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Theme

A study of some nonlinear elliptic differential equations in $L^m(\Omega)$

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Dedications

I dedicate this modest work :

*-To my dear parents **Kaddour** and **Hannach Louiza**,*

*-To my beloved brothers **Hicham**, **Sadam**, and **Salim**,*

*-To my dear sister **Soundous**,*

-To all my family,

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To all my adorable ones that I have known during all my life.

Contents

Notation

We introduce the necessary notations and definition which are used in the sequel..

H	Hilbert space.
E	Banach space.
X'	Topological dual X.
Ω	Open set of $\mathbb{R}$.
$\partial\Omega$	Border of Ω.
$L^p(\Omega)$	$= \left\{ u : \Omega \to \mathbb{R} \text{ mesurable} : \int_\Omega \lvert u\rvert^p dx \leq \infty \text{ with } \quad 1 \leq p < \infty. \right\}$
$\lVert u\rVert_{L^p}$	$= \left(\int_\Omega \lvert u(x)\rvert^p dx \right)^{\frac{1}{p}}.$
$L^\infty(\Omega)$	$= \{u : \Omega \to \mathbb{R} \text{ mesurable and } \exists C : \lvert u(x)\rvert \leq C\ a.e\ in\ \Omega\}.$
$\lVert u\rVert_{L^\infty}$	$= \inf\{C : \lvert u(x)\rvert \leq C \text{ a.e in } \Omega\}.$
$C(\Omega)$	Continuous function space in Ω.
$D(A)$	Domain of definition of a bounded operator A.
$B(x_0, r)$	open ball of radius r centered at $x0$.
$< ., . >$	Define a scalar product.
$\mathcal{L}(X, Y)$	Set of continuous linear applications.
a.e.	Almost everywhere.
$\mathbb{R}^N$	Euclidean space of dimension N, where N is a nonzero natural number.
x	Vector in $\mathbb{R}^N, x = (x_1, x_2, ..., x_N), x_i \in \mathbb{R}, 1 \leq i \leq N$.
dx	Lebesgue measure in N-dimensional space.
$\lvert E\rvert$	Or $mes(E)$ measure of the set E.
$\lvert.\rvert$	Hilbert norm .
χ_E	Characteristic function of set E.
$\frac{\partial u}{\partial n}$	outward normal derivative.
div u	divergence of the vector u, $divu = \frac{\partial u}{dx_1} + \frac{\partial u}{dx_2} + ... + \frac{\partial u}{dx_N}$.
$\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u.
$f_n \to f$	Denotes that the sequence $\{f_n\}$ converge to f .
$f_n \rightharpoonup f$	Denotes that the sequence $\{f_n\}$ converge weakly to f.
Supp u	Support of the function u.
$\int_\Omega f(x)dx$	Integral of f in Ω with respect to the Lebesgue measure .

$\mathcal{D}(\Omega)$ Space of infinitely differentiable functions on Ω with compact support in Ω.

General Introduction

Partial Differential Equations (PDEs) constitute one of the most important branches of modern mathematics. They provide a rigorous mathematical framework for describing and analyzing numerous phenomena arising in physics, engineering, biology, chemistry, and many other scientific fields. Several natural processes such as heat conduction, fluid flow, diffusion, elasticity theory, and material sciences can be modeled through partial differential equations.

The study of PDEs is mainly concerned with the existence, uniqueness, and qualitative properties of solutions. While some linear equations admit explicit solutions, most models encountered in practical applications are nonlinear and therefore require sophisticated analytical techniques. Consequently, the development of Functional Analysis and Sobolev Space Theory has played a crucial role in the modern treatment of nonlinear partial differential equations.

Among the different classes of PDEs, nonlinear elliptic equations occupy a central position because of their theoretical significance and their numerous applications. Their analysis relies on powerful tools from variational methods, compactness theory, weak convergence techniques, and nonlinear operator theory.

In recent years, increasing attention has been devoted to anisotropic elliptic problems. Unlike isotropic models, anisotropic equations allow different growth conditions in different spatial directions. Such models arise naturally in several applications and provide a more realistic description of many physical phenomena. However, this anisotropic structure introduces additional mathematical difficulties and requires the use of anisotropic Sobolev spaces and appropriate embedding results.

In this work, we study the following nonlinear anisotropic elliptic problem

$$(P)\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, the datum f belongs to $L^m(\Omega)$, and the nonlinear operator $a(x,u,\nabla u)$ satisfies suitable assumptions of growth, coercivity, and monotonicity.

The analysis of this problem presents several mathematical difficulties. The nonlinear character of the operator prevents the direct application of classical linear methods. Furthermore, the anisotropic structure of the equation requires the use of anisotropic Sobolev spaces and specific compactness results. In addition, the low regularity of the datum $f \in L^m(\Omega)$ makes the derivation of *a priori* estimates and the study of convergence more delicate.

The main objective of this dissertation is to establish the existence of weak solutions to the problem

(P). To achieve this goal, we employ techniques from nonlinear functional analysis, particularly the theory of monotone and pseudo-monotone operators. The proof is based on approximation procedures, uniform estimates, compactness arguments, and suitable convergence results in anisotropic Sobolev spaces.

This dissertation is organized into three chapters.

Chapter 1 is devoted to the presentation of the preliminary notions and fundamental mathematical tools used throughout this work. We recall several results concerning Lebesgue spaces, anisotropic Sobolev spaces with variable exponents, embedding theorems, and convergence concepts.

In Chapter 2, we introduce the principal notions of nonlinear operator theory. In particular, we study bounded, coercive, monotone, hemicontinuous, and pseudo-monotone operators. We also present the surjectivity result for pseudo-monotone operators, which constitutes the theoretical framework of our analysis.

Finally, Chapter 3 is devoted to the study of the nonlinear anisotropic elliptic problem under consideration. We establish suitable *a priori* estimates, analyze an approximation scheme, prove the coercivity and pseudo-monotonicity of the associated operator, and then apply the abstract existence theorem to obtain the existence of weak solutions.

Preliminaries

The purpose of this first chapter is to present a number of analytical tools from the theory of Sobolev spaces that will be used throughout this thesis. We will also take the opportunity to introduce the main notations.

1.1 Reminders

Throughout this chapter Ω will denote a bounded domain in $\mathbb{R}^N$,$N \geq 1$, i.e.a connected and bounded open subset of $\mathbb{R}^N$. Its boundary will be denoted by Γ or $\partial\Omega$ and its closure by $\overline{\Omega}$.
Let $u = u(x_1, ..., x_N)$ be a function defined in $\Omega \subset \mathbb{R}^N$. Assuming that exists, the gradient of u at point x is defined as the vector:

$$\nabla u(x) = \Big(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x)\Big)$$

The Euclidean norm of ∇u is denoted by $|\nabla u|$:

$$|\nabla u| = \left[\left|\frac{\partial u}{\partial x_1}\right|^2 + ... + \left|\frac{\partial u}{\partial x_N}\right|^2\right]^{1/2}$$

Definition 1.1. ***(Space of test functions)****: The space $\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$ is the set of infinitely differentiable functions $\varphi : \Omega \longrightarrow \mathbb{R}$ (of class $C^\infty(\Omega)$) with compact support in Ω.*
$\mathcal{D}(\Omega)$ is a vector space, and each element of this space is called a test function.

- *supp* $\varphi = \overline{\{x \in \Omega,\ \varphi(x) \neq 0\}}$,
- $\mathcal{D}(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$.

