

COMPARISON BETWEEN TAYLOR AND PERTURBED METHOD FOR VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND

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ABSTRACT. As it is known the equation $A\varphi = f$ with injective compact operator has a unique solution for all f in the range $R(A)$. Unfortunately, the right-hand side f is never known exactly, so we can take an approximate data f_δ and used the perturbed problem $\alpha\varphi + A\varphi = f_\delta$ where the solution $\varphi_{\alpha\delta}$ depends continuously on the data f_δ , and the bounded inverse operator $(\alpha I + A)^{-1}$ approximates the unbounded operator A^{-1} but not stable. In this work we obtain the convergence of the approximate solution of $\varphi_{\alpha\delta}$ of the perturbed equation to the exact solution φ of initial equation provided α tends to zero with $\frac{\delta}{\sqrt{\alpha}}$.

1. Introduction. The theory of inverse and ill-posed problems has a considerable part in the domain of algebra, geometry, differential equations and mathematical physics. In particular, integral equations where we find problems of quantum mechanics, acoustics, optimal control theory and financial mathematics. In this domain many researchers study the stability of the approximation of the unstable mappings as the discretization of the continuous problem and reduce it to a finite system of linear or nonlinear equations or the projection methods and give a necessary and sufficient condition for the convergence. In [2, 4, 6] the authors use the Legendre wavelets basis and wavelet collocation method in order to reduce an integral equation into a set of algebraic equations. The paper [1] shows that various (a discrete) methods for the approximate solution of Volterra integral equations of the first kind correspond to some discrete version of the method of (recursive) collocation in the space of (continuous) piecewise polynomials.

A discrete methods for solving Volterra integral equation of the first kind correspond to special discretization of recursive collocation methods in the space of piecewise polynomials which possess discontinuities.

Let A be a linear compact operator between Hilbert spaces H_1 and H_2 over the field $\mathbb{R}$. We formulate the inverse problem as operator equations of the form.

$$A\varphi = f. \tag{1}$$

Assuming that the compact operator A is injective so, there exists a unique solution $\varphi \in H_1$ of the unperturbed equation (1) for all $f \in R(A)$. Unfortunately,

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the right-hand side $f \in H_2$ is never known exactly, so we can take an approximate data $f_\delta \in H_2$ of f with

$$\|f - f_\delta\| \leq \delta. \quad (2)$$

Note that, the problem (1) is transformed to the perturbed problem

$$A\varphi_\delta = f_\delta, \quad (3)$$

which is not solvable due to the fail of information of the approximate right-hand side f_δ is in the range $R(A)$ or not. For the solution of the problem (3) we replace this latter by a perturbed well posed problem

$$\alpha\varphi + A\varphi = f_\delta, \quad \alpha > 0, \quad (4)$$

where the solution $\varphi_{\alpha\delta}$ depends continuously on the data f_δ , and the bounded inverse operator $(\alpha I + A)^{-1}$ from H_2 to H_1 approximates the unbounded operator A^{-1} from $R(A)$ to H_1 when α tends to zero. In other words, the operator $(\alpha I + A)^{-1}A$ converges to the identity

$$\lim_{\alpha \rightarrow 0} (\alpha I + A)^{-1}A\varphi = \varphi,$$

for all $\varphi \in H_1$

In this work we orient our study of the Volterra integral equations of the first kind

$$A\varphi(x) = \int_a^x k(x, y)\varphi(y)dy = f(x), \quad a \leq x \leq b$$

where $k(x, y)$ is a continuous function on $[a, b] \times [a, b]$, $f(x) \in H_2([a, b])$ denote the data on the Hilbert space H_2 of the inverse problem and $\varphi(x) \in H_1([a, b])$ is the potential function on the Hilbert space H_1 to be determined.

The equation (1) with a compact operator has a unique solution if $f(a) = 0$ and $k(x, x) \neq 0$ but with the small errors in the data f can lead to large deviation in the solution φ . Hence one has to look for a stable approximation method based on the Taylor expansion

$$\varphi(x+h) = \varphi(x) + h\varphi'(x) + \frac{h^2}{2}\varphi''(x) + O(h^3). \quad (5)$$

Differentiating (1) three times, we obtain

$$\begin{aligned} \varphi(x) &= \frac{1}{k(x, x)} \left(f'(x) - \int_a^x k_x(x, y)\varphi(y)dy \right) \\ \varphi'(x) &= \frac{1}{k(x, x)} \left(f''(x) - \left(2\varphi(x)k_x(x, x) + \int_a^x k_{xx}(x, y)\varphi(y)dy \right) \right) \\ \varphi''(x) &= \frac{1}{k(x, x)} \left(f'''(x) - (3\varphi'(x)k_x(x, x) + 3\varphi(x)k_{xx}(x, x) \right. \\ &\quad \left. + \int_a^x k_{xxx}(x, y)\varphi(y)dy) \right), \end{aligned}$$

where the integrals are calculated by trapezoidal or modified Simpson methods [8]

