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Lipschitz closed injective ideals and interpolative Lipschitz ideals

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Dedication

I dedicate this work to:

To my dear ***parents****,*

To my ***sisters*** *and my* ***brother****,*

And to all my ***friends****,*

for their constant support and encouragement.

Contents

Notaions

$\mathbb{K}$	The field of real or complex numbers.
p'	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p'} = 1$
E^*	The topological dual of E.
B_E	The closed unit ball of E
$L(E;F)$	The set of all linear operators.
$\mathcal{I}$	The ideal of all linear operator.
$\mathcal{K}$	The set of all compact linear operators.
$\mathcal{W}$	The set of all weakly compact linear operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.
$\Pi_p(X,Y)$	The ideal linear p-summing
$\mathcal{I}_{Lip}$	The ideal of Lipschitz operators.
Lip_0	The set of all Lipschitz operators that vanish at 0.
$X^\#$	The Lipschitz dual of the pointed metric space X.
$\mathcal{M}(X)$	The linear space of all molecules on the metric space X.
$m_{xx'}$	The molecule defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$, where χ_A is the characteristic function of the set A.
Æ(X)	The Arens-Eells space of X.
T_L	The linearization of the operator T.
$\mathcal{L}_f$	The set of all finite rank linear operators.
Lip_{0F}	The set of Lipschitz finite rank operators.
Π_p	The set of all linear p-summing operators $(1 \leq p < \infty)$.
Π_p^L	The set of all Lipschitz p-summing operator $(1 \leq p < \infty)$.
$\mathcal{I}$	The ideal of all linear operator.
$\mathcal{I}^{inj}$	The injective of the linear operators ideal $\mathcal{I}$.
$\overline{\mathcal{I}}^{inj}$	The closed injective of the linear operators ideal $\mathcal{I}$.
$(\mathcal{I}_{Lip})_\sigma$	The interpolative of the Lipschiz operators ideal $\mathcal{I}_{Lip}$

Introduction

The theory of operator ideals, pioneered by A. Pietsch [14, 15], provides a powerful structural framework for classifying bounded linear operators between Banach spaces. Pietsch's seminal work established the fundamental classes of operators, most notably the ideal of absolutely p-summing operators, which he characterized through his celebrated Domination and Factorization Theorems. These results demonstrated that p-summing operators admit an integral representation via a probability measure on the dual unit ball, revealing a deep connection between summability properties and measure theory. This foundational theory has since become a cornerstone of modern Banach space theory, influencing areas ranging from local theory to harmonic analysis.

The systematic study of operator ideals has led to various methods for generating new ideals from existing ones. Among these, the *closed injective hull* procedure, extensively studied by Jarchow [11], extends a given ideal by allowing a controlled perturbation up to an arbitrary ε-term. Let $\mathcal{I}$ be complete ideal-quasinorm, A linear operator $T \in \mathcal{I}(E, F)$ belongs to $\overline{\mathcal{I}}^{inj}(E, F)$ if and only if, for each $\varepsilon > 0$, there exist a Banach space G and an operator $S \in \mathcal{I}(E, G)$ exist such that

$$\|Tx\| \leqslant N(\varepsilon) \cdot \|Sx\| + \varepsilon \cdot \|x\|, \quad \forall \varepsilon > 0, \ \forall x \in E. \tag{1}$$

More germane to this work is the *interpolative ideal procedure*, introduced by Matter [13] in 1987. by taking $N(\varepsilon) = \varepsilon^{-r}$ for a constant $1 \leq r < +\infty$ and computing the minimum of the right-hand side of (1) with respect to ε, which demonstrates that (1), is equivalent to the interpolation formula:

$$\|Tx\| \leq C \, \|Sx\|^{1-\sigma} \|x\|^{\sigma},$$

where σ is a parameter in $(0, 1)$. This elegantly generates a new operator ideal, denoted $\mathcal{I}_{\sigma}$, from a given ideal $\mathcal{I}$, effectively interpolating between the original ideal and the trivial ideal of bounded operators.

Building upon this linear theory, the primary objective of this memoir is to extend the interpolative ideal procedure to the non-linear category of Lipschitz operators between pointed metric spaces and Banach spaces. Following the work of Farmer and Johnson [10], who introduced Lipschitz p-summing operators, we develop a comprehensive framework for *interpolative Lipschitz operator ideals*. Our work is based on the recent article by Yahi, R., Achour, D., and Dahia, E. [20], which systematically studies the closed injective hull of Lipschitz ideals and the interpolative ideals of ipschitz operators. We establish fundamental properties, including the Banach ideal structure, injectivity, and the critical relationship between the Lipschitz injective hull, the closed injective hull, and the interpolative Lipschitz operator ideals. In particular, we show that $\mathcal{I}_{Lip}^{inj} \subset (\mathcal{I}_{Lip})_\sigma \subset \overline{\mathcal{I}_{Lip}}^{inj}$ Furthermore, we provide a detailed analysis of the concrete class $(\Pi_p \circ Lip_0)_\sigma$, which consists of Lipschitz operators whose linearization is a (p, σ)-absolutely continuous linear operator [13]. This class is characterized by Pietsch-type domination inequalities and summability conditions, unifying and extending classical results.

The contents of this memory are organized as follows. The preliminary chapter covers the essential background on p-summing linear operators, operator ideals, metric spaces, Lipschitz spaces, the molecule space Æ(X) [7], and the linearization process [18]. The second chapter systematically develops the injective hull and the closed injective hull for linear operator ideals and then introduces the interpolative ideals. The last chapter presents and analyzes to the study of the closed injective hull for Lipschitz operator ideals and the interpolative Lipschitz operator ideals.

Chapter 1

Preliminaries

We briefly review the theory of p-summing linear operators between Banach spaces, which plays a central role in the linear theory of operator ideals. After recalling the definition and Pietsch's Domination Theorem, we present the Factorization Theorem and basic examples. The class Π_p is shown to form an injective Banach operator ideal, a property that will later be extended to the Lipschitz setting.

This chapter is based on three key references: *Absolutely Summing Operators* by J. Diestel, H. Jarchow, and A. Tonge [9]; A. Pietsch's book *Operator Ideals* [14]; and N. Weaver, *Lipschitz Algebras* (second edition), World Scientific Publishing, Singapore, 2018, for the theory of metric spaces and Lipschitz operators [18].

1.1 Linear Operator Ideals

1.1.1 Finite Rank Operators

Recall that a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite-dimensional subspace of F. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only if it is of the form:

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle y.$$

Thus, if $T \in \mathcal{L}_f(E, F)$, it can be written as:

$$T = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ (see [14, Page 25]).

1.1.2 Linear Operator Ideals

Definition 1.1 *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F, its components $\mathcal{I}(E,F) := \mathcal{L}(E,F) \cap \mathcal{I}$ satisfy:*

(i) $\mathcal{I}(E,F)$ is a linear subspace of $\mathcal{L}(E,F)$ which contains all finite rank operators.

(ii) The ideal property: If $v \in \mathcal{L}(G,E)$, $u \in \mathcal{I}(E,F)$, and $w \in \mathcal{L}(F,H)$, then the composition $w \circ u \circ v$ belongs to $\mathcal{I}(G,H)$.

If there exists a function $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfying:

(i') $(\mathcal{I}(E,F), \|\cdot\|_{\mathcal{I}})$ is a Banach space for all Banach spaces E and F,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') $\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\|\|u\|_{\mathcal{I}}\|v\|$ for operators v, u, w as defined above,

then $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be closed if each $\mathcal{I}(E,F)$ is a closed subspace of $\mathcal{L}(E,F)$ under the standard supremum norm.

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal, whereas $\mathcal{L}$ is the largest one [14, Theorem 1.2.2].

1.2 The injective hull procedure for linear operator ideals

Definition 1.2 *Following ([11, page 421]) an ideal $\mathcal{I}$ is called injective if , for all Banach spaces E, F, G and every isometric embedding $J \in \mathcal{L}(F,G)$, we have that $T \in \mathcal{L}(E,F)$ belongs to $\mathcal{I}(E,F)$ whenever $J \circ T$ is in $\mathcal{I}(E,G)$.*

Definition 1.3 *The injective of an operator ideal denoted by $\mathcal{I}^{inj}$ is defined by*

$$\mathcal{I}^{inj}(E,F) := \left\{ S \in \mathcal{L}(E,F) ; J_F \circ S \in \mathcal{I}\left(E, \ell_{B_F^*}\right) \right\}.$$

$\mathcal{I}^{inj}$ is the smallest injective ideal .Consequently , $\mathcal{I}$ is injective iff $\mathcal{I}^{inj} = \mathcal{I}$.

The following Theorem characterize the injective hull of a linear operators Ideal:

Theorem 1.1 *Let $\mathcal{I}$ be an ideal of linear operators, E and F be two Banach spaces, then $T \in \mathcal{I}^{inj}(E,F)$ if and only if there exist a Banach space G and an operator*

$S \in \mathcal{I}(E,G)$ such that

$$\|T(x)\| \leqslant \|S(x)\|, \quad \forall x \in E.$$

1.3 Linear p-Summing Operators

Definition 1.4 *Let $1 \leq p < \infty$. A linear operator $T : E \longrightarrow F$ is said to be p-summing if there exists a constant $C \geq 0$ such that for every finite sequence $(x_i)_{1\leq i\leq n}$ in E,*

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*}\leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}}. \tag{1.1}$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p(T)$. The collection of all p-summing operators between E and F is denoted by $\Pi_p(E,F)$.

Theorem 1.2 *[9, Page 39] If $1 \leq p \leq q < \infty$, then $\Pi_p(E,F) \subset \Pi_q(E,F)$. Moreover, $\pi_q(T) \leq \pi_p(T)$ for every $T \in \Pi_p(E,F)$.*

Proposition 1.1 *The classes of p-summing operator ($1 \leq p < \infty$) is an injective linear operator ideals that is:*

$$T \in \Pi^p(E,F) \iff J \circ T \in \Pi^p(E,G)$$

for every isometric embedding $J \in \mathcal{L}(F,G)$. Moreover $(\Pi^p)^{inj} = \Pi^p$

The following basic result about p-summing operators is due to A. Pietsch. It characterizes p-summability by means of a domination theorem.

Theorem 1.3 *(Pietsch Domination Theorem) [9, page 44]*
Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E,F)$. Then T is p-summing if and only if there exist a constant C and a regular Borel probability measure μ on B_{E^*} (equipped with the weak-star topology) such that:

$$\|T(x)\| \leq C \left(\int_{B_{E^*}} |\langle x, x^*\rangle|^p d\mu(x^*)\right)^{\frac{1}{p}}, \quad \forall x \in E. \tag{1.2}$$

In this case, $\pi_p(T)$ is the infimum of all constants C for which (1.2) holds.

1.4 Ideals of Compact and Weakly Compact Linear Operators

We say that a bounded linear operator $T : E \to F$ is compact (or weakly compact) if $T(B_E)$ is relatively compact (respectively, relatively weakly compact) in F. We denote the sets of compact and weakly compact linear operators from E to F by $\mathcal{K}(E, F)$ and $\mathcal{W}(E, F)$, respectively.
Schauder's theorem regarding the compactness of the adjoint of a compact linear operator is well known and essential for our subsequent analysis.

Theorem 1.4 *Let $T \in \mathcal{L}(E, F)$. Then T is compact if and only if its adjoint T^* is compact.*

The weak counterpart to this theorem is due to Gantmacher:

Theorem 1.5 *Let $T \in \mathcal{L}(E, F)$. Then T is weakly compact if and only if T^* is weakly compact.*

Proposition 1.2 *([14]) The classes $\mathcal{K}$ and $\mathcal{W}$ constitute closed, injective Banach operator ideals under the standard operator norm.*

1.5 Lipschitz Spaces

In this section, we present the general concepts and principles of Lipschitz spaces. To do this, we begin by recalling fundamental definitions and properties related to metric spaces and Lipschitz functions.

1.6 Metric Spaces

A metric space consists of a set—whose elements are considered points—equipped with a function that measures the distance between any two points.

Definition 1.5 *Let X be an arbitrary set and $d : X \times X \longrightarrow \mathbb{R}_+$ be a mapping. We say that d is a metric (or distance) on X if it satisfies the following conditions for all $x, y, z \in X$:*

1. $d(x, y) = 0 \Leftrightarrow x = y$.
2. ***Symmetry:*** $d(x, y) = d(y, x)$.

