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Theme

Filters and Ideals in a Fuzzy Lattice

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Filters and Ideals in a Fuzzy Lattice

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بسم الله الرحمن الرحيم

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Contents

Introduction

In 1965, L.A. Zadeh introduced the fuzzy sets concept. This field received great interest from researchers

In [8,9] Chon defined a fuzzy lattice and a fuzzy relation, in [14,15,16] Mezoomo defined the fuzzy ideal in a fuzzy lattice and in [1,2,3,4,5,6,18] Amroune defined filters in a fuzzy lattice

On this memory, we will study

In chapter 1, we recall some notions and definitions of classical lattices, fuzzy set and fuzzy lattice (Definition, Properties, Fuzzy Homomorphism and Operation on Bounded Fuzzy Lattices)

In chapter 2, we study definitions of filters and ideals in a fuzzy lattices and some their properties and types of filters and ideals (α-filter, prime filter, α-ideal and prime ideal) in a Fuzzy Lattice

In chapter 3, we study definitions fuzzy filters and fuzzy ideals in a fuzzy lattice and some their properties and types of fuzzy filters and fuzzy ideals (fuzzy t-filter, fuzzy α-filter, fuzzy prime fillter, fuzzy α-ideal and fuzzy prime ideal) in a Fuzzy Lattice
We are also studying theories the converse image of fuzzy filters and fuzzy ideals via fuzzy lattices isomorphism.

CHAPTER 1

PRELIMINARIES

1.1 Classical Lattices

1.1.1 Partial Orders

Definition 1.1.1 [2]

When P is a nonempty set a partial order (order) on P is a binary relation $\leq$ on P such that for all $x, y, z \in P$

(i) $x \leq x$ *(reflexivity);*

(ii) $x \leq y$ *and* $y \leq x$ *imply* $x = y$ *(antisymmetry);*

(iii) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$ *(transitivity).*

A set $P \neq \emptyset$ *equipped with a partial order relation* $\leq$ *is said to be a partially order set or poset;*
When we need to specify the order relation, we write $P = (P, \leq)$.

Example 1.1.1

$\wp(X)$ *is a family of parts of X equipped with a partial order relation* $\subseteq$ *is poset.*

Definition 1.1.2 (chain) [5]

Let P be an ordered set, Then P is a chain if, for all $x, y \in P$, *either* $x \leq y$ *or* $y \leq x$ *(That is any two elements of P are comparable).*

Eample 1.1.2

1. The sets $\mathbb{N}, \mathbb{R}, \mathbb{Z}$ *equipped with the usual order* $\leq$ *are chains.*

2. The set $D(16) = \{1, 2, 4, 8, 16\}$ *of all divisors of the integer 16 equipped with the relation divide | is a chain.*

Definition 1.1.3 (Bottom and Top) [10]

Let $P = (P, \leq)$ be an order set, we say P has a Bottom element if there is $\perp \in P$ (celled bottom) with the property that $\perp \leq x$ for all $x \in P$.

Dually P has a Top element if there is $\top \in P$ such that $x \leq \top$ for all $x \in P$.

Example 1.1.3

1. In $(\wp(X), \subseteq)$, we have $\perp = \emptyset$ and $\top$ = X.

2. A finite chain always has Bottom and Top elements.

Definition 1.1.4 (upper bound and lower bound) [10]

When $P = (P, \leq)$ is a poset and let $S \subseteq P$

An element $x \in P$ is an upper bound of S if $y \leq x$ for all $y \in S$ of all upper bound of S denoted:

$$S^u = \{x \in P : y \leq x \quad forall \qquad y \in S\}.$$

.

An element $x \in P$ is a lower bound of S if $y \geq x$ for all $y \in S$ of all lower bound of S denoted:

$$S^l = \{x \in P : y \geq x \quad forall \quad y \in S\}.$$

Example 1.1.4

The set $D(180) = \{1, 2, 3, 5, 4, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180\}$ of all divisors of the integer 180 equipped with the relation divide $|$.

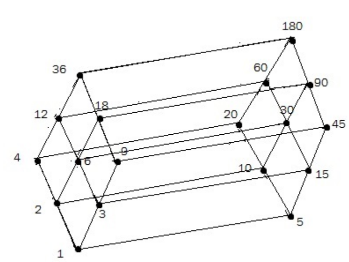

Figure 1.1:

$$\{3,5\}^u = \{3\}^u \cap \{5\}^u,$$
$$\{3,5\}^u = \{6,9,12,15,18,30,36,45,60,90,180\} \cap \{10,15,30,45,60,90,180\},$$
$$\{3,5\}^u = \{15,30,45,60,90,180\}.$$

1.1.2 Lattices

Definition 1.1.5 *[10]*

Let $P = (P, \leq)$ is a nonempty poset

(i) *$(P, \leq)$ is called a lattice if $sup\{x, y\}$ and $inf\{x, y\}$ exist for all $x, y \in P$;*

(ii) *$(P, \leq)$ is called a complete lattice if $supS$ and $infS$ exist for all $S \subseteq P$.*

Given a lattice L, we may define binary operations join and meet on the nonempty set L by for all $x, y \in L$ $x \wedge y = inf\{x, y\}$ and $x \vee y = sup\{x, y\}$.

Example 1.1.5

The diagrams are lattices

Figure 1.2:

Proposition 1.1.1

1. Let L a lattice and let $x, y, \in L$, Then the following are equivalent:

(i) *$x \leq y$;*

(ii) *$x \wedge y = x$;*

(iii) *$x \vee y = y$.*

2. Let L be a lattice. Then for all $x, y, z, t \in L$,

(i) $x \leq y$ *implies* $x \wedge z \leq y \wedge z$ *and* $x \vee z \leq y \vee z$;

(ii) $x \leq y$ *and* $z \leq t$ *imply* $x \wedge z \leq y \wedge t$ *and* $x \vee z \leq y \vee t$.

Definition 1.1.6 *[10]*

An algebraic structure $L = (L, \wedge, \vee)$ where L is a nonempty set and $\wedge, \vee$ are binary operations is a lattice if for each $x, y, z \in L$ the following properties are verified:

(i) $x \wedge y = y \wedge x$ *(commutativity);*

(ii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *and* $x \vee (y \vee z) = (x \vee y) \vee z$ *(associativity);*

(iii) $x \wedge (x \vee y) = x$ *and* $x \vee (x \wedge y) = x$ *(obsorption low);*

(iv) $x \wedge x = x$ *and* $x \vee x = x$ *(indempotency).*

Definition 1.1.7 *(Bounded lattice) [10]*

Let L be a lattice we say L has a top element if there is $1 \in L$ such that $a = a \wedge 1$ for all $a \in L$. Dually, we say L has a bottom element if there is $0 \in L$ such that $a = a \vee 0$ for all $a \in L$.

The lattice $(L, \wedge, \vee, 0, 1)$ a bounded lattice such that 0 and 1 are the bottom and top elements respectively.

Example 1.1.6

The set $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ of all divisors of the integer 30 equipped with the relation divide $|$ is a bounded lattice

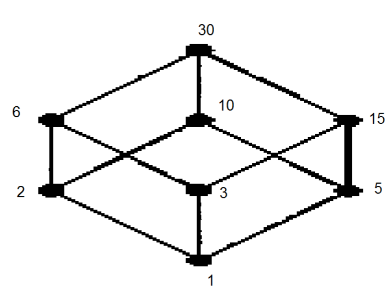

Figure 1.3:

Definition 1.1.8 *(sublattice) [10]*

Let L be a lattice and let $\emptyset \neq M \subseteq L$, Then M ba a sublattice of L if, for all $x, y \in M$ implies $x \wedge y \in M$ and $x \vee y \in M$.

Example 1.1.7

In the diagrams the shaded elements in lattice (i) and (ii) from sublattice, while those in (iii) and (iv) do not.

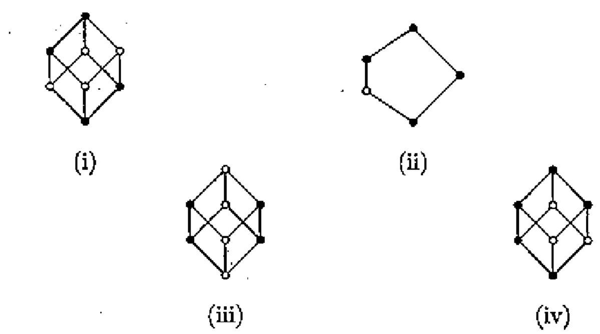

Figure 1.4:

1.1.3 Lattice Homomorphism

Definition 1.1.9 (Homomorphism)

Let $L = (L, \wedge_l, \vee_l, 0_l, 1_l)$ and $M = (M, \wedge_m, \vee_m, 0_m, 1_m)$ be bounded lattices, A mapping $h : L \longrightarrow M$ is said to be a lattice homomorphism if for all $x, y \in L$

(i) $h(x \wedge_l y) = h(x) \wedge_m h(y)$;

(ii) $h(x \vee_l y) = h(x) \vee_m h(y)$;

(iii) $h(0_l) = 0_m$ *and* $h(1_l) = 1_m$.

A bijective homomorphism is a lattice isomorphism.