Definition 1.2. ***(Separable Spaces)*** *A Banach space E is said to be separable if there exists a countable set that is dense in E.*

Definition 1.3 (Dual Space). . *The topological dual of a vector space E over the field $\mathbb{K}$, $(where \mathbb{K} = \mathbb{R}$ or $\mathbb{C})$is defined as the set of continuous linear forms on E,i.e. continuous linear maps form E to $\mathbb{K}$. We denote by $E' = \mathbb{L}(E, \mathbb{K})$.*
E' equipped with the norm $\|.\|$ defined by :

$$\|u\|_{E'} = \sup_{x\in E, \|x\|=1} |u(x)| = \sup_{x\in E, \|x\|\leq 1} |u(x)| = \sup_{x\in E, \|x\|\neq 0} \frac{|u(x)|}{\|x\|}$$

Proposition 1.1.

1. $E_1 \subset E_2$ so $E_2' \subset E_1'$,
2. $\langle u, x\rangle_{E'\times E} \leq \|u\|_{E'}\|x\|_E$.

1.2 Functional spaces

1.2.1 L^p Spaces

Let Ω be an open subset of $\mathbb{R}^N$ and let p be a real number greater than or equal to 1. The Lebesgue Space $L^p(\Omega)$ is defined as:

$$L^p(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \quad \text{is measurable and} \quad \int_\Omega \mid u(x) \mid^p < \infty \Big\}.$$

We equip it with the following norm

$$\|u\|_{L_p(\Omega)} = \|u\|_p = \Big(\int_\Omega |u(x)|^p \Big)^{\frac{1}{p}}, \quad p \geq 1.$$

By $L^\infty(\Omega)$ we denote the set

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable}, \exists M > 0 \mid |u(x)| \leq M \quad \text{a.e.} \}$$

It is equipped with the following norm:

$$\|u\|_{\infty,\Omega} = \inf \quad \{ M > 0 : \mid u \mid \leq M \quad \text{a.e.in} \quad \Omega \}$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{2,\Omega} = \int_\Omega uv dx.$$

Remark 1.1. Let $1 \leq p \leq \infty$, be denoted by p' the conjugate exponent of p

$$\text{i.e.} \quad \frac{1}{p} + \frac{1}{p'} = 1$$

Let $1 \leq p < \infty$. Then the dual space of $L^p(\Omega)$ is $L^{p'}(\Omega)$.

1.2.2 Main inequalities

Let $1 \leq p \leq q \leq \infty$ be two real numbers; and let p' be the conjugate exponent of p.

Lemma 1.1 (**Hölder's inequality**). *[5] If $f \in L^p(\Omega)$, and $g \in L^{p'}(\Omega)$, then $f \cdot g \in L^1(\Omega)$ and*

$$\| f \cdot g \|_{L^1(\Omega)} \leq \| f \|_{L^p(\Omega)} \| g \|_{L^{p'}(\Omega)}$$

If more $\mid \Omega \mid < \infty$ and $f \in L^q(\Omega)$, so $f \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leq \mid \Omega \mid^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\Omega)}$$

In particular,

$$L^q(\Omega) \subset L^p(\Omega), \quad \forall \ 1 \leq p \leq q < \infty$$

Lemma 1.2. *Let $p_i \in [1, +\infty]$ be exponents with $1 \leq i \leq k$ such that:*
$1/p = 1/p_1 + ... + 1/p_k \leq 1$. *Then, for all functions $f_i \in L^{p_i}(\Omega)$, We have $f = f_1 ... f_k \in L^p(\Omega)$ and generalized Hölder inequality*

$$\|f\|_p \leq \| f \|_{p_1} \dots \| f \|_{p_k} .$$

Lemma 1.3 (Young's inequality). *[5] For all non-negative real numbers a, b and every $1 < p < \infty$, then we have:*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad p' = \frac{p}{p-1}. \tag{1.1}$$

Corollarry 1.1 (Generalized Young's inequality). *.[5] for $1 < p < \infty$, $p' = \frac{p}{p-1}$ and any positive ϵ we have:*

$$ab \leq \epsilon^p \frac{a^p}{p} + \frac{1}{\epsilon^{p'}} \frac{b^{p'}}{p'}.$$

Theorem 1.1 (Fisher-Riesz). *L^p is a Banach space for all $1 \leq p \leq \infty$.*

Lemma 1.4 (Divergence and Green's formulas). *[17] Let Ω be a domain in $\mathbb{R}^N$, and $n(x)$ its exterior normal. Let u and v be two regular functions, and w be a vector field defined on Ω. Then,*

$$\int_\Omega \mathrm{div} w \quad dx = \int_{\partial\Omega} w \cdot n d\sigma \quad \textit{(divergence formula)}$$

$$\int_\Omega (\Delta u) v \quad dx = -\int_\Omega \nabla u \cdot \nabla v \quad dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \quad d\sigma \quad \textit{(green formula)}$$

Theorem 1.1 (Rellich-Kondrachov). *[2] Let Ω be a bounded open set of $\mathbb{R}^N$ and $p \geq 1$. If $p < N$ then for any q such that $1 \leq q \leq p^*$ (with $1/p^* = 1/p - 1/N$), the injection of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ is continuous and for all q such that $1 \leq q < p^*$ the injection is compact, that is, the bounded of $W_0^{1,p}(\Omega)$ are relatively compact in $L^q(\Omega)$.*

1.2.3 Convergence theorems

In this section, we present some definitions and results on the convergence sequences of measurable functions

Definition 1.4. *Let (u_n) be a sequence of measurable functions on Ω and u a measurable function on Ω.*

- *The sequence (u_n) converges almost everywhere on Ω to u if and only if*

$$meas\{x \in \Omega : u_n(x) \textit{ does not converge to } \quad u(x)\} = 0,$$

- *The sequence (u_n) is said to be Cauchy in measure if for every $\varepsilon > 0$ and every $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, so*

$$meas\{x \in \Omega : |u_n(x) - u(x)| > \eta\} \leq \varepsilon$$

Lemma 1.5. *[24] Let (u_n) be a sequence of measurable functions of in Ω and $\mathbb{R}^N$. If (u_n) of Cauchy in measure then there exists a sub-sequence of (u_n) converge almost everywhere*

Lemma 1.6. *[23] Let f be a strictly positive measurable function. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable $A \subset \Omega$, we have*

$$\int_A f dx \leq \delta \quad \Rightarrow \quad meas(A) \leq \varepsilon. \tag{1.2}$$

Lemma 1.7. *Let (u_n) the sequence in $L^p(\Omega)$ with $1 < p < \infty$.*
Suppose that

- *(u_n) is bounded in $L^p(\Omega)$;*
- *$u_n \to u$ a.e. on Ω.*

Then, $u_n \to u$ in $L^q(\Omega)$, for all $1 \leq q < p$ and weakly in $L^p(\Omega)$, i.e.,

$$\lim_{n\to+} \int_\Omega u_n v dx = \int_\Omega uv dx, \textit{ for all } \quad v \in L^{p'}(\Omega)$$

Theorem 1.2 (Giuseppe Vitali convergence theorem). *Let $p \in [1, \infty[$ and (f_n) be a sequence of functions in $L^p(\Omega)$ such that*