3. ***Triangle Inequality:*** $d(x,y) \leq d(x,z) + d(z,y)$.

From these, the following properties inherently follow:

- ***Reverse triangle inequality:*** $|d(x,y) - d(x,z)| \leq d(y,z)$ *for all* $x,y,z \in X$.
- ***Generalized triangle inequality:*** $d(x_1,x_n) \leq \sum_{j=1}^{n-1} d(x_j,x_{j+1})$ *for all* $x_1,\ldots,x_n \in X$.

Definition 1.6 *A metric space is a pair* (X,d) *where* X *is a set and* d *is a metric on* X*.*

Example 1.1 *The set of real numbers equipped with the distance* $d(x,y) = |x-y|$ *is a metric space.*

Notation: Let (X,d_X,e) be a pointed metric space (i.e., e is a distinguished base point; we typically use 0 if X is a normed space).

1.7 Lipschitz Spaces $\mathrm{Lip}_0(X,Y)$

Definition 1.7 *The Lipschitz space* $\mathrm{Lip}_0(X,E)$ *is the Banach space of all Lipschitz mappings* T *from a pointed metric space* X *to a Banach space* E *that preserve the base point (i.e.,* $T(0)=0$*), equipped with the Lipschitz norm:*

$$\mathrm{Lip}(T) := \sup_{x \neq y} \frac{\|T(x) - T(y)\|}{d(x,y)}. \tag{1.3}$$

When $E = \mathbb{K}$*, we write* $X^\# = \mathrm{Lip}_0(X,\mathbb{K})$*.*

Definition 1.8 (Lipschitz Dual) *Let* X *be a pointed metric space. The Lipschitz dual, denoted by* $X^\#$*, is the Banach space of all Lipschitz scalar-valued forms on* X*. Specifically,* $X^\# = \mathrm{Lip}_0(X,\mathbb{R}) = \mathrm{Lip}_0(X)$*, equipped with the norm:*

$$\mathrm{Lip}(x^\#) = \sup_{x \neq y} \frac{|x^\#(x) - x^\#(y)|}{d_X(x,y)}.$$

1.8 The Predual of $\mathrm{Lip}_0(X)$

In this section, we introduce the predual of $\mathrm{Lip}_0(X)$. This means that there exists a Banach space Z such that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to Z^*.

Definition 1.9 *Let (X, d_X) be a metric space. A molecule on X is a finitely supported scalar function $m : X \to \mathbb{R}$ (where the support is $\sigma(m) = \{x \in X : m(x) \neq 0\}$) that satisfies:*

$$\sum_{x \in \sigma(m)} m(x) = 0.$$

We denote the space of all molecules on X by $M(X)$. We can express m as $m = \sum_{x\in\sigma(m)} m(x)1_{\{x\}}$, where $1_{\{x\}}$ denotes the characteristic (indicator) function of the singleton $\{x\}$.

Remark 1.1 *It can be shown that any $m \in M(X)$ can be written in the form:*

$$m = \sum_{i=1}^{n} \lambda_i (1_{\{x_i\}} - 1_{\{x_i'\}}),$$

though this representation is not unique.

Proposition 1.3 *Let (X, d_X) be a metric space. We equip the space of molecules $M(X)$ with the following norm:*

$$||m||_{M(X)} = \inf \left\{ \sum_{i=1}^{n} |\lambda_i| d_X(x_i, x_i') \;\middle|\; m = \sum_{i=1}^{n} \lambda_i (1_{\{x_i\}} - 1_{\{x_i'\}}) \right\}.$$

Let $E(X)$ denote the completion of the normed space $(M(X), ||.||_{M(X)})$. The space $E(X)$ is often called the Lipschitz-free space over X , and is also frequently denoted by $\mathcal{F}(X)$.

Theorem 1.6 *Let X be a pointed metric space (i.e., $X \in \mathcal{M}_0$). Then $(X)^* = \mathrm{Lip}_0(X)$.*

Corollary 1.1 *Let X be a pointed metric space.*

(1) For any molecule m, we have $||m||_{(X)} = \sup_{f \in B_{X^\#}} |\langle m, f\rangle|$.

(2) $||.||_E$ is a norm on the space of molecules that satisfies $\forall x, y \in X : ||1_{\{x\}} - 1_{\{y\}}||_{(X)} = d_X(x, y)$.

The Banach space Æ(X) possesses several remarkable universal properties.

Theorem 1.7 *([18, Theorem 2.2.4])*
Let X be a pointed metric space, E be a Banach space, and $T : X \longrightarrow E$ be a base-point preserving

Lipschitz map ($T(0) = 0$). There exists a unique bounded linear map $T_L : \text{Æ}(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$. In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "\delta_X"'] & & E \\ & \text{Æ}(X) \arrow[ur, "T_L"'] & \end{tikzcd}$$

Furthermore, $\|T_L\| = \mathrm{Lip}(T)$.

The operator T_L is referred to as the linearization of T. The correspondence:

$$T \longleftrightarrow T_L$$

establishes an isometric isomorphism between the vector spaces $\mathrm{Lip}_0(X, E)$ and $\mathcal{L}(\text{Æ}(X), E)$.

Chapter 2

The closed injective hull and the interpolative linear operator ideals

In this chapter, we present a fundamental result due to Jarchow concerning the closed injective hull of an operator ideal. This result provides a characterization of operators belonging to $\overline{\mathcal{I}}^{i}(E,F)$ in terms of a quantitative norm inequality.

2.1 The Closed Injective hull

The Closed Injective hull procedure $\mathcal{I} \mapsto \overline{\mathcal{I}}^{inj}$ was introduced by Jarchow in 1981. This procedure allow us to produce a new operator ideal from a given operators ideal.
The following theorem, due to Jarchow, characterizes the closed injective hull of an operator ideal via a norm inequality involving an arbitrary ε-perturbation.

Theorem 2.1 *Let $\mathcal{I}$ be complete ideal-quasinorm, then $T \in \mathcal{I}(E,F)$ belongs to $\overline{\mathcal{I}}^{inj}(E,F)$ if and only if, for each $\varepsilon > 0$, there exist a Banach space G and an operator $S \in \mathcal{I}(E,G)$ exist such that*

$$\|Tx\| \leqslant N(\varepsilon) \cdot \|Sx\| + \varepsilon \cdot \|x\|, \quad \forall \varepsilon > 0,\ \forall x \in E. \tag{ii}$$

Example 2.1 *$T \in \mathcal{L}(E,F)$ belongs to $\overline{\Pi_1}^{inj}(E,F)$ if and only if there exist a positive measure $\mu \in \mathcal{M}(B_E^*)$ and a map $N : \mathbb{R}_* \to \mathbb{R}_*$ such that*

$$\|Tx\| \leq N(\varepsilon) \cdot \left(\int_{B_{E^*}} |\langle a, x \rangle| \, d\mu(a) \right) + \varepsilon \cdot \|x\|, \quad \forall \varepsilon > 0, \forall x \in E.$$

Theorem 2.2 *Let $\mathcal{I}$ be an ideal and E and F Banach spaces. $T \in \mathcal{L}(E,F)$ belongs to $\overline{\mathcal{I}}^{i}(E,F)$ iff, for each $\varepsilon > 0$, a Banach space G_ε and an operator $S_\varepsilon \in \mathcal{I}(E,G_\varepsilon)$ exist such that*

$$\|Tx\| \leqslant \|S_\varepsilon x\| + \varepsilon \|x\|, \quad \forall \varepsilon > 0,\ \forall x \in E. \tag{i}$$

If $\mathcal{I}$ admits a complete ideal-quasinorm, then we may choose in addition $G_\varepsilon = G$ and $S_\varepsilon = N(\varepsilon) \cdot S$ for some $S \in \mathcal{I}(E, G)$ and some $N(\varepsilon) > 0$, i.e. we may replace (i) by

$$\|Tx\| \leqslant N(\varepsilon) \cdot \|Sx\| + \varepsilon \cdot \|x\|, \quad \forall \varepsilon > 0, \ \forall x \in E. \tag{ii}$$

Proof 2.1 *(a) The implication that (i) holds for every $T \in \overline{\mathcal{I}}^i(E, F)$ is straightforward. Conversely, assume that (i) is satisfied. Fix $\varepsilon > 0$ and consider the Banach space $H_\varepsilon := (G_\varepsilon \oplus E)_{\ell_1}$ (see 17.2.10). Define the linear operator $R_\varepsilon : E \to H_\varepsilon$ by $R_\varepsilon x := (S_\varepsilon x, \varepsilon x)$. Clearly, R_ε is linear, continuous, and injective. By virtue of (i), the map $A_\varepsilon : R(R_\varepsilon) \to F$ given by $A_\varepsilon(R_\varepsilon x) := Tx$ is well defined, linear, continuous, and satisfies $\|A_\varepsilon\| \leq 1$. Since F^∞ has the extension property, $J_F \circ A_\varepsilon$ extends to an operator $\widetilde{A}_\varepsilon \in \mathcal{L}(H_\varepsilon, F^\infty)$ with $\|\widetilde{A}_\varepsilon\| = \|A_\varepsilon\|$. Now define*

$$B_\varepsilon \in \mathcal{I}(E, F^\infty) \quad by \quad B_\varepsilon x := \widetilde{A}_\varepsilon(S_\varepsilon x, 0),$$

and

$$C_\varepsilon \in \mathcal{L}(E, F^\infty) \quad by \quad C_\varepsilon x := \widetilde{A}_\varepsilon(0, \varepsilon x),$$

for all $x \in E$. Then $J_F \circ T = B_\varepsilon + C_\varepsilon$, and hence

$$\|J_F \circ T - B_\varepsilon\| = \|C_\varepsilon\| \leq \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude that $T \in \overline{\mathcal{I}}^i(E, F)$.

(b) Now suppose additionally that $\mathcal{I}$ admits a complete ideal quasinorm α. We may assume that α is an ideal-r-norm for some $0 < r \leq 1$. From (i), there exist Banach spaces G_n and operators $S_n \in \mathcal{I}(E, G_n)$ such that for every $n \in \mathbb{N}$ and all $x \in E$,

$$\|Tx\| \leq \|S_n x\| + 2^{-n}\|x\|. \tag{iii}$$

Set $G := \left(\bigoplus_{n \in \mathbb{N}} G_n\right)_{\ell_1}$ and let $I_k : G_k \to G$ be the k-th canonical injection. Then (iii) implies

$$\|Tx\| \leq 2^{n/r}\alpha(S_n) \sum_{k=1}^{\infty} \left\| [2^{k/r}\alpha(S_k)]^{-1} I_k S_k x \right\| + 2^{-n}\|x\|, \tag{iv}$$

for all $n \in \mathbb{N}$ and $x \in E$. Indeed, note that

$$\sum_{k=1}^{\infty} \left\| [2^{k/r}\alpha(S_k)]^{-1} I_k S_k x \right\| \le \sum_{k=1}^{m} 2^{-k/r} \frac{\|S_k\|}{\alpha(S_k)} \|x\| < \|x\|,$$

for every $x \in E$ and $m \in \mathbb{N}$, so the series on the right-hand side of (iv) converges. Moreover, for all $i, j \in \mathbb{N}$,

$$\alpha\left(\sum_{k=i}^{i+j} [2^{k/r}\alpha(S_k)]^{r} I_k S_k\right) \le \sum_{k=i}^{i+j} 2^{-k},$$

which shows that the sequence $\left(\sum_{k=1}^{m} [2^{k/r}\alpha(S_k)]^{-1} I_k S_k\right)_{m\in\mathbb{N}}$ is Cauchy in $[\mathcal{I}(E,G),\alpha]$. By hypothesis, the limit

$$S := \sum_{k=1}^{\infty} [2^{k/r}\alpha(S_k)]^{-1} I_k S_k$$

exists in $[\mathcal{I}(E,G),\alpha]$. From (iv) and the definition of G, we obtain

$$\|Tx\| \le 2^{n/r}\alpha(S_n)\|Sx\| + 2^{-n}\|x\|, \quad \forall x \in E,\ \forall n \in \mathbb{N}. \tag{v}$$

Define $N(\varepsilon) := 2^{1/r}\alpha(S_1)$ for $\varepsilon > 1$, and $N(\varepsilon) := 2^{n/r}\alpha(S_n)$ for $2^{-n} < \varepsilon \le 2^{-n+1}$ with $n \in \mathbb{N}$. Then (v) immediately yields (ii). $\square$

We observe that $\overline{\mathcal{I}}^{i} = \mathcal{K}$ whenever $\mathcal{I} \subseteq \mathcal{K}$. Moreover, $\overline{\mathcal{I}}^{i} = \mathcal{I}$ holds for $\mathcal{I} = \mathcal{K}, \mathcal{W}, \mathcal{V}$, since these ideals are injective and closed. Of particular interest are the ideals $\mathcal{I} = \Pi_p$ with $1 < p < \infty$. From $\Pi_p \subset \mathcal{V} \cap \mathcal{W}$, we obtain the following.