1.1.4 Filters and Ideals in a Lattice

Definition 1.1.11 (Filter) *[15]*

Let L be a nonempty set and let $L = (L, \wedge, \vee)$ a lattice. A nonempty subset F is called a Filter of L if for all $x, y \in L$

(i) *If $y \in F$ and $y \leq x$, then $x \in F$;*

(ii) *If $x, y \in F$, implies $x \wedge y \in F$.*

Example 1.1.8

In the N5 lattices, the parts in bold are filters

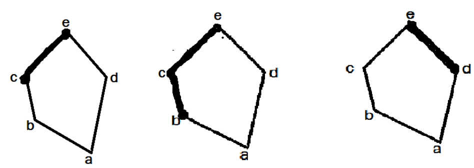

Figure 1.5: the N5 lattices

Proporitie 1.1.1

Let L be a lattice

. *A filter F of L is called proper if it does not coincide with L.*

. *A filter F of a lattice with 0 is proper if and only if $0 \notin F$.*

. *A filter F of L is prime if and only if $x, y \in L$ and $x \vee y \in F$ imply that $x \in F$ or $y \in F$.*

. *For each $x \in L$ the set $\uparrow x = \{y \in L, y \geq x\}$ is a filter.*

Definition 1.1.12 (Ideal) *[12]*

Let L be a nonempty set and let $L = (L, \wedge, \vee)$ a lattice. A nonempty subset I is called an Ideal of L if for all $x, y \in L$

(i) *If $y \in I$ and $y \geq x$, then $x \in I$;*

(ii) *If $x, y \in I$, implies $x \vee y \in I$.*

Example 1.1.9

In the N5 lattices, the parts in bold are ideals

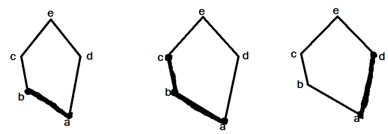

Figure 1.6: the N5 lattices

Proporitie 1.1.2

Let L be a lattice

. *An ideal I of L is called proper if it does not coincide with L.*

. *An ideal I of a lattice with 1 is proper if and only if* $1 \notin I$.

. *An ideal I of L is prime if and only if* $x, y \in L$ *and* $x \wedge y \in I$ *imply that* $x \in I$ *or* $y \in I$.

. *For each* $x \in L$ *the set* $\downarrow x = \{y \in L, x \geq y\}$ *is an ideal.*

1.2 Fuzzy Sets

1.2.1 Definitions and Properties

Definition 1.2.1 *[19]*

Let X be a non-empty set. A fuzzy set μ_A *on X is any application from X to the unite interval. i.e.,*

$$\mu_A : X \longrightarrow [0, 1].$$

where $[0, 1]$ *means real numbers between 0 and 1.*

The fuzzy set A in X may be represented as a set of ordered pairs of a generic element $x \in X$ *and its grade of membership, i.e.,*

$$A = \{(x, \mu_A(x)) : x \in X\}.$$

Definition 1.2.2 (Subset of fuzzy set) *[13]*

Suppose there are two fuzzy sets A and B. When their degrees of membership are same, we say A and B are equivalent. That is, A = B if and only if $\mu_A(x) = \mu_B(x)$ *for all* $x \in X$.

If $\mu_A(x) \neq \mu_B(x)$ *for any element, then A* $\neq$ *B. If the following relation is satisfied in the fuzzy set A and B, A is a subset of B.* $\mu_A(x) \leq \mu_B(x)$ *for all* $x \in X$.

This relation is expressed as $A \subseteq B$. *We call that A is a subset of B.*

1.2.2 Support and $\alpha-$Cut set

Definition 1.2.3 (support) *[13]*

Support of a fuzzy set A denoted by S(A) we mean all elements of X that belong to A to a non-zero degree, that is S(A) is a classical set defined by:

$$S(A) = \{x \in X : \mu_A(x) \geq 0\}.$$

Definition 1.2.4 ($\alpha - cut$***)*** *[10]*

$\alpha - cut$ *set of A, denoted by* A_α *is a set consisting of those elements of a universe X whose membership values are equal* α

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha, \alpha \in]0, 1]\}.$$

When two cut sets A_{α_1} *and* A_{α_2} *exist and if* $\alpha_1 \leq \alpha_2$ *for them, then* $A_{\alpha_2} \subseteq A_{\alpha_1}$.

Example 1.2.1 *[1]*

Consider a universal set X which is defined on the age domain.

X = {5, 15, 25, 35, 45, 55, 65, 75, 85}, Table of fuzzy set

age(element)	*infant*	*young*	*adult*	*senior*
5	0	0	0	0
15	0	0.2	0.1	0
25	0	1	0.9	0
35	0	0.8	1	0
45	0	0.9	1	0.1
55	0	0.1	1	0.2
65	0	0	1	0.6
75	0	0	1	1
85	0	0	1	1

$support_{\{young\}} = \{15, 25, 35, 45, 55\}$,

$young_{\{0.2\}} = \{15, 25, 35, 45\}, young_{\{0.8\}} = \{25, 35, 45\}, young_{\{0.8\}} \subseteq young_{\{0.2\}}$.

1.2.3 Operations on fuzzy sets

Complement

We can defined complement set of fuzzy set A likewise in crisp set. We denote the complement set of A as $\overline{A}$. Membership degree can be calculated as following

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x).$$

Intersection

Intersection of fuzzy sets A and B takes smaller value of membership function between A and B

$$\mu_{A \cap B}(x) = min\{\mu_A(x), \mu_B(x)\}.$$

union

Membership value of member x in the union takes the greater value of membership between A and B

$$\mu_{A\cup B}(x) = max\{\mu_A(x), \mu_B(x)\}.$$

Cartesian Product

The Cartesian product applied to fuzzy sets can be defined as

let $\mu_{A_1}, \mu_{A_2}, ..., \mu_{A_n}$ *be membership function of* $A_1, A_2, ..., A_n$. *Then, the membership degree of* $(x_1, x_2, ..., x_n) \in A_1 \times A_2 \times ... \times A_n$ *on the fuzzy set* $A_1 \times A_2 \times ... \times A_n$ *is*

$$\mu_{A_1\times A_2\times ...\times A_n}(x_1, x_2, ..., x_n) = min(\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)).$$

Characteristics of fuzzy set

Involution	$\overline{\overline{A}} = A.$
Commutativity	$A\cap B = B\cap A, \qquad A\cup B = B\cup A.$
Associativity	$A\cap(B\cap C) = (A\cap B)\cap C, \qquad A\cup(B\cup C) = (A\cup B)\cup C.$
Distributivity	$A\cap(B\cup C) = (A\cap B)\cup(A\cap C), \qquad A\cup(B\cap C) = (A\cup B)\cap(A\cup C).$
Idempotency	$A\cap A = A, \qquad A\cup A = A.$
Identity	$A\cap X = A, \qquad A\cap\emptyset = \emptyset, \qquad A\cup X = X, \qquad A\cup\emptyset = A.$
Transitivity	*if* $A\subseteq B$ *and* $B\subseteq C$, *then* $A\subseteq C$.
De Morgan's law	$\overline{A\cap B} = \overline{A}\cup\overline{B}, \qquad \overline{A\cup B} = \overline{A}\cap\overline{B}.$

1.2.4 Fuzzy Relations

Let X be a nonempty set. A fuzzy relation is any application R: $X^2 \longrightarrow [0,1]$

(i) *R is a fuzzy reflexive relation if* $R(x,x) = 1$ *for all* $x \in X$*;*

(ii) *R is a fuzzy symmetrical relation if* $R(x,y) = R(y,x)$*for all* $x, y \in X$*;*

(iii) *R is a fuzzy antisymmetrical relation if* $R(x,y) \geq 0$ *and* $R(x,y) \geq 0$ *implies* $x = y$ *for all* $x, y \in X$*;*

(iv) *R is a fuzzy transitive relation if* $R(x,y) \geq sup_{z\in X} min\{R(x,z), R(z,y)\}$.

. *A fuzzy relation R is called fuzzy partial order relation if R is reflexive, antisymmetric and transitive.*

. *(X,R) is called a fuzzy partial order set (fuzzy poset).*

. *A fuzzy partial order relation R is a fuzzy total order relation if either $R(x,y) > 0$ or $R(y,x) > 0$ for all $x, y \in X$.*

. *If R is a fuzzy total order relation on a set X, then (X, R) is called a fuzzy total ordered set.*

1.2.5 T-norm and T-conorms

Definition 1.2.5 (T-norm) *[20]*

A triangular norm (T-norm) is a binary operation T: $[0,1] \times [0,1] \longrightarrow [0,1]$, satisfying the following conditions:

(i) *Commutativity T(x,y)=T(y,x);*

(ii) *Associativity T(x,T(y,z))=T(T(x,y),z);*

(iii) *Monotonicity if $y \leq z$ then $T(x,y) \leq T(x,z)$;*

(iv) *Identity T(1,x)=x.*

Notice that for all T-norm T(0,x)=0.

Definition 1.2.6 (T-conorm) *[20]*

A triangular conorm (T-conorm) is a binary operation S: $[0,1] \times [0,1] \longrightarrow [0,1]$, satisfying the following conditions:

(i) *Commutativity S(x,y)=S(y,x);*

(ii) *Associativity S(x,S(y,z))=S(S(x,y),z);*

(iii) *Monotonicity if $y \leq z$ then $S(x,y) \leq S(x,z)$;*

(iv) *Identity S(1,x)=1.*

Notice that for all T-conorm S(0,x)=x.

We can see that duality exists between t-norm and t-conorm,

$$T(x,y) = 1 - S(1-x, 1-y);$$
$$S(x,y) = 1 - T(1-x, 1-y).$$

1.3 Fuzzy Lattice

1.3.1 Definitions and Properties

Definition 1.3.1 *[3]*

Let (X, R) be a fuzzy poset and let A be a nonempty subset of X. An element $u \in X$ is said to be an upper bound of a subset A if and only if $R(a, u) > 0$, for all $a \in A$. An upper bound u_0 of A is the least upper bound of A if and only if $R(u_0, u) > 0$, for every upper bound u of A. An element $l \in X$ is said to be a lower bound of a subset A if and only if $R(l, a) > 0$, for all $a \in A$. A lower bound l_0 of A is the greatest lower bound of A if and only if $R(l, l_0) > 0$, for every lower bound l of A. A least upper bound of A will be denoted by $supA$ and a greatest lower bound by $infA$. We denote the least upper bound of the set { x,y} by $x \vee y$ and denote the greatest lower bound of the set { x,y} by $x \wedge y$.

Definition 1.3.2 *[8]*

A fuzzy poset (X, R) is called a fuzzy lattice if $x \wedge y$ and $x \vee y$ exist for all $x, y \in X$.