- *$f_n \to f$ a.e. in Ω*
- *(f_n) equal-integration in Ω, i.e. $\forall\varepsilon > 0, \exists\delta > 0, \forall E \subset \Omega$ with $|E| \leq \delta$, such that*

$$\int_E |f_n(x)|^p dx \leq \varepsilon, \quad \forall n \in \mathbb{N}$$

Or

$$\limsup_{|E|\to 0} \int_E |f_n(x)|^p dx = 0, \quad \forall n \in \mathbb{N}$$

Then

$$f \in L^p(\Omega) \quad \textit{and} \quad f_n \to f \quad \textit{in} \quad L^p(\Omega) \quad \textit{strongly.}$$

Theorem 1.3 (Lebesgue's Dominated Convergence Theorem). *[5]Let $p \in [1, \infty)$, and (u_n) be a sequence of functions in $L^p(\Omega)$ such that*

- *$u_n \to u$ a.e in Ω*
- *there is a function $v \in L^p(\Omega)$ satisfying*

$$|u_n(x)| \leq v(x) \quad a.e \textit{ in} \quad \Omega \quad \forall n \in \mathbb{N}$$

Then $u_n \to u$ in $L^p(\Omega)$, that is

$$u \in L^p(\Omega), \quad \textit{and} \quad \|u_n - u\|_{L^p(\Omega)} \to 0 \quad as \quad n \to \infty.$$

1.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.5. *The Sobolev space $W^{1,p}(\Omega)$, is defined by*

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega), |\ \nabla u \in (L^p(\Omega))^N\right\}$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\| u \|_{W^{1,p}(\Omega)} = \| u \|_p + \| \nabla u \|_p \tag{1.3}$$

and for $p = \infty$

$$\|u\|_{W^{1,\infty}} = \max(\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}).$$

Definition 1.6. *the space $D(\Omega)$ denotes the set of functions in $C^\infty(\Omega)$ with compact support in Ω, we define $1 \leq p < \infty$*

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega), \quad with \quad u = 0 \quad on \quad \partial\Omega\right\}.$$

Remark 1.1. *Since, for $1 \leq p < \infty$, the space $D(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ by definition, we can identify the dual of $W_0^{1,p}(\Omega)$ with a subspace of the space of distributions $D'(\Omega)$. we denote this as: $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$*

In the particular case where $p = 2$, $W^{1,2}(\Omega)$, $W_0^{1,2}(\Omega)$, $W^{-1,2}(\Omega)$, are respectively denoted $H^1(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$

The space H^1 is a Hilbert space equipped with the scalar product.

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$$

The associated norm

$$\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})^{1/2}$$

is a norm equivalent to that of $W^{1,2}(\Omega)$

The following proposition represents the fundamental properties of the spaces $W^{1,p}$, $W_0^{1,p}$. For the proof we refer the reader to reference [**?**].

Proposition 1.1.
- *The space $W^{1,p}$ is a Banach space for $1 \leq p \leq \infty$.*
- *The space $W^{1,p}$ is a separable space for $1 \leq p < \infty$.*
- *The space $W^{1,p}$ is a reflexive space for $1 < p < \infty$.*
- *The space $W_0^{1,p}$ is a separable space . Also reflexive for $1 < p < \infty$.*
- *The space H_0^1 is a separable Hilbert space.*

1.4 Sobolev injections

Sobolev injections are widely used. they provide inequalities between the norms of sobolev spaces and L^p norms. This compactness result is a powerful tool in the study of PDEs, allowing us to transition from a sobolev space to a space of Lebesgue.

Theorem 1.4 (compactness theorem). *Let Ω a bounded open subset of $\mathbb{R}^N$ with $N \geq 1$ and $1 \leq p < \infty$, any bounded subset of $W_0^{1,p}(\Omega)$ is relatively compact in $L^p(\Omega)$, this implies that for any bounded sequence in $W_0^{1,p}(\Omega)$, we can extract a sub-sequence that converges in $L^p(\Omega)$.*

Theorem 1.5 (continuous Injection). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N \geq 1$ and $1 \leq p < \infty$.*

- *If $1 \leq p < N$ then $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-p} > p$ and the injection is continues, i.e $\exists C \geq 0$ (depending only on p, N) such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W_0^{1,p}(\Omega)}$$

- *If $p > N$ then $W_0^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{N}{p}}$. where for $\alpha > 0$, $\quad C^{0,\alpha}$ is the set of Holder continuous functions with exponent α, defined as*

$$C^{0,\alpha} = \{u \in C(\Omega, \mathbb{R})/\exists K \geq 0, \quad |u(x) - u(y)| \leq K\|x - y\|^{\alpha},\ \forall (x, y) \in \Omega^2\}$$

- *If $p = N$ then $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega),\ \forall q \geq 1$*

Lemma 1.8 (Poincare's inequality). *Suppose that $1 \leq p < \infty$, and Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that*

$$\| u \|_{L^p(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)} . \tag{1.4}$$

Moreover , by the density of $\mathcal{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, the inequality (1.4) stay true for any function $u \in W_0^{1,p}(\Omega)$.

1.5 Strong convergence and Weak convergence

1.5.1 Strong convergence

Definition 1.7. *Convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ in a normed vector space $(E, \|\cdot\|_E)$ to an element $x \in E$, defined in the following way:*

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow \|x_n - x\|_E \leq \varepsilon.$$

Or,

$$\lim_{n\to+\infty} \|x_n - x\|_E = 0.$$

We denote by $x_n \to x$ as $n \to +\infty$.

1.5.2 Weak convergence

Definition 1.8. *We say that the sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $x \in E$ if and only if :*

$$\langle f, x_n \rangle \to \langle f, x \rangle, \quad \forall f \in E'.$$

This weak convergence is written as $x_n \rightharpoonup x$, for $n \to +\infty$.

Chapter 2

Nonlinear elliptic equation in bounded domain

This chapter is devoted to the study the Dirichlet problem for the nonlinear anisotropic elliptic equation with degenerate coercivity in variable exponent. We focus particularly on the theory of monotone and pseudo-monotone operators, which we will use later in the approximation framework in Chapter 3. For that purpose, we introduce fundamental notions such as monotonicity, hemicontinuity, coercivity, and pseudo-monotonicity, with simple illustrative examples for each.

2.1 The Operators

2.1.1 Bounded operator

Definition 2.1. *Let V and W be two Banach spaces and let $A : V \to W$ be an operator. We say that A is bounded if it maps every bounded set in V to a bounded set in W; i.e*

$$\forall \rho > 0, \quad \exists C_\rho > 0 : \quad A(B_V(0,\rho)) \subset B_W(0, C_\rho)$$

2.1.2 Monotone Operators

V denotes a real Banach space, $V^{'}$ denotes its topological dual of V.

Definition 2.2. *An operator $A : V \to V^{'}$ is said to be:*

- *Monotone if* $\quad \langle Au - Av, u - v\rangle \geq 0, \forall u, v \in V$
- *strictly monotone if* $\quad \langle Au - Av, u - v\rangle > 0, \forall u, v \in V, u \neq v$

Example 2.1. *The operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined by $Au = -\Delta u$ is monotone, where $H_0^1(\Omega)$ is equipped with the norm of the gradient. Indeed, for $u, v \in H_0^1(\Omega)$,*
we have:

$$\begin{aligned} \langle Au - Av, u - v\rangle &= \int_\Omega \nabla(u-v).\nabla(u-v) \\ &= \|u-v\|^2_{H_0^1(\Omega)} \\ &\geq 0. \end{aligned}$$

2.1.3 Hemicontnuous Operators

Definition 2.3. *An operator $A : V \to W$ is said to be hemicontinuous at the point u_∞ in V if, for any sequence $\{u_n\}$ converging to u_∞, the sequence $\{Au_n\}$ converges weakly to Au_∞ in W, in other words :*

$$\forall v \in V, \quad \forall \{\lambda_n\} \subset \mathbb{R}, \quad \lambda_n \to 0, \quad A(u_\infty + \lambda_n v) \rightharpoonup Au_\infty.$$

If A is hemicontnuous at every point in V, we say that it is hemicontinuous on V. In reflexive spaces and when $W = V'$, and passing from sequential to continuous, we can define hemicontinuitly on V by requiring that:

$$\forall u, v, w \in V \quad \text{the map} \quad \mathbb{R} \ni \lambda \mapsto \langle A(u + \lambda v), w\rangle \in \mathbb{R}$$

is continuous .

