الجمهـــورية الجزائــــرية الديمقــــراطية الشعــــبية
People's Democratic Republic of Algeria
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research
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Faculty of Technology
كـلية التكنـــولـــوجيـــا

Vice Deanship of Post-Graduation, Scientific Research and External Relations
نيابة العمادة لما بعد التدرج والبحث العلمي والعلاقات الخارجية

رقم: 194/ ن.ع.ب.ع /ك.ت /2025

المسيلة في: 27 ماي 2025

مستخلص من محضر إجتماع المجلس العلمي لكلية التّكنولوجيا

رقم الدورة :25/03، المنعقد بتاريخ: 2025/05/27.

في يوم الثلاثاء السابع والعشرين من شهر ماي عام ألفين وخمسة وعشرين، وعلى الساعة الواحدة ونصف (13:30) بعد الزوال، إنعقد إجتماع للمجلس العلمي لكليّة التّكنولوجيا، في جلسة عادية بقاعة المناقشات بالكلية، ومن بين النقاط المطروحة للنقاش في جدول الأعمال:

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وتمت الموافقة بالاجماع على هذه المطبوعة.

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Lecture Notes on Mathematics 2

Domain: Exact Sciences and Engineering

Field: Siences and technology

Level: First Year (L1)

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Lecture Notes on Mathematics 2 for First-Year STEM Students

Technology, Computer Science, and Natural Sciences

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May 26, 2025

Abstract

These lecture notes are specifically designed for first-year undergraduate students pursuing degrees in Technology, Computer Science, or Natural Sciences. The material covers foundational concepts essential for STEM disciplines, with a focus on building both theoretical understanding and practical problem-solving skills.

These notes are primarily intended for:

- Freshmen in Computer Science and Information Technology
- First-year Engineering and Technology students
- Natural Sciences undergraduates

Contents

Introduction

Mathematics is the universal framework through which we decode the complexities of the natural and engineered world. Mathematics 2 builds upon the foundational concepts introduced in your first year, equipping you with advanced tools to model, analyze, and solve multifaceted problems across science, technology, engineering, and mathematics (STEM). This course is structured around four chapters including integrals, ordinary differential equations (ODEs), matrices, and multivariable functions. Each chapter carefully designed to bridge abstract theory with real world applications. Whether you aspire to optimize energy systems, simulate biological processes, or design resilient structures, these topics will serve as the bedrock of your analytical toolkit, fostering both critical thinking and computational fluency.

We begin with integrals, expanding your understanding of calculus to quantify accumulation, area, and volume. Beyond calculating the work done by variable forces or the total charge in an electrical circuit, integration underpins probability theory, signal processing, and even machine learning algorithms. You will master techniques like substitution, integration by parts, and partial fractions, empowering you to tackle problems ranging from fluid dynamics in mechanical engineering to data analysis in environmental science.

Next, ordinary differential equations introduce the language of dynamic systems, essential for modeling phenomena such as circuit behavior, population growth, or heat dissipation. By solving first and second-order ODEs, you will learn to predict system evolution over time, a skill critical for designing control systems in robotics, predicting chemical reaction rates, or stabilizing civil infrastructure against dynamic loads.

The course then transitions to matrices, the backbone of linear algebra. From solving systems of equations to representing geometric transformations, matrices are indispensable in computer graphics, structural analysis, and cryptography. You will explore matrix operations, determinants, and inverses, enabling you to decode network flows in electrical grids or optimize resource allocation in industrial processes. Finally, multivariable functions extend calculus to higher dimensions, where variables like temperature, pressure, and velocity interact in complex systems. Through partial derivatives, gradients, and optimization, you will model 3D phenomena—such as stress distribution in materials or airflow over aerodynamic surfaces—and learn to navigate saddle points, maxima, and minima in multidimensional landscapes.

Together, these chapters form a cohesive narrative, illustrating how mathematics transcends disciplinary boundaries to solve modern challenges. By integrating theory with coding exercises, visualizations, and case studies, this course prepares you not only to excel in advanced STEM coursework but also to innovate in research, design, and technology. Welcome to a journey where abstract equations become the blueprints for progress.

Chapter 1

Integrals

Integration, the inverse operation of differentiation, lies at the heart of calculus, unifying the study of accumulation, area, and volume. This chapter begins with simple integrals, exploring the concept of primitives (indefinite integrals) as functions that reverse differentiation, enabling the reconstruction of original functions from their derivatives. We then transition to defined integrals, which quantify accumulated quantities such as the area under a curve or the volume of a solid of revolution using limits of Riemann sums. Through the Fundamental Theorem of Calculus, we bridge these two ideas, revealing how primitives facilitate the exact computation of defined integrals. Applications span geometry (calculating planar regions and 3D volumes) and physics (work, displacement), showcasing integration's power to transform abstract antiderivatives into tangible solutions. Prepare to master techniques like substitution, integration by parts, and numerical approximations, while deepening your intuition for continuous summation.

1.1 Indefinite Integral

1.1.1 General Concepts

Definition 1.1. *Let f be a continuous function on an interval I. A primitive of f is any differentiable function F satisfying $F' = f$ on I.*

Example 1.2. *The function f defined on R by $f(x) = 2x$ has a primitive F defined on R by $F(x) = x^2$.*

Proposition 1.3. *Let f be a continuous function with a primitive F. Then any function $F + c$, where c is a constant, is also a primitive of f.*

Definition 1.4. *Let f be a continuous function on $I \subseteq R$. The **Indefinite Integral** of f is the set of all primitives of f.*

- The indefinite integral of f is denoted by $f(x)dx$, where is the integral sign, $f(x)$ is the integrand, and dx is the differential notation.

Note that x is the variable of integration. Thus, we write:

$$f(x)dx = F(x) + c, \text{ where } F \text{ is a particular primitive and } c \in R.$$

Example 1.5. *$2xdx = x^2 + c$, where $c \in R$.*

Remark 1.6. *The variable x is a dummy variable. Replacing x with t, for example, does not change the result.*

$$\cos xdx = \sin x + c, \text{ and } \cos tdt = \sin t + c,\ c \in R.$$

Properties Let f and g be continuous functions:

1. $f'(x)dx = f(x) + c$, $c \in R$
2. $(f(x)dx)' = f(x)$
3. $(f(x) + g(x))\, dx = f(x)dx + g(x)dx$
4. $\lambda f(x)dx = \lambda f(x)dx$ for any scalar λ

Properties 3 and 4 establish the linearity of the integral operator.

1.1.2 Techniques for Computing Primitives

Integration by Recognition

Recognize the integrand as the derivative of a known function (or composite function). In other words:

$$\text{If } f(x) = F'(x), \text{ then } \int f(x)\, dx = F(x) + c.$$

Example 1.7. ***Example 2***

1) $\int \cos(x)dx =?$
We know $(\sin(x))' = \cos(x)$.
Thus, $\int \cos(x)dx = \int (\sin(x))'\, dx = \sin(x) + c$, $\forall c \in \mathbb{R}$.
2) $\int \cos(x^2)2xdx =?$
We know $(\sin(x^2))' = (\sin(u))' = u' \cos(u)$.
Hence, $(\sin(x^2))' = 2x\cos(x^2)$, so $\int \cos(x^2)2xdx = \sin(x^2) + c$.

Integration by Parts

Proposition 1.8. *Let $f, g : I \to \mathbb{R}$ be two C^1 functions on I. Then:*

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Proof. Consider the derivative of the product $f(x)g(x)$:
$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
Integrate both sides:
$\int (f(x)g(x))'\, dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$.
Therefore:
$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$. □

The idea is to choose f' and g such that f' and g' are easier to integrate.

Example 1.9. $I = \int x\sin(x)dx$

Apply integration by parts:
Let: $f(x) = x \quad f'(x) = 1$
$g'(x) = \sin(x) \quad g(x) = -\cos(x)$
Then:
$I = -x\cos(x) + \int \cos(x)dx$
$I = -x\cos(x) + \sin(x) + c$, where c is the constant of integration.

Remark 1.10. *Integration by parts is frequently used for integrals of the form:*

$$\int x^k \sin(x)dx,\ \int x^k \cos(x)dx,\ \int x^k e^{\alpha x}dx,\ \int x^k \ln(x)dx.$$

Integration by Substitution

If computing $\int f(x)dx$ is difficult, replace x with $\varphi(t)$, a differentiable function. Then $dx = \varphi'(t)dt$, leading to:

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt$$

Example 1.11. $I = \int \sin(x)\cos(x)dx$

Let $t = \sin(x)$, so $dt = \cos(x)dx$.
Then $I = \int tdt = \frac{1}{2}t^2 + c = \frac{1}{2}\cos^2(x) + c$.