Corollary 2.1 *For every $1 < p < \infty$,*

$$\mathcal{K} = (\overline{\Pi}_p^{i})^2 \subset \overline{\Pi}_p^{i} \subset \mathcal{V} \cap \mathcal{W}.$$

Corollary 2.2 *An operator $T \in \mathcal{L}(E,F)$ belongs to $\overline{\Pi}_p^{i}(E,F)$ if and only if there exist a positive measure $\mu \in \mathscr{M}(\mathbf{B}_{E'})$ and a function $N : \mathbb{R}_* \to \mathbb{R}_*$ such that*

$$\|Tx\| \le N(\varepsilon)\left(\int_{\mathbf{B}_{E'}} |\langle a, x\rangle|^p d\mu(a)\right)^{1/p} + \varepsilon\|x\|,$$

for every $\varepsilon > 0$ and every $x \in E$.

We will later see that $\overline{\mathscr{P}}_1^i = \overline{\mathscr{P}}_p^i$ *for all* $1 < p < \infty$. *This relies on the following central theorem, .*

Theorem 2.3 *Let K be any compact space and F any Banach space. Then*

$$\mathscr{W}(\mathscr{C}(K), F) = \overline{\Pi}_1^i(\mathscr{C}(K), F).$$

2.2 The interpolative for linear operator ideals

Now we talk about the interpolative procedure for operator ideal which introduced by Matter in 1987. We consider the inequality

$$\|Tx\| \leqslant N(\varepsilon) \cdot \|Sx\| + \varepsilon \cdot \|x\|, \quad \forall \varepsilon > 0,\ \forall x \in E. \tag{ii}$$

We set $N(\varepsilon) = \varepsilon^{-r}$ *for a fixed* $0 < \sigma < 1$ *and* $r = \frac{\sigma}{1-\sigma}$. *Let* f *the function from* $\mathbb{R}_+$ *to* $\mathbb{R}_+$ *defined by* $f(\varepsilon) = \varepsilon^{-r} \cdot \|Sx\| + \varepsilon \cdot \|x\|$,
We now construct the table of variation f.

ϵ	0		ϵ_1		$+\infty$
$f'(\epsilon)$		$-$	0	$+$	
$f(\epsilon)$	$+\infty$	$\searrow$	$f(\epsilon_1) = c\Vert Sx\Vert^{1-\sigma} \cdot \Vert x\Vert^{\sigma}$	$\nearrow$	$+\infty$

where c *is a constant depending only on* σ *with*

$$\left(c = \left(\frac{1-\sigma}{\sigma}\right)^{\sigma} + \left(\frac{\sigma}{1-\sigma}\right)^{1-\sigma} \right).$$

Then (ii) is equivalent to the "interpolation forimula"

$$\boxed{\|T(x)\| \leq f(\epsilon_1) = c\|Sx\|^{1-\sigma} \cdot \|x\|^{\sigma}, \forall x \in E},$$

where c *is a constant depending only on* σ.

Definition 2.1 *Let* $0 < \sigma < 1$. *An operator* $T : E \to F$ *belongs to* $\mathcal{I}_\sigma(E, F)$ *if there exist a Banach space* G *and an operator* $S \in \mathcal{I}(E, G)$ *such that*

$$\|Tx\| \leq \|Sx\|^{1-\sigma}\|x\|^\sigma \quad \forall x \in E.$$

For each $T \in (\mathcal{I})_\sigma(E, F)$, *we set*

$$\alpha_\sigma(T) := \inf \alpha(S)^{1-\sigma}$$

where the infimum is taken over all operators S *admitted above.*
Note that $\|T\| \leq \alpha_\sigma(T)$, *by definition.*

Throughout this section, $[\mathcal{I}, \alpha]$ *will denote a complete quasinormed operator ideal.*

Theorem 2.4 $[\mathcal{I}_\sigma, \alpha_\sigma]$ *is an injective complete quasinormed operator ideal. If* α *is even a norm, then the same is true for* α_σ.

Proof. *We verify only the triangle inequality. Let* $T_1, T_2 \in \mathcal{I}_\sigma(E, F)$. *Given* $\varepsilon > 0$, *there is a Banach space* G_i *and an operator* $S_i \in \mathcal{I}(E, G_i)$ *such that*

$$\|T_i x\| \leq \|\alpha(S_i)^{-\sigma} S_i x\|^{1-\sigma}\|\alpha(S_i)^{1-\sigma}x\|^\sigma \quad \forall x \in E$$

and $\alpha(S_i)^{1-\sigma} \leq (1+\varepsilon)\alpha_\sigma(T_i)$ $(i = 1, 2)$. *Introducing the* l_1*-sum* $G := G_1 \oplus G_2$ *and the operator* $S := \alpha(S_1)^{-\sigma}J_1S_1 + \alpha(S_2)^{-\sigma}J_2S_2 \in \mathcal{I}(E, G)$ *(*J_1, J_2 *the canonical injections) and applying HÖLDER's inequality, we get for all* $x \in E$

$$\begin{aligned}
\|(T_1 + T_2)x\| &\leq \|T_1x\| + \|T_2x\| \\
&\leq \sum_{i=1}^{2} \|\alpha(S_i)^{-\sigma}S_ix\|^{1-\sigma}\|\alpha(S_i)^{1-\sigma}x\|^\sigma \\
&\leq \left(\sum_{i=1}^{2} \|\alpha(S_i)^{-\sigma}S_ix\|\right)^{1-\sigma}\left(\sum_{i=1}^{2} \|\alpha(S_i)^{1-\sigma}x\|\right)^\sigma \\
&= (\alpha(S_1)^{1-\sigma} + \alpha(S_2)^{1-\sigma})\|Sx\|^{1-\sigma}\|x\|^\sigma.
\end{aligned}$$

Hence $T_1 + T_2 \in \mathcal{I}_\sigma(E, F)$ *and furthermore, if* χ *is the quasinorm constant of* α,

$$\alpha_\sigma(T_1 + T_2) \leq \chi^{1-\sigma}(\alpha(S_1)^{1-\sigma} + \alpha(S_2)^{1-\sigma})$$

$$\leq \chi^{1-\sigma}(1+\varepsilon)(\alpha_\sigma(T_1)+\alpha_\sigma(T_2)).$$

In particular, if α *is a norm, then so is* α_σ. ■

The ideal $\mathcal{I}_0$ *is just the injective hull of* $\mathcal{I}$ *and the quasinorms are equal. Further properties are given in*

Proposition 2.1 *Let* $0 \leq \sigma, \sigma_1, \sigma_2 < 1$. *Then the following holds:*

(a) $\mathcal{I}_{\sigma_1} \subset \mathcal{I}_{\sigma_2}$ *if* $\sigma_1 \leq \sigma_2$.

(b) $\mathcal{I}^{inj} \subset \mathcal{I}_\sigma \subset \overline{\mathcal{I}}^{inj}$.

(c) $(\mathcal{I}_{\sigma_1})_{\sigma_2} \subset \mathcal{I}_{\sigma_1+\sigma_2-\sigma_1\sigma_2}$.

Here all inclusion maps have norm 1.

Proof. To verify (a), let $T \in \mathcal{I}_{\sigma_1}(E,F)$ *and* $\varepsilon > 0$. *Then*

$$\|Tx\| \leq \|Sx\|^{1-\sigma_1}\|x\|^{\sigma_1} \quad \forall x \in E$$

holds for a suitable BANACH space G *and an operator* $S \in \mathcal{I}(E,G)$ *with* $\alpha(S)^{1-\sigma_1} \leq (1+\varepsilon)\alpha_{\sigma_1}(T)$. *Since*

$$\|Tx\| \leq \|S\|^{\sigma_2-\sigma_1}\|Sx\|^{1-\sigma_2}\|x\|^{\sigma_2} \quad \forall x \in E,$$

we get $T \in \mathcal{I}_{\sigma_2}(E,F)$ *and*

$$\alpha_{\sigma_2}(T) \leq \|S\|^{\sigma_2-\sigma_1}\alpha(S)^{1-\sigma_2 \leq \alpha(S)^{1-\sigma_1} \leq (1+\varepsilon)\alpha_{\sigma_1}(T)}.$$

(c) is straightforward whereas (b) is a simple consequence of the introductory remarks to this section. ■

Remark*. The concept introduced above can be generalized as follows. Given two complete quasi-normed operator ideals* $[\mathcal{I},\alpha]$ *and* $[\mathfrak{B},\beta]$, *the class of operators* $T: E \to F$ *satisfying*

$$\|Tx\| \leq \|S_0x\|^{1-\sigma}\|S_1x\|^{\sigma} \quad \forall x \in E \tag{3c}$$

for suitable BANACH *spaces* G_0, G_1 *and operators* $S_0 \in \mathcal{I}(E,G_0)$ *and* $S_1 \in \mathfrak{B}(E,G_1)$ *again constitutes an injective operator ideal* $(\mathcal{I},\mathfrak{B})_\sigma$. *A complete ideal quasinorm on* $(\mathcal{I},\mathfrak{B})_\sigma$ *is given by*

$$(\alpha,\beta)_\sigma(T) := \inf \alpha(S_0)^{1-\sigma}\beta(S_1)^{\sigma}$$

where the infimum is taken over all operators S_0, S_1 admitted in (3c).

2.3 (p,σ)-absolutely continuous operators

Throughout this section, let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. If we apply definition 3.1 to the ideal $[\Pi_p, \pi_p]$ of absolutely p-summing operators, we obtain the injective operator ideal

$$(\Pi_p)_\sigma$$

which is complete with respect to the ideal norm $(\pi_p)_\sigma$. The subsequent characterizations of $(\Pi_p)_\sigma$ are derived from the corresponding properties of Π_p. As usual, the dual unit ball B_{E^*} of a given Banach *space E is considered as a compact space with respect to the $\sigma(E^*, E)$-topology.*

Theorem 2.5 *For every operator $T : E \to F$, the following are equivalent:*

(i) $T \in (\Pi_p)_\sigma(E, F)$

(ii) There exist a constant $C > 0$ and a probability measure μ on B_{E^*} such that

$$\|Tx\| \leq C \left(\int_{B_{E^*}} (|\langle x, a\rangle|^{1-\sigma} \|x\|^\sigma)^{\frac{p}{1-\sigma}} d\mu(a) \right)^{\frac{1-\sigma}{p}} \quad \forall x \in E\,.$$

(iii) There exists a constant $C > 0$ such that for every finite sequence $x_1, \dots, x_n$ in E,

$$\left(\sum_{i=1}^n \|Tx_i\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq C \sup_{a \in B_{E^*}} \left(\sum_{i=1}^n (|\langle x_i, a\rangle|^{1-\sigma} \|x_i\|^\sigma)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} .$$

In addition, $(\pi_p)_\sigma(T)$ is the smallest number C for which, respectively, (ii) and (iii) hold.

The operators in $(\Pi_p)_\sigma$ will be called (p,σ)-absolutely continuous operators. For $\sigma = 0$, the above theorem reduces to the well-known properties of Π_p.