In [3] The author sees that for the second-kind Volterra integral equations (VIEs) with weakly singular kernel at mesh points, the convergence order can be improved, and it is better and better as n increasing. In [5] Authors studied finite difference scheme with temporal nonuniform mesh for time-fractional Benjamin–Bona–Mahony equations with non-smooth solutions. Their approximation bases on an integral equation equivalent to the nonlinear problem under consideration. They

employ high-order interpolation formulas to obtain a linearized scheme on a nonuniform mesh and, by using a modified Gronwall inequality established.

We can deduce by differentiability of equation (1) if the function $k(x, x)$ is nonvanishing a linear Volterra integral equation of the second kind

$$\varphi(x) - \frac{1}{k(x, x)} \int_a^x \frac{\partial k(x, y)}{\partial x} \varphi(y) dy = \frac{f'(x)}{k(x, x)}.$$

On the other hand, if $k(x, x)$ vanishes, but $f''(t)$ and $\frac{\partial k(x, x)}{\partial x}$ is nonvanishing, we can obtain an equation of the second kind by a further differentiation.

2. Main Results.

Theorem 2.1. *The problem (4) is well posed with the norm $\|(\alpha I + A)^{-1}\| = O\left(\frac{1}{\sqrt{\alpha}}\right)$ provided A is a positive operator.*

Proof.

$$\begin{aligned} \langle \varphi, \varphi \rangle &= \langle (\alpha I + A)(\alpha I + A)^{-1} \varphi, (\alpha I + A)(\alpha I + A)^{-1} \varphi \rangle \\ &= \langle \alpha(\alpha I + A)^{-1} \varphi + A(\alpha I + A)^{-1} \varphi, \alpha(\alpha I + A)^{-1} \varphi + A(\alpha I + A)^{-1} \varphi \rangle \\ &= \langle \alpha(\alpha I + A)^{-1} \varphi, \alpha(\alpha I + A)^{-1} \varphi \rangle + \langle \alpha(\alpha I + A)^{-1} \varphi, A(\alpha I + A)^{-1} \varphi \rangle \\ &\quad + \langle A(\alpha I + A)^{-1} \varphi, \alpha(\alpha I + A)^{-1} \varphi \rangle + \langle A(\alpha I + A)^{-1} \varphi, A(\alpha I + A)^{-1} \varphi \rangle \\ \|\varphi\|^2 &= \alpha^2 \|(\alpha I + A)^{-1} \varphi\|^2 + 2\alpha \|A\| \|(\alpha I + A)^{-1} \varphi\|^2 + \|A(\alpha I + A)^{-1} \varphi\|^2. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \|\varphi\|^2 &\geq 2\alpha \|(\alpha I + A)^{-1} \varphi\| \|A(\alpha I + A)^{-1} \varphi\| + 2\alpha \|A(\alpha I + A)^{-1} \varphi, (\alpha I + A)^{-1} \varphi\| \\ &= 2\alpha \|(\alpha I + A)^{-1} \varphi\|^2 \left\| A \left(\frac{(\alpha I + A)^{-1} \varphi}{\|(\alpha I + A)^{-1} \varphi\|} \right) \right\| \\ &\quad + 2\alpha \|(\alpha I + A)^{-1} \varphi\|^2 \left\| A \left(\frac{(\alpha I + A)^{-1} \varphi}{\|(\alpha I + A)^{-1} \varphi\|} \right), \left(\frac{(\alpha I + A)^{-1} \varphi}{\|(\alpha I + A)^{-1} \varphi\|} \right) \right\|, \end{aligned}$$

or still, for all $\varphi \in H_1$, we write

$$\|\varphi\|^2 \geq 2\alpha \|(\alpha I + A)^{-1} \varphi\|^2 \|A(\psi) + A(\psi, \psi)\|, \quad \forall \|\psi\| = 1.$$

Hence

$$\|(\alpha I + A)^{-1}\| = O\left(\frac{1}{\sqrt{\alpha}}\right).$$

□

Lemma 2.2. *Let $\varphi \in H_1([0, 1])$ be the solution of the equation (1), and $\varphi_{\alpha\delta}$ be the solution of the equation (4) where $\|f(x) - f^\delta(x)\| \leq \delta$, and for all $x \in [0, 1]$. it follows that*

$$\lim_{\alpha \rightarrow 0 \text{ and } \frac{\delta}{\sqrt{\alpha}} \rightarrow 0} \|\varphi - \varphi_{\alpha\delta}\| = 0, \text{ on } [0, 1],$$

provided A is a positive operator.

