximation of $f(4.5)$:</u>

Since the polynomial is of degree 2, we need only 3 points. We choose points with abscissas close to 4.5. These points are presented in the following table:

x_i	$x_0 = 3$	x_1=5	x_2=7
$f(x_i)$=y_i	y_0=1.2528	$y_1 = 1.6094$	y_2=1.9459

We have 3 points, so the degree of the polynomial is $n = 2$. The Newton polynomial passing through these 3 points is:

$P_2(x)$=$y_0 + \frac{\Delta y_0}{1!h^1}(x-x_0)$+$\frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1)$

where: h = x_1 - x_0 = $x_2 - x_1$= 2

x_0 = 3 → y_0 = 1.2528

$$\Delta y_0 = y_1 - y_0 = 0.3566$$

x_1 = 5 → y_1 = 1.6094

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = -0.0201$$

$$\Delta y_1 = y_1 - y_0 = 0.3566$$

x_2 = 7 → y_2 = 1.9459

Thus:

$P_2(x)$=1.2528 + $\frac{0.3566}{2}$ $(x - 3)$ + $\frac{-0.0201}{2!2^2}$ $(x - 3)\ (x - 5)$

$$P_2(x) = 1.2528 + 0.1783\ (x - 3) - 0.0025\ (x - 3)(x - 5)$$

The approximation of $f(4.5)$ is:

$$f(4.5) \approx P_2(4.5) = 1.2528 + 0.1783\ (4.5 - 3) - 0.0025\ (4.5 - 3)(4.5 - 5)$$

$P_2(4.5)$ = 1.5221

Exercise 4:

1. Calculation of the Approximation of $\sqrt{1.6}$

x_i	$x_0 = 1$	x_1=2	x_2=3
$f(x_i)$=y_i	y_0=1	$y_1 = 1.4142$	y_2=1.7320

The Newton polynomial passing through these 3 points is:

$P_2(x)$=$y_0 + \frac{\Delta y_0}{1!h^1}(x - x_0)$+$\frac{\Delta^2 y_0}{2!h^2}(x - x_0)(\ x - x_1)$

where: h = x_1 - x_0 = $x_2 - x_1$= 2

x_0 = 1 → y_0 = 1

$$\Delta y_0 = y_1 - y_0 = 0.4142$$

x_1 = 2 → y_1 = 1.4142

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = -0.0964$$

$$\Delta y_1 = y_1 - y_0 = 03178$$

x_2 = 3 → y_2 = 1.7320

Thus:

$P_2(x)$=1 + $\frac{0.4142}{1!1^1}$ $(x - 1)$ + $\frac{-0.0946}{2!1^2}$ $(x - 1)\ (x - 2)$

$$P_2(x) = 1 + 0.4142\ (x - 1) - 0.0482\ (x - 1)(x - 2)$$

The approximation of $f(\sqrt{\mathbf{1.6}})$ is:

$$f(1.6) = = 1 + 0.4142\ (1.6 - 1) - 0.0482\ (1.6 - 1)(1.6 - 2)$$

$P_2(1.6)$ = 1.2601

2. Error Committed:

The error committed in approximating $\sqrt{\mathbf{1.6}}$ byP_2 (1.6) is:

$$E_{exact} = |f(1.6){-}P_2(x)| = \left|\sqrt{1.6} - 1.2601\right| = 0.0048.$$

Exercise 5 (Supplementary):

x_i	$x_0 = -1$	x_1=-0.5	x_2=0	x_2=0.5	x_2=1
$f(x_i)$= y_i	y_0=1	$y_1 = 0.5$	y_2=0	y_2=0.5	y_2=1

1. Lagrange Interpolation Polynomial of the Function $\boldsymbol{f(x)}$ Passing Through the Above Points:

We have 5 points, so the degree of the polynomial is $n \leq 4$.

$$P_4(x) = \sum_{k=0}^{2} \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2 + \frac{L_3(x)}{L_3(x_3)} y_3 + \frac{L_4(x)}{L_4(x_4)} y_4$$

For $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)}$$

$$= \frac{(x + 0.5)(x - 0)(x - 0)(x - 0)}{(-1 + 0.5)(-1 - 0)(-1 - 0.5)(-1 - 1)}$$

$$= \frac{2}{3}(x + 0.5)x(x - 0.5)(x - 1)$$

For $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)}$$

$$= \frac{(x + 1)(x - 0)(x - 0.5)(x - 1)}{(-0.5 + 1)(0.5 - 0)(-0.5 - 0.5)(-0.5 - 1)}$$

$$= -\frac{8}{3}(x + 1)x(x - 0.5)(x - 1)$$

For $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}$$

$$= \frac{(x + 1)(x + 0.5)(x - 0.5)(x - 1)}{(0 + 0.5)(0 + 0.5)(0 - 0.5)(0 - 1)}$$

$$= 4(x + 1)(x + 0.5)(x - 0.5)(x - 1)$$

For $K = 3$:

$$\frac{L_3(x)}{L_3(x_3)} = \frac{(x - x_1)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)}$$

$$= \frac{(x + 1)(x + 0.5)(x - 0)(x - 1)}{(0.5 + 1)(0.5 + 0.5)(-0.5 - 1)(0.5 - 1)}$$

$$= -\frac{8}{3}(x + 1)(x + 0.5)x(x - 1)$$

For $K = 4$:

$$\frac{L_4(x)}{L_4(x_4)} = \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}$$

$$= \frac{(x + 1)(x + 0.5)(x - 0)(x - 0.5)}{(1 + 1)(1 + 0.5)(1 - 0)(1 - 0.5)}$$

$$= \frac{2}{3}(x + 1)x(x + 0.5)x(x - 0.5)$$

Thus,

$P_4(x)$=$\frac{2}{3}(x + 0.5)x(x - 0.5)(x - 1)(1) - \frac{8}{3}(x + 1)x(x - 0.5)(x - 1)(0.5) + 4(x + 1)(x + 0.5)(x - 0.5)(x - 1)(0) - \frac{8}{3}(x + 1)(x + 0.5)x(x - 1)(0.5)$+$\frac{2}{3}(x + 1)x(x + 0.5)x(x - 0.5)(1)$

$P_4(x) = -\frac{4}{3}x^4 + \frac{7}{3}x^2$.

Conclusion

Conclusion

This document provides a comprehensive summary of the methods studied across the six chapters of the Numerical Methods course. The first chapter introduced methods for solving nonlinear equations. The second chapter focused on techniques for function interpolation. The third chapter we addressed methods for computing definite integrals of functions. In the fourth chapter, examined basic methods for solving ordinary differential equations. The fifth chaptercovered direct methods for solving systems of linear equations. Finally, the sixth chapter discussed indirect methods for solving such systems.

Once these methods are thoroughly understood and applied manually, it is advisable to automate the computational steps for each method using Excel. Additionally, it is highly recommended to implement the associated algorithms in MATLAB or, alternatively, in another high-level programming language such as C or Pascal.

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