Remark 1.12. $\int \dfrac{g'(x)}{g(x)}dx = \int \dfrac{dt}{t} = \ln|t| + c = \ln|g(x)| + c$

Here, set $t = g(x) \Rightarrow dt = g'(x)dx$. Success depends on choosing the right substitution to simplify calculations.

1.1.3 Integration of Rational Functions

Definition 1.13. *A rational function is of the form* $f(x) = \dfrac{p(x)}{q(x)}$*, where* $p(x)$ *and* $q(x)$ *are polynomials with* $q(x) \neq 0$.

1.1.4 Integration of Rational Functions

Let $f(x) = \dfrac{\alpha x + \beta}{ax^2 + bx + c}$, where $\alpha, \beta, a, b, c \in \mathbb{R}$ and $a \neq 0$. Three cases arise:

First case: If $\Delta = b^2 - 4ac > 0$, the denominator $ax^2 + bx + c$ has two distinct real roots x_1, x_2.

Then $f(x)$ decomposes as:

$$f(x) = \frac{A}{x - x_1} + \frac{B}{x - x_2}$$

Thus:

$$\int f(x)dx = A\ln|x - x_1| + B\ln|x - x_2| + c$$

Example 1.14. $\int \frac{2x+3}{x^2-x-2}dx = \int \frac{2x+3}{(x+1)(x-2)}dx = \int \frac{A}{x+1}dx + \int \frac{B}{x-2}dx$

Determine A and B:
For A, multiply by $x+1$ and set $x=-1$:
$A = \frac{2(-1)+3}{-1-2} = -\frac{1}{3}$.
For B, multiply by $x-2$ and set $x=2$:
$B = \frac{2(2)+3}{2+1} = \frac{7}{3}$.
Thus:
$\int \frac{2x+3}{x^2-x-2}dx = -\frac{1}{3}\ln|x+1| + \frac{7}{3}\ln|x-2| + c$.
Second case: If $\Delta = 0$, the denominator has a repeated root x_0.
Then $f(x)$ decomposes as:

$$f(x) = \frac{A}{(x-x_0)^2} + \frac{B}{x-x_0}$$

Thus:

$$\int f(x)dx = -\frac{A}{x-x_0} + B\ln|x-x_0| + c$$

Example 1.15. *Compute* $\int \frac{5x+6}{x^2+2x+1}dx$

Decompose:
$\int \frac{5x+6}{(x+1)^2}dx = \int \frac{1}{(x+1)^2}dx + \int \frac{5}{x+1}dx = -\frac{1}{x+1} + 5\ln|x+1| + c$.
Third case: If $\Delta < 0$, the denominator has no real roots. Complete the square:

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)$$

Example 1.16. $I = \int \frac{x+4}{x^2+2x+5}dx$

Rewrite:
$I = \frac{1}{2}\int \frac{2x+8}{x^2+2x+5}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}dx + 3\int \frac{dx}{x^2+2x+5}$.
Compute $I_2 = \int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right)$.
Thus:

$$I = \frac{1}{2}\ln|x^2+2x+5| + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + c$$

Partial Fraction Decomposition

Proposition 1.17. *A rational function* $\frac{p(x)}{q(x)}$ *decomposes into a polynomial part (by performing polynomial division if* $\deg p \geq \deg q$*) and partial fractions such that:*

- Each irreducible linear factor $(ax+b)^k$ *in the denominator generates k partial fractions of the form:*

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}, \quad \text{with } A_i \in \mathbb{R}.$$

- Each irreducible quadratic factor $(ax^2+bx+c)^k$ *in the denominator generates k partial fractions of the form:*

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}, \quad \text{with } A_i, B_i \in \mathbb{R}.$$

Example 1.18. $\frac{x}{x(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$

$$\frac{2x}{x^2+2x+1} = \frac{2x}{(x+1)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2}$$

$$\frac{x-3}{x(x^2+1)^2} = \frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2}$$

$$\frac{x^5}{(x^2-1)(x^2-4x+5)} = \frac{x^5}{(x-1)(x+1)(x^2-4x+5)} = x+4+\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2-4x+5}$$

A) Integration of Simple Elements $\frac{A}{(x-x_0)^n}$

- If $n = 1$:

$$\int \frac{A}{x-x_0}dx = A\ln|x-x_0| + c$$

- If $n \geq 2$:

$$\int \frac{A}{(x-x_0)^n}dx = \frac{A}{(n-1)(x-x_0)^{n-1}} + c$$

B) Integration of Simple Elements $\frac{Bx+c}{(ax^2+bx+c)^n}$

Rewrite the integrand as:

$$\frac{Bx+c}{(ax^2+bx+c)^n} = \alpha\frac{2ax+b}{(ax^2+bx+c)^n} + \delta\frac{1}{(ax^2+bx+c)^n}$$

Integrate using substitution $u = ax^2+bx+c$:

$$\int \frac{2ax+b}{(ax^2+bx+c)^n}dx = \int \frac{u'}{u^n}dx = \frac{1}{(n-1)u^{n-1}} + c$$

For $\int \frac{dx}{(ax^2+bx+c)^n}$:

- If $n = 1$, complete the square:

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c}{a}-\frac{b^2}{4a^2}\right)} = \frac{1}{a}\arctan\left(\frac{x+\frac{b}{2a}}{\sqrt{\frac{c}{a}-\frac{b^2}{4a^2}}}\right) + c$$

- If $n \geq 2$, use reduction formulas via integration by parts to relate I_n to I_{n-1}.

Integrating $p(x)e^{\alpha x}$

To compute $\int p(x)e^{\alpha x}dx$, where p is a polynomial and α is a scalar:
- Use successive integration by parts if $\deg p(x)$ is small. - Alternatively, assume a primitive of the form $\Phi(x)e^{\alpha x}$ with $\deg p(x) = \deg \Phi(x)$ and solve for coefficients.

1.1.5 Integration of Trigonometric Functions

Integrals of the form $\int \sin(\alpha x)\cos(\beta x)dx$ Use trigonometric identities:

$$\sin(\alpha x)\cos(\beta x) = \frac{1}{2}\left[\sin((\alpha+\beta)x) + \sin((\alpha-\beta)x)\right]$$

Integrals of the form: $\int \cos^p(x)\sin^q(x)dx$, $p, q \in \mathbb{N}$

Case 1: If p is odd, substitute $t = \sin(x)$.
Case 2: If q is odd, substitute $t = \cos(x)$.
Case 3: If both are even, use power-reduction formulas.

Bioche's Rules

For $\int f(\sin x, \cos x)dx$, use substitution based on symmetry:
- $u = \cos x$ if f is even in $\cos x$. - $u = \sin x$ if f is even in $\sin x$. - $u = \tan x$ if periodic in π.

Example 1.19. $\int \frac{\cos x}{2-\cos^2 x}dx = \arctan(\sin x) + c$

1.2 Definite Integral

Let $f : I \to \mathbb{R}$ and $a, b \in I$. Then:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Example 1.20. $\int_0^{\frac{3}{2}} x^2 dx = \left[\frac{x^3}{3}\right]_0^{\frac{3}{2}} = \frac{9}{8}$

1.2.1 Riemann Integral

In the presentation of the Riemann integral, step functions play a pivotal role. We begin by outlining their properties and defining their integrals.

Step Functions

Definition 1.21. *A* ***subdivision*** *of a compact interval (i.e., closed and bounded)* $[a, b]$ *in* $\mathbb{R}$ *is a finite set of points* $x_0, x_1, x_2, \ldots, x_n$ *such that:*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Note that $x_i - x_{i-1} > 0$ for all $i = 1, \ldots, n$. The **mesh** of the subdivision is the real number $\max_{1 \leq i \leq n}(x_i - x_{i-1})$.

Definition 1.22. *A function* $f : [a, b] \to \mathbb{R}$ *is called a* ***step function*** *if there exists a subdivision* $\{x_0, x_1, \ldots, x_n\}$ *of* $[a, b]$ *such that* f *is constant on each subinterval* (x_{i-1}, x_i) *for* $i = 1, 2, \ldots, n$.

A function is called a step function on $\mathbb{R}$ *if there exists an interval* $[a, b]$ *such that* f *is zero outside* $[a, b]$ *and a step function on* $[a, b]$.

Example 1.23. *1) The function* f *defined on* $[0, 1]$ *by:*

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \in \left(0, \frac{1}{2}\right) \\ 3 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x \in \left(\frac{1}{2}, 1\right) \end{cases}$$

is a step function on $[0, 1]$.