Proof. In this proof we imitate the technical of ([3]) where the authors intervene the auxiliary equation given by

$$(\alpha I + A)\varphi_\alpha = f, \quad (6)$$

and

$$(\alpha I + A)\varphi_{\alpha\delta} = f_\delta, \quad (7)$$

estimate the difference between the equation (1) and the equation (6), we get

$$\alpha(\varphi - \varphi_\alpha) + A(\varphi - \varphi_\alpha) = \alpha\varphi$$

or still

$$\|\varphi - \varphi_\alpha\| \leq \alpha\|\varphi\| \left\| (\alpha I + A)^{-1} \right\|.$$

Hence

$$\begin{aligned} \|\varphi - \varphi_\alpha\| &\leq \alpha\|\varphi\| \left\| (\alpha I + A)^{-1} \right\| \\ &= O(\sqrt{\alpha}) \end{aligned}$$

For the difference between the equation (6) and the equation (7), we obtain

$$(\alpha I + A)(\varphi_\alpha - \varphi_{\alpha\delta}) = f - f_\delta,$$

therefore

$$\begin{aligned} \|\varphi_\alpha - \varphi_{\alpha\delta}\| &= \|f - f_\delta\| \left\| (\alpha I + A)^{-1} \right\| \\ &= O\left(\frac{\delta}{\sqrt{\alpha}}\right) \left\| (\alpha I + A)^{-1} \right\| \end{aligned}$$

Finally, for the difference given by

$$\|\varphi - \varphi_{\alpha\delta}\| \leq \|\varphi - \varphi_\alpha\| + \|\varphi_\alpha - \varphi_{\alpha\delta}\|, \quad (8)$$

the first term in the right-hand side of (8) vanishes as $\alpha \rightarrow 0$, the second term in the right-hand side of (8) vanishes as $\frac{\delta}{\sqrt{\alpha}} \rightarrow 0$. So the convergence of the approximate solution of $\varphi_{\alpha\delta}$ of the perturbed equation (7) to the exact solution φ of initial equation (1). $\square$

3. Illustrating Examples. In this section, we present several examples.

Example 1. Consider the first-kind integral equations of Volterra

$$\int_0^x \cos(x-y)\varphi(y)dy = x \sin x,$$

where $0 \leq x, y \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = 2 \sin x.$$

Table 1. We present the two approximate solutions $\varphi_{\alpha\delta}(x)$ and φ_T of $\varphi(x)$ obtained by the modification Lavrentiev's classical method and the approximation method based on the Taylor expansion respectively, in some arbitrary points, the error is calculated for $N = 10$.

Val of x	Ex sol φ	Ap sol φ_T	Error $_T$	Ap sol $\varphi_{\alpha\delta}$	Error $_\delta$
0.000	0.00e+00	0.00e+00	0.00e+00	0.00e+00	0.00e+00
0.200	3.97e-01	3.97e-01	3.31e-04	3.97e-01	5.55e-17
0.400	7.78e-01	7.79e-01	3.17e-04	7.78e-01	5.55e-16
0.600	1.12e+00	1.13e+00	9.36e-04	1.12e+00	2.44e-15
0.800	1.43e+00	1.43e+00	1.52e-03	1.43e+00	3.99e-15
1.000	1.68e+00	1.68e+00	2.08e-03	1.68e+00	4.21e-15

Example 2. Consider the first-kind integral equations of Volterra

$$\int_0^x \exp(x+y)\varphi(y)dy = x \exp(x),$$

where $0 \leq x, y \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \exp(-x).$$

Table 2. We present the two approximate solutions $\varphi_{\alpha\delta}(x)$ and φ_T of $\varphi(x)$ obtained by the modification Lavrentiev's classical method and the approximation method based on the Taylor expansion respectively, in some arbitrary points, the error is calculated for $N = 10$.

Val of x	Ex sol φ	Ap sol φ_T	Error $_T$	Ap sol $\varphi_{\alpha\delta}$	Error $_\delta$
0.000	1.00e+00	1.00e+00	0.00e+00	1.00e+00	0.00e+00
0.200	8.18e-01	8.10e-01	7.84e-03	8.18e-01	3.89e-011
0.400	6.70e-01	6.63e-01	6.38e-03	6.70e-01	5.14e-011
0.600	5.48e-01	5.43e-01	5.39e-03	5.48e-01	5.17e-011
0.800	4.49e-01	4.44e-01	4.73e-03	4.49e-01	4.67e-011
1.000	3.67e-01	3.63e-01	4.29e-03	3.67e-01	4.01e-011

Example 3. Consider the first-kind integral equations of Volterra

$$\int_0^x (x^2 - y + 2)\varphi(y)dy = (x^2 - x + 2) \sin x - \cos x + 1,$$

where $0 \leq x, y \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \cos x.$$

Table 3. We present the two approximate solutions $\varphi_{\alpha\delta}(x)$ and φ_T of $\varphi(x)$ obtained by the modification Lavrentiev's classical method and the approximation method based on the Taylor expansion respectively, in some arbitrary points, the error is calculated for $N = 20$.