Example 2.2. *The operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ defined by $Au = -\Delta u = -\mathrm{div}(\nabla u)$ is hemicontinuous. Indeed, let $u, v, w \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$, we have :*

$$\begin{aligned}
\langle A(u + \lambda v), w\rangle &= \int_\Omega \nabla(u + \lambda v).\nabla w \\
&= \int_\Omega \nabla u.\nabla w + \lambda \int_\Omega \nabla v \nabla w \\
&= a + b\lambda.
\end{aligned}$$

This shows that $\lambda \longrightarrow \langle A(u + \lambda v), w\rangle$ is continuous.

Coercive Operators

Definition 2.4. *Let V be a reflexive Banach space. An operator $A : V \to V'$ is coercive if,*

$$\frac{\langle Av, v\rangle}{\|v\|_V} \to +\infty, \quad as \quad \|v\|_V \to +\infty.$$

Example 2.3. *Let $Au = -\Delta u + a(x)u$, where $a(x) \geq 0$. The operator A maps $H_0^1(D)$ into its dual $H^{-1}(D)$. It is coercive.*

2.1.4 Pseudo-monotonic Operators

Definition 2.5. *Let V be a reflexive Banach space. An operator $A : V \to V'$ is pseudomonotone if*

1. *A is bounded, that is, the image of a bounded subset of V is a bounded subset of V';*
2. *if $u_j \rightharpoonup u$ weakly in V and if*

$$\limsup_{j\to+\infty} \langle Au_j, u_j - u\rangle \leq 0,$$

then

$$\liminf_{j\to+\infty} \langle Au_j, u_j - v\rangle \geq \langle Au, u - v\rangle,$$

for every v in V, where $\langle\cdot, \cdot\rangle$ refers to the duality product between V' and V.

Proposition 2.1. *If $A : V \to V'$ be bounded, hemicontinuous and monotonic, then A is pseudo-monotone.*

Proof. Let $(u_j)_{j\geq 0}$ a sequence weakly converging to u in V. Suppose that

$$\limsup_{j\to+\infty} \langle Au_j, u_j - u\rangle \leq 0,$$

we have A is monotone, we obtain

$$\lim_{j\to+\infty} \langle Au_j, u_j - u\rangle = 0. \tag{2.1}$$

Indeed, the monotonicity of A and the weak convergence of u_j to wards u implies that

$$\langle Au_j, u_j - u\rangle \geq \langle Au, u_j - u\rangle \longrightarrow 0 \qquad \text{as } j \longrightarrow +\infty.$$

So

$$0 \geq \limsup_{j\to+\infty} \langle Au_j, u_j - u\rangle \geq \liminf_{j\to+\infty} \langle Au_j, u_j - u\rangle \geq \lim_{j\to+\infty} \langle Au, u_j - u\rangle = 0.$$

Hence (2.1).

On the other hand, for $t \in]0, 1[$, let

$$w = (1 - t)u + tv.$$

We have

$$\langle Au_j - Aw, u_j - w\rangle \geq 0$$

so that

$$t\langle Au_j, u - v\rangle \geq -\langle Au_j, u_j - u\rangle + \langle Aw, u_j - u\rangle - t\langle Aw, v - u\rangle.$$

From which, thanks to (3.1):

$$\liminf_{j\to+\infty} t\langle Au_j, u - v\rangle \geq -t\langle Aw, v - u\rangle,$$

from which, dividing by t and taking into account (2.1):

$$\liminf_{j\to+\infty} \langle Au_j, u - v\rangle \geq \langle Aw, u - v\rangle. \tag{2.2}$$

$$w = (1 - t)u + tv, \qquad \forall t \in]0, 1[.$$

By making t tend towards 0 in (2.2), and using hemicontinuity, we deduce

$$\liminf_{j\to+\infty} \langle Au_j, u - v\rangle \geq \langle Au, u - v\rangle, \qquad \forall v \in V.$$

Which means that A is pseudo-monotonic. □

Theorem 2.1. *Let V be a reflexive separable Banach space. Let $A : V \to V'$ be a pseudomonotone coercive operator. Then A is surjective, that is, for every f in V' there exists u in V such that*

$$Au = f.$$

Non-linear elliptic equation in bounded domain

We consider the following elliptic problem:

$$\begin{cases} -\operatorname{div}\Big(a(x,u,\nabla u)\Big) = f & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{2.3}$$

where D is a smooth bounded open set of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary denoted by ∂D, the function f belongs to $(W_0^{1,p(\cdot)}(D))'$ the dual space of $W_0^{1,p(\cdot)}(D)$. And we suppose $a : D\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function satisfy the conditions:

For all $x \in D$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$

$$a(x,u,\xi)\cdot\xi \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \quad (\alpha > 0) \tag{2.4}$$

where $a(x,u,\xi) = \big(a_1(\cdot), a_2(\cdot), \ldots, a_N(\cdot)\big)$.

We define the following operator:

$$A : W_0^{1,\bar{p}(\cdot)}(D) \longrightarrow \Big(W_0^{1,\bar{p}(\cdot)}(D)\Big)',$$

$$u \longmapsto Au,$$

by

$$Au = -\operatorname{div}\big(a(x,u,\nabla u)\big). \tag{3.9}$$

The proof is based on the assertion that the operator A is pseudo-monotone and coercive.

2.1.4.1 The coercivity of the operator

Let A defined by (3.9), we have

$$\langle Au, v\rangle = \sum_{i=1}^{N} \int_D a_i(x,u,\nabla u)\, D_i v\, dx,$$

for any

$$v \in W_0^{1,\bar{p}(\cdot)}(D).$$

For any $u = v$, we get

$$\langle Av, v\rangle = \sum_{i=1}^{N} \int_D a_i(x,v,\nabla v)\nabla_i v\, dx.$$

Using the assumptions (2.4) and (**??**), we find

$$\langle A(v), v\rangle \geq \sum_{i=1}^{N} \int_D a_i(x, v, \nabla v)\nabla_i v\, dx \tag{3.10}$$

$$\geq \alpha \sum_{i=1}^{N} \int_D |\nabla_i v|^{p_i(x)}\, dx. \tag{3.11}$$

In view of Proposition 1.15, we obtain

$$\langle A(v), v\rangle \geq \alpha \sum_{i=1}^{N} \min\Big\{\|\nabla_i v\|^{p_i^-}_{L^{p_i(\cdot)}(D)}, \|\nabla_i v\|^{p_i^+}_{L^{p_i(\cdot)}(D)}\Big\},$$

$$\geq \alpha \sum_{i=1}^{N} \|\nabla_i v\|^{\lambda_i}_{L^{p_i(\cdot)}(D)},$$

where

$$\lambda_i = p_i^+ \quad \text{if} \quad \|\nabla_i v\|_{L^{p_i(\cdot)}(D)} \leq 1,$$

or

$$\lambda_i = p_i^- \quad \text{if} \quad 1 \leq \|\nabla_i v\|_{L^{p_i(\cdot)}(D)} < \infty.$$

So

$$\langle A(v), v\rangle \geq \alpha \sum_{i=1}^{N} \|\nabla_i v\|^{\lambda}_{L^{p_i(\cdot)}(D)} - \alpha N$$

$$\geq \frac{\alpha}{N^\lambda} \|v\|^{\lambda}_{W_0^{1,\bar{p}(\cdot)}(D)} - \alpha N,$$

which implies

$$\frac{\langle A(v), v\rangle}{\|v\|_{W_0^{1,\bar{p}(\cdot)}(D)}} \longrightarrow +\infty$$

as

$$\|v\|_{W_0^{1,\bar{p}(\cdot)}(D)} \longrightarrow +\infty,$$

since $\lambda > 1$.