$L = (X, R)$ is a bounded fuzzy lattice if there exist $\perp$ and $\top$ in X such that for any $x \in X$ we have that $R(\perp, x) > 0$ and $R(x, \top) > 0$.

Remark 1.3.1 *[18]*

Since R is antisymmetric, it follows that if the least upper (greatest lower) bound exists, then it is unique.

Example 1.3.1

Let X = {a, b, c, d, e } and let R be the fuzzy relation on X defined by:

R	*a*	*b*	*c*	*d*	*e*
a	*1*	*0*	*0.3*	*0.3*	*0.3*
b	*0*	*1*	*0*	*0*	*0.4*
c	*0*	*0*	*1*	*0*	*0.4*
d	*0*	*0*	*0*	*1*	*0.6*
e	*0*	*0*	*0*	*0*	*1*

(X,R) is a fuzzy lattice.

Proposition 1.3.1 *[9]*

Let (X, R) be a fuzzy lattice and let $x, y, z \in X$, then

1/ $R(x, x \vee y) > 0, R(y, x \vee y) > 0, R(x \wedge y, x) > 0, R(x \wedge y, y) > 0$;

2/ $R(x, z) > 0$ *and* $R(y, z) > 0$ *iff* $R(x \vee y, z) > 0$;

3/ $R(z, x) > 0$ *and* $R(z, y) > 0$ *iff* $R(z, x \wedge y) > 0$;

4/ $R(x, y) > 0$ *iff* $x \vee y = y$;

5/ $R(x, y) > 0$ *iff* $x \wedge y = x$;

6/ *If* $R(x, y) > 0$ *then* $R(x \wedge z, y \wedge z) > 0$ *and* $R(x \vee z, y \vee z) > 0$.

Definition 1.3.3 *[18]*

Let (X,R) be a fuzzy lattice. (X,R) is distributive if and only if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

. ***Proposition 1.3.2***

Let R: $X \times X \longrightarrow [0, 1]$ *be a fuzzy relation. Then* R *is a fuzzy partial order relation on X if and only* *if each $\alpha \in]0,1]$ the $\alpha - cut$ set R_α is a partial order relation in X.*

Proof

Let R be a fuzzy partial order relation on X and $\alpha \in]0,1]$.

Since R(x,x)=1, then $R(x,x) \geq \alpha$ for all $x \in X$, hence R_α is reflexive.

To show that R_α is antisymmetric, let $x, y \in X$ withe $R(x,y) \geq \alpha$ and $R(y,x) \geq \alpha$, then $R(x,y) \wedge R(y,x) \geq \alpha$ because R is fuzzy antisymmetric hence x=y.

Finally, we show that R_α is transitive, suppose $(x,y) \in R_\alpha$ and $(y,z) \in R_\alpha$, then $R(x,y) \geq \alpha$ and $R(y,z) \geq \alpha$. Since $R(x,z) \geq supmin\{R(x,y), R(y,z)\} \geq \alpha$, then $R(x,z) \geq \alpha$ that is $(x,z) \in R_\alpha$

Conversely, let R_α be a partial order relation for all $\alpha \in]0,1]$.

$R_1(x,x) \geq 1$, then R(x,x)=1. Hence R is reflexive.

To show that R is antisymmetric, suppose $R(x,y) \wedge R(y,x) \neq 0$ and $x \neq y$. $R(x,y) \wedge R(y,x) \neq 0$, then $R(x,y) \geq \alpha$ and $R(y,x) \geq \alpha$, since $(x,y) \in R_\alpha$ and $(y,x) \in R_\alpha$, then x=y contradiction. Therefore $R(x,y) \wedge R(y,x) \neq 0$ and $x = y$.

Finally, we show that R is transitive, let $R(x,y) > \alpha$ and $R(y,z) > \alpha$, then $(x,y) \in R_\alpha$ and $(y,z) \in R_\alpha$, hence $(x,z) \in R_\alpha$, since $R(x,z) \geq supmin\{R(x,y), R(y,z)\} \geq \alpha$ that is R is fuzzy transitive

Definition 1.3.4

A fuzzy poset (X,R) is called a fuzzy sup-lattice if each pair of elements has supremum on X, Duality a fuzzy poset (X,R) is called a fuzzy inf-lattice if each pair of elements has infimum on X.

Remark 1.3.2

Notice that a fuzzy poset is a fuzzy semi lattice if and only if it is either fuzzy sup-lattice or fuzzy inf-lattice.

1.3.2 Fuzzy Homomorphism on Bounded Fuzzy Lattices

$L = (X, R)$ and $M = (Y, P)$ be bounded fuzzy lattices. A mapping h: $X \longrightarrow Y$ is a fuzzy homomorphism from L into M if, for all $x, y \in X$, it satisfies the following conditions:

(i) $h(x \wedge_L y) = h(x) \wedge_M h(y)$;

(ii) $h(x \vee_L y) = h(x) \vee_M h(y)$;

(iii) $h(0_L) = 0_M, h(1_L) = 1_M$.

A fuzzy isomorphism is a bijective fuzzy homomorphism.

Definition 1.3.6

Let $L = (X, R)$ and $M = (Y, P)$ be bounded fuzzy lattices and let h: $X \longrightarrow Y$ be a fuzzy order homomorphism from L into M if, for all $x, y \in X$, satisfies the following conditions:

(i) *If $R(x, y) > 0$, then $P(h(x), h(y)) > 0$;*

(ii) $h(\top_L) = \top_M$ *and* $h(\bot_L) = \bot_M$.

If h is bijective, then it is called a fuzzy order isomorphism.

CHAPTER 2

FILTERS AND IDEALS IN A FUZZY LATTICE

2.1 Definitions and Properties

Definition 2.1.1

Let $L = (X, R)$ be a fuzzy lattice and F be a nonempty set of X. F is a filter of X if for all $x, y \in X$

(i) *If $y \in F$ and $R(y, x) > 0$, then $x \in F$;*

(ii) *If $x, y \in F$, then $x \wedge y \in F$.*

We denoted the set of all filters by elements of X by F(X).

Definition 2.1.2

Let $L = (X, R)$ be a fuzzy lattice and I be a nonempty set of X. I is an ideal of X if for all $x, y \in X$

(i) *If $y \in I$ and $R(y, x) > 0$, then $x \in I$;*

(ii) *If $x, y \in I$, then $x \vee y \in I$.*

We denoted the set of all idealss by elements of X by I(X).

Example 2.1.1

Let $X = \{a, b, c, d, e\}$ and R be the fuzzy relation on X defined by

R	a	b	c	d	e
a	1	0.3	0.3	0.3	0.3
b	0	1	0	0	0.4
c	0	0	1	0	0.5
d	0	0	0	1	0.6
e	0	0	0	0	1

$F_1 = \{b, e\}, F_2 = \{c, e\}$ *and* $F_3 = \{d, e\}$ *are filters of X.*

$I_1 = \{a, b\}, I_2 = \{a, c\}$ *and* $I_3 = \{a, d\}$ *are ideals of X.*

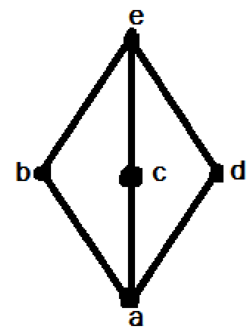

Figure 2.1:

Lemma 2.1.1 *[18]*

Let S be a nonempty subset of a fuzzy lattice (X,R), then

$$\uparrow S = \{x \in X | R(a_1 \wedge ... \wedge a_n, x) > 0 \quad for \quad some \quad a_1, ..., a_n \in S\}.$$

Is the filter generated by S.

Proof

Let $\uparrow S = \{x \in X | R(a_1 \wedge ... \wedge a_n, x) > 0, a_1, ..., a_n \in S\}$*. First, we show that* $\uparrow S$ *is nonempty set, let* $a \in S$*. Since* $R(a, a) = 0$*, then* $a \in \uparrow S$*, if follows that* $\uparrow S \neq \emptyset$*. To show that* $\uparrow S$ *is a filter, let* $x \in \uparrow S$ *and* $y \in X$ *such that* $R(x, y) > 0$*, There exist* $a_1, a_2, ..., a_n$ *such that* $R(a_1 \wedge ... \wedge a_n, x) > 0$*. It follows that* $R(a_1 \wedge ... \wedge a_n, y) > 0$*, i.e.,*$y \in F$

On the other hand, let $x, y \in \uparrow S$*. There exist* $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m$ *such that* $R(a_1 \wedge ... \wedge a_n, x) > 0$ *and* $R(b_1 \wedge ... \wedge b_m, y) > 0$*, this implies that* $R(a_1 \wedge ... \wedge a_n \wedge b_1 \wedge ... \wedge b_m, x \wedge y) > 0$*. Hence,* $x \wedge y \in \uparrow S$*. Next let* $a \in S$*. Since* $R(a, a) > 0$*, it follows that* $a \in \uparrow S$*, this implies that* $S \subseteq \uparrow S$*. Finally, suppose that F is a filter with* $S \subseteq F$*. It follows that, for any* $x \in \uparrow S$*, there exist* $a_1 \wedge ... \wedge a_n$ *such that* $R(a_1 \wedge ... \wedge a_n, x) > 0$*, i.e.,*$x \in F$*. Hence,* $\uparrow S \subseteq F$

Lemma 2.1.2

Let S be a nonempty subset of a fuzzy lattice (X,R), then

$$\downarrow S = \{x \in X | R(x, a_1 \vee ... \vee a_n) > 0 \quad for \quad some \quad a_1, ..., a_n \in S\}.$$

Is the ideal generated by S.

Proposition 2.1.1 *[7]*

(i) $S \subseteq \uparrow S$;

(ii) $S \subseteq T \Rightarrow \uparrow S \subseteq \uparrow T$;

(iii) $\uparrow\uparrow S = \uparrow S$;

(iv) $S \subseteq \downarrow S$;

(v) $S \subseteq T \Rightarrow \downarrow S \subseteq \downarrow T$;

(vi) $\downarrow\downarrow S = \downarrow S$.