Proof. *The equivalence of (i) and (ii) is immediate from the definition of $(\Pi_p)_\sigma$ and* Pietsch*'s characterization of absolutely p-summing operators. If $T \in (\Pi_p)_\sigma(E, F)$, then $(\pi_p)_\sigma(T)$ is just the infimum over all constants C such that (ii) holds. The implication (ii) $\Rightarrow$ (iii) is obvious. As for the converse, exactly the same arguments (separation theorem,* Riesz *representation theorem)* *apply as the proof of [10], 19.6.1, provided the functional $f_M(a)$ given there is replaced by*

$$f_M(a) = C^{\frac{p}{1-\sigma}} \sum_{x\in M} (|\langle x,a\rangle|^{1-\sigma}\|x\|^{\sigma})^{\frac{p}{1-\sigma}} - \sum_{x\in M} \|Tx\|^{\frac{p}{1-\sigma}}\,.$$

Furthermore, the respective sets of constants C satisfying (ii) and (iii) are identical, and the minimum which exists in view of (iii) equals $(\pi_p)_\sigma(T)$. ■

We are going now to compare $(\Pi_p)_\sigma$ with other known ideals. By 3.3(b), we immediately get

$$\Pi_p \subset (\Pi_p)_\sigma \subset \mathcal{L}\,.$$

Here the left hand inclusion can be improved. As for the right hand inclusion, $\mathcal{L}$ may be substituted by the ideal $[\Pi_{q,p},\pi_{q,p}]$ of absolutely (q,p)-summing operators, for q suitably chosen.

Proposition 4.2. $\Pi_{\frac{p}{1-\sigma}} \subset (\Pi_p)_\sigma \subset \Pi_{\frac{p}{1-\sigma},p}$. *Both inclusion maps have norm* 1.

As will be seen in section 5, each of the above inclusions is strict and the index $\frac{p}{1-\sigma}$ can not be improved.

Proof. *Let $T\in\Pi_{\frac{p}{1-\sigma}}(E,F)$. Then there exists a probability measure μ on B_{E^*} such that*

$$\|Tx\| \le \pi_{\frac{p}{1-\sigma}}(T)\left(\int_{B_{E^*}} |\langle x,a\rangle|^{\frac{p}{1-\sigma}} d\mu(a)\right)^{\frac{1-\sigma}{p}} \quad \forall x\in E\,.$$

It follows that 4.1 (ii) holds with $C=\pi_{\frac{p}{1-\sigma}}(T)$. Thus $T\in(\Pi_p)_\sigma(E,F)$ and $(\pi_p)_\sigma(T)\le\pi_{\frac{p}{1-\sigma}}(T)$.

To verify the second inclusion, let $T\in(\Pi_p)_\sigma(E,F)$. Again by 4.1 we have

$$\left(\sum_{i=1}^n \|Tx_i\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \le (\pi_p)_\sigma(T) \sup_{a\in B_{E^*}} \left(\sum_{i=1}^n (|\langle x_i,a\rangle|^{1-\sigma}\|x_i\|^{\sigma})^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$

for all finite sequences $x_1,\dots,x_n$ in E. Since $\|x_k\|\le\varepsilon_p(x_i)$ for each $1\le k\le n$, where

$$\varepsilon_p(x_i) := \sup_{a\in B_{E^*}} \left(\sum_{i=1}^n |\langle x_i,a\rangle|^p\right)^{\frac{1}{p}},$$

we conclude

$$\left(\sum_{i=1}^n \|Tx_i\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \le (\pi_p)_\sigma(T)\cdot\varepsilon_p(x_i)^{\sigma}\cdot \sup_{a\in B_{E^*}} \left(\sum_{i=1}^n |\langle x_i,a\rangle|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$

$$= (\pi_p)_\sigma(T) \cdot \varepsilon_p(x_i)\,.$$

Hence $T \in \Pi_{\frac{p}{1-\sigma}}(E,F)$ *and* $\pi_{\frac{p}{1-\sigma}}(T) \leq (\pi_p)_\sigma(T)$. ■

Chapter 3

The closed injective hull and the interpolative Lipschitz operator ideals

In this chapter we present two procedures, the Lipschitz closed injective hull and interpolative Lipschitz ideals. These procedures aim to construct new Lipschitz operator ideals between pointed metric spaces and Banach spaces. The new ideals are characterized by specific criteria that determine whether a Lipschitz operator belongs to them, using summability properties and interpolation formulas. This chapter based on the article of Rachid Yahi et al [20].

3.1 Lipschitz operator ideals

In this section we present a useful criterion for a class ofter Lipschitz operators to form a Banach Lipschitz operator ideal. The Theorem 3.1, which is the Lipschitz analogue of a classical result in the linear theory, provides necessary and sufficient conditions in terms of completeness, ideal property, and a summability condition. Subsequently,

The notion of linear operator ideals is extended to Lipschitz version by Achour et al. in [5].

Definition 3.1 *A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components*

$$\mathcal{I}_{Lip}(X,E) := Lip_0(X,E) \cap \mathcal{I}_{Lip}$$

satisfy

1. *$\mathcal{I}_{Lip}(X,E)$ is a linear subspace of $Lip_0(X,E)$.*
2. *$vg \in \mathcal{I}_{Lip}(X,E)$ for $v \in E$ and $g \in X^{\#}$.*
3. *The ideal property: if $S \in Lip_0(Y,X)$, $T \in \mathcal{I}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then the composition wTS is in $\mathcal{I}_{Lip}(Y,F)$.*

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ *is a normed (Banach) Lipschitz operator ideal if there is* $\|.\|_{\mathcal{I}_{Lip}}$: $\mathcal{I}_{Lip} \longrightarrow [0,+\infty[$ *that satisfies*

1. *For every pointed metric space* X *and every Banach space* E*, the pair* $(\mathcal{I}_{Lip}(X,E), \|.\|_{\mathcal{I}_{Lip}})$ *is a normed (Banach) space and* $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ *for all* $T \in \mathcal{I}_{Lip}(X,E)$.
2. $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.
3. *If* $S \in Lip_0(Y,X)$, $T \in \mathcal{I}_{Lip}(X,E)$ *and* $w \in \mathcal{L}(E,F)$*, then* $\|wTS\|_{\mathcal{I}_{Lip}} \leq Lip(S)\,\|T\|_{\mathcal{I}_{Lip}}\,\|w\|$.

Now, we recall the composition method introduced in [5] to produce the Lipschitz operator ideal $\mathcal{I}\circ Lip_0$*, from a given linear operator ideal* $\mathcal{I}$*. The Lipschitz operator* $T : X \longrightarrow E$ *belongs to* $\mathcal{I}\circ Lip_0(X,E)$ *if there are a Banach space* G*, a Lipschitz operator* $S \in Lip_0(X,G)$*, and a linear operator* $u \in \mathcal{I}(G,E)$ *such that* $T = u \circ S$*. This definition is equivalent to* $T_L \in \mathcal{I}(Æ(X), E)$*. If* $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ *is a normed operator ideal, we write*

$$\|T\|_{\mathcal{I}\circ Lip_0} = \inf \|u\|_{\mathcal{I}}\, Lip(S) = \|T_L\|_{\mathcal{I}},$$

where the infimum is taken over all u, S *as described (see [5, Proposition 3.2]).*

Theorem 3.1 *The class* $\mathcal{I}_{Lip}$ *endowed with a map* $\|\cdot\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0,+\infty)$ *is a Banach Lipschitz operator ideal if and only if the following conditions are satisfied*

i) $Lip(\cdot) \leq \|\cdot\|_{\mathcal{I}_{Lip}}$ *and* $id_{\mathbb{K}} \in \mathcal{I}_{Lip}$ *with* $\|id_{\mathbb{K}}\|_{\mathcal{I}_{Lip}} = 1$.

ii) $(\mathcal{I}_{Lip}, Lip(\cdot))$ *enjoys the ideal property and the ideal inequality.*

iii) If $(T_n)_n \subset \mathcal{I}_{Lip}(X,E)$ *with* $\sum_{n\geq 1} \|T_n\|_{\mathcal{I}_{Lip}} < \infty$*, then*

$$T := \sum_{n\geq 1} T_n \in \mathcal{I}_{Lip}(X,E) \ \ \text{and}\ \|T\|_{\mathcal{I}_{Lip}} \leq \sum_{n\geq 1} \|T_n\|_{\mathcal{I}_{Lip}}.$$

Proof 3.1 *Firstly, we will show that conditions (i), (ii) and (iii) implies* $\mathcal{I}_{Lip}$ *is a Banach Lipschitz operator ideal. By the inequality* $Lip(\cdot) \leq \|\cdot\|_{\mathcal{I}_{Lip}}$*, if* $\|T\|_{\mathcal{I}_{Lip}} = 0$ *then* $T \equiv 0 \in \mathcal{I}_{Lip}$*. Also by (iii) we have* $S + T \in \mathcal{I}_{Lip}$ *for every* $S, T \in \mathcal{I}_{Lip}$*. For* $\alpha \in \mathbb{K}$,

$$\alpha T = u_\alpha \circ T \circ id_X \in \mathcal{I}_{Lip}(X,E),$$

where $u_\alpha \in \mathcal{L}(E,E)$ *given by* $u_\alpha(x) = \alpha x$. *Thus, we have shown that* $\mathcal{I}_{Lip}(X,E)$ *is a vector subspace of* $Lip_0(X,E)$. *The triangular inequality and the homogeneity of* $\|\cdot\|_{\mathcal{I}_{Lip}}$ *are derived directly from (iii). Now let* $0 \neq f \in X^{\#}$ *and* $e \in E$. *Then*

$$f \cdot e = u \circ id_{\mathbb{K}} \circ f \in \mathcal{I}_{Lip}(X,E),$$

where $u \in \mathcal{L}(\mathbb{K},E)$ *given by* $u(\alpha) = \alpha e$. *In order to prove that* $\left(\mathcal{I}_{Lip}(X,E), \|\cdot\|_{\mathcal{I}_{Lip}}\right)$ *is a Banach space, we will consider a Cauchy sequence* $(T_n)_{n\geq 1} \subset \mathcal{I}_{Lip}(X,E)$. *Then, it is possible to obtain a subsequence* $(T_{n_k})_{k\geq 1}$ *of* $(T_n)_{n\geq 1}$ *such that* $\left\|T_{n_k+1} - T_{n_k}\right\|_{\mathcal{I}_{Lip}} < \frac{1}{2^k}$, *which means that* $\sum_{k\geq 1}\left\|T_{n_k+1} - T_{n_k}\right\|_{\mathcal{I}_{Lip}} < \infty$. *By (iii) we have* $S_j = \sum\limits_{k\geq j}(T_{n_k+1} - T_{n_k})$ *belongs to* $\mathcal{I}_{Lip}(X_1, X_2; E)$ *and* $\left\|S_j\right\|_{\mathcal{I}_{Lip}} \leq \frac{1}{2^{j-1}}$ *for all* $j \geq 1$. *Now, note that* $S_j = (S_1 + T_{n_1}) - T_{n_j}$, *and then*

$$\lim_{j\longrightarrow+\infty}\left\|T_{n_j} - (S_1 + T_{n_1})\right\|_{\mathcal{I}_{Lip}} = \lim_{j\longrightarrow+\infty}\left\|S_j\right\|_{\mathcal{I}_{Lip}} \leq \lim_{j\longrightarrow+\infty}\frac{1}{2^{j-1}} = 0.$$

This shows that $\left(T_{n_j}\right)_{j\geq 1}$ *converges to* $S_1 + T_{n_1} \in \mathcal{I}_{Lip}(X,E)$ *with respect to the norm* $\|\cdot\|_{\mathcal{I}_{Lip}}$. *Which means that the Cauchy sequence* $(T_n)_{n\geq 1}$ *has a convergent subsequence* $\left(T_{n_j}\right)_{j\geq 1}$. *Then* $(T_n)_{n\geq 1}$ *is convergent in* $\left(\mathcal{I}_{Lip}(X,E), \|\cdot\|_{\mathcal{I}_{Lip}}\right)$.
Conversely, If $\left(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}}\right)$ *is a Banach Lipschitz operator ideal, then the conditions (i), (ii) are valid and (iii) is derived directly from the completeness of* $\left(\mathcal{I}_{Lip}(X,E), \|\cdot\|_{\mathcal{I}_{Lip}}\right)$.

3.2 The injective hull of Lipschitz operator ideals

In this section we characterize the injective hull of a Banach Lipschitz operator ideal via a domination inequality, and we give a quantitative description of its closed injective hull, we present the following result, which can be seen as a generalization of [16, Satz 4.1] to the Lipschitz case.