2.1.4.2 The pseudomonotonicity of the operator

Proof. (a) A is bounded. Indeed, let u be a bounded function in $W_0^{1,\bar{p}(\cdot)}(D)$, that is

$$u \in B(0,\rho), \qquad \rho > 0,$$

and for all $v \in W_0^{1,\bar{p}(\cdot)}(D)$, we have

$$\|A(u)\|_{\left(W_0^{1,\bar{p}(\cdot)}(D)\right)'} = \sup_{\substack{v\in W_0^{1,\bar{p}(\cdot)}(D)\\ \|v\|\leq 1}} |\langle A(u), v\rangle|.$$

then, we have

$$\langle Au, v\rangle = \sum_{i=1}^{N}\int_D a_i(x,u,\nabla u)\nabla_i v\, dx,$$

which implies

$$|\langle Au, v\rangle| \leq \sum_{i=1}^{N}\int_D |a_i(x,u,\nabla u)|\,|\nabla_i v|\, dx.$$

Since (3.5) and the fact that $g \in L^\infty(D)$, we have

$$\|A(u)\|_{\left(W_0^{1,\bar{p}(\cdot)}(D)\right)'} = \sup_{\substack{v\in W_0^{1,\bar{p}(\cdot)}(D)\\ \|v\|\leq 1}} |\langle Au, v\rangle|$$

$$\leq \sup_{\substack{v\in W_0^{1,\bar{p}(\cdot)}(D)\\ \|v\|\leq 1}} \int_D |a_i(x,u,\nabla u)|\,|\nabla_i v|\, dx.$$

Using the Holder's inequality and (3.5), we obtain

$$\|A(u)\|_{\left(W_0^{1,\bar{p}(\cdot)}(D)\right)'} \leq 2 \sup_{\substack{v\in W_0^{1,\bar{p}(\cdot)}(D)\\ \|v\|\leq 1}} \|a_i(x,u,\nabla u)\|_{L^{p_i'(\cdot)}(D)}\|\nabla_i v\|_{L^{p_i(\cdot)}(D)}.$$

$$\leq 2 \sup_{\substack{v\in W_0^{1,\bar{p}(\cdot)}(D)\\ \|v\|\leq 1}} \|a_i(x,u,\nabla u)\|_{L^{p_i'(\cdot)}(D)}\Big(\|\nabla_i v\|_{L^{p_i(\cdot)}(D)} + \|v\|_{L^{p_i(\cdot)}(D)}\Big).$$

$$\leq 2\,\|a_i(x,u,\nabla u)\|_{L^{p_i'(\cdot)}(D)}.$$

$$\leq 2\max\left\{\left(\int_D |a_i(x,u,\nabla u)|^{p_i'(\cdot)}dx\right)^{\frac{1}{p_i^+}}, \left(\int_D |a_i(x,u,\nabla u)|^{p_i'(\cdot)}dx\right)^{\frac{1}{p_i^-}}\right\}.$$

$$\leq \left(\int_D |a_i(x,u,\nabla u)|^{p_i'(\cdot)}dx\right)^{\frac{1}{p_i^+}} + \left(\int_D |a_i(x,u,\nabla u)|^{p_i'(\cdot)}dx\right)^{\frac{1}{p_i^-}}.$$

We recall that

$$\forall a \geq 0, \quad \alpha \leq \beta \Longrightarrow a^{\alpha} \leq a^{\beta} + 1.$$

Then, we have

$$\langle \mathcal{A}(u), v \rangle = \sum_{i=1}^{N} \int_D a_i(x, u, \nabla u) \nabla_i v \, dx, \tag{2.5}$$

which implies

$$|\langle \mathcal{A}(u), v \rangle| \leq \sum_{i=1}^{N} \int_D |a_i(x, u, \nabla u)| \, |\nabla_i v| \, dx. \tag{2.6}$$

Since (3.5) and the fact that $g \in L^{\infty}(D)$, we have

$$\begin{aligned} \|\mathcal{A}(u)\|_{\left(W_0^{1,\bar{p}(\cdot)}(D)\right)'} &= \sup_{\substack{v \in W_0^{1,\bar{p}(\cdot)}(D) \\ \|v\| \leq 1}} |\langle \mathcal{A}(u), v \rangle| \\ &\leq \sup_{\substack{v \in W_0^{1,\bar{p}(\cdot)}(D) \\ \|v\| \leq 1}} \sum_{i=1}^{N} \int_D |a_i(x, u, \nabla u)| \, |\nabla_i v| \, dx. \end{aligned} \tag{2.7}$$

Using the HÃfÂ¶lder's inequality (3.6) and (3.8), we obtain

$$\begin{aligned} \|\mathcal{A}(u)\|_{\left(W_0^{1,\bar{p}(\cdot)}(D)\right)'} &\leq 2 \sum_{i=1}^{N} \sup_{\substack{v \in W_0^{1,\bar{p}(\cdot)}(D) \\ \|v\| \leq 1}} \|a_i(x, u, \nabla u)\|_{L^{p_i'(\cdot)}(D)} \|\nabla_i v\|_{L^{p_i(\cdot)}(D)} \\ &\leq 2 \sum_{i=1}^{N} \sup_{\substack{v \in W_0^{1,\bar{p}(\cdot)}(D) \\ \|v\| \leq 1}} \|a_i(x, u, \nabla u)\|_{L^{p_i'(\cdot)}(D)} \Big(\|\nabla_i v\|_{L^{p_i(\cdot)}(D)} + \|v\|_{L^{p_i(\cdot)}(D)} \Big) \\ &\leq 2 \sum_{i=1}^{N} \|a_i(x, u, \nabla u)\|_{L^{p_i'(\cdot)}(D)}, \end{aligned} \tag{2.8}$$

since $\|\nabla_i v\|_{L^{p_i(\cdot)}(D)} \leq \|v\|_{W_0^{1,\bar{p}(\cdot)}(D)} \leq 1$.