Proof

(i) *Let* $y \in S$, *since* $R(y, y) > 0$, *then* $y \in \uparrow S$.

(ii) *Suppose that* $S \subseteq T$ *and* $x \in \uparrow S$, *then there exists* $a_1, .., a_n \in T$ *such that* $R(a_1 \wedge ... \wedge a_n, x) > 0$, *as* $S \subseteq T$, *and* $R(a_1 \wedge ... \wedge a_n, x) > 0$, *so,* $x \in \uparrow T$.

(iii) *In (i) and (ii) we have* $\uparrow\uparrow S \subset \uparrow S$;

Suppose that $x \in \uparrow\uparrow S$, *then there exists* $a_1, .., a_n \in \uparrow S$ *such that* $R(a_1 \wedge ... \wedge a_n, x) > 0$.

Since $a_1, .., a_n \in \uparrow S$ *and* $\uparrow S$ *is a filter, then* $a_1 \wedge ... \wedge a_n \in \uparrow S$.

So, there exists $b_1, .., b_m \in S$ *such that* $R(b_1 \wedge ... \wedge b_m, a_1 \wedge ... \wedge a_n) > 0$. *By the transitivity, there exists* $b_1, .., b_m \in S$ *such that* $R(b_1 \wedge ... \wedge b_m, x) > 0$. *Therefore,* $x \in \uparrow S$.

In the same manner, we proof the properties $(iv), (v)$ *and* (vi)**.**

Proposition 2.1.2

Let (X, R) be a fuzzy lattice. Then,

(i) $X \in F(X)$ *and* $X \in I(X)$;

(ii) $\cap W \in F(X)$, *for all* $W \in F(X)$;

(iii) $\cap Z \in I(X)$, *for all* $Z \in I(X)$.

Proof

(i) *Clear;*

(ii) *Suppose* $y \in \cap W$, *then* $y \in W$ *for all* $W \in F(X)$. *If* $R(y, x) > 0$, *for some* $x \in X$, *then* $x \in W$, *for all* $W \in F(X)$, *since* W *is a filter, and hence* $x \in \cap W$. *Thus,* $\cap W \in F(X)$;

If $x, y \in \cap W$, *then* $x, y \in W$, *for all* $W \in F(X)$, *since each* W *is a filter, then* $x \wedge y \in W$. *Therefore,* $x \wedge y \in W$;

(iii) *Similar to* (ii) .

Definition 2.1.3 (principal filter)

Let x is an element of the fuzzy lattice (X,R), $(\uparrow x) = \{y \in X | R(x, y) > 0\}$ *is the principal filter generated by x.*

Definition 2.1.4 (principal ideal)

Let x is an element of the fuzzy lattice (X,R), $(\downarrow x) = \{y \in X | R(y, x) > 0\}$ *is the principal ideal generated by x.*

Proposition 2.1.3

Let (X, R) *be a fuzzy lattice,* $F \subseteq X$ *is a filter (ideal) of* (X, R) *if and only if* F *is a filter (ideal) of* $(X, S(R))$, *where S(R) is the support of R.*

Proof

Suppose that F *is a filter of* (X, R). *Then,*

(i) $x \in X, y \in F$, *if* $(y, x) \in S(R)$, *then* $R(y, x) > 0$, *so,* $x \in F$;

(ii) *$x, y \in F$, then $x \wedge y \in F$.*

Hence, F is a filter of $(X, S(R))$.

Conversely, let F be a filter of $(X, S(R))$.

(i) *$x \in X, y \in F$, if $R(y, x) > 0$, then $(y, x) \in S(R)$ and $x \in F$;*

(ii) *Trivially.*

Hence, F is a filter of (X,R).

Similarly, we can prove that I is an ideal of (X,R) if and only if I is an ideal of (X,S(R)).

2.2 Types of Filters and Ideals in a Fuzzy Lattice

2.2.1 α-Filter and α-Ideal

Definition 2.2.1 *[3]*

Let (X,R) be a fuzzy lattice and $\alpha \in]0, 1]$. F be a nonempty set of X, F as an $\alpha - filter$ of X, if the following conditions hold

(i) *If $y \in X, x \in F$ and $R(x, y) \geq \alpha$, then $y \in F$;*

(ii) *If $x, y \in F$, then $x \wedge y \in F$.*

Definition 2.2.2

Let (X,R) be a fuzzy lattice, $\alpha \in]0, 1]$ and let I be a nonempty set of X, I as an $\alpha - ideal$ of X, if the following conditions hold

(i) *If $y \in X, x \in I$ and $R(x, y) \geq \alpha$, then $y \in I$;*

(ii) *If $x, y \in I$, then $x \vee y \in I$.*

Example 2.2.1

Let $X = \{a, b, c, d, e\}$ and R be the fuzzy relation on X defined by:

R	a	b	c	d	e
a	1	0.5	0.5	0.5	0.5
b	0	1	0	0	0.6
c	0	0	1	0	0.7
d	0	0	0	1	0.8
e	0	0	0	0	1

$F_2 = \{c, e\}$ *and* $F_3 = \{d, e\}$ *are 0.7-filters.*

$I_2 = \{a, c\}$ *and* $I_3 = \{a, d\}$ *are 0.7-ideals.*

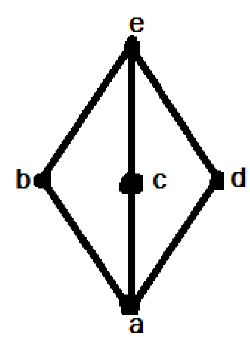

Figure 2.2:

Proposition 2.2.1 *[14]*

Let (X,R) be a fuzzy lattice. If $\alpha \leq \beta$, then any $\alpha - filter$ is a $\beta - filter$.

Proof

Let F be a fuzzy $\alpha - filter$ of X and $\alpha \leq \beta$. It holds that for any $x, y \in X$

$R(x, y) \geq \beta \Longrightarrow R(x, y) \geq \alpha \Longrightarrow F(x) \leq F(y)$.

2.2.2 Prime Filters and Prime Ideals

Definition 2.2.3 *[17]*

Let (X,R) be a fuzzy lattice and F be a filter in X. Then a proper $(F \neq X)$ filter F is called prime filter if $x, y \in X$ and $x \vee y \in F$ imply $x \in F$ or $y \in F$.

Definition 2.2.4 *[7]*

Let (X,R) be a fuzzy lattice and I be an ideal in X. Then a proper $(I \neq X)$ ideal I is called prime ideal if $x, y \in X$ and $x \wedge y \in I$ imply $x \in I$ or $y \in I$.

Definition 2.2.5

Let (X,R) be a fuzzy lattice and F be a filter in X. Then a proper $(F \neq X)$ filter F is called maximal filter if for any filter J in X and $F \subseteq J$ imply $J = F$ or $J = X$.

Definition 2.2.6 *[11]*

Let (X,R) be a fuzzy lattice and I be an ideal in X. Then a proper $(I \neq X)$ ideal I is called maximal ideal if for any ideal J in X and $I \subseteq J$ imply $J = I$ or $J = X$.

Example 2.2.2

Let $X = \{a, b, c, d, e, f\}$ and R be the fuzzy relation on X defined by

R	a	b	c	d	e	f
a	1	0.4	0.4	0.4	0.4	0.4
b	0	1	0.5	0	0	0.5
c	0	0	1	0	0	0.6
d	0	0	0	1	0	0.7
e	0	0	0	0	1	0.8
f	0	0	0	0	0	1

$F_1 = \{b, c, f\}$ *and* $F_2 = \{c, f\}$ *are filters and* $F_2 \subset F_1, F_1$ *is a maximal filter.*
$I_1 = \{a, b\}$ *and* $I_2 = \{a, b, c\}$ *are ideals and* $I_1 \subset I_2, I_2$ *is a maximal ideal.*

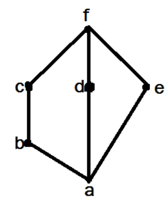

Figure 2.3:

Theorem 2.2.1 *[3]*

Let (X,R) be a fuzzy lattice and F be a proper filter in X. F is a prime filter if and only if for any filters H and G,$H \cap G \subseteq F$ *imply* $H \subseteq F$ *or* $G \subseteq F$.

Proof

Suppose that F is a prime filter and H,G two filters satisfies $H \cap G \subseteq F$. *If* $H \nsubseteq F$*, then exist* $x \in H$ *and* $x \notin F$. *Let* $y \in G$*, then* $x \vee y \in H$ *and* $x \vee y \in G$. *Since, H and G are filters,* $R(x, x \vee y) > 0$ *and* $R(y, x \vee y) > 0$*, imply that* $x \vee y \in H \cap G$. *From* $H \cap G \subseteq F$*, it follows that* $x \vee y \in F$. *since F is a prime filter, it holds that* $y \in F$. *Hence* $G \subseteq F$
Conversely, let F be a proper filter in X and $x, y \in X$*, such that* $x \vee y \in F, < \{x\} >$ *and* $< \{y\} >$ *are filters by lemma 2.1.1. Since* $< \{x\} > \cap < \{y\} >=< \{x \vee y\} >\subseteq F$*, it follows that* $< \{x\} >\subseteq F$ *or* $< \{y\} >\subseteq F$. *Thus* $x \in F$ *or* $y \in F$. *This show that F is a prime filter.*

Theorem 2.2.2

Let (X,R) be a fuzzy lattice and I be a proper ideal in X. I is a prime ideal if and only if for any ideals H and G, $H \cap G \subseteq I$ *imply* $H \subseteq I$ *or* $G \subseteq I$.