Theorem 3.2 *Let* $\mathcal{I}_{Lip}$ *be a Banach Lipschitz operator ideal. A Lipschitz operator* $T : X \longrightarrow E$ *belongs to* $\mathcal{I}_{Lip}^{inj}(X,E)$ *if and only if there are a Banach space* G *and a Lipschitz operator* $S \in \mathcal{I}_{Lip}(X,G)$ *such that*

$$\left\|\sum_{i=1}^{n} a_i\left(T(x_i) - T(x_i')\right)\right\| \leq \left\|\sum_{i=1}^{n} a_i\left(S(x_i) - S(x_i')\right)\right\|, \tag{3.1}$$

for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ *in* X *and* $(a_i)_{i=1}^n$ *in* $\mathbb{R}$*. Moreover,* $\|T\|_{\mathcal{I}_{Lip}^{inj}} = \inf \|S\|_{\mathcal{I}_{Lip}}$*, where the infimum is taken over all* G *and* S *in (3.1).*

Proof 3.2 *For the "if" part, suppose that inequality (3.1) holds. Consider the normed space* $G_0 \subset G$ *spanned by* $S(X)$ *and the linear operator* $R_0 : G_0 \longrightarrow E$ *given by* $R_0\left(\sum_{i=1}^n a_i S(x_i)\right) = \sum_{i=1}^n a_i T(x_i)$*. By using (3.1), we can check that* R_0 *is well-defined, continuous with* $\|R_0\| \leq 1$ *and then has a unique extension* R *to the completion* $\overline{G_0}$ *of* G_0*. The metric extension property of* $\ell_\infty(B_{E^*})$*, (see [?, Page 33]), guarantees the existence of a linear operator* $\widetilde{R} : G \longrightarrow \ell_\infty(B_{E^*})$*, satisfies that*

$$J_E \circ T = \widetilde{R} \circ S \in \mathcal{I}_{Lip}(X, \ell_\infty(B_{E^*})).$$

This means that $T \in \mathcal{I}_{Lip}^{inj}(X, E)$*. In addition, we have*

$$\|T\|_{\mathcal{I}_{Lip}^{inj}} \leq \left\|\widetilde{R}\right\| \|S\|_{\mathcal{I}_{Lip}} \leq \|S\|_{\mathcal{I}_{Lip}}.$$

To prove the "only if" part, it is enough to take $G = \ell_\infty(B_{E^*})$ *and* $S = J_E \circ T$*.*

The next result provides a characterization of the closed injective hull $\overline{\mathcal{I}_{Lip}}^{inj}$ *of a given ideal* $\mathcal{I}_{Lip}$ *of Lipschitz operators by means of a norm inequality.*
Recall that if $p \geq 1$ *and* $(E_i)_{i=1}^\infty$ *is a family of Banach spaces, then the* ℓ_p *direct sum* $\left(\bigoplus_{i=1}^\infty E_i\right)_{\ell_p}$ *is the space consists of all* $(x_i)_{i=1}^\infty \in \prod_{i=1}^\infty E_i$ *such that* $\sum_{i=1}^\infty \|x_i\|^p < \infty$*. This is a Banach space under the norm* $\left\|(x_i)_{i=1}^\infty\right\| = \left(\sum_{i=1}^\infty \|x_i\|^p\right)^{\frac{1}{p}}$ *(see [?, Page 35]).*

Theorem 3.3 *Let* $\mathcal{I}_{Lip}$ *be a Lipschitz operator ideal. The Lipschitz operator* $T : X \longrightarrow E$ *belongs to* $\overline{\mathcal{I}}_{Lip}^{inj}(X, E)$ *if and only if, for each* $\varepsilon > 0$ *there are a Banach space* G_ε *and* $S_\varepsilon \in \mathcal{I}_{Lip}(X, G_\varepsilon)$ *such that*

$$\left\|\sum_{i=1}^n a_i(T(x_i) - T(x_i'))\right\| \leq \left\|\sum_{i=1}^n a_i(S_\varepsilon(x_i) - S_\varepsilon(x_i'))\right\| + \varepsilon \sum_{i=1}^n |a_i| d(x_i, x_i') \tag{3.2}$$

for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ *in* X *and* $(a_i)_{i=1}^n$ *in* $\mathbb{R}$*.*

Proof 3.3 *Take* $T \in \overline{\mathcal{I}}_{Lip}^{inj}(X, E)$ *and* $\varepsilon > 0$*. Then* $J_E \circ T$ *is a limit of a sequence* $(S_m)_m \subset \mathcal{I}_{Lip}(X, \ell_\infty(B_{E^*})$ *with respect to the norm* $Lip(\cdot)$*. We can choose positive integer* m_ε *such that*

$Lip(J_E \circ T - S_{m_\varepsilon}) \leq \varepsilon$. On the other hand, for each $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$ we have

$$\begin{aligned}
&\left\| \sum_{i=1}^n a_i(T(x_i) - T(x_i')) \right\| \\
&= \left\| \sum_{i=1}^n a_i(J_E \circ T(x_i) - J_E \circ T(x_i')) \right\| \\
&\leq \left\| \sum_{i=1}^n a_i(S_{m_\varepsilon}(x_i) - S_{m_\varepsilon}(x_i')) \right\| \\
&\quad + \left\| \sum_{i=1}^n a_i \left((J_E \circ T - S_{m_\varepsilon})(x_i) - (J_E \circ T - S_{m_\varepsilon})(x_i') \right) \right\| \\
&\leq \left\| \sum_{i=1}^n a_i(S_{m_\varepsilon}(x_i) - S_{m_\varepsilon}(x_i')) \right\| + \varepsilon \sum_{i=1}^n |a_i| d(x_i, x_i').
\end{aligned}$$

Conversely, assume that (3.2) is holds for a given $\varepsilon > 0$ and a Banach space G_ε. By using the linearization of the Lipschitz operators T and S_ε we obtain $\|T_L(\mathfrak{m})\| \leq \|(S_\varepsilon)_L(\mathfrak{m})\| + \varepsilon \sum_{i=1}^n |a_i| d(x_i, x_i')$, for all molecule $\mathfrak{m} \in \mathcal{M}(X)$ with representation $\mathfrak{m} = \sum_{i=1}^n a_i \mathfrak{m}_{x_i, x_i'}$. Therefore,

$$\|T_L(\mathfrak{m})\| \leq \|(S_\varepsilon)_L(\mathfrak{m})\| + \varepsilon \|\mathfrak{m}\|_{Æ(X)}.$$

Consider $H_\varepsilon = (G_\varepsilon \oplus Æ(X))_{\ell_1}$. Define the injective linear operator $I_\varepsilon : Æ(X) \longrightarrow H_\varepsilon$ by $I_\varepsilon(\mathfrak{m}) := ((S_\varepsilon)_L, \varepsilon.m)$. According to the injectivity of I_ε, we can define a linear mapping $A_\varepsilon : R(I_\varepsilon) \longrightarrow E$ by $A_\varepsilon (I_\varepsilon(\mathfrak{m})) := T_L(\mathfrak{m})$. This mapping is continuous with $\|A_\varepsilon\| \leq 1$, indeed, for all $m \in \mathcal{M}(X)$ we have

$$\|A_\varepsilon(I_\varepsilon(\mathfrak{m}))\| \leq \|(S_\varepsilon)_L(\mathfrak{m})\| + \varepsilon \|\mathfrak{m}\|_{Æ(X)} = \|I_\varepsilon(\mathfrak{m})\|.$$

Using the metric extension property of $\ell_\infty(B_{E^*})$, we can extend $J_E \circ A_\varepsilon$ to the linear operator $\widetilde{A_\varepsilon} \in \mathcal{L}(H_\varepsilon, \ell_\infty(B_{E^*}))$ with $\left\|\widetilde{A_\varepsilon}\right\| \leq 1$.

$$\begin{tikzcd}
X \arrow[r, "\delta_X"] \arrow[dr, "S_\varepsilon"'] & Æ(X) \arrow[r, "T_L"] \arrow[d, "(S_\varepsilon)_L"] \arrow[dr, "I_\varepsilon"] & E \arrow[r, "J_E"] & \ell_\infty(B_{E^*}) \\
& G_\varepsilon \arrow[r, "j_\varepsilon"] & H_\varepsilon \arrow[ur, "\hat{A}_\varepsilon"] &
\end{tikzcd}$$

We put

$$B_\varepsilon := \widetilde{A_\varepsilon} \circ J_\varepsilon \circ S_\varepsilon \in \mathcal{I}_{Lip}(X, \ell_\infty(B_{E^*})),$$

where $J_\varepsilon : G_\varepsilon \to H_\varepsilon$ *is the linear operator defined by* $J_\varepsilon(t) := (t, o)$. *Define* $C_\varepsilon : X \longrightarrow \ell_\infty(B_{E^*})$ *by* $C_\varepsilon(x) := \widetilde{A_\varepsilon}(0, \varepsilon.\mathfrak{m}_{x,0})$. *Because of*

$$\|C_\varepsilon(x) - C_\varepsilon(x')\| \leq \left\|\widetilde{A_\varepsilon}\right\| \|(0, \varepsilon.\mathfrak{m}_{x,x'})\|_{H_\varepsilon} \leq \varepsilon d(x, x'),$$

for all $x, x' \in X$, *the mapping* C_ε *belongs to* $Lip_0(X, \ell_\infty(B_{E^*}))$ *with* $Lip(C_\varepsilon) \leq \varepsilon$. *Taking into account that* $B_\varepsilon(x) = \widetilde{A_\varepsilon}\left((S_\varepsilon)_L(\mathfrak{m}_{x,0}), 0\right)$ *for every* $x \in X$, *we obtain* $B_\varepsilon + C_\varepsilon = J_E \circ T$ *and then* $Lip(J_E \circ T - B_\varepsilon) = Lip(C_\varepsilon) \leq \varepsilon$. *Since this holds for all* $\varepsilon > 0$, *it follows that* $J_E \circ T \in \overline{\mathcal{I}}_{Lip}(X, E)$. *The proof is concluded.*

In the following result, we characterize the closed injective hull of a given Banach Lipschitz operator ideal $(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}})$, *which will be used in the sequel.*

Corollary 3.1 *Let* $(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}})$ *be a Banach Lipschitz operator ideal. The Lipschitz operator* $T : X \longrightarrow E$ *belongs to* $\overline{\mathcal{I}}_{Lip}^{inj}(X, E)$ *if and only if, for each* $\varepsilon > 0$ *there are a Banach space* G, *a Lipschitz operator* $S \in \mathcal{I}_{Lip}(X, G)$, *and a function* $N : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that*

$$\left\|\sum_{i=1}^{n} a_i(T(x_i) - T(x_i'))\right\| \leq N(\varepsilon) \left\|\sum_{i=1}^{n} a_i(S(x_i) - S(x_i'))\right\| + \varepsilon \sum_{i=1}^{n} |a_i| d(x_i, x_i'), \tag{3.3}$$

for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ *in* X, $\varepsilon > 0$ *and* $(a_i)_{i=1}^n$ *in* $\mathbb{R}$.