By utilizing the relationship between the Luxemburg norm and the modular fluid integral, it follows that:

$$\begin{aligned} \|\mathcal{A}(u)\|_{\left(W_0^{1,\bar{p}(\cdot)}(D)\right)'} &\leq 2 \sum_{i=1}^{N} \max \left\{ \left(\int_D |a_i(x, u, \nabla u)|^{p_i'(\cdot)} dx \right)^{\frac{1}{(p_i')^+}}, \left(\int_D |a_i(x, u, \nabla u)|^{p_i'(\cdot)} dx \right)^{\frac{1}{(p_i')^-}} \right\} \\ &\leq \sum_{i=1}^{N} \left[\left(\int_D |a_i(x, u, \nabla u)|^{p_i'(\cdot)} dx \right)^{\frac{1}{(p_i')^+}} + \left(\int_D |a_i(x, u, \nabla u)|^{p_i'(\cdot)} dx \right)^{\frac{1}{(p_i')^-}} \right], \end{aligned} \tag{2.9}$$

because $(p_i')^- \leq (p_i')^+ \implies \frac{1}{(p_i')^+} \leq \frac{1}{(p_i')^-}$. And we recall that

$$\forall a \geq 0, \quad \alpha \leq \beta \implies a^{\alpha} \leq a^{\beta} + 1. \tag{2.10}$$

we obtain

$$\|\mathcal{A}_i(u)\|_{W^{-1,p'(\cdot)}(D)} \leq C_5 \left(1 + \|u\|_{W_0^{1,\bar{p}(\cdot)}(D)} \right)^{p_*^+ - 1} + 1.$$

So,

$$\|\mathcal{A}(u)\|_{W^{-1,p'(\cdot)}(D)} \leq C_5(1+r)^{p_*^+-1} + 1 = r'.$$

Then $\mathcal{A}_i$ is bounded.

(b) If $u_m \rightharpoonup u$ weakly in $W_0^{1,\vec{p}(\cdot)}(D)$, as $m \to +\infty$, and for any $v \in W_0^{1,\vec{p}(\cdot)}(D)$

$$\begin{aligned} 0 &\geq \limsup_{m\to+\infty} \langle \mathcal{A}u_m, u_m - v\rangle \\ &= \limsup_{m\to+\infty} \left(\sum_{i=1}^{N} \int_D a_i(x, u_m, \nabla u_m)(u_m - v)dx \right. \\ &\quad \left. + \int_D H(x, u_m)(u_m - v)dx \right). \end{aligned} \tag{3.12}$$

Then,

$$\liminf_{m\to+\infty} \langle \mathcal{A}u_m, u_m - v\rangle \geq \langle \mathcal{A}u, u - v\rangle.$$

Indeed, the compact embedding (1.15) yields that $u_m \to u$ in $L^{q(\cdot)}(D)$ for a subsequence still denoted as (u_m). Moreover, we assume that $u_m \to u$ a.e. in D.

Let us first prove that

$$\sum_{i=1}^{N} \int_D [a_i(x, u_m, \nabla u_m) - a_i(x, u, \nabla u)]\nabla_i(u_m - u)dx \to 0.$$

—

We observe that $\int_D H(x, u_m)(u_m - u)dx \to 0$ since $u_m \to u$ in $L^{q(\cdot)}(D)$ and the sequence $(H(x, u_m))_m$ is bounded in $L^{q'(\cdot)}(D)$. By invoking (3.12) and using the fact that $D_i u_m \rightharpoonup D_i u$ weakly in $L^{p_i(\cdot)}(D)$, we get

$$\limsup_{m\to+\infty} \sum_{i=1}^{N} \int_D [a_i(x, u_m, \nabla u_m) - a_i(x, u, \nabla u)]\nabla_i(u_m - u)dx \leq 0.$$

We have for all $i = 1, \ldots, N$

$$\int_D |a_i(x, u_m, \nabla_i u_m) - a_i(x, u, \nabla_i u)| D_i(u_m - u)dx \geq \int_D [a_i(x, u_m, \nabla u) - a_i(x, u, \nabla u)] D_i(u_m - u)dx.$$

Now, let us prove that

$$\liminf_{m\to+\infty} \langle \mathcal{A}u_m, u_m - v\rangle \geq \langle \mathcal{A}u, u - v\rangle, \quad \forall v \in W_0^{1,\vec{p}(\cdot)}(D).$$

Because that, from (**??**), we deduce up to a subsequence

$$D_i u_m \to D_i u \text{ a.e in } D,\ i = 1, \ldots, N.$$

Therefore, for each $i = 1, \ldots, N$

$$a_i(x, u_m, \nabla u_m) \rightharpoonup a_i(x, u, \nabla u),$$

weakly in $L^{p_i'(\cdot)}(D)$ and a.e in D. Thus

$$\lim_{m\to+\infty}\int_D a_i(x,u_m,\nabla u_m)\nabla_i v dx = \int_D a_i(x,u,\nabla u)\nabla_i v dx,$$

for all $v \in W_0^{1,\vec{p}(\cdot)}(D)$. By virtue of Fatou's lemma, we get

$$\liminf_{m\to+\infty}\int_D a_i(x,u_m,\nabla u_m)dx \geq \int_D a_i(x,u,\nabla u)dx. \tag{3.13}$$

On the other hand, we have

$$\int_D H(x,u_m)(u_m - v)dx \to \int_D H(x,u)(u-v)dx, \quad \forall v \in W_0^{1,\vec{p}(\cdot)}(D). \tag{3.14}$$

—

Finally, combining (3.13), and (3.14), we obtain

$$\begin{aligned}\liminf_{m\to+\infty}\langle \mathcal{A}u_m, u_m - v\rangle &\geq \sum_{i=1}^{N}\int_D a_i(x,u_m,\nabla u_m)\nabla_i(u-v)dx + \int_D H(x,u)(u-v)dx\\ &= \langle \mathcal{A}u, u-v\rangle.\end{aligned}$$

Therefore $\mathcal{A}$ is pseudomonotone. Then, according to Theorem 3.7, there exists at least one weak solution $u \in W_0^{1,\vec{p}(\cdot)}(D)$ to Problem (3.3). □

Chapter 3

Anisotropic elliptic equations in $L^m(\Omega)$

3.1 Elliptic problems with $L^m(\Omega)$

Let Ω be an open bounded set of $\mathbb{R}^N$ $(N \geq 2)$, $p_i > 1$, $(i = 1, 2, \ldots, N)$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function.

We assume that there exist two real positive constants α, β and a nonnegative function $h \in L^1(\Omega)$ such that for any $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$ and for almost every $x \in \Omega$, every component $a_j(x, s, \xi)$ of a,

$$a(x, s, \xi)\xi \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{1}$$

$$|a_j(x, s, \xi)| \leq \beta \left(h(x) + |s|^p + \sum_{i=1}^{N} |\xi_i|^{p_i} \right)^{1-\frac{1}{p_j}}, \tag{2}$$

where p satisfies

$$\frac{1}{p} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}.$$

Also,

$$[a(x, s, \xi) - a(x, s, \eta)][\xi - \eta] > 0, \qquad \xi \neq \eta. \tag{3}$$

The aim of this paper is to obtain a solution of the anisotropic elliptic equation

$$(P) \begin{cases} -\operatorname{div}(a(x, u, Du)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense of distributions.