2) A constant function on $[a, b]$ *is a step function on* $[a, b]$.

Integral of Step Functions

Definition 1.24. *Let* f *be a step function on* $[a, b]$*, where* $f(x) = c_i$ *for* $x \in (x_{i-1}, x_i)$ *and* $i = 1, 2, \ldots, n$*. The* ***integral of*** f *over* $[a, b]$ *is defined as:*

$$I(f) = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$$

This sum S_n is called the Riemann sum of f over $[a, b]$, and we denote it by $I(f) = \int_a^b f(x)dx$.

Remark 1.25. *1) The value* $I(f)$ *depends only on* f *and not on the choice of subdivision.*

2) $I(f)$ *is independent of* x.

3) If f *is a step function on* $[a, b]$ *with* $f \geq 0$*, then* $\int_a^b f(x)dx = 0 \not\Rightarrow f = 0$.

4) The integral $\int_a^b f(x)dx$ *does not depend on the values of* f *at the subdivision points.*

Example 1.26. *1) Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = c$. Then:*

$$\int_a^b f(x)dx = (b - a)c$$

2) For the function f defined on $[0, 1]$ by:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \in \left(0, \frac{1}{2}\right) \\ 3 & \text{if } x = \frac{1}{2} \\ 1 & \text{if } x \in \left(\frac{1}{2}, 1\right) \end{cases}$$

the integral is:

$$\int_0^1 f(x)dx = \frac{1}{2}\left(\frac{1}{2} - 0\right) + 1\left(1 - \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

1.2.2 Properties

Proposition 1.27 (Linearity of the Integral)**.** *Let f and g be step functions on $[a, b]$, and let λ be a real constant. Then:*

1. $\int_a^b (\lambda f)(x)dx = \lambda \int_a^b f(x)dx$
2. $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Proposition 1.28 (Monotonicity of the Integral)**.** *Let f and g be step functions on $[a, b]$. Then:*

1. *If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$.*
2. *If $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.*
3. *For any $u \in [a, b]$, $\int_u^u f(x)dx = 0 \nRightarrow f = 0$.*
4. *For $u, v \in [a, b]$ with $u < v$, $\int_u^v f(x)dx = -\int_v^u f(x)dx$.*
5. *If $c \in (a, b)$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.*

Chapter 2

Ordinary differential Equations

Ordinary differential equations (ODEs) are the cornerstone of modeling dynamic systems, describing relationships between functions and their derivatives to capture phenomena such as growth, motion, decay, and equilibrium. These equations classified by order, linearity, and solution techniques serve as governing principles in physics, engineering, biology, and economics. This chapter introduces the foundational theory of ODEs, focusing on first-order equations, linear second-order equations, and methods like separation of variables, integrating factors, and characteristic equations. We explore initial and boundary value problems, emphasizing analytical solutions. By linking theoretical frameworks to real world applications from population dynamics to mechanical oscillations we unravel how ODEs translate abstract mathematics into predictive tools for understanding continuous change.

2.1 Definitions and Properties

Definition 2.1. *An ordinary differential equation, denoted (ODE), of order n is a relation between the real variable x, an unknown function $x \mapsto y(x)$, and its derivatives $y', y'', \ldots, y^{(n)}$ at x, defined by:*

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

where: $y' = \dfrac{dy}{dx}$ and $y^{(n)} = \dfrac{d^n y}{dx^n}$.

If $n = 1$, the function $F(x, y, y') = 0$ is called a **first-order differential equation**.

If $n = 2$, the function $F(x, y, y', y'') = 0$ is called a **second-order differential equation**.

$$\begin{array}{rl} y'(x) - x = 0: & \text{first-order differential equation.} \\ y''(x) - y'(x) = 2x\sin(x): & \text{second-order differential equation.} \\ y^{(4)}(x) + 2y''(x) - y(x) = x: & \text{fourth-order differential equation} \end{array}$$

Notation: We write y instead of $y(x)$, and y' instead of $y'(x)$.

For example: $y' = \cos x$ means $y'(x) = \cos x$.

2.2 First-Order Differential Equations

Definition: An equation of the form $F(x, y, y') = 0$, where y is a function of x, is called a first-order differential equation.

2.2.1 Separable Differential Equations (SDE)

They are of the form:

$$y' f(y) = g(x)$$

where f and g are given functions with known antiderivatives F and G. We have:

$$\int y' f(y)\, dy = \int g(x)\, dx \quad (y' = \tfrac{dy}{dx}) \quad \Longrightarrow \ F(y) = G(x) + C \quad \text{where } C \in \mathbb{R}.$$

Example 2.2. *1.*

Example 2.3. *Solve the equation:* $y' = x^2 + 1$.

We have $y' = x^2 + 1 \implies \int y'\, dx = \int (x^2 + 1)\, dx$.
$\implies \int dy = \int (x^2 + 1)\, dx$
$\implies y = \frac{x^3}{3} + x + C, \quad \forall C \in \mathbb{R}$.

2 Integrate the following equation: $y' = xy$.

We have $y' = xy \implies \dfrac{y'}{y} = x$
$\implies \int \frac{1}{y}\, dy = \int x\, dx$
$\implies \ln|y| = \frac{x^2}{2} + C$ with $C \in \mathbb{R}$.
Thus, any non-zero solution is of the form:

$$y(x) = Ke^{\frac{x^2}{2}}, \quad \text{with } K \in \mathbb{R}^*.$$

2.2.2 Homogeneous Differential Equations (HDE)

They are of the form:

$$y' = F\left(\frac{y}{x}\right)$$

To solve this equation, we set $t = \frac{y}{x}$ (i.e., $y = xt$ and $y' = t'x + t$), where t is a function of x. This transforms the equation into a separable differential equation.

Example 2.4. $xy' = x + y$

$xy' = x + y$ is a homogeneous equation, as it can be written as $y' = 1 + \frac{y}{x}$.
Setting $\frac{y}{x} = t$ (i.e., $y = xt$), we obtain the equation $t'x + t = 1 + t$.
Hence, $t' = \frac{1}{x}$ (SDE).
The general solution is $t = \ln|x| + C, \quad C \in \mathbb{R}$.
Thus, the general solution of the homogeneous equation is:

$$y = x \ln|x| + K, \quad K \in \mathbb{R}.$$

2.2.3 Linear Differential Equations (LDE)

They are of the form:

$$y' + f(x)y = g(x). \tag{(E)}$$

where f and g are given functions.

The equation E is called homogeneous (HE) or without a right-hand side if $g = 0$, i.e.,

$$y' + f(x)y = 0. \tag{(E)}$$

The general solution of the complete equation E is of the form:

$$y_g = y_0 + y_p$$

where y_p is a particular solution of E and y_0 is the general solution of E_0.

Method of Variation of Constants

The method of variation of constants is used to determine solutions of a nonhomogeneous differential equation by using the solution of the homogeneous equation.

If y_0 is a solution of the homogeneous equation, we seek a particular solution in the form $y(x) = C(x)y_0(x)$.

Example: Solve:

$$xy' - y = x^2 e^x \quad \text{on } (0, +\infty).$$

1. Solve the homogeneous equation $xy' - y = 0$:

$$\begin{aligned} x\tfrac{dy}{dx} - y = 0 \ &\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \\ &\Rightarrow \ln y = \ln x + \ln C \\ &\Rightarrow y = Cx. \end{aligned}$$

2. Seek a particular solution of the form $y = C(x)x$. Then $y' = C'(x)x + C(x)$. Substituting into the complete equation:

$$C'(x)x^2 + C(x)x - C(x)x = x^2 e^x \Leftrightarrow C'(x) = e^x$$

Thus, $C(x) = e^x + \lambda$, and the general solution is:

$$y = x(e^x + \lambda)$$

2.2.4 Bernoulli Differential Equation

A Bernoulli differential equation is any differential equation of the form:

$$y' + yf(x) = y^n g(x)$$

If $n = 0$, it becomes a complete linear equation. If $n = 1$, it becomes a homogeneous linear equation.

For $n \neq 0$ and $n \neq 1$, and $y \neq 0$, we set $z = y^{1-n}$. This transforms the equation into a linear form:

$$\frac{1}{1-n} z' + z f(x) = g(x).$$

Example 2.5. *Integrate the equation:* $y' + 2y - (x+1)\sqrt{y} = 0$.