Val of x	Ex sol φ	Ap sol φ_T	Error $_T$	Ap sol $\varphi_{\alpha\delta}$	Error $_\delta$
0.000	1.00e+00	1.00e+00	0.00e+00	1.00e+00	0.00e+00
0.200	9.80e-01	1.00e+01	2.15e-02	9.79e-01	7.12e-05
0.400	9.21e-01	9.51e-01	3.06e-02	9.20e-01	1.51e-04
0.600	8.25e-01	8.60e-01	3.55e-02	8.25e-01	2.26e-04
0.800	6.96e-01	7.31e-01	3.52e-02	6.96e-01	2.89e-04
1.000	5.40e-01	5.71e-01	3.09e-02	5.39e-01	3.42e-04

Example 4. Consider the first-kind integral equations of Volterra

$$\int_0^x \exp(x+y)\varphi(y)dy = \frac{1}{2}(\exp 2x) - 1),$$

where $0 \leq x, y \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \frac{1}{2}(\exp(-2x) + 1).$$

Table 4. We present the two approximate solutions $\varphi_{\alpha\delta}(x)$ and φ_T of $\varphi(x)$ obtained by the modification Lavrentiev's classical method and the approximation method based on the Taylor expansion respectively, in some arbitrary points, the error is calculated for $N = 10$.

Val of x	Ex sol φ	Ap sol φ_T	Error $_T$	Ap sol $\varphi_{\alpha\delta}$	Error $_{\delta}$
0.000	1.00e+00	1.00e+00	0.00e+00	1.00e+00	0.00e+00
0.200	8.35e-01	8.28e-01	7.06e-03	8.35e-01	1.36e-05
0.400	7.24e-01	7.19e-01	5.18e-03	7.24e-01	4.52e-05
0.600	6.50e-01	6.46e-01	4.01e-03	6.50e-01	8.47e-05
0.800	6.00e-01	5.97e-01	3.29e-03	6.00e-01	1.26e-04
1.000	5.67e-01	5.64e-01	2.83e-03	5.67e-01	1.66e-04

Example 5. Consider the first-kind integral equations of Volterra

$$\int_0^x \sin(x+y)\varphi(y)dy = \cos 2x + \frac{1}{2}(\sin x - \cos x) - \frac{1}{2}\exp(-x)(\sin 2x + \cos 2x),$$

where $0 \leq x, y \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \exp(-x) - 1.$$

Table 5. We present the two approximate solutions $\varphi_{\alpha\delta}(x)$ and φ_T of $\varphi(x)$ obtained by the modification Lavrentiev's classical method and the approximation method based on the Taylor expansion respectively, in some arbitrary points, the error is calculated for $N = 20$.

The kernel $k(x, y) = \sin(x+y)$ vanishes at the point $(x, y) = (0, 0)$ and so we can't obtain the approximate Taylor solution φ_T of the above equation.

Val of x	Ex sol φ	Ap sol φ_T	Error $_T$	Ap sol $\varphi_{\alpha\delta}$	Error $_{\delta}$
0.000	0.00e+00	NaN	NaN	0.00e+00	0.00e+00
0.200	-1.81e-01	NaN	NaN	-1.84e-01	3.51e-03
0.400	-3.29e-01	NaN	NaN	-3.32e-01	2.83e-03
0.600	-4.51e-01	NaN	NaN	-4.53e-01	2.62e-03
0.800	-5.50e-01	NaN	NaN	-5.53e-01	2.65e-03
1.000	-6.32e-01	NaN	NaN	-6.35e-01	2.91e-03

4. Conclusion. A numerical method for solving Volterra linear integral equations of first kind, based on the technical modified Lavrentiev classical method and the approximation method based on the Taylor expansion, the approximate solutions $\varphi_{\alpha\delta}(x)$ and φ_T will be measurably close to the solution $\varphi(x)$ on the entire interval $[0, 1]$. The efficiency of the modified Lavrentiev classical method is tested by solving

some examples for which the exact solution is known. This allows us to estimate the exactness of our numerical results.

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