When $f \in L^m(\Omega)$ with m satisfying

$$1 < m < \overline{m} = \frac{N\overline{p}}{N\overline{p} - N + \overline{p}}. \tag{4}$$

We also assume

$$2 - \frac{1}{N} < p_i < \frac{(N-1)\bar{p}}{N - \bar{p}}, \qquad \overline{p} < N, \qquad i = 1, 2, \ldots, N. \tag{5}$$

Set

$$m^* = \frac{Nm}{N-m}, \tag{6}$$

and

$$W_0^{1,(r_i)}(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) \ ; \ D_i u \in L^{r_i}(\Omega) \right\}, \ (r_i \geq 1, i = 1, 2, \cdots, N). \tag{7}$$

If a does not depend on x and s, namely

$$a(x, s, \xi) \equiv a(\xi),$$

where $a(\xi)$ is the vector field whose components are

$$|\xi_i|^{p_i - 2} \xi_i, \qquad (i = 1, 2, \cdots, N; p_i > 1),$$

then in [**?**] it has been proved that there exists a weak solution

$$u \in \bigcap_{i=1}^{N} W_0^{1,(r_i)}(\Omega)$$

with

$$1 \leq r_i < \frac{p_i(\overline{p} - 1)N}{\overline{p}(N-1)},$$

when $f \in M_b(\Omega)$.

Moreover there exists a weak solution

$$u \in \bigcap_{i=1}^{N} W_0^{1,(q_i)}(\Omega)$$

with

$$q_i = \frac{p_i(\overline{p} - 1)N}{\overline{p}(N-1)}$$

when

$$f \in L^1 \log L^1(\Omega).$$

If

$$p_1 = p_2 = \cdots = p_N = p,$$

the existence results have been proved in [**?**] when $f \in M_b(\Omega)$, $f \in L^1 \log L^1(\Omega)$ and $f \in L^m(\Omega)$ with

$$1 < m < \frac{Np}{Np - N + p}.$$

We consider the existence of weak solutions to problem (P) when $f \in L^m(\Omega)$ $(m > 1)$.

If $\overline{p} = N$, then $\overline{m} = 1$, and if $f \in L^m(\Omega)$ with $m > \overline{m} = 1$, problem (P) is known to have a weak solution in

$$\bigcap_{i=1}^{N} W_0^{1,(p_i)}(\Omega)$$

by [?].

Let us now assume that $\overline{p} < N$. Then $\bar{m} > 1$ and if $f \in L^m(\Omega)$ with $m \geq \overline{m}$, problem (P) is known to have a weak solution in

$$\bigcap_{i=1}^{N} W_0^{1,(p_i)}(\Omega).$$

The only case of interest is when f is in $L^m(\Omega)$ with

$$1 < m < \overline{m},$$

and we prove the following theorem.

Theorem 3.1. *Assume that (1)–(3) and (5). Let*

$$1 < m < \overline{m} = \frac{N\overline{p}}{N\overline{p} - N + \overline{p}}$$

and

$$f \in L^m(\Omega).$$

Then problem (P) admits a weak solution

$$u \in \bigcap_{i=1}^{N} W_0^{1,(q_i)}(\Omega),$$

with

$$q_i = \frac{p_i(\overline{p} - 1)m^*}{\overline{p}}.$$

Remark 3.1. *The theorem extends the results of Proposition 1 in [?] and Theorem 3 in [?]. Furthermore it can even be seen as a regularity theorem regarding the solution u obtained in Theorem 1 of [?].*

3.2 Proof of Theorem 3.1

In order to prove Theorem 1.1, we need the following nonisotropic Sobolev inequality.

Lemma 3.1. *If*

$$u \in \bigcap_{i=1}^{N} W_0^{1,(r_i)}(\Omega), \qquad r_i \geq 1, \ (i = 1, 2, \cdot, N),$$

then

$$\|u\|_{L^s(\Omega)} \leq C_1 \left(\prod_{i=1}^{N} \|D_i u\|_{L^{r_i}(\Omega)} \right)^{1/N}, \tag{8}$$

where

$$s = \overline{r}^* = \frac{N\overline{r}}{N - \overline{r}}$$

if $\overline{r} < N$, and $\overline{r}$ satisfies

$$\frac{1}{\overline{r}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{r_i}.$$

C_1 is a positive constant depending only on N and r_i, $(i = 1, 2, \ldots, N)$; if $\overline{r} \geq N$, then (8) is satisfied for every $s \in [1, +\infty)$ and C_1 depends also on s and meas Ω.
By the density property, we may choose a sequence $\{f_k\} \subset C_0^\infty(\Omega)$ such that

$$f_k \to f \quad \text{strongly in } L^m(\Omega), \qquad k \to \infty, \tag{9}$$

and

$$\|f_k\|_{L^m(\Omega)} \leq \|f\|_{L^m(\Omega)}, \qquad k = 1, 2, \ldots \tag{10}$$

3.3 Approximation problem

We consider the following approximation problem:

$$(P_k) \begin{cases} -\operatorname{div}(a(x, u_k, Du_k)) = f_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

3.4 Uniform estimates

Lemma 3.2. *Assume (1)–(3), (9)–(10) and (5). Let*

$$1 < m < \overline{m}.$$

Then for any given $k \geq 1$, there exists a weak solution

$$u_k \in \bigcap_{i=1}^{N} W_0^{1,(p_i)}(\Omega)$$

to problem (P_k). Moreover,

$$\|D_i u_k\|_{L^{q_i}(\Omega)} \leq C_2, \qquad q_i = \frac{p_i(p-1)m^*}{p}, \qquad i = 1, 2, \dots, N, \tag{11}$$

and

$$\|u_k\|_{L^{q^*}(\Omega)} \leq C_2, \tag{12}$$

where

$$q^* = \frac{Nq}{N-q}, \qquad \frac{1}{q} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{q_i},$$

and C_2 is a positive constant independent of k.

Proof. For any given $k \geq 1$, by [**?**], problem (P_k) admits a weak solution

$$u_k \in \bigcap_{i=1}^{N} W_0^{1,(p_i)}(\Omega)$$

such that

$$\int_\Omega a(x, u_k, Du_k)Dv\,dx = \int_\Omega f_k v\,dx, \qquad \forall v \in \bigcap_{i=1}^{N} W_0^{1,(p_i)}(\Omega). \tag{13}$$

For $0 < s < 1$, define

$$\varphi(y) = \int_0^y (1 + |t|)^{-s}\,dt, \qquad \forall y \in \mathbb{R}. \tag{14}$$

It is easy to see that

$$\varphi(u_k) \in \bigcap_{i=1}^{N} W_0^{1,(p_i)}(\Omega).$$

Taking

$$v = \varphi(u_k)$$

in (13), we obtain

$$\int_\Omega a(x, u_k, Du_k)\varphi'(u_k)Du_k\,dx = \int_\Omega f_k\varphi(u_k)\,dx. \tag{15}$$

Using (1) and (14), we get

$$\sum_{i=1}^{N}\int_\Omega |D_iu_k|^{p_i}(1+|u_k|)^{-s}\,dx \le \frac{1}{\alpha(1-s)}\int_\Omega |f_k|(1+|u_k|)^{1-s}\,dx. \tag{16}$$

For any $q_i < p_i$, HÃfÂ¶lder’s inequality and (16) imply

$$\int_\Omega |D_iu_k|^{q_i}\,dx \le \left(\int_\Omega |D_iu_k|^{p_i}(1+|u_k|)^{-s}\,dx\right)^{q_i/p_i}$$

$$\times\left(\int_\Omega (1+|u_k|)^{\frac{sq_i}{p_i-q_i}}\,dx\right)^{1-\frac{q_i}{p_i}}. \tag{17}$$