Proof

Consider two ideals H *and* G *verifying* $H \cap G \subseteq I$ *and suppose that* $H \subsetneq I$ *and* $G \subsetneq I$*, then there exist* x, y *in* X *such that* $x \in H$, $x \notin I$ *and* $y \in G$*, and* $y \notin I$. *Since H,G are ideals,* $R(x \wedge y, x) > 0$ *and* $R(x \wedge y, y) > 0$*, hence* $x \wedge y \in H$ *and* $x \wedge y \in G$*, so,* $x \wedge y \in H \cap G$. *From* $H \cap G \subseteq F$ *we have* $x \wedge y \in I$. *Since* I *is a prime ideal, it holds that* $x \in I$ *or* $y \in I$*, this mean* $H \subseteq I$ *or* $G \subseteq I$*, contradiction. So,* I

is a prime ideal implies then, for any ideals H *and* G *such that* $H \cap G \subseteq I$, *implies* $H \subseteq I$ *or* $G \subseteq I$. *conversely, let* I *be an ideal of* X *verifying for any ideals* H *and* G, $H \cap G \subseteq I$ *implies* $H \subseteq I$ *or* $G \subseteq I$, *and show that* I *is a prime ideal. For this, let* x, y *in* X *with* $x \wedge y \in I$.

$x \wedge y \in I$, *this implies* $\downarrow x \cap \downarrow y = \downarrow (x \wedge y) \subseteq I$, *this implies* $\downarrow x \in I$ *or* $\downarrow y \in I$, *i.e.,* $x \in I$ *or* $y \in I$.

Hence I is prime.

CHAPTER 3

FUZZY FILTERS AND FUZZY IDEALS IN A FUZZY LATTICE

3.1 Definitions and Properties

Definition 3.1.1 *[4]*

Let (X,R) be a fuzzy lattice. A fuzzy sublattice F in X is called a fuzzy filter of X if it hold that

(i) *For all* $x, y \in X, F(x \wedge y) \geq F(x) \wedge F(y)$*;*

(ii) *For all* $x, y \in X, R(x, y) > 0$ *implies* $F(x) \leq F(y)$*.*

Definition 3.1.2 *[12]*

Let (X,R) be a fuzzy lattice. A fuzzy set I on X is called a fuzzy ideal in (X,R) if, for all $x, y \in X$ *the following conditions verified:*

(i) *If* $\mu_I(y) > 0$ *and* $R(x, y) > 0$*, then* $\mu_I(x) > 0$*;*

(ii) *If* $\mu_I(x) > 0$ *and* $\mu_I(y) > 0$*, then* $\mu_I(x \vee y) > 0$*.*

Theoreme 3.1.1 *[4]*

Let (X,R) be a fuzzy lattice and $(F_i)_{i\in I}$ *be a family of fuzzy filters of X. Then* $\bigcap_{i\in I} F_i$ *is a fuzzy filter of X.*

Proof

Let $(F_i)_{i\in I}$ *be a family of fuzzy filters of X. For all* $x, y \in X$*, it follows that* $F_i(x \wedge y) \geq F_i(x) \wedge F_i(y)$ *for all* $i \in I$*, which implies* $\wedge_{i\in I} F_i(x \wedge y) \geq \wedge_{i\in I}(F_i(x) \wedge F_i(y)) \geq \wedge_{i\in I}(F_i(x)) \wedge \wedge_{i\in I}(F_i(y))$ *on the other hand, let* $x, y \in X$*, if* $R(x, y) > 0$*, then* $F_i(y) \geq F_i(x)$ *for all* $i \in I$*. Hence,* $\wedge_{i\in I}(F_i(y)) \geq \wedge_{i\in I}(F_i(x))$*.*

Theoreme 3.1.2

Let (X,R) be a fuzzy lattice and $(I_i)_{i \in N}$ be a family of fuzzy ideals of X. Then $\bigcap_{i \in N} I_i$ is a fuzzy ideal of X.

3.2 Types of Fuzzy Filters in a Fuzzy Lattice

3.2.1 Fuzzy t-filter

Definition 3.2.1 *[4]*

Let (X,R) be a fuzzy lattice and t be a t-norm. A fuzzy subset F in X is called a fuzzy t-filter of X if it hold that

(F1) For all $x, y \in X, F(x \wedge y) \geq t(F(x), F(y))$;

(F2) For all $x, y \in X, R(x, y) > 0$ implies $F(x) \leq F(y)$.

Example 3.2.1

Let $X = [0, 1]$ and R be the fuzzy relation on X defined by

$$R(x, y) = \begin{cases} 1, & if x = y, \\ \lambda, & (\lambda \in [0, 1]) \quad if \quad x \leq y, \\ 0, & otherwise. \end{cases}$$

(X,R) is a fuzzy lattice, F is a fuzzy set defined by: F: $[0, 1] \longrightarrow [0, 1], F(x) = x$, F is a fuzzy t-filter.

Theoreme 3.2.1 *[4]*

Let (X,R) be a fuzzy lattice and F non-constant fuzzy subset of X, if F_α is a filter for every $\alpha \in]0, 1]$, such that the cut $F_\alpha \neq \emptyset$, then F is a fuzzy t-filter.

Proof

Let F be a non-constant fuzzy subset of X, such that for every $\alpha \in]0, 1], F_\alpha \neq \emptyset$, suppose that the cut F_α is a filter of X. For all $x, y \in X$, if $t(F(x), F(y) \neq \emptyset$, we denote $t(F(x), F(y)) = \alpha$, *then* $F(x) \geq \alpha$ *and* $F(y) \geq \alpha$, *since* $t(F(x), F(y)) \leq F(x) \wedge F(y)$, *hance* $x, y \in F_\alpha$ *which imply* $x \wedge y \in F_\alpha$,*i.e.*,$F(x \wedge y) \geq \alpha$. *Hence*,$F(x \wedge y) \geq t(F(x), F(y))$. *If* $t(F(x), F(y)) = 0$, *then the relation* $F(x \wedge y) \geq t(F(x), F(y))$ *is cleary true.*

On other hand, let $F(x) = \alpha$. *If* $R(x, y) > 0$ *and* $x \in F_\alpha$, *then* $y \in F_\alpha$, *hance* $\alpha = F(x) \leq F(y)$, *it is easy to see that if F(x)=0, we obtain* $F(x) \leq F(y)$.

Example 3.2.2

Let $X = \{a, b, c, d, e, f\}$ *and R be the fuzzy relation on X defined by:*

R	a	b	c	d	e	f
a	1	1	1	1	1	1
b	0	1	0.4	0.4	0.4	1
c	0	0	1	0	0.4	1
d	0	0	0	1	0.4	1
e	0	0	0	0	1	1
f	0	0	0	0	0	1

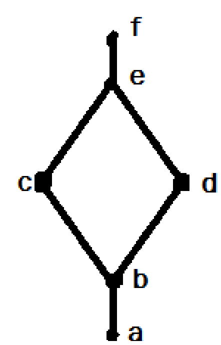

Figure 3.1:

Let F be a fuzzy set defined on X by F(a)=0.1, F(b)=0.2, F(c)=0.2, F(d)=0.5, F(e)=0.6, F(f)=0.6. (X,R) is a fuzzy lattice, F is a fuzzy t-filter in X. $F_{0.1}, F_{0.2}, F_{0.5}, F_{0.6}$ *are filters in X.*

Theoreme 3.2.2 *[4]*

Let (X,R) be a fuzzy lattice, A fuzzy set F in X is fuzzy t-filter in X if and only if it satisfies

(F1) For all $x, y \in X$, $F(x \wedge y) \geq t(F(x), F(y))$;

(F3) For all $x, y \in X, F(x) \geq F(x \wedge y) \wedge F(y)$.

Proof

Suppose that (F3) is satisfied i.e., for all $x, y \in X, F(x) \geq F(x \wedge y) \wedge F(y)$ *if* $R(x, y) > 0$, *it follows that* $x \wedge y = x$ *i.e.,* $F(y) \geq F(x \wedge y) \wedge F(y) = F(x)$. *Hence, (F2) satisfied.*

Conversely, if (F2) is satisfied, then it holds that $R(x, y) > 0$. *Hence,* $F(x \wedge y) \wedge F(y) \leq F(x) \wedge F(y) \leq F(x)$.

Theoreme 3.2.3 *[4]*

Let (X,R) be a fuzzy lattice, A fuzzy set F in X is fuzzy t-filter in X if it satisfied

(F4) $\forall x_1, x_2, x_3 \in X, R(x_1 \wedge x_2, x_3) > 0 \Longrightarrow F(x_3) \geq F(x_1) \wedge F(x_2)$.

Proof

Let $x_1, x_2, x_3 \in X$. *Suppose that* $R(x_1 \wedge x_2, x_3) > 0 \Longrightarrow F(x_3) \geq F(x_1) \wedge F(x_2)$

If we choose $x_1 = x_2 = x$ *and* $x_3 = y$, *then* $R(x, y) > 0 \Longrightarrow F(y) \geq F(x)$. *Hence, (F2) is satisfied.*

On other hand, since $R(x \wedge y, x \wedge y) > 0$ *it follows that* $F(x \wedge y) \geq F(x) \wedge F(y) \geq t(F(x), F(y))$. *Hence, (F1) is satisfied.*

Theoreme 3.2.4 *[4]*

Let (X,R) be a fuzzy lattice, A fuzzy set F in X is fuzzy filter in X if and only if it satisfies:

(F1) For all $x, y \in X, F(x \wedge y) \geq t(F(x), F(y))$;

(F5) For all $x, y \in X, F(x \vee y) \geq F(x)$.

Proof

Suppose that F is a fuzzy filter in X. Let $x, y \in X, R(x, x \vee y) > 0$ *implies* $F(x \vee y) \geq F(x)$. *Hence, (F5) is satisfied.*

Conversely, suppose that (F5) is satisfied, Let $x, y \in X$ *such that* $R(x, y) > 0$, *since* $F(x \vee y) \geq F(x)$, *it follows that* $F(y) \geq F(x)$.