Set $z = y^{1-\frac{1}{2}} = \sqrt{y}$. The equation becomes:

$$2z' + 2z = x + 1 \quad \text{(Linear differential equation)}.$$

2.3 Second-Order Differential Equations

2.4 General Concepts

A second-order differential equation is any relation of the form:

$$F(x, y, y', y'') = 0$$

between the variable x, the function $y(x)$, and its first two derivatives.

Example 2.6. $y'' + 2y' + y = 0$; $y'' + 4y' + 3y = e^{-2x}$.

2.5 Incomplete Second-Order Differential Equations (Not Containing y)

These equations are of the form:

$$F(x, y', y'') = 0.$$

To solve, substitute $y' = t$, reducing the equation to a first-order differential equation:

$$F(x, t, t') = 0.$$

Example 2.7. $xy'' + 2y' = 0$ *(Equation E).*

Substitute $y' = t \implies xt' + 2t = 0$ (separable first-order equation):

$$\begin{aligned}
&\implies x\tfrac{dt}{dx} = -2t \\
&\implies \textstyle\int \tfrac{dt}{t} = -2\int \tfrac{dx}{x} \\
&\implies \ln|t| = -2\ln|x| + \ln|C| \\
&\implies t = \tfrac{C}{x^2} = y'.
\end{aligned}$$

Thus, the general solution of E is:

$$y = -\frac{C}{x} + k, \quad k \in \mathbb{R}.$$

2.5.1 Linear Second-Order Differential Equations with Constant Coefficients

These equations are of the form:

$$ay'' + by' + cy = f(x) \tag{(EC)}$$

where a, b, c are constants, and $f(x)$ is the nonhomogeneous term. The associated homogeneous equation is:

$$ay'' + by' + cy = 0 \quad (E_0).$$

The general solution of EC is:

$$y = y_H + y_P,$$

where y_H is the general solution of E_0, and y_P is a particular solution of EC.

Solution Method:

a) Finding y_H: Solve E_0 using the characteristic equation $ar^2 + br + c = 0$. The solution depends on the discriminant $\Delta = b^2 - 4ac$:

1. Case 1: $\Delta > 0$ Two distinct real roots r_1 and r_2:

$$y_H = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \quad C_1, C_2 \in \mathbb{R}.$$

2. Case 2: $\Delta = 0$ One double root $r = -\frac{b}{2a}$:

$$y_H = (C_1 x + C_2) e^{rx}, \quad C_1, C_2 \in \mathbb{R}.$$

3.Case 3: $\Delta < 0$ Complex roots $\alpha \pm \beta i$:

$$y_H = e^{\alpha x} \left(C_1 \cos \beta x + C_2 \sin \beta x \right), \quad C_1, C_2 \in \mathbb{R}.$$

Example 2.8. *Solve the following differential equations:*

$$1)\ y'' + y' - 2y = 0, \quad 2)\ y'' + 2y' + y = 0, \quad 3)\ y'' + y = 0.$$

1. Equation 1: Characteristic equation: $r^2 + r - 2 = 0$ has roots $r_1 = 1$, $r_2 = -2$. General solution:

$$y = C_1 e^x + C_2 e^{-2x}, \quad C_1, C_2 \in \mathbb{R}.$$

2. Equation 2: Characteristic equation: $r^2 + 2r + 1 = 0$ has a double root $r = -1$. General solution:

$$y = e^{-x}(C_1 + C_2 x), \quad C_1, C_2 \in \mathbb{R}.$$

3. Equation 3: Characteristic equation: $r^2 + 1 = 0$ has roots $r = \pm i$. General solution:

$$y = C_1 \cos x + C_2 \sin x, \quad C_1, C_2 \in \mathbb{R}.$$

b) Finding y_P: Particular Solution of EC

$$ay'' + by' + cy = f(x).$$

b.1) Case where $f(x) = e^{\alpha x} P_n(x)$: Here, $P_n(x)$ is a polynomial of degree n, and $\alpha \in \mathbb{R}$. The particular solution depends on α:

- i. If α is not a root of E_0: $y_P = e^{\alpha x} Q_n(x)$, where $Q_n(x)$ is a polynomial of degree n.
- ii. If α is a simple root of E_0: $y_P = x e^{\alpha x} Q_n(x)$.
- iii. If α is a double root of E_0: $y_P = x^2 e^{\alpha x} Q_n(x)$.

Example 2.9. *Solve:* $y'' + 2y' + 4y = xe^x$.

1.Homogeneous solution y_H: Characteristic equation $r^2+2r+4=0$ has complex roots $r = -1 \pm \sqrt{3}i$. General solution:

$$y_H = e^{-x}\left(C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x\right).$$

2.Particular solution y_P: Here, $f(x) = xe^x$, $\alpha = 1$ (not a root of E_0). Assume $y_P = e^x(ax+b)$. Substituting into EC, solve for coefficients $a = \frac{1}{7}$, $b = -\frac{4}{49}$. Thus:

$$y_P = e^x\left(\frac{1}{7}x - \frac{4}{49}\right).$$

Final general solution:

$$y_G = e^{-x}\left(C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x\right) + e^x\left(\frac{1}{7}x - \frac{4}{49}\right).$$

b.2) Case where $f(x) = P_n(x)$: If $f(x)$ is a polynomial $P_n(x)$, the particular solution form depends on c:
- i. If $c \neq 0$: $y_P = Q_n(x)$. -ii. If $c = 0$, $b \neq 0$: $y_P = xQ_n(x)$. -iii.If $c = 0$, $b = 0$: $y_P = x^2Q_n(x)$.
Where $Q_n(x)$ is a polynomial of degree n.

2.5.2 General Method

Method of Variation of Parameters

We aim to solve a differential equation of the form:

$$ay'' + by' + cy = f(x) \quad \text{with } a \neq 0.$$

When the nonhomogeneous term $f(x)$ does not match standard forms, we use the method of variation of parameters. Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation:

$$ay'' + by' + cy = 0.$$

Assume a particular solution of the complete equation in the form:

$$y_P = C_1(x)y_1 + C_2(x)y_2,$$

where $C_1(x)$ and $C_2(x)$ are differentiable functions. Impose the condition:

$$C_1'(x)y_1 + C_2'(x)y_2 = 0.$$

Differentiating y_P, we find:

$$y_P' = C_1y_1' + C_2y_2',$$

and differentiating again:

$$y_P'' = C_1'y_1' + C_2'y_2' + C_1y_1'' + C_2y_2''.$$

Substituting into the original equation and simplifying (using y_1, y_2 as homogeneous solutions), we derive the system:

$$\begin{cases} C_1'y_1' + C_2'y_2' = \dfrac{f(x)}{a}, \\ C_1'y_1 + C_2'y_2 = 0. \end{cases}$$

Example 2.10. *Solve:* $y'' + y = \cos(x)$.

1. Homogeneous solution: The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$. General solution:

$$y_H = C_1\cos(x) + C_2\sin(x).$$

2. Particular solution via variation of parameters: Let $y_1 = \cos(x)$, $y_2 = \sin(x)$. Solve the system:

$$\begin{cases} -C_1'\sin(x) + C_2'\cos(x) = \cos(x), \\ C_1'\cos(x) + C_2'\sin(x) = 0. \end{cases}$$

Solving for C_1' and C_2':

$$C_1'(x) = -\frac{\sin(2x)}{2}, \quad C_2'(x) = \frac{\cos(2x)+1}{2}.$$

Integrate to find:

$$C_1(x) = \frac{\cos(2x)}{4}, \quad C_2(x) = \frac{\sin(2x)}{4} + \frac{x}{2}.$$

Thus, the particular solution is:

$$y_P = \frac{\cos(x)}{4} + \frac{x\sin(x)}{2}.$$

3. General solution:

$$y(x) = C_1\cos(x) + C_2\sin(x) + \frac{\cos(x)}{4} + \frac{x\sin(x)}{2}.$$

—

Example 2.11. *Solve:* $y'' - 3y' + 2y = 2x^2 - 5x + 2$.

1. Homogeneous solution: Characteristic equation: $r^2 - 3r + 2 = 0$ has roots $r_1 = 1$, $r_2 = 2$. General solution:

$$y_H = C_1e^x + C_2e^{2x}.$$

2. Particular solution: Assume $y_P = C_0 + C_1x + C_2x^2$. Substituting into the equation and solving for coefficients:

$$y_P = x^2 + \frac{1}{2}x + \frac{3}{4}.$$

3.General solution:

$$y_G = C_1e^x + C_2e^{2x} + x^2 + \frac{1}{2}x + \frac{3}{4}.$$

Chapter 3

Matrices

Matrices, rectangular arrays of numbers or functions arranged in rows and columns, serve as foundational tools in linear algebra and computational mathematics. This chapter explores their core concepts, including types such as square, diagonal, and identity matrices, as well as essential operations like matrix multiplication, where the product of two matrices is computed through row-column dot products. We delve into key properties such as the determinant, a scalar value encoding a matrix's invertibility, and the inverse of a matrix, which allows solving systems of linear equations when it exists. These concepts underpin applications ranging from geometric transformations to data science, making them indispensable in both theoretical and applied disciplines.