If

$$q^* = \frac{sq_i}{p_i - q_i}, \tag{18}$$

then by HÃfÂ¶lder’s inequality,

$$\int_\Omega |D_iu_k|^{q_i}\,dx \le C_3\left(\int_\Omega (1+|u_k|)^{(1-s)m'}dx\right)^{\frac{q_i}{m'p_i}}$$

$$\times\left(\int_\Omega (1+|u_k|)^{q^*}dx\right)^{1-\frac{q_i}{p_i}}, \tag{19}$$

where

$$m' = \frac{m}{m-1}.$$

If

$$m'(1-s) = q^*, \tag{20}$$

then

$$\int_\Omega |D_iu_k|^{q_i}\,dx \le C_4 + C_5\left(\int_\Omega |u_k|^{q^*}\,dx\right)^{1-\frac{q_i}{p_i}+\frac{q_i}{m'p_i}}. \tag{21}$$

From (18) and (20),

$$q = (p-1)m^*, \qquad q_i = \frac{p_i}{p}(p-1)m^*. \tag{22}$$

Applying Lemma 2.1 with $r_i = q_i$ and $s = q^*$,

$$\left(\int_\Omega |u_k|^{q^*}\,dx\right) \leq C_1^{q^*}\left(\prod_{j=1}^{N}\|D_j u_k\|_{L^{q_j}(\Omega)}\right)^{q^*/N}. \tag{23}$$

Substituting (23) into (21),

$$\int_\Omega |D_i u_k|^{q_i}\,dx \leq C_4 + C_5 C_1^{q^*\left(1-\frac{q_i}{p_i}+\frac{q_i}{m' p_i}\right)}$$

$$\times\left(\prod_{j=1}^{N}\|D_j u_k\|_{L^{q_j}(\Omega)}\right)^{\frac{q^*}{N}\left(1-\frac{q_i}{p_i}+\frac{q_i}{m' p_i}\right)}. \tag{24}$$

Therefore there exist two positive constants C_6 and C_7 independent of k such that

$$\|D_i u_k\|_{L^{q_i}(\Omega)} \leq C_6 + C_7\left(\prod_{j=1}^{N}\|D_j u_k\|_{L^{q_j}(\Omega)}\right)^{\frac{q^*}{N}\left(\frac{1}{q_i}-\frac{1}{mp_i}\right)}, \tag{25}$$

for all $i = 1, 2, \ldots, N$.

Let

$$d = \prod_{j=1}^{N}\|D_j u_k\|_{L^{q_j}(\Omega)}. \tag{26}$$

Then

$$d \leq C_8 + C_9 d^{q^*\left(\frac{1}{q}-\frac{1}{mp}\right)}. \tag{27}$$

By (22) and the assumptions on m and p,

$$q^*\left(\frac{1}{q}-\frac{1}{mp}\right) < 1. \tag{28}$$

Hence there exists a positive constant C_{10} independent of k such that

$$d \leq C_{10}. \tag{29}$$

Thus (11) follows from (29) and (25).

Lemma 2.1 together with (11) yields (12). □

3.5 passage to the limit

Proof of Theorem 3.1.

By (11) and (12), there exists a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$) such that

$$D_i u_k \rightharpoonup D_i u \quad \text{weakly in } L^{q_i}(\Omega), \qquad i = 1, 2, \dots, N, \tag{30}$$

$$u_k \to u \quad \text{strongly in } L^q(\Omega), \tag{31}$$

and

$$u_k \to u \quad \text{a.e. in } \Omega. \tag{32}$$

Using the same method as in [**?**],

$$D_i u_k \to D_i u \quad \text{a.e. in } \Omega, \qquad i = 1, 2, \dots, N. \tag{33}$$

Since a is a CarathÃfÂ©odory function, (32) and (33) imply

$$a_i(x, u_k(x), Du_k(x)) \to a_i(x, u(x), Du(x)), \quad \text{a.e. in } \Omega. \tag{34}$$

By (2), (11) and (12), there exists a positive constant C_{11} independent of k such that

$$\|a_i(\cdot, u_k, Du_k)\|_{L^{\frac{p_i(p-1)m^*}{(p_i-1)p}}(\Omega)} \leq C_{11}. \tag{35}$$

Hence

$$a_i(\cdot, u_k, Du_k) \rightharpoonup a_i(\cdot, u, Du)$$

weakly in

$$L^{\frac{p_i(p-1)m^*}{(p_i-1)p}}(\Omega). \tag{36}$$

Passing to the limit in (13),

$$\int_\Omega a(x, u, Du) Dv \, dx = \int_\Omega f v \, dx, \qquad \forall v \in C_0^\infty(\Omega). \tag{37}$$

Therefore u is a weak solution to problem (P) and

$$u \in \bigcap_{i=1}^{N} W_0^{1,(q_i)}(\Omega),$$

with

$$q_i = \frac{p_i(p-1)}{p} m^*.$$

Conclusion

In this thesis, we have successfully investigated the existence, qualitative properties, and regularity of weak solutions for some classes of nonlinear and anisotropic elliptic differential equations with data in Lebesgue spaces $L^m(\Omega)$.

Through this work, we demonstrated how the abstract theory of operators—specifically pseudo-monotone, coercive, and monotone operators—can be effectively applied to establish the existence of solutions in bounded domains even when the data lacks high regularity. Furthermore, by employing nonisotropic Sobolev inequalities and fine a priori estimation techniques, we established existence and regularity results within the challenging framework of anisotropic spaces.

While the results presented in this work provide robust answers to several questions regarding elliptic problems with L^m data, they also open up new perspectives for future research. Interesting extensions could include:

- Investigating the evolution counterparts of these problems, such as anisotropic parabolic equations with variable exponents and low-regularity initial data.
- Exploring the system case, where multiple coupled nonlinear elliptic equations interact under similar nonisotropic diffusion conditions.
- Analyzing the behavioral features of solutions when the domain is unbounded or when the equations involve critical Sobolev exponents.

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Abstract

This thesis is devoted to the study of some classes of nonlinear and anisotropic elliptic differential equations with data belonging to Lebesgue spaces $L^m(\Omega)$.

In the first part, we recall the essential preliminaries, including properties of classical Lebesgue and Sobolev spaces, convergence theorems, and fundamental topological tools such as the Faedo-Galerkin method and Minty's lemma.

The second part focuses on the existence and uniqueness of weak solutions for nonlinear elliptic equations in bounded domains, utilizing the properties of pseudo-monotone, hemicontinuous, and coercive operators.

Finally, we investigate the framework of anisotropic elliptic equations, establishing the priori estimates and regularity of solutions under nonisotropic Sobolev embeddings when the data is locally integrable or lies in $L^m(\Omega)$.

Résumé

Cette thèse est consacrée à l'étude de certaines classes d'équations différentielles elliptiques non linéaires et anisotropes avec des données appartenant aux espaces de Lebesgue $L^m(\Omega)$.

Dans la première partie, nous rappelons les préliminaires essentiels, y compris les propriétés des espaces classiques de Lebesgue et de Sobolev, les théorèmes de convergence, ainsi que les outils topologiques fondamentaux tels que la méthode de Faedo-Galerkin et le lemme de Minty.

La deuxième partie se concentre sur l'existence et l'unicité des solutions faibles pour les équations elliptiques non linéaires dans des domaines bornés, en utilisant les propriétés des opérateurs pseudo-monotones, hémicontinus et coercifs.

Enfin, nous étudions le cadre des équations elliptiques anisotropes, en établissant les estimations a priori et la régularité des solutions sous des injections de Sobolev non isotropes lorsque les données sont localement intégrables ou appartiennent à $L^m(\Omega)$.