Colloray 3.2.1 *[4]*

Let (X,R) be a fuzzy lattice, A fuzzy set F in X is a fuzzy filter in X if for any $x, a_1, a_2, ..., a_n \in X, R(a_1 \wedge a_2 \wedge ... \wedge a_n, x) > 0$ *implies* $F(x) \geq F(a_1) \wedge F(a_2) \wedge ... \wedge F(a_n)$.

Proof

Let F be a fuzzy set in X. Suppose that for any $x, a_1, a_2, ..., a_n \in X, R(a_1 \wedge a_2 \wedge ... \wedge a_n, x) > 0$ *implies* $F(x) \geq F(a_1) \wedge F(a_2) \wedge ... \wedge F(a_n)$. *For any* $x, a_1, a_2 \in X, R(a_1 \wedge a_2, a_1 \wedge a_2) > 0$ *implies* $F(a_1 \wedge a_2) \geq F(a_1) \wedge F(a_2) \geq t(F(a_1), F(a_2))$*.Hence (F1) is satisfied.*

On other hand, Since $R(a_1, a_2) > 0$ *implies that* $R(a_1 \wedge a_2, a_2) > 0$*, it follows that* $F(a_2) \geq F(a_1) \wedge F(a_2)$*. Hence,* $F(a_2) \geq F(a_1)$*.*

Theoreme 3.2.5 *[4]*

Let (X,R) be a fuzzy lattice and F be a nonempty crisp set in X. If F is a filter, then φ_F *is a fuzzy t-filter of X, such that*

$$\varphi_F(x) = \begin{cases} \alpha, & if x \in F, \\ \beta, & otherwise. \end{cases}$$

Proof

Let F be a filter in X and $x, y \in X$

1. If $x \wedge y \in F$*, then* $\varphi_F(x \wedge y) \geq \varphi_F(x) \wedge \varphi_F(y)$*;*

2. If $x \wedge y \notin F$*, then* $x \notin F$ *or* $y \notin F$*. Hence,* $t(\varphi_F(x), \varphi_F(y)) \leq t(\alpha, \beta) \leq \alpha \wedge \beta \leq \beta = \varphi_F(x \wedge y)$*.*

On other hand, let $x, y \in X, R(x, y) > 0$

1.If $x \in F$*, then If* $y \in F$*. Hence,* $\varphi_F(x) \leq \varphi_F(y) = \alpha$*;*

2. If $x \notin F$*, then* $\beta = \varphi_F(x) \leq \varphi_F(y)$

Finally, let $x \in X$*.*

$$\begin{aligned} x \in (\varphi_F)_\alpha \iff & \varphi_F(x) = \alpha, \\ \iff & x \in F. \end{aligned}$$

Definition 3.2.2 *[4]*

Let (X,R) be a fuzzy lattice and S be a fuzzy set in X. A fuzzy filter F in X is saide to be generated by S, if $S \subseteq F$ *and for any fuzzy filter T in X,* $S \subseteq T$ *implies* $F \subseteq T$*. The fuzzy filter generated by S will be denoted by* $< S >$*.*

Theoreme 3.2.6 *[4]*

If S is a fuzzy set in X, then the fuzzy filter generated by S is given by:

$< S > (x) = \vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n)/R(a_1, a_2, ..., a_n, x) > 0, a_1, a_2, ..., a_n \in X\}.$

Proof

Let $< S > (x) = \vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n)/R(a_1, a_2, ..., a_n, x) > 0, a_1, a_2, ..., a_n \in X\}$ *First, we show that* $< S >$ *is a fuzzy filter. Let* $x, y \in X$ *such that* $R(x, y) > 0$. *If there exist* $a_1, a_2, ..., a_n$ *such that* $R(a_1 \wedge ... \wedge a_n, x) > 0$. *Then* $R(a_1 \wedge ... \wedge a_n, y) > 0$ *and*

$$\begin{aligned} < S > (x) = & \vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n)/R(a_1, a_2, ..., a_n, x) > 0, a_1, a_2, ..., a_n \in X\} \\ \leq & \vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n)/R(a_1, a_2, ..., a_n, y) > 0, a_1, a_2, ..., a_n \in X\} \\ = & < S > (y). \end{aligned}$$

On other hand, let $x, y \in X$. *If there exist* $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m$ *suct that* $R(a_1 \wedge ... \wedge a_n, x) > 0$ *and* $R(b_1 \wedge ... \wedge b_m, y) > 0$, *then it follows that* $R(a_1 \wedge ... \wedge a_n \wedge b_1 \wedge ... \wedge b_m, x \wedge y) > 0$. *Then it follows that,*

$$\begin{aligned} t(< S > (x), < S > (y)) = & \ t(\vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n)/R(a_1 \wedge a_2 \wedge ... \wedge a_n, x) > 0, a_1, a_2, ..., a_n \in X\} \\ & \vee\{S(b_1) \wedge S(b_2) \wedge ... \wedge S(b_m)/R(b_1 \wedge b_2 \wedge ... \wedge b_m, y) > 0, b_1, b_2, ..., b_m \in X\}), \\ \leq & \ \vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n)/R(a_1 \wedge a_2 \wedge ... \wedge a_n, x) > 0, a_1, a_2, ..., a_n \in X\} \\ & \wedge \vee \{S(b_1) \wedge S(b_2) \wedge ... \wedge S(b_m)/R(b_1 \wedge b_2 \wedge ... \wedge b_m, y) > 0, b_1, b_2, ..., b_m \in X\} \\ = & \ \vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n) \wedge S(b_1) \wedge S(b_2) \wedge ... \wedge S(b_m)/R(a_1 \wedge a_2, \\ & \wedge ... \wedge a_n, x) > 0, R(b_1, b_2, ..., b_m, y) > 0, a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in X\} \\ \leq & \ \vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n) \wedge S(b_1) \wedge S(b_2) \wedge ... \wedge S(b_m)/R(a_1 \wedge a_2 \\ & \wedge ... \wedge a_n \wedge b_1 \wedge b_2 \wedge ... \wedge b_m, x \wedge y) > 0, a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in X\} \\ = & \ < S > (x \wedge y). \end{aligned}$$

It follows that $< S > (x \wedge y) \geq t(< S > (x), < S > (y))$. *Hence,* $< S >$ *is a fuzzy t-filter.*

To show $S \subseteq < S >$, *since* $R(x \wedge x, x) > 0$, *it follows that* $< S > (x) \geq S(x) \wedge S(y) = S(x)$.

Finally, suppose that T is a fuzzy filter $S \subseteq T$, then it holds that for any $x \in X$.

$$\begin{aligned} < S > (x) = \; & \vee\{S(a_1) \wedge S(a_2) \wedge ... \wedge S(a_n)/R(a_1 \wedge a_2 \wedge ... \wedge a_n, x) > 0, a_1, a_2, ..., a_n \in X\} \\ \leq \; & \vee\{T(a_1) \wedge T(a_2) \wedge ... \wedge T(a_n)/R(a_1 \wedge a_2 \wedge ... \wedge a_n, x) > 0, a_1, a_2, ..., a_n \in X\} \\ \leq \; & \vee\{T(x)\}, \\ = \; & T(x). \end{aligned}$$

3.2.2 Fuzzy α-Filter and Fuzzy α-Ideal

Definition 3.2.3

Let (X,R) be a fuzzy lattice and$\alpha \in]0, 1]$. A fuzzy set F in X is called a fuzzy $\alpha - filter$ of X if it satisfies

(F1) For all $x, y \in X, F(x \wedge y) \geq F(x) \wedge F(y)$;

(FA) For all $x, y \in X, R(x, y) \geq \alpha \Longrightarrow F(x) \leq F(y)$.

Example 3.2.3

In Example 3.2.1 F is a $\lambda - filter$.

***Definition 3.2.4** [4]*

Let (X,R) be a fuzzy lattice and $\alpha \in]0, 1]$. A fuzzy set F in X is called a fuzzy $\alpha - t - filter$ of X if it satisfies

(TF1) For all $x, y \in X, F(x \wedge y) \geq t(F(x), F(y))$;

(FA) For all $x, y \in X, R(x, y) \geq \alpha \Longrightarrow F(x) \leq F(y)$.

Example 3.2.4

In Example 3.2.1 F is $\lambda - t - filter$.

***Proposition 3.2.1** [4]*

Let (X,R) be a fuzzy lattice. If $\alpha \leq \beta$, then any fuzzy $\alpha - t - filter$ is a fuzzy $\beta - t - filter$.

Proof

Let F be a fuzzy $\alpha - filter$ of X and $\alpha \leq \beta$. It holds that for any $x, y \in X$

$R(x, y) \geq \beta \Longrightarrow R(x, y) \geq \alpha \Longrightarrow F(x) \leq F(y)$.