Proof 3.4 *It is enough to show that (3.3) is holds for* $T \in \overline{\mathcal{I}}_{Lip}^{inj}(X, E)$. *According to (3.2), for each* $m \in \mathbb{N}$ *there are a Banach space* G_m *and* $S_m \in \mathcal{I}_{Lip}(X, G_m)$ *such that*

$$\left\|\sum_{i=1}^{n} a_i(T(x_i) - T(x_i'))\right\| \leq \left\|\sum_{i=1}^{n} a_i(S_m(x_i) - S_m(x_i'))\right\| + 2^{-m} \sum_{i=1}^{n} |a_i| d(x_i, x_i') \tag{3.4}$$

for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ *in* X *and* $(a_i)_{i=1}^n$ *in* $\mathbb{R}$. *Putting* $G = \left(\bigoplus_{n \in \mathbb{N}} G_n\right)_{\ell_1}$ *and for each* k *consider the canonical inclusion* $I_k : G_k \longrightarrow G$, *then* $I_k \circ S_k \in \mathcal{I}_{Lip}(X, G)$ *and*

$$\sum_{k=1}^{\infty} \frac{1}{2^k \|S_k\|_{\mathcal{I}_{Lip}}} \|I_k \circ S_k\|_{\mathcal{I}_{Lip}} \leq \sum_{k=1}^{\infty} \frac{1}{2^k \|S_k\|_{\mathcal{I}_{Lip}}} \|I_k\| \|S_k\|_{\mathcal{I}_{Lip}} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty,$$

which means that the series $\sum_{k=1}^{\infty} 2^{-k} \|S_k\|_{\mathcal{I}_{Lip}}^{-1} I_k \circ S_k$ *converges in the Banach space* $\left(\mathcal{I}_{Lip}(X, G), \|\cdot\|_{\mathcal{I}_{Lip}}\right)$

to the Lipschitz operator

$$S = \sum_{k=1}^{\infty} \frac{1}{2^k \|S_k\|_{\mathcal{I}_{Lip}}} I_k \circ S_k \in \mathcal{I}_{Lip}(X, G).$$

It follows from $\|\cdot\| \leq \|\cdot\|_{\mathcal{I}_{Lip}}$ *that such series is pointwise convergent. The inequality (3.4) implies*

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} a_i (T(x_i) - T(x_i')) \right\| \\
&\leq 2^m \|S_m\|_{\mathcal{I}_{Lip}} \left\| \sum_{i=1}^{n} 2^{-m} \|S_m\|_{\mathcal{I}_{Lip}}^{-1} a_i \left(S_m(x_i) - S_m(x_i') \right) \right\| + 2^{-m} \sum_{i=1}^{n} |a_i| \, d(x_i, x_i') \\
&\leq 2^m \|S_m\|_{\mathcal{I}_{Lip}} \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} 2^{-k} \|S_k\|_{\mathcal{I}_{Lip}}^{-1} a_i \left(S_k(x_i) - S_k(x_i') \right) \right\| + 2^{-m} \sum_{i=1}^{n} |a_i| \, d(x_i, x_i') \\
&= 2^m \|S_m\|_{\mathcal{I}_{Lip}} \left\| \sum_{i=1}^{n} a_i \left(S(x_i) - S(x_i') \right) \right\| + 2^{-m} \sum_{i=1}^{n} |a_i| \, d(x_i, x_i').
\end{aligned}$$

Finally, for all $m \in \mathbb{N}$, *if we consider the function* $N : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *defined by* $N(\varepsilon) = 2 \|S_1\|_{\mathcal{I}_{Lip}}$ *for* $\varepsilon > 1$ *and* $N(\varepsilon) = 2^m \|S_m\|_{\mathcal{I}_{Lip}}$ *for* $2^{-m} < \varepsilon \leq 2^{-m+1}$, *we get the result.*

The next result describes the closed injective hull of a Banach Lipschitz operator ideal of composition type.

Proposition 3.1 *For a Banach linear operator ideal* $\mathcal{I}$, *we have*

$$\overline{\mathcal{I} \circ Lip_0}^{inj} = \overline{\mathcal{I}}^{inj} \circ Lip_0.$$

Proof 3.5 *Since* $\overline{\mathcal{I}}^{inj}$ *is closed then* $\overline{\mathcal{I}}^{inj} \circ Lip_0$ *is closed. On the other hand, for a given* T *in* $Lip_0(X, E)$, *suppose that* $J_E \circ T$ *belongs to* $\overline{\mathcal{I}}^{inj} \circ Lip_0(X, \ell_\infty(B_{E^*}))$. *The uniqueness of the linearization maps gives*

$$(J_E \circ T)_L = J_E \circ T_L \in \overline{\mathcal{I}}^{inj}(Æ(X), \ell_\infty(B_{E^*})).$$

Then, the injectivity of $\overline{\mathcal{I}}^{inj}$ *gives* $T_L \in \overline{\mathcal{I}}^{inj}(Æ(X), E)$, *so* T *is in* $\overline{\mathcal{I}}^{inj} \circ Lip_0(X, E)$, *shows that* $\overline{\mathcal{I}}^{inj} \circ Lip_0$ *is closed and injective Lipschitz operator ideal. Now, since* $\mathcal{I} \subset \overline{\mathcal{I}}^{inj}$, *we obtain*

$$\overline{\mathcal{I} \circ Lip_0}^{inj} \subset \overline{\overline{\mathcal{I}}^{inj} \circ Lip_0}^{inj} = \overline{\mathcal{I}}^{inj} \circ Lip_0.$$

For the converse inclusion, given T *in* $\overline{\mathcal{I}}^{inj} \circ Lip_0(X, E)$ *then there are a Banach space* G*, a Lipschitz operator* $S \in Lip_0(X, G)$ *and a linear operator* $u \in \overline{\mathcal{I}}^{inj}(G, E)$ *such that* $T = u \circ S$*. For* $\varepsilon > 0$*, choose* $v \in \mathcal{I}(G, \ell_\infty(B_{E^*}))$ *such that* $\|v - J_E \circ u\| \leq \dfrac{\varepsilon}{Lip(S)}$ *and* $v \circ S$ *belongs to* $\mathcal{I} \circ Lip_0(X, \ell_\infty(B_{E^*}))$*. We have*

$$Lip(v \circ S - J_E \circ T) \leq Lip(S)\|v - J_E \circ u\| \leq \varepsilon.$$

Therefore $J_E \circ T \in \overline{\mathcal{I} \circ Lip_0}(X, \ell_\infty(B_{E^*}))$*, which means that* $T \in \overline{\mathcal{I} \circ Lip_0}^{inj}(X, E))$*.*

The subsequent result provides several illustrations of the notion of a closed injective hull. Indeed, Jiménez-Vargas et al. [12] introduced $Lip_{0\mathcal{F}}$*,* $Lip_{0\mathcal{K}}$*, and* $Lip_{0\mathcal{W}}$*, the ideals of Lipschitz finite-dimensional rank, Lipschitz compact operators, and Lipschitz weakly compact operators, respectively, as an extension of a linear case.*

Proposition 3.2 *We have* $\overline{Lip_{0\mathcal{F}}}^{inj} = Lip_{0\mathcal{K}}$*.*

Proof 3.6 *Firstly, note that the ideals* $\mathcal{K}$ *and* $\mathcal{W}$ *of linear compact and linear weakly compact operators are injective. From [5, Proposition 3.6] and [1, Proposition 2.4], we get the injectivity of* $Lip_{0\mathcal{K}}$ *and* $Lip_{0\mathcal{W}}$*, indeed*

$$(Lip_{0\mathcal{K}})^{inj} = (\mathcal{K} \circ Lip_0)^{inj} = \mathcal{K}^{inj} \circ Lip_0 = Lip_{0\mathcal{K}},$$
$$(Lip_{0\mathcal{W}})^{inj} = (\mathcal{W} \circ Lip_0)^{inj} = \mathcal{W}^{inj} \circ Lip_0 = Lip_{0\mathcal{W}}.$$

Now, since $\ell_\infty(B_{E^*})$ *has the approximation property, then it follows from [5, Proposition 2.8] that* $J_E \circ T \in \overline{Lip_{0\mathcal{F}}}(X, \ell_\infty(B_{E^*}))$ *if and only if* $J_E \circ T \in Lip_{0\mathcal{K}}(X, \ell_\infty(B_{E^*}))$*. This is equivalent to the fact that the operator* T *belongs to* $Lip_{0\mathcal{K}}(X, E)$*.*

Taking into account that $Lip_{0\mathcal{K}}, Lip_{0\mathcal{W}}$ *are closed ([5, Corollary 3.3 and Proposition 3.6]), we obtain the following.*

Proposition 3.3 *We have* $\overline{Lip_{0\mathcal{K}}}^{inj} = Lip_{0\mathcal{K}}$ *and* $\overline{Lip_{0\mathcal{W}}}^{inj} = Lip_{0\mathcal{W}}$*.*

3.3 The interpolative Lipschitz operator ideal

In this section, we present another method for generating a new Lipschitz operators ideals from a given Lipschitz operators ideals, which is called the interpolative method. From (3.3), by taking

$N(\varepsilon) = \varepsilon^{-r}$ for a constant $1 \leq r < +\infty$ and computing the minimum of the right-hand side with respect to ε, which demonstrates that condition (3.3) is equivalent with the following inequality

$$\left\| \sum_{i=1}^{n} a_i (T(x_i) - T(x_i')) \right\| \leq$$

$$\leq \left(r^{-\frac{r}{r+1}} + r^{\frac{1}{r+1}} \right) \left\| \sum_{i=1}^{n} a_i (S(x_i) - S(x_i')) \right\|^{\frac{1}{r+1}} \left(\sum_{i=1}^{n} |a_i| d(x_i, x_i') \right)^{\frac{r}{r+1}}.$$

Definition 3.2 *Let $\mathcal{I}_{Lip}$ be a Lipschitz operators ideals and $0 \leq \sigma < 1$. A Lipschitz operator $T \in Lip_0(X, E)$ belongs to $\left(\mathcal{I}_{Lip}\right)_\sigma (X, E)$ if there are Banach space G and a Lipschitz operator $S \in \mathcal{I}_{Lip}(X, G)$ such that*

$$\left\| \sum_{i=1}^{n} a_i (T(x_i) - T(x_i')) \right\| \leq \left\| \sum_{i=1}^{n} a_i (S(x_i) - S(x_i')) \right\|^{1-\sigma} \left(\sum_{i=1}^{n} |a_i| d(x_i, x_i') \right)^{\sigma}, \tag{3.5}$$

for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X and $(a_i)_{i=1}^n$ in $\mathbb{R}$. For each $T \in \left(\mathcal{I}_{Lip}\right)_\sigma (X, E)$ denoting

$$\|T\|_{\left(\mathcal{I}_{Lip}\right)_\sigma} := \inf \|S\|_{\mathcal{I}_{Lip}}^{1-\sigma},$$

where the infimum is taken over all Banach spaces G and operators $S \in \mathcal{I}_{Lip}(X, G)$ working in the above inequality.

Note that if we take $n = 1$ in (3.5) we get $Lip(\cdot) \leq \|\cdot\|_{\left(\mathcal{I}_{Lip}\right)_\sigma}$.

Proposition 3.4 *If $(\mathcal{I}_{Lip}, \|\cdot\|_{\mathcal{I}_{Lip}})$ is an injective Banach Lipschitz operator ideal then $((\mathcal{I}_{Lip})_\sigma, \|\cdot\|_{(\mathcal{I}_{Lip})_\sigma})$ is injective Banach Lipschitz operator ideal.*

Proof 3.7 *We check the conditions (ii) and (iii) in Theorem 3.1.*
(ii) Let $R \in Lip_0(Y, X)$, $T \in \left(\mathcal{I}_{Lip}\right)_\sigma (X, E)$ and $w \in \mathcal{L}(E, F)$. For all $\varepsilon > 0$ there are a Banach space G and $S \in \mathcal{I}_{Lip}(X, G)$ satisfies (3.5) and $\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma} \leq (1 + \varepsilon) \|T\|_{\left(\mathcal{I}_{Lip}\right)_\sigma}$. For every $(y_i)_{i=1}^n, (y_i')_{i=1}^n$ in Y and $(a_i)_{i=1}^n$ in $\mathbb{R}$,

$$\left\| \sum_{i=1}^{n} a_i (w \circ T \circ R(y_i) - w \circ T \circ R(y_i')) \right\| \leq$$

$$\leq \|w\| \, Lip(R)^{\sigma} \left\| \sum_{i=1}^{n} a_i (S \circ R(y_i)) - S \circ R(y'_i)) \right\|^{1-\sigma} \left(\sum_{i=1}^{n} |a_i| d(y_i, y'_i) \right)^{\sigma} .$$