3.1 Definitions and Properties

Definition 3.1. *A matrix with entries in a field $\daleth$ is defined by a number p of columns, a number n of rows, and a set of np elements from $\daleth$ arranged in a table of n rows and p columns.*

The entries are numbered with two indices: the first indicates the row number (numbered from top to bottom), and the second indicates the column number (numbered from left to right). Thus, the entry a_{ij} is located at the intersection of the i^{th} row and the j^{th} column. We denote the matrix by $(a_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq p}}$.

Size of a matrix: A matrix with n rows and p columns, denoted $(a_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq p}}$, is said to be of size $n \times p$ (read "n by p" in order).

Example 3.2. *If $A = (a_{ij})_{\substack{1\leq i\leq 3\\ 1\leq j\leq 2}} = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & 6 \end{pmatrix}$, then $a_{11} = 1$, $a_{21} = -2$, $a_{31} = 3$, and $a_{32} = 6$.*

$$(i+2j)_{\substack{1\leq i\leq 2\\ 1\leq j\leq 3}} = \begin{pmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{pmatrix}$$

Notation 3.3. *The set of $n \times p$ matrices with entries in $\daleth$ is denoted by $M_{n,p}(\daleth)$.*

3.1.1 Special Matrices

Column Matrices

These are matrices with one column: $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Row Matrices

These are matrices with one row: $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$

Square Matrices

These are matrices with the same number of rows and columns. This number is called the order of the matrix. The entries with equal row and column indices (denoted a_{ii}) are called diagonal entries.

For example, $A = \begin{pmatrix} 2 & 0 & 1 \\ 6 & 3 & 5 \\ 7 & 0 & -4 \end{pmatrix}$ is a square matrix of order 3. The diagonal entries are 2, 3, and -4.

The set of square matrices of order n with entries in $\daleth$ is denoted by $M_n(\daleth)$.

Lower Triangular Matrices

A square matrix $(a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq n}}$ is called lower triangular if $a_{ij} = 0$ for all $j > i$, i.e., all entries strictly above the diagonal are zero.

Example 3.4. $\begin{pmatrix} 2 & 0 & 0 \\ -2 & 3 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

Upper Triangular Matrices

A square matrix $(a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq n}}$ is called upper triangular if $a_{ij} = 0$ for all $i > j$, i.e., all entries strictly below the diagonal are zero.

Example 3.5. $\begin{pmatrix} 4 & 1 & 6 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Diagonal Matrices

These are square matrices that are both upper and lower triangular. The only non-zero entries are on the diagonal.

We denote by $\mathrm{Diag}(a_1, a_2, \ldots, a_n)$ the diagonal matrix with entries $(a_1, a_2, \ldots, a_n)$.

Example 3.6. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Scalar Matrices

These are diagonal matrices where all diagonal entries are equal.

Example 3.7. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}$

Identity Matrix

This is the scalar matrix with all diagonal entries equal to 1. We denote the identity matrix of order n by I_n.

Example 3.8. $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Zero Matrix

This is a matrix (not necessarily square) where all entries are zero. We denote the zero matrix with n rows and p columns by $0_{n,p}$. For example, $0_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3.2 Matrix Operations

3.2.1 Equality of Matrices

Two matrices $A = (a_{ij})_{\substack{1\le i\le n \\ 1\le j\le p}}$ and $B = (b_{ij})_{\substack{1\le i\le m \\ 1\le j\le q}}$ are equal, denoted $A = B$, if:

- They have the same number of rows $(n = m)$. - They have the same number of columns $(p = q)$. - Corresponding entries are equal $(a_{ij} = b_{ij})$ for all $1 \le i \le n$ and $1 \le j \le p$.

3.2.2 Matrix Addition

Let $A = (a_{ij})_{\substack{1\le i\le n \\ 1\le j\le p}}$ and $B = (b_{ij})_{\substack{1\le i\le n \\ 1\le j\le p}}$ be two matrices in $M_{n,p}(\daleth)$. The sum $A + B$ is the matrix in $M_{n,p}(\daleth)$ where each entry is the sum of the corresponding entries of A and B. Thus, $A + B = (a_{ij} + b_{ij})_{\substack{1\le i\le n \\ 1\le j\le p}}$.

Proposition 3.9. *If A, B, and C are matrices in $M_{n,p}(\daleth)$:*

-**Associativity:** $(A + B) + C = A + (B + C)$. - **Additive Identity:** $A + 0_{n,p} = A$, where $0_{n,p}$ is the zero matrix. - **:Additive Inverse:** For $A = (a_{ij})$, define $-A = (-a_{ij})$. Then $A + (-A) = 0_{n,p}$. - **Commutativity**: $A + B = B + A$.

Remark 3.10. $A - B = A + (-B)$.

3.2.3 Scalar Multiplication

Let $A = (a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}$ be a matrix in $M_{n,p}(\daleth)$, and let $\lambda \in \daleth$. The scalar product λA is defined as:

$$\lambda A = (\lambda a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}.$$

Example 3.11. $3\begin{pmatrix} 2 & 1 & 4 \\ -2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 12 \\ -6 & 0 & 9 \end{pmatrix}.$

Proposition 3.12. *Let $\alpha, \beta \in \daleth$, and $A, B \in M_{n,p}(\daleth)$. Then:*

1. $\alpha(A + B) = \alpha A + \alpha B$
2. $(\alpha + \beta)A = \alpha A + \beta A$
3. $(\alpha\beta)A = \alpha(\beta A)$
4. $1 \cdot A = A$
5. $0 \cdot A = 0_{n,p}$
6. $(-1) \cdot A = -A$

3.2.4 Matrix Multiplication

Row Matrix and Column Matrix Product

Let $A = (a_1, a_2, \ldots, a_p)$ be a row matrix in $M_{1,p}(\daleth)$ and $B = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}$ be a column matrix in $M_{p,1}(\daleth)$. **Note**: A must have the same number of columns as B has rows. The product AB is the scalar:

$$AB = a_1b_1 + a_2b_2 + \cdots + a_pb_p.$$

Example 3.13. $\begin{pmatrix} 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = (2 \times 1) + (3 \times 4) + (-2 \times 3) = 8.$

Matrix and Column Matrix Product

Let $A = (a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}$ be a matrix in $M_{n,p}(\daleth)$, and $B = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}$ be a column matrix in $M_{p,1}(\daleth)$. The product AB is a column matrix in $M_{n,1}(\daleth)$, where the i-th entry is the dot product of the i-th row of A and B:

$$AB = \begin{pmatrix} a_{11}b_1 + \cdots + a_{1p}b_p \\ \vdots \\ a_{n1}b_1 + \cdots + a_{np}b_p \end{pmatrix}.$$

Example 3.14. $\begin{pmatrix} 3 & 2 & 0 \\ -2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \times 2 + 2 \times 3 + 0 \times 1 \\ -2 \times 2 + 1 \times 3 + 4 \times 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \end{pmatrix}.$

Matrix-Matrix Product

Let $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$ be a matrix in $M_{n,p}(\daleth)$, and $B = (b_{ij})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}}$ be a matrix in $M_{p,q}(\daleth)$. **Note**: A must have the same number of columns as B has rows.

The product AB is a matrix in $M_{n,q}(\daleth)$, where the entry in the i-th row and j-th column is:

$$(AB)_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Example 3.15. $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \times 4 + 0 \times 2 + 1 \times 1 & 1 \times 3 + 0 \times 0 + 1 \times 2 \\ 2 \times 4 + 3 \times 2 + 4 \times 1 & 2 \times 3 + 3 \times 0 + 4 \times 2 \\ 2 \times 4 + 1 \times 2 + 0 \times 1 & 2 \times 3 + 1 \times 0 + 0 \times 2 \end{pmatrix} =$ $\begin{pmatrix} 5 & 5 \\ 18 & 14 \\ 10 & 6 \end{pmatrix}.$

Properties

1. Matrix multiplication is defined only if the number of columns of A equals the number of rows of B.

2. **Associativity**: If $A \in M_{n,p}(\daleth)$, $B \in M_{p,m}(\daleth)$, and $C \in M_{m,q}(\daleth)$, then $(AB)C = A(BC)$.