Theoreme 3.2.7 *[4]*

Let (X,R) be a fuzzy lattice and $(F_i)_{i\in I}$ *a family of fuzzy sets such that* F_i *is a fuzzy* $\alpha_i - filters$ *of X. Then,* $\bigcap\limits_{i\in I} F_i$ *is a fuzzy* $\underset{i\in I}{\vee}\alpha_i - filter$ *of X.*

Proof

Let (X,R) be a fuzzy lattice and suppose that for all $i \in I$, F_i *is a fuzzy* $\alpha_i - filters$ *of X*

1/ For all $x, y \in X$, *since* $F_i(x \wedge y) \geq F_i(x) \wedge F_i(y)$ *for all* $i \in I$, *it follows*

$$\underset{i\in I}{\wedge} F_i(x \wedge y) \geq \underset{i\in I}{\wedge}(F_i(x) \wedge F_i(y)) \geq (\underset{i\in I}{\wedge} F_i(x)) \wedge (\underset{i\in I}{\wedge} F_i(y))$$

. *2/ For all* $x, y \in X$,

$$\begin{aligned} R(x,y) \geq \underset{i\in I}{\vee}\alpha_i &\Longrightarrow F_i(x) \leq F_i(y), \forall i \in I, \\ &\Longrightarrow \underset{i\in I}{\wedge} F_i(x) \leq \underset{i\in I}{\wedge} F_i(y), \\ &\Longrightarrow (\underset{i\in I}{\wedge} F_i)(x) \leq (\underset{i\in I}{\wedge} F_i)(x). \end{aligned}$$

Hence, $\underset{i\in I}{\wedge} F_i$ *is a fuzzy* $\alpha_i - filters$ *of X.*

Theoreme 3.2.8 *[4]*

Let (X,R) be a fuzzy lattice and $\alpha \in]0,1]$. *A non-constant mapping F: X* $\longrightarrow [0,1]$ *is a fuzzy* $\alpha - t - filter$ *of X. For all* $\alpha_i - filters$, *if* $F_\alpha = \emptyset$ *or* $F_\alpha = X$ *or* F_α *is an* $\alpha - filter$ *of X.*

Proof

Suppose that F_α *is an* $\alpha - filter$ *of X for all* $\alpha \in]0,1]$. *Let* $x, y \in X$, *if t(F(x),F(y))=0, then we have* $F(x \wedge y) \geq t(F(x), F(y))$. *Otherwise, setting* $t(F(x), F(y)) = \lambda(\lambda \in]0,1])$, *it follows that* $F(x) \wedge F(y) \geq \lambda$, *i.e.,* $x \in F_\alpha$ *and* $y \in F_\alpha$, *which implies* $x \wedge y \in F_\alpha$, *i.e.,* $F(x \wedge y) \geq \lambda$. *Hence,* $F(x \wedge y) \geq t(F(x), F(y))$.

On other hand, if $R(x,y) > \alpha$ *and F(x)=0, we have* $F(x) \leq F(y)$. *If* $R(x,y) > \alpha$ *and* $F(x) \neq 0$, *then* $F(x) = \lambda$, *it follows that* $x \in F_\lambda$ *and* $R(x,y) > \alpha$, *which implies* $y \in F_\lambda$. *Hence,* $\lambda = F(x) \leq F(y)$.

3.2.3 Fuzzy Prime Filters and Fuzzy Prime Ideals

Definition 3.2.5 *[4]*

Let (X,R) be a fuzzy lattice and F be a non-constant fuzzy t-filter in X. Then F is called a fuzzy prime t-filter if for any $x, y \in X$, $F(x \vee y) = F(x) \vee F(y)$.

Example 3.2.4

In Example 3.2.1 F is a fuzzy prime t-filter.

Theoerem 3.2.9 *[4]*

Let (X,R) be a fuzzy lattice and F be a non-constant fuzzy t-filter in X. If $F_\alpha = \emptyset$ or $F_\alpha = X$ or F_α is a prime filter in X for all $\alpha \in]0,1]$, then F is a fuzzy prime t-filter in X.

Proof

Let F be a non-constant fuzzy subset in X such that for every $\alpha \in]0,1], F_\alpha$ is a prime filter in X. For all $x, y \in X$, setting $F(x \vee y) = \lambda$, it follows that $x \vee y \in F_\lambda$ which imply $x \in F_\lambda$ or $y \in F_\lambda$, i.e., $F(x) \vee F(y) \geq \lambda$, it follows that $F(x) \vee F(y) \geq F(x \vee y)$.

Hence, $F(x) \vee F(y) = F(x \vee y)$. It follows that F is a fuzzy prime t-filter in X.

Theoerem 3.2.10 *[4]*

Let (X,R) be a fuzzy lattice and F be a non-constant fuzzy t-filter in X. Setting $M_F = \bigvee_{x \in X} F(x)$. Then, F is a fuzzy prime t-filter if and only if it satisfies, for all $x, y \in X$

$$F(x \vee y) = M_F \Longrightarrow \begin{cases} F(x) = & M_F, \\ & \textit{or} \\ F(y) = & M_F. \end{cases}$$

Proof

Let F be a fuzzy t-filter in X,

$$\begin{aligned} F(x \vee y) = M_F \Longrightarrow & \quad F(x) \vee F(y) = M_F, \\ \Longrightarrow & \quad F(x) = M_F \quad \textit{or} \quad F(y) = M_F. \end{aligned}$$

Conversely, let F be a fuzzy t-filter in X such that for all $x, y \in X, F(x \vee y) = M_F$ imply $F(x) = M_F$ or $F(y) = M_F$. It follows that

$$
\begin{aligned}
F(x \vee y) = M_F \Longrightarrow \quad & F(x) = M_F \quad \textit{or} \quad F(y) = M_F, \\
\Longrightarrow \quad & F(x) \vee F(y) = M_F.
\end{aligned}
$$

Hence, $F(x \vee y) = F(x) \vee F(y)$*.*

Collary 3.2.2 *[4]*

Let (X,R) be a bounded fuzzy lattice and F be a non-constant fuzzy t-filter in X. Then F is a fuzzy prime filter if and only if it satisfies, fo rall $x, y \in X$*,*

$$
F(x \vee y) = F(1) \Longrightarrow \begin{cases} F(x) = & F(1), \\ & \textit{or} \\ F(y) = & F(1). \end{cases}
$$

Proof

Let F be a fuzzy t-filter in X,

$$
\begin{aligned}
F(x \vee y) = F(1) \Longrightarrow \quad & F(x) \vee F(y) = F(1), \\
\Longrightarrow \quad & F(x) = F(1) \quad \textit{or} \quad F(y) = F(1).
\end{aligned}
$$

Conversely, let F be a fuzzy t-filter in X such that for all $x, y \in X$, $F(x \vee y) = F(1)$ *imply* $F(x) = F(1)$ *or* $F(y) = F(1)$*. It follows that*

$$
\begin{aligned}
F(x \vee y) = F(1) \Longrightarrow \quad & F(x) = F(1) \quad \textit{or} \quad F(y) = F(1), \\
\Longrightarrow \quad & F(x) \vee F(y) = F(1).
\end{aligned}
$$

Hence, $F(x \vee y) = F(x) \vee F(y)$*.*

Theoerem 3.2.11 *[4]*

Let (X,R) be a fuzzy lattice and $(F_i)_{i \in I}$ *be a family of fuzzy prime t-filters in X. Then* $\bigcap_{i \in I} F_i$ *is a fuzzy prime t-filter in X.*

Proof

For any $x, y \in X$

$$\begin{aligned}(\bigcap_{i\in I} F_i)(x \vee y) = & \bigwedge_{i\in I} F_i(x \vee y), \\ = & \bigwedge_{i\in I}(F_i(x) \vee F_i(y)), \\ = & \bigwedge_{i\in I} F_i(x) \vee \bigwedge_{i\in I} F_i(y), \\ = & (\bigcap_{i\in I} F_i)(x) \vee (\bigcap_{i\in I} F_i)(y).\end{aligned}$$

Theoerem 3.2.12 *[3]*

Let (X,R) be a fuzzy lattice and F is a non-constant t-filter in X. If F is a prime filter in X then φ_F *is a fuzzy prime t-filter in X.*

Proof

Let F is a prime filter in X and $x, y \in X$

If $x \in X$ *or* $y \in X$*, then* $\varphi_F(x \vee y) = \alpha = \varphi_F(x) \vee \varphi_F(y)$.

If $x \notin X$ *or* $y \notin X$*, then* $\varphi_F(x \vee y) = \beta = \varphi_F(x) \vee \varphi_F(y)$.

Theoerem 3.2.13 *[4]*

Let (X,R) be a fuzzy lattice and F be a non-constant fuzzy filter in X. F is a fuzzy prime t-filter if and only if F_{M_F} *is a prime filter.*

Proof

Suppose that F is a prime t-filter,let $x, y \in X$*, if* $x \vee y \in F_{M_F}$*, it follows that* $F(x \vee y) = F(x) \vee F(y) = F_{M_F}$*, hence* $x \in F_{M_F}$ *or* $y \in F_{M_F}$.

Conversely, suppose that F_{M_F} *is a fuzzy prime filter, if* $F(x \vee y) = M_F$ *it follows that* $x \vee y \in F_{M_F}$*, which implies* $x \in F_{M_F}$ *or* $y \in F_{M_F}$*, hence* $F(x) = F_{M_F}$ *or* $F(y) = F_{M_F}$ *by Theorem 3.2.10 F is a fuzzy prime t-filter.*

Collary 3.2.3 *[4]*

Let (X,R) be a bounded fuzzy lattice and F be a non-constant fuzzy filter in X. F is a fuzzy prime filter if and only if $F_{F(1)}$ *is a prime filter.*

Proof

Suppose that F is a prime t-filter, let $x, y \in X$, if $x \vee y \in F_{F(1)}$, it follows that $F(x \vee y) = F(x) \vee F(y) = F_{F(1)}$, hence $x \in F_{F(1)}$ or $y \in F_{F(1)}$.
Conversely, suppose that $F_{F(1)}$ is a fuzzy prime filter, if $F(x \vee y) = F(1)$ it follows that $x \vee y \in F_{F(1)}$, which implies $x \in F_{F(1)}$ or $y \in F_{F(1)}$, hence $F(x) = F_{F(1)}$ or $F(y) = F_{F(1)}$ by collary 2.4.1 F is a fuzzy prime t-filter.

Theoerem 3.2.14 *[4]*

Let (X,R) be a fuzzy lattice and F be a non-constant fuzzy t-filter satisfying the following condition for any fuzzy filter H and G: $H \cap_t G \subseteq F$ implies $H \subseteq F$ or $G \subseteq F$. Then F is a fuzzy prime t-filter.