Which means that $w \circ T \circ R$ *is in* $(\mathcal{I}_{Lip})_{\sigma}(Y, F)$ *and*

$$\begin{aligned} \|w \circ T \circ R\|_{(\mathcal{I}_{Lip})_{\sigma}} &\leq \|w\| \, Lip(R)^{\sigma} \, \|S \circ R\|_{\mathcal{I}_{Lip}}^{1-\sigma} \\ &\leq \|w\| \, Lip(R)(1+\varepsilon) \|T\|_{(\mathcal{I}_{Lip})_{\sigma}}, \end{aligned}$$

as ε *is arbitrary we obtain (ii).*
(iii) Let $(T_n)_n$ *be a sequence in* $(\mathcal{I}_{Lip})_{\sigma}(X, E)$ *such that* $\sum\limits_{n=1}^{\infty} \|T_n\|_{(\mathcal{I}_{Lip})_{\sigma}} < \infty$. *Since* $Lip(\cdot) \leq \| \cdot \|_{(\mathcal{I}_{Lip})_{\sigma}}$ *and* $Lip_0(X, E)$ *is a Banach space, there exists* $T = \sum\limits_{n=1}^{\infty} T_n \in Lip_0(X, E)$. *For each* $k \in \mathbb{N}$ *and all* $\varepsilon > 0$ *choose Banach space* G_k *and* $S_k \in \mathcal{I}_{Lip}(X, G_k)$ *satisfies (3.5) and* $\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} \leq \|T_k\|_{(\mathcal{I}_{Lip})_{\sigma}} + \varepsilon/2^k$. *Hence*

$$\sum_{k=1}^{\infty} \|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} \leq \sum_{k=1}^{\infty} \|T_k\|_{(\mathcal{I}_{Lip})_{\sigma}} + \frac{\varepsilon}{2^k} < \infty.$$

Introducing $G = \left(\bigoplus\limits_{n \in \mathbb{N}} G_n \right)_{\ell_1}$ *and the k-th canonical injection* $I_k : G_k \longrightarrow G$. *Then the series* $\sum_{k=1}^{\infty} \|S_k\|_{\mathcal{I}_{Lip}}^{-\sigma} I_k S_k$ *converge to* $S \in \mathcal{I}_{Lip}(X, G)$. *Now select* $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ *in* X *and* $(a_i)_{i=1}^n$ *in* $\mathbb{R}$. *An application of Hölder inequality reveals that*

$$\begin{aligned} \left\| \sum_{i=1}^{n} a_i (T(x_i) - T(x'_i)) \right\| &\leq \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} a_i (T_k(x_i) - T_k(x'_i)) \right\| \\ &\leq \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} a_i \|S_k\|_{\mathcal{I}_{Lip}}^{-\sigma} (S_k(x_i) - S_k(x'_i)) \right\|^{1-\sigma} \\ &\quad \times \left(\sum_{i=1}^{n} |a_i| \|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} d(x_i, x'_i) \right)^{\sigma} \\ &\leq \left(\sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} a_i \|S_k\|_{\mathcal{I}_{Lip}}^{-\sigma} (S_k(x_i) - S_k(x'_i)) \right\| \right)^{1-\sigma} \\ &\quad \times \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} |a_i| \|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma} d(x_i, x'_i) \right) \right)^{\sigma} \end{aligned} \tag{3.6}$$

We have to estimate the latter expression of the right-hand side of (3.6), we get

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\left\|\sum_{i=1}^{n}a_i\|S_k\|_{\mathcal{I}_{Lip}}^{-\sigma}(S_k(x_i)-S_k(x_i'))\right\|\right)^{1-\sigma} &\times\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n}|a_i|\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}d(x_i,x_i')\right)\right)^{\sigma}\\
&\leq\left\|\sum_{i=1}^{n}a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma}\\
&\times\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n}|a_i|\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}d(x_i,x_i')\right)\right)^{\sigma}\\
&\leq\left(\sum_{k=1}^{\infty}\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}\right)^{\sigma}\left\|\sum_{i=1}^{n}a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma}\\
&\quad\times\left(\sum_{i=1}^{n}|a_i|d(x_i,x_i')\right)^{\sigma}.\end{aligned}$$

Then

$$\left\|\sum_{i=1}^{n}a_i(T(x_i)-T(x_i'))\right\|\leq\left(\sum_{k=1}^{\infty}\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}\right)^{\sigma}\left\|\sum_{i=1}^{n}a_i(S(x_i)-S(x_i'))\right\|^{1-\sigma}\times\left(\sum_{i=1}^{n}|a_i|d(x_i,x_i')\right)^{\sigma}.$$

So $\sum\limits_{n=1}^{\infty}T_n$ *converge to* $T=\sum\limits_{n=1}^{\infty}T_n\in(\mathcal{I}_{Lip})_{\sigma}(X,E)$, *moreover*

$$\begin{aligned}\|T\|_{(\mathcal{I}_{Lip})_{\sigma}}&\leq\left(\sum_{k=1}^{\infty}\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}\right)^{\sigma}\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma}\leq\sum_{k=1}^{\infty}\|S_k\|_{\mathcal{I}_{Lip}}^{1-\sigma}\\
&\leq\sum_{k=1}^{\infty}\|T_k\|_{(\mathcal{I}_{Lip})_{\sigma}}+\frac{\varepsilon}{2^k}=\sum_{k=1}^{\infty}\|T_k\|_{(\mathcal{I}_{Lip})_{\sigma}}+\varepsilon,\end{aligned}$$

and the proof follows.

Next, we show that the inclusion between a pair of Lipschitz operator ideals of the class with different parameters behaves as expected.

Proposition 3.5 *Let* $0\leq\sigma_1,\sigma_2<1$ *such that* $\sigma_1\leq\sigma_2$ *and let* $\mathcal{I}_{Lip}$ *be a normed Lipschitz operator ideal. Then* $(\mathcal{I}_{Lip})_{\sigma_1}\subset(\mathcal{I}_{Lip})_{\sigma_2}$ *with* $\|\cdot\|_{(\mathcal{I}_{Lip})_{\sigma_2}}\leq\|\cdot\|_{(\mathcal{I}_{Lip})_{\sigma_1}}$.

Proof 3.8 *Let* $T\in(\mathcal{I}_{Lip})_{\sigma_1}(X,E)$ *and* $\varepsilon>0$. *Choose a Banach space* G *and* $S\in\mathcal{I}_{Lip}(X,G)$ *such* *that* $\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma_1} \leq (1+\varepsilon)\|T\|_{(\mathcal{I}_{Lip})_{\sigma_1}}$. *For all* $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ *in* X *and* $(a_i)_{i=1}^n$ *in* $\mathbb{R}$, *we have*

$$\begin{aligned}
&\left\|\sum_{i=1}^n a_i(T(x_i) - T(x_i'))\right\| \\
&\leq \left\|\sum_{i=1}^n a_i(S(x_i) - S(x_i'))\right\|^{1-\sigma_1} \left(\sum_{i=1}^n |a_i| d(x_i, x_i')\right)^{\sigma_1} \\
&\leq Lip(S)^{\sigma_2-\sigma_1} \left\|\sum_{i=1}^n a_i(S(x_i) - S(x_i'))\right\|^{1-\sigma_2} \left(\sum_{i=1}^n |a_i| d(x_i, x_i')\right)^{\sigma_2}.
\end{aligned}$$

Hence $T \in (\mathcal{I}_{Lip})_{\sigma_2}(X, E)$ *and*

$$\|T\|_{(\mathcal{I}_{Lip})_{\sigma_2}} \leq Lip(S)^{\sigma_2-\sigma_1}\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma_2} \leq \|S\|_{\mathcal{I}_{Lip}}^{\sigma_2-\sigma_1}\|S\|_{\mathcal{I}_{Lip}}^{1-\sigma_2} \leq (1+\varepsilon)\|T\|_{(\mathcal{I}_{Lip})_{\sigma_1}}.$$

The result follows.

Let us establish the relationship between the Lipschitz injective hull, the Lipschitz closed injective hull, and the interpolative Lipschitz operator ideals.

Proposition 3.6 *If* $\mathcal{I}_{Lip}$ *is a normed Lipschitz operator ideal, then*

$$\mathcal{I}_{Lip}^{inj} \subset (\mathcal{I}_{Lip})_{\sigma} \subset \overline{\mathcal{I}_{Lip}}^{inj}.$$

In addition, $\|\cdot\|_{\overline{\mathcal{I}_{Lip}}^{inj}} \leq \|\cdot\|_{(\mathcal{I}_{Lip})_{\sigma}} \leq \|\cdot\|_{\mathcal{I}_{Lip}^{inj}}$.

Proof 3.9 *For a pointed metric space* X *and a Banach space* E *take* $T \in \mathcal{I}_{Lip}^{inj}(X, E)$ *and fix* $\varepsilon > 0$. *According to Theorem 3.2 for all* $\varepsilon > 0$ *choose Banach space* G *and* $S \in \mathcal{I}_{Lip}(X, G)$ *such that* $\|S\|_{\mathcal{I}_{Lip}} \leq (1+\varepsilon)\|T\|_{\mathcal{I}_{Lip}^{inj}}$ *and for every* $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ *in* X *and* $(a_i)_{i=1}^n$ *in* $\mathbb{R}$, *we have*

$$\begin{aligned}
&\left\|\sum_{i=1}^n a_i(T(x_i) - T(x_i'))\right\| \\
&\leq Lip(S)^{\sigma} \left\|\sum_{i=1}^n a_i(S(x_i) - S(x_i'))\right\|^{1-\sigma} \left(\sum_{i=1}^n |a_i| d(x_i, x_i')\right)^{\sigma}
\end{aligned}$$

from which we deduce that $T \in (\mathcal{I}_{Lip})_\sigma(X, E)$ *with*

$$\|T\|_{(\mathcal{I}_{Lip})_\sigma} \leq Lip(S)^\sigma \|S\|^{1-\sigma}_{\mathcal{I}_{Lip}} \leq (1+\varepsilon)\|T\|_{\mathcal{I}^{inj}_{Lip}}.$$

The second inclusion is a straightforward consequence derived from the introductory statements in this section.

The following outcome provides the interpolative Lipschitz operator ideal of a normed Lipschitz operator ideal of composition type.

Proposition 3.7 *Let* $\mathcal{I}$ *be a normed linear operator ideal and* $0 \leq \sigma < 1$*. Then*

$$(\mathcal{I} \circ Lip_0)_\sigma = (\mathcal{I})_\sigma \circ Lip_0. \tag{3.7}$$

Moreover, for any pointed metric spaces X *and a Banach space* E *we have* $\|T\|_{(\mathcal{I})_\sigma \circ Lip_0} = \|T\|_{(\mathcal{I} \circ Lip_0)_\sigma}$.

Proof 3.10 *Take* T *in* $(\mathcal{I} \circ Lip_0)_\sigma(X, E)$, $\mathfrak{m} \in \mathcal{M}(X)$ *and fix* $\varepsilon > 0$*. Choose a representation* $\sum_{i=1}^n a_i \mathfrak{m}_{x_i x'_i}$ *of* m*, where* $x_i, x'_i \in X$*, and* $a_i \in \mathbb{R}$*, such that* $\left(\sum_{i=1}^n |a_i| d(x_i, x'_i)\right) \leq (1+\varepsilon)\|\mathfrak{m}\|$*. Then there are a Banach space* G *and* $S \in \mathcal{I} \circ Lip_0(X, G)$ *with* $\|S\|^{1-\sigma}_{\mathcal{I}\circ Lip_0} \leq (1+\varepsilon)\|T\|_{(\mathcal{I}\circ Lip_0)_\sigma}$ *and*

$$\begin{aligned}
\|T_L(\mathfrak{m})\| &= \left\| \sum_{i=1}^n a_i (T(x_i) - T(x'_i)) \right\| \\
&\leq \left\| \sum_{i=1}^n a_i (S(x_i) - S(x'_i)) \right\|^{1-\sigma} \left(\sum_{i=1}^n |a_i| d(x_i, x'_i) \right)^\sigma \\
&\leq \|S_L(\mathfrak{m})\|^{1-\sigma} (1+\varepsilon) \|\mathfrak{m}\|^\sigma.
\end{aligned}$$

Since $S_L \in \mathcal{I}(Æ(X), G)$ *and the above inequality is holds for every* $\varepsilon > 0$*, we get* $T_L \in (\mathcal{I})_\sigma(Æ(X), E)$ *with*

$$\|T_L\|_{(\mathcal{I})_\sigma} \leq \|S_L\|^{1-\sigma}_{\mathcal{I}} = \|S\|^{1-\sigma}_{\mathcal{I}\circ Lip_0} \leq (1+\varepsilon)\|T\|_{(\mathcal{I}\circ Lip_0)_\sigma}.$$