3. **Non-Commutativity**: Matrix multiplication is generally not commutative.

4. **Identity Matrix**: For $A \in M_{n,p}(\daleth)$, $AI_p = A$ and $I_n A = A$.

5. **Distributivity**: If $A, B \in M_{n,p}(\daleth)$ and $C \in M_{p,q}(\daleth)$, then $(A+B)C = AC+BC$.

Remark 3.16. *The product of two non-zero matrices can be zero. For example:* $A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{pmatrix}$, *then* $AB = 0_{2,3}$, *but* $A \neq 0$ *and* $B \neq 0$.

Matrix Powers

For an integer $k \geq 0$ and a square matrix $A \in M_n(\daleth)$, the k-th power of A is defined as:

$$A^k = \begin{cases} I_n & \text{if } k = 0 \\ A^{k-1}A = \underbrace{A \cdots A}_{k \text{ times}} & \text{if } k \geq 1. \end{cases}$$

Proposition 3.17. *For a square matrix A and integers k, l:*

1. $A^k A^l = A^{k+l}$

2. $(A^k)^l = A^{kl}$
3. $(\lambda A)^k = \lambda^k A^k$ for $\lambda \in \daleth$.
4. If $A = \mathrm{Diag}(a_1, a_2, \ldots, a_n)$ is diagonal, then $A^k = \mathrm{Diag}(a_1^k, a_2^k, \ldots, a_n^k)$.

3.3 Inverse of a Matrix

3.3.1 Determinant

The determinant of a matrix is only defined if the matrix is square.

Definition 3.18. *Let $A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\daleth)$ be a square matrix. The determinant of A, denoted $\det(A)$ or $|A|$, is the element of $\daleth$ defined recursively as follows:*

- If $n = 1$ then $A = (a_{11})$ and $\det(A) = a_{11}$
- If $n \geq 2$ then $A = (a_{ij})_{1\leq i,j\leq n}$ and

$$\det(A) = a_{11}\Delta_{11} - a_{21}\Delta_{21} + \cdots + (-1)^{n-1}a_{n1}\Delta_{n1}$$

Where Δ_{i1} is the determinant of the $M_{n-1}(\daleth)$ matrix obtained by removing the i^{th} row and first column from A.

Example 3.19. *We calculate* $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example 3.20. *The determinant of the matrix* $\begin{pmatrix} 2 & 4 & -2 \\ 1 & -1 & 5 \\ 0 & 2 & 3 \end{pmatrix}$ *is*

$$\begin{vmatrix} 2 & 4 & -2 \\ 1 & -1 & 5 \\ 0 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} -1 & 5 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix} + 0\begin{vmatrix} 4 & -2 \\ -1 & 5 \end{vmatrix} = 2(-3-10) - 1(12+4) + 0 = -42$$

Proposition 3.21. *If we multiply one row of a matrix by an element of $\daleth$, then the determinant of this matrix is multiplied by the same element of $\daleth$.*

3.3.2 Matrix Transpose

Definition 3.22. *If $A = (a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}$ is an $n\times p$ matrix, we call the transpose of A (denoted A^t) the $p\times n$ matrix whose entry in the i^{th} row and j^{th} column is a_{ji}. Thus, $((a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}})^t = (a_{ji})_{\substack{1\leq i\leq p \\ 1\leq j\leq n}}$.*

The columns of A^t are the rows of A, or equivalently, the rows of A^t are the columns of A.

Example 3.23. *The transpose of* $A = \begin{pmatrix} 7 & 3 & 2 \\ 0 & -4 & 5 \end{pmatrix}$ *is* $A^t = \begin{pmatrix} 7 & 0 \\ 3 & -4 \\ 2 & 5 \end{pmatrix}$.

Proposition 3.24. *Let A and B be matrices and $\lambda \in \daleth$. Then:*

1. $(A^t)^t = A$ $(A \in M_{n,p}(\daleth))$
2. $(\lambda A)^t = \lambda A^t$ $(A \in M_{n,p}(\daleth))$
3. $(A+B)^t = A^t + B^t$ $(A, B \in M_{n,p}(\daleth))$
4. $(AB)^t = B^t A^t$ $(A \in M_{n,p}(\daleth), B \in M_{p,q}(\daleth))$. Note the order reversal when taking the transpose of a product.

3.3.3 Matrix Inverse

Definition and Properties

Definition 3.25. *Let $A \in M_n(\daleth)$ be a square matrix of order n. A square matrix B of order n is called a right inverse of A if $AB = I_n$, and a left inverse of A if $BA = I_n$, where I_n is the identity matrix of order n.*

If a matrix A has both a right inverse B and a left inverse C, then $B = C$ and we can unambiguously call B the inverse of A.

Definition 3.26. *A square matrix A is called invertible if it has both left and right inverses. This inverse is then unique. We denote the inverse of an invertible matrix A by A^{-1}. For $A \in M_n(\daleth)$, we have $AA^{-1} = A^{-1}A = I_n$.*

Proposition 3.27. *Let A be an invertible matrix and $\lambda \in \daleth^*$.*

1. The matrix A^{-1} is invertible with inverse A.
2. The matrix λA is invertible with inverse $\frac{1}{\lambda}A^{-1}$.

Proof. (Exercise) □

Theorem 3.28. *Let A and B be two invertible square matrices of the same size. Then the product AB is invertible and its inverse is $(AB)^{-1} = B^{-1}A^{-1}$.*

Theorem 3.29. *For a square matrix A, $\det(A) \neq 0 \iff A$ is invertible. Moreover, if A is invertible then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.*

Inverse Calculation

There are various methods to calculate the inverse of a matrix A.

Cofactor Method

Definition 3.30. *The cofactor matrix $C = com(A)$ is the matrix with entries*

$$C_{ij} = (-1)^{i+j}\det(A_{ij})$$

Where A_{ij} is the matrix obtained by removing the i^{th} row and j^{th} column from A.

Example 3.31. *For* $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix}$

$$\text{com}(A) = \begin{pmatrix} +\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & 0 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -4 & -7 & 12 \\ 5 & 2 & -6 \\ 1 & 4 & -3 \end{pmatrix}.$$

Theorem 3.32. *If A is invertible then*

$$A^{-1} = \frac{1}{\det(A)}(\text{com}A)^t$$

Example 3.33. *For* $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix}$

$\det(A) = 2\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 2 \times (-4) - 3 \times (-5) + 2 \times 1 = 9$

Since $\det(A) \neq 0$, the matrix A is invertible.

$\text{com}(A) = \begin{pmatrix} -4 & -7 & 12 \\ 5 & 2 & -6 \\ 1 & 4 & -3 \end{pmatrix}$

Thus $A^{-1} = \dfrac{1}{\det(A)}(\text{com}A)^t = \dfrac{1}{9}\begin{pmatrix} -4 & 5 & 1 \\ -7 & 2 & 4 \\ 12 & -6 & -3 \end{pmatrix}$.

We verify that $AA^{-1} = A^{-1}A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Chapter 4

Functions of Several Variables

Functions of several real variables, or multivaraible functions form the cornerstone of multivariable calculus, extending the principles of single-variable analysis to higher dimensions. These functions, which map inputs from $\mathbb{R}^{\ltimes}$ to $\mathbb{R}$, model complex phenomena where multiple interdependent factors coexist—from physical fields like temperature distributions to economic optimization problems. This chapter explores foundational concepts such as partial derivatives, gradients, and multiple integrals, which enable the study of continuity, differentiability.

4.1 Definitions and Properties

4.2 Review of Topological Concepts in $\mathbb{R}^n$

For a non-zero integer n, we define $\mathbb{R}^n$ as the Cartesian product of $\mathbb{R}$ with itself n times. Thus, $\mathbb{R}^n$ is the set of vectors with n real components. Whether in theoretical or numerical analysis, as well as in computer science, we primarily manipulate vectors and apply treatments analogous to those used with numbers. We need to measure the proximity of two vectors to discuss limits, continuity, and differentiability of functions whose domain is not an interval of $\mathbb{R}$ but a set of vectors.

Distance

In $\mathbb{R}$; the distance between two real numbers x and y is given by $d(x, y) = |x - y|$.

In $\mathbb{R}^2$; the distance between two points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ is given by $d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

In $\mathbb{R}^n$; the distance between two points $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ is given by $d(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$.