Proof

Let F be a non-constant fuzzy t-filter. Let H,G be two filters in X such that $H \cap G \subseteq F_{M_F}$. Suppose that $H \nsubseteq F_{M_F}, G \nsubseteq F_{M_F}$, then exist $x_0, y_0 \in X$ such that $x_0 \in H, x_0 \notin F_{M_F}, y_0 \in G, y_0 \notin F_{M_F}$. Hence $F(x_0) < M_F, F(y_0) < M_F$, let us define the fuzzy set h,g as follows:

$$g(x) = \begin{cases} M_F, & if \quad x \in G, \\ 0, & if \quad x \notin G. \end{cases} \qquad h(x) = \begin{cases} M_F, & if x \in H, \\ 0, & if x \notin H. \end{cases}$$

To show that, g and h are fuzzy t-filter let $x, y \in x$. If $x \notin G$ or $y \notin G$ we have $g(x \wedge y) \geq t(g(x), g(y)) =$ 0(since t(0,0)=0). If $x \in G$ and $y \in G$ imply $x \wedge y \in G$, it follows that $g(x \wedge y) \geq t(g(x), g(y))$. On other hand, suppose that $R(x, y) > 0$, if $x \in G$ implies $y \in G$, it follows that g(x)=g(y) if $x \notin G, g(y) \geq g(x) = 0$.
At the same way we can show that h is a fuzzy t-filter. Therefore,
1/ if $x \in G \bigcap H$ then, $(g \bigcap_t h)(x) = t(M_F, M_F) \leq M_F \leq F(x)$.
2/ if ($x \notin G$ and $x \in H$) or ($x \in G$ and $x \notin H$) then, $(g \bigcap_t h)(x) = t(M_F, 0) = 0 \leq F(x)$.
3/if $x \notin G$ and $x \notin H$ then, $(g \bigcap_t h)(x) = t(0, 0) = 0 \leq F(x)$.
Hence, $g \bigcap_t h < F$, since $h(x_0) = M_F > F(x_0)$ and $h(y_0) = M_F > F(y_0)$, it follows that $h \nsubseteq F$ and $g \nsubseteq F$, contradiction. Hence, $H \subseteq F_{M_F}$ or $G \subseteq F_{M_F}$. Thus, F_{M_F} is a prime filter follows from Theorem 3.2.10 by Theorem 3.2.11 F is a prime t-filter.

3.3 Fuzzy Lattice Isomorphisms

Definition 3.3.1 *[16]*

Let (X, R) *and* (Y, P) *two fuzzy lattices. The function h:* $X \to Y$ *is said to be a homomorphism if it satisfies*

(i) $h(x \vee y) = h(x) \vee h(y)$ *and* $h(x \wedge y) = h(x) \wedge h(y)$*, for all* $x, y \in X$*;*

(ii) $R(x, y) \leq P(h(x), h(y))$*, for all* $x, y \in X$*.*

If h is a bijection, then h is said to be an isomorphism.

Theoerem 3.3.1 *[4]*

Let (X,R) and (Y,P) be two fuzzy lattice and f: $X \longrightarrow Y$ *be an isomorphism. Then F is a fuzzy t-filter in Y if and only if* $f^{-1}(F)$ *is a fuzzy t-filter in X (*$f^{-1}(S) = \{(x, S(f(x)))/x \in X\}$*).*

Proof

Let f: $X \longrightarrow Y$ *be an isomorphism and F be a fuzzy t-filter in Y. For all* $x, y \in X$*,*

$$\begin{aligned} F(f(x \wedge y)) = &\ F(f(x) \wedge f(y)), \\ \geq &\ t(F(f(x)), F(f(y))). \end{aligned}$$

On other hand, let $x, y \in X, R(x, y) > 0 \Longrightarrow F(f(x)) \geq F(f(y))$ *(since* $P(f(x), f(y)) > 0$*).*

Conversely, let $f^{-1}(F)$ *be a fuzzy t-filter and* $a, b \in Y$*, then there exist a unique element* $(x, y) \in X^2$ *such that a=f(x) and b=f(y), it follows that*

$$\begin{aligned} F(a \wedge b) = &\ F(f(x) \wedge f(y)), \\ = &\ F(f(x \wedge y)), \\ \geq &\ t(F(f(x)), F(f(y))), \\ = &\ t(F(a), F(b)). \end{aligned}$$

On other hand,

$$\begin{aligned} P(a, b) > 0 \Longrightarrow &\ P(f(x), f(y)) > 0, \\ \Longrightarrow &\ F(f(y)) \geq F(f(x))(since R(x, y) > 0), \\ \Longrightarrow &\ F(b) \geq F(a). \end{aligned}$$

If h is a bijection, then h is said to be an isomorphism.

Theoerem 3.3.2

Let (X,R) and (Y,P) be two fuzzy lattice and f: $X \longrightarrow Y$ be an isomorphism. Then I is a fuzzy ideal in Y if and only if $f^{-1}(I)$ is a fuzzy ideal in X ($f^{-1}(S) = \{(x, S(f(x)))/x \in X\}$).

***Theoerem 3.3.3** [4]*

Let (X,R) and (Y,P) be two fuzzy lattice and f: $X \longrightarrow Y$ be an isomorphism. Then F is a fuzzy α-t-filter in Y if and only if $f^{-1}(F)$ is a fuzzy α-t-filter in X.

Proof

Let f: $X \longrightarrow Y$ be an isomorphism and F be a fuzzy α-t-filter in Y. For all $x, y \in X$,

$$\begin{aligned} R(x,y) > 0 \Longrightarrow \quad & P(f(x) \wedge f(y)) > \alpha, \\ \Longrightarrow \quad & F(f(y)) \geq F(f(x)). \end{aligned}$$

Conversely, let $f^{-1}(F)$ be a fuzzy t-filter and $a, b \in Y$, then there exists a unique element $(x, y) \in X^2$ such that a=f(x) and b=f(y), it follows that

$$\begin{aligned} P(a,b) > 0 \Longrightarrow \quad & P(f(x), f(y)) > \alpha, \\ \Longrightarrow \quad & R(x,y) > \alpha, \\ \Longrightarrow \quad & F(f(y)) \geq F(f(x)), \\ \Longrightarrow \quad & F(b) \geq F(a). \end{aligned}$$

Theoerem 3.3.4

Let (X,R) and (Y,P) be two fuzzy lattice and f: $X \longrightarrow Y$ be an isomorphism. Then I is a fuzzy α-ideal in Y if and only if $f^{-1}(I)$ is a fuzzy α-ideal in X.

***Theoerem 3.3.5** [4]*

Let (X,R) and (Y,P) be two fuzzy lattice and f: $X \longrightarrow Y$ be an isomorphism. Then F is a fuzzy prime t-filter in Y if and only if $f^{-1}(F)$ is a fuzzy prime t-filter in X ($f^{-1}(S) = \{(x, S(f(x)))/x \in X\}$).

Proof

D Let f: $X \longrightarrow Y$ *be an isomorphism and F be a fuzzy prime t-filter in Y. For all* $x, y \in X$,

$$\begin{aligned} F(f(x \vee y)) = M_{f^{-1}(F)} &\Longrightarrow F(f(x) \vee f(y)) = M_{f^{-1}(F)}, \\ &\Longrightarrow F(f(x)) \vee F(f(y)) = M_{f^{-1}(F)}, \\ &\Longrightarrow F(f(x)) = M_{f^{-1}(F)} \quad or \quad F(f(y)) = M_{f^{-1}(F)}. \end{aligned}$$

such that $M_{f^{-1}(F)} = \bigvee_{i \in I} F(f(x))$

Conversely, let $f^{-1}(F)$ *be a fuzzy t-filter and* $a, b \in Y$, *then there exist a unique element* $(x, y) \in X^2$ *such that a=f(x) and b=f(y), it follows that*

$$\begin{aligned} F(a \vee b) = M_F \Longrightarrow &\quad F(f(x) \vee f(y)) = M_F, \\ \Longrightarrow &\quad F(f(x \vee y) = M_F, \\ \Longrightarrow &\quad F(f(x)) \vee F(f(y)) = M_F, \\ \Longrightarrow &\quad F(a) \vee F(b) = M_F, \\ \Longrightarrow &\quad F(a) = M_F \quad or \quad F(b) = M_F. \end{aligned}$$

Hence, F is a fuzzy prime t-filter in X.

Theoerem 3.3.6

Let (X,R) and (Y,P) be two fuzzy lattice and f: $X \longrightarrow Y$ *be an isomorphism. Then I is a fuzzy prime ideal in Y if and only if* $f^{-1}(F)$ *is a fuzzy prime ideal in X* ($f^{-1}(S) = \{(x, S(f(x)))/x \in X\}$).

Conclusion

In this memory entitled (Filters and Ideals in a Fuzzy Lattice) we have studied some definitions and properties of the fuzzy lattice and we have seen some types of fuzzy filters and fuzzy ideals in a fuzzy lattice.

we noticed that the fuzzy lattice isomorphism preserves the nature and type of fuzzy filters (fuzzy ideals).

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ملخص

في هذه المذكرة قمنا بدراسة بعض التعريفات وخصائص الشبكة الضبابية ورأينا بعض أنواع المرشحات الضبابية والمثل العليا في شبكة ضبابية.
لاحظنا أن التشابه الشبكي الغامض يحافظ على طبيعة ونوع المرشحات الغامضة (المثل العليا الغامضة).

كلمات مفتاحية

المجموعات الضبابية، المرشحات الضبابية، المثل العليا الضبابية، التشابه الشبكي الغامض.

Abstract

In this memory, we have studied some definitions and properties of the fuzzy lattice and we have seen some types of fuzzy filters and fuzzy ideals in a fuzzy lattice.
we noticed that the fuzzy lattice isomorphism preserves the nature and type of fuzzy filters (fuzzy ideals).

Key words

Fuzzy sets, fuzzy filters, fuzzy ideals, fuzzy lattice isomorphism.

Résumé

Dans ce mémoire, nous avons étudié quelques définitions et propriétés du réseau flou et nous avons vu certains types de filtres flous et d'idéaux flous dans un réseau flou.
nous avons remarqué que l'isomorphisme du réseau flou préserve la nature et le type des filtres flous (idéaux flous).

Mot-clés

Ensembles flous, filtres flous, idéaux flous, isomorphisme de treillis flou.