Which means that T *belongs to* $(\mathcal{I})_\sigma \circ Lip_0(X, E)$ *and* $\|T\|_{(\mathcal{I})_\sigma \circ Lip_0} \leq \|T\|_{(\mathcal{I}\circ Lip_0)_\sigma}$*. Conversely, if* $T \in (\mathcal{I})_\sigma \circ Lip_0(X, E)$ *then for a fix* $\varepsilon > 0$ *there are a Banach space* F*, a Lipschitz operator* $w \in Lip_0(X, F)$ *and* $u \in (\mathcal{I})_\sigma(F, E)$ *such that* $T = u \circ w$ *with* $Lip(w)\|u\|_{(\mathcal{I})_\sigma} \leq (1+\varepsilon)\|T\|_{(\mathcal{I})_\sigma \circ Lip_0}$*. The definition of* $(\mathcal{I})_\sigma$ *assures the existence of a Banach space* G *and* $v \in \mathcal{I}(F, G)$ *such that*

$\|v\|_{\mathcal{I}}^{1-\sigma} \leq (1+\varepsilon)\|u\|_{(\mathcal{I})_\sigma}$. *We have*

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} a_i (T(x_i) - T(x_i')) \right\| \\
&= \left\| u \left(\sum_{i=1}^{n} a_i (w(x_i) - w(x_i')) \right) \right\| \\
&\leq Lip(w)^{\sigma} \left\| \sum_{i=1}^{n} a_i (v \circ w(x_i) - v \circ w(x_i')) \right\|^{1-\sigma} \left(\sum_{i=1}^{n} |a_i| d(x_i, x_i') \right)^{\sigma},
\end{aligned}$$

for all $x_i, x_i' \in X$, *and* $a_i \in \mathbb{R}, (1 \leq i \leq n)$. *Since* $v \circ w$ *is in* $\mathcal{I} \circ Lip_0(X, G)$, *the Lipschitz operator* T *belongs to* $(\mathcal{I} \circ Lip_0)_\sigma (X, E)$ *and*

$$\begin{aligned}
\|T\|_{(\mathcal{I} \circ Lip_0)_\sigma} &\leq Lip(w)^{\sigma} \|v \circ w\|_{\mathcal{I} \circ Lip_0}^{1-\sigma} \\
&\leq Lip(w) \|v\|_{\mathcal{I}}^{1-\sigma} \leq Lip(w)(1+\varepsilon)\|u\|_{(\mathcal{I})_\sigma} \\
&\leq (1+\varepsilon)^2 \|T\|_{(\mathcal{I})_\sigma \circ Lip_0}.
\end{aligned}$$

Then it follows that $\|T\|_{(\mathcal{I} \circ Lip_0)_\sigma} \leq \|T\|_{(\mathcal{I})_\sigma \circ Lip_0}$.

By the following example, let us complete the panorama of summability for Lipschitz opeartors defined between pointed metric spaces and Banach spaces.

Example 3.1 *Let* $(\Pi_p, \|\cdot\|_{\Pi_p})$ *be the famous Banach operator ideal of* p*-summing linear operators* $(p \geq 1)$ *which was introduced by Pietsch in [15]. When we utilize the composition method for* Π_p, *and subsequently apply the interpolative Lipschitz ideal procedure for the Banach Lipschitz operator ideal* $\Pi_p \circ Lip_0$, *we acquire the Banach Lipschitz operator ideal* $(\Pi_p)_\sigma \circ Lip_0$. *Note that* $(\Pi_p)_\sigma$ *is exactly the Banach ideal* $\Pi_{p,\sigma}$ *of* (p,σ)*-absolutely continuous operators building from* Π_p *by the interpolative ideal procedure (see [13, Section 4]).*

Taking into consideration that $T \in \left(\Pi_p \circ Lip_0\right)_\sigma (X, E)$ *if and only if* T_L *belongs to* $\Pi_{p,\sigma}(Æ(X), E)$ *and by using [13, Theorem 4.1], we characterize the mappings that belongs to* $\left(\Pi_p \circ Lip_0\right)_\sigma$ *by means of integral domination inequality and summability inequality.*
Note that the unit ball $B_{X^\#}$ *of* $X^\#$ *is a compact Hausdorff space in the topology of pointwise convergence on* X.

Theorem 3.4 *Let* $1 \leq p < \infty$, $0 \leq \sigma < 1$. *The following are equivalent for a Lipschitz operator* $T \in Lip_0(X, E)$.

(a)(a)(a)(a)

1. $T \in \left(\Pi_p \circ Lip_0\right)_\sigma (X, E)$.

2. There exist a constant $C > 0$ and a regular Borel probability measure μ on $B_{X^\#}$ such that

$$\left\| \sum_{i=1}^{n} a_i (T(x_i) - T(x_i')) \right\| \leq$$
$$C \left(\int_{B_{X^\#}} \left(\left| \sum_{i=1}^{n} a_i \left(f(x_i) - f(x_i') \right) \right|^{1-\sigma} \left(\sum_{i=1}^{n} |a_i| d(x_i, x_i') \right)^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu\,(f) \right)^{\frac{1-\sigma}{p}},$$

for any $x_i, x_i' \in$ X, $a_i \in \mathbb{R}$ $(1 \leq i \leq n)$.

3. There is a constant $C > 0$ such that for any $x_i^j, x_i'^j$ in X and all a_i^j in $\mathbb{R}$ $(1 \leq i \leq n$, $1 \leq j \leq s)$, we have

$$\left(\sum_{j=1}^{s} \left\| \sum_{i=1}^{n} a_i^j (T(x_i^j) - T(x_i^{j\prime})) \right\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}$$
$$\leq C \sup_{f \in B_{X^\#}} \left(\sum_{j=1}^{s} \left(\left| \sum_{i=1}^{n} a_i^j (f(x_i^j) - f(x_i'^j)) \right|^{1-\sigma} \left(\sum_{i=1}^{n} |a_i^j| d(x_i^j, x_i'^j) \right)^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$

Furthermore, the infimum of the constants $C > 0$ *in* (b) *and* (c) *is equal to* $\|T\|_{\left(\Pi_p \circ Lip_0\right)_\sigma}$.

Remark 3.1 *Note that if* T *belongs to* $\left(\Pi_p \circ Lip_0\right)_\sigma (X, E)$, *then setting* $n = 1$ *in* (c) *we have* $T \in \Pi_{p,\sigma}^L(X, E)$, *the space of* (p, σ)*-absolutely Lipschitz operators was introduced by Achour et al. in [4]. It is currently unknown whether this latter class satisfies the linearization theorem. However, based on the above, the class* $\left(\Pi_p \circ Lip_0\right)_\sigma$ *indeed fulfills it. This means that this class preserves nearly all the fundamental properties of the linear theory of* (p, σ)*-absolutely continuous operators.*

Bibliography

[1] D. Achour, E. Dahia and P. Turco, *The Lipschitz injective hull of Lipschitz operator ideals and applications, Banach J. Math. Anal. 14 (2020), 1241–1257.*

[2] D. Achour, E. Dahia and P. Turco, *Lipschitz p-compact mappings, Monatshefte für Mathematik, 189 (2019), 595–609.*

[3] D. Achour, E. Dahia and M. A. S, Saleh, *Multilinear mixing operators and Lipschitz mixing operator ideals, Operators and Matrices. 12 (4) (2018), 903–931.*

[4] D. Achour, P. Rueda and R. Yahi, *(p,σ)-absolutely Lipschitz operators, Ann. Funct. Anal. 8 (1) (2017), 38–50.*

[5] D. Achour, P. Rueda, E.A. Sànchez-Pèrez and R. Yahi, *Lipschitz operator ideals and the approximation property. Math. Anal. Appl. 436 (2016), 217–236*

[6] D. Achour and E. Dahia, *Building ideals of two-Lipschitz operators between mertic and Banach spaces, arXiv:2312.03104v2.*

[7] R. F. Arens and J. Eells Jr, *On embedding uniform and topological spaces, Pacific J. Math 6 (1956), 397–403.*

[8] D. Chen and B. Zheng, *Lipschitz p-integral operators and Lipschitz p-nuclear operators, Nonlinear Anal. 75 (2012), 5270–5282.*

[9] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators. Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.*

[10] J. D. Farmer and W. B. Johnson, *Lipschitz p-summing operators, Proc. Amer. Math. Soc. 137, 9 (2009), 2989-2995.*

[11] H. Jarchow, *Locally convex spaces, B. G. Teubner, Stuttgart, 1981.*

[12] *A. Jiménez-Vargas, J. M. Sepulcre and Moisés Villegas-Vallecillos, Lipschitz compact operators. J. Math. Anal. Appl. 415 (2) (2014), 889–901.*

[13] *U. Matter, Absolutely continuous operators and super-reflexivity, Math. Nachr. 130 (1987), 193-216.*

[14] *A. Pietsch, Operator ideals. Deutsch. Verlag Wiss., Berlin, 1978; North-Holland, Amsterdam-London-New York-Tokyo, 1980.*

[15] *A. Pietsch, Absolute p-summierende abbildungen in normierten räumen, Studia Math.* ***28*** *(1967), 333–353.*

[16] *I. Stephani, Injektive operatorenidealeüber der gesamtheit aller Banachräume und ihre topologische Erzeugung, Studia Math. 38 (1970), 105–124.*

[17] *T. Tiaiba and D. Achour, The ideal of Lipschitz classical p-compact operators and its injective hull, Moroccan J. of Pure and Appl. Anal., 8 (1), (2022), 28–43.*

[18] *N. Weaver, Lipschitz Algebras (second edition), World Scientific Publishing Co. Pte. Ltd., Singapore, 2018.*

[19] *R. Yahi, D. Achour and P. Rueda, Absolutely summing Lipschitz conjugates, Mediterr. J. Math., 13 (2016), 1949–1961.*

[20] *Yahi, R., Achour, D., Dahia, E. (2025). Lipschitz closed injective hull ideals and Lipschitz interpolative ideals. Quaestiones Mathematicae, 48(4), 685–701.*

Summary and Keywords

ملخص

في هده المدكرة قمنا بدراسة طريقتين لتوليد مثالي ليشيزي انطلاقا من مثالي ليبشتيزي كيفي $\mathcal{I}_{Lip}$ ، الاولى تمثل في طريقة الغلاف المغلق $\overline{\mathcal{I}_{Lip}}^{inj}$. أما الطريقة الثانية هي طريقة الاستقطاب، ويرمز لها بالرمز $(\mathcal{I}_{Lip})_\sigma$ حيث درسنا مختلف الخاصئص المتعلقة بالطريقتين وقمنا بدراسة العلاقة بين المثاليات المتعلقة بالطريقتين حيث وجدنا الاحتواء التالي $\mathcal{I}_{Lip}^{inj} \subset (\mathcal{I}_{Lip})_\sigma \subset \overline{\mathcal{I}_{Lip}}^{inj}$ ، بالنسبة للصنف التطبيقي $(\Pi_p \circ Lip_0)_\sigma$ المكون من مؤثرات ليبشيتز التي خطيتها هي مؤثر خطي (p,σ)- مطلق الاستمرارية، نتحصل على مبرهنات من نوع *Pietsch* للهيمنة والتفكيك

الكلمات المفتاحية: مثالية المؤثرات، مؤثرات ليبشيتز، المؤثرات pالجمعية، الغلاف المغلق، الاستقطاب، مؤثرات (p,σ)مطلقة الاستمرارية، مبرهنة الهيمنة لبيتش .

Abstract

In this memoir, we study two methods for generating a Lipschitz operator ideal from a given Lipschitz ideal $\mathcal{I}_{Lip}$. The first method is the closed injective hull procedure, denoted by $\overline{\mathcal{I}_{Lip}}^{inj}$. The second method is the interpolative procedure, denoted by $(\mathcal{I}_{Lip})_\sigma$. We investigate various properties related to both procedures and study the relationship between the resulting ideals, establishing the following inclusion:

$$\mathcal{I}_{Lip}^{inj} \subset (\mathcal{I}_{Lip})_\sigma \subset \overline{\mathcal{I}_{Lip}}^{inj}.$$

For the concrete class $(\Pi_p \circ Lip_0)_\sigma$, consisting of Lipschitz operators whose linearization is a (p,σ)-absolutely continuous linear operator, we obtain Pietsch-type domination and factorization theorems.

Keywords: *Operator ideals, Lipschitz operators, p-summing operators, closed injective hull, interpolative ideals, (p,σ)-absolutely continuous operators, Pietsch domination theorem.*