Norm

Definition 4.1. *Let $(E, +, .)$ be a $\mathbb{R}$-vector space. A norm on E is any application N from E to $\mathbb{R}^+$ such that:*

- $\forall x \in E, N(x) = 0 \Longleftrightarrow x = 0.$
- $\forall \alpha \in \mathbb{R}, \forall x \in E, N(\alpha x) = |\alpha| N(x).$
- $\forall x \in E, \forall y \in E, N(x + y) \leq N(x) + N(y)$ (Triangle inequality).

A normed space is a vector space equipped with a norm.

Remark 4.2. *The norm of an element X in $\mathbb{R}^n$ is its distance to the origin (Note that $\mathbb{R}^n$ is a normed $\mathbb{R}$-vector space).*

Usual Norms on $\mathbb{R}^n$ For any element $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$:

- **Euclidean norm** denoted $N_2(X) = \|X\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.
- **Maximum norm** denoted $N_\infty(X) = \|X\|_\infty = \max_{1 \le i \le n} |x_i|$.
- **Sum norm** denoted $N_1(X) = \|X\|_1 = \sum_{i=1}^n |x_i|$.

These three norms are equivalent. (Two norms N_1 and N_2 on a space E are equivalent if and only if there exist positive real numbers a and b such that $\forall x \in E, aN_1(x) \le N_2(x) \le bN_1(x)$).

Open and Closed Balls in $\mathbb{R}^n$

Definition 4.3. *The open ball centered at $X_0 \in \mathbb{R}^n$ with radius $r > 0$, denoted $B(X_0, r)$, is the set of points in $\mathbb{R}^n$ whose distance to X_0 is strictly less than r, i.e.,*

$$B(X_0, r) = \{X \in \mathbb{R}^n, \|X - X_0\| < r\}$$

- In $\mathbb{R}$, the usual norms coincide with the absolute value, so the open ball $B(x_0, r)$ in $\mathbb{R}$ is the open interval $(x_0 - r, x_0 + r)$.
- In $\mathbb{R}^n$ using the Euclidean norm, the ball $B(X_0, r) = \left\{X = (x, y) \in \mathbb{R}^2, \sqrt{(x - x_0)^2 + (y - y_0)^2} < r\right.$ is the open disk centered at X_0 with radius r.
- The open ball for the maximum norm in $\mathbb{R}^2$ is a square. Indeed,

$$\|X - X_0\|_\infty < r \iff \max(|x - x_0|, |y - y_0|) < r \iff |x - x_0| < r \text{ and } |y - y_0| < r$$

Here, $|x - x_0| < r$ characterizes a vertical band of width $2r$, bounded by the lines $x = x_0 - r$ and $x = x_0 + r$. The intersection of these bands is a square centered at X_0.

Definition 4.4. *The closed ball centered at $X_0 \in \mathbb{R}^n$ with radius $r > 0$, denoted $\overline{B}(X_0, r)$, is the set of points in $\mathbb{R}^n$ whose distance to X_0 is less than or equal to r, i.e.,*

$$\overline{B}(X_0, r) = \{X \in \mathbb{R}^n, \|X - X_0\| \le r\}$$

Neighborhood of a Point A neighborhood of a point $X_0 \in \mathbb{R}^n$ is any subset of $\mathbb{R}^n$ containing an open ball centered at X_0.

A neighborhood of infinity is a annulus $\{X \in \mathbb{R}^n, \|X\| > r \text{ with } r > 0 \text{ very large}\}$.

Open Set in $\mathbb{R}^n$ A subset U of $\mathbb{R}^n$ is called open if it is a neighborhood of each of its points. In other words,

$$U \text{ open } \iff (\forall X_0 \in U; \exists r > 0, B(X_0, r) \subset U)$$

- By convention, the empty set $\emptyset$ and $\mathbb{R}^n$ are open sets in $\mathbb{R}^n$.
- The arbitrary union of open sets is open.
- The finite intersection of open sets is open.

Example 4.5. *The set $U = \{(x, y) \in \mathbb{R}^2, x > y\}$ is open; it is the half-plane below the first bisector (excluding the line).*

Closed Set in $\mathbb{R}^n$ A subset F of $\mathbb{R}^n$ is closed if its complement in $\mathbb{R}^n$ is open.

Example 4.6. *The set $F = \{(x, y) \in \mathbb{R}^2, x \leq y\} = \mathbb{R}^2 \setminus U$ is closed.*

- The empty set $\emptyset$ is open, so its complement $\mathbb{R}^n$ is closed. Similarly, since $\mathbb{R}^n$ is open, its complement $\emptyset$ is closed.
- $\mathbb{R}^n$ and $\emptyset$ are the only subsets of $\mathbb{R}^n$ that are both open and closed.
- A general subset of $\mathbb{R}^n$ need not be open or closed; for example, a half-open interval is neither open nor closed, and the set $\mathbb{Q} \times \mathbb{Q}$ is neither open nor closed in $\mathbb{R}^2$.

Bounded Set A subset A of $\mathbb{R}^n$ is bounded if it is contained within a ball. In other words, A is bounded $\Longleftrightarrow \exists r > 0, \forall x \in A, \|X\| \leq r$.

Compact Set in $\mathbb{R}^n$ Any non-empty, closed, and bounded subset K of $\mathbb{R}^n$ is called compact.

4.3 Real-Valued Functions of Several Variables

Definition 4.7. *A real-valued function of several variables is a mapping*

$$\begin{array}{rl} f: & D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R} \\ & X = (X_1, X_2, ..., X_n) \longmapsto f(X) \end{array}$$

D is the domain of definition of f.

- $f(x, y) = 2(x + y)$ is a function of two variables representing the perimeter of a rectangle with length x and width y, defined on $\mathbb{R}^2$.
- $f(x, y) = \dfrac{xy}{x^2 + y^2}$, a function of two variables defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$.
- $f(P, V, T) = PV - nRT$ is a function of three variables representing the ideal gas law, where n is the amount of substance, R is a constant, V is the volume, P is the pressure, and T is the temperature.

Graphical Representation
For $n = 1$; $y = f(x)$ is represented by a curve in the plane $\mathbb{R}^2$.
For $n = 2$; $z = f(x, y)$ is represented by a surface in the space $\mathbb{R}^3$.
For $n \geq 3$; graphical representation is difficult to visualize.

Remark 4.8. *By fixing $z = f(x, y) = k$, we obtain curves called level lines (contour lines) of f, denoted $L_k = \{(x, y) \in \mathbb{R}^2, f(x, y) = k \ (k \text{ real})\}$. These reflect physical realities:*

- On a topographic map, they indicate altitude.
- On a nautical chart, they indicate depth (soundings).
- On a weather map, isobars connect points of equal atmospheric pressure.

Example 4.9. $z = \sin x + \cos(x + y)$

4.4 Continuity of a Real-Valued Function of Several Variables

Definition 4.10. *Let f be a function defined in a neighborhood of a point $A \in \mathbb{R}^n$. We say f is continuous at A if $\lim_{X \to A} f(X) = f(A)$, i.e., $\forall \varepsilon > 0; \forall X \in D_f; \exists \alpha > 0; \|X - A\| < \alpha \Longrightarrow |f(X) - f(A)| \leq \varepsilon$.*

Here, X and $A \in \mathbb{R}^n$, so $X = (x_1, x_2, ..., x_n)$ and $A = (a_1, a_2, ..., a_n)$. Then $X \to A$ means:

$$\begin{cases} x_1 \to a_1 \\ x_2 \to a_2 \\ \vdots \\ x_n \to a_n \end{cases}$$

Note that in $\mathbb{R}$, f is continuous at $A \Longleftrightarrow \lim_{X \overset{>}{\to} A} f(X) = \lim_{X \overset{<}{\to} A} f(X) = f(A)$; there is only one path to approach A. But in $\mathbb{R}^n$, $X \to A$ means all coordinates of X approach those of A simultaneously and independently; there are infinitely many paths to approach A, making the definition non-trivial to apply (e.g., $\lim_{(x,y) \to (a,b)} f(X) \neq \lim_{x \to a} \lim_{y \to b} f(X)$ in general).

- f is said to be continuous on a subset $D \subset \mathbb{R}^n$ if it is continuous at every point of D.

Practical Techniques

- Applying the definition: Let $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Show f is continuous at $(0, 0)$. Let $M = (x, y)$, $\|\overrightarrow{OM}\| = \sqrt{x^2 + y^2}$. We have $|x| \leq \sqrt{x^2 + y^2} = \|\overrightarrow{OM}\|$ and $|y| \leq \sqrt{x^2 + y^2} = \|\overrightarrow{OM}\|$. Thus, $|f(x, y) - f(0, 0)| = \left|\dfrac{xy}{x^2 + y^2}\right| \leq \|\overrightarrow{OM}\|^2$, which tends to 0 as M approaches the origin. Hence, $\forall \varepsilon > 0; \exists \alpha = \sqrt{\varepsilon} > 0$ such that $\|\overrightarrow{OM}\| < \alpha \Longrightarrow |f(x, y) - f(0, 0)| \leq \varepsilon$, proving continuity at $(0, 0)$.
- Variable substitution using polar coordinates (for two variables).

Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r > 0$, $\theta \in [0, 2\pi]$.

Example 4.11. $\lim\limits_{(x,y)\longrightarrow(0,0)} \dfrac{x^3y}{x^2+y^2} = \lim\limits_{r\longrightarrow 0} \dfrac{r^4\cos\theta\sin\theta}{r^2} = 0$

Example 4.12. $\lim\limits_{(x,y)\longrightarrow(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{r\longrightarrow 0} \cos\theta\sin\theta$, *depend on θ. Thus the limit does not exist.*

- **Using Taylor expansions:** Consider the function $f(x,y) = \frac{x\sin(y)-y}{x^2+y^2}$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Near $y = 0$, we know $\sin(y) - y = \circ(y^2)$, i.e., there exists a function ε such that $\forall y \in V(0)$, $\sin(y) - y = y^2\varepsilon(y)$ with $\varepsilon(y) \to 0$ as $y \to 0$. Since $|xy| \leq \frac{1}{2}(x^2+y^2)$, we have:

$$|f(x,y)| = \left|\frac{xy^2\varepsilon(y)}{x^2+y^2}\right| \leq \frac{|y|\varepsilon(y)}{2} \underset{(x,y)\to(0,0)}{\longrightarrow} 0.$$

In this case, f is **extendable by continuity** at $(0,0)$.

- To show that a limit does not exist, it suffices to find two distinct paths yielding different limit values.

Example 4.13. *The limit* $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ *does not exist. Approaching along the $y = 0$ axis gives $f(x,0) = 1 \to 1$, while approaching along the $x = 0$ axis gives $f(0,y) = -1 \to -1$. Hence, the limit does not exist.*

Theorem 4.14. *Let $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$.*

- If f and g are continuous at $A \in \mathbb{R}^n$, then $f + g$ and $f \cdot g$ are continuous at A.
- If f is continuous at A and $f(A) \neq 0$, then $\frac{1}{f}$ is continuous at A.

4.5 Partial Derivatives

Definition 4.15. *Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and $A = (a_1, a_2, \ldots, a_n)$ be an interior point of D_f. Define:*

$$g_i : \mathbb{R} \to \mathbb{R}, \quad t \mapsto f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n).$$

The first-order partial derivative of f with respect to its i^{th} variable at A, denoted $\frac{\partial f}{\partial x_i}(A)$, is the derivative of g_i at a_i:

$$\frac{\partial f}{\partial x_i}(A) = \lim_{h\to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(A)}{h}.$$

For a function of two variables, the partial derivatives (if they exist) are:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Example 4.16. *Consider* $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. *The partial derivatives at* $(0,0)$ *are:*

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0, \quad \frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0.$$

Practical Method: Compute the partial derivative with respect to the i^{th} variable by fixing the others and differentiating the resulting single-variable function.

Example 4.17. *Let* $f(x,y) = y^x$ *defined on* $\mathbb{R} \times \mathbb{R}_+^*$. *Then:*

$$\frac{\partial f}{\partial x}(x,y) = (\ln y)y^x, \quad \frac{\partial f}{\partial y}(x,y) = xy^{x-1}.$$

Remark 4.18. *For single-variable functions, differentiability implies continuity. For multivariable functions, the existence of partial derivatives does not guarantee continuity.*

Example 4.19. *The function* $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ *is not continuous at* $(0,0)$, *as* $\lim_{(x,y)\to(0,0)} f(x,y)$ *depends on* θ *in polar coordinates. However, both partial derivatives at* $(0,0)$ *exist and equal zero.*

Definition 4.20. *Let* $U \subset \mathbb{R}^n$ *be open. A function* $f : U \to \mathbb{R}$ *is of* ***class*** $\mathcal{C}^1(U)$ *if all first-order partial derivatives exist and are continuous on* U.

Corollary 4.21. *If* $f : U \subset \mathbb{R}^n \to \mathbb{R}$ *is of class* $\mathcal{C}^1(U)$, *then* f *is continuous on* U.

4.6 Higher-Order Partial Derivatives

For a function $f(x,y)$, the second-order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Generally, the number of partial derivatives equals (number of variables)$^{\text{order}}$.

Definition 4.22. *A function* f *is of* ***class*** $\mathcal{C}^2$ *if all its first-order partial derivatives are of class* $\mathcal{C}^1$.

Theorem 4.23 (Schwarz's Theorem)**.** *Let* $U \subset \mathbb{R}^n$ *be open and* $f : U \to \mathbb{R}$ *be of class* $\mathcal{C}^2(U)$. *Then for all* $i \neq j$:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

A function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is of **class** $\mathcal{C}^k(U)$ if all its partial derivatives up to order k exist and are continuous.

4.7 Partial Derivatives of Composite Functions

Theorem 4.24. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be of class $\mathcal{C}^1(U)$ and $g : \mathbb{R} \to U$, $t \mapsto (x_1(t), \ldots, x_n(t))$, be of class $\mathcal{C}^1$. The composite function $f \circ g(t) = f(x_1(t), \ldots, x_n(t))$ is of class $\mathcal{C}^1$, and its derivative is:*

$$(f \circ g)'(t) = \sum_{i=1}^{n} x_i'(t) \frac{\partial f}{\partial x_i}(x_1(t), \ldots, x_n(t)).$$

Example 4.25. *Let $F(t) = f(\cos t, \sin t)$ with $f(x,y) = x^2 - y^2$. Then:*

$$F'(t) = -\sin t \cdot 2x(t) + \cos t \cdot (-2y(t)) = -4\cos t \sin t.$$

4.7.1 Change of Variables in $\mathbb{R}^2$

Let $f(x,y)$ be of class $\mathcal{C}^1$, and consider a change of variables $x = x(u,v)$, $y = y(u,v)$. The partial derivatives of $g(u,v) = f(x(u,v), y(u,v))$ are:

$$\begin{cases} \frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \\ \frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \end{cases}$$

Example 4.26. *For polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, simplify $y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$. Let $f(x,y) = g(r,\theta)$. Then:*

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = r\sin\theta\left(\frac{\partial g}{\partial r}\cos\theta - \frac{\partial g}{\partial \theta}\frac{\sin\theta}{r}\right) + r\cos\theta\left(\frac{\partial g}{\partial r}\sin\theta + \frac{\partial g}{\partial \theta}\frac{\cos\theta}{r}\right) = r\frac{\partial g}{\partial \theta}.$$

Solutions to $y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$ satisfy $\frac{\partial g}{\partial \theta} = 0$, implying g depends only on r.

4.7.2 Directional Derivative

Definition 4.27. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined near $A = (a_1, \ldots, a_n)$, and let $\mathbf{u} = (u_1, \ldots, u_n)$. The **directional derivative** of f at A in the direction $\mathbf{u}$ is:*

$$f'_{\mathbf{u}}(A) = \lim_{t \to 0} \frac{f(a_1 + tu_1, \ldots, a_n + tu_n) - f(A)}{t}.$$

For $n = 2$:

$$f'_{\mathbf{u}}(A) = \lim_{t \to 0} \frac{f(a_1 + tu_1, a_2 + tu_2) - f(a_1, a_2)}{t}.$$

Remark 4.28. *Partial derivatives are directional derivatives along the canonical basis vectors.*

Example 4.29. *For $f(x,y) = x^2 + xy$ and direction $\mathbf{u} = (\cos\theta, \sin\theta)$ at $(1,-1)$:*

$$f'_{\mathbf{u}}(1,-1) = \cos\theta + \sin\theta.$$

The maximum slope occurs at $\theta = \frac{\pi}{4}$.

4.7.3 Differential

Definition 4.30. *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$, and $A \in U$. f is **differentiable** at A if:*

$$\lim_{\|\mathbf{h}\|\to 0} \frac{f(A+\mathbf{h}) - f(A) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(A) h_i}{\|\mathbf{h}\|} = 0.$$

For $n = 2$:

$$\lim_{\|(h_1,h_2)\|\to 0} \frac{f(a_1+h_1, a_2+h_2) - f(a_1,a_2) - \left(\frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2\right)}{\|(h_1,h_2)\|} = 0